Diffeomorphisms of discs

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Why study diffeomorphism groups?

Diff(M) acts on any space of differential-geometric data on M
e.g. vector fields, metrics, operators, ...
$\Rightarrow$ can use *Diff*(M) to probe such spaces

Theorem. [Hitchin '74]
(M, g) Riemannian spin manifold with $scal(g) > 0$ then
$\pi_0 \mathcal{R}^{scal>0}(M) \neq 0$ if $dim(M) \equiv 0, 1 \mod 8$
$\pi_1 \mathcal{R}^{scal>0}(M) \neq 0$ if $dim(M) \equiv 0, 7 \mod 8$.

Theorem. [Botvinnik–Ebert–R-W '17]
(M, g) Riemannian spin manifold, $scal(g) > 0$ and $dim(M) \geq 6$ then
$\pi_k(\mathcal{R}^{scal>0}(M)) \neq 0$ if $k + dim(M) \equiv 0, 1, 3, 7 \mod 8$.

Theorem. [Krannich–Kupers–R-W '20]
$\pi_3(\mathcal{R}^{sec>0}(\mathbb{HP}^2)) \otimes \mathbb{Q} \neq 0$. Similarly for $Ric > 0$, $scal > 0$.

Why study diffeomorphism groups?

$\mathit{Diff}(M) =$ structure group for smooth fibre bundles with fibre M
$\Rightarrow \mathit{BDiff}(M)$ classifies such bundles
$\Rightarrow H^*(\mathit{BDiff}(M)) =$ characteristic classes of such bundles

It is part of the mandate of algebraic topology to understand fibre bundles and their invariants.

It is rarely possible to obtain complete answers (none known for any compact manifold of dimension ≥ 4).

Results can be very surprising:

Theorem. [Furusawa–Tezuka–Yagita '88, Morita '87]
We have $H^i(\mathit{BDiff}^+(S^1 \times S^1); \mathbb{Q}) = 0$ for $i \not\equiv 1 \mod 4$ and

$$\dim_{\mathbb{Q}} H^{4n+1}(\mathit{BDiff}^+(S^1 \times S^1); \mathbb{Q}) = 1 + 2 \cdot \dim_{\mathbb{C}} \left\{ \substack{\text{cusp forms} \\ \text{of weight } 2n+2} \right\} \sim \frac{n}{3}.$$

Smoothing and discs

A scaling trick

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism

Consider $g_t(x) = \frac{f(t \cdot x) - (1-t^2) \cdot f(0)}{t}$:

$$g_1(x) = f(x)$$
$$g_t(x) = \frac{f(tx) - f(0)}{t} + t \cdot f(0) \underset{t \to 0}{\longrightarrow} D_0 f(x)$$

This deforms the topological group $Diff(\mathbb{R}^d)$ of all diffeomorphisms into the subgroup $GL_d(\mathbb{R})$ of linear diffeomorphisms.

The Gram–Schmidt process deforms $GL_d(\mathbb{R})$ into the subgroup $O(d)$ of orthogonal diffeomorphisms

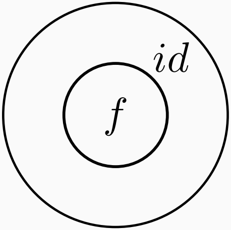
$$Diff(\mathbb{R}^d) \simeq O(d)$$

Another scaling trick

Let $f : D^d \to D^d$ be a *homeo*morphism which is the identity on the boundary

"Alexander trick"

Consider $g_t(x) = \begin{cases} x & |x| \geq t \\ t \cdot f(x/t) & |x| \leq t \end{cases}$



$$g_1(x) = f(x)$$
$$g_t(x) \xrightarrow[t \to 0]{} x$$

(If f is smooth then all g_t are smooth, but convergence is only C^0)

$$Homeo_\partial(D^d) \simeq *$$

Smoothing theory

M a topological d-manifold, maybe with smooth boundary ∂M

$$\mathcal{S}m(M) = \{ \text{ space of smooth structures on } M, \text{ fixed near } \partial M \ \}$$

Recording germs of smooth structure near each point gives a map

$$\mathcal{S}m(M) \longrightarrow \Gamma_\partial(\mathcal{S}m(TM) \to M)$$

(the space of sections of the bundle with fibre $\mathcal{S}m(T_m M) \cong \mathcal{S}m(\mathbb{R}^d)$)

Theorem. [Hirsch–Mazur '74, Kirby–Siebenmann '77]
For $d \neq 4$ this map is a homotopy equivalence.

$\mathit{Homeo}_\partial(M)$ acts on $\mathcal{S}m(M)$, giving

$$\mathcal{S}m(M) \cong \coprod_{[W]} \mathit{Homeo}_\partial(W)/\mathit{Diff}_\partial(W)$$

Similarly, $\mathcal{S}m(\mathbb{R}^d) \cong \mathit{Homeo}(\mathbb{R}^d)/\mathit{Diff}(\mathbb{R}^d)$

A consequence of smoothing theory

Write $Top(d) := Homeo(\mathbb{R}^d)$, so $\mathcal{S}m(\mathbb{R}^d) \simeq Top(d)/O(d)$.

Applied to D^d, $d \neq 4$, smoothing theory gives a map

$$Homeo_\partial(D^d)/Diff_\partial(D^d) \longrightarrow \Gamma_\partial(\mathcal{S}m(TD^d) \to D^d) = map_\partial(D^d, Top(d)/O(d))$$

which is a homotopy equivalence to the path components it hits.

The Alexander trick $Homeo_\partial(D^d) \simeq *$ implies

$$BDiff_\partial(D^d) \simeq \Omega^d_0 Top(d)/O(d),$$

or if you prefer

$$Diff_\partial(D^d) \simeq \Omega^{d+1} Top(d)/O(d).$$

$O(d)$ is "well understood" so $Diff_\partial(D^d)$ and $Top(d)$ are equidifficult.

But $Diff_\partial(D^d)$ is more approachable: can *use* smoothness.

What do we know?

The theorem of Farrell and Hsiang

The classical approach to studying $Diff_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $hAut_\partial(M)$ analysed by homotopy theory.
2. Comparison $hAut_\partial(M)/\widetilde{Diff}_\partial(M)$ with "block-diffeomorphisms" analysed by surgery theory.
3. Comparison $\widetilde{Diff}_\partial(M)/Diff_\partial(M)$ with diffeomorphisms analysed by pseudoisotopy theory (and hence K-theory), but only valid in the "pseudoisotopy stable range".
   [Igusa '84]: this is at least $min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$.
   [RW '17]: it is at most $d-2$.

Theorem. [Farrell–Hsiang '78]

$$\pi_*(BDiff_\partial(D^d)) \otimes \mathbb{Q} = \begin{cases} 0 & d \text{ even} \\ \mathbb{Q}[4] \oplus \mathbb{Q}[8] \oplus \mathbb{Q}[12] \oplus \cdots & d \text{ odd} \end{cases}$$

in the pseudoisotopy stable range for d (so certainly for $* \lesssim \frac{d}{3}$).

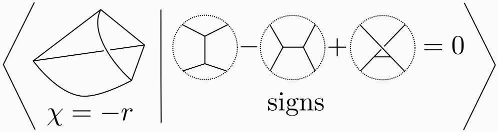
The theorems of Watanabe

Theorem. [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_{\partial}(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

where

$$\mathcal{A}_r^{odd} =$$



has $dim(\mathcal{A}_r^{odd}) = 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 9, \ldots$

Theorem. [Watanabe '18]
There is a surjection

$$\pi_r(BDiff_{\partial}(D^4)) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{even}$$

where $dim(\mathcal{A}_r^{even}) = 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$ (so $\pi_2(BDiff_{\partial}(D^4)) \neq 0$)

The theorem of Weiss

Closely related to the classical story is the fact that the stable map

$$O = \operatorname*{colim}_{d \to \infty} O(d) \longrightarrow Top = \operatorname*{colim}_{d \to \infty} Top(d)$$

is a $\mathbb{Q}$-equivalence, and hence

$$H^*(BTop; \mathbb{Q}) \cong H^*(BO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

In $H^*(BO(2n); \mathbb{Q})$ the usual definition of Pontrjagin classes shows

$$p_n = e^2 \text{ and } p_{n+i} = 0 \text{ for all } i > 0. \qquad (!)$$

Theorem. [Weiss '15]
For many n and $i \geq 0$ there are classes $w_{n,i} \in \pi_{4(n+i)}(BTop(2n))$ which pair nontrivially with p_{n+i} (i.e. (!) does not hold on $BTop(2n)$).

$\Rightarrow \pi_{2n-1+4i}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \neq 0$ for such n and i.

A pattern

A pattern

Inspired by Weiss' argument, Alexander Kupers and I have begun a programme to determine

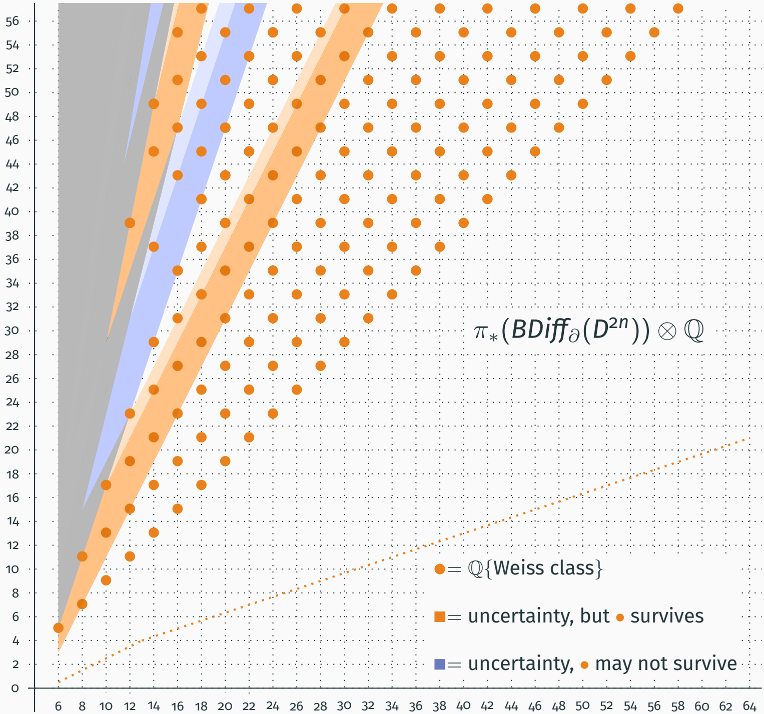
$$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$$

as completely as possible. The first installment just came out:

A. Kupers, O. R-W, *On diffeomorphisms of even-dimensional discs* (arXiv:2007.13884)

Here we

1. fully determine these groups in degrees $* \leq 4n - 10$,
2. determine them in higher degrees outside of certain "bands",
3. understand something about the structure of these bands.


$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$
= $\mathbb{Q}${Weiss class}
= uncertainty, but ● survives
= uncertainty, ● may not survive
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56
6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64

A pattern

Theorem. [Kupers–R-W]
Let $2n \geq 6$.

(i) If $d < 2n - 1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ vanishes, and
(ii) if $d \geq 2n - 1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ is

$$\begin{cases} \mathbb{Q} & \text{if } d \equiv 2n-1 \bmod 4 \text{ and } d \notin \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-1], \\ 0 & \text{if } d \not\equiv 2n-1 \bmod 4 \text{ and } d \notin \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-1], \\ ? & \text{otherwise.} \end{cases}$$

A pattern

Using $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

Reformulation (slightly stronger).
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

$$* \in \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-2].$$

Reflecting D^{2n} or $\mathbb{R}^{2n}$ induces compatible involutions on

$$\Omega_0^{2n+1}\frac{Top}{Top(2n)} \longrightarrow BDiff_\partial(D^{2n}) \simeq \Omega_0^{2n}\frac{Top(2n)}{O(2n)} \longrightarrow \Omega_0^{2n}\frac{Top}{O(2n)}.$$

We show this acts as -1 on

$$\pi_*(\Omega_0^{2n}\frac{Top}{O(2n)}) \otimes \mathbb{Q} = \mathbb{Q}[2n-1] \oplus \mathbb{Q}[2n+3] \oplus \mathbb{Q}[2n+7] \oplus \cdots$$

and acts on $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ as $(-1)^r$ in the rth band.

The orange/blue colours in the chart are the $+1/-1$ eigenspaces.

The first uncertainty

We also determine to some extent what happens in the first band shown in the chart: the groups $\pi_*(\Omega^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ in degrees $[4n-9, 4n-4]$ are calculated by a chain complex of the form

$$\mathbb{Q}^2 \longleftarrow \mathbb{Q}^4 \longleftarrow \mathbb{Q}^{10} \longleftarrow \mathbb{Q}^{21} \longleftarrow \mathbb{Q}^{15} \hookleftarrow \mathbb{Q}^3$$

We don't know the differentials, but it has Euler characteristic 1 so has some homology.

It lies in the $+1$-eigenspace, so injects into $\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$.

By analogy with Watanabe's theorem for D^4 one expects

$$dim\, \pi_{4n-6}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \geq 1$$

which is compatible with the above.

Remarks on the proof

Philosophy

As we have seen several times, many results in this flavour of geometric topology are *relative*: they describe the difference between

1. topological/smooth manifolds (smoothing)
2. homotopy equivalences/block diffeomorphisms (surgery)
3. block diffeomorphisms/diffeomorphisms (pseudoisotopy)

Weiss suggested a new kind of relativisation:

for M with $\partial M = S^{d-1}$ and $\frac{1}{2}\partial M := D^{d-1} \subset S^{d-1}$ he showed

$$\frac{\mathit{Diff}_\partial(M)}{\mathit{Diff}_\partial(D^d)} \simeq \mathit{Emb}^{\cong}_{1/2\partial}(M).$$

Under mild conditions on M such an embedding space can be analysed using the theory of embedding calculus.

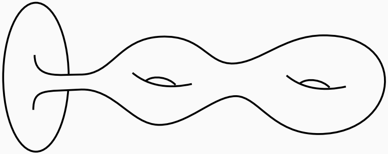
Strategy: find a manifold M for which one can understand $\mathit{Emb}^{\cong}_{1/2\partial}(M)$ and $\mathit{Diff}_\partial(M)$, then deduce things about $\mathit{Diff}_\partial(D^d)$.

The manifold $W_{g,1}$

A good choice is

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$

especially for "arbitrarily large" g.



Theorem. [Madsen–Weiss '07 $2n = 2$, Galatius–R-W '14 $2n \geq 4$]

$$\lim_{g \to \infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$$

Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

Theorem. [Berglund–Madsen '20 $2n \geq 6$]

$$\lim_{g \to \infty} H^*(\widetilde{BDiff}_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\tilde{\kappa}_c^\xi \,|\, (c, \xi) \in \mathcal{B}']$$

$$\lim_{g \to \infty} H^*(BhAut_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\tilde{\kappa}_c^\xi \,|\, (c, \xi) \in \mathcal{B}'']$$

Here $\mathcal{B}'$ and $\mathcal{B}''$ are much more complicated than $\mathcal{B}$, and we will probably never be able to enumerate them completely.

Difficulties I

The embedding calculus machine calculates $\pi_*(Emb^{\cong}_{1/2\partial}(W_{g,1}))$.

A "machine" in the sense of algebraic topology ($=$ many spectral sequences) is not an algorithm, and at each step there is no guarantee of being able to proceed.

The main issue is to determine/estimate the characters of

$$H^i(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q}) \quad \text{and} \quad \pi_i(Conf(k, W_{g,1})) \otimes \mathbb{Q}$$

as representations of $\mathfrak{S}_k \times \pi_0(\mathrm{Diff}_\partial(W_{g,1}))$.

The first can be done easily using a theorem of Petersen '20.

The second is much more complicated, but possible.

We are able to completely determine the layers of the embedding calculus tower, but unfortunately not their interaction.

Nonetheless this lets us prove that $\pi_*(Emb^{\cong,fr}_{1/2\partial}(W_{g,1}))\otimes\mathbb{Q}$ is supported in degrees $* \in \cup_{r\geq 1}[r(n-2)-1, r(n-1)]$. This is the darkly shaded region in the chart.

Difficulties II

While we have very good understanding of $H^*(BDiff_\partial(W_{g,1}); \mathbb{Q})$, the strategy requires $\pi_*(BDiff_\partial(W_{g,1})) \otimes \mathbb{Q}$.

$\pi_1(BDiff_\partial(W_{g,1})) \sim Sp_{2g}(\mathbb{Z})$ (n odd) or $O_{g,g}(\mathbb{Z})$ (n even)
$\Rightarrow$ wildly complicated and not nilpotent: cannot expect to calculate the rational homotopy of $BDiff_\partial(W_{g,1})$ from cohomology.

Can pass to the Torelli subgroup

$$Tor_\partial(W_{g,1}) := ker(Diff_\partial(W_{g,1}) \to Aut(H_n(W_{g,1}; \mathbb{Z})))$$

to eliminate the arithmetic group, but this changes the cohomology.

In two companion papers we prove that $BTor_\partial(W_{g,1})$ is nilpotent, and determine $H^*(BTor_\partial(W_{g,1}); \mathbb{Q})$ as $g \to \infty$.

Adapting this to framed case, we find

$$\pi_*(BTor^{fr}_\partial(W_{g,1})) \otimes \mathbb{Q} = \left(\bigoplus_{i \in \mathbb{Z}} \mathbb{Q}[2n-1+4i]\right) \text{ “}\oplus\text{” } \begin{pmatrix} \text{something supported in} \\ * \in \bigcup_{r \geq 0} [r(n-1)+1, rn-2] \end{pmatrix}$$

The second piece is the lightly shaded region in the chart.

Optimism

Divergent embedding calculus

Can apply embedding calculus to diffeomorphisms, considered as codimension 0 embeddings. It need not converge and in fact does not converge: by work of Fresse, Turchin, and Willwacher '17 it predicts (modulo a subtlety) that $\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ should be

$$\left(\bigoplus_{i>0} \mathbb{Q}[2n-4i]\right) \oplus \mathbb{Q}[4n-6] \oplus \mathbb{Q}[8n-10] \oplus \mathbb{Q}[10n-15] \oplus \cdots$$

so misses the Weiss classes and starts with some spurious classes. But apart from this it has classes supported in our bands, and here is given precisely by Kontsevich's graph complex GC^2_{2n}.

Could there be a rational fibration

$$BDiff_\partial(D^{2n}) \longrightarrow BT_\infty Diff_\partial(D^{2n}) \longrightarrow \Omega^{\infty+2n} L(\mathbb{Z})?$$

Evidence

Could there be a rational fibration

$$BDiff_\partial(D^{2n}) \longrightarrow BT_\infty Diff_\partial(D^{2n}) \longrightarrow \Omega^{\infty+2n} L(\mathbb{Z})?$$

Evidence.
It is consistent with everything we know, and would explain Watanabe's and Weiss' results.

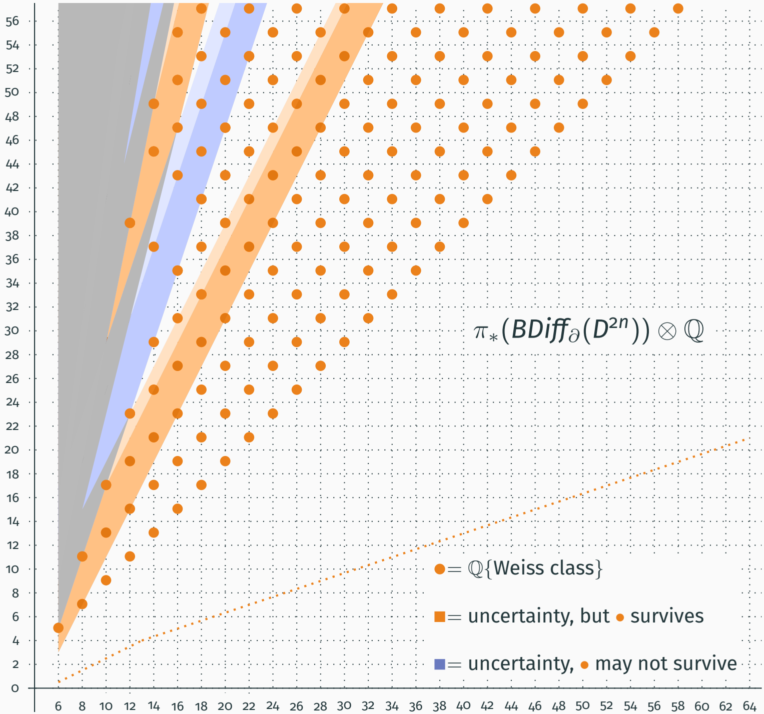
Evidence. [Knudsen–Kupers '20]
If $d \geq 6$, M^d 2-connected, $\partial M = S^{d-1}$ then

$$hofib(BDiff_\partial(M) \to BT_\infty Diff_\partial(M))$$

is independent of M.

Evidence. [Prigge '20]
The family signature theorem does not hold on $BT_2 Diff_\partial(M)$.

Questions?



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