Higher codimension MCF and the search for stable structures

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Joint work with Bill Minicozzi.

- Much less is known than for hypersurfaces.
- Some of the new ideas from function theory on manifolds.

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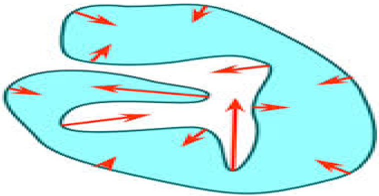
Mean curvature flow

Mean curvature flow: n-dimensional submanifolds $M_t \subset \mathbf{R}^N$ evolving by

$$\frac{\partial x}{\partial t} = -\mathbf{H}\,.$$

$\mathbf{H}$ is the mean curvature vector of M_t at x.



MCF and the heat equation

If $M_t \subset \mathbf{R}^N$ is a MCF, then the position vector $\mathbf{x}$ satisfies

$$\partial_t \mathbf{x} = \Delta_{M_t} \mathbf{x} \, .$$

This is a heat equation with a time-varying metric.

It is nonlinear since Δ_{M_t} depends on M_t.

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Producing new flows

- Translating in space and time.
- Parabolic scaling:

$$\tilde{M}_t = c\, M_{c^{-2}\, t}\,.$$

- Limits of these, for instance, as $c \to \infty$.

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Shrinkers

- Blow ups around a fixed point in space-time give a shrinker (**Huisken, Ilmanen, White**).
- Shrinker evolves by rescaling $M_t = \sqrt{-t}\, M_{-1}$.
- Shrinkers (M_{-1}) satisfies the elliptic eq.:

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- Round cylinders: $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k} \subset \mathbf{R}^N$.
- Minimal submanifolds of spheres:

  If $\Sigma^n \subset \partial B_{\sqrt{2n}} \subset \mathbf{R}^N$ is minimal in $\partial B_{\sqrt{2n}}$, then Σ^n is a shrinker.

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One consequence:

- Generic shrinkers have:

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- $u(x, y)$ is *harmonic* if $\Delta u = 0$. (Laplace eqn).

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The space $\mathcal{H}_d$

$\mathcal{H}_d(M) =$ harmonic functions of polynomial growth at most d:

$$\Delta u = 0 \textit{ and for some } p \in M \textit{ and constant } C_u$$

$$\sup_{B_R(p)} |u| \leq C_u \, (1+R)^d \textit{ for all } R\,.$$

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Yau's polynomial growth conjecture

In 1974, S.T. Yau conjectured the stronger Liouville property:

Yau conjectured: If $\text{Ric}_M \geq 0$, then $\dim \mathcal{H}_d(M) < \infty$ for each d.

CM97: Yau's conjecture holds: $\dim \mathcal{H}_d(M^n) \leq C\, d^{n-1}$.

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Finite dimensionality

CM97 proved dim $\mathcal{H}_d(M) < \infty$ under weaker assumptions:

(1) A volume doubling bound.

(2) A scale-invariant Poincaré inequality.

- (1) and (2) hold for Ric ≥ 0.
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Results prior to **CM97**

Earlier work related to Yau's conjecture by Avellaneda-Lin, Cheeger-Colding-Minicozzi, Donnelly-Fefferman, Kasue, Kazdan, Li-Tam, Moser-Struwe and many others.

Later applications of ideas of **CM97**

- Kleiner (2010) used, in part, ideas from **CM97** in his new proof of Gromov's classification of groups of polynomial growth.
- Shalom-Tao (2010) used this for a finitary version of Gromov's theorem.

The space $\mathcal{P}_d$

$u \in \mathcal{P}_d(M)$ if $\partial_t u = \Delta u$ and for some constant C_u

$$\sup_{B_R(p)\times[-R^2,0]} |u| \leq C_u \, (1+R)^d \; \text{ for all } R\,.$$

Calle's 2006 thesis.

Caloric polynomials

$\mathcal{P}_d(\mathbf{R}^n) =$ caloric polynomials.

Generalizations of Hermite polynomials.

dim $\mathcal{P}_d(\mathbf{R}^n) \approx d^n$.

Bounding dim $\mathcal{P}_d$

CM, 2019: If $\mathrm{Vol}(B_R(p)) \leq C\,(1+R)^{d_V}$ for all $R > 0$, then

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Related results

Recently, F.H. Lin and Q.S. Zhang, adapted the methods of **CM97** to get the bound d^{n+1}.

$\mathcal{P}_d$ and MCF

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- $\dim \mathcal{P}_1$ bounds the codimension of the MCF.
- The Laplacian is now time-varying.

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Gaussian area

- *Gaussian area F* of Σ:

$$F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}} .$$

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Entropy:

- *Entropy* $\lambda = \sup F$ over all translations and dilations

$$\lambda(\Sigma) = \sup_{c,x_0} F(c\,\Sigma + x_0)\,.$$

- **CM12** using **Huisken**: $\lambda(M_t) \downarrow$ for a MCF.
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- **Stone**: F, and thus λ, of spheres $\downarrow$ in dim

$$\sqrt{2} \leq \lambda(\mathbf{S^n}) < \lambda(\mathbf{S^{n-1}}) < \cdots < \lambda(\mathbf{S^1}) = \sqrt{\frac{\mathbf{2}\,\pi}{\mathrm{e}}} \approx \mathbf{1.52}\,.$$

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- CM-Ilmanen-White '13: $\mathbf{S^n}$ least λ for closed shrinker in $\mathbf{R}^{n+1}$.
- Bernstein-Wang '17: In $\mathbf{R}^3$ spheres and cylinders are the lowest λ shrinkers.
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Ancient flows

- $M_t \subset \mathbf{R}^N$ MCF.
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Finite dimensionality of $\mathcal{P}_d$

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When $d = 1$, we get a bound for the codimension:

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Shrinkers and drift Laplacian

Let $\Sigma^n \subset \mathbf{R}^N$ be a shrinker.

- Gaussian L^2 inner product $\int_\Sigma u\, v\, e^{-\frac{|x|^2}{4}}$.
- Drift Laplacian (Ornstein-Uhlenbeck) $\mathcal{L} = \Delta - \frac{1}{2}\nabla_{x^T}$ is Gaussian self-adjoint.
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Shrinking curves are planar

$\gamma^1 \subset \mathbf{R}^N$ a shrinking curve.

- Coordinates x_i satisfy ODE $\mathcal{L}\, x_i = -\frac{1}{2}\, x_i$.
- 2nd order ODE $\to$ 2-dim'l space of solutions.
- Thus, only two linearly independent x_i's $\to \gamma$ in a plane.

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Sharp polynomial bounds for eigenfunctions

Let $\Sigma^n \subset \mathbf{R}^N$ be a shrinker.

CM19: L^2 eigenfunctions grow polynomially with degree twice the eigenvalue.

Precisely: If $\mathcal{L}\, u = -\mu\, u$ and $\|u\|_{L^2} < \infty$, then

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Rigidity for cylinders, I

CM19: Cylinders are rigid in a very strong sense:

> *Any shrinker, even in a large dimensional space, that is sufficiently close to a cylinder on a large enough, but compact, set is itself a cylinder.*

Rigidity for cylinders, II: Two steps

1. Shrinker close to cylinder must be a hypersurface.
2. By **CIM15**, cylinders are rigid as hypersurfaces.

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- White (2000) constructed *ancient ovals* asymptotic to cylinders at $-\infty$ (cf. Haslhofer-Hershkovits).

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There is no analog of this result for minimal surfaces even in $\mathbf{R}^4$; cf. results of Micallef.

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