Higher codimension MCF and the search for stable structures

Tobias Holck Colding

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Overview

Joint work with Bill Minicozzi.

- Much less is known than for hypersurfaces.

- Some of the new ideas from function theory on manifolds.
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Mean curvature flow: $n$-dimensional submanifolds $M_t \subset \mathbb{R}^N$ evolving by

$$\frac{\partial x}{\partial t} = -H.$$ 

$H$ is the mean curvature vector of $M_t$ at $x$. 

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This is a heat equation with a time-varying metric.

It is nonlinear since $\Delta_{M_t}$ depends on $M_t$. 
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Producing new flows

- Translating in space and time.

- Parabolic scaling:
  \[ \tilde{M}_t = c M_{c^{-2} t} . \]

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Shrinkers

- Blow ups around a fixed point in space-time give a shrinker (Huisken, Ilmanen, White).

- Shrinker evolves by rescaling $M_t = \sqrt{-t} M_{-1}$.

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- Minimal submanifolds of spheres:
  
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Caloric and Harmonic functions

- $u(x, y, t)$ is *caloric* if $u_t = \Delta u$. (Heat eqn).

- $u(x, y)$ is *harmonic* if $\Delta u = 0$. (Laplace eqn).
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Classical Liouville theorem: A bounded harmonic function on all of $\mathbb{R}^n$ must be constant.

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The space $\mathcal{H}_d$

$\mathcal{H}_d(M) = \text{harmonic functions of polynomial growth at most } d:\$

$$\Delta u = 0 \text{ and for some } p \in M \text{ and constant } C_u$$

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Harmonic polynomials

- $\mathcal{H}_d(\mathbb{R}^n)$ consists of harmonic polynomials of degree $d$.

- In particular, $\mathcal{H}_d(\mathbb{R}^n)$ is finite dimensional for each $d$. 
Harmonic polynomials

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In 1974, S.T. Yau conjectured the stronger Liouville property:

**Yau conjectured**: If $\text{Ric}_M \geq 0$, then $\dim \mathcal{H}_d(M) < \infty$ for each $d$.

**CM97**: Yau’s conjecture holds: $\dim \mathcal{H}_d(M^n) \leq C d^{n-1}$.

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CM97 proved \( \dim \mathcal{H}_d(M) < \infty \) under weaker assumptions:

1. A volume doubling bound.


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Earlier work related to Yau’s conjecture by Avellaneda-Lin, Cheeger-Colding-Minicozzi, Donnelly-Fefferman, Kasue, Kazdan, Li-Tam, Moser-Struwe and many others.

Shalom-Tao (2010) used this for a finitary version of Gromov’s theorem.
The space $\mathcal{P}_d$

\[ u \in \mathcal{P}_d(M) \text{ if } \partial_t u = \Delta u \text{ and for some constant } C_u \]

\[
\sup_{B_R(p) \times [-R^2,0]} |u| \leq C_u \left(1 + R\right)^d \text{ for all } R.
\]

Calle’s 2006 thesis.
Caloric polynomials

$\mathcal{P}_d(\mathbb{R}^n) =$ caloric polynomials.

Generalizations of Hermite polynomials.

$\dim \mathcal{P}_d(\mathbb{R}^n) \approx d^n.$
**CM, 2019:** If $\text{Vol}(B_R(p)) \leq C (1 + R)^{d_V}$ for all $R > 0$, then
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\dim \mathcal{P}_{2k}(M) \leq (k + 1) \dim \mathcal{H}_{2k}(M).
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Recently, F.H. Lin and Q.S. Zhang, adapted the methods of CM97 to get the bound $d^{n+1}$. 
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$\dim \mathcal{P}_1$ bounds the codimension of the MCF.

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Gaussian area $F$ of $\Sigma$:

$$F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x|^2}{4}}.$$

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Entropy $\lambda = \sup F$ over all translations and dilations

$$\lambda(\Sigma) = \sup_{c,x_0} F(c\Sigma + x_0).$$

- CM12 using Huisken: $\lambda(M_t) \downarrow$ for a MCF.

- CM12: If $\Sigma$ is a shrinker, then $\lambda(\Sigma) = F(\Sigma)$. 
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\textbf{Stone: }$F$, and thus $\lambda$, of spheres $\downarrow$ in dim

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\sqrt{2} \leq \lambda(S^n) < \lambda(S^{n-1}) < \cdots < \lambda(S^1) = \sqrt{\frac{2\pi}{e}} \approx 1.52.
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\[\lambda(\Sigma \times \mathbb{R}) = \lambda(\Sigma).\]
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\( \lambda(\Sigma \times \mathbb{R}) = \lambda(\Sigma) \).
CM-Ilmanen-White ’13: $\mathbb{S}^n$ least $\lambda$ for closed shrinker in $\mathbb{R}^{n+1}$.

Bernstein-Wang ’17: In $\mathbb{R}^3$ spheres and cylinders are the lowest $\lambda$ shrinkers.

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Ancient flows

- $M_t \subset \mathbb{R}^N$ MCF.

- $\lambda(M_t) \leq \lambda$ at the initial time $= \lambda_0$.

- Any blow-up limit gives an ancient flow (exists for all prior times) with $\lambda \leq \lambda_0$. 
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CM19: If $M^n_t \subset \mathbb{R}^N$ is an ancient MCF with $\lambda(M_t) \leq \lambda_0$, then
\[ \dim \mathcal{P}_d \leq C_n \lambda_0 d^n . \]

Dependence on $d$ is sharp on $\mathbb{R}^n$. 
Finite dimensionality of $\mathcal{P}_d$

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Dependence on $d$ is sharp on $\mathbb{R}^n$. 
When $d = 1$, we get a bound for the codimension:

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Let $\Sigma^n \subset \mathbb{R}^N$ be a shrinker.

- Gaussian $L^2$ inner product $\int_{\Sigma} u v e^{-\frac{|x|^2}{4}}$.

- Drift Laplacian (Ornstein-Uhlenbeck) $\mathcal{L} = \Delta - \frac{1}{2} \nabla x^T$ is Gaussian self-adjoint.

- Coordinates are eigenfunctions $\mathcal{L} x_i = -\frac{1}{2} x_i$. 
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Shrinking curves are planar

$\gamma^1 \subset \mathbb{R}^N$ a shrinking curve.

- Coordinates $x_i$ satisfy ODE $\mathcal{L} x_i = -\frac{1}{2} x_i$.
- 2nd order ODE $\rightarrow$ 2-dim’l space of solutions.
- Thus, only two linearly independent $x_i$’s $\rightarrow \gamma$ in a plane.
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**CM19**: $L^2$ eigenfunctions grow polynomially with degree twice the eigenvalue.

Precisely: If $\mathcal{L} u = -\mu u$ and $\|u\|_{L^2} < \infty$, then

$$u^2(x) \leq C \|u\|_{L^2}^2 (4 + |x|^2)^2 \mu.$$ 

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When $\Sigma$ is minimal in a sphere, follows from Cheng-Li-Yau ’84.
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CM19: Cylinders are rigid in a very strong sense:

Any shrinker, even in a large dimensional space, that is sufficiently close to a cylinder on a large enough, but compact, set is itself a cylinder.
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We will call the shrinker a tangent flow at $-\infty$. 
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Examples

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$$\lambda(\Sigma) \leq C (1 + \gamma).$$

There is no analog of this result for minimal surfaces even in $\mathbb{R}^4$; cf. results of Micallef.
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