FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

**Bachelor Thesis in Mathematics. Department of Mathematical Sciences, University of Copenhagen**
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Mads Christian Hansen

# Towards Seifert Fibrations

Construction by manifolds and circle bundles

Advisor: Richard A. Hepworth

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**Abstract**

The purpose of the present paper is an investigation of Seifert fibered manifolds. These are constructed from scratch through definitions and deductions of the fundamental objects: manifolds and fibre-bundles. The main results are (1) The base space of a Seifert manifold can be given the structure of a smooth surface where the projection map is smooth thus obtaining a form analogous to a generalized smooth circle bundle. (2) Given a base surface with finitely many points marked, one can construct a Seifert fibered manifold whose space of fibres is precisely this surface, and whose multiple fibres produce the prescribed labelled points. The notation $M(g;\alpha_1/\beta_1,\ldots,\alpha_k/\beta/k)$ for a Seifert manifold is introduced. The theory is exemplified through the Lens spaces.

**Resumé**

Formålet med nærværende skrivelse er en undersøgelse af Seifert fibrerede mangfoldigheder. Disse opbygges fra grunden gennem definitioner og udledninger om de grundlæggende objekter: mangfoldigheder og fiberbundter. Hovedresultaterne er: (1) Baserummet for en Seifert mangfoldighed kan udstyres med en struktur for en glat flade, hvor projektionsafbildningen er glat, hvorved man opnår en fremstilling som er analogt med et generaliseret glat cirkelbundt. (2) Givet en baseflade med endeligt mange punkter markeret, kan man konstruere en Seifert fibreret mangfoldighed hvis rum af fibre præcis er denne flade, og hvis multiple fibre giver de foreskrevne afmærkede punkter. Notationen $M(g;\alpha_1/\beta_1,\ldots,\alpha_k/\beta/k)$ for en Seifert mangfoldighed introduceres. Teorien er eksemplificeret gennem Linserummene.

# Preface

Seifert manifolds, or Seifert fibered spaces, are a special class of manifolds consisting of spaces which are among the best understood 3-manifolds, that is spaces which locally resembles regular Euclidean three-dimensional space. A Seifert manifold consists of disjoint simple closed curves called circles or fibres, arranged in a specific way. The purpose of looking at these spaces originates from the classification problem of 3-manifolds, in which the famous Poincaré conjecture belongs. Instead of investigating a complete system of topological invariants of 3-dimansional manifolds, one could search for a system of invariants for fiber preserving maps of fibered 3-manifolds. This was carried out in the 1930's mainly by Herbert Seifert in his dissertation 'Topologie 3-dimensionaler gefaserter Räume' from 1933 [Sei33] where he explicitly classified all Seifert manifolds [Sei80]. It turns out that most 'small' 3-manifolds and all spherical manifolds are Seifert fibered spaces, and they account for all compact oriented manifolds in 6 of the 8 Thurston geometries of the geometrization conjecture [Sco83, p. 403], [Mil04]. This paper will focus on the creation of these Seifert fibered spaces.

Section 1 covers abstract manifolds. These are given a smooth structure, which allows us to define smooth maps and diffeomorphisms between such spaces, and we show a number of important properties needed for the construction of Seifert manifolds. Thereafter we focus on how to construct compact connected 2-manifolds, or surfaces, by polygonal regions. In particular the $g$-fold torus is considered, and its fundamental group is computed. We end the section by considering group actions on manifolds and show that, under certain conditions, this is again a manifold. One such object is the Lens space $L_{p/q}$ which we introduce to illustrate the theory and show is a 3-manifold.

The theme of section 2 is fibre bundles. We encounter the Möbius strip as a first example and show that covering spaces are fibre bundles with discrete fibres, arming us with a number of new examples. Then we focus on circle bundles, which play a prominent role in the theory of Seifert fibered spaces. We give a couple of examples including the Klein Bottle and the historically important Hopf fibration.

Section 3 gives an introduction to Seifert manifolds. We show how to construct the model Seifert fibration, functioning as the local model and work out a couple of its important properties. Then we go on to define Seifert manifolds emphasizing its relationship with smooth circle bundles and work out a couple of results on central concepts such as multiplicities and embeddings of model Seifert fibrations. This is followed by an investigation of what happens when fibres are projected to a base space. We show that this space can be given the structure of a smooth surface such that the projection map is smooth.

Finally section 4 focusses on how to create new Seifert fibered manifolds from given base surfaces. We show that given such a surface with finitely many multiple fibres marked, there exists a Seifert fibered manifold whose space of fibres is precisely the given surface, and work out how to construct such a Seifert manifold explicitly. This allows us to introduce the notation $M(g, \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$ for a Seifert fibered space. We define the notion of isomorphic Seifert manifolds and give a summary of the proof that every orientable Seifert manifold is isomorphic to one of the models $M(g, \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$, including a sufficient and necessary condition for this to happen, thus giving a complete classification of Seifert fiberings. The theory developed is then exemplified for the Lens spaces.

The material is aimed to be self contained, however it is recommended that the reader is familiar with point set topology, abstract algebra, basic complex analysis and beginning algebraic topology. Each section as well as subsection will contain explicit information of where the material is drawn from. Where such description is not given, the results are due to the author.

I would like to express my wholehearted gratitude to my advisor, without whom this project would have never gotten its present depth and form, let alone dared being commenced in the first place. Thanks for keeping the spirits high, always taking the time to listen to my questions and for showing an inspiring approach to mathematics in general. I owe a great deal of appreciation to my fellow students Birger Brietzke, Jóhan V. Gunnarsson and Sune Jakobsen for having made it a joy to study mathematics over the last three years, and to professor Ian Stewart for giving me 'Math Hysteria'.

# Contents

# 1 Manifolds

In this section we introduce the concept of manifolds which, in the simplest terms, are spaces that locally look like some Euclidean space $\mathbb{R}^m$. We furthermore impose a smooth structure on these spaces which allows one to apply calculus thereon thus making the notion of diffeomorphisms well defined. Definitions are inspired from Schlichtkrull [Sch07], Madsen [MT97] and Milnor [Mil65]. In the following let $M$ be a Hausdorff topological space, and let $m \in \mathbb{N}$ be fixed.

## 1.1 Charts, atlases and definition

**Definition 1.** An $m$-dimensional **smooth atlas** on $M$ is a collection of triples $\mathcal{A} = \{(V_i, U_i, \sigma_i)\}_{i \in I}$ where $(V_i)_{i\in I}$ is a collection of open sets $V_i$ in $M$ such that $M = \bigcup_{i\in I} V_i$ and $(U_i)_{i\in I}$ is a collection of open sets in $\mathbb{R}^m$ and furthermore $\sigma_i : V_i \to U_i = \sigma_i(V_i)$ is a collection of homeomorphisms, called **charts**, with the property of smooth transitions on overlaps: For each pair $i, j \in I$ the map $\sigma_j \circ \sigma_i^{-1} : \sigma_i(V_i \cap V_j) \to \sigma_j(V_i \cap V_j) \subset \mathbb{R}^m$ is smooth. Two smooth atlases of $M$ are said to be **compatible** if their union is again an atlas[1]. Compatibility is an equivalence relation and an equivalence class is called a **smooth structure** on $M$.

**Definition 2.** An abstract **manifold** $M$ of dimension $m$ is a Hausdorff topological space equipped with a smooth structure. In the following we simply denote these as $m$-manifolds.

**Lemma 1.** *A given $m$-dimensional smooth atlas can be replaced by one in which all the $U_i$ are $B(0,\epsilon)$ for some $\epsilon > 0$ or $\mathbb{R}^m$. The new atlas is compatible with the old.*

*Proof.* Let $x \in M$ be given. By definition of a smooth atlas on $M$ we can find $V_{i'} \in (V_i)_{i\in I}$ such that $x \in V_{i'}$. But then this is homeomorphic to $U_{i'} \subset \mathbb{R}^m$ by $\sigma_{i'}$. Since $U_{i'}$ is open we can find an $\epsilon > 0$ such that $K = B_d(\sigma_{i'}(x), \epsilon) \subset U_{i'}$. The restriction $\sigma_{i'}^{-1}|_K$ of the homeomorphism $\sigma_{i'}^{-1}$ to this ball is still a homeomorphism [Mun00, 18.4 (d)]. Furthermore $N_{i'} = \sigma_{i'}^{-1}|_K(K) \subset V_{i'}$ is a neighborhood of $x$ in $M$. Therefore we can define $\phi_{i'} : N_{i'} \to B(0,\epsilon)$ by $\phi_{i'}(t) = (\sigma_{i'}^{-1}|_K)^{-1}(t) - \sigma_{i'}(x)$ which is then a homeomorphism from a neighborhood of $x$ to $B(0,\epsilon)$ as required and clearly these neighborhoods constitute $M$.

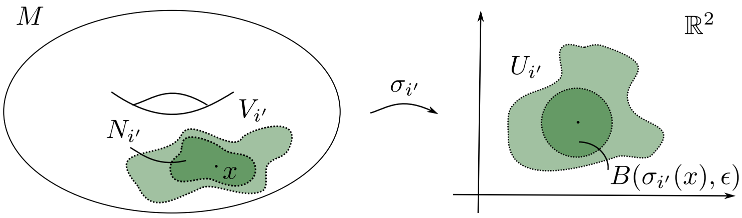



Figure 1: The chart $\sigma_i'$ carrying neighborhoods of $M$ to neighborhoods of $\mathbb{R}^2$.

We must show that the collection $(\phi_i)_{i\in I}$ satisfies the property of smooth transition on overlaps. The map $\phi_j \circ \phi_i^{-1} : \phi_i(N_i \cap N_j) \to \phi_j(N_i \cap N_j)$ is given by

$$\phi_j(\phi_i^{-1}(t)) = \phi_j\big(\sigma_i^{-1}|_K(t + \sigma_i(x))\big) = (\sigma_j^{-1}|_K)^{-1}(\sigma_i^{-1}|_K(t + \sigma_i(x))) - \sigma_i(x) \tag{1}$$

$$= \sigma_j|_{N_j}(\sigma_i^{-1}|_K(t + \sigma_i(x))) - \sigma_j(x). \tag{2}$$

Since we have $\phi_i(N_i \cap N_j) + \sigma_i(x) = \sigma_i|_{N_i}(N_i \cap N_j) - \sigma_i(x) + \sigma_i(x) \subset \sigma_i(N_i \cap N_j) \subset \sigma_i(V_i \cap V_j)$ and $\sigma_j \circ \sigma_i^{-1}$ is smooth on this set by definition, the restriction to $\phi_i(N_i \cap N_j) + \sigma_i(x)$ must be smooth as well and addition of the constant in (2) does not alter this hence $\phi_j \circ \phi_i^{-1}$ satisfies the desired property. Compatibility follows by the same reasoning by noting that given some $i, j \in I^2$ the map $\sigma_j \circ \phi_i^{-1} : \phi_i(N_i \cap V_j) \to \sigma_j(N_i \cap V_j)$ is given by $\sigma_j(\sigma_i^{-1}|_K(t + \sigma_i(x)))$ and hence smooth. We conclude that these atlases defines the same smooth structure on M as required.

To see that this also holds for $\mathbb{R}^m$, define the smooth homeomorphism $h : B(0,\epsilon) \to \mathbb{R}^m$ by $h(t) = \frac{t}{\epsilon - \|t\|^2}$. Then the map $\varphi_{i'} : N_{i'} \to \mathbb{R}^m$ given by $\varphi_{i'}(t) = h \circ \phi_{i'}(t)$ is the required homeomorphism. To see that the collection $(\varphi_i)_{i\in I}$ satisfies the property of transition on overlaps, it is sufficient to note that compositions of smooth maps preserves smoothness and we have $\varphi_j \circ \varphi_i^{-1} = h \circ \phi_j \circ \phi_i^{-1} \circ h^{-1}$ which by the first part is seen to be smooth on $\varphi_i(N_i \cap N_j) = h \circ \phi_i(N_i \cap N_j)$ as required. □

[1] Equivalently every chart from one atlas has smooth transition on its overlap with every chart from the other.

**Example 1.** *For $m \in \mathbb{N}$ the $m$-sphere $S^m$ is a compact $m$-manifold.* To see this note that by definition $S^m = \{\mathbf{x} \in \mathbb{R}^{m+1} \mid \|\mathbf{x}\| = 1\}$ which is a closed and bounded subset of a metric space and therefore Hausdorff and compact [Mun00, 27.3]. Let $n = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$ and $s = (0, \dots, 0, -1)$ be the 'north' and 'south' pole of $S^m$ respectively. Define the **stereographic projection** $f : (S^m - n) \to \mathbb{R}^m$ given by

$$f(\mathbf{x}) = f(x_1, \dots, x_{m+1}) = \frac{1}{1 - x_{m+1}}(x_1, \dots, x_m) \tag{3}$$

which is then clearly well-defined and continuous. The mapping $f^{-1} : \mathbb{R}^m \to (S^m - n)$ given by

$$f^{-1}(\mathbf{y}) = f^{-1}(y_1, \dots, y_m) = \Big(\frac{2}{1 + \|\mathbf{y}\|^2} y_1, \dots, \frac{2}{1 + \|\mathbf{y}\|^2} y_m, 1 - \frac{2}{1 + \|\mathbf{y}\|^2}\Big) \tag{4}$$

is also continuous and it is the inverse of $f$. Therefore $(S^m - n)$ is homeomorphic to $\mathbb{R}^m$. Similarly we can define the projection from the 'south' pole $g : (S^m - s) \to \mathbb{R}^m$ given by $g(\mathbf{x}) = g(x_1, \dots, x_{m+1}) = \frac{1}{1+x_{m+1}}(x_1, \dots, x_m)$ with inverse $g^{-1} : \mathbb{R}^m \to (S^m - s)$ given by $g^{-1}(\mathbf{y}) = g(y_1, \dots, y_m) = (\frac{2}{1+\|\mathbf{y}\|^2} y_1, \dots, \frac{2}{1+\|\mathbf{y}\|^2} - 1)$ and we conclude that $(S^m - s)$ is homeomorphic to $\mathbb{R}^m$ as well. We now have the smooth atlas $\mathcal{A} = \{((S^m - n), \mathbb{R}^m, f), ((S^m - s), \mathbb{R}^m, g)\}$. The transition map is $g \circ f^{-1} : \mathbb{R}^m - \{0\} \to \mathbb{R}^m - \{0\}$ given by

$$g \circ f^{-1}(\mathbf{y}) = g\Big(\frac{2}{1 + \|\mathbf{y}\|^2} y_1, \dots, \frac{2}{1 + \|\mathbf{y}\|^2} y_m, 1 - \frac{2}{1 + \|\mathbf{y}\|^2}\Big) \tag{5}$$

$$= \frac{1}{1 + (1 - \frac{2}{1+\|\mathbf{y}\|^2})}\Big(\frac{2}{1 + \|\mathbf{y}\|^2} y_1, \dots, \frac{2}{1 + \|\mathbf{y}\|^2} y_m\Big) \tag{6}$$

$$= \Big(\frac{y_1}{1 + \|\mathbf{y}\|^2 - 1}, \dots, \frac{y_m}{1 + \|\mathbf{y}\|^2 - 1}\Big) = \frac{\mathbf{y}}{\|\mathbf{y}\|^2} \tag{7}$$

hence it is clearly smooth as required.

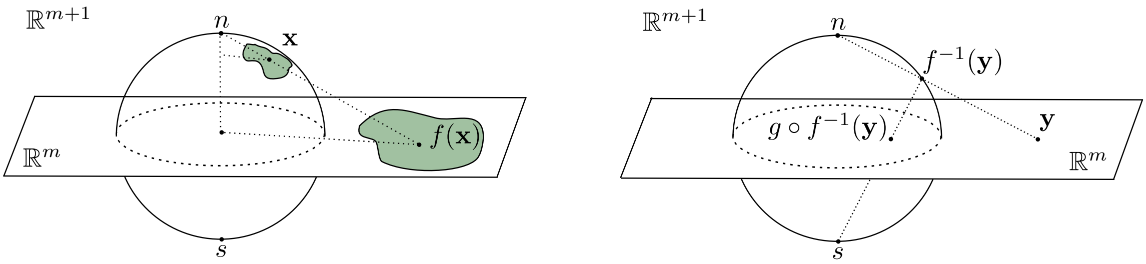



Figure 2: Geometric interpretation of stereographic projection and the transition map.

**Lemma 2.** *Let $M_1$ be an $m_1$-manifold and $M_2$ an $m_2$-manifold. Then the product $M_1 \times M_2$ is an $m_1 + m_2$-manifold.*

*Proof.* Since $M_1$ and $M_2$ are manifolds they are in particular Hausdorff hence the product $M_1 \times M_2$ is Hausdorff as well [Mun00, 17.11]. It also follows that we have collections of open sets $(V_i)_{i \in I}$ and $(V_j)_{j \in J}$ in $M_1$ and $M_2$ respectively where $M_1 = \cup_{i \in I} V_i$ and $M_2 = \cup_{j \in J} V_j$ along with collections of open sets $(U_i)_{i \in I}$ and $(U_j)_{j \in J}$ in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ respectively. Furthermore there are charts $\sigma_i : V_i \to U_i$ and $\sigma_j : V_j \to U_j$ with the property of smooth transitions on overlaps. We can now simply for each chart $\sigma_i : U_i \to V_i$ and each chart $\sigma_j : V_j \to U_j$ define the map

$$\sigma_i \times \sigma_j : V_i \times V_j \to U_i \times U_j \qquad \text{by} \qquad \sigma_i \times \sigma_j(x, y) = (\sigma_i(x), \sigma_j(y)). \tag{8}$$

Then clearly $(V_i \times V_j)_{(i,j) \in I \times J}$ is a collection of open sets such that $\cup_{(i,j) \in I \times J} V_i \times V_j = M_1 \times M_2$ and $(U_i \times U_j)_{(i,j) \in I \times J}$ is a collection of open sets in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} = \mathbb{R}^{m_1 + m_2}$ with the defined maps as charts already fulfilling the property of smooth transitions since we have for $i' \in I$ and $j' \in J$

$$(\sigma_{i'} \times \sigma_{j'}) \circ (\sigma_i \times \sigma_j)^{-1} = (\sigma_{i'} \circ \sigma_i^{-1}) \times (\sigma_{j'} \circ \sigma_j^{-1}) \tag{9}$$

which is a product of smooth maps hence smooth. Thus $M_1 \times M_2$ is Hausdorff and has an $m_1 + m_2$-dimensional smooth atlas. It is in other words it is an $m_1 + m_2$-manifold. ☐

## 1.2 Smooth maps and diffeomorphisms

The smooth structure given to the manifold makes it possible for us to give the following definitions and lemmas which will turn out to be useful when constructing Seifert fibered manifolds.

**Definition 3.** Let $M_1$ and $M_2$ be manifolds with atlases $\mathcal{A}_1 = \{(V_i, U_i, \sigma_i)\}$ and $\mathcal{A}_2 = \{(W_i, O_i, \tau_i)\}$ respectively. The map $f : M_1 \to M_2$ is said to be **smooth** if for any charts $\sigma_i$ and $\tau_j$ the map

$$\tau_j \circ f \circ \sigma_i^{-1} : \sigma_i(V_i \cap f^{-1}(W_j)) \to \tau_j(W_j) \tag{10}$$

is smooth. If in addition $f$ is a homeomorphism with smooth inverse, it is called a **diffeomorphism**.

Smoothness is in fact a *local* property as the next lemma shows.

**Lemma 3.** *Let $f : M_1 \to M_2$ be a map between manifolds, and suppose each point $p \in M_1$ has a neighborhood $N$ such that $f|_N$ is smooth. Then $f$ is smooth.*

*Proof.* We have an open cover of $M_1$ by the neighborhoods $N$; call this $\mathcal{N}$. Let $\mathcal{A}_1 = \{(V_i, U_i, \sigma_i)\}$ and $\mathcal{A}_2 = \{(W_i, O_i, \tau_i)\}$ be atlases for $M_1$ and $M_2$ respectively. We must show that the map $\tau_j \circ f \circ \sigma_i^{-1}|_{\sigma_i(f^{-1}(W_j)\cap V_i)}$ is smooth. Since $\{\sigma_i(N \cap V_i) \,|\, N \in \mathcal{N}\}$ is an open cover of $U_i \subset \mathbb{R}^m$ it suffices, by locality of smoothness in Euclidean space, to show that, for some given $N \in \mathcal{N}$, the map $\tau_j \circ f \circ \sigma_i^{-1}|_{\sigma_i(f^{-1}(W_j)\cap V_i \cap N)}$ is smooth. We have an atlas for $N$ given by $\{(V_i \cap N, U_i \cap \sigma_i(N), \sigma_i|_N)\}$. Since $f|_N$ is smooth we have by definition that the map

$$\tau_j \circ f|_N \circ (\sigma_i|_N)^{-1} : \sigma_i|_N(V_i \cap N \cap f|_N^{-1}(W_j)) \to \tau(W_j) \tag{11}$$

is smooth. Since $f = f|_N$ on this domain, $(\sigma_i|_N)^{-1} = \sigma_i^{-1}|_{\sigma_i(N)}$ and $f|_N^{-1}(W_j) = f^{-1}(W_j) \cap N$ we can rewrite 11 as follows

$$\tau_j \circ f \circ \sigma_i^{-1} : \sigma_i|_N(V_i \cap N \cap f^{-1}(W_j)) \to \tau(W_j) \tag{12}$$

which we recognize as the map to be proven smooth. □

**Note 1.** One could, due to lemma 3 equivalently state that $f$ is **smooth at** $x \in M_1$ if there exists charts $\sigma : V_1 \to U_1$ and $\tau : V_2 \to U_2$ on $M_1$ and $M_2$ respectively with $x \in V_1$ and $f(x) \in V_2$ such that the composite map

$$\tau \circ f \circ \sigma^{-1} : \sigma(f^{-1}(V_2)) \to U_2 \tag{13}$$

is smooth, and then declare $f$ smooth, if it satisfies the property at all $x$, compare with [MT97]. We also see that smoothness of real-valued functions is a special case: Taking $M_2 = \mathbb{R}^n$ and $\tau = id$ we would get the function $f : M_1 \to \mathbb{R}^n$ which is smooth if for all charts $\sigma$ in a smooth atlas for $M_1$ the composite map $f \circ \sigma^{-1}$ is smooth.

**Lemma 4.** *Let $M$ be a Hausdorff topological space and let $U, V \subset M$ be open with $U \cup V = M$. Let $\lambda : M_1 \to U$ be a homeomorphism from a manifold $M_1$ to $U$, and let $\mu : M_2 \to V$ be a homeomorphism from a manifold $M_2$ to $V$. Suppose that the 'transition map'*

$$\lambda^{-1}(\mu(M_2)) \xrightarrow{\lambda|} \lambda(M_1) \cap \mu(M_2) \xrightarrow{\mu^{-1}|} \mu^{-1}(\lambda(M_1)) \tag{14}$$

*is a diffeomorphism. Then $M$ admits the structure of a smooth manifold. The resulting smooth structures on $U$ and $V$ make both $\lambda$ and $\mu$ into diffeomorphisms.*

*Proof.* Since $M_1$ and $M_2$ are manifolds we can find smooth atlases $\mathcal{A}_1 = \{(V_i, U_i, \sigma_i)\}_{i \in I_1}$ and $\mathcal{A}_2 = \{(W_i, O_i, \tau_i)\}_{i \in I_2}$ say for these respectively. We claim that we can turn them into a smooth atlas of $M$ by the homeomorphisms $\lambda$ and $\mu$, which gives us $\mathcal{B} = \{(\lambda(V_i), U_i, \sigma_i \circ \lambda^{-1})\}_{i \in I_1} \cup \{(\mu(W_i), O_i, \tau_i \circ \mu^{-1})\}_{i \in I_2}$. The $\lambda(V_i)$ and $\mu(W_i)$ constitute an open cover of $M$ since $\bigcup_{i \in I_1} \lambda(V_i) \cup \bigcup_{i \in I_2} \mu(W_i) = \lambda(\bigcup_{i \in I_1} V_i) \cup \mu(\bigcup_{i \in I_2}(W_i) = \lambda(M_1) \cup \mu(M_2) = U \cup V = M$. It is smooth because for $i, j \in I_1$,

$$(\sigma_j \circ \lambda^{-1}) \circ (\sigma_i \circ \lambda^{-1})^{-1} = \sigma_j \circ \sigma_i^{-1} \tag{15}$$

which is smooth by the assumption that $\mathcal{A}_1$ is a smooth atlas. In the same way we see that for $i, j \in I_2$ the transition map is smooth. Finally if $i \in I_1$ and $j \in I_2$ we have

$$(\tau_j \circ \mu^{-1}) \circ (\sigma_i \circ \lambda^{-1})^{-1} = \tau_j \circ \mu^{-1} \circ \lambda \circ \sigma_i^{-1} \tag{16}$$

which is smooth by the assumption that $\mu^{-1} \circ \lambda$ is a diffeomorphism cf. definition 3. To see that $\lambda$ is a diffeomorphism we note that the composite map becomes $\sigma_i \circ \lambda^{-1} \circ \lambda \circ \sigma_j^{-1} = \sigma_i \circ \sigma_j^{-1}$ which is smooth and likewise the inverse map is smooth. Similarly we see that $\mu$ is a diffeomorphism as required. □

## 1.3 Surfaces as manifolds

Before turning to 3-manifolds in generality let's get a feeling of how 2-manifolds, called **surfaces**, are looking. We give a detailed example of the torus and then move on to a general scheme for creating compact oriented surfaces. These will play a key role in the theory of Seifert fibered spaces.

**Example 2.** *The torus $T^2 = I \times I/\!\sim$ is a 2-manifold.* To see this note that the equivalence relation $\sim$ is given by pasting opposite edges of the unit square together hence the equivalence classes are

$$\{(x,y)\}, \quad (x,y) \in \mathrm{Int}\,(I \times I) \qquad \{(x,0),(x,1)\}, \quad x \in (0,1) \tag{17}$$

$$\{(0,y),(1,y)\}, \quad y \in (0,1) \qquad \{(0,0),(0,1),(1,1),(1,0)\}. \tag{18}$$

Define the quotient map $p : I \times I \to I \times I/\!\sim$ by $p(x,y) = [(x,y)]$. Furthermore define the sets:

$$U_1 = \mathrm{Int}\,(I \times I), \qquad U_2 = (0,1) \times [[0,1/2) \cup (1/2,1]] \tag{19}$$

$$U_3 = [[0,1/2) \cup (1/2,1]] \times (0,1), \qquad U_4 = [[0,1/2) \cup (1/2,1]] \times [[0,1/2) \cup (1/2,1]]. \tag{20}$$

These are all quickly seen to be open in $I \times I$ with the subspace topology and saturated with respect to $p$. Since $p$ is a quotient map the spaces $p(U_1), \dots p(U_4)$ are open in $I \times I/\!\sim$. Furthermore since $I \times I = \bigcup_{i=1}^4 U_i$ we have $I \times I/\!\sim = \bigcup_{i=1}^4 p(U_i)$. We now show that every $p(U_i)$ is homeomorphic to $\mathrm{Int}\,(I \times I) \subset \mathbb{R}^2$. Define the map $f_1$ as the identity and maps $f_i$ for $i = 2,3$ by shuffling around two sections of $\mathrm{Int}\, I \times I$ in a similar way as the following last one, $f_4$, with four sections:

$$f_4 : U_4 \to \mathrm{Int}\,(I \times I) \qquad \text{by} \qquad f_4(x,y) = \begin{cases} (x+1/2, y-1/2) & \text{if } x < 1/2, y > 1/2 \\ (x-1/2, y-1/2) & \text{if } x > 1/2, y > 1/2 \\ (x+1/2, y+1/2) & \text{if } x < 1/2, y < 1/2 \\ (x-1/2, y+1/2) & \text{if } x > 1/2, y < 1/2 \end{cases} \tag{21}$$

One can easily verify that these are all well defined, continuous and constant on each $p^{-1}([x,y])$ thus we get induced continuous maps $\bar{f}_i : p(U_i) \to \mathrm{Int}\,(I \times I)$ [Mun00, 22.2].

$$\begin{array}{ccc} U_i & & \\ {\scriptstyle p}\downarrow & \searrow^{f_i} & \\ p(U_i) & \xrightarrow[\bar{f}_i]{} & \mathrm{Int}\,(I \times I) \end{array}$$

These have inverse maps $\bar{g}_i$, $i = 1\dots,4$ of forms quite similar to

$$\bar{g}_4 : \mathrm{Int}\,(I \times I) \to p(U_2) \qquad \text{by} \qquad \bar{g}_4(x,y) = \begin{cases} [(x+1/2, y-1/2)] & \text{if } x \le 1/2, y \ge 1/2 \\ [(x-1/2, y-1/2)] & \text{if } x \ge 1/2, y \ge 1/2 \\ [(x+1/2, y+1/2)] & \text{if } x \le 1/2, y \le 1/2 \\ [(x-1/2, y+1/2)] & \text{if } x \ge 1/2, y \le 1/2 \end{cases} \tag{22}$$

which is well defined since $\bar{g}_4(1/2,1/2) = [(1,0)] = [(0,0)] = [(1,1)] = [(0,1)]$, while for $x < 1/2$ we have $\bar{g}_4(x,1/2) = [(x+1/2,0)] = [(x+1/2,1)]$ and for $x > 1/2$ we have $\bar{g}_4(x,1/2) = [(x-1/2,0)] = [(x-1/2,1)]$. Exactly the same argument holds for the cases $y < 1/2$ and $y > 1/2$. We furthermore conclude that $\bar{g}_i$ is continuous by the pasting lemma hence $\bar{f}_i$ is a homeomorphism. We now have a 2-dimensional atlas given by $\mathcal{A} = \{p(U_i), \mathrm{Int}\,(I \times I), \bar{f}_i\}_{i \in \{1,\dots,4\}}$. To see that it is smooth we must show that the maps $\bar{f}_i$, $i = 1,\dots,4$ satisfies the property of smooth transition on overlaps. Again we only look at one example of the twelve potentially interesting cases. Pick $i = 4$ and $j = 2$ say. Then the map $\bar{f}_4 \circ \bar{f}_2^{-1} : \bar{f}_2(p(U_4) \cap p(U_2)) \to \bar{f}_4(p(U_4) \cap p(U_2))$ is given by

$$\bar{f}_4 \circ \bar{f}_2^{-1}(x,y) = \begin{cases} (x+1/2, y) & \text{if } x < 1/2, y > 1/2 \\ (x-1/2, y) & \text{if } x > 1/2, y > 1/2 \\ (x+1/2, y) & \text{if } x < 1/2, y < 1/2 \\ (x-1/2, y) & \text{if } x > 1/2, y < 1/2 \end{cases} \tag{23}$$

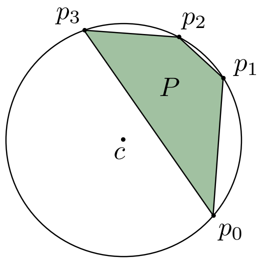
which is smooth on each of the disjoint parts and hence smooth. The other transition maps follows the same pattern of shuffling around parts of $\mathrm{Int}\,(I \times I)$. That $I \times I/\!\sim$ is Hausdorff follows by lemma 5 below thus $T^2$ is a 2-manifold.

**Note 2.** We actually knew this from earlier. The torus $T^2$ can be written as $S^1 \times S^1$ and since by example 1, $S^1$ is a 1-manifold we conclude by lamma 2 that $T^2$ is a 2-manifold. The point of including the example is that this is the simplest case of a more general way to construct new 2-manifolds.

### 1.3.1 Construction by polygonal regions

We will now show how to construct a number of compact connected surfaces in a similar way to example 2 and we compute their fundamental groups. This closely follows [Mun00, §74].

**Definition 4.** Given a point $c \in \mathbb{R}^2$ and $r > 0$ consider the circle of radius $r$ and center $c$. Given a finite sequence $\theta_0 < \cdots < \theta_n$ of real numbers, where $n \geq 3$ and $\theta_n = \theta_0 + 2\pi$, consider the points $p_i = c + r(\cos\theta_i, \sin\theta_i)$ on the circle. Denote the half-plane containing the points $p_k$, made by the line through $p_{i-1}$ and $p_i$, by $H_i$. Then the space $P = H_1 \cap \cdots \cap H_n$ is called the **polygonal region** determined by the points $p_i$.




The positive linear map $h : [a,b] \to [c,d]$ defined by $(1-s)a + sb \mapsto (1-s)c + sd$ is a homeomorphism. If $Q$ is also a polygonal region with the same number of edges as $P$, we can find a homeomorphism of $P$ to $Q$ by using the fact that $P$ is star-convex with respect to any point in $\operatorname{Int} P$ hence $P$ is the union of all the line segments from $p \in \operatorname{Int} P$ to points in $\operatorname{Bd} P$ and we can then just expand the positive linear map to these going from $[p,x]$ to $[q,h(x)]$ for a given $q \in \operatorname{Int} Q$ and $x \in \operatorname{Bd} P$. It is therefore only the number of points, not their position which matters.

**Definition 5.** A **labeling** of the edges of $P$ is a map $\ell : P \to S$ where $S$ is the set of labels. An **orientation** of a line segment is an ordering of its end points. Given a labeling and orientation of the edges of $P$, we define an equivalence relation $\sim$ on $P$ as follows: Each point of $\operatorname{Int} P$ is equivalent only to itself. Given any two edges of $P$ that have the same label, let $h$ be the positive linear map of one onto the other and relate each point $x$ of the first to $h(x)$ on the other. The quotient space $X$ obtained from this equivalence class is said to have been obtained by **pasting the edges** of $P$ together according to the labels and orientations.

**Definition 6.** Let $P$ be a polygonal region with successive vertices $p_0, \ldots, p_n$, where $p_0 = p_n$. Given orientations and labeling of the edges of $P$ let, write $a_k$ for the label assigned to $p_{k-1}p_k$, and write $\epsilon_k = +1$ or $-1$ according to the orientation. Then the number of edges, their orientation and labeling are completely specified by

$$w = (a_1)^{\epsilon_1}(a_2)^{\epsilon_2} \ldots (a_n)^{\epsilon_n} \tag{24}$$

called the **labeling scheme** of length $n$ for the edges of $P$. Clearly a cyclic permutation in the scheme will only change the space $X$ up to homeomorphism.

**Example 3.** The labeling scheme $aba^{-1}b^{-1}$ gives rise to the torus in example 2, $abac$ to the Möbius band and $abab^{-1}$ to the Klein bottle both of which we will encounter later in sections 2.1 and 2.2 respectively. It is in fact possible to create surfaces of all **genus** $g$, see appendix B.4, by labeling a $4g$ sided polygonal region $P$ according to the following scheme $(a_1b_1a_1^{-1}b_1^{-1}) \ldots (a_gb_ga_g^{-1}b_g^{-1})$. The space is called the $g$**-fold connected sum of tori** and is denoted $T\sharp \ldots \sharp T$, cf. B.3. The famous classification theorem for surfaces states, that this does in fact account for all compact orientable surfaces, see appendix B.4.

**Note 3.** Notice that in the case of the $g$-fold torus every corner of the polygon gets sent to the same point under the quotient map. To see that these are manifolds we could go through exactly the same procedure as in example 2. It is however intuitively clear if one defines $U_1 = \operatorname{Int} P$, the set $U_2$ as the polygonal region with straight lines going between opposite corners subtracted and $U_3$ as the polygonal region with straight lines going between midpoints of opposite edges subtracted. These

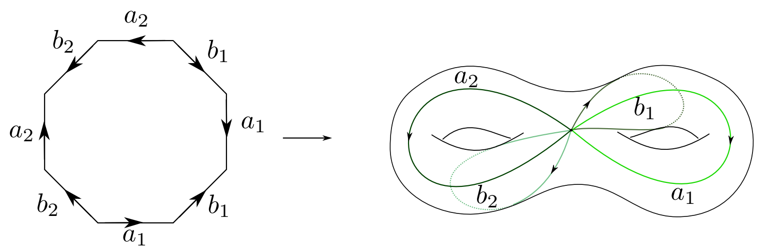



Figure 3: The labeled octagon and the resulting double torus $T\sharp T$

are all open and saturated and we can find continuous maps going to open subsets of $\mathbb{R}^2$ by shuffling the 'slices' around, just as in example 2 such that it is constant on each $p^{-1}([x, y])$. This would give us the desired induced homeomorphisms, and one could then easily check that the transition property is satisfied to conclude that they are indeed manifolds, since they are also Hausdorff by the following theorem.

**Lemma 5.** *Let $X$ be the space obtained from a finite collection of polygonal regions by pasting edges together according to some labelling scheme. Then $X$ is a compact Hausdorff space. Cf. [Mun00].*

*Proof.* For simplicity we treat the case where $X$ is formed by a single polygonal region. Since any polygonal region is compact, being a closed and bounded set of a metric space, we know that since the quotient map is continuous, $X$ is compact [Mun00]. To show that $X$ is Hausdorff it suffices by lemma 21, since $P$ is a compact Hausdorff space, to show that the quotient map $\pi$ is a closed map. We must therefore show that for each closed set $C$ of $P$, the set $\pi^{-1}\pi(C)$ is closed in $P$. Now $\pi^{-1}\pi(C)$ consists of the points of $C$ and all the points of $P$ which are pasted to points of $C$ by $\pi$. These points can be determined; for each edge $e$ of $P$, let $C_e$ denote the compact subspace $C \cap e$ of $P$. If $e_i$ is an edge of $P$ that is pasted to $e$, and if $h_i : e_i \to e$ is the pasting homeomorphism, then the set $D_e = \pi^{-1}\pi(C) \cap e$ contains the space $h_i(C_{e_i})$. Indeed $D_e$ equals the union of $C_e$ and the spaces $h_i(C_{e_i})$ as $e_i$ ranges over the edges of $P$ that are pasted to $e$. This (finite) union is compact hence closed in $e$ and in $P$. Since $\pi^{-1}\pi(C)$ is the union of the set $C$ and the sets $D_e$, as $e$ ranges over all edges of $P$, it is closed in $P$ as required. □

**Theorem 1.** *Let $P$ be a polygonal region; let $w = (a_1)^{\epsilon_1} \dots (a_n)^{\epsilon_n}$ be a labeling scheme for the edges of $P$. Let $X$ be the resulting quotient space and $\pi : P \to X$ the quotient map. If $\pi$ maps all the vertices of $P$ to a single point $x_0$ of $X$ and if $a_1, \dots, a_k$ are the distinct labels that appear in the labeling scheme, then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on $k$ generators $\alpha_1, \dots, \alpha_k$ by the least normal subgroup containing the element $(\alpha_1)^{\epsilon_1} \dots (\alpha_n)^{\epsilon_n}$*

*Proof.* By lemma 5 we see that $X$ is Hausdorff. Since $\pi$ maps all vertices of $P$ to a single point $x_0$ of $X$ the set $W = \pi(\mathrm{Bd}\, P)$ is a so called wedge of $k$ circles, see appendix C.1, and it is clearly a path connected subspace of $X$. Furthermore $\pi$ is by definition of the quotient topology continuous and it maps $\mathrm{Int}\, P$ bijectively, by the identity, onto $X - W$ and $\mathrm{Bd}\, P$ to $W$. Let $p \in \mathrm{Bd}\, P$ be a vertice.

For each $i$, choose an edge of $P$ labelled $a_i$. Define the map $f_i$ from $I$ to this edge by the positive linear map and furthermore define $g_i = \pi \circ f_i$. Then $g_i$ is a loop and hence represents a generator for $\pi_1(W, x_0)$ and the loops $g_1 \dots g_k$ represent a system of free generators for the free group $\pi_1(W, x_0)$ [Mun00, 71.1]. The loop $f$ running around $\mathrm{Bd}\, P$ once in the positive direction generates the fundamental group of $\mathrm{Bd}\, P$ and the loop $\pi \circ f$ equals the loop $(g_1)^{\epsilon_1} * \dots * (g_k)^{\epsilon_n}$. In other words the map $k : (\mathrm{Bd}\, P, p) \to (W, x_0)$ defined by restricting $\pi$ has image $[g_1]^{\epsilon_1} * \dots * [g_k]^{\epsilon_n}$.

Then lemma 26 tells us that the homomorphism $i_* : \pi_1(W, x_0) \to \pi_1(X, x_0)$ induced by inclusion is surjective and its kernel, $N$, is the least normal subgroup of $\pi_1(W, x_0)$ containing the image of $k_* : \pi_1(\mathrm{Bd}\, P, p) \to \pi_1(W, x_0)$. This can be reformulated using the first isomorphism theorem [Tho07, p. 98] giving us the isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(W, x_0)/N \tag{25}$$

and by the above calculations $\pi_1(W, x_0)$ is the free group on the $k$ generators $[g_1] \dots, [g_k]$ while the image of $k_*(\pi_1(P, p))$ is $[g_1]^{\epsilon_1} \dots [g_n]^{\epsilon_n}$ which is the required statement with $g$'s replaced by $\alpha$'s. □

**Corollary 1.** *Let $X$ denote the $g$-fold torus, which is the space obtained from the labeling scheme $(a_1 b_1 a_1^{-1} b_1^{-1}) \ldots (a_g b_g a_g^{-1} b_g^{-1})$. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the $2g$ generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ by the least normal subgroup containing $[\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_g, \beta_g]$.*

*Proof.* It is fairly easy to check that all vertices in a polygonal region $P$ with the labeling scheme $(a_1 b_1 a_1^{-1} b_1^{-1}) \ldots (a_g b_g a_g^{-1} b_g^{-1})$ are mapped by $\pi$ to the same point in $X$; one could simply note, that since this is the case for the torus it follows by induction for the general $g$-fold torus. The distinct labels are two for each genus, $a_j, b_j$ say, hence there are $2g$ i all. Theorem 1 now tells us that $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on $2g$ generators $\alpha_1 \beta_1 \ldots \alpha_g \beta_g$ by the least normal subgroup containing the element $(\alpha_1)^{\epsilon_1} \ldots (\alpha_n)^{\epsilon_n} = (\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}) \ldots (\alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}) = [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]$ as required. □

**Note 4.** Since every orientable surface has the form of a $g$-fold torus by the classification theorem, cf. appendix B.4, this gives us a recipe for computing the genus of any such space, since $g$ can be extracted from the presentation of $\pi_1$.

## 1.4 Lens spaces

We will now turn to an investigation of the so called Lens spaces $L_{p/q}$, introduced by Heinrich Tietze in 1908 [Tie08]. These will serve as our main examples of Seifert manifolds, and we shall return to them later on as well. The original geometric formulation goes as follows: A lens is a region of 3-space bounded by two spherical caps which meet in an equatorial circle. Divide the equatorial circle into $q$ equal segments. Its two caps then become $q$-gons. Then $L_{p/q}$ is obtained from this by a reflection in the plane containing the rim of the lens, taking one face to the other, followed by a rotation of this face through the angle $2\pi p/q$. See Hatcher [Hat02] and Seifert [ST80] for further details on this geometric representation.

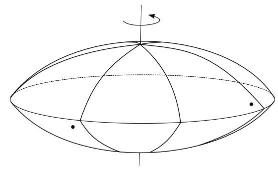


Figure 4: The solid lens-shaped 3-ball with two equivalent points.

We will not go further into the geometric description, instead we define the lens space formally in the following way, which can be shown to be equivalent to the one sketched out above, see [Hat02].

**Definition 7.** The **Lens space** $L_{p/q}$ is defined by $L_{p/q} = S^3/\sim$ where $p, q$ are coprime and $\sim$ is generated by $(u, v) \sim (e^{2\pi i/q} u, e^{2\pi i p/q} v)$. Compare with [Hat07, p.12].

Note that $L_{p/1} \simeq S^3$ for all $p \in \mathbb{Z}$ by insertion. We are going to use the language of group actions to show that the Lens space $L_{p/q}$ is a 3-manifold. Definitions are from [Tho07].

### 1.4.1 Group action terminology

**Definition 8.** Let $\Gamma$ be a fixed group. It is said to **act on** the set $M$ if there is a map $\Gamma \times M \to M$ denoted $(g, x) \mapsto g.x$ such that for $x \in M$ and $g, h \in \Gamma$ we have

$$e.x = x \qquad \text{and} \qquad (gh).x = g.(h.x). \tag{26}$$

For a given $g \in \Gamma$ we define the bijective map $\rho_g : M \to M$ by $x \mapsto g.x$. Suppose $\Gamma$ acts on $M$. Two elements $x, y \in M$ are said to be **$\Gamma$-equivalent**, written $x \sim y$, if $x = g.y$ for some $g \in \Gamma$. The equivalence classes are called **orbits**. For a given $x \in M$ we can consider the orbit through $x$ given by $\Gamma.x = \{g.x \,|\, g \in \Gamma\}$. A group $\Gamma$ **acts freely** if the only element of $\Gamma$ that fixes any element of $M$ is the identity: $g.x = x$ for some $x \in M$ implies $g = e$.

**Definition 9.** If $M$ is a space we say that the action is continuous if the corresponding maps $\rho_g : M \to M$ for $g \in \Gamma$ are continuous (thus homeomorphism with inverse $\rho_{g^{-1}}$). We give the **orbit space** $M/\Gamma$ the quotient topology and write $p : M \to M/\Gamma$ for the quotient map.

**Definition 10.** Let $M$ be a manifold. Then the action is said to be **smooth** if the map $\Gamma \times M \to M$ is smooth, that is if $\rho_g(x)$ depends smoothly on $(g, x)$. If this is the case then for each $g \in \Gamma$ the map $\rho_g : M \to M$ is a diffeomorphism.

### 1.4.2 The orbit space as a manifold

The central theorem is the following which requires some lemmas before it can be proved. This will in principle enable us to create a vast number of new 3-manifolds, and we start by applying it to show that the Lens spaces belong to this category.

**Theorem 2.** *Let $\Gamma$ be a finite group acting smoothly and freely on a compact $m$-manifold $M$. Then $M/\Gamma$ is a compact $m$-manifold.*

**Corollary 2.** *The Lens space $L_{p/q}$ is a compact 3-manifold.*

*Proof.* $S^3$ is a compact 3-manifold by example 1. The group $\langle e^{2\pi i/q}\rangle \simeq C_q$ is finite and acts freely on $S^3$: Define the map $C_q \times S^3 \to S^3$ by $(g, (u, v)) \mapsto g.(u, v) = (e^{2\pi i g/q}u, e^{2\pi i g p/q}v)$. Using additive notation we have

$$0.(u, v) = (e^{2\pi i 0/q}u, e^{2\pi i 0p/q}v) = (u, v) \tag{27}$$

$$(g + h).(u, v) = (e^{2\pi i(g+h)/q}u, e^{2\pi i(g+h)p/q}v) = g.(h.(u, v)) \tag{28}$$

hence it is an action and we can define $\rho_g : S^3 \to S^3$ by $\rho_g(u, v) = (e^{2\pi i g/q}u, e^{2\pi i g p/q}v)$. If $\rho_g(u, v) = (u, v)$ then $e^{2\pi i g/q}u = u$ hence $g \equiv 0 \pmod q$ or $u = 0$. In the last case we must have $|v| = 1$ in particular $v \neq 0$. Therefore $e^{2\pi i g p/q} = 1$ hence $gp \equiv 0 \pmod q$ which implies $g \equiv 0 \pmod q$ since $q$ and $p$ are coprime. In any case we see that $g = e$ thus $C_q$ acts freely. Since $S^3$ is a manifold and $\rho_g$ is clearly smooth, the action is smooth as well. The equivalence relation $(u, v) \sim (e^{2\pi i/q}u, e^{2\pi i p/q}v)$ gives rise to precisely the equivalence classes $[(u, v)] = \{g.(u, v) \mid g \in C_q\} = C_q.(u, v)$ in other words the orbits. By theorem 2 we conclude that $L_{p/q} = S^3/C_q = S^3/\sim$ is a compact 3-manifold. □

We now move back to show a series of lemmas leading to the proof of theorem 2. These will furthermore find applications later on when we return to examine the Lens spaces further.

**Lemma 6.** *If the group $\Gamma$ is finite then the quotient map $p : M \to M/\Gamma$ is open and closed.*

*Proof.* Let $U \subset M$ be open. We must show that $p(U)$ is open in $M/\Gamma$ or equivalently that $p^{-1}(p(U))$ is open in $M$. But we have

$$p^{-1}(p(U)) = \{g.(U) \mid g \in \Gamma\} = \bigcup_{g\in\Gamma} g.(U) = \bigcup_{g\in\Gamma} \rho_g(U) \tag{29}$$

and since $\rho_g$ is a homeomorphism we conclude that $p^{-1}(p(U))$ is a union of open sets hence open. By the same argument we conclude, since the union is finite and $p$ is a quotient map, that $p^{-1}(p(V))$ is closed for $V \subset M$ closed. □

**Lemma 7.** *If $M$ is a compact Hausdorff space and $\Gamma$ is finite, then $M/\Gamma$ is compact Hausdorff.*

*Proof.* The quotient map $p : M \to M/\Gamma$ defined by $p(x) = [x]$ is, by definition of the quotient topology, continuous [Mun00, p. 138]. Since $M/\Gamma = p(M)$ it is a continuous image of the compact space $M$ hence compact [Mun00, 26.5]. We must show that it fulfills the Hausdorff condition. Pick $[x], [y] \in M/\Gamma$ where $[x] \neq [y]$. Then $p^{-1}([x]) = \{g.x \mid g \in \Gamma\} = \Gamma.x$ and $p^{-1}([y]) = \{g.y \mid g \in \Gamma\} = \Gamma.y$ are disjoint in $M$. Since $\Gamma$ is finite so is $\Gamma.x$ and $\Gamma.y$ and they are therefore also compact. By lemma 20 it follows that we can find disjoint open sets $U$ and $V$ containing $\Gamma.x$ and $\Gamma.y$ respectively. Therefore the complements $M - U$ and $M - V$ are closed and by lemma 6 we see that $p(M - U)$ and $p(M - V)$ are closed. Define

$$A = M/\Gamma - p(M - U) \qquad \text{and} \qquad B = M/\Gamma - p(M - V) \tag{30}$$

which are then obviously open and contain $[x]$ and $[y]$ respectively. Finally we have

$$A \cap B = M/\Gamma - p((M - U) \cup (M - V)) = M/\Gamma - p(M - (U \cap V)) = M/\Gamma - p(M) = \emptyset \tag{31}$$

hence the Hausdorff condition is satisfied as required. □

**Lemma 8.** *Let $\Gamma$ be a finite group acting freely on a Hausdorff space $M$. For any $x \in M$ there exists a neighborhood, $A$, such that for $g \neq e$ we have $A \cap g.A = \emptyset$.*

*Proof.* Without loss of generality we can write the finite group as $\Gamma = \{g_1, \ldots, g_k\}$ where $g_1 = e$. Since $\Gamma$ acts freely $g_j.x = x$ if and only if $g_j = e$ and we conclude, as $\rho_g$ is bijective, that $g_1.x, \ldots, g_k.x$ are distinct in $M$. We can now, due to $M$ being Hausdorff, find pairwise disjoint neighborhoods $A_1, \ldots A_k$ of these points (use lemma 20 $k$ times and take intersections). Define

$$A = \bigcap_{i=1}^{k} g_i^{-1}.A_i = \bigcap_{i=1}^{k} \rho_{g_i^{-1}}(A_i) \tag{32}$$

which, due to $\rho_{g_i^{-1}}$ being a homeomorphism, is a neighborhood of $x$ in $M$. We see that $A = A_1 \cap (\bigcap_{i=2}^{k} g_i^{-1}.A_i)$ hence $A \subset A_1$. Letting $g_i$ act on (32) furthermore leaves us with $g_i.A = \bigcap_{j=1}^{k} g_i.g_j^{-1}.A_j$ hence $g_i.A \subset A_i$. But then $A \cap g_i.A \subset A_1 \cap A_i = \emptyset$. □

*Proof of theorem 2.* The compact $m$-manifold $M$ is by definition Hausdorff hence by lemma 7 $M/\Gamma$ is compact Hausdorff. We need to find a smooth atlas of $M/\Gamma$, that is show that every point of $M/\Gamma$ has a neighborhood homeomorphic to an open set in $\mathbb{R}^m$. Let $[x] \in M/\Gamma$ be given. By lemma 8 we can find a neighborhood $A$ of the representative $x$ such that $A \cap g.A = \emptyset$ for $g \neq e$. Since $M$ is a manifold we can by lemma 1 find a neighborhood $V$ of $x$ which is diffeomorphic to $\mathbb{R}^m$: $\sigma : V \to \mathbb{R}^m$. Now $W = V \cap A$ is open in $V$ hence $\sigma(W)$ is open in $\mathbb{R}^m$. Furthermore $W$ is an open neighborhood of $x$ in $A$.

We wish to show that $p(W)$ is the desired neighborhood of $[x]$ in $M/\Gamma$. The mapping $p|_W : W \to p(W)$ is by definition surjective. To see that it is injective suppose $p|_W(y_1) = p|_W(y_2)$ for $y_1, y_2 \in W$. Then $[y_1] = [y_2]$ which by definition implies that $y_1 = g_t.y_2$ for some $g_t \in \Gamma$. Therefore $y_1 \in W \cap g_t.W \subset A \cap g_t.A$ hence we must have $g_t = e$ by the above assumption which gives us $y_1 = y_2$ as required for injectivity.

Since $p : M \to M/\Gamma$ is open by lemma 6 and continuous the restriction $p|_W$ is open and continuous [Mun00, 18.2 (d)] and therefore a homeomorphism. But then $p(W)$ is homeomorphic to $W$ which by the above is homeomorphic to $\mathbb{R}^m$ and clearly $[x] \in p(W)$ as required.

Finally we need to check that the property of smooth transitions on overlaps is satisfied. Define $\psi_i : p(W_i) \to \mathbb{R}^m$ by $\psi_i(x) = \sigma_i \circ p|_{W_i}^{-1}(x)$. Then we have an atlas of $M/\Gamma$ given by $\{(p(W_i), \mathbb{R}^m, \psi_i)\}_{i \in M}$ and the transition map

$$\psi_j \circ \psi_i^{-1} : \psi_i(p(W_i) \cap p(W_j)) \to \psi_j(p(W_i) \cap p(W_j)) \qquad \text{is given by} \tag{33}$$

$$\psi_j \circ \psi_i^{-1}(x) = (\sigma_j \circ p|_{W_j}^{-1}) \circ (\sigma_i \circ p|_{W_i}^{-1})^{-1}(x) = \sigma_j \circ \sigma_i^{-1}(x) \tag{34}$$

and thus seen to be smooth since by definition $\sigma_j \circ \sigma_i^{-1}$ satisfies the property. □

**Lemma 9.** *Let $M$ be a set and $\Gamma$ a finite group acting freely on $M$. Then any quotient map $\pi : M \to M/\Gamma$ is a covering map.*

*Proof.* We claim that the quotient map $\pi : M \to M/\Gamma$ given by $\pi(x) = [x]$ is a covering map. To see this note by lemma 8 we can for any $x \in M$ find a neighborhood $A$ such that for $g \neq e$ we have $A \cap g.A = \emptyset$. Thus for $x, y \in A$ we see that if $\pi(x) = \pi(y)$ then $x = y$ in other words the restriction $\pi|_A : A \to M/\Gamma$ is injective and therefore bijective. Furthermore we see from lemma 6 that $\pi|_A$ is an open map thus we conclude that $\pi|_A$ is a homeomorphism. We have

$$\pi^{-1}(\pi(A)) = \bigcup_{g \in \Gamma} g.A = \bigsqcup_{g \in \Gamma} g.A \tag{35}$$

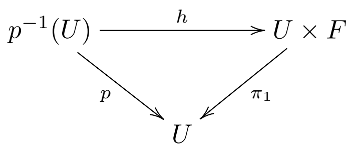
which is a disjoint union of open sets, since it follows by lemma 8 that $g_1.A \cap g_2.A = \emptyset$ for $g_1 \not\equiv g_2 \pmod{q}$ and $g.A = \rho_g(A)$. We now note that for any $g \in C_q$ the map $\pi|_{g.A} : g.A \to \pi(A)$ is given by $\pi|_{g.A}(x) = \pi|_A \circ \rho_{g^{-1}}(x)$ which is a composition of homeomorphisms hence $\pi|_{g.A}$ is a homeomorphism as well. We conclude that the open set $\pi(A)$ is evenly covered thus $\pi$ is a covering map as required. □

# 2 Fibrations and fibre bundles

In order to impose some structure on a given space we introduce the concept of a fibre bundle. This is intuitively a space $E$ which locally looks like a product space $B \times F$, but may have a global structure which differs. We give a number of examples including the class of circle bundles, among which we encounter the Hopf fibration, named after Heinz Hopf who first described it in 1931 [Hop31]. These will furthermore illustrate the general thought behind the theory of Seifert manifolds. Definitions are inspired from Hatcher [Hat02] and Madsen and Tornehave [MT97].

## 2.1 Definitions and examples

**Definition 11.** A **fibre bundle** structure on a **total space** $E$ with **fibre** $F$ and **base space** $B$, consists of a projection map $p : E \to B$ such that each point of $B$ has a neighborhood $U$ for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the diagram below commute.




Commutativity of the diagram means that $h$ carries each fibre $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of $F$. Thus fibres are arranged locally as the product $B \times F$. The homeomorphism $h$ is called a **local trivialization** of the bundle, compare with [Hat02]. A fibre bundle is said to be **smooth**, if $E, F$ and $B$ are manifolds, $p$ is a smooth map, and $h$ above can be chosen to be a diffeomorphism, see also [MT97]. We write the fibre bundle as $F \to E \to B$.

### 2.1.1 The Möbius strip

As one of the most simple non-trivial examples of a fibre bundle we have the **Möbius strip**. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We then define

$$M = \{(z, w) \mid z \in S^1, w \in \mathbb{C}, w^2 = \lambda z, \lambda \in [0, 1]\}. \tag{36}$$

Then $M$ is the Möbius strip. We claim that it is a fibre bundle $[-1, 1] \to M \to S^1$. To see this define $p : M \to S^1$ by $p(z, w) = z$. Define $U = S^1 - \{1\}$. We can choose the function $\rho_2(z) = \sqrt{|z|}e^{i\frac{Arg_0(z)}{2}}$ which is a holomorphic branch of the square root on the cut plane $\mathbb{C}_0$, see [Ber10, § 5]. Define $\sqrt{z} = \rho_2(z)$. By restriction to $U$ we get a continuous square root function $U \to S^1$. Define $h : p^{-1}(U) \to U \times [-1, 1]$ by $h(z, w) = (z, w/\sqrt{z})$. Note that $w/\sqrt{z} \in [-1, 1]$ since $(w/\sqrt{z})^2 = w^2/z = \lambda \in [0, 1]$. Furthermore $h$ is a homeomorphism with inverse $(z, t) \mapsto (z, t\sqrt{z})$. Similarly we can trivialize over $V = S^1 - \{-1\}$ by using the usual branch of the square root. Clearly $S^1 = U \cup V$ thus $M$ is the desired fibre bundle. In other words a Möbius strip is locally a product space. This is intuitively right since we can cut it open along a fibre $F_b = \{b\} \times [-1, 1]$ and untwist it to a rectangle. The Möbius strip is *not* globally a product space however since this would result in a cylinder.

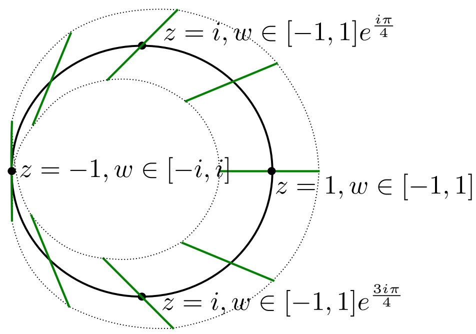



Figure 5: The Möbius strip, $M$, seen as a subset of $\mathbb{C}^2$

### 2.1.2 Covering spaces as fibre bundles

It turns out that we actually know a lot of fibre bundles since any **covering space** whose fibres have constant cardinality, is a fibre bundle. We state this as a theorem and give a couple of examples.

**Theorem 3.** *Let $p : E \to B$ be a covering map whose fibres $p^{-1}(b)$ are all bijective with a given set $F$. Then $p : E \to B$ is a fibre bundle with discrete fibre $F$.*

*Proof.* Let $b \in B$. By definition of a covering space [Mun00, p. 336] we can find a neighborhood $U$ of $b$ which is evenly covered by $p$. Again by definition we can then write $p^{-1}(U)$ as the disjoint union of open sets $V_\alpha$, such that $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for each $\alpha$. In particular each $V_\alpha$ contains precisely one point of $p^{-1}(b)$ which is then seen to be discrete. We now define $h$ as

$$\begin{array}{ccc} h : p^{-1}(U) & \longrightarrow & U \times p^{-1}(b) \\ \| & \nearrow & \\ \sqcup V_\alpha & & \end{array}$$

that is $h$ is the map sending $v \in V_\alpha$ to $(p(v), \tilde{b})$ where $\tilde{b}$ is the element of $p^{-1}(b)$ inside $V_\alpha$. Then $h$ is continuous [Mun00, Theorem 18.4]. Furthermore it has the inverse map $g : U \times p^{-1}(b) \to \sqcup V_\alpha$ given by $g(u, \tilde{b}) = (p|_{V_\alpha})^{-1}(u)$ where $\alpha$ is chosen such that $\tilde{b} \in V_\alpha$. Therefore $h$ is a homeomorphism. We now see that the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad h \quad} & U \times p^{-1}(b) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U & \end{array}$$

commutes: $\pi_1 \circ h(v) = \pi_1(p(v), \tilde{b}) = p(v)$ as required. Finally choose a bijection $\varphi_b : p^{-1}(b) \to F$ which is possible by assumption. Then the composite

$$p^{-1}(U) \xrightarrow{h} U \times p^{-1}(b) \xrightarrow{Id \times \varphi_b} U \times F$$

is the required trivialization and the bijection $\varphi_b$ insures that $F$ is discrete. □

**Example 4.** The map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map [Mun00]. For any $b \in S^1$ we have the fibre $p^{-1}(b) \simeq \mathbb{Z}$. By theorem 3 we conclude that $p : \mathbb{R} \to S^1$ is a fibre bundle with fibre $\mathbb{Z}$. Alternatively we can write this fibre bundle as $\mathbb{Z} \to \mathbb{R} \to S^1$.

**Example 5.** *The map $q : S^1 \to S^1$ given by $q(z) = z^n$ is a fibre bundle with fibre $F = \{w \in \mathbb{C} \mid w^n = 1\}$.* To see this it is enough by theorem 3 to show that $p$ is a covering map whose fibres are all bijective with $F$. Let $p$ be the covering map from example 4 and define the homeomorphism $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = nx$. Pick $w \in S^1$ and define $U = S^1 - \{w\}$. Choose $x \in p^{-1}(a)$ and define $W_\alpha = (x + \alpha, x + \alpha + 1)$, $Z_\alpha = (\frac{x+\alpha}{n}, \frac{x+\alpha+1}{n})$ and $V_\alpha = p(Z_\alpha)$. Then $q^{-1}(U) = \{z \in S^1 \mid z^n \neq w\} = \bigsqcup_{\alpha=0}^{n-1} V_\alpha$ and we have the commutative diagrams:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{h} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow[q]{} & S^1 \end{array} \qquad\qquad \begin{array}{ccc} Z_\alpha & \xrightarrow{h|_{Z_\alpha}} & W_i \\ {\scriptstyle p|_{Z_\alpha}}\downarrow & & \downarrow{\scriptstyle p|_{W_\alpha}} \\ V_i & \xrightarrow[q|_{V_\alpha}]{} & U \end{array}$$

The restriction $q|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for each $\alpha$, being the composite of three homeomorphisms. It is clear that every point $b \in S^1$ has such a neighborhood $U$ we just have to pick $w \neq b$ hence $q$ is a covering map. If we let $b = e^{i\theta}$ for some $\theta \in [0, 2\pi)$ then we see that the fibres are $q^{-1}(b) = \{z \in S^1 \mid z^n = b\} = \{e^{i\theta_k} \mid \theta_k = \frac{\theta}{n} + k\frac{2\pi}{n}, k = 0, \ldots, n-1\}$ which is homeomorphic to $F$ by the homeomorphism $g : q^{-1}(b) \to F$ defined by $g(z) = ze^{-i\theta/n}$. We conclude that $S^1$ is a fibre bundle over $S^1$ with fibre $F$, that is $F \to S^1 \to S^1$ as required.

**Example 6.** Define $E = \{(A, \mathbf{v}) \mid A \in GL(n, \mathbb{R}), \mathbf{v} \in \mathbb{R}^n, A^{-1}\mathbf{v} \in \mathbb{R}e_1\}$. Then $p : E \to GL(n, \mathbb{R})$ given by $p(A, \mathbf{v}) = A$ is a fibre bundle with fibre $\mathbb{R}e_1$. To see this we simply note that since the determinant map is continuous and $GL(n, \mathbb{R}^n)$ is the inverse image of this map on the open set $\mathbb{R} - 0$, it is itself open. We can write $E = GL(n, \mathbb{R}) \times \{v \in \mathbb{R}^n \mid A^{-1}v \in \mathbb{R}e_1\}$ since for any matrix we can simply pair it with the 0-vector. Now it is clear that we have a fibre bundle with the homeomorphism $h : E \to GL(n, \mathbb{R}^n) \times \mathbb{R}e_1$ given by $h(A, v) = (A, A^{-1}v)$. We can write it as $\mathbb{R}e_1 \to E \to GL(n, \mathbb{R}^n)$.

## 2.2 Circle-bundles and the Hopf map

Circle bundles are, as the name suggests, fibre bundles where the fibres are circles. They will play a prominent role in the following sections, since circle bundles over surfaces are a special case of Seifert fibered spaces.

**Example 7.** As a first example we will look at the **Klein bottle**, $K^2$. We claim that $K^2$ is the total space of a circle bundle $S^1 \to K^2 \to S^1$. To see this we will exploit our knowledge of polygonal regions. Let $K^2 = I \times I/\sim$ where $\sim$ is given by pasting the edges together according to the following scheme

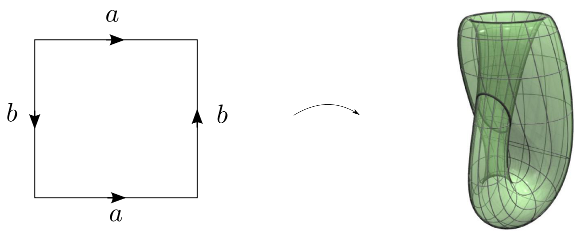



Figure 6: The Klein bottle immersed in 3 dimensions

Then $K^2$ is the Klein Bottle. Let $I/\sim$ be the unit interval (mod 1) which is clearly homeomorphic to the circle $S^1$, one could simply note that the continuous and surjective map $f_S : I \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$ descends to a homeomorphism $\bar{f}_S : I/\sim \to S^1$. Let the projection maps $I \times I \to K^2$ and $I \to I/\sim$ be denoted $\pi$ and $\pi'$ respectively. Define the map $p : K^2 \to I/\sim$ by $p([(x, y)]) = [x]$. Now let $U = \{[x] \mid x \neq 1/2\}$ and $V = \{[x] \mid x \in (0, 1)\}$. Then $I/\sim = U \cup V$ and we claim that these are both homeomorphic to the open unit interval $(0, 1)$. To see this we note that we have the diagrams

$$\begin{array}{ccc} [0,1/2) \cup (1/2,1] & & \\ \downarrow{\scriptstyle \pi'} & \searrow^{f_U} & \\ U & \xrightarrow[\bar{f}_U]{} & (0,1) \end{array} \qquad \begin{array}{ccc} (0,1) & & \\ \downarrow{\scriptstyle \pi'} & \searrow^{f_V} & \\ V & \xrightarrow[\bar{f}_V]{} & (0,1) \end{array}$$

where $f_U(x) = 1_{[0,1/2)}(x)\frac{1}{2} - 1_{(1/2,1]}(x)\frac{1}{2}$ and $f_V$ is the identity. It is easy to verify, by the same methods as in example 2, that these do in fact induce homeomorphisms $\bar{f}_U : U \to (0, 1)$ and $\bar{f}_V : V \to (0, 1)$ [Mun00, 22.2]. Furthermore we have

$$p^{-1}(U) = \{[(x, y)] \in K^2 \mid x \neq 1/2\} \qquad \text{and} \qquad p^{-1}(V) = \{[(x, y)] \in K^2 \mid x \in (0, 1)\}. \tag{37}$$

These are both homeomorphic to $\operatorname{Int} I \times I/\sim$ since we have the diagrams

$$\begin{array}{ccc} [0,1/2) \cup (1/2,1] \times [0,1] & & \\ \downarrow{\scriptstyle \pi} & \searrow^{f} & \\ p^{-1}(U) & \xrightarrow[\bar{f}]{} & \operatorname{Int} I \times I/\sim \end{array} \qquad \begin{array}{ccc} (0,1) \times [0,1] & & \\ \downarrow{\scriptstyle \pi} & \searrow^{g} & \\ p^{-1}(V) & \xrightarrow[\bar{g}]{} & \operatorname{Int} I \times I/\sim \end{array}$$

where $f(x, y) = (x + 1/2, \pi'(y))1_{[0,1/2)}(x) + (x - 1/2, \pi'(y))1_{(1/2,1]}$ and $g(x, y) = (x, \pi'(y))$ which are clearly continuous [Mun00, 18.2 (f)]. Along the lines of the previous verifications, one can easily see that we get induced homeomorphisms as stated.

Now we define the homeomorphisms $h_U : p^{-1}(U) \to U \times S^2$ by $h_U(x, y) = (\bar{f}_U^{-1} \times \bar{f}_S) \circ \bar{f}$ and $h_V : p^{-1}(V) \to V \times S^1$ by $h_V(x, y) = (\bar{f}_V^{-1} \times \bar{f}_S) \circ \bar{g}$ which are the desired local trivializations. Thus we have a circle bundle $S^1 \to K^2 \to S^2$ as required.

**Note 5.** Notice that the Klein bottle is *not* a product space - the corresponding one would be $S^1 \times S^1$, the torus. This relationship can in fact be exploited to give a broader definition of Seifert manifolds, than the one we are going to use, see [Sco83, p. 428]. Furthermore we note, that the Klein bottle is an example of a closed nonorientable manifold – a 'horizontal' path in the square corresponds to a Möbius strip.

### 2.2.1 The Hopf Fibration

In this section we will show that $S^3$ is the total space of a fibre bundle $S^1 \to S^3 \to S^2$ called the **Hopf fibration**. In order to define the projection $\phi : S^3 \to S^2$, which is called the **Hopf map**, it will be useful to decompose the base space $S^2$, which we recognize as the Riemann sphere, into the two subsets $U_p = S^2 - \{p\}$ and $U_q = S^2 - \{q\}$ where $p = (0, 0, 1)$ and $q = (0, 0, -1)$ are the 'north' and 'south' pole respectively. Recall that the 3-sphere can be written as $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Then we can define the functions

$$f_p : U_p \to \mathbb{C} \quad \text{by} \quad f_p(x, y, z) = \frac{x + iy}{1 - z} \qquad \text{and} \qquad f_q : U_q \to \mathbb{C} \quad \text{by} \quad f_q(x, y, z) = \frac{x - iy}{1 + z}. \tag{38}$$

We recognize these as slight variations of the stereographic projection maps from example 1 hence just as we did there one can show that they are homeomorphisms. Now the transition map $f_q \circ g_p : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$

$$\mathbb{C} - \{0\} \xrightarrow{g_p} S^2 - \{p, q\} \xrightarrow{f_q} \mathbb{C} - \{0\}$$

is given by $z \mapsto z^{-1}$ as can be verified by inserting in the functions from example 1, hence it is just an inversion and therefore a Möbius transformation [Ber10, §9]. We can now define the Hopf map, $\phi : S^3 \to S^2$ by

$$\phi(z, w) = \begin{cases} g_p(z/w) & \text{if } w \neq 0 \\ g_q(w/z) & \text{if } z \neq 0 \end{cases} \tag{39}$$

It is well defined since for $w, z \neq 0$ we have

$$g_q(w/z) = g_q\Big(\frac{1}{z/w}\Big) = g_q \circ f_q \circ g_p(z/w) = g_p(z/w) \tag{40}$$

as required. Furthermore it is continuous and surjective [Mun00, 18.2 (f)]. One can easily check that the fibres are circles. For example

$$\phi^{-1}(1, 0, 0) = \phi^{-1}(g_p(1)) = \{(z, w) \in S^3 \mid z/w = 1\} = \{(z/\sqrt{2}, z/\sqrt{2}), \mid |z| = 1\}. \tag{41}$$

In fact $\phi$ makes $S^3$ into a circle bundle over $S^2$ since we have

$$\phi^{-1}(U_p) = \{(z, w) \in S^3 \mid g_p(z/w) \in S^2 - \{p\}, w \neq 0\} = \{(z, w) \in S^3 \mid w \neq 0\} \tag{42}$$

and we can define the following function

$$h_p : \{(z, w) \in S^3 \mid w \neq 0\} \to U_p \times S^1 \qquad \text{by} \qquad h_p(z, w) = (g_p(z/w), w/|w|) \tag{43}$$

which is continuous since $g_p$ is a homeomorphism [Mun00, 18.4]. Furthermore it has inverse given by $h_p^{-1}(g_p(u), v) = (\frac{uv}{\sqrt{1+|u|^2}}, \frac{v}{\sqrt{1+|u|^2}})$ which is also continious hence $h_p$ is a homeomorphism. We now see that the diagram

$$\begin{array}{ccc} \phi^{-1}(U_p) & \xrightarrow{\quad h_p \quad} & U_p \times S^1 \\ & {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U_p & \end{array}$$

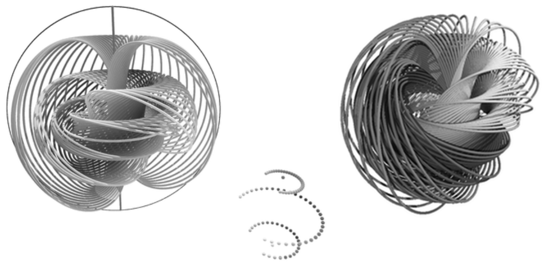


Figure 7: The Hopf Fibration, http://www.nilesjohnson.net/ppics.html

commutes: $\pi_1 \circ h_p(z,w) = \pi_1(g_p(z/w), w/|w|) = g_p(z/w) = \phi(z,w)$. Likewise we have for the remaining neighborhood $U_q$ that $\phi^{-1}(U_q) = \{(z,w) \in S^3 \,|\, z \neq 0\}$. Following the approach above with the function $h_q$ defined in an analogous way as $h_p$ we see that $\phi^{-1}(U_q)$ and $U_q \times S^1$ are homeomorphic. We have now obtained the Hopf fibration $S^1 \to S^3 \to S^2$. Every point of the two-sphere comes from a distinct circle on the 3-sphere.

It can be shown that the stereographic projection sends circles to circles. Therefore the preimage under the Hopf map of each circle of lattitude on the 2-sphere results in a union of circles on the 3-sphere which is still a union of circles after stereographic projection. It turns out that this union is actually a torus, see figure 7. By changing the lattitude of the circle on the 2-sphere we get nested tori which fill up 3-dimensional space. See [Lyo03] for more implications of the Hopf map, as well as a quick and dirty exposition of the Hopf Fibration using quaternions.

The 3-sphere $S^3$ is not globally a product space $S^2 \times S^1$ though since we have $\pi_1(S^3) = e$ because $S^3$ is simply connected [Mun00, 59.3] and $\pi_1(S^2 \times S^1) = e \times \mathbb{Z} \simeq \mathbb{Z}$ by the fact that the fundamental group of a product space is isomorphic to the product of the fundamental groups of each space [Mun00, 60.1].

# 3 Seifert Manifolds

A Seifert fibered space is a 3-manifold which is a union of pairwise disjoint circles, called fibres, arranged in a specific way. It is thus like an ordinary smooth circle bundle, but allows for a finite number of 'multiple' fibres where the local model incorporates a 'twist'. We are therefore capable of creating a much broader family of manifolds than just smooth circle bundles. These spaces were first investigated by Herbert Seifert in the 1930's [Sei80]. We will take a more modern approach inspired by Hatcher [Hat07].

## 3.1 Model Seifert fibration

Just as an $m$-manifold is defined by each *point* being homeomorphic to the local model $\mathbb{R}^m$, a Seifert fibred space is defined using a collection of local models for neighborhoods of *fibres* each of which is a circle. The local models are called 'model Seifert fiberings' which we will now construct explicitly followed by a derivation of several of their important properties.

### 3.1.1 Construction

First choose $p, q \in \mathbb{Z}$, $q \neq 0$. Let $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$ be the unit complex disc and decompose $[0,1] \times D^2$ into the segments $[0,1] \times \{z\}$ for $z \in D^2$. Define $\tau : D^2 \to D^2$ by $\tau(z) = ze^{2\pi i p/q}$ so that $\tau$ turns $D^2$ by $p/q$ of a full circle. Clearly $\tau$ is a homeomorphism and it is completely determined by the rational number $p/q$ (mod 1). Therefore we can without loss of generality assume that $p$ and $q$ are co-prime and $0 \leq p < q$. Define the equivalence relation $\sim$ on $[0,1] \times D^2$ by

$$(t_1, z_1) \sim (t_2, z_2) \Leftrightarrow \begin{cases} (t_1, z_1) = (t_2, z_2) \\ (0, z_1) = (1, \tau^{-1}(z_2)) \end{cases} \tag{44}$$

Then the **model Seifert fibering** is defined as the quotient space $[0,1] \times D^2/\sim$.

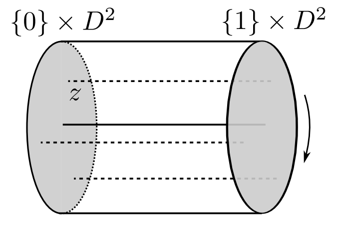



Figure 8: The space $[0,1] \times D^2$ with a segment $[0,1] \times \{z\}$

For a given $z \in D^2 - \{0\}$ there is a finite number of images under powers of $\tau$. To see this note that $\tau^k \circ \tau^n(z) = \tau^l(z)$ where $l = k + n$ (mod $q$) since the equation is satisfied if and only if for $a \in \mathbb{Z}$ we have $\frac{p(k+n)}{q} + a = \frac{pl}{q}$ which since $p$ and $q$ are coprime is if and only if $l - (k+n) = qa$ which is $l \equiv k + n$ (mod $q$) by definition. Furthermore $\tau^q(z) = z$. We conclude that there are exactly $q$ different elements in an orbit. For $z = 0$ the orbit is clearly just 0. We can now describe the model Seifert fibering as a union of circles. Define arcs $A_z = \{[(t, z)] \mid t \in [0,1]\}$ for fixed $z \in D^2$. Then $A_0$ is a circle and for $z \neq 0$ the union of arcs $\bigcup_{i=0}^{q-1} A_{\tau^i(z)}$ constitute a circle and $A_{\tau^i(z)} = A_{\tau^j(z)} \Leftrightarrow i \equiv j$ (mod $q$). These circles are called **fibres**.

**Lemma 10.** *The model Seifert fibering $[0,1] \times D^2/\sim$ is homeomorphic to the solid torus $S^1 \times D^2$.*

*Proof.* Define the quotient map $p : [0,1] \times D^2 \to [0,1] \times D^2/\sim$ by $p(t,z) = [(t,z)]$ and the continuous map $g : [0,1] \times D^2 \to S^1 \times D^2$ by $g(t,z) = (e^{2\pi i t}, ze^{2\pi i t p/q})$. Then $g$ is constant on each set $p^{-1}([t,z])$ since we have

$$g(1, z_1) = (e^{2\pi i \cdot 1}, z_1 e^{2\pi i p/q \cdot 1}) = (1, \tau(z_1)) = (e^{2\pi i \cdot 0}, \tau(z_1)e^{2\pi i p/q \cdot 0}) = g(0, \tau(z_1)) \tag{45}$$

It now follows by [Mun00, 22.2] that $g$ induces a continuous map $f_{p/q} : [0,1] \times D^2/\sim \to S^1 \times D^2$ such that $f_{p/q} \circ p = g$ hence $f_{p/q}([t,z]) = (e^{2\pi i t}, ze^{2\pi i t p/q})$. Define $h : S^1 \times D^2 \to [0,1] \times D^2/\sim$ by $h(w,z) = [(\frac{Log(w)}{2\pi i}, ze^{-2\pi i p/q})]$ where $Log$ denotes the principal logarithm. This is continuous, see [Ber10, §5] and one easily checks that it is inverse to $f_{p/q}$. $\square$

**Note 6.** The fibres in $[0,1]\times D^2/\sim$ are as noted earlier the union of the arcs $A_z = \{[(t,z)]\,|\,t\in[0,1]\}$ for fixed $z\in D^2$. Now $A_z$ gets sent by $f_{p/q}$ to $f_{p/q}(A_z) = \{(e^{2\pi i t}, ze^{2\pi i t p/q})\,|\,t\in[0,1]\}$ and similarly,

$$f_{p/q}\Big(\bigcup_{i=0}^{q-1} A_{\tau^i(z)}\Big) = \{f_{p/q}([(t,z)])\,|\,t\in[0,q]\} = \{(e^{2\pi i t}, ze^{2\pi i t p/q})\,|\,t\in[0,q]\}. \tag{46}$$

These sets are called the **circles of slope** $p/q$. Therefore the model Seifert fibration is exactly $S^1\times D^2$ fibered by the circles of slope $p/q$. We can now define the model Seifert fibering in another more concise manner:

**Definition 12.** The **model Seifert fibering** with parameter $p/q$ where $p,q\in\mathbb{Z}$ are coprime is the decomposition of $S^1\times D^2$ into the circles

$$C_z = \{(e^{2\pi i t}, e^{2\pi i t p/q}z)\,|\,t\in[0,q]\} \tag{47}$$

for $z\in D^2$. $C_z$ is the fibre through $(1,z)$ and $C_0$ is called the **core**. The **open** model Seifert fibering with parameter $p/q$ is $S^1\times \operatorname{Int} D^2$ with the induced decomposition into circles.

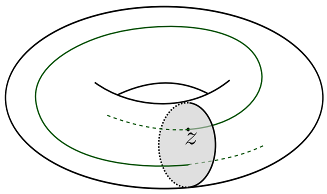



Figure 9: The model Seifert fibration with the circle $C_z$

**Note 7.** In definition 12 we have omitted the condition that $0\le p<q$. This is due to the following obervation. Let $p/q, p'/q'\in\mathbb{Q}$ where each pair $p,q$ and $p',q'$ are coprime. Suppose that $p/q\equiv p'/q'$ (mod 1) but that $p/q\neq p'/q'$. Then the model Seifert fibration with parameter $p/q$, $M_{p/q}$, and the model Seifert fibration with parameter $p'/q'$, $M_{p'/q'}$, are not strictly equal. However there is a diffeomorphism $\varphi: M_{p/q}\to M_{p'/q'}$ sending fibres to fibres. Writing $p'/q' = p/q + r$ for $r\in\mathbb{Z}$ let it be given by $\varphi(x,z)\to(x,x^r z)$. Then we have $\varphi(e^{2\pi i t}, e^{2\pi i p/q}z) = (e^{2\pi i t}, e^{2\pi i p'/q'})$ thus $\varphi$ does send circles of slope $p/q$ to circles of slope $p'/q'$. The point is however that $\varphi$ does *not* fix the boundary. This does not matter for the model Seifert fibration alone, but it does impact the general setting. We shall return to this in section 4.2.

**Lemma 11.** *The **boundary circle of slope** $p/q$ in $S^1\times S^1$, defined as the circle that lifts to the line $y=\frac{p}{q}x$ in $\mathbb{R}^2$, is uniquely determined by the set of pairs $(e^{2\pi i x}, e^{2\pi i p/q x})$ for $x\in\mathbb{R}$.*

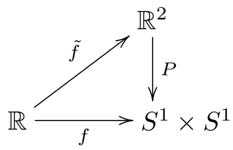
*Proof.* To see this we note that by [Mun00] we have a covering map of the torus given by

$$P = p\times p : \mathbb{R}^2 \to S^1\times S^1 \tag{48}$$

where $p:\mathbb{R}\to S^1$ is the usual covering map of the circle given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. We thus have the following using Eulers formulas [Ber10, §1]

$$P(x_1,x_2) = (\cos 2\pi x_1, \sin 2\pi x_1, \cos 2\pi x_2, \sin 2\pi x_2) \simeq (e^{2\pi i x_1}, e^{2\pi i x_2}). \tag{49}$$

Define the map $\tilde f:\mathbb{R}\to\mathbb{R}^2$ by $\tilde f(x) = (x, \frac{p}{q}x)$. Now by definition of a **lifting** the diagram




must commute hence we must have $P\circ\tilde f = f$ where $f:\mathbb{R}\to S^1\times S^1$ is given by $f(x) = P(x,\frac{p}{q}x) = (e^{2\pi i x}, e^{2\pi i p/q x})$ which was what we wanted. Uniqueness follows by the uniqueness of liftings upon specification of a point $e_0\in\mathbb{R}^2$ [Mun00, 54.1]. $\square$

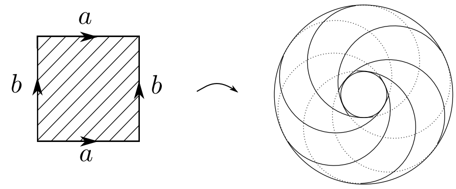



Figure 10: The lines and circles of slope $p/q$ in the special case of slope 1.

## 3.2 Definition and properties of Seifert fibered manifolds

We will now define Seifert fibered manifolds as well as some central notions for these including multiplicities of fibres. Then we exploit our knowledge of model Seifert fibrations to obtain some structure on a Seifert fibered space through embeddings.

### 3.2.1 Definition, multiplicities and imbedding

**Definition 13.** A **Seifert fibered manifold** is a 3-manifold $M$ decomposed into circles called **fibres** such that each circle has a neighborhood diffeomorphic to an *open* model Seifert fibering and the diffeomorphism sends fibres to fibres.

**Definition 14.** Each fibre circle $C_z$ in a Seifert fibering has a **multiplicity** given by the number of times a small disk transverse to $C_z$ meets each nearby fibre. Fibres of multiplicity 1 are called **regular** while the others are called **multiple**. Compare with [Hat07].

**Lemma 12.** *The multiplicity of a fibre is well defined and the multiple fibres are isolated in the interior of the Seifert fibered manifold.*

*Proof.* By definition of a Seifert fibered manifold, each neighborhood of a fibre is diffeomorphic to the open model Seifert fibering hence it is sufficient to look at this case. Given $C_{z'}$, for $z' \in \operatorname{Int} D^2 - \{0\}$, in the model Seifert fibration we must find an open disc with centre $z'$, which only intersects each nearby fibre once. We can write $z' = r'e^{2\pi i t'}$. Since applying $\tau$ to $z'$ corresponds to multiplication by $e^{2\pi i p/q}$, that is with a turn in $p/q$ of a full circle, we can define the slice

$$S = \{re^{2\pi i t} \,|\, t \in (t' - \frac{1}{2q}, t' + \frac{1}{2q}), r \in (0,1)\} \subset \operatorname{Int} D^2. \tag{50}$$

This is an open set hence by definition we can find an open ball, $B(z', \epsilon)$ for some $\epsilon > 0$, contained in $S$. Furthermore $S$ satisfies that for any point in $S$ the remaining corresponding orbit is disjoint from $S$ i.e. $\tau^k(S) \cap \tau^n(S) = \emptyset$ or $\tau^k(S) = \tau^n(S)$ for $k \not\equiv n \pmod q$ and $k \equiv n \pmod q$ respectively. To see this we first note that $S$ 'spans' an argument of $1/q$ of a full circle thus $\tau^j(S)$ is clearly either $S$ or $\emptyset$. It is therefore sufficient to see when the 'endpoint' $t' - \frac{1}{2q} + \frac{kp}{q} \equiv t' - \frac{1}{2q} + \frac{np}{q} \pmod 1$. This happens if and only if $\frac{kp}{q} \equiv \frac{np}{q} \pmod 1$. By definition of the equivalence class this happens if and only if $1|\frac{p(k-n)}{q}$ that is if and only if $q|(k-n)$ since $p, q$ are coprime, thus $k \equiv n \pmod q$ as stated. We conclude that every point in $\operatorname{Int} D^2 - \{0\}$ has a small disc transverse to it which meets every nearby fibre exactly once. The centre $z' = 0$ has a special role. Since $\tau$ preserves the modulus, every orbit in the disc around 0 is fully contained there hence every fibre is intersected exactly $q$ times which gives $z'$ multiplicity $q$.

A given multiple fibre has in particular a neighborhood diffeomorphic to an open model Seifert fibering hence we conclude that any multiple fibre must be the core fibre, and since this is the only multiple one it is clearly isolated. □

**Lemma 13.** *Every fibre can be considered as the core of a possibly trivial model Seifert fibration.*

*Proof.* Given $z \in D^2 - \{0\}$ define the **imbedding** [Mun00, p. 105] $\lambda_z : S^1 \times D^2 \hookrightarrow S^1 \times D^2$ of a trivial open model Seifert fibering in the open model Seifert fibering with parameter $p/q$ by $\lambda_z(u, v) = (u^q, v^q\epsilon + z)$ where $\epsilon$ is the radius of the disc found in lemma 12. Then $\lambda_z(C_0) = \lambda_z(u, 0) = (u^q, z) = (e^{2\pi i q t}, ze^{2\pi i p}) = (e^{2\pi i t'}, ze^{2\pi i t' p/q}) = C_z$ where $t' = t/q$. We thus see that the core gets sent to the fibre through $(1, z)$. We furthermore see that $\{w\epsilon + z | w \in D^2\} = B(z, \epsilon)$ hence in general $\lambda_z$ sends fibres to fibres and $\lambda_z(\{(u, w) \,|\, w \in D^2\}) = S^1 \times D^2$ is the trivial model Seifert fibration. □

## 3.3 The base surface

In this section we will show that by projecting each fibre of a Seifert fibered manifold to a point in the base space $B$, we obtain a surface. We can then mark this manifold with certain points, namely the points corresponding to multiple fibres, which are isolated by lemma 12. The label given is the parameter $p/q$ of the local model for the fibre. We will furthermore use this to explain the connection to general smooth circle bundles.

### 3.3.1 Projecting fibres

**Definition 15.** Let $M$ be a Seifert fibered manifold. Define $B$ to be the space of fibres of $M$, that is $B = M/\sim$ where $\sim$ is defined by $m \sim m' \Leftrightarrow m, m'$ lie on the same fibre. We furthermore define the quotient map $\pi : M \to B$ by $\pi(m) = [m]$.

**Lemma 14.** *The base space for the open model Seifert fibering can be identified with* $\operatorname{Int} D^2$.

*Proof.* Let $M$ be the open model Seifert fibering with parameter $p/q$. Since $M$ clearly is diffeomorphic to a an open subset of a Seifert manifold, it is itself Seifert fibered by lemma 24. Define

$$\pi_{p/q} : S^1 \times \operatorname{Int} D^2 \to \operatorname{Int} D^2 \qquad \text{by} \qquad \pi_{p/q}(u, v) = u^{-p} v^q \tag{51}$$

Then $\pi_{p/q}(e^{2\pi it}, e^{2\pi itp/q} z) = z^q$ hence we see that $\pi_{p/q}(C_z) = z^q$. In other words $\pi_{p/q}$ is a quotient map since it is surjective and sends saturated closed sets to closed sets. We can therefore identify $\pi$ with $\pi_{p/q}$ and $B$ with $\operatorname{Int} D^2$. The same argument applies for model Seifert fiberings, the base space being $D^2$. □

We see that in this case $B$ is a smooth surface. In fact this happens in general:

**Theorem 4.** *The base space $B$ can be given the structure of a smooth surface in such a way that the projection mapping $\pi$ becomes a smooth map.*

Before proving the theorem, let us show the following technical lemma guaranteeing the existence of a map from the base space to the open model Seifert fibration.

**Lemma 15.** *For any $x \in \operatorname{Int} D^2 - \{0\}$ there exists a neighborhood $U$ on which there is a smooth function $\eta : U \to S^1 \times \operatorname{Int} D^2$ with the property that $\pi_{p/q} \circ \eta = id_U$.*

*Proof.* Pick $x \in D^2 - \{0\}$ and write $x = |x|e^{i\alpha+\pi}$. Define the cut plane $\mathbb{C}_\alpha = \mathbb{C} - \{re^{i\alpha} \mid r \geq 0\}$ and the neighborhood $U = \mathbb{C}_\alpha \cap \operatorname{Int} D^2$. Then we have a holomorphic branch of the $q$'th root function:

$$\rho_q|_U : U \to \left\{ z \in \mathbb{C} - \{0\} \,\middle|\, |Arg_\alpha z| < \frac{\pi}{q} \right\} \cap \operatorname{Int} D^2 \qquad \text{defined by} \qquad \rho_q|_U(z) = \sqrt[q]{|z|} e^{i \frac{Arg_\alpha z}{q}}. \tag{52}$$

This is furthermore bijective with inverse $z \mapsto z^q$ hence it is a homeomorphism. Now choose $t' \in [0, 1)$ and define $\eta|_U : U \to S^1 \times \operatorname{Int} D^2$ by

$$\eta|_U(x) = (e^{2\pi it'}, e^{2\pi it'p/q} \rho_q|_U(x)). \tag{53}$$

Then we obviously have $\pi_{p/q}(\eta|_U(x)) = x$ as required. Since both coordinates of $\rho_U$ are smooth so is $\eta|_U$. We thus see that every point $x \in \operatorname{Int} D^2 - \{x\}$ has a neighborhood $U$ such that $\eta|_U$ is smooth and has the desired property. □

*Proof of theorem 4.* To prove Hausdorffness of $B$ let $a, b$ be distinct fibres in $M$. Since fibres are compact and $M$ is Hausdorff it follows by lemma 20 that we can find disjoint open sets $U, V$ containing $a$ and $b$ respectively. By definition of a Seifert fibered manifold we can find a neighborhood $N$ of $a$ such that $N$ is diffeomorphic by $\varphi$ to an open model Seifert fibration, and by lemma 13 we can without loss of generality assume that $a$ is the core. Now define $\Omega = N \cap U$ which is then a neighborhood of $a$ in $M$ thus $\varphi(\Omega)$ is a neighborhood of the core $S^1 \times \{0\}$ in $S^1 \times \operatorname{Int} D^2$. We claim that there is a $\delta > 0$ such that $S^1 \times B(0, \delta)$ is contained in $\varphi(\Omega)$. To see this note that $S^1$ is compact and the open set $\varphi(\Omega)$ of the product space $S^1 \times \operatorname{Int} D^2$ contains the slice $S^1 \times \{0\}$ thus by [Mun00, Tube lemma] $\varphi(\Omega)$ contains some 'tube' $S^1 \times B(0, \delta)$ for $\delta > 0$. Note that this 'tube' is saturated by construction of the model Seifert fibration. But then $\Omega = N \cap U$ contains the set $U' = \varphi^{-1}(S^1 \times B(0, \delta))$ which is saturated since $\varphi$ sends fibres to fibres. In the same way we can find a saturated neighborhood

$V'$ of $b$ contained in $V$. It now follows by definition of the quotient topology that $\pi(U')$ and $\pi(V')$ are open subsets of $B$ containing $a$ and $b$ respectively and that $\pi(U') \cap \pi(V') = \emptyset$ thus $B$ is Hausdorff.

Pick $[x] \in B$. We must find a neighborhood $W_x$ of $[x]$ which is diffeomorphic to an open set in $\mathbb{R}^2$. Since $M$ is Seifert fibered we can find a neighborhood $V_x$ of $x$ consisting of points constituting fibres which is diffeomorphic by $\varphi_x$ to the open model Seifert fibration. By lemma 13 we can without loss of generality assume that $x$ is contained in the core fibre. $V_x$ is saturated since it contains the entire fibres and since $\pi$ is a quotient map we conclude that $\pi(V_x) = W_x$ is an open neighborhood of $[x] \in B$. We furthermore see that $g_x = \pi_{p/q} \circ \varphi_x$ is continuous and surjective and that

$$W_x = \{g_x^{-1}(\{z\}) \mid z \in \operatorname{Int} D^2\}. \tag{54}$$

We conclude, since $g_x$ is a quotient map, that $g_x$ induces a homeomorphism $f_x : W_x \to \operatorname{Int} D^2$ by $f_x([y]) = \pi_{p/q} \circ \varphi_x(y)$ and because $\operatorname{Int} D^2$ is Hausdorff so is $W_x$ [Mun00, 22.3].

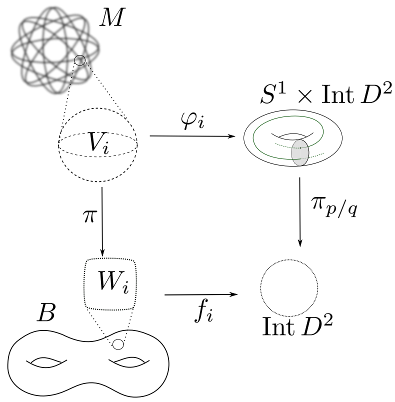



Figure 11: The Seifert fibered space $M$ and the base space $B$ with corresponding maps.

We must show that the collection $\{f_i \mid i \in B\}$ satisfies the transition on overlaps criterion, namely that the following function, remembering that $W_i = \pi(V_i)$, is smooth for any $i, j \in B$:

$$f_j \circ f_i^{-1} : f_i((\pi(V_i) \cap \pi(V_j)) \to f_j((\pi(V_i) \cap \pi(V_j)). \tag{55}$$

The open set $V_i \cap V_j$ consists of fibres, since by definition of a model Seifert manifold the fibres are disjoint. Furthermore the intersection does not contain any multiple fibre if $i \neq j$, since an open model Seifert fibering has only one multiple fibre. We have $f_i(\pi(V_i) \cap \pi(V_j)) = f_i(\pi(V_i \cap V_j)) = \pi_{p/q}(\varphi_i(V_i \cap V_j)) \subset \operatorname{Int} D^2 - \{0\}$. By lemma 15 we see that for any point $x$ here, we can find a neighborhood $U_x$ on which there exists a smooth function $\eta|_U$ such that $\pi_{p/q} \circ \eta|_U = id_U$ hence we have the diagram:

$$\begin{array}{ccccc}
 & S^1 \times \operatorname{Int} D^2 & \xleftarrow{\varphi_i} & M & \xrightarrow{\varphi_j} & S^1 \times \operatorname{Int} D^2 \\
 \nearrow^{\eta|_U} & \downarrow \pi_{p/q} & & \downarrow \pi & & \downarrow \pi_{p'/q'} \\
U \subset & \operatorname{Int} D^2 & \xleftarrow[f_i]{} & B & \xrightarrow[f_j]{} & \operatorname{Int} D^2
\end{array}$$

It is now evident that the transition map can be rewritten as follows

$$(f_j \circ f_i^{-1})|_U = f_j \circ f_i^{-1} \circ \pi_{p/q} \circ \eta|_U = f_j \circ \pi \circ \varphi_i^{-1} \circ \eta_U = \pi_{p'/q'} \circ \varphi_j \circ \varphi_i^{-1} \circ \eta|_U \tag{56}$$

which is smooth since it is the composition of $\eta|_U$ and $\pi_{p'/q'}$ which are smooth by definition, and $\varphi_k$ for $k \in B$ are smooth being diffeomorphisms. Since smoothness is a local property by lemma 3 we see that $f_j \circ f_i^{-1}$ is smooth as desired. We thus conclude that $\mathcal{A} = \{(W_i, \operatorname{Int} D^2, f_i)\}_{i \in B}$ is a smooth atlas hence $B$ has the structure of a smooth surface as required. It follows by commutativity of the diagram above, that $\pi$ is a composition of smooth maps $\pi = f^{-1} \circ \pi_{p/q} \circ \varphi$ hence smooth itself as required. □

**Lemma 16.** *$M$ is connected if and only if $B$ is connected. $M$ is compact if and only if $B$ is compact in which case there are only finitely many multiple fibres.*

*Proof.* If the Seifert manifold $M$ is connected, then $B$ is connected since the projection map $\pi$ is continuous. To prove the converse assume that $M$ is not connected. Then by definition we can write $M = M_1 \sqcup M_2$ for $M_1, M_2$ open and nonempty. Since the fibres of $M$ are connected, each one lies entirely within $M_1$ or $M_2$. Therefore $M_1$ and $M_2$ are saturated, open and disjoint sets, making $\pi(M_1)$ and $\pi(M_2)$ open and disjoint sets in $B$. We conclude that $B$ is not connected.

If $M$ is compact, so is $B$ since $\pi$ is continuous. Suppose $B$ is compact. Let $\mathcal{A}$ be a covering of $M$ consisting of all open model Seifert fibrations. The projection of these gives us an open covering of $B$, and since $B$ is compact it follows that we can find a finite subcollection still covering $B$. Therefore finitely many open model Seifert fibrations cover $M$, and in particular the corresponding model Seifert fibrations cover $M$. We see that $M$ is now the union of finitely many compact sets hence itself compact. In particular we see that since the open model Seifert fibrations only contains one multiple fibre, $M$ does only contain finitely many multiple fibres and $B$ only finitely many marked points. □

### 3.3.2 Connections with circle bundles

**Note 8.** We can now explain what the differences between smooth circle bundles and Seifert fibered spaces are. In theorem 4 we showed that the base space $B$ is a surface, hence the definition of a Seifert manifold can be stated as the following commutative diagram, where $\varphi$ is a diffeomorphism and $\pi_{p/q}$ is the map from note 14:

$$\begin{array}{ccc} M \supset p^{-1}(\operatorname{Int} D^2) & \xrightarrow{\quad\varphi\quad} & S^1 \times \operatorname{Int} D^2 \\ & \searrow^{\pi} \quad \swarrow^{\pi_{p/q}} & \\ & \operatorname{Int} D^2 & \\ & \cap & \\ & B & \end{array}$$

Since $\pi_{0/1} = \pi_2$ we see by comparing with the definition of smooth circle bundles that the Seifert fibered spaces are more general. In other words any smooth circle bundle over a surface is a Seifert manifold. In particular a space of the form $\Sigma \times S^1$ is a trivial circle bundle, the homeomorphism being the identity, hence Seifert fibered. We state this as a lemma, compare with [Sco83]:

**Lemma 17.** *Let $\Sigma$ be a surface. Then $\Sigma \times S^1$ is a Seifert fibered manifold. In fact any circle bundle over a surface is a Seifert fibre space.*

Furthermore the projection $\pi : M \to B$ is an ordinary fibre bundle on the complement of the multiple fibres. In the model Seifert fibration case we simply have

$$\begin{array}{ccc} M - C_0 & \xrightarrow{\qquad h \qquad} & S^1 \times (\operatorname{Int} D^2 - \{0\}) \\ & \searrow^{\pi_{p/q}|} \quad \swarrow^{\pi_2} & \\ & \operatorname{Int} D^2 - \{0\} & \end{array}$$

Where $C_0$ denotes the core fibre and the homeomorphism $h : M - C \to S^1 \times (\operatorname{Int} D^2 - \{0\})$ is given by $h(u, v) = (u, u^{-p}v^q)$. It follows that this will be the general case for any Seifert fibered manifold since by definition we can find a neighborhood of any fibre which is diffeomorphic to $M - C$ above.

# 4 Creating Seifert fibered manifolds from the base surfaces

Let $M$ be a Seifert fibered manifold and let $B$ be its space of fibres, which by theorem 4 is a smooth surface. We can mark the surface $B$ with certain points, namely the points corresponding to multiple fibres. These are isolated by lemma 12, and we can label them with the parameter $p/q$ of the local model for the fibre. We will reverse this to construct new Seifert fibered spaces.

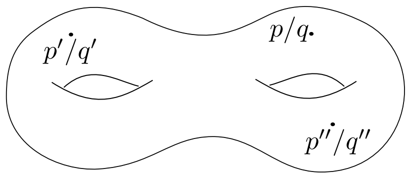



Figure 12: The base surface $B$ with labeled points corresponding to multiple fibres.

## 4.1 The main theorem

**Theorem 5.** *Suppose we are given a (compact, oriented) surface $B$ marked with finitely many points, each with a label of the form $p/q$ for $p, q$ coprime and $0 < q$. Then there exists a Seifert fibered manifold $M$ whose space of fibres is precisely $B$, and whose multiple fibres produce the prescribed labelled points on $B$.*

Let us begin with the following preliminary lemma.

**Lemma 18.** *Let $D \subset B$ be a closed disc around $x$. There is a homeomorphism*

$$\bar{\psi} : S^1 \times (D - \{x\}) \to S^1 \times (D^2 - \{0\}) \tag{57}$$

*with the following properties:*

*i. It restricts to a diffeomorphism $\psi : S^1 \times \partial D \to S^1 \times S^1$.*

*ii. It sends circles $S^1 \times \{d\}$ to circles of slope $p/q$.*

*iii. It restricts to a diffeomorphism $\psi' : S^1 \times (\mathrm{Int}\, D - \{x\}) \to S^1 \times (\mathrm{Int}\, D^2 - \{0\})$.*

*Proof.* Since $B$ is a smooth surface we can find a diffeomorphism $\omega : D^2 \to D$ such that $\omega(0) = x$, $\mathrm{Int}\, D = \omega(\mathrm{Int}\, D^2)$ and $\partial D = \omega(S^1)$. Given $p, q$ coprime and $0 < q$ we can by corollary 3.9 in Thorup [Tho07], find $b, d \in \mathbb{Z}$ such that the condition $qd - bp = 1$ is satisfied, that is such that the matrices

$$A = \begin{pmatrix} q & p \\ b & d \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \begin{pmatrix} d & -p \\ -b & q \end{pmatrix} \tag{58}$$

have determinant 1. In practice $b$ and $d$ can be found explicitly using Euclid's algorithm backwards. Now define the map $\bar{\psi} : S^1 \times (D - \{x\}) \to S^1 \times (D^2 - \{0\})$ by $\bar{\psi}(u, v') = \bar{\psi}(u, \omega(rv)) = (u^q v^b, r u^p v^d)$ where $v \in S^1$ and $r \in (0, 1]$. By $A^{-1}$ above we see that it has an inverse given by $\bar{\psi}^{-1}(u, v) = (u^d v^{-b} r^b, u^{-p} v^q r^{-q})$ thus it is a homeomorphism.

i. The restriction to $r = 1$ gives us $ry \in S^1$ and since $\partial D = \omega(S^1)$ we have the restricted map $\psi : S^1 \times \partial D \to S^1 \times S^1$ given by $\psi(x, y') = \psi(x, \omega(y)) = (x^q y^b, x^p y^d)$ which is quickly seen to be smooth, thus it is a diffeomorphism as required.

ii. We can write $S^1 \times \{d\} = S^1 \times \{\omega(rv^q)\}$ for some unique $v^q \in S^1$ and we see that $\psi(e^{2\pi it}, \omega(rv^q)) = (e^{2\pi iqt} v^{bq}, re^{2\pi ipt} v^{qd})$. Write $v = e^{2\pi i\Lambda}$, possible since $v \in S^1$. Then using $qd - bp = 1$ we see that $(e^{2\pi iqt} v^{bq}, re^{2\pi ipt} v^{qd}) = (e^{2\pi iqt} e^{2\pi i\Lambda bq}, re^{2\pi ipt} e^{2\pi i\Lambda qd}) = (e^{2\pi iq(t+\Lambda b)}, re^{2\pi ip(t+b\Lambda)} v) = (e^{2\pi it'}, re^{2\pi ip/qt'} v)$, letting $t' = (t + \Lambda b)q$. We recognize this as $C_{rv}$ hence $\bar{\psi}$ does actually send 'meridians' $S^1 \times \{d\}$ to circles of slope $p/q$ as required.

iii. The restriction to $r \in (0, 1)$ gives us $ry \in \mathrm{Int}\, D^2$ and since $\mathrm{Int}\, D = \omega(\mathrm{Int}\, D^2)$ we have the restricted map $\psi' : S^1 \times (\mathrm{Int}\, D - \{x\}) \to S^1 \times (\mathrm{Int}\, D^2 - \{0\})$ which is a diffeomorphism by the same arguments as in (i). $\square$

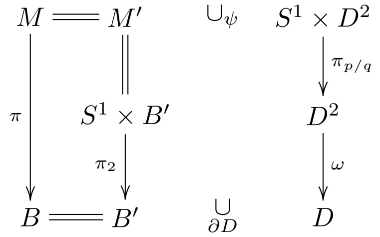
Before proving theorem 5 in its full generality, we will tackle the case where the base surface $B$ has precisely one marked point, corresponding to a single multiple fibre.

**Theorem 6.** *Given a surface $B$ with a single point labelled with the parameter $p/q$ for $p, q$ coprime, $q < 0$. Then one can construct a Seifert fibered manifold $M$ by attaching a model Seifert fibration.*

*Proof.* Take a surface $B$ with a point $x$ labeled with multiplicity $p/q$ and choose a closed disc $D \subset B$ around $x$. Since $B$ is a 2-manifold we can by definition find a diffeomorphism $\omega : D^2 \to D$ such that $\omega(0) = x$, $\operatorname{Int} D = \omega(D^2)$ and $\partial D = \omega(S^1)$. Define $B' = B - \operatorname{Int} D$. Then $B = B' \underset{\partial D}{\cup} D$. Define the circle bundle $M' = S^1 \times B'$. This is by lemmas 2 and 24 a compact manifold everywhere except along $S^1 \times \partial D$, thus by lemma 20 it is normal. By the same lemma we see that the space $S^1 \times D^2$ is normal as well. Now form the space $M$ by adjoining a Seifert fibered torus to $M'$ as follows

$$M = (S^1 \times B') \cup_\psi (S^1 \times D^2) \tag{59}$$

where $\psi$ is the diffeomorphism from lemma 18 (i), that is we identify $(x, y') \in S^1 \times \partial D$ with $\psi(x, y') \in S^1 \times S^1$. Then $M$ is an adjunction space, see appendix A.2, and thus normal itself by lemma 22, in particular Hausdorff. We claim that this identifies the part of $M'$ over $\partial D$ with the part of $S^1 \times D^2$ over $\partial D$ such that fibres are identified with fibres.




The part of $M'$ over $\partial D$ is $S^1 \times \partial D$ while the part of $S^1 \times D^2$ over $\partial D$ is $S^1 \times S^1$. The part of $M'$ over $d \in \partial D$ is then $S^1 \times \{\omega(v^q)\} = S^1 \times \{d\} \subset S^1 \times \partial D$ for some unique $v^q \in S^1$, in other words they are 'meridians'. The part of $S^1 \times S^1$ over $d = \omega(v^q)$ is by the above diagram seen to be the fibre or the circle of slope $p/q$ if one prefers:

$$C_v = \{(e^{2\pi i t}, v e^{2\pi i t p/q}) \mid t \in [0, q]\}. \tag{60}$$

It follows by lemma 18 (ii) that $\psi$ precisely sends the fibres $S^1 \times \{d\}$ to the fibres $C_v$ as required.


Figure 13: A meridian and the corresponding circle of slope $p/q$.

We need to show that $M$ is a smooth manifold. Take $U \subset M$ given by $U = (S^1 \times B') \cup_\psi (S^1 \times (D^2 - \{0\}))$ which is open in $M$ since we can write $U = M - (S^1 \times \{0\})$, and define

$$\lambda : S^1 \times (B - \{x\}) \to (S^1 \times B') \cup_\psi (S^1 \times (D^2 - \{0\})) \quad \text{by} \quad \lambda(u, z) = \begin{cases} (u, z) & \text{if } z \in B' \\ \bar{\psi}(u, z) & \text{if } z \in D - \{x\} \end{cases}$$

This is well defined since for $z \in \partial D$ we have $\bar{\psi}(u, z) \sim (u, z)$ by definition of the adjunction space. We furthermore see that the sets $S^1 \times B'$ and $S^1 \times (D - \{x\})$ are both closed in the domain with the subspace topology thus we conclude that $\lambda$ is continuous [Mun00, Pasting lemma]. Furthermore, since both branches are homeomorphisms so is $\lambda$. Define $V \subset M$ as an open model Seifert fibration $V = S^1 \times \operatorname{Int} D^2$ and let $\mu : V \to V$ be the identity map. Then $U \cup V = M$ and the 'transition property' from lemma 4 boils down to

$$\lambda^{-1}(S^1 \times (\operatorname{Int} D^2 - \{0\})) \xrightarrow{\lambda|} \lambda(S^1 \times (B - \{x\})) \cap S^1 \times \operatorname{Int} D^2 \tag{61}$$

which we can rewrite to the following

$$S^1 \times (\operatorname{Int} D - \{x\}) \xrightarrow{\psi'} S^1 \times (\operatorname{Int} D^2 - \{0\}) \tag{62}$$

which is a diffeomorphism by lemma 18 (iii). Thus $M$ is a smooth manifold by lemma 4. Finally we need to show that $M$ is Seifert fibered. To see this it is sufficient to note that the map $\lambda : S^1 \times (B - \{x\}) \hookrightarrow M - (S^1 \times \{0\})$ is a diffeomorphism sending trivial fibres $S^1 \times \{b\}$ for $b \in B - \{x\}$ to fibres. Thus all fibres except $S^1 \times \{0\}$ are the cores of a trivial model Seifert fibration. It remains to show that $S^1 \times \{0\}$ is the core of a model Seifert fibration, but this follows form the construction. □

*Proof of theorem 5.* Take a surface $B$ labelled with points $x_1, \dots x_k$ each with some given multiplicity and choose disjoint discs $D_1, \dots, D_k$ containing these respectively. Define $B' = B - \bigcup_{i=1}^k \operatorname{Int} D_i$ and the circle bundle $M' = S^1 \times B'$. Then we form the space $M = (S^1 \times B') \cup_{\psi_1} (S^1 \times D^2) \cup_{\psi_2} \cdots \cup_{\psi_k} (S^1 \times D^2)$ where $\psi_i$ is the restriction of $\bar{\psi}_i$ defined by $\bar{\psi}_i : S^1 \times (D_i - \{x_i\}) \to S^1 \times (D^2 - \{0\})$ similar to lemma 18. Using lemma 22 inductively $k$ times we see that $M$ is Hausdorff. It now follows by slight modifications, obvious from what we have already defined, that the procedure in theorem 6 goes through thus showing that $M$ is Seifert fibered. □

## 4.2 Isomorphic Seifert fibered spaces

**Definition 16.** We use the notation $M(g; \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$ for a Seifert-fibered manifold $M$ constructed as in section 4 where $g$ is the genus of the surface $B$ and $\alpha_i/\beta_i$ is the unique parameter corresponding to the $i$'th adjoined model Seifert fibration. Two Seifert fiberings are **isomorphic** if there is a diffeomorphism carrying fibers of the first to fibres of the second.

**Note 9.** The notation used in definition 16 differs from standard. Normally one would write $M(\pm g, b; \alpha_1/\beta_1, \dots \alpha_k/\beta_k)$ for a Seifert fibered manifold with $+$ if $B$ is orientable and $-$ if $B$ is nonorientable, see appendix B.2, and where $b$ is the number of boundary components of $B$, compare with [Hat07]. Since we are only dealing with orientable surfaces, the notation introduced in the definition is sufficient. Seifert originally used the notation $Oo, On, No, Nn, NnI, NnII$ and $NnIII$, see [Sei80, p. 391] for details.

**Theorem 7.** *Every compact and orientable Seifert manifold is isomorphic to one of the models $M(g, \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$. Seifert fiberings $M(g, \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$ and $M(g, \alpha'_1/\beta'_1, \dots, \alpha'_k/\beta'_k)$ are isomorphic by an orientation-preserving diffeomorphism if and only if, after possibly permuting indices, $\alpha_i/\beta_i \equiv \alpha'_i/\beta'_i$ (mod 1) for each $i$ and $\sum_i \alpha_i/\beta_i = \sum_i \alpha'_i/\beta'_i$. Cf. [Hat07, Prop. 2.1].*

*Proof.* The proof of this theorem lies well outside the scope of this paper, however we will give a summary of the main ideas from Hatcher [Hat07]. Given a compact oriented Seifert manifold $M$, it has by note 16 a finite number of multiple fibres, $C_1, \dots, C_k$. We can by definition find disjoint open model Seifert fibrations around these with the multiple fibre as the core, call these $S_1, \dots, S_k$.

Choose a cross section $s$ of $M' \to B'$ i.e. a continuous map $s : B' \to M'$ such that $\pi(s(x)) = x$ for all $x \in B'$. It is a 'standard' fact from algebraic topology, that an oriented circle bundle over a connected surface with nonempty boudary is trivial. Therefore we can choose a diffeomorphism $\Delta : M' \to B' \times S^1$. Thus

$$M \cong M' \underset{\partial S_1, \dots, \partial S_k}{\cup} (S_1 \sqcup \cdots \sqcup S_k) \cong S^1 \times B' \underset{\partial \varphi_1, \dots, \partial \varphi_k}{\cup} (S^1 \times D^2 \sqcup \cdots \sqcup S^1 \times D^2) \tag{63}$$

where $\varphi_1, \dots, \varphi_k$ are now *some* diffeomorphisms $\varphi_i : S^1 \times \partial D_i \to S^1 \times S^1$. It turns out that any such diffeomorphism $\varphi_i$ is isotopic to one determined by a matrix as in the proof of theorem 5, and that this is sufficient to show that they have the required form. Thus $M \cong M(g; p_1/q_1, \dots, p_k/q_k)$.

Suppose that $M(g; \alpha_1/\beta_1, \dots, \alpha_k/\beta_k) \cong M(g'; \alpha'_1/\beta'_1, \dots, \alpha'_k/\beta'_k)$ by some diffeomorphism $\Phi$. Then $k = k'$ since the number of multiple fibres must be the same. Moreover $B \simeq B'$, since $\Phi$ send fibres to fibres, and thus $g = g'$ since genus is a topological invariant. Without loss we can assume that $\Phi$ sends the $i$'th multiple fibre to the $i$'th multiple fibre. Now by an examination of the local structure of the model Seifert fibration it is possible to show that we must have $\alpha_i/\beta_i \equiv \alpha'_i/\beta'_i$ (mod 1). It remains to argue that $\sum_i \alpha_i/\beta_i = \sum_i \alpha'_i/\beta'_i$. This, we believe, follows by computing the fundamental groups of $M$ and $M'$. □

**Note 10.** This gives the complete isomorphism classification of Seifert fiberings since $g$ is determined by the isomorphism class of a fibering, which determines the base surface $B$, and the Seifert fiberings $M(g, \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k)$ and $M(g, \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k, 0)$ are the same.

### 4.2.1 Classification of Seifert Manifolds

One can, in the general setting described in note 9, prove the following main theorem, see [Hat07] for details and (part of) proof.

**Theorem 8.** *Seifert fiberings of orientable Seifert manifolds are unique up to isomorphism, with the exception of the following fiberings:*

*i.* $M(0, 1; \alpha/\beta)$*, the various Seifert fiberings of* $S^1 \times D^2$*.*

*ii. M(0,1;1/2,1/2)=M(-1,1;), two fiberings of* $S^1 \times S^1 \times I$*.*

*iii.* $M(0; \alpha_1/\beta_1, \alpha_2/\beta_2)$*, various fiberings of* $S^3$*,* $S^1 \times S^2$ *and Lens spaces*

*iv.* $M(0, 0; 1/2, -1/2, \alpha/\beta) = M(-1, 0; \beta/\alpha)$

*v.* $M(0, 0; 1/2, 1/2, -1/2, -1/2) = M(-2, 0; )$*, two fiberings of* $S^1 \times S^1 \times S^1$

**Note 11.** Suppose one is interested in classifying all 3-manifolds up to diffeomorphism. To make life easier one could decide to concentrate on just those 3-manifolds that admit a Seifert fibering, and ask for a classification of these up to diffeomorphism. Now theorem 7 tells us how to create an exhaustive list of all of these manifolds on the form $M(g; \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k)$. It tells us by a specific condition exactly when two of these objects are isomorphic, but not when they are diffeomorphic. Theorem 8 fills this gap, by telling us when two non-isomorphic Seifert fibered manifolds are in fact diffeomorphic. Accidentally this only happens for short list of deviants. We shall look at one such case now.

**Example 8.** *Different fibrations of* $S^3$. Let $\alpha, \beta \in \mathbb{Z}$ be coprime. We claim that we can fibre $S^3$ by the circles $(e^{2\pi i\alpha t}u, e^{2\pi i\beta t}v)$ such that $S^3 = M(0, \alpha/\beta, \beta/\alpha)$. Recall that $S^3 = \{(u, v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}$. Define $S^3_L = \{(u, v) \in S^3 \mid |u| \geq |v|\}$ and $S^3_R = \{(u, v) \in S^3 \mid |u| \leq |v|\}$. These are both diffeomorphic to a solid torus since we have

$$\varphi_L : S^3_L \to S^1 \times D^2 \qquad \text{defined by} \qquad \varphi_L(u, v) = \Big(\frac{u}{|u|}, \sqrt{2}v\Big) \tag{64}$$

$$\varphi_R : S^3_R \to D^2 \times S^1 \qquad \text{defined by} \qquad \varphi_R(u, v) = \Big(\sqrt{2}u, \frac{v}{|v|}\Big) \tag{65}$$

where the inverses are $\varphi_L^{-1}(x, z) = (\sqrt{1 - \frac{|z|}{2}}x, \frac{z}{\sqrt{2}})$ and similarly for $\varphi_R^{-1}$. Furthermore we define the map on the overlap $\varphi_{LR} : S^3_L \cap S^3_R \to S^1 \times S^1$ by $\varphi_{LR}(u, v) = (\sqrt{2}u, \sqrt{2}v)$. Then we have

$$S^3 = S^1 \times D^2 \underset{S^1 \times S^1}{\cup} D^2 \times S^1 \tag{66}$$

because the inverse map of $\varphi_{LR}$ is the inclusion $\varphi_{LR}^{-1} : S^1 \times S^1 \hookrightarrow S^3$ defined by $\varphi_{LR}^{-1}(x, z) = (\frac{x}{\sqrt{2}}, \frac{z}{\sqrt{2}})$. Note that this is well defined since we have $|\frac{x}{\sqrt{2}}|^2 + |\frac{z}{\sqrt{2}}|^2 = 1$. Therefore $\varphi_{LR}^{-1}(S^1 \times S^1) = \{(u, v) \in S^3 \mid |u| = |v|\}$.

We furthermore define the modified open sets $\check{S}^3_L = \{(u, v) \in S^3 \mid |u| > |v|\}$, $\check{S}^3_R = \{(u, v) \in S^3 \mid |u| < |v|\}$ and let $\check{\varphi}_L$ and $\check{\varphi}_R$ be the restricted diffeomorphisms going onto $S^1 \times \operatorname{Int} D^2$. Finally we define the open set $\check{S}^3_N = \{(u, v) \in S^3 \mid |u| \neq 0\}$ and the diffeomorphism $\check{\varphi}_N : \check{S}^3_N \to S^1 \times \operatorname{Int} D^2$ by $\check{\varphi}_N(u, v) = (\frac{u}{|u|}, v)$ with inverse $\check{\varphi}_N^{-1} : S^1 \times \operatorname{Int} D^2 \to \check{S}^3_N$ given by $\check{\varphi}_N(x, z) = (\sqrt{1 - |z|^2}x, z)$ which is then clearly well defined. Now the circle $(e^{2\pi it\alpha}u, e^{2\pi it\beta}v) \in \check{S}^3_L$ gets sent by $\check{\varphi}_L$ to

$$\check{\varphi}_L(e^{2\pi it\alpha}u, e^{2\pi it\beta}v) = (e^{2\pi it\alpha}\frac{u}{|u|}, \sqrt{2}e^{2\pi it\beta}v) = (e^{2\pi it\alpha}x, e^{2\pi it\beta}z) \tag{67}$$

for some $(x, y) \in S^1 \times D^2$. Letting $t = t'\alpha$ we recognize this as the circle of slope $\beta/\alpha$ thus $\check{S}^3_L$ is diffeomorphic to the open model Seifert fibration with parameter $\beta/\alpha$. The same argument using $\check{\varphi}_N$ tells us that this holds for $\check{S}_N$. Similarly the circle $(e^{2\pi it\alpha}u, e^{2\pi it\beta}v) \in \check{S}^3_L$ gets sent by $\check{\varphi}_R$ to

$$\check{\varphi}_R(e^{2\pi it\alpha}u, e^{2\pi it\beta}v) = (\sqrt{2}e^{2\pi it\alpha}u, e^{2\pi it\beta}\frac{v}{|v|}) = (e^{2\pi it\alpha}z, e^{2\pi it\beta}x) \tag{68}$$

for some $(x, z) \in S^1 \times D^2$. By transposing the factors and letting $t' = t\beta$ we see that this is actually $(e^{2\pi it}x, e^{2\pi it\alpha/\beta}z) \in S^1 \times \operatorname{Int} D^2$ which we recognize as the circle of slope $\alpha/\beta$ thus $\check{S}^3_R$ is diffeomorphic to the open model Seifert fibration with parameter $\alpha/\beta$. Since $\check{S}^3_N \cup \check{S}^3_R = S^3$ we see that every fibre has a neighborhood diffeomorphic to an open model Seifert fibration sending fibres to fibres thus $S^3$ is a Seifert fibred manifold by definition.

$$\begin{array}{ccccc}
S^3 & = & S^3_L & \cup & S^3_R \\
 & & \downarrow \varphi_L & & \downarrow \varphi_R \\
\pi \downarrow & & S^1 \times D^2 & \cup_\mu & D^2 \times S^1 \\
 & & \downarrow \pi_{\beta/\alpha} & & \downarrow \pi^t_{\alpha/\beta} \\
S^2 & = & D^2 & \cup_\nu & D^2
\end{array}$$

We have seen that $S^3$ is Seifert fibred. Now let us show that it is in fact the manifold $M(0, \alpha/\beta, \beta/\alpha)$. The map $\mu : S^1 \times S^1 \to S^1 \times S^1$ which respects the identifications in the diagram is just $\varphi^{-1}_{LR} \circ \varphi_{LR} = id_{S^1 \times S^1}$. Define the tansposed map $\pi^t_{p/q} : D^2 \times S^1 \to D^2$ by $\pi^t_{p/q}(u, v) = \pi_{p/q}(v, u)$. We can now similarly find a map $\nu : S^1 \to S^1$ respecting the diagram, that is such that $\pi_{\beta/a}(u, v) = u^{-\beta}v^\alpha$ is sent to $\pi^t_{\alpha/\beta} = v^{-\alpha}u^\beta$. We conclude that $\nu : S^1 \to S^1$ given by $\nu(z) = z^{-1}$ is the desired map. Since this corresponds to a flip it is clear that pasting the two discs together along their boundary according to $\nu$ results in a space homeomorphic to $S^2$, which is then the base surface. The multiple fibres are by definition of the projection maps from the model Seifert firbations sent to the centres of the two discs $D^2$ thus they lie at the 'south' and 'north' pole respectively. The genus of $S^2$ is 0 hence we can write $S^3 = M(0; \alpha/\beta, \beta/\alpha)$.

## 4.3 Lens spaces revisited

In this section we will return to the Lens spaces defined in section 1.4, in order to give an example of the theory derived above. In particular we will show that $L_{p/q}$ is a Seifert manifold, that the base surface $B$ is $S^2$ and that we can give it a representation of the form $M(g; \alpha_1/\beta 1, \ldots, \alpha_k/\beta_k)$. We will furthermore work out the fundamental group of these spaces.

**Lemma 19.** *There exists an $a \in \mathbb{Z}$ such that the orbits of the action determined by $\theta : (u, v) \mapsto (e^{2\pi i/q}u, e^{2\pi ip/q}v)$ are the same as the orbits of the action defined by $\phi : (u, v) \mapsto (e^{2\pi ia/q}u, e^{2\pi i/q}v)$.*

*Proof.* Since $p, q$ are coprime we can find $a, b \in \mathbb{Z}$ such that $ap + bq = 1$. Then $a \equiv p^{-1} \pmod q$ and $\frac{a}{q} + \frac{b}{a} = \frac{1}{qa}$. Therefore we have, letting $w = e^{2\pi i/q}$:

$$(wu, w^p v) = (w^{ap}w^{bq}u, w^p v) = (w^{ap}u, w^p v) = ((w^a)^p, (w^p)v) \tag{69}$$

hence $\theta = \phi^p$ and similarly $\phi = \theta^a$ because both $w$ and $w^p$ are primitive $q$-th roots of unity. By definition we have $x \sim y \Leftrightarrow x = \theta^i(y)$ for some $i$ which, by inserting the previous expressions, is if and only if $x = \phi^j(y)$ for some $j$. Define this equivalence relation $\approx$. We conclude that the orbits emanating from the two actions are the same. □

**Theorem 9.** *The Lens space $L_{p/q}$ is Seifert fibred with circles $[(e^{2\pi it/q}u, e^{2\pi itp/q}v)]$ for fixed $u, v$.*

*Proof.* By definition of the Lens space $L_{p/q} = S^3/\!\sim$ we have $(u, v) \sim (wu, w^p v)$ for $w = e^{2\pi i/q}$. Assume $|u| \geq |v|$ then we obtain the following, using the function $\varphi_L$ from example 8:

$$\begin{array}{ccc}
(u, v) & \xmapsto{\varphi_L} & (\frac{u}{|u|}, \sqrt{2}v) \\
\wr & & \wr \\
(wu, w^p v) & \longmapsto & (\frac{wu}{|u|}, \sqrt{2}w^p v)
\end{array}$$

This gives us a homeomorphism $\psi_L : S^3_L/\!\sim\, \to S^1 \times D^2/\!\sim$ given by $\psi_L([u,v]) = \varphi_L(u,v)$. We claim that $S^1 \times D^2/\!\sim$ is in fact homeomorphic to $S^1 \times D^2$. To see this define the map $f : S^1 \times D^2 \to S^1 \times D^2$ by $f(x,y) = (x^q, x^{-p}y)$. Then $f(x,y) = f(x',y') \Leftrightarrow (x,y) \sim (x',y')$. Assume $(x,y) \sim (x',y')$ then we must have $(x',y') = (wx, w^p y)$ hence

$$f(x',y') = f(wx, w^p y) = ((wx)^q, (wx)^{-p} w^p y) = (w^q x^q, x^{-p} y) = (x^q, x^{-p}y) = f(x,y) \tag{70}$$

On the other hand suppose $f(x,y) = f(x',y')$ then $(x^q, x^{-p}y) = (x'^q, x'^{-p}y')$ hence $x' = w^i x$ for some $i \in \{0,\dots,q-1\}$. Therefore $x^{-p}y = (w^i x)^{-p} y' = w^{-ip} x^{-p} y'$ hence $y' = w^{ip} y$ and we conclude that $(x,y) \sim (x',y')$ as required. It is immediately seen that $f$ is continuous and surjective hence by [Mun00, 22.3] we conclude that $f$ descents to a homeomorphism $\hat{f} : S^1 \times D^2/\!\sim\, \to S^1 \times D^2$ as stated. Assume $|u| \le |v|$. By lemma 19 we see that $S^3_R/\!\sim\, = S^3_R/\!\approx$ thus by the same approach as above we obtain a homeomorphism between $S^3_R/\!\sim$ and $D^2 \times S^1/\!\sim$. To see that it is homeomorphic to $D^2 \times S^1$ define the map $h : D^2 \times S^1 \to D^2 \times S^1$ by $h(x,y) = (y^{-a}x, y^q)$. Then exactly the same approach as above gives the required homeomorphism.

As in example 8 we define the modified open sets $\check{S}^3_L/\!\sim\, = \{[u,v] \in L_{p/q} \,|\, |u| > |v|\}$, $\check{S}^3_R/\!\sim\, = \{[u,v] \in L_{p/q} \,|\, |u| < |v|\}$ and $\check{S}^3_N/\!\sim\, = \{[u,v] \in L_{p/q} \,|\, |u| \neq 0\}$. Now the circle $(e^{2\pi i t/q}u, e^{2\pi i t p/q}v) \in \check{S}^3_L/\!\sim$ gets sent by $\hat{f} \circ \psi_L$ to

$$\hat{f} \circ \psi_L(e^{2\pi i t/q}u, e^{2\pi i t p/q}v) = (e^{2\pi i t}\frac{u^q}{|u|^q}, \sqrt{2}v\frac{u^{-p}}{|u|^{-p}}) = (e^{2\pi i t}x, y) \tag{71}$$

for $(x,y) \in S^2 \times \operatorname{Int} D^2$. We recognize this as a circle of slope $0/1$. This actually holds on $\check{S}^3_N/\!\sim$ using the map $\hat{f} \circ \psi_N$ thus we conclude that $\check{S}^3_N/\!\sim$ is diffeomorphic to the open model Seifert fibration with parameter $0/1$ with fibres going to fibres. Similarly we see that the circle $(e^{2\pi i t/q}u, e^{2\pi i t p/q}v) \in S^3_R/\!\sim$ gets sent by $\hat{h} \circ \psi_R$ to

$$\hat{h} \circ \psi_R(e^{2\pi i t/q}u, e^{2\pi i t p/q}v) = (e^{2\pi i t(1-aq)/q}y, e^{2\pi i t p}x) = (e^{2\pi i t b}y, e^{2\pi i t p}x) \tag{72}$$

for some $(x,y) \in S^1 \times D^2$, where we have used that $ap + bq = 1$. By transposing the factors and letting $t' = tp$ we see that this is actually $(e^{2\pi i t}x, e^{2\pi i t b/p}y) \in S^1 \times \operatorname{Int} D^2$ which we recognize as the fibre $C_y$ in the open model Seifert fibration with parameter $b/p$. We conclude that $S^3_R/\!\sim$ is diffeomorphic to the model Seifert fibration with parameter $b/p$ with fibres going to fibres. Now $L_{p/q} = \check{S}^3_N/\!\sim\, \cup \check{S}^3_R/\!\sim$ hence every circle in $L_{p/q}$ has a neighborhood diffeomorphic to an open model Seifert fibration where the diffeomorphism sends fibres to fibres, thus $L_{p/q}$ is by definition a Seifert manifold as required. □

**Corollary 3.** *The Lens space $L_{p/q}$ can be written $M(0; p^{-1}/q)$.*

*Proof.* By the previous theorem we see that the only a multiple fibre is at $[(0,v)]$ where $|v| = 1$, and that it has multiplicity $b/p$. Since we have $ap + bq = 1$ we see that $b \equiv q^{-1} \pmod p$ thus we can write the multiplicity as $q^{-1}/p$. From example 8 we see that we can write

$$L_{p/q} = S^3/\!\sim\, = S^3_L/\!\sim\, \cup S^3_R/\!\sim\, \simeq S^1 \times D^2 \cup_\mu D^2 \times S^1 \tag{73}$$

for some function $\mu : S^1 \times S^1 \to S^1 \times S^1$ sending fibres to fibres. The part of $S^3_L/\!\sim$ over $v^q \in S^1$ is then $\psi_L^{-1} \circ \hat{f}^{-1}(C_v) = [(\frac{1}{\sqrt{2}}e^{2\pi i t/q}, \frac{v}{\sqrt{2}}e^{2\pi i t p/q})] \in S^3_L/\!\sim\, \cap S^3_R/\!\sim$. This gets sent by $\hat{h} \circ \psi_R$ to

$$\hat{h} \circ \psi_R([(\frac{1}{\sqrt{2}}e^{2\pi i t/q}, \frac{v}{\sqrt{2}}e^{2\pi i t p/q})]) = (v^{-a}e^{2\pi i t b}, v e^{2\pi i t p}). \tag{74}$$

Since $v \in S^1$ we recognize this as the circle with slope $b/p$, thus the map $\mu : S^1 \times S^1 \to S^1 \times S^1$ given by $\mu(\alpha,\beta) = \hat{h} \circ \varphi_{LR} \circ (\hat{f} \circ \varphi_{LR})^{-1}(\alpha,\beta) = \hat{h} \circ \hat{f}^{-1}(\alpha,\beta) = (\alpha^b\beta^{-a}, \alpha^p\beta^q)$ is a diffeomorphism sending fibres to fibres; the inverse is $\mu^{-1}(\alpha,\beta) = (\alpha^q\beta^a, \alpha^{-1}\beta^b)$.

$$\begin{array}{ccccc}
M = & S^3_L/\sim & \cup & S^3_R/\sim \\
& \downarrow{\scriptstyle \hat{f}\circ\psi_L} & & \downarrow{\scriptstyle \hat{h}\circ\psi_R} \\
\pi\downarrow & S^1 \times D^2 & \cup_\mu & D^2 \times S^1 \\
& \downarrow{\scriptstyle \pi_{0/1}} & & \downarrow{\scriptstyle \pi^t_{b/p}} \\
B = & D^2 & \cup_\nu & D^2
\end{array}$$

To find the map $\nu : S^1 \to S^1$ satisfying the diagram we calculate the image of a point $(x, z) \in S^1 \times S^1$ going to the two discs:

$$\pi_{0/1}(x, z) = z \qquad \text{and} \qquad \pi^t_{b/p} \circ \mu(x, z) = \pi^t_{b/p}(x^b z^{-a}, x^p z^q) = z^{-1} \tag{75}$$

This has to hold for all $(x, z)$ thus we conclude that $\nu$ is given by $\nu(z) = z^{-1}$. This just corresponds to a flip of one of the discs thus we see that by adjoining these along the their boundary, $S^1$, according to $\nu$, gives us $S^2$. The only multiple fibre is sent to the centre of one of the discs thus we can depict it as being at the 'south' pole on $S^2$. Since the genus of $S^2$ is 0 we have $L_{p/q} = M(0; q^{-1}/p)$. □

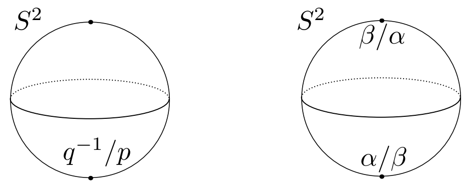



Figure 14: The base space $B = S^2$ of the Lens space, $L_{p/q}$, and the three-sphere, $S^3$.

**Note 12.** We could alternatively show directly that $B$ is homeomorphic to $S^2$ by defining the map

$$\xi : L_{p/q} \to S^2 \qquad \text{by} \qquad \xi([u, v]) = \begin{cases} g_p(u^{-p} v) & \text{if } u \neq 0 \\ g_p(u^p v^{-1}) & \text{if } v \neq 0 \end{cases} \tag{76}$$

where $g_p : \mathbb{C} \to S^2$ is the inverse map of the stereographic projection defined in section 2.2.1. We would furthermore need to verify that $\bar{\xi} : B \to S^2$ is a diffeomorphism. This is not an obvious fact since it has been shown that two homeomorphic differentiable manifolds are not necessarily diffeomorphic, see [Mun60]. By definition 3 we would have to look at the composite of three rather unpleasant functions hence we stick with the first approach.

**Theorem 10.** *The fundamental group of the Lens space $L_{p/q}$ is isomorphic to $\mathbb{Z}/q$.*

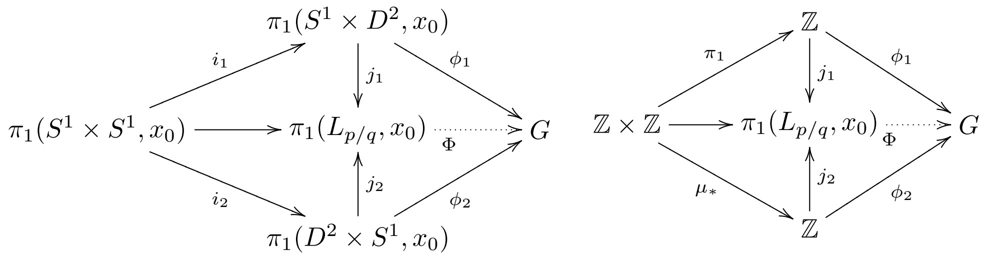
*Proof.* One way to go is by the Seifert-Van Kampen theorem. We have from the proof of the preceding theorem that we can write

$$L_{p/q} \simeq S^1 \times D^2 \cup_\mu D^2 \times S^1 \tag{77}$$

thus it is sufficient to examine the adjunction space. Define $U = S^1 \times D^2$ and $V = D^2 \times S^1$. Although these are not open in $L_{p/q}$, they admit open neighborhoods of which they are deformation retracts, and thus the Seifert-van Kampen theorem applies without modifications. Clearly $U$ and $V$ are, as well as the intersection $S^1 \times S^1$, path connected. Let $x_0 \in U \cap V$. Let $G$ be a group, and let

$$\phi_1 : \pi_1(S^1 \times D^2, x_0) = \mathbb{Z} \to G \qquad \text{and} \qquad \phi_2 : \pi_1(D^2 \times S^1, x_0) = \mathbb{Z} \to G \tag{78}$$

be homomorphism given by $\phi_1(1) = g_1$ and $\phi_2(1) = g_2$. Let $i_1, i_2, j_1, j_2$ be the homomorphisms indicated in the following diagram, each induced by inclusion.




Then we have $\mu_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\mu_*(x, y) = xp + yq$ and $i_*(x, y) = \pi_1(x, y) = x$. We see that $\phi_1 \circ i_*(x, y) = \phi_2 \circ \mu_*(x, y)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ if and only if $g_1^x = g_2^{xp+yq}$. In particular we have $1 = g_2^q$ and $g_1 = g_2^p$. By the Seifert-van Kampen theorem we see that a homomorphism $\pi_1(L_{p/q}, x_0) \to G$ determines and is determined by $g_2 \in G$ such that $g_2^q = 1$. We conclude that $\pi_1(L_{p/q}, x_0) \cong C_q$ as required. □

*Alternative proof of theorem 10.* We can give a more intuitive and thorough proof using lifting correspondences. Since $C_q$ acts freely on $S^3$ as shown in the proof of corollary 2, we conclude by lemma 9 that the quotient map $\pi : S^3 \to S^3/C_q = L_{p/q}$ given by $\pi(x) = [x]$ is a covering map. The fact that $C_q$ acts freely on $S^3$ furthermore tells us that the fibres $f^{-1}(b_0)$ for any $b_0 \in L_{p/q}$ have cardinality $q$. Choose $e_0 \in S^3$ such that $\pi(e_0) = b_0$ and let $\tilde{f}$ be the lifting of $f$ to a path in $S^3$ beginning at $e_0$. Then we have a well defined **lifting correspondence**

$$\phi : \pi_1(L_{p/q}, b_0) \to \pi^{-1}(b_0) \tag{79}$$

sending $[f]$ to the endpoint $\tilde{f}(1)$ of $\tilde{f}$. Since $S^3$ is simply connected [Mun00, 59.3] we see that the lifting correspondence $\phi$ is bijective [Mun00, 54.4]. Therefore we have, letting $\lambda$ be the generator of $C_q$ such that $\lambda.(u, v) = (wu, w^p v)$, the following:

$$\pi_1(L_{p/q}, b_0) \cong \pi^{-1}(b_0) = \{e_0, \lambda.e_0 \ldots, \lambda^{q-1}.e_0\} \tag{80}$$

Define $g_0, \ldots, g_{q-1}$ by $g_i \mapsto \lambda^i.e_0$. We claim that $g_i g_j = g_k$ where $k \equiv i + j \pmod q$. To see this let $\tilde{f}_i$ and $\tilde{f}_j$ be paths from $e_0$ to $\lambda^i.e_0$ and $\lambda^j.e_0$ respectively. Since the lifting correspondence is bijective we have $g_i = [\pi \circ \tilde{f}_i]$ and $g_j = [\pi \circ \tilde{f}_j] = [\pi \circ \lambda^i.\tilde{f}_j]$. Note that $\lambda^i.\tilde{f}_j$ is now a path from $\lambda^i.e_0$ to $\lambda^i.\lambda^j.e_0 = \lambda^k.e_0$. We now have $g_k = [\pi \circ \tilde{f}_k] = [\pi \circ (\tilde{f}_i * \lambda^i.\tilde{f}_j)] = [\pi \circ \tilde{f}_i * \pi \circ \lambda^i.\tilde{f}_j] = [\pi \circ \tilde{f}_i] * [\pi \circ \lambda^i.\tilde{f}_j] = g_i * g_j$. Since the lifting correspondence is an isomorphism we might as well write this as $g_i g_j = g_k$ thus $\pi_1(L_{p/q}, b_0) \cong \mathbb{Z}/q$. $\square$

# A Miscellaneous topological results

We need a couple of technical topological results which we cover here. These center around the properties of Hausdorffness, normality and regularity which are important features in proving that a given space, including adjunction spaces, is a candidate for a manifold.

## A.1 Some useful definitions and lemmas

**Definition 17.** Suppose that one-point sets are closed in $X$. Then $X$ is said to be **regular** if for each pair consisting of a point $x$ and a closed set $B$ disjoint form $x$, there exists disjoint open sets containing $x$ and $B$ respectively. The space $X$ is said to be **normal** if for each pair $A, B$ of disjoint closed sets of $X$, there exists disjoint open sets containing $A$ and $B$ respectively. Clearly a normal space is regular, and a regular space is Hausdorff.

**Lemma 20.** *Let $A$ and $B$ be disjoint compact subspaces of the Hausdorff space $X$. Then there exists disjoint open sets $U$ and $V$ containing $A$ and $B$, respectively.*

*Proof.* Since $B$ is a compact subspace of the Hausdorff space $X$ and $A \cap B = \emptyset$ it follows from lemma 26.4 [Mun00] that for any $a \in A$ there exists *disjoint* open sets $U_a$ and $V_a$ of $X$ containing $a$ and $B$ respectively.

The collection $\mathcal{A} = \{U_a \,|\, a \in A\}$ is an open cover of $A$. Since $A$ is assumed compact it follows by definition that $\mathcal{A}$ contains a finite subcover, that is finitely many $U_a$'s cover A:

$$U = U_{a_1} \cup \cdots \cup U_{a_n} \supset A. \tag{81}$$

Clearly $U$ is open since it is a union of open sets. We claim that the intersection of the corresponding $V_a$'s contains $B$ and is disjoint from $U$. To see this we define

$$V = V_{a_1} \cap \cdots \cap V_{a_n} \tag{82}$$

which is an open set because it is a *finite* intersection of open sets. Since each $V_{a_i}$ contains $B$ it is clear that $V \supset B$. Furthermore $U \cap V = \emptyset$ since if $z \in U$ then $z \in U_{a_i}$ for some $i$ but we have $U_{a_i} \cap V_{a_i} = \emptyset$ hence $z \notin V_{a_i}$ and we conclude that $z \notin V$. We are now left with the two disjoint and open sets $U$ and $V$ containing $A$ and $B$ respectively as required. □

**Note 13.** If $X$ is compact, then any two disjoint closed sets $A$ and $B$ are compact [Mun00, 26.2] and since every compact subspace of a Hausdorff space is closed [Mun00, 26.3], this lemma states that if $X$ is a compact Hausdorff space, then it is normal.

**Lemma 21.** *Let $\pi : E \to X$ be a closed quotient map. If $E$ is normal, then so is $X$. Cf. [Mun00].*

*Proof.* Assume that $E$ is normal. One-point sets are closed in $X$ since one-point sets are by definition closed in the normal space $E$. Now let $A$ and $B$ be disjoint closed sets of $X$. Then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are disjoint closed sets of $E$ since $\pi$ is continuous. Choose disjoint open sets $U$ and $V$ of $E$ containing $\pi^{-1}(A)$ and $\pi^{-1}(B)$ respectively.

Let $C = E - U$ and $D = E - V$. Because $C$ and $D$ are closed sets of $E$, the sets $\pi(C)$ and $\pi(D)$ are closed in $X$. Since $C \cap \pi^{-1}(A) = \emptyset$ we see that $\pi(C) \cap A = \emptyset$. Then $U_0 = X - \pi(C)$ is an open set of $X$ containing $A$ and similarly $V_0 = X - \pi(D)$ is an open set of $X$ containing $B$. Furthermore $U_0$ and $V_0$ are disjoint: If $x \in U_0$, then $\pi^{-1}(x)$ is disjoint from $C$ hence it is contained in $U$. Similarly if $x \in V_0$ then $\pi^{-1}(x) \in V$. Since $U$ and $V$ are disjoint, so are $U_0$ and $V_0$. □

## A.2 Adjunction space

**Definition 18.** Let $X$ and $Y$ be disjoint normal spaces with $A \subset X$. Let $f : A \to Y$ be continuous. Then the **adjunction space**, denoted $X \cup_A Y$ or $X \cup_f Y$, is defined to be the quotient space obtained from $X \sqcup Y$ by identifying each point $a$ of $A$ with the point $f(a)$ and all the points $f^{-1}(\{f(a)\})$, compare with [Mun00, p. 224].

**Theorem 11** (Tietze Theorem)**.** *If $X$ is normal and $A \subset X$ is closed, then any continuous function $f : A \to [0, 1]$ extends to a continuous function $g : X \to [0, 1]$.*

**Corollary 4.** *If $P$ and $Q$ are disjoint closed subsets of a normal space $X$, then there is a continuous function $g : X \to [0, 1]$ with the property that $P \subset g^{-1}(0)$ and $Q \subset g^{-1}(1)$.*

*Proof.* The set $A = P \cup Q$ is a closed subset of $X$, and the function $f : A \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & x \in P \\ 1, & x \in Q \end{cases} \tag{83}$$

is continuous [Mun00, Pasting lemma]. By the Tietze theorem, $f$ extends to a continuous function $g : X \to [0, 1]$ which clearly has the claimed properties. □

**Lemma 22.** *The adjunction space $X \cup_f Y$ is normal.*

*Proof.* We claim that one-point sets in $X \cup_f Y$ are closed. To see this note that since $X$ and $Y$ are normal spaces they are in particular Hausdorff, thus one-point sets are closed in these [Mun00, 17.8]. Pick $[z] \in X \cup_f Y$ such that $z \notin A \cup f(A)$. Then the singleton $\{z\}$ is closed and equals $[z]$. Pick $[z] \in X \cup_f Y$ such that $z \in A \cup f(A)$. Then we can find an $a \in A$ such that $[z] = f^{-1}(\{f(a)\}) \cup f(a)$, and since $f$ is continuous this is closed.

Let $P$ and $Q$ be disjoint closed sets in $X \cup_f Y$. We must by definition find disjoint open subsets $U$ and $V$ such that $P \subset U$ and $Q \subset V$. It will suffice to find a continuous function $g : X \cup_f Y \to [0, 1]$ such that $P \subset g^{-1}(0)$ and $Q \subset g^{-1}(1)$ because then the sets $U = g^{-1}[0, 1/2)$ and $V = g^{-1}(1/2, 1]$ has the wanted properties.

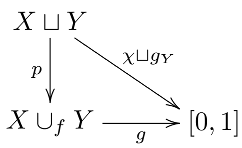
Let $P_X, Q_X \subset X$ denote the preimages of $P$ and $Q$ in $X$, and let $P_Y, Q_Y \subset Y$ denote the preimages of $P$ and $Q$ in $Y$. Since $P$ and $Q$ are disjoint and closed so are, by definition of the quotient map, $P_Y$ and $Q_Y$. It now follows by 4 that there is a continuous function $g_Y : Y \to [0, 1]$ with the property that $P_Y \subset g_Y^{-1}(0)$ and $Q_Y \subset g_Y^{-1}(1)$. Now the function $g_Y \circ f : A \to [0, 1]$, being a composition of continuous maps, is continuous with the properties that

$$g_Y \circ f(A \cap P_X) = \{0\} \qquad \text{and} \qquad g_Y \circ f(A \cap Q_X) = \{1\} \tag{84}$$

since $f(A \cap P_X) \subset P_Y$ and $f(A \cap Q_X) \subset Q_Y$. It follows that the function

$$\psi : P_X \cup A \cup Q_X \to [0, 1] \qquad \text{given by} \qquad \psi(x) = \begin{cases} 0 & x \in P_X \\ g_Y \circ f(x) & x \in A \\ 1 & x \in Q_X \end{cases} \tag{85}$$

is well defined and continuous [Mun00, Pasting lemma]. Since $P_X \cup A \cup Q_X$ is a closed subset of $X$, the map $\psi$ extends by the Tietze theorem to a continuous map $\chi : X \to [0, 1]$. Now the map $\chi \sqcup g_Y : X \sqcup Y \to [0, 1]$ clearly respects the equivalence relation on $X \sqcup Y$ hence we have an induced map $g$ [Mun00, 22.2] making the diagram




commute and since $\chi \sqcup g_Y$ is continuous so is $g$. We have $g^{-1}(0) = \chi^{-1}(0) \cup g_Y^{-1}(0) \supset P_X \cup P_Y = P$ and similarly $g^{-1}(1) \supset Q$ as required. Now take $U$ and $V$ as defined above. □

# B Manifold vocabulary

In this section we will, for the sake of completeness, look at some of the central notions when dealing with manifolds in generality. This will mostly be an informal survey touching upon orientability, manifolds with boundary, connected sums and genus. We will furthermore show some essential lemmas supporting the approach taken in section 1.

**Lemma 23.** *Compatibility of atlases is an equivalence relation.*

*Proof.* Reflexibility follows easily by noting that $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$ and symmetry follows by commutativity of the union; if $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ hence $\mathcal{B} \sim \mathcal{A}$. We can thus focus on transitivity. Let $\mathcal{A} = (V_i, U_i, \sigma_i)_{i \in I_1}$, $\mathcal{B} = (W_i, O_i, \tau_i)_{i \in I_2}$ and $\mathcal{C} = (P_i, N_i, \varphi_i)_{i \in I_3}$ be smooth atlases of a manifold $M$. Assume $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ then it is enough to show that the transition map

$$\varphi_j \circ \sigma_i^{-1}|_{\sigma(O_k)} : \sigma_i(V_i \cap P_j \cap O_k) \to \varphi_j(V_i \cap P_j \cap O_k) \tag{86}$$

is smooth, since we have $U_i \subset \bigcup_{k \in I_2} \sigma(O_k)$ and by lemma 3 smoothness is a local property. Furthermore choosing charts originating from the same atlas would result in a smooth map by assumption. We can write the transition map as follows

$$\varphi_j \circ \sigma_i^{-1}|_{\sigma(O_k)} = (\varphi_j \circ \tau_k^{-1}) \circ (\tau_k \circ \sigma_i^{-1}|_{\sigma(O_k)}) \tag{87}$$

which is smooth by assumption that $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$. Thus $\mathcal{A} \sim \mathcal{C}$ as required. □

**Lemma 24.** *Any open subset of an $m$-manifold is an $m$-manifold.*

*Proof.* The proof is straightforward and therefore omitted. See [Sch07, p. 19] for details. □

## B.1 Manifold with boundary and closed manifold

**Definition 19.** An **$m$-manifold with boundary** is a Hausdorff space such that each point has an open neighborhood homeomorphic to either an open disc $\operatorname{Int} D^m$ or to the space

$$\{(x_1, \ldots, x_m) \in \operatorname{Int} D^m : x_1 \geq 0\} \tag{88}$$

The set of all points that have an open neighborhood homeomorphic to $\operatorname{Int} D^m$ is called the **interior** points, while the set of points $p$ that have an open neighborhood $V$ such that there exists a homeomorphism $h$ of $V$ onto $\{\mathbf{x} \in \operatorname{Int} D^m : x_1 \geq 0\}$ with $h(p) = (0, \ldots, 0)$ is called the boundary of the manifold, compare with [Mas67]. A **closed manifold** is a compact manifold without boundary.

**Note 14.** Throughout this paper we will make it explicit if we are not dealing with closed manifolds. One should be aware of the contrasting terms. The notion of closed manifold must not be confused with a closed set. A closed disc for example is a closed set, but not a closed manifold[2]. A compact manifold is a topological space which is compact, but could have a boundary. A manifold with boundary is intuitively a manifold with an edge. The boundary of an $m$-manifold with boundary can be shown to be an $(m-1)$-manifold.

## B.2 Orientability

We will first give an intuitive description of what orientability is, which can be found in [Mas67]. This is then followed by a more formal approach in a reduced setting. Connected $m$-manifolds are, for $m > 1$ divided into two kinds: orientable and nonorientable. In the plane we can, at a given point, define which of the two possible kinds of coordinate systems we will consider as right-handed and which we will consider left-handed. This can then by a relevant homeomorphism be transferred to any connected 2-manifold. An **orientation-reversing path** is a path with the property that going along it will reverse the original orientation. We then define a connected 2-manifold as **nonorientable** if an orientation reversing path exists. The same approach can be taken for 3-manifolds – where a path is orientation-reversing if it mixes up left and right – but for general $m$-dimensional space we need to take a more formal stand:

[2]When cosmologists speak of the universe as being 'open' or 'closed', they most commonly are referring to yet another property namely its curvature.

**Definition 20.** Let $V$ be a finite dimensional vector space. Two ordered bases are said to be **equally oriented** if the transition matrix $S$ has positive determinant. Being equally oriented is an equivalence relation among bases, for which there are precisely two equivalence classes. The space $V$ is said to be oriented if a specific class has been chosen. This class is then called the orientation of $V$, and its member bases are called **positive**.

**Note 15.** Tthe **tangent space** at $p$, $T_pM$, is a linear subspace which, if $M \subset \mathbb{R}^n$ can be seen as the $m$-dimensional hyperplane through the origin in $\mathbb{R}^n$ which is parallel to the hyperplane which best approximates $M$ near $p$. The linear mapping $df_p : T_pM_1 \to T_{f(p)}M_2$ is called the derivative. See [Mil65] for details. By the Whitney embedding theorem, see [Sch07, ch. 2] or [MT97, 8.11], the restriction to manifolds in $\mathbb{R}^n$ is still quite general, however for an arbitrary abstract manifold another approach is required, see [Sch07, ch. 3] for further details.

**Definition 21.** An **orientation** of a manifold $M$ is an orientation of each tangent space $T_pM$, $p \in M$, such that there exists an atlas of $M$ in which all charts induce the given orientation on each tangent space. The manifold is called **orientable** if there exists an orientation. If an orientation has been chosen we say that $M$ is an **oriented manifold** and we call a chart positive if it induces the proper orientation on each tangent space. A smooth map $f : M_1 \to M_2$ between oriented $m$-manifolds is said to be **orientation preserving** if for each $p \in M$, the differential $df_p$ maps positive bases for $T_pM_1$ to positive bases for $T_{f(p)}M_2$.

**Note 16.** As examples of orientable manifolds we have all the $g$-fold tori explicitly created in section 1.3.1 along with the sphere and in general the $m$-sphere. Examples of nonorientable manifolds include the Klein bottle, see example 7, which is a closed 2-manifold, and the Möbius strip which is a 2-manifold with boundary, the boundary being $S^1$, a 1-manifold.

## B.3 Connected sum

**Definition 22.** Let $S_1$ and $S_2$ be disjoint surfaces. Their **connected sum**, denoted by $S_1 \sharp S_2$ is constructed as follows. Choose closed discs $D_1 \subset S_1$ and $D_2 \subset S_2$. Let $S_i' = S_i - D_i$ for $i = 1, 2$. Choose a homeomorphism $h : \partial D_2 \to \partial D_2$. Then $S_1 \sharp S_2$ is the adjunction space $S_1' \cup_h S_2$ cf. [Mas67]. This can be done smoothly as in [KM63].

**Note 17.** It is clear that $S_1 \sharp S_2$ is again a surface and it can be proven that it is independent of the choice of the discs $D_1$ and $D_2$ of the homomorphism $h$. Furthermore we see that $S^2$ is a neutral element with respect to this composition. The connected sum of two oriented surfaces is again orientable. If either one of the surfaces is nonorientable then so is $S_1 \sharp S_2$, see [Mas67].

## B.4 Genus

The **classification theorem** for compact surfaces states that any compact surface is either homeomorphic to a sphere or to a connected sum of tori or to a connected sum of projective planes. Since projective planes are nonorientable we have in other words actually, in section 1.3.1, shown how to compute any compact orientable surface.

**Definition 23.** The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.

**Note 18.** We see that the genus for a surface can be described as the number of handles on it, which again is equal to the number of tori attached under the connected sum. As examples we have the sphere $S^2$ with genus 0, the tori with genus 1, the double torus with genus 2 and so on. In this way the genus completely determines the shape of the orientable surface up to homeomorphism.

# C The free product and the free group

In this section we give a brief definition of the free product and the free group, which appear in the Seifert-van Kampen theorem and thus also in the derivation of fundamental groups for the compact surfaces. This follows [Mun00, §69] closely, where more details can be found.

**Definition 24.** Let $G$ be a group. If $\{G_\alpha\}_{\alpha\in J}$ is a family of subgroups of $G$ we say that these groups **generate** $G$ if every element $x \in G$ can be written as a finite product of elements of the groups $G_\alpha$. Such a finite sequence $(x_1, \dots, x_n) = x$ is called a **word**. It is said to represent $x \in G$.

Similarly if $\{a_\alpha\}_{\alpha\in J}$ is a family of elements of $G$ we say that the elements **generate** $G$ if every element in $G$ can be written as a product of powers of the elements $a_\alpha$.

**Note 19.** We do not assume commutativity, but if $x_i$ and $x_{i+1}$ belongs to the same $G_\alpha$ we can group the elements thus obtaining the word $(x_1, \dots, x_i x_{i+1}, \dots, x_n)$. If $x_i = e$ we can erase it from the word. In this way we gat a so called **reduced word**.

**Definition 25.** Let $G$ be a group, let $\{G_\alpha\}_{\alpha\in J}$ be a family of subgroups of $G$ that generates $G$. Suppose $G_\alpha \cap G_\beta = e$ whenever $\alpha \neq \beta$. $G$ is the **free product** of the groups $G_\alpha$, written

$$G = \prod_{\alpha\in J}^{*} G_\alpha, \tag{89}$$

if for each $x \in G$, there is only one reduced word in the groups that represents $x$.

**Definition 26.** Let $\{a_\alpha\}$ be a family of elements of a group $G$. Suppose each $a_\alpha$ generates an infinite cyclic subgroup $G_\alpha$ of $G$. If $G$ is the free product of the groups $\{G_\alpha\}$, then $G$ is said to be a **free group**, and the family $\{a_\alpha\}$ is called a **system of free generators**.

## C.1 Wedge of circles and adjoining a 2-cell

The basic lemmas needed to show the results in section 1.3.1 are covered here. We only state these as the detailed proofs can be found in [Mun00].

**Definition 27.** Let $X$ be a Hausdorff space that is the union of the subspaces $S_1, \dots, S_n$, each of which is homeomorphic to the unit circle $S^1$. Assume that $S_i \cap S_j = \{p\}$ for some $p \in X$ whenever $i \neq j$. Then $X$ is called the wedge of the circles $S_1, \dots, S_n$.

**Lemma 25.** *Let $X$ be the wedge of the circles $S_1, \dots, S_n$; let $p$ be the common point. Then $\pi_1(X, p)$ is a free group. If $f_i$ is a loop in $S_i$ that represents a generator of $\pi_1(S_i, p)$, then the loops $f_1, \dots, f_n$ represents a system of free generators for $\pi_1(X, p)$.*

**Example 9.** This gives us a way to calculate the fundamental group of the **figure-eight space** which can be seen as a union of two circles with a point in common. The fundamental group is thus a free group with two generators which we can write as $\mathbb{Z} * \mathbb{Z}$.

**Lemma 26.** *Let $X$ be a Hausdorff space; let $A$ be a closed path-connected subspace of $X$. Suppose there is a continuous map $h : B^2 \to X$ that maps $Int\, B^2$ bijectively onto $X - A$ and maps $S^1 = Bd\, B^2$ into $A$. Let $p \in S^1$ and let $a = h(p)$; let $k : (S^1, p) \to (A, a)$ be the map obtained by restricting $h$. Then the homomorphism*

$$i_* : \pi_1(A, a) \to \pi_1(X, a) \tag{90}$$

*induced by inclusion is surjective and its kernel is the least normal subgroup of $\pi_1(A, a)$ containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(A, a)$.*

*Proof.* Munkres pp. 439-441. Note that it uses the Seifert-van Kampen theorem. □

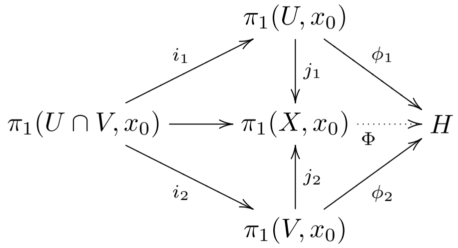
# D The Seifert-van Kampen theorem

In this section we will give a proof of the Seifert-van Kampen theorem, which states that if $X$ is the union of open path connected sets, then $\pi_1(X, x_0)$ is in fact completely determined by the groups $\pi_1(U, x_0)$ and $\pi_2(V, x_0)$, and the various homeomorphisms of these groups induced by inclusion. It will enable us to compute the fundamental groups of a number of spaces including the compact 2-manifolds, the Lens space $L_{p/q}$ and certain circle bundles. The proof is due to J. Munkres, see [Mun00] for details and alternative formulations.

**Theorem 12** (Seifert-van Kampen)**.** *Let $X = U \cup V$, where $U$ and $V$ are open in $X$; assume $U, V$ and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let $H$ be a group, and let*

$$\phi_1 : \pi_1(U, x_0) \to H \qquad \text{and} \qquad \phi_2 : \pi_1(V, x_0) \to H \tag{91}$$

*be homomorphisms. Let $i_1, i_2, j_1, j_2$ be the homomorphisms indicated in the following diagram, each induced by inclusion.*




*If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$, and $\Phi$ is completely determined this way.*

*Proof.* We show uniqueness of $\Phi$ first. We know that $\pi_1(X, x_0)$ is generated by the images of $j_1$ and $j_2$ [Mun00, 59.1]. The value of $\Phi$ on the generator $j_1(g_1)$ must be equal to $\phi_1(g_1)$ and on the generator $j_2(g_2)$ it must equal $\phi_2(g_2)$. Therefore $\Phi$ is completely determined from the given homomorphisms $\phi_1$ and $\phi_2$.

In order to show existence of $\Phi$ we introduce the following notation: Given a path $f$ in $X$, $[f]$ denotes its path-homotopy class in $X$. If $f$ happens to lie in $U$, then $[f]_U$ is the path-homotopy class in $U$ and likewise for $[f]_V$ and $[f]_{U\cap V}$.

*Step 1.* Define the set map $\rho$ that to each loop $f$ based at $x_0$ and lying in either $U$ or $V$, assigns an element of the group $H$ by

$$\rho(f) = \phi_1([f]_U) \quad \text{if } f \text{ lies in } U \tag{92}$$

$$\rho(f) = \phi_1([f]_V) \quad \text{if } f \text{ lies in } V \tag{93}$$

The map is well defined since if $f$ lies in both $U$ and $V$ then

$$\phi_1([f]_U) = \phi_1(i_1([f]_{U\cap V})) = \phi_1 \circ i_1([f]_{U\cap V}) \overbrace{=}^{hypothesis} \phi_2 \circ i_2([f]_{U\cap V}) = \phi_2(i_2([f]_{U\cap V})) = \phi_2([f]_V)$$

as required. The set map $\rho$ satisfies the following conditions

i. If $[f]_U = [g]_U$ or if $[f]_V = [g]_V$ then $\rho(f) = \rho(g)$

ii. If both $f$ and $g$ lies in $U$, or if they both lie in $V$ then $\rho(f * g) = \rho(f) \cdot \rho(g)$

The first follows since $\rho(f) = \phi_1([f]_U) = \phi_1([g]_U) = \rho(g)$ and likewise for $V$. The second follows since if whithout loss of generalization we have $f$ and $g$ in $U$ and hence $\rho(f * g) = \phi_1([f * g]_U) = \phi_1([f]_U * [g]_U) = \phi_1([f]_U) \cdot \phi_1([g]_U) = \rho(f) \cdot \rho(g)$ where we use that $\phi_1$ is a homomorphism.

*Step 2.* We extend $\rho$ to a set map $\sigma$ that assigns, to each path $f$ in $U$ or $V$, an element of $H$, such that (i) is satisfied and (ii) holds when $f * g$ is defined.

Choose for each $x \in X$ a path $\alpha_x$ from $x_0$ to $x$ as follows: If $x = x_0$, let $\alpha_x$ be the constant path at $x_0$. If $x \in U \cap V$, let $\alpha_x$ be a path in $U \cap V$. If $x$ is in $U$ or $V$ but not in $U \cap V$ let $\alpha_x$ be a path in $U$ or $V$ respectively. Then for any path $f$ in $U$ or in $V$ we define a loop $L(f)$ in $U$ or $V$, respectively by the equation

$$L(f) = \alpha_x * (f * \bar{\alpha}_y) \tag{94}$$

where $x$ is the initial point of $f$ and $y$ the final point. This is possible since $U, V$ and $U \cap V$ are assumed path connected. Finally define

$$\sigma(f) = \rho(L(f)) \tag{95}$$

First we show that $\sigma$ is an extension of $\rho$. If $f$ is a loop based at $x_0$ lying in either $U$ or $V$ then

$$L(f) = e_{x_0} * (f * e_{x_0}) \tag{96}$$

where $\alpha_{x_0}$ is the constant path at $x_0$. Then $L(f)$ is clearly path homotopic to $f$ in either $U$ or $V$, so that $\rho(L(f)) = \rho(f)$ by condition (i) and hence $\sigma(f) = \rho(f)$.

To check condition (i), let $f$ and $g$ be path homotopic in $U$ or in $V$. Without loss of generalization we assume $[f]_U = [g]_U$. Then we have $[L(f)]_U = [\alpha_x * (f * \bar{\alpha}_y)]_U = [\alpha_x]_U * [f]_U * [\bar{\alpha}_y]_U = [\alpha_x]_U * [g]_U * [\bar{\alpha}_y]_U = [L(g)]_U$ hence $\sigma(f) = \sigma(g)$ i.e. condition (i) applies. To check (ii), let $f$ and $g$ be arbitrary paths in $U$ or in $V$ such that $f(1) = g(0)$. We have

$$L(f) * L(g) = (\alpha_x * (f * \bar{\alpha}_y)) * (\alpha_y * (g * \bar{\alpha}_z)) \tag{97}$$

for appropriate points $x, y$ and $z$. This is a loop which clearly is path homotopic to $L(f * g)$. Then

$$\rho(L(f * g)) \overbrace{=}^{(i)} \rho(L(f) * L(g)) \overbrace{=}^{(ii)} \rho(L(f)) \cdot \rho(L(g)) \tag{98}$$

Therefore we conclude that $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$ i.e. condition (ii) is satisfied.

*Step 3.* Finally we extend $\sigma$ to a set map $\tau$ that assigns, to an arbitrary path $f$ of $X$, an element of $H$. We want it to satisfy

i. If $[f] = [g]$ then $\tau(f) = \tau(g)$.

ii. $\tau(f * g) = \tau(f) \cdot \tau(g)$ if $f * g$ is defined.

Given $f$, choose a subdivision $s_0 < \cdots < s_n$ of $[0, 1]$. This can be done as follows: Since $f : [0, 1] \to X$ is continuous and $\{U, V\}$ is an open covering of $X$ we see that $\mathcal{A} = \{f^{-1}(U), f^{-1}(V)\}$ is an open covering of the compact and metric space $[0, 1]$. By the Lebesgue number lemma [Mun00], we can then find a $\delta > 0$ such that for each subset of $[0, 1]$ having diameter less that $\delta$, there exists an element of the open covering $\mathcal{A}$ containing it. Thus we can choose a subdivision $s_0, \dots, s_n$ of $[0, 1]$ such that for each $i$ the set $f([s_{i-1}, s_i])$ is contained in either $U$ or $V$ by letting $|s_{i-1} - s_i| < \delta$. Let $f_i$ denote the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$ followed by $f$. Then $f_i$ is a path in $U$ or $V$ and

$$[f] = [f_1] * \cdots * [f_n] \tag{99}$$

[Mun00, 51.2]. If $\tau$ is to be an extension of $\sigma$ and satisfy (i) and (ii) we must have

$$\tau(f) = \tau(f_1 * \cdots * f_n) = \tau(f_1) \cdot \cdots \cdot \tau(f_n) = \sigma(f_1) \cdot \cdots \cdot \sigma(f_n) \tag{100}$$

We use this as our definition of $\tau$. We must show that it is well defined i.e. that it dies not depend on the choice of subdivision. It suffices to show that $\tau(f)$ is unchanged if we adjoin a single additional point $p$ to the subdivision. Let $i$ be the index such that $s_{i-1} < p < s_i$. If we compute $\tau(f)$ using this subdivision we have $\sigma(f_i)$ is substituted with $\sigma(f_i') \cdot \sigma(f_i'')$ where $f_i'$ and $f_i''$ are equal to the positive linear map from $[s_{i-1}, p]$ and $[p, s_i]$ respectively to $[0, 1]$ followed by $f$. But $f_i$ is path homotopic to $f_i' * f_i''$ in $U$ or $V$ hence $\sigma(f_i) = \sigma(f_i') \cdot \sigma(f_i'')$ by conditions (i) and (ii). Therefore $\tau$ is well defined. Furthermore it is an extension of $\sigma$ since if $f$ lies in $U$ or $V$ we can use the trivial subdivision of $[0, 1]$ to define $\tau(f)$; then $\tau(f) = \sigma(f)$ by definition.

*Step 4.* We prove condition *(i)* for $\tau$. First we look at a special case: Let $f$ and $g$ be paths in $X$ from $x$ to $y$ and let $F$ be a path homotopy between them. Assume that there is a subdivision $s_0, \dots, s_n$ of $[0, 1]$ such that $F$ carries each rectangle $R_i = [s_{i-1}, s_i] \times I$ into either $U$ or $V$. We show that in this case $\tau(f) = \tau(g)$.

Given $i$ consider the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$ followed by $f$ or by $g$; call these paths $f_i$ and $g_i$ respectively. The restriction of $F$ to $R_i$ gives a homotopy[3] from $f_i$ to $g_i$ in either $U$

[3] Not a path homotopy since the endpoints may differ

or $V$. Let us consider the pats traced out by the endpoints during the homotopy. Define $\beta_i$ to be the path $\beta_i(t) = F(s_i, t)$. Then $\beta_i$ is a path in $X$ from $f(s_i)$ to $g(s_i)$. The paths $\beta_0$ and $\beta_n$ are the constant paths at $x$ and $y$ respectively. We show that for each $i$,

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i \tag{101}$$

with the path homotopy taking place in either $U$ or $V$. In the rectangle $R_i$ take the broken-line path that runs along the bottom and right edges of $R_i$ from $(s_{i+1}, 0)$ to $(s_i, 0)$ to $(s_i, 1)$; taking $F$ on this we obtain $f_i * \beta_i$. Similarly we get $\beta_{i-1} * g_i$ by taking the broken-line along the left and top of $R_i$ followed by $F$. Since $R_i$ is convex we can find a path homotopy between these broken lines and hence if we follow by $F$ we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ that takes place in $U$ or $V$ as desired.

We furthermore have

$$\sigma(f_i) \cdot \sigma(\beta_i) = \sigma(f_i * \beta_i) \overbrace{=}^{(i)} \sigma(\beta_{i-1} * g_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \quad \text{hence} \quad \sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1}$$

And similarly we have, since $\beta_0$ and $\beta_n$ are constant paths that

$$\sigma(\beta_0) = \sigma(\beta_0 * \beta_0) = \sigma(\beta_0) \cdot \sigma(\beta_0) \qquad \text{hence} \qquad \sigma(\beta_0) = 1 \tag{102}$$

$$\sigma(\beta_n) = \sigma(\beta_n * \beta_n) = \sigma(\beta_n) \cdot \sigma(\beta_n) \qquad \text{hence} \qquad \sigma(\beta_n) = 1 \tag{103}$$

We can now insert this in our definition of $\tau$:

$$\tau(f) = \sigma(f_1) \cdot \dots \cdot \sigma(f_n) = \sigma(g_1) \cdot \dots \cdot \sigma(g_n) = \tau(g) \tag{104}$$

as wanted. To prove it in the general case we do as follows: Given $f$ and $g$ and a path homotopy $F$ between them, let us choose subdivisions $s_9, \dots, s_n$ and $t_0, \dots, t_m$ of $[0, 1]$ such that $F$ maps each subrectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into either $U$ or $V$. This is possible due to the Lebesgue number lemma. Let $f_j$ be the path $f_j(s) = F(s, t_j)$; then $f_0 = f$ and $f_m = g$. The pair of paths $f_{j-1}$ and $f_j$ satisfy the requirements of our special case hence $\tau(f_{j-1}) = \tau(f_j)$ for each $j$. It follows that $\tau(f) = \tau(f_0) = \tau(f_m) = \tau(g)$ as required.

*Step 5.* Now we prove condition (ii) for $\tau$. Given a path $f * g$ in $X$, let us choose a subdivision $s_0 < \dots < s_n$ of $[0, 1]$ containing the point $1/2$ as a subdivision point, such that $f * g$ carries each subinterval into either $U$ or $V$, again possible due to the Lebesgue number lemma. Let $k$ be the index such that $s_k = 1/2$. For $i = 1, \dots, k$ the positive linear map of $[0, 1]$ to $[s_{i-1}, s_i]$ followed by $f * g$, is the same as the positive linear map of $[0, 1]$ to $[2s_{i-1}, 2s_i]$ followed by $f$; call this map $f_i$. Similarly for $i = k+1, \dots, n$ the positive linear map of $[0, 1]$ to $[s_{i-1}, s_i]$ followed by $f * g$ is the same as the positive linear map of $[0, 1]$ to $[2s_{i-1} - 1, 2s_i - 1]$ followed by $g$; call this map $g_{i-k}$. Using the subdivision $s_0, \dots, s_n$ for the domain of the path $f * g$, we have

$$\tau(f * g) = \tau(f_1 * \dots * f_k * g_1 * \dots * g_{n-k}) = \sigma(f_1) \dots \sigma(f_k) \cdot \sigma(g_1) \dots \sigma(g_{n-k}) \tag{105}$$

Using the subdivision $2s_0, \dots, 2s_k$ for $f$ and $2s_k - 1, \dots, 2s_n - 1$ for $g$ we have

$$\tau(f) = \sigma(f_1) \dots \sigma(f_k) \qquad \text{and} \qquad \tau(g) = \sigma(g_1) \dots \sigma(g_{n-k}) \tag{106}$$

Hence (ii) is also satisfied. *Step 6.* We prove the theorem. For each loop $f$ in $X$ based at $x_0$, define

$$\Phi([f]) = \tau(f) \tag{107}$$

Conditions (i) and (ii) shows that $\Phi$ is a well defined homomorphism. To show that $\Phi \circ j_1 = \phi_1$, let $f$ be a loop in $U$. Then

$$\Phi(j_1([f]_U)) = \Phi([f]) = \tau(f) = \rho(f) = \phi_1([f]_U) \tag{108}$$

Similarly for a loop $g$ in $V$ we have $\Phi \circ j_2 = \phi_2$

$$\Phi(j_2([f]_V)) = \Phi([f]) = \tau(f) = \rho(f) = \phi_2([f]_V) \tag{109}$$

which was what we wanted. □

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