Homotopy Theory of Topological Spaces and Simplicial Sets

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Bachelorthesis

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Summary

In this article we will study a generalization of the homotopy theory we know from algebraic topology. We discuss the abstract tools needed for this generalization, namely model categories and their homotopy categories. We will apply our general setting to topological spaces to find the familiar homotopy theory. Afterwards we will look at the application to simplicial sets and see that their homotopy category is equivalent to that of topological spaces.

Data

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Contents

1 Introduction

The aim of this article is to study a general setting for homotopy theories, by the use of category theory. Specifically by defining model categories and their homotopy categories. The author was not familiar with category theory before writing this article, but due to the formal character of the subject decided to add this material as an appendix. So the reader might decide for himself whether or not to read this before starting on the article. Furthermore the reader is assumed to know some algebraic topology.

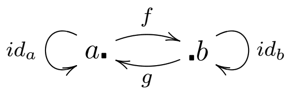
In algebraic topology we know the definition of *homotopic* maps and *homotopy equivalences*. In this article we will study homotopy theories at a more general level and we will see that this abstract definition applied to the category of topological spaces coincides with the familiar notion. Let us first recall those definitions from algebraic topology. Two maps $f_0, f_1 : X \to Y$ are called *homotopic* if we can deform them continuously into each other. In other words if we can find a map $H : X \times I \to Y$ with $H(x, 0) = f_0$ and $H(x, 1) = f_1$. Furthermore recall that a map $f : X \to Y$ is called a *homotopy equivalence* if it has a 'homotopy inverse', $g : Y \to X$, a map such that the compositions are homotopic to the relative identity maps. The notion of being homotopic is a generalization of being homeomorphic (e.g. X, Y homeomorphic if there are maps f and g as before where the compositions *are* the identities). When the maps f and g form a homotopy equivalence, g is the inverse of f up to homotopy.

The main example of a model category that we will discuss in this article is the example on topological spaces. The general construction of homotopy that we will use allows us to talk about homotopy theories in many other settings though. We will only describe one of these setting, namely the case of simplicial sets. Other examples can for example be found in [Hov99] and [DS95].

As named before we will be using category theory to make this generalization possible. A category is a collection of objects plus a collection of morphisms between all those objects that satisfy some rather weak axioms. For the precise definition of a category we refer to the appendix. The appendix deals with basic concepts and examples from category theory. It is meant to be self-contained and we will refer to it when necessary.

Since the axioms for a category are weak, we can find them everywhere in mathematics. Categories (and functors) make it possible to compare different fields in mathematics in a formal way. Lets look at some examples of categories first. In topology, we are interested in topological spaces and continuous maps. In differential geometry we study manifolds and smooth maps. In basic set theory we study sets and maps between them. In the three situations above, we would in categorical language say, that we study objects and morphisms between objects. A *category* is defined to be a collection of objects with a collection of arrows between each two objects.

In a category each object a has an identity arrow, id_a, if we are in the following setting;



and if $fg = id_a$ and $gf = id_b$ we will call the objects a and b isomorphic, as usual. Isomorphic objects will have some common properties, depending on the category. For example in the setting of sets as before isomorphisms are just bijections and the sets that are isomorphic will have the same cardinality. The isomorphic objects in the homotopy category that we are going to construct will be 'homotopic' objects. We use commas because we will actually define those objects as homotopic objects, and afterwards we will see that this notion of homotopic applied to topological spaces coincides with the concept familiar to us. To be able to define the homotopy category of a model category, we of course need to define what a model category is first. We start section one by introducing the axioms for a model category, afterwards we will study certain objects in a model category that will be suitable for defining homotopy relations. This will enable us to finish the first section by defining the homotopy category.

All this is very general and abstract, so we will continue the article by studying two applications. We will put model category structures on both the category of topological spaces and simplicial sets. The verifying of the axioms of a model category will turn out to be a big task. This is why we will only verify the model category structure on topological spaces in detail. We will do this in section two. Then in the third and last section we will define the model category structure on simplicial sets. Unfortunately the proof of this lies outside the scope of this project. The model category structure on simplicial sets is defined with help of topological spaces. Simplicial sets can be used as an approximation to topological spaces. This is useful because it is easier to work with simplicial sets since they are purely combinatorial objects. We will end the article by stating a well known theorem that tells us that the homotopy categories of simplicial sets and topological spaces are equivalent.

2 Model Categories and their Homotopy Category

In this section we start by giving the axioms for a model category. These axioms are rather strong, as mentioned in the introduction. Since checking them in different situations will be a hard task, we will only look at some formal examples in this section. After these examples we will see that the axioms for a model category are over determined. This allows us to introduce *cofibrantly* generated model categories. It is easier to check that a category is a cofibrantly generated model category so this gives us a more convenient tool for proving that a category has a model category structure on it.

In a model category there are three distinguished types of maps, weak equivalences, fibrations and cofibrations. We would like to find an inverse for the weak equivalences, we will see that the homotopy category $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is suitable for this in the second part of this section. The weak equivalences in the model category will map to the isomorphisms in the homotopy category, under a functor γ. The homotopy category will have the same objects as the model category we constructed it from, so will differ from it concerning the morphisms only. In $Ho(C)$ the morphisms will be classes of morphisms from $\mathcal{C}$. For $Hom_{Ho(\mathcal{C})}(A, X)$ the morphisms will be classes of maps between these objects itself or closely related objects. This depends, as we will see, on the properties of A and X. The classes will be our abstractly defined homotopy classes. In case of topological spaces these will be the conventional homotopy classes, but we will see this in section three. At the end of this section we will state a theorem that will give a condition for homotopy categories to be equivalent.

2.1 Model Categories

Model categories were introduced by Quillen in 1967 as closed model categories. We will study one type only, so we simply call them model categories.

We start by giving the following definition of lifting properties, which we will need to define the model category structure;

Definition 2.1.1: Given a commutative diagram;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

a *lift* or a *lifting* is a map $h : B \to X$ such that the resulting diagram commutes;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Thus such that $hi = f$ and $ph = g$. The map i is said to have the *left lifting property*, LLP with respect to p, and p has the *right lifting property* RLP with respect to i.

Definition 2.1.2: A *model category* is a category $\mathcal{C}$ with three distinguished types of maps;

(i) weak equivalences ($\xrightarrow{\sim}$)

(ii) fibrations ($\twoheadrightarrow$)

(iii) cofibrations ($\hookrightarrow$)

The sets of those maps are closed under composition and all contain the identity maps. A map which is both a fibration/cofibration is called an acyclic fibration/cofibration. The category $\mathcal{C}$ has to obey the following axioms;

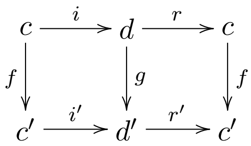
MC1: Finite limits and colimits exist in $\mathcal{C}$
MC2: If f and g are maps in $\mathcal{C}$ such that fg is defined, then if two of the three maps are weak equivalences, so is the third. *(2 out of 3)*
MC3: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f. *(retracts)*
MC4: Given a commutative diagram of the form (*), a lift exists in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration. *(lifting)*
MC5: Any map f can be factored as $f = pi$ in the following two ways: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration. *(factorization)*

Note: If we let $\mathcal{J}$ be the empty category, then the colimit [A.27] of a functor $\mathcal{J} \to \mathcal{C}$ is an initial object in $\mathcal{C}$. We get a unique map to every object in $\mathcal{C}$. Similarly, the limit of F is a terminal object. So the first axiom guaranties that a model category $\mathcal{C}$ has an initial and a terminal object, denoted by $\emptyset$ and $*$ relatively. We call an object A in $\mathcal{C}$ *cofibrant* if the map $\emptyset \to A$ is a cofibration and dually we call an object X *fibrant* if the map $X \to *$ is a fibration.

Example 2.1.3: Let $\mathcal{C}$ be a category with all finite colimits and limits. Let all morphisms in $\mathcal{C}$ be fibrations and cofibrations and let $f : X \to Y$ in $\mathcal{C}$ be a weak equivalence if it is an isomorphism. With these choices of maps $\mathcal{C}$ is a model category. The first axiom we assumed, the second axiom is obvious since isomorphisms have the 2 out of 3 property. The third axiom is clear for the cofibrations and fibrations. We will state the case of weak equivalences as a lemma;

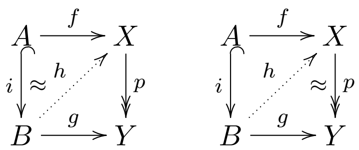
Lemma 2.1.4: If g is an isomorphism in $\mathcal{C}$ and f is a retract of g, then f is an isomorphism as well.

Proof: By definition of a retract [A.21] we have the following commuting diagram for f and g;



where $ri = id_c$ and $r'i' = id_{c'}$, now since g is an isomorphism we have an arrow in the opposite direction g^{-1}. We claim that $rg^{-1}i'$ is the inverse of f. We find $rg^{-1}i'f = ri = id_c$ and $frg^{-1}i' = r'i' = id_{c'}$ $\square$.

The fourth axiom requires liftings in the following two diagrams;



In the diagram at the left we take $h(b) = fi^{-1}(b)$ and in the right diagram we take $h(b) = p^{-1}g(x)$.

Now the fifth and last axiom is easy since all maps are cofibrations and fibrations we can factor $f : X \to Y$ as $f \circ i_X$ and $i_Y \circ f$. So we found a model category structure on $\mathcal{C}$.

Equally easy model category structures on $\mathcal{C}$ can be found by either letting all maps be weak equivalences and fibrations and letting the cofibrations be the isomorphisms or dually, letting all maps be weak equivalences and cofibrations and letting the fibrations be the isomorphisms.

Our next example shows that the axioms of a model category are self dual. That is if $\mathcal{C}$ is a model category then $\mathcal{C}^{op}$ is a model category as well. This means that every theorem about model categories has a dual theorem. We can prove the dual theorem by simply dualizing the original proof. See appendix A for a more elaborate discussion of duality.

Example 2.1.5: If $\mathcal{C}$ is a model category then so is $\mathcal{C}^{op}$ if we let $f^{op} : d \to c$ in $\mathcal{C}^{op}$ be

(i) a weak equivalence if $f : c \to d$ is a weak equivalence in $\mathcal{C}$,

(ii) a fibration if $f : c \to d$ is a cofibration in $\mathcal{C}$, and

(iii) a cofibration if $f : c \to d$ is a fibration in $\mathcal{C}$.

Let's check the axioms,

MC1: A colimit in $\mathcal{C}^{op}$ corresponds to a limit in $\mathcal{C}$ and a limit in $\mathcal{C}^{op}$ corresponds to a colimit in $\mathcal{C}$. Thus the fact that all finite limits and finite colimits exist in $\mathcal{C}$ directly implies that the same holds for $\mathcal{C}^{op}$.
MC2: Let $f^{op} : d \to c, g^{op} : e \to d$ in $\mathcal{C}^{op}$ such that $f^{op}g^{op} : e \to c$ is defined. Now $f^{op}g^{op} = (gf)^{op}$. And;

$$\begin{aligned} f^{op} \text{ is a weak equivalence} &\Leftrightarrow f \text{ is a weak equivalence} \\ g^{op} \text{ is a weak equivalence} &\Leftrightarrow f \text{ is a weak equivalence} \\ f^{op}g^{op} \text{ is a weak equivalence} &\Leftrightarrow gf \text{ is a weak equivalence} \end{aligned}$$

So if two out of three of the maps are weak equivalences in $\mathcal{C}^{op}$ so are two out of three in $\mathcal{C}$ and thus all of them in $\mathcal{C}$ and then also all three in $\mathcal{C}^{op}$.
MC3: The morphism f^{op} is a retract of g^{op} if and only if f is a retract of g, since you only change the direction of the vertical arrows in the diagram in [A.16] formally. When g^{op} is a weak equivalence, fibration or cofibration, so is g by definition, and thus f and also f^{op}.
MC4: Now given the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f_0^{op}} & B \\ {\scriptstyle g_0^{op}}\downarrow & & \downarrow{\scriptstyle g_1^{op}} \\ C & \xrightarrow{f_1^{op}} & D \end{array}$$

where g_0^{op} is a cofibration and g_1^{op} an acyclic fibration, this gives the following diagram in $\mathcal{C}$

$$\begin{array}{ccc} A & \xleftarrow{f_0} & B \\ {\scriptstyle g_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ C & \xleftarrow{f_1} & D \end{array}$$

With g_0 a fibration and g_1 an acyclic cofibration, thus there exist a lift $h : B \to C$ and this gives a lift in the first diagram $h^{op} : C \to B$. The proof for the second condition is similar and left as an exercise.

MC5: Since any f in $\mathcal{C}$ can be factored as pi, any f^{op} in $\mathcal{C}^{op}$ can be factored as $i^{op}p^{op}$. In case (i) i is a cofibration and p an acyclic fibration, so p^{op} is an acyclic cofibration and i^{op} is a fibration, so it gives (i) for the opposite category. Similarly in case (ii) p^{op} is a cofibration and i^{op} is an acyclic fibration.

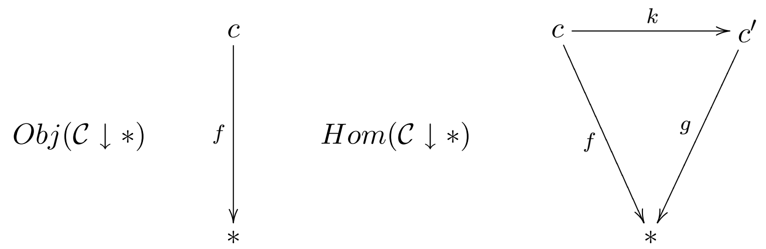
Example 2.1.6: If $\mathcal{C}$ is a model category then for $* \in \mathcal{C}$ a terminal object, the over category $\mathcal{C} \downarrow *$ [A.23] can be given a model category structure by defining $\langle k, id_* \rangle : (c \to *) \to (c' \to *)$ in $\mathcal{C} \downarrow *$ to be

(i) a weak equivalence if $k : c \to c'$ is a weak equivalence in $\mathcal{C}$,

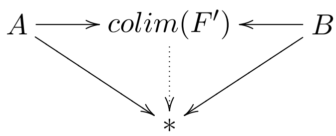
(ii) a fibration if $k : c \to c'$ is a fibration in $\mathcal{C}$, and

(iii) a cofibration if $k : c \to c'$ is a cofibration in $\mathcal{C}$.

This over category can be visualized as follows;



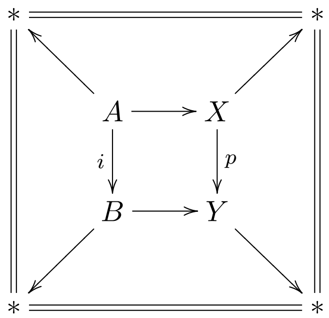
see example [A.23] as well. Now let us start checking the model category axioms. Let $F : \mathcal{J} \to \mathcal{C} \downarrow *$ be a functor from a finite category $\mathcal{J}$. To find a colimit of F we can look at the functor $F' : \mathcal{J} \to \mathcal{C}$ that sends an object in J to the object X instead of $X \to *$. Then $colim(F')$ is an object in $\mathcal{C} \downarrow *$ by the universal property of colimits



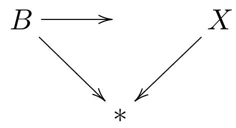
then the object $colim(F') \to *$ in $\mathcal{C} \downarrow *$ is the colimit for F. For the limit we have to be a bit more careful because we do not get a map to $*$ automatically.

Again we are given a functor $F : \mathcal{J} \to \mathcal{C} \downarrow *$. Now let $\mathcal{J}'$ be $\mathcal{J}$ with an extra terminal object $*$. Let $G : \mathcal{J}' \to \mathcal{C}$ again be a functor induced by F and $G(*) = *$, $G(A \to *) = F(A) \to *$. Now $lim(G) \to *$ is the limit of F.

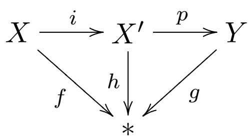
The second and third axiom follow easily from the fact that $\mathcal{C}$ is a model category. Now for the fourth axiom we have to find a lift in the following commutative diagram;



when i is a cofibration, p is a fibration and one of them is acyclic, we can find the lift in $\mathcal{C}$ and we see from the above diagram that we find a commutative triangle;



so we have a morphism in $\mathcal{C} \downarrow *$.
For the fifth axiom we want to find a factorization for a morphism in $\langle f, i_* \rangle$ in $\mathcal{C} \downarrow *$. We can factor f as pi with p a fibration and i a cofibration, and one of them acyclic. Now we want a map h to make the following diagram to commute;



But this is forced, since $*$ is a terminal object so the maps are unique.

Now we have seen a couple of abstract examples of model categories. In section two and three we will study 'real' examples as promised before. As we said before the axioms of a model category are over determined in some sense. If we know the weak equivalences and the fibrations/cofibrations then the cofibrations/fibrations are determined as well. The following proposition states this formally.

Proposition 2.1.7: Let $\mathcal{C}$ be a model category.
(i) The fibrations in $\mathcal{C}$ are the maps that have the RLP with respect to all acyclic cofibrations.
(ii) The acyclic fibrations in $\mathcal{C}$ are the maps that have the RLP with respect to all cofibrations.

Dually:
(iii) The cofibrations in $\mathcal{C}$ are the maps that have the LLP with respect to all acyclic fibrations.
(iv) The acyclic cofibrations in $\mathcal{C}$ are the maps that have the LLP with respect to all fibrations.

Proof: We will only prove the first two statements since the other two follow by duality. By the lifting axiom we see that the (acyclic) fibrations have the RLP with respect to the (acyclic) cofibrations so we have to prove that a map that has the RLP with respect to (acyclic) cofibrations is a (acyclic) fibration.
Now let $f : X \to Y$ have the RLP with respect to all acyclic cofibrations. We can factor $f = pi$ as $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow Y$ by the factorization axiom. Now we can find a lift in the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i_K} & K \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ K' & \xrightarrow[p]{} & L \end{array}$$

This makes f into a retract of p;

$$\begin{array}{ccccc} K & \longrightarrow & K' & \longrightarrow & K \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ L & \xrightarrow{i_L} & L & \xrightarrow{i_L} & L \end{array}$$

so with the retract axiom we find that f is a fibration. The proof when f has the RLP with respect to all fibrations is similar, we take the other factorization of f, $X \hookrightarrow Y' \overset{\sim}{\twoheadrightarrow} Y'$ and then continue the argument in the same way.

Now that we know that the model category structure is over determined we can look at *cofibrantly generated model categories.* We could as well look at fibrantly generated model categories, but according to Hovey [Hov99] this definition is much less useful. The idea of a cofibrantly generated model category is that we pick a subcategory of $\mathcal{C}$ which will contain all weak equivalences, and we pick two sets of maps, I and J, the generating cofibrations and the generating acyclic cofibrations. So we even get away with picking a collection of the cofibrations instead of all of them. The class of fibrations will be the class of maps that have the RLP with respect to all maps in J, the generating acyclic cofibrations. Similarly the class of acyclic fibrations will be the class of maps with the RLP with respect to all maps in I. Now by the proposition above we see that the (acyclic) cofibrations have to be the class of maps that have the LLP with respect to the (acyclic) fibrations. In general this class is bigger then the class of generating (acyclic) cofibrations. The author did not study cofibrantly generated categories thoroughly and will not use them while constructing the model category structure on topological spaces. In case of simplicial sets we will use them though. But as said before we will only define the model category structure there. We will defer the further discussion of cofibrantly generated model categories until section three. This is because some of the terminology that comes up there we will use while proving the model category structure on topological spaces. The abstract definitions will be more readable after having seen the application to topological spaces.

Before we can continue this section with discussing the homotopy category we need to define homotopy relations on the model category structure. We will use the axioms of a model category $\mathcal{C}$ to construct suitable homotopy relations on the sets of morphisms $hom_{\mathcal{C}}(A, X)$. We will define a notion of right homotopy and a dual notion of left homotopy. To define right homotopy we will introduce path objects and for left homotopy we will construct cylinder objects. We will give the definitions of left, right and normal homotopy now and discuss some important properties that we need in order to be able to construct the homotopy category.

Definition 2.1.8: A *path object* for $X \in \mathcal{C}$ is an object X^I of C together with a diagram

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

which factors the diagonal map $(id_X, id_X) : X \to X \times X$. So the following diagram commutes;

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X^I \\ & \searrow^{id_X + id_X} & \downarrow p \\ & & X \times X \end{array}$$

Definition 2.1.9: Two maps $f, g : A \to X$ are said to be *right homotopic* if the product map $(f, g) : A \to X \times X$ can be lifted to a map $H : A \to X^I$ as illustrated in the diagram;

$$\begin{array}{ccc} & & X^I \\ & \nearrow^{H} & \downarrow \\ A & \xrightarrow{(f,g)} & X \times X \end{array}$$

Notation; $f \overset{r}{\sim} g$.

Now dually we define;

Definition 2.1.10: A *cylinder object* for A is an object $A \wedge I$ of $\mathcal{C}$ together with a diagram;

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the map $id_A + id_A : A \coprod A \to A$.

Definition 2.1.11: Two maps $f, g : A \to X$ are said to be *left homotopic* if the sum map $f + g : A \coprod A \to X$ can be extended to a map $H : A \wedge I \to X$ as illustrated in the diagram;

$$\begin{array}{ccc} A \coprod A & \xrightarrow{f+g} & X \\ \downarrow{\scriptstyle i} & \overset{H}{\nearrow} & \\ A \wedge I & & \end{array}$$

Notation; $f \overset{l}{\sim} g$.

Both left and right homotopy are relations on maps and we denote the equivalence classes of $Hom(A, X)$ under the equivalence relation generated by these relation as $\pi^l(A, X)$ and $\pi^r(A, X)$. When A is cofibrant and X is fibrant left and right homotopy coincide and we write $\pi(A, X)$. [B.13]

In the construction of a homotopy category of a model category $\mathcal{C}$ we want to look at a subcategory where the objects are all cofibrant and the morphisms are right homotopy classes of maps. To be able to do this we need the composition of right homotopic maps to be defined. This is the case for cofibrant objects, see [B.10]. We can also form the dual subcategory, with fibrant objects and classes of left homotopic morphisms. Finally we can look at the subcategory with objects that are both fibrant and cofibrant and classes of homotopic maps by the note above.

Two final observations we will need for the construction of the homotopy category is that when A is cofibrant (X is fibrant) and $p : Y \to X$ is an acyclic fibration ($i : A \to B$ is an acyclic cofibration), then composition with p (i) induces a bijection on left (right) homotopy groups. And in this situation $f : A \to X$ is a weak equivalence if and only if it has a homotopy inverse.

The proofs of the above statements and more can be found in appendix B. Now we are ready for the construction of the homotopy category.

2.2 Homotopy Categories

Here we will construct the homotopy category of a model category. The objects will be the same but the morphisms will be classes of objects. Because of the strong structure that a model category has we find that there is a functor γ from the model category to its homotopy category that is a localization of the model category with respect to the weak equivalences. So we can formally invert the weak equivalences in our model category and get a nice category.

We just saw that for a morphism between objects that are both fibrant and cofibrant the notion of being a weak equivalences and homotopic coincides. We want to define a category where the isomorphisms are 'homotopy equivalences'. Recall that in **Top** those are defined to be the maps with a homotopy inverse fg and gf are homotopic to the relative identities. We can define homotopy equivalences in a model category in the precise same way, now with the abstract definition of homotopic maps.

So to define a homotopy category of a model category it should not be surprising that we are interested in cofibrant and fibrant objects. Not all objects in a model category, $\mathcal{C}$, are both fibrant and cofibrant though. We start this paragraph by associating a fibrant and a cofibrant object to every object in X.

Recall that a model category has an initial and terminal object. There is a morphism $\emptyset \to X$ in $\mathcal{C}$ for each object $X \in \mathcal{C}$. By **MC5** we can factor this morphism as a composition of an acyclic fibration with a cofibration. Let QX be the cofibrant object in this factorization, furthermore write $p_X : QX \overset{\sim}{\twoheadrightarrow} X$. If X already was cofibrant we let $QX = X$. In similar fashion we define RX to be the fibrant object in the factorization $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$, and when X was already fibrant $RX = X$. We call the first map in the factorization i_X.

Now for the construction of the homotopy category we want to use the following categories, associated to a model category $\mathcal{C}$;
$\mathcal{C}_c$ - the full subcategory [A.3] of $\mathcal{C}$ generated by the cofibrant objects.
$\mathcal{C}_f$ - the full subcategory of $\mathcal{C}$ generated by the fibrant objects.
$\mathcal{C}_cf$ - the full subcategory of $\mathcal{C}$ generated by the objects of $\mathcal{C}$ that are both fibrant and cofibrant.
$\pi\mathcal{C}_c$ - the category with objects the cofibrant objects and whose morphisms are right homotopy classes of maps [B.7].
$\pi\mathcal{C}_f$ - the category with objects the fibrant objects and whose morphisms are left homotopy classes of maps.
$\pi\mathcal{C}_cf$ - the category with objects the objects of $\mathcal{C}$ that are both fibrant and cofibrant, whose morphisms are the homotopy classes of maps.

As mentioned before we want the homotopy category to have the homotopy equivalences as its isomorphisms, this means that we want the morphisms in $Ho(C)$ to be classes of homotopy equivalent maps. We will construct a functor $F : \mathcal{C} \to \pi C_{cf}$, and then let the morphisms between X and Y in $Ho(C)$ be $Hom_{\pi C_{cf}}(X, Y)$. To get this functor we will first construct two functors $R : \mathcal{C} \to \pi\mathcal{C}_f$ and $Q : \mathcal{C} \to \pi\mathcal{C}_c$. The following lemmas tell us how to do this.

Lemma 2.2.1: Given a map $f : X \to Y$ in $\mathcal{C}$ there exists a map $\overline{f} : RX \to RY$ such that the following diagram commutes;

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle i_Y}\downarrow{\scriptstyle \sim} \\ RX & \xrightarrow{\overline{f}} & RY \end{array}$$

The map $\overline{f}$ depends up to right homotopy only on f, and is a weak equivalence if and only if f is. If X is cofibrant, then $\overline{f}$ depends up to right homotopy or up to left homotopy only on the right homotopy class of f.

Proof: We get the map $\overline{f}$ by applying **MC4** to the following diagram;

$$\begin{array}{ccc} X & \xrightarrow{i_Y f} & RY \\ {\scriptstyle i_X}\downarrow{\scriptstyle \sim} & \nearrow & \downarrow \\ RX & \longrightarrow & * \end{array}$$

Indeed we have $\overline{f} i_X = i_Y f$. The uniqueness statements follow from some properties of the homotopy relations. [B.8][B.9][B.13].

Lemma 2.2.2: Given a map $f : X \to Y$ in $\mathcal{C}$ there exists a map $\tilde{f} : QX \to QY$ such that the following diagram commutes;

$$\begin{array}{ccc} QX & \xrightarrow{\tilde{f}} & QY \\ {\scriptstyle p_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle p_Y}\downarrow{\scriptstyle \sim} \\ X & \xrightarrow{f} & Y \end{array}$$

The map $\tilde{f}$ depends up to right homotopy only on f, and is a weak equivalence if and only if f is. If Y is fibrant, then $\tilde{f}$ depends up to right homotopy or up to left homotopy only on the right homotopy class of f.

Proof: The proof is completely dual to the previous one.

Now we can construct the following functors; $R : \mathcal{C} \to \pi C_f$ and $Q : \mathcal{C} \to \pi C_c$ as follows. Send an object X of $\mathcal{C}$ to the fibrant object RX defined before. Send a map $f : X \to Y$ to the left homotopy class $[\overline{f}] \in \pi^l(RX, RY)$. We have to check that this indeed is a functor. Let $f = id_X$, then id_{RX} makes the diagram commute;

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ {\scriptstyle i_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle i_X}\downarrow{\scriptstyle \sim} \\ RX & \xrightarrow{id_{RX}} & RX \end{array}$$

thus for any $\overline{f} \overset{r}{\sim} id_X$. So indeed $R(id_X) = [id_Q X]$. Now let $f : X \to Y$ and $g : Y \to Z$, and $h = gf$. Similarly, since both $\overline{h}$ and $\overline{gf}$ make the diagram with gf on top commute, they are left homotopic to each other thus and thus $R(h) = R(g)R(f)$. Analogously we define the functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$, which sends objects X to cofibrant objects QX and maps $f : X \to Y$ to the right homotopy classes $[\tilde{f}] \in \pi^r(QX, QY)$. These functors are called the *fibrant replacement functor* and the *cofibrant replacement functor* respectively

Lemma 2.2.3: The restriction of the functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$ to $\mathcal{C}_f$ induces a functor $Q' : \pi\mathcal{C}_f \to \pi\mathcal{C}_{cf}$. And dually the restriction of the functor $R : \mathcal{C} \to \pi\mathcal{C}_f$ to $\mathcal{C}_c$ induces a functor $Q' : \pi\mathcal{C}_c \to \pi\mathcal{C}_{cf}$.

We state this lemma without proving it. Now we can finally define the homotopy category.

Definition 2.2.4: The *homotopy category* $Ho(\mathcal{C})$, of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and with;

$$Hom_{Ho(\mathcal{C})}(X, Y) = Hom_{\pi\mathcal{C}_cf}(R'QX, R'QY) = \pi(RQX, RQY)$$

The classes of homotopic maps from objects closely related to X and Y. When X and Y are both fibrant and cofibrant $\pi(RQX, RQY) = \pi(X, Y)$

The functor γ that will give a localization of a model category $\mathcal{C}$ with respect to the weak equivalences in $\mathcal{C}$ can now be defined. Let $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ be the identity on objects and let it send a morphism $f : X \to Y$ to the morphism $R'Q(f) : R'Q(X) \to R'Q(Y)$.

By theorem 6.2 from [DS97] we see that γ is a localization of $\mathcal{C}$ with respect to the weak equivalences. This means that we can control the formal inversion of the weak equivalences neatly by the structure we get from the fibrations and cofibrations.

2.3 Quillen Equivalence

In this paragraph we will define what is meant by a Quillen equivalence. We will state a theorem that gives us a condition for a Quillen equivalence, but we will not prove this. We will only explain what the conditions are. In order to do this we will introduce total left and right derived functors, $\mathbf{L}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ and $\mathbf{R}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$, of a functor $F : \mathcal{C} \to \mathcal{D}$.

Definition 2.3.1: A *Quillen equivalence* between $\mathcal{C} \leftrightarrow \mathcal{D}$ is an adjoint pair [A.15] of functors $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ such that:
(i) F preserves cofibrations and G preserves fibrations and,
(ii) the derived functors $\mathbf{L}F : Ho(\mathcal{C}) \Longleftrightarrow Ho(\mathcal{D}) : \mathbf{R}G$ induce an equivalence of categories [A.14].

The first condition implies that the total left and right derived functors exist, as stated in the following theorem. In addition the theorem gives a condition on $\mathcal{C}$ and $\mathcal{D}$ to get a Quillen equivalence.

Theorem 2.3.2: Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and;

$$F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$$

be a pair of adjoint functors. Suppose that;
(i) F preserves cofibrations and G preserves fibrations, then the total derived functors exist and form an adjoint pair; $\mathbf{L}F : Ho(\mathcal{C}) \Longleftrightarrow Ho(\mathcal{D}) : \mathbf{R}G$
(ii) If in addition we have that the following condition holds for each cofibrant object $A \in \mathcal{C}$ and fibrant object $X \in \mathcal{D}$ and a map $f : A \to G(X)$; f is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\flat : F(A) \to X$ is a weak equivalence in $\mathcal{D}$. Then $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ gives a Quillen equivalence.

Here the adjoint map comes from the bijection between hom sets that an adjunction of functors gives us [A.15.]. The proof of this theorem can be found in [DS95][pg. 43-45]. We continue by defining those total left and right derived functors. To do this we first have to define the left and right derived functors. The definitions are each others duals, so we will focus on right derived functors.
Definition 2.3.3: Let $\mathcal{C}$ be a model category, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Look at pairs (G,σ) with;

$$G : Ho(\mathcal{C}) \to \mathcal{D}, \text{ a functor, and } \quad \sigma : F \to G\gamma, \text{ a natural transformation.}$$

Now a *right derived functor* is an object (RF, τ) like this that is universal from the right. By universal we mean that for every other pair (G, σ) there is a unique natural transformation

$\sigma' : RF \to G$ such that the composite natural transformation;

$$F \xrightarrow{\tau} RF\gamma \xrightarrow{\sigma'\gamma} G\gamma$$

is the natural transformation σ.

Definition 2.3.4: A *left derived functor* is an object (LF, τ) as above but now with a universal property from the left. So for every other pair (G, σ) there is a unique natural transformation $\sigma' : G \to LF$ such that the composite natural transformation;

$$G\gamma \xrightarrow{\sigma'\gamma} LF\gamma \xrightarrow{\tau} F$$

is the natural transformation σ.

Definition 2.3.5: A *total left derived functor* $\mathbf{L}F$ for a functor $F : \mathcal{C} \to \mathcal{D}$ between model categories, is a functor

$$\mathbf{L}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$$

which is a left derived functor for the composition $\gamma_{\mathcal{D}}F : \mathcal{C} \to Ho(\mathcal{D})$. Similarly a *total right derived functor* $\mathbf{R}F$ is a functor $\mathbf{R}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ that is a right derived functor for $\gamma_{\mathcal{D}}F$.

With this theorem as a tool, the Quillen equivalence between the **Top** and **sSet** can be established, as we will see at the end of section three. But first we will study the application to topological spaces in the next section.

3 The Model Category Structure on Topological Spaces

In this chapter we assume that the reader has some knowledge of algebraic topology. We start by recalling some definitions. A good introduction to the subject is given in [AH01]. Afterwards we will continue by pointing out collections of maps in **Top** that are our weak equivalences, fibrations and cofibrations. We proof that those choices define a model category structure on **Top** and finally show that this model category structure leads to the conventional homotopy relations in the homotopy category.

3.1 Definitions from Algebraic Topology

We will just recall some definitions from algebraic topology that we are going to use directly, for further reading we refer to [AH01].

Definition 3.1.1: A map between topological spaces, $f : A \to B$ is called a *weak homotopy equivalence* if it induces isomorphisms on homotopy groups $f_*\pi_i(A, a_0) \to \pi_i(B, f(a_0))$, for $i \geq 0$.

Definition 3.1.2: In the following diagram;

$$\begin{array}{ccc} A & \xrightarrow{k_0} & X \\ {\scriptstyle f}\downarrow & \exists h \nearrow & \downarrow{\scriptstyle g} \\ B & \xrightarrow{k_1} & Y \end{array}$$

where there exist a lift h, the map f is said to have the *right lifting property* also *RLP* with respect to g and similarly, g has the *left lifting property* or *LLP* with respect to f.

Definition 3.1.3: A map $p : X \to Y$ is called a *Serre fibration* if it has the RLP with respect to the inclusion $i : A \times \{0\} \to A \times [0, 1]$, for each CW-complex A.

3.2 Model Category Structure on Topological Spaces

Before we start with pointing out the special maps in **Top**, we want to note that there are several possible choices. We want to get a model category structure on **Top** such that the homotopy category consist of equivalence classes of maps, with respect to regular homotopy. It turns out that there are different model category structures realizable on **Top** with the same homotopy category. We will focus on one choice only, as described in chapter 8 of [DS95].

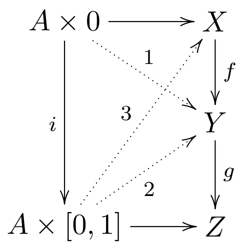
Proposition 3.2.1: The category **Top** of topological spaces can be given a model category structure by defining $f : A \to B$ to be;

(i) a weak equivalence if f is a weak homotopy equivalence,

(ii) a fibration if f is a Serre fibration, and

(iii) a cofibration if it has the LLP with respect to acyclic cofibrations.

Proof: The first thing we need to do is to check if the just defined classes all contain identity maps and are closed under composition. The identity map is a weak homotopy equivalence. Furthermore weak homotopy equivalences are closed under composition since π_i is a functor and thus $\pi_i(g \circ f) = \pi_i(g) \circ \pi_i(f)$. The identity map has the RLP with respect to every map, by just taking $h = k_1$ in (*). To show that a Serre fibrations are closed under composition look at the following diagram;



Where A is a CW-complex and i the inclusion map, both f and g are Serre fibrations. We start out with the diagram without the dotted arrows. Now in step 1 we construct a map from $A \times \{0\}$ to Y by composition. Since g is a Serre fibration we find a lift of this map in step 2, and finally in step 3 we use the fact that f is a Serre fibration too, and we get the wanted lift from $A \times [0, 1]$ to X. It should be clear now that the class of cofibrations contains the identities and is closed under composition too.

MC1: This axiom follows from the fact that the category of topological spaces, **Top** contains all small limits and colimits. [ML71][pg. 128]

MC2: For $f : X \to Y$, $g : Y \to Z$ and gf we have as before, $\pi_i(gf) = \pi_i(g)\pi_i(f)$ so when two out of three of the those are isomorphisms so is the third.

MC3: Lets look at the case of weak equivalences first. By lemma [2.1.4] we know that a retract of an isomorphism is an isomorphism. Now by functorality of π_i the commuting diagram from definition [A.19] induces a commuting diagram like this;

$$\begin{array}{ccccc}
\pi_i(c) & \xrightarrow{\pi_i(i)} & \pi_i(d) & \xrightarrow{\pi_i(r)} & \pi_i(c) \\
\downarrow{\scriptstyle \pi_i(f)} & & \downarrow{\scriptstyle \pi_i(g)} & & \downarrow{\scriptstyle \pi_i(f)} \\
\pi_i(c') & \xrightarrow{\pi_i(i')} & \pi_i(d') & \xrightarrow{\pi_i(r')} & \pi_i(c')
\end{array}$$

So when g is a weak equivalence the middle map is an isomorphism for all i and thus $\pi_i(f)$ is an isomorphism for all i, so f is a weak equivalence too.

Now the cases of cofibrations and fibrations have a similar proof, so we will only look at fibrations. For each CW-complex A we get the following commutative diagram since g is a fibration;

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\tilde{k_0}} & d \\ {\scriptstyle f}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle g} \\ A \times [0,1] & \xrightarrow{\tilde{k_1}} & d' \end{array}$$

Now look at the following diagram;

$$\begin{array}{ccccccc} A \times \{0\} & \xrightarrow{k_0} & c & \xrightarrow{i} & d & \xrightarrow{r} & c \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ A \times [0,1] & \xrightarrow{k_1} & c' & \xrightarrow{i'} & d' & \xrightarrow{r'} & c' \end{array}$$

Again we get h since g is a fibration and the diagram commutes. Claim $rh : A \times [0,1] \to c$ is a lift in the left diagram. We find $rhj = rik_0 = k_0$ and $frh = r'gh = r'i'k_1 = k_1$. So we found a lift for k_1 with respect to f and f indeed is a fibration.

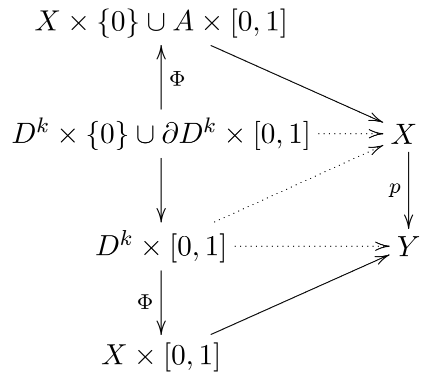
The proofs of **MC4** and **MC5** are more complicated. We will need to introduce the Infinite Gluing Construction and to proof a couple of lemmas before we are ready to proof those axioms.

Lemma 3.2.2: Let $p : X \to Y$ be a map in **Top**, then p is a Serre fibration if and only if p has the RLP with respect to the inclusions $D^n \times \{0\} \to D^n \times [0,1]$

Proof: " $\Rightarrow$ " is trivial since D^n is a CW-complex.

" $\Leftarrow$ " We will proof a slightly more general lemma here of which the previous statement is a special case. Namely the RLP with respect to inclusions $D^n \times \{0\} \to D^n \times [0,1]$ implies the RLP with respect to inclusions $X \times \{0\} \cup A \times [0,1] \to X \times [0,1]$ where (X, A) is a CW-pair [AH01][pg. 7]. Then the special case where we take $(D^n, \emptyset)$ as our CW-'pair' reduces to our lemma.

Note that the following pairs are homeomorphic $(D^n \times [0,1], D^n \times \{0\})$, $(D^n \times [0,1], D^n \times \{0\} \cup \partial D^n \times [0,1])$, which means that the RLP with respect to the two inclusions is equivalent. We can use induction over the skeleta of X and just lift one cell of $X - A$ at a time. Those lifts reduce to the lifting of disks by composing with the characteristic map [AH01][pg. 7] of the cell $\Phi : D^n \to X$, as we can see in the following diagram;



The upper and lower left map being restrictions of the characteristic map, where we know $\Phi(\partial D^k \times [0,1]) \subset A \times [0,1]$ since the boundary of an $n+1$-cell attaches to the n-skeleta and we are making A bigger in every step of the induction. The dotted horizontal maps are just composition maps and we find a lift. Since Φ is an inclusion we can inductively construct a lift for the whole of X with this construction $\Box$.

Definition 3.2.3: A *relative CW-pair* (X, A) is a pair of topological spaces such that X is obtained from A by attaching a finite number of cells.

Now we continue by defining the Gluing Construction and the Infinite Gluing Construction, we will use this construction afterwards to proof the following lemma;

Lemma 3.2.4: Every map $p : X \to Y$ in **Top** can be factored as $p_\infty i_\infty$, where i_∞ is a weak homotopy equivalence which has the LLP with respect to all Serre fibrations and p_∞ is a Serre fibration.

So we would like to construct a factorization of $p : X \to X' \to Y$ where the second map has the RLP with respect to a certain collection of maps, namely the inclusion maps from $A \times \{0\} \to A \times [0,1]$ or equally for the inclusions $D^n \times \{0\} \to D^n \times [0,1]$. To achieve this we could of course just choose X' to be Y, but we also want X' to resemble X quite a lot, we want those spaces to be weakly homotopy equivalent. So we need to be more subtle, this aim will be reached by using the previously named Infinite Gluing Construction that we now finally will define.

Definition 3.2.5: Let $\mathcal{F} = \{f_i : A_i \to B_i\}$ be a set of maps in $\mathcal{C}$. Let $p : X \to Y$ be the map we want to factor as a composite $X \to X' \to Y$. Now for each $i \in \mathcal{I}$ consider the set $S(i)$ of pairs of maps (g, h) that make the following diagram commute;

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B_i & \xrightarrow{h} & Y \end{array}$$

Now the *Gluing Construction*, $G^1(\mathcal{F}, p)$, is the pushout of the following diagram;

$$\begin{array}{ccc} \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} A_i & \xrightarrow{+_i+_{(g,h)}g} & X \\ {\scriptstyle \coprod f_i}\big\downarrow & & \big\downarrow{\scriptstyle i_1} \\ \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} B_i & \xrightarrow{+_i+_{(g,h)}h} & G^1(\mathcal{F}, p) \end{array}$$

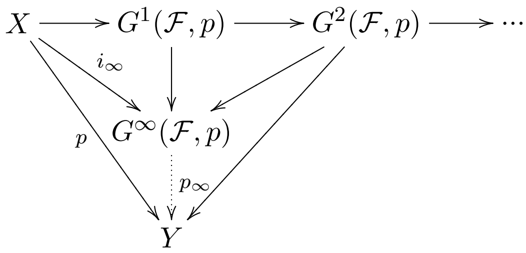
So what we are doing is gluing a copy of B_i to X along A_i for every commuting diagram as above. The map i_1 is natural, by definition of a pushout [A.27]. Now by universality of colimits [A.27] we find a map p_1 such that the following diagram commutes;

$$\begin{array}{cccc} \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} A_i & \xrightarrow{+_i+_{(g,h)}g} & X & \\ {\scriptstyle \coprod f_i}\big\downarrow & & \big\downarrow{\scriptstyle i_1} & \searrow{\scriptstyle p} \\ \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} B_i & \xrightarrow{+_i+_{(g,h)}h} & G^1(\mathcal{F}, p) & \\ & \searrow{\scriptstyle +_i+_{(g,h)}h} & \searrow{\scriptstyle p_1} & \\ & & & Y \end{array}$$

Now to get the Infinite Gluing Construction we are going to repeat the process to construct $G^k(\mathcal{F}, p)$ and maps p_k from this space to Y. We repeat the Gluing Construction but now replacing p by p_1, so let $G^2(\mathcal{F}, p) = G^1(\mathcal{F}, p_1)$, $p_2 = (p_1)_1$ and continue in this fashion. More generally $G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$ and $p_k = (p_{k-1})_1$. This results in the following commutative diagram;

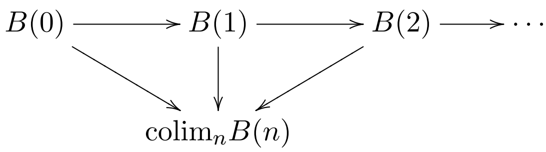
$$\begin{array}{ccccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & G^2(\mathcal{F}, p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_k} & G^k(\mathcal{F}, p) & \xrightarrow{k+1} & \cdots \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_2}\big\downarrow & & & & {\scriptstyle p_k}\big\downarrow & & \\ Y & = & Y & = & Y & = & \cdots & = & Y & = & \cdots \end{array}$$

Now define $G^\infty(\mathcal{F}, p)$, the *Infinite Gluing Construction* to be the colimit of the upper row, thus by universality of the colimit, there are natural maps $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ such that $p_\infty i_\infty = p$

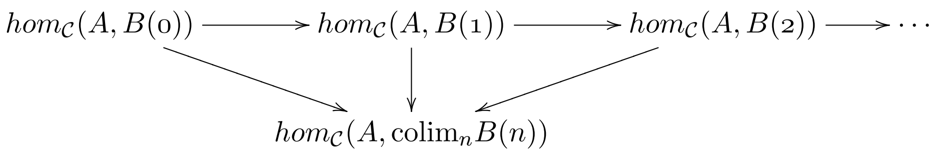


Now for p_∞ to obey the desired lifting property, we have to put an extra condition on the set of maps $\mathcal{F} = \{f_i : A_i \to B_i\}$. To do this, let us first look at the following canonical map.

Let $\mathcal{C}$ be a small category, e.g. a category that contains all small limits and colimits. For B a functor from $\{0 \to 1 \to 2 \to 3 \to ...\}$ to $\mathcal{C}$, we can take the sequential colimit [A.31], $\text{colim}_n B(n)$, this gives us a commuting diagram;



For A an object of $\mathcal{C}$ we can define the covariant hom-functor [A.12], $hom_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{sSet}$. By functorality this induces a new commutative diagram;



Now we can take the colimit of the upper row in this diagram, $\text{colim}_n hom_{\mathcal{C}}(A, B(n))$. By universality of the colimit we find a map;

$$\text{colim}_n hom_{\mathcal{C}}(A, B(n)) \to hom_{\mathcal{C}}(A, \text{colim}_n B_n)$$

Now we will use this map in the following proposition.

Proposition 3.2.6: When we are in the situation of [3.2.5] and suppose that for each $i \in \mathcal{I}$, the object A_i has the property that,

$$\text{colim}_n hom_{\mathcal{C}}(A_i, G^n(\mathcal{F}, p)) \to hom_{\mathcal{C}}(A_i, \text{colim}_n G^n(\mathcal{F}, p)) = hom_{\mathcal{C}}(A_i, G^\infty(\mathcal{F}, p))$$

is a bijection, then the map p_∞ has the RLP with respect to each of the maps in the family $\mathcal{F}$.

Proof: The lifting problem is illustrated in the following diagram for a map from $\mathcal{F}$

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^{\infty}(\mathcal{F},p) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y \end{array}$$

The bijection from $\text{colim}_n hom_{\mathcal{C}}(A_i, G^n(\mathcal{F},p)) \to hom_{\mathcal{C}}(A_i, \text{colim}_n G^n((F,p))$ gives us an integer k, for which there is a map $g' : A_i \to G^k(\mathcal{F},p)$ such that composing with the natural map $G^k(\mathcal{F},p) \to G^{\infty}(\mathcal{F},p)$ gives us g.
We can enlarge the previous commutative diagram to;

$$\begin{array}{ccccccc} A_i & \xrightarrow{g'} & G^k(\mathcal{F},p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathcal{F},p) & \longrightarrow & G^{\infty}(\mathcal{F},p) \\ \downarrow{\scriptstyle f_i} & & {\scriptstyle p_k}\downarrow & & {\scriptstyle p_{k+1}}\downarrow & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y & = & Y & = & Y \end{array}$$

Where g is the composite of the top row. The pair (g', h) is contained in the set of maps $S(i)$ in the construction of $G^{k+1}(\mathcal{F},p)$ from $G^k(\mathcal{F},p)$. So there is a map from B_i to $G^{k+1}(\mathcal{F},p)$ making the diagram commute. We can compose this map with the last map in the upper row, to get the wanted lift $\square$.

Lemma 3.2.7: Suppose that $X : \{0 \to 1 \to 2 \to 3 \to ...\} \to \textbf{Top}$ has the property that for each $n \geq 0$ the space X_n is a subspace of X_{n+1} and the pair (X_{n+1}, X_n) is a relative CW-complex. If A is a finite CW-complex then

$$\text{colim}_n hom_{\textbf{Top}}(A, X_n) \to hom_{\textbf{Top}}(A, \text{colim}_n X_n)$$

is a bijection.

Proof: The colimit of a sequence of topological spaces or of sets is a quotient of the disjoint union of them. The identifications are being made by identifying points in X_i to their image in X_{i+1}. For $\text{colim}_n(X_n)$ the maps are inclusions and we just get the union of all the spaces X_n. All inclusions are between CW-complexes, so actually the colimit is the CW-complex X with n-skeletons X_n. On the left hand side we have $\text{colim}_n hom_{\textbf{Top}}(A, X_n) = \coprod_n hom_{\textbf{Top}}(A, X_n)/\sim$.
Now we start by showing that the map, call it u is surjective. If we have a map $f : A \to \text{colim}_n(X_n)$ then since A is finite, A is compact and so $f(A)$ is compact. Now by [AH01, A.1] we find that $f(A)$ is contained in a finite skeleton of X, so for a certain k, f can be factored as $A \to X_k \to X$ where the last map is gotten while taking the colimit. For the first map we write $\overline{f}$. This map lies in the disjoint union of hom sets on the left hand side, and the first map is u. So u is surjective.

$$
\begin{array}{c}
hom_{\mathbf{Top}} \ni \overline{f} \\
\downarrow \\
\mathrm{colim}_n hom_{\mathbf{Top}}(A, X_n) \ni \overline{f} \\
\vdots\, u \\
\downarrow \\
hom_{\mathbf{Top}}(A, \mathrm{colim}_n X_n)
\end{array}
$$

Now for injectivity, we take two maps $\overline{f_i}, \overline{g_i} : A \to X_i$, they lie in the quotient of the coproduct of homsets (e.g. in the colimit). We can take them to the same set X_i cause we can always compose the one with lowest index with all the inclusions to make it a map to the same space. Now assume that they take the same value on X when composed with u. So $\forall a \in A$. Call those functions; $f_i, g_i : A \to X_i \to X$, again we see that the image of f_i and g_i lies in a finite skeleton, say X_n. But this means that the maps take the same value after inclusions and thus are the same in the quotient, that is the colimit.

Lemma 3.2.8: Let $p : X \to Y$ be a map between spaces, then the following conditions are equivalent;
(i) p is both a Serre fibration and a weak equivalence,
(ii) p has the RLP with respect to inclusion maps $A \to B$ where (B, A) is a relative CW-pair,
(iii) p has the RLP with respect to the maps $j_n : S^{n-1} \to D^n$ for $n \geq 0$

Proof: '(ii) $\Rightarrow$ (iii)': (D^n, S^{n-1}) is a relative CW-pair so this implication is obvious.
'(iii) $\Rightarrow$ (ii)': p has the RLP with respect to all $j_n : S^{n-1} \to D^n$, just as in the proof of [3.2.2] we can use induction on the skeleton of $B - A$ and lift by using the characteristic maps;

$$
\begin{array}{ccccc}
S^{n-1} & \xrightarrow{\Phi} & A & \longrightarrow & X \\
\downarrow & & \downarrow k & & \downarrow p \\
D^n & \xrightarrow{i} & B & \longrightarrow & Y
\end{array}
$$

We can use these lifts to get a lift h from B to X by taking $h(b) = ki^{-1}(b)$ since i is injective.
'(iii),(ii) $\Rightarrow$ (i)': p has the RLP with respect to inclusions of CW-pairs, in particular with respect to $(D^n \times [0,1], D^n)$ and thus is a Serre fibration by lemma [3.2.2]. To see that it induces bijections on homotopy groups we check injectivity first. A map $f : S^{n-1} \to X$ get send to $pf : S^{n-1} \to Y$, now if pf is homotopic to a constant map, we have to show f is too. But this means that pf can be extended to a map from D^n to Y so we get the following diagram;

$$
\begin{array}{ccc}
S^{n-1} & \xrightarrow{f} & X \\
\downarrow & & \downarrow p \\
D^n & \longrightarrow & Y
\end{array}
$$

There is a lift in the diagram by condition (iii), so f extends to a map from D^n and is homotopic to a constant map. Now for surjectivity we start with a map $f : S^{n-1} \to Y$ and we want to show that there is a map $\tilde{f} : S^{n-1} \to X$ such that $p\tilde{f} \sim f$. We can find $\tilde{f}$ as a lift in the following diagram;

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ S^{n-1} & \xrightarrow{f} & Y \end{array}$$

this lift exists since $(S^{n-1}, *)$ is relative CW-pair, and so by condition (ii) the map on the right has the RLP with respect to p.

'(i) $\Rightarrow$ (iii)': If we have a commutative diagram like this;

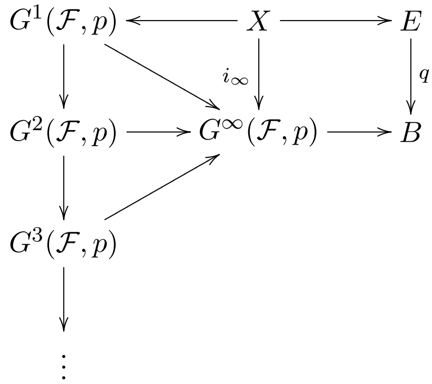
$$\begin{array}{ccc} S^{i-1} & \xrightarrow{f} & X \\ i\downarrow & & \downarrow p \\ D^i & \xrightarrow{g} & Y \end{array}$$

with p a Serre fibration and a weak homotopy equivalence, and i the inclusion. The map $pf : S^{i-1} \to Y$ is homotopic to a constant map since it extends over the disc D^i. Since p is a weak homotopy equivalence, this means that $f : S^{i-1} \to X$ is homotopic to a constant map as well, and thus extends over the disc $D^i + 1$, so p has the RLP with respect to all maps j_i.

Proof of lemma [3.2.4]: Let $\mathcal{F}$ be the set of maps $\{D^n \times \{0\} \to D^n \times [0,1]\}_{n \geq 0}$ and construct $G^1(\mathcal{F}, p)$. We obtain $G^1(\mathcal{F}, p)$ by gluing many solid cylinders $D^n \times [0,1]$ to X along one end of those cylinders. So it follows that $(G^1(\mathcal{F}, p), X)$ is a relative CW-pair and in fact i_1 is a deformation retraction since a solid cylinder is contractible, so i_1 is a homotopy equivalence. This map i_1 is a relative CW inclusion and a weak homotopy equivalence so it follows from the definition of a Serre fibration that it has the LLP with respect to all Serre fibrations.
By the same arguments the map $i_{k+1} : G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p)$ is a homotopy equivalence and has the LLP with respect to all Serre fibrations for each k. Now consider the factorization;

$$X \overset{i_\infty}{\to} G^\infty(\mathcal{F}, p) \overset{p_\infty}{\to} Y$$

obtained by the Infinite Gluing Construction. To see that i_∞ has the LLP with respect to all Serre fibrations we look at the following commuting diagram, where q is a Serre fibration;



We can find a lift from $G^1(\mathcal{F},p)$ to E, since i_1 has the LLP with respect to Serre fibrations, but then we can use this to get a lift from $G^2(\mathcal{F},p)$ to E, and we can repeat this process to get maps from all $G^n(\mathcal{F},p)$ to E, which define a natural transformation. Thus by the universal property of $G^\infty(\mathcal{F},p)$ we finally get the wanted lift, from $G^\infty(\mathcal{F},p)$ to E.
Now to show that p_∞ is a Serre fibration, we use [3.2.6]. The problem reduces to showing that;

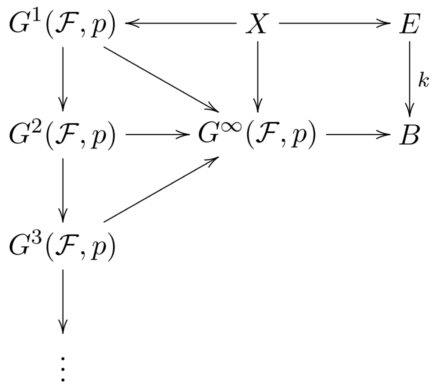
$$\text{colim}_n hom_{\mathbf{Top}}(D^i, G^n(\mathcal{F},p) \to hom_{\mathbf{Top}}(D^i, G^\infty(\mathcal{F},p))$$

is a bijection. This is the case by lemma [3.2.7].
Now the last thing we need to show is that i_∞ is a weak homotopy equivalence. We again use lemma [3.2.7], but now with S^i as our finite CW-complex. So we see that every map $S^i \to G^\infty(\mathcal{F},p)$ lands in one of the subsets $G^k(\mathcal{F},p)$ for a certain k. Now since all the maps i_k are weak equivalences, so is i_∞.

Now we are finally ready to proof **MC4** and **MC5**.

*Proof of **MC5**:* Part (ii) is an immediate consequence from the just proved lemma, we saw that we can construct a factorisation pi for every map $f \in \mathbf{Top}$ where p is a Serre fibration (thus a fibration) and i is a weak homotopy equivalence and has the LLP with respect to all Serre fibrations (and thus is an acyclic cofibration).

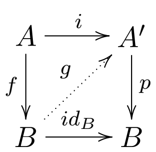
To prove (i) we use a similar construction. Let p be a map in **Top** and let $\mathcal{F} = \{j_n : S^{n-1} \to D^n\}$. Now use the Infinite Gluing Construction to find the factorization $p_\infty i_\infty$. We see that $G^{n+1}(\mathcal{F},p)$ is obtained from $G^n(\mathcal{F},p)$ by attaching n-cells along there boundary, so $(G^{n+1}(\mathcal{F},p), G^{n+1}(\mathcal{F},p))$ is a relative CW-pair. From lemma [3.2.8] we observe that the maps $i_{n+1} : G^n(\mathcal{F},p) \to G^{n+1}(\mathcal{F},p)$ have the LLP with respect to all Serre fibration that are also weak equivalences, since those have the RLP with respect to inclusion maps of relative CW-pairs. Now let k in the following diagram be a weak homotopy equivalence and a Serre fibration;



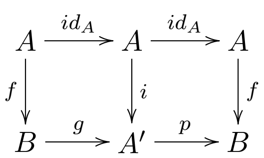
Again by induction we can find lifts from $G^k(\mathcal{F},p)$ to E for all k and by the universality of the colimit we get a lift from $G^k(\mathcal{F},p)$ to E, so i_∞ has the LLP with respect to all Serre fibrations that are also weak homotopy equivalences and thus is a cofibration. And again by lemma [3.2.7] and proposition [3.2.6] we find that p_∞ has the RLP with respect to all maps in the set $\mathcal{F}$ and thus is a Serre fibration and a weak homotopy equivalence, which is a acyclic fibration in our model category structure. □

*Proof of **MC4**:* Condition (i) is clear, by definition the cofibrations have the RLP with respect to acyclic Serre fibrations. Now for condition (ii) let $f : A \to B$ be an acyclic cofibration. We have to show that it has the LLP with respect to all fibrations. By lemma [3.2.4] we can factor f as pi, where p is a fibration and i a weak equivalence that has the LLP with respect to all fibrations. We want to show that f is a retract of i, since this lifting property is closed under taking of retracts. So if we manage to show this, f has the LLP with respect to all fibrations and we will be done.

To achieve this we look at the following diagram;



We can find a lift g that makes the diagram commute since f is a cofibration and p is an acyclic fibration, where p is a weak equivalence because i and f are. So we find the following commutative diagram;



The composition $p \circ g$ equals the identity on B by the first diagram, so indeed f is a retract of i.

So after some work we now have established a model category structure on **Top**. We want to look at the associated homotopy category, and we will prove that for maps $f : A \to X$, with A a CW-complex, we get the same homotopy relations as we normally work with in usual homotopy relations from algebraic topology [see introduction].

Proposition 3.2.9: Let $A, X \in \mathbf{Top}$, where A is a CW-complex. The set $hom_{Ho(\mathbf{Top})}(A, X)$ is in natural bijective correspondence with the set of usual homotopy classes of maps $f : A \to X$.

Proof: First we want to establish that A is cofibrant and X is cofibrant, because in that case $hom_{Ho(\mathbf{Top})}(A, X)$ is naturally isomorphic to $\pi(A, X)$. So then the proof reduces to showing that this is equal to the usual homotopy classes.
Since $(A, \emptyset)$ is a relative CW-pair, we find by [3.2.8] that the inclusion map $\emptyset \to A$ has the LLP w.r.t. weak homotopy equivalences that are also Serre fibration, thus acyclic fibrations, this means that $\emptyset \to A$ is a cofibration, so indeed A is cofibrant.
To see that X is fibrant we need to show that $X \to *$ is a Serre fibration. So we want a lift h in the following diagram;

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ A \times [0,1] & \longrightarrow & * \end{array}$$

(A a CW-complex). We can just take $h(a, t) = f(a, 0)$. This obviously makes the upper triangle commute. And since the maps to $*$ are all unique the whole diagram commutes. So indeed X is fibrant.
Now we want to show that $A \times [0, 1]$ is a good cylinder object [B.11], for A. This comes down to showing that (id_A, id_A) factors trough $A \coprod A \to A \times [0, 1] \to A$ with the first map a cofibration and the second a weak equivalence. For the first map we just include the two copies of A at the bottom and the top of the cylinder. The second map will send (a, t) to a, for every a. This indeed factors (id_A, id_A). The first map is a cofibration because it is an inclusion of $CW - complexes$ and we can use lemma [3.2.8], the second map is a homotopy equivalence, so it is definitely a weak homotopy equivalence. Now since A is cofibrant and X is fibrant we find for this fixed cylinder object that maps $f, g : A \to X$ are homotopic if and only if they are left homotopic through this cylinder object. So in our case $f \sim g$ iff there is an extension of the map $f + g : A \coprod A \to X$ like this;

$$\begin{array}{ccc} A \coprod A & \xrightarrow{f+g} & X \\ \downarrow & \overset{H}{\nearrow} & \\ A \times [0,1] & & \end{array}$$

This is exactly what it means to be homotopic in the conventional case, there is a map $H : A \times [0, 1] \to X$ such that $H(a, 0) = f$ and $H(a, 1) = g$. This is the case here, since we chose the map $A \coprod A \to A \times [0, 1]$ to be the inclusions at the bottom and top.

Another thing that we find is Whiteheads theorem. When A and X are CW-complexes, they are both fibrant and cofibrant objects. This implies that $f : A \to X$ is a weak equivalence if and only if it has a homotopy inverse. Whiteheads theorem [AH01, 4.5] tells us that a weak homotopy equivalence between CW-complexes is a homotopy equivalence, so this is the 'only if' direction.

4 The Model Category Structure on Simplicial Sets

In this section we are going to study simplicial sets. We start by giving two definitions of simplicial sets and show that they are equal. We will point out the weak equivalences and a set of generating cofibrations and acyclic cofibrations. We will see that the model structure on simplicial sets resembles the model structure on topological spaces and state the theorem which says that the categories are Quillen equivalent. For the proof of the model structure of the simplicial sets and the Quillen equivalence we refer to [Hov99] and [GJ99].

4.1 Simplicial Sets

We will give the two definitions of simplicial sets in this paragraph. The first one will be rather explicit whereas the second will be categorical and more abstract. Afterwards we will discuss some examples of simplicial sets and we end the paragraph with showing that the two definitions are equal.

Definition 4.1.1: A *simplicial set* is a graded set X, indexed on the non-negative integers, together with maps, $d_i : X_k \to X_{k-1}$ and $s_i : X_k \to X_{k+1}$ which satisfy:

(i)	$d_i d_j = d_{j-1} d_i$	$i < j$
(ii)	$s_i s_j = s_{j+1} s_i$	$i \leq j$
(iii)	$d_i s_j = s_{j-1} d_i$	$i < j$
	$d_j s_j = \mathrm{id} = d_{j+1} s_j$	
	$d_i s_j = s_{j+1} d_i$	$i > j + 1$

Where we'll call the maps, d_i, s_i boundary and degeneracy maps respectively. We will call a simplex y degenerate if there is a simplex x such that $y = s_i(x)$ for some i.

Definition 4.1.2: A *simplicial map* is a map between simplicial sets, $f : X \to Y$, that sends n-simplices of X to n-simplices of Y and commutes with the face and degeneracy maps. So we can write f as a collection of functions $f_n : X_n \to Y_n$ and the following identities hold: $f_{n-1} d_i = d_i f_n$ and $f_{n+1} s_i = s_i f_n$.

Example 4.1.3: Let K be a simplicial complex, thus a set of finite subsets of a set $\overline{K}$ with the property that every nonempty subset of a complex in K is again a complex in K. A simplicial complex already resembles a simplicial set in ways, the complex is a graded set where K_n is the set of n-simplices. A boundary map can be defined by simply removing the i-th element of a simplex. Problems arise though when we try to define the degeneracy maps, we can't be sure if we land in K again.

But we can construct a simplicial set in the following way. Let an m-simplex in $\tilde{K}$ be a set

$(a_0, ...a_m)$ of simplices in K such that the union $\bigcup_{0 \leq i \leq m} a_i$ is a n-simplex of K where $n \leq m$. Now we can define the boundary and degeneracy maps to be;

$$\begin{aligned} d_i((a_0, ..., a_m)) &= (a_0, ..., a_{i-1}, a_{i+1}, ..., a_m) \\ s_i((a_0, ..., a_m)) &= (a_0, ..., a_i, a_i, ..., a_m) \end{aligned}$$

It's easy to check that those maps satisfy the wanted identities (from def 1.1.1). So we have constructed a simplicial set $\tilde{K}$.

Let us first introduce the simplicial category;

Definition 4.1.4: Δ, the *simplicial category* is a subcategory of **Set** as well. Now the objects are subsets of the natural numbers in the following form, $[n] = (0, 1, ..., n)$, with $n \geq 0$. The morphism are all monotonic weakly increasing functions, (thus $i < j$ implies $f(i) \leq f(j)$).

Definition 4.1.5: The category of *simplicial sets* is a functor category for the functors, $X : \Delta^{op} \to \textbf{Set}$, where Δ^{op} is the opposite category of Δ [A.5],[A.7]. A *simplicial set* is an object in this category, a covariant functor $X : \mathbf{\Delta^{op}} \to \textbf{Set}$, or equally a contravariant functor $X : \Delta \to \textbf{Set}$.

Note that the morphisms in **sSet** are simplicial maps [4.1.2]. The commutative diagrams bellow illustrate this.

$$\begin{array}{c} [n] \\ \Big\downarrow{\scriptstyle \partial_i^{op}} \\ [n-1] \end{array} \qquad \begin{array}{ccc} X[n] & \xrightarrow{f_n} & Y[n] \\ \Big\downarrow{\scriptstyle d_i} & & \Big\downarrow{\scriptstyle d_i} \\ X[n-1] & \xrightarrow{f_{n-1}} & Y[n-1] \end{array} \qquad \begin{array}{c} [n] \\ \Big\downarrow{\scriptstyle \sigma_i^{op}} \\ [n+1] \end{array} \qquad \begin{array}{ccc} X[n] & \xrightarrow{f_n} & Y[n] \\ \Big\downarrow{\scriptstyle s_i} & & \Big\downarrow{\scriptstyle s_i} \\ X[n+1] & \xrightarrow{f_{n+1}} & Y[n+1] \end{array}$$

Lemma 4.1.6: The two definitions of simplicial sets in 3.1.2 and 3.1.6 are equivalent.

Proof: A graded set is a union of sets indexed on the integers

$$X = \cup_{n \in \mathbb{N}} X_n$$

In the case of definition 3.1.2. we the set is indexed by the nonnegative integers. So X assigns a set to each nonnegative integer. This is exactly what the functor in definition 3.1.6. does to the objects in Δ.

Lemma: all the morphisms in Δ are generated by morphisms of the following two types:

$$\begin{aligned} \partial_i \quad &: \quad [n-1] \to [n] \\ &(0, 1, ..., n-1) \mapsto (0, 1, ..., i-1, i+1, ..., n) \end{aligned}$$

$$\begin{aligned} \sigma_i \quad &: \quad [n+1] \to [n] \\ &(0, 1, ..., n+1) \mapsto (0, 1, ..., i, i, ..., n) \end{aligned}$$

Proof: Recall that the morphisms in Δ are the monotonic weakly increasing functions $f : [n] \to [m]$ in Δ. The function f is determined by its image, a subset of $[m]$ and by the subset of $[n]$ of the points where f does not increase. Let $i_1...i_s$ be the points in $[m]$ that are not in the image of f, and let $j_1, ..., j_t$ be the points in $[n]$ with the property $f(j_i) = f(j_{i+1})$. Now;

$$\partial_{i_s} \circ ... \circ \partial_{i_1} \circ \sigma_{j_1} \circ ... \circ \sigma_{j_t}$$

It's easy to check that the maps above obey the following identities:

(i)	$\partial_j \partial_i = \partial_i \partial_{j-1}$	$i < j$
(ii)	$\sigma_j \sigma_j = \sigma_i \sigma_{j+1}$	$i \leq j$
(iii)	$\sigma_j \partial_i = \partial_i \sigma_{j-1}$	$i < j$
	$\sigma_j \partial_j = \text{id} = \sigma_j \partial_{j+1}$	
	$\sigma_j \partial_i = \partial_i \sigma_{j+1}$	$i > j+1$

Those conditions are dual to the ones in definition 3.1.2, and are sometimes called the *cosimplicial identities.* Now we check the first identity of the third conditions. The checking of the others is similar and left to the reader.

$$\begin{array}{ccc} (0, ..., n) & \xrightarrow{d_i} & (0, ..., i-1, i+1, ..., n+1) \\ \downarrow{\scriptstyle s_{j-1}} & & \downarrow{\scriptstyle s_j} \\ (0, ..., i-1, i+1, ..., j-1, j-1, ..., n) & \xrightarrow[d_i]{} & (0, ..., j-1, j-1, ..., n-1) \end{array}$$

So when we start out with a functor, the morphisms in **Set** that are the images of ∂_i and σ_i are the boundary and degeneracy maps. And if we start out with a graded set we can define the functor on those generating maps only, and send them to our face and degeneracy maps.

Example 4.1.7: The *standard n-simplex*, Δ^n is defined by;

$$\Delta^n = hom_{(\Delta, -)}[n]$$

Where $hom.(-,[n])$ is the contravariant hom-functor from example [A.10] This is a contravariant functor from Δ to **Set** indeed.

Example 4.1.8: Another important example of a simplicial set is $\partial\Delta^n$, *boundary* of Δ^n. The functor $\partial\Delta^n$ takes $[k]$ to the set of non identity injective order preserving maps $[k] \to [n]$ in Δ.

Example 4.1.9: For $0 \leq r \leq n$ we define the *r-horn* of Δ^n. This functor Λ_r^n sends $[k]$ to the set of all non identity injective order preserving maps $[k] \to [n]$ except the map $[n-1] \xrightarrow{d^r} [n]$ whose image does not contain r.

We can interpreter the above three examples geometrically, strictly speaking by taking the geometric realization, we will define the geometric realization functor in the next paragraph. But we can already think about for example the standard 3-simplex as a solid tetraheder, the boundary as an empty tetraheder and the r-horn as an empty tetraheder missing the interior of its r^{th} face.

Example 4.1.10: A final example of a simplicial set is the nerve of a small category $\mathcal{N}(\mathcal{C})$. Again we'll give two definitions, one more explicit and one more formal.

(i) Let the zero-simplices be the objects in $\mathcal{C}$, $C_0 = Obj(\mathcal{C})$

and $\qquad C_1 = Hom(\mathcal{C})$

$\qquad C_2 = C_1 \times_{C_0} C_1$

and similarly $\qquad C_n = C_1 \times_{C_0} ... \times_{C_0} C_1$, ($n$ times)

where $C_1 \times_{C_0} C_1$ is the pullback in the following diagram;

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow t \\ C_1 & \xrightarrow{s} & C_0 \end{array}$$

So the C_2 is the set of pairs of composable arrows in $\mathcal{C}$, since $C_2 = \{(f_1, f_2) \in C_1 \times C_1 | s(f_1) = t(f_2)\}$. Now C_n is the iterated pull-back. This is well defined since composition of morphism is associative.

Since $\mathcal{C}$ is a small category we got a graded set now. We just have to define boundary and degeneracy maps. Those are:

$$\begin{aligned}
\partial_i \quad : \quad & C_n \to C_{n-1} \\
& (f_1, ..., f_n) \mapsto \begin{cases} (f_2, ..., f_n), & \text{if } i = 1 \\ (f_1, ..., f_i \circ f_{i+1}, ..., f_n), & \text{if } 1 < i < n \quad \sigma_i \\ (f_1, ..., f_{n-1}), & \text{if } i = n \end{cases} \qquad :
\end{aligned}$$

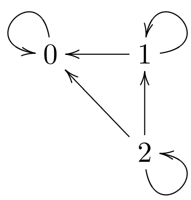
$$\begin{aligned}
C_n \to C_{n+1} & \\
& (f_1, ..., f_n) \mapsto \begin{cases} (id_{t(f_1)}, f_1, ..., f_n) & \text{if } i = 1 \\ (f_1, ..., f_i, id_{s(f_i)}, ..., f_n) & \text{if } i > 1 \end{cases}
\end{aligned}$$

again it is an easy exercise, although quite long, to check that the maps satisfy the equations from definition 4.1.1.

(ii) The second definition is more formal but gives a nice visualization of the construction. We need to define another functor $H : \Delta \to \mathbf{Cat}$ first.

$$H : [n] \to H[n]$$

$H[n]$ is the category with objects the elements from $[n]$ so the numbers 0 till n and a morphism $a \to b$ whenever $a \geq b$. Below we visualize $H[2]$



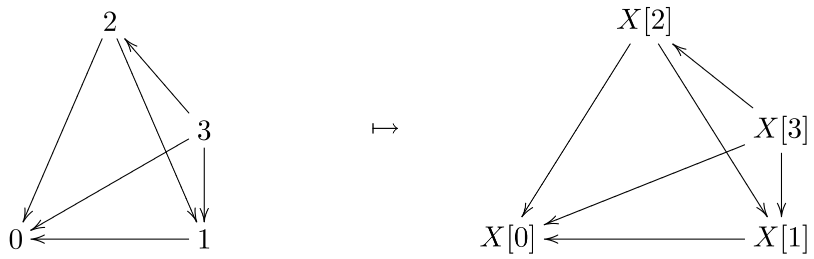
Furthermore H maps the nondecreasing functions of Δ to functors in $\mathbf{Cat}$ in the following way:

$$\begin{aligned}
[m] \xrightarrow{\alpha} [n] \mapsto A \quad : \quad & H[m] \to H[n] \\
& a \mapsto \alpha(a) \\
& a \to b \mapsto \alpha(a) \to \alpha(b)
\end{aligned}$$

This is well defined since we only have an arrow from a to b when $a \geq b$ and this implies that $\alpha(a) \geq \alpha(b)$ so there is an arrow in $H[n]$ from $\alpha(a)$ to $\alpha(b)$. Now the nerve of $\mathcal{C}$, is a functor;

$$\begin{aligned}
\mathcal{N}(\mathcal{C}) : \Delta^{\text{op}} \quad & \to \quad \mathbf{Set} \\
[n] \quad & \mapsto \quad \{f : H[n] \to X(\text{in } \mathcal{C})\} \\
[n] \xrightarrow{\alpha^*} [m] \quad & \mapsto \quad f \circ \alpha_*, \ \alpha_* = H(\alpha)
\end{aligned}$$

So it sends the objects of Δ^{op} to functors in the category of categories. Below we see a drawing of such a functor (for [3]):



So here we get a serie of composable morphisms, plus some extra morphisms which might or might not be the composition of the other ones.

4.2 Model Category Structure on Simplicial Sets

In this paragraph we will define a model category structure on the category of simplicial sets. First of all we will discuss a cofibrantly generated model structure in some more detail. Afterwards we will define the geometric realization functor $|\,\bullet\,| : \mathbf{sSet} \to \mathbf{Top}$ and what it means for a map in **sSet** to be a Kan fibration. Then we have defined the right terms to pick the weak equivalences and sets of generating cofibrations and acyclic cofibrations. Finally we will define the singular functor and proof that this functor is adjoint to the geometric realization functor, a modest start to proving the Quillen equivalence between **Top** and **sSet**.

So we start out by giving a tool for checking that a model category is cofibrantly generated. We will use some new terms to do this, which we will explain roughly afterwards. For a detailed discussion we refer to [Hov99].

Proposition 4.2.1: Let $\mathcal{C}$ be a category with all finite limits and colimits. Suppose $\mathcal{W}$ is a subcategory and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model category structure on $\mathcal{C}$ as described above if and only if the following conditions are satisfied.

(i) The subcategory $\mathcal{W}$ has the two out of three property and is closed under retracts.

(ii) The domains of I are small relative to I-cell.

(iii) The domains of J are small relative to J-cell.

(iv) J-cell $\subset \mathcal{W} \cup I$-cof.

(v) I-inj $\subset \mathcal{W} \cup J$-inj.

(vi) Either $\mathcal{W} \cup I$-cof $\subset J$-cell or $\mathcal{W} \cup J$-inj $\subset I$-inj.

In the definition we use I-cell, this is a class of maps that are called relative I-cell complexes. A relative I-cell complex is a generalization of a relative CW-complex in several ways. For a relative CW-complex (X, A) we are attaching disks along there boundary to A in order of their dimension. That is, to construct the 'n-skeleton' of (X, A), we take the pushout in the following diagram;

$$\begin{array}{ccc} \coprod_\alpha S^{n-1}_\alpha & \longrightarrow & A \cup (X - A)_{n-1} \\ {\scriptstyle +\phi_\alpha}\downarrow & & \downarrow \\ \coprod_\alpha D^n_\alpha & \longrightarrow & X_n \end{array}$$

For an I-cell we take an arbitrary collection of maps, I, instead of just $S^{n-1} \to D^n$ and there is no ordering involved. Besides that Hovey allows to take transfinite compositions while with CW-complexes the compositions are countable. So I-cell is a collection of relative I-cells, a map from an object X to an object constructed from X by taking pushouts with maps from I.

For an object A in $\mathcal{C}$ to be small relative to I-cell, means that for a transfinite sequence and some ordinal λ;

$$X_0 \to X_1 \to X_2 \to ...$$

such that the map $X_\beta \to X_{\beta+1} \in I$-cell if $\beta + 1 < \lambda$, the following map [3.2.7];

$$\text{colim}_{\beta<\lambda}\mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \text{colim}_{\beta<\lambda}X_\beta)$$

is an isomorphism.

The other new terms in our definition are easier to grasp, I-cof stands for the collection of maps that have the LLP with respect to all maps that have the RLP with respect to all maps in I. So this will be our collection of cofibrations. I-inj is the collection of maps that have the RLP with respect to all maps in I, or our acyclic cofibrations. And similar J-inj will be our fibrations.

The first property in the proposition is needed for axiom two and three of a model category. The second and third property are being used in the small object argument to construct functorial factorizations. The argument is similar to the one we used with topological spaces. Then the fourth property implies that the maps in $J - cell$ are acyclic cofibrations. In the case of topological spaces this means that the relative CW-inclusions that are weak homotopy equivalences have the RLP with respect to the Serre fibrations. The fifth property implies that the acyclic fibrations are weak equivalences and fibrations. The last property gives the converse statement to the fourth or fifth property, so the last three properties are needed for axiom four.

Now before picking our weak equivalences and generating (acyclic) cofibrations we need the definition of the geometric realization functor.

Definition 4.2.2: The *geometric realization* $|X|$ of a simplicial set X is the topological space constructed in the following way. Give X the discrete topology and let $\overline{X} = \bigcup_{n \geq 0} X_n \times |\Delta^n|$, where $|\Delta^n|$ is the standard topological n-simplex;

$$|\Delta^n| = \Big\{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | t_i \geq 0, \sum_{i=0}^{n} t_i = 1\Big\}$$

Now define an equivalence relation by:
$(d_i x_n, u_{n-1}) \sim (x_n, \partial_i u_n - 1)$ for $x_n \in X_n$ and $u_{n-1} \in |\Delta^{n-1}|$
$(s_i x_n, u_{n+1}) \sim (x_n, \sigma_i u_{n+1})$ for $x_n \in X_n$ and $u_{n+1} \in |\Delta^{n+1}|$
Now $|X| = \overline{X}/\sim$.

We can make the geometric realization into a functor; $|.| : \mathbf{sSet} \to \mathbf{Top}$. We let the simplicial sets $X \in \mathbf{sSet}$ map to the just defined geometric realization $|X|$ and the simplicial maps $f : X \to Y$ get mapped to

$$\begin{aligned} |f| \quad : \quad & |X| \to |Y| \\ & |x_n, u_n| \mapsto |f(x_n), u_n| \end{aligned}$$

Since we gave $\overline{X}$ and $\overline{Y}$ the discrete topology, this is a continuous map. Furthermore $|i_X|\big(|x_n, u_n|\big) = |x_n, u_n| = i_{|X|}\big(|x_n, u_n|\big)$. And when $f : X \to Y, g : Y \to Z$, we find that;

$$|f \circ g|\big(|x_n, u_n|\big) = |f \circ g(x_n), u_n| = |f|\big(|g|(x_n), u_n\big) = |f| \circ |g|\big(x_n, u_n\big)$$

So $|.|$ indeed is a functor.

Theorem 4.2.3: Let **sSet** be the category of simplicial sets, define I to consist of the inclusions $\partial\Delta^n \to \Delta^n$ for $n \geq 0$. Define J to consist of the inclusions $\Lambda^n_r \to \Delta^n$ for $n > 0$ and $0 \leq r \leq n$. And let $f \in \mathcal{W}$ if and only if $|f|$ is a weak equivalence in **Top**. These definitions give **sSet** a cofibrantly generated model category structure.

The fibrations of **sSet** are called the *Kan fibrations*, the maps that have the LLP with respect to the inclusions $\Lambda^n_r \to \Delta^n$. The acyclic cofibrations, the maps that have the RLP with respect to all Kan fibrations are called the *anodyne extensions.* In chapter 3 of [Hov99] the above theorem is proved. We will only point out some resemblances to topological spaces and leave the proof.

As mentioned in the previous part of this section we can view the 3-simplex as a solid tetraheder. This is homotopic to a solid cylinder. The boundary of the tetraheder is homotopic to the empty cylinder and the r-horn is homotopic to the empty cylinder without a top. This is again homotopic to the 2-sphere. So the Kan fibrations are similar to the Serre fibrations since they have the LLP w.r.t. $\Lambda^n_r \to \Delta^n$ which in a topological setting is equivalent to having the RLP w.r.t. $S^{n-1} \to D^n$ which is the property of a Serre fibration. The strict setting of simplicial sets makes the proofs a lot more difficult though.
Recall the following theorem from section 2.3.;
Theorem 2.3.2: Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and;

$$F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$$

be a pair of adjoint functors. Suppose that;
(i) F preserves cofibrations and G preserves fibrations, then the total derived functors exist and form an adjoint pair; $\mathbf{L}F : Ho(\mathcal{C}) \Longleftrightarrow Ho(\mathcal{D}) : \mathbf{R}G$
(ii) If in addition we have that the following condition holds for each cofibrant object $A \in \mathcal{C}$ and fibrant object $X \in \mathcal{D}$ and a map $f : A \to G(X)$; f is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\flat : F(A) \to X$ is a weak equivalence in $\mathcal{D}$. Then $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ gives a Quillen equivalence.

Definition 4.2.4: The *singular complex functor* is the functor **Sing** : **Top** $\to$ **sSet** that sends a topological space A to the graded set;

$$\mathbf{Sing}(A) = \big\{ f : |\Delta^n| \to A \mid f \text{continuous}, n \in \mathbb{N}^+ \big\}.$$

So $\mathbf{Sing}_n(A)$ are all the continuous maps from $|\Delta^n|$ to A.
The functor sends $g : A \to B$ to;

$$\begin{aligned} \mathbf{Sing}(g) \quad : \quad & \mathbf{Sing}(A) \to \mathbf{Sing}(B) \\ & \big(f : |\Delta^n| \to A\big) \mapsto \big(g \circ f : |\Delta^n| \to A \to B\big) \end{aligned}$$

So indeed $\mathbf{Sing}(i_A) = i_{\mathbf{Sing}(A)}$ and $\mathbf{Sing}(h \circ g) = \mathbf{Sing}(h) \circ \mathbf{Sing}(g)$. Thus **Sing** indeed is a (covariant) functor.

Proposition 4.2.5: The geometric realization and singular complex functor are adjoint functors [A.11]

Proof: We will construct functions $\phi : Hom_{\mathbf{sSet}}(X, \mathbf{Sing}(A)) \to Hom_{\mathbf{Top}}(|X|, A)$, and $\psi : Hom_{\mathbf{Top}}(|X|, A) \to Hom_{\mathbf{sSet}}(X, \mathbf{Sing}(A))$ to get the natural bijection on hom sets. Let;

$$\phi(f)|x_n, u_n| = f(x_n)(u_n)$$

So $f : X \to \mathbf{Sing}(A)$ gets send to $\phi(f) : |X| \to A$ which send $|x_n, u_n|$ to the value of $f(x_n) : |\Delta^n| \to A$ at u_n.

$$\psi(g)(x_n)(u_n) = g|x_n, u_n|$$

Here $g : |X| \to A$ gets send to $\psi(g) : X \to \mathbf{Sing}(A)$ which sends $x_n \in X$ to the function $\psi(g)(x_n) : |\Delta^n| \to A$ and this function $\psi(g)(x_n)$ sends u_n to $g|x_n, u_n|$.

To check that those function indeed give an adjunction, we have to check that they give rise to a bijection, this is easy;

$$\begin{aligned} \phi \circ \psi\big((g)(x_n)(u_n)\big) &= \phi(g)|x_n, u_n| = g(x_n)(u_n) \\ \psi \circ \phi\big((f)|x_n, u_n|\big) &= \psi(f)(x_n)(u_n) = f|x_n, u_n| \end{aligned}$$

Now we just want to show that ϕ is natural in X and A. So we have to check that for $h : Y \to X$ and $k : A \to B$ the following diagrams commute:

$$\begin{array}{ccc} Hom_{\mathbf{sSet}}(X, \mathbf{Sing}(A)) & \xrightarrow{\phi} & Hom_{\mathbf{Top}}(|X|, A) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle |h|^*} \\ Hom_{\mathbf{sSet}}(Y, \mathbf{Sing}(A)) & \xrightarrow{\phi} & Hom_{\mathbf{Top}}(|Y|, A) \end{array}$$

$$\begin{array}{ccc} Hom_{\mathbf{sSet}}(X, \mathbf{Sing}(A)) & \xrightarrow{\phi} & Hom_{\mathbf{Top}}(|X|, A) \\ \downarrow{\scriptstyle \mathbf{Sing}(k)_*} & & \downarrow{\scriptstyle k_*} \\ Hom_{\mathbf{sSet}}(X, \mathbf{Sing}(B)) & \xrightarrow{\phi} & Hom_{\mathbf{Top}}(|X|, B) \end{array}$$

Now for the first diagram, let us walk down and then right first, we find $f \mapsto f \circ h \mapsto \phi(f \circ h)$. Now when we walk right first and then down, we find $f \mapsto \phi(f) \mapsto \phi(f) \circ |h|$. Now $\phi(f) \circ |h||y_n, v_n| = \phi(f)|h(y_n), v_n| = (f \circ h)(y_n)(u_n) = \phi(f \circ h)|y_n, u_n|$. So the first diagram does indeed commute.

For the second diagram we will walk down and then right to get, $f \mapsto \mathbf{Sing}(k) \circ f \mapsto$

$\phi(\mathbf{Sing}(k) \circ f)$, since $\mathbf{Sing}(k)$ is just composition with k we find $f \mapsto k \circ f(k_n) : |\Delta^n| \to A \to B$. And when we now walk first right and the down we get $f \mapsto \phi(f) \mapsto k \circ \phi(f)$, thus the diagram commutes. So indeed $|.| : \mathbf{sSet} \Leftrightarrow \mathbf{Top} : \mathbf{Sing}$

Theorem 4.2.6: The adjoint pair of functors $|.| : \mathbf{sSet} \Leftrightarrow \mathbf{Top} : \mathbf{Sing}$ gives a Quillen equivalence between the category of simplicial sets and topological spaces.

The proof of this theorem can be found in [GJ99]. This implies that the homotopy categories of **sSet** and **Top** are equivalent. The theorem shows that the the category of simplicial sets is a good category of combinatoric models for the study of ordinary homotopy theory.

5 Popular Summary

The project that I have written is within the field of 'algebraic topology'. This might sound like Chinese to you at the moment, so I will try to explain what we do in this field. We are studying so called 'topological' spaces, this can for example be just a plane or a sphere. More complicated examples involve the Mobiüs band or even the Klein bottle. (see pictures)

We study those spaces by assigning algebraic structures to them, this can be groups or more complicated structures. In algebraic topology we are often interested in spaces that have the same shape in a rather broad sense. Most people will disagree completely on what we call spaces with the same shape, or formally homotopic spaces. For example we will say that a scissor has the same shape as the number eight, a towel has the same shape as a plate and a soccer ball is similar to a closed bottle. As you can see we do not really care about the shape, we actually care about the holes in our space, in the first example our objects have two holes, in the second example there are no holes and in the last there is just one hole. We can pretend that our objects are made out of very flexible rubber. We can stretch and shrink all parts as much as we like and as long as we do not rip the material we will call the objects homotopic. We sometimes call those space homotopy equivalent too.

In algebraic topology there is a strict definition for homotopy equivalent spaces. If we call our two spaces A and B, so for example A is our scissor and B the number eight. Then to show that the spaces are homotopic we will need a map from A to B and a map from B to A, we can call these f and g relatively. The definition does not require those maps to be each others inverse, so for a point a in the scissor the point $g(f(a))$ may be another point of the scissor.

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

But the maps $gf : A \to A$ and $fg : B \to B$ should be similar to the identity map. We call such maps homotopic to the identity map.

In other fields of mathematics we sometimes want to do the same thing. We want to have an inverse map that is not really an inverse, but that is an inverse up to homotopy. That is when you compose the maps you get a map that is similar to the identity map. In my project I studied techniques that make a generalization of these different definitions of 'homotopy' possible. The tools needed to do this are category theory and model categories. They allow us to talk about homotopy theories in several different settings. But the main example that

I discussed is still the example of topological spaces. And after all the abstractness it again turned out that the towel and the plate are homotopic.

A Categories

In this appendix we will give a short introduction to category theory with some elementary definitions that are used in the article. For a more elaborate introduction, see for example [ML71].

Definition A.1: A category, $\mathcal{C}$ is a collection of objects, $Obj(\mathcal{C})$, together with a set of morphisms, $hom_{\mathcal{C}}(X,Y)$, for each two objects X, Y in C. We denote the collection of all these morphisms by $Hom(\mathcal{C})$. The category has four structure maps:
The target and source maps

$$Hom(\mathcal{C}) \stackrel{t,s}{\to} Obj(\mathcal{C})$$

which assign a source $s(f)$ and target $t(f)$ to each morphism f in $\mathcal{C}$.
Now let $Hom(\mathcal{C}) \times_{Obj(\mathcal{C})} Hom(\mathcal{C}) = \{(f,g)|t(f) = s(g)\}$ be the set of composable morphisms in $\mathcal{C}$. The category has a composition map,

$$\begin{aligned} \circ \quad : \quad & Hom(\mathcal{C}) \times_{Obj(\mathcal{C})} Hom(\mathcal{C}) \to Hom(\mathcal{C}) \\ & (f,g) \mapsto g \circ f \end{aligned}$$

further, there is an identity morphism id_X for each object $X \in \mathcal{C}$, such that $f \circ id_X = f$ and $id_X \circ g = g$ for any two morphisms f and g with $s(f) = X$ and $t(g) = X$. This gives rise to an identity map:

$$\begin{aligned} id \quad : \quad & Obj(\mathcal{C}) \to Hom(\mathcal{C}) \\ & X \mapsto id_X \end{aligned}$$

Those maps obey the following identities

$$\begin{aligned} s(i_X) = X &= t(id_X) \\ s(g \circ f) &= s(f) \\ t(g \circ f) &= t(g) \end{aligned}$$

for all objects and composable morphisms in $\mathcal{C}$. Furthermore, the maps are subject to the following axiom:
Associativity: For pairs of composable morphisms (f,g) and (g,h) the following equality holds

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Notation: we write

$$X \in \mathcal{C} \qquad f \text{ in } \mathcal{C}$$

where X is an object of $\mathcal{C}$ and f is a morphism in $\mathcal{C}$. And

$$hom_{\mathcal{C}}(X, Y)$$

for the set of morphisms with source X and target Y. We sometimes call the morphism of a category the arrows or maps of the category.

There are many examples of categories, however we will focus on the categories of topological spaces and simplicial sets. To define the category of simplicial sets we will need the simplicial category and the category of sets. So this we post phone to later.

Example A.2: **Set** is a category where the objects are sets and the morphism are all functions between sets. Composition is just the usual function composition.

Definition A.3: A *subcategory* $\mathcal{S}$, of a category $\mathcal{C}$, is a category with objects and morphisms in $\mathcal{C}$. A *full subcategory* of $\mathcal{C}$ has objects in $\mathcal{C}$ and all morphisms between those objects as its morphisms.

Example A.4: **Top**, the category of topological spaces is a subcategory of the category **Set**. The objects are now sets with a topology on them, or topological spaces. And the morphisms become the continuous maps between them.

Example A.5: Δ, the simplicial category is a subcategory of **Set** as well. Now the objects are subsets of the natural numbers in the following form, $[n] = (0, 1, ..., n)$, with $n \geq 0$. The morphism are all monotonic weakly increasing functions, (thus $i < j$ implies $f(i) \leq f(j)$).

Example A.6: **Grp**, the category of groups is a third subcategory of **Set**, with objects the sets with group structure and morphism the homomorphisms between them.

Definition A.7: The *opposite category*, $\mathcal{C}^{op}$ of a category $\mathcal{C}$ has the same objects as C and for each morphism $X \to Y$ in $\mathcal{C}$ it has a morphism $Y \to X$. The morphisms compose according to the formula $f^{op}g^{op} = (gf)^{op}$

Definition A.8: An *initial object* of a category $\mathcal{C}$ is if it exists an object $\emptyset$ for which there is for every object $X \in \mathcal{C}$ an unique morphism $\emptyset \to X$. An *terminal object* $*$ is an object of C such that for every object X in $\mathcal{C}$ there is an unique morphism $X \to *$.

Example A.9: The category of topological spaces has both an initial and a terminal object, the initial object being the empty set, and all one-point sets being terminal objects.

Definition A.10: A *covariant functor*, $F : \mathcal{C} \to \mathcal{D}$ is a map between categories. It maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$ and morphisms $f : X \to Y$ in $\mathcal{C}$ to morphism $F(f) : F(X) \to F(Y)$

in $\mathcal{D}$, such that;

$$F(id_X) = id_{F(X)}$$
$$F(g \circ f) = F(g) \circ F(f)$$

A *contravariant functor* is a map between categories that reverses morphisms, it again sends objects to objects but it sends a morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$ such that;

$$F(id_X) = id_{F(X)}$$
$$F(g \circ f) = F(f) \circ F(g)$$

Example A.11: For the readers familiar with algebraic topology, the fundamental group is a functor from the category of base pointed topological spaces (that is a subcategory of **Top**, with objects topological spaces with a distinct base point) to the category **Grp**.

$$\begin{aligned} \pi_1 \quad : \quad & \mathbf{Top}_* \to \mathbf{Grp} \\ & (X, x) \to \pi_1(X, x) \\ & f \to f_* \end{aligned}$$

The notion of functors has become really common in modern mathematics. They were first explicitly recognized in algebraic topology. The founders of category theory, Eilenberg and Mac Lane, actually introduced categories just with the purpose of introducing functors and natural transformations [A.13] of functors. It is not surprising that functors show up in algebraic topology, in this field we want to associate algebraic structures to topological spaces, mostly groups, but also more complicated structures like for example rings or chain complexes. Functors are very convenient tools to do this, because they save a lot of the structure, since they do not just map the objects but also the maps between those. As we just realized in the above example, the fundamental group is a functor from **Top** to **Grp**. There are plenty other examples to be found in algebraic topology, some of the functors that come up while construction the homology of a space can be found in [AH01][pg. 164].

Example A.12: Let $\mathcal{C}$ be a *small* category, e.g. $hom_{\mathcal{C}}(X, Y)$ is a set for all objects in C. We can form two hom-functors, $hom_{\mathcal{C}}(X, -), hom_{\mathcal{C}}(-, Y) : \mathcal{C} \to \mathbf{Set}$, by the following means;

$$\begin{aligned} hom_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set} & \\ A \mapsto hom_{\mathcal{C}}(X, A) & \end{aligned}$$

$$A \xrightarrow{f} B \mapsto hom_{\mathcal{C}}(X, A) \to hom_{\mathcal{C}}(X, B)$$
$$X \xrightarrow{g} A \mapsto X \xrightarrow{f \circ g} B$$

$$hom_{\mathcal{C}}(-, Y) : \mathcal{C} \to \textbf{Set}$$
$$A \mapsto hom_{\mathcal{C}}(A, Y)$$
$$A \xrightarrow{f} B \mapsto hom_{\mathcal{C}}(B, Y) \to hom_{\mathcal{C}}(A, Y)$$
$$B \xrightarrow{g} Y \mapsto A \xrightarrow{g \circ f} Y$$

Where the first functor is an example of a covariant functor while the second is contravariant.

Definition A.13: Let $F, G : \mathcal{C} \to \mathcal{D}$ be two covariant functors, and let $\tau : F \to G$ be a function that assigns to each $X \in \mathcal{C}$ an function $\tau_X : FX \to GX$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes;

$$\begin{array}{ccccc} X & & FX & \xrightarrow{\tau_X} & GX \\ {\scriptstyle f}\downarrow & & {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Y & & FY & \xrightarrow{\tau_Y} & GY \end{array}$$

the function τ is called a *natural transformation* from F to G. A natural transformation is called a *natural equivalence* if for every $X \in \mathcal{C}$ the function τ_X is an isomorphism in $\mathcal{D}$.

Definition A.14: An *equivalence of categories* is given by a pair of functors, $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, together with natural equivalences between GF and $id_{\mathcal{C}}$ and between FG and $id_{\mathcal{D}}$.

If two categories are equivalent this means that they are very similar, the gain of this is a better understanding of both categories and translation of theorems from the one to the other.

Definition A.15: Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be covariant functors. Those functors together with a function ϕ give an *adjunction* from $\mathcal{C}$ to $\mathcal{D}$, where ϕ is a function that assigns to each pair of objects, $X \in \mathcal{C}, Y \in \mathcal{D}$ a bijection;

$$\phi = \phi_{X,Y} : hom_{\mathcal{D}}(FX, Y) \cong hom_{\mathcal{C}}(X, GY)$$

which is natural in X and Y. Here naturality in X and Y means that the for all $f : X' \to X$ and $g : Y \to Y'$, the following diagrams commute;

$$\begin{array}{ccc} hom_{\mathcal{D}}(FX, Y) & \xrightarrow{\phi} & hom_{\mathcal{C}}(X, GY) \\ {\scriptstyle (Ff)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ hom_{\mathcal{D}}(FX', Y) & \xrightarrow{\phi} & hom_{\mathcal{C}}(X', GY) \end{array} \qquad \begin{array}{ccc} hom_{\mathcal{D}}(FX, Y) & \xrightarrow{\phi} & hom_{\mathcal{C}}(X, GY) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (Gg)_*} \\ hom_{\mathcal{D}}(FX, Y') & \xrightarrow{\phi} & hom_{\mathcal{C}}(X, GY') \end{array}$$

Where f^* and g_* are the obvious compositions.

We just saw some examples of duality, statements where we have 'reversed all arrows'. A contravariant functor is the dual of a covariant functor. We often find such dual definitions or objects in mathematics. Other examples are the projection map and the injection map or a surjective and an injective map. In category theory there is an exact description of duality. We won't go into much detail here, but the main reason that we mention it is because we can use duality later on. It is namely the case that for any proof of a theorem, the dual proof is valid as well (for the dual statement). This is because for each axiom for a category, the dual statement is also an axiom. We obtain dual statements by reversing all arrows and composites. Here are some examples of statements and there duals;

statement	**dual statement**
$f : X \to Y$	$f : Y \to X$
$s(f) = X$	$t(f) = X$
u is a right inverse of h	u is a left inverse of h
id_X	id_X

Now we will continue by defining the functor category.

Definition A.16: Let $\mathcal{B}$ and $\mathcal{C}$ be categories. We may construct the *functor category*, $\mathcal{B}^{\mathcal{C}}$ with objects the functors $\mathcal{C} \to \mathcal{B}$ and morphisms the natural transformations between them. We have to check that the natural transformations contain an identity map, are closed under composition and associative. Let $R, S, T : \mathcal{C} \to \mathcal{B}$ be three functors, with $\sigma : R \to S$, $\tau : S \to T$ natural transformations [A.13]. So we have the following two commutative diagrams for each morphism $f : X \to Y$:

$$\begin{array}{ccc} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad \begin{array}{ccc} RX & \xrightarrow{\sigma_X} & SX \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Sf} \\ RY & \xrightarrow{\sigma_Y} & SY \end{array} \qquad \begin{array}{ccc} SX & \xrightarrow{\tau_X} & TX \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SY & \xrightarrow{\tau_Y} & TY \end{array}$$

This gives rise to a big commutative diagram:

$$\begin{array}{ccc} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad \begin{array}{ccccc} RX & \xrightarrow{\sigma_X} & SX & \xrightarrow{\tau_X} & TX \\ {\scriptstyle Rf}\downarrow & & {\scriptstyle Sf}\downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle Tf} \\ RY & \xrightarrow{\sigma_Y} & SY & \xrightarrow{\tau_Y} & TY \end{array}$$

So we see that we can compose the components of the natural transformation, to get a new natural transformation $\tau \cdot \sigma : R \to T$ with $(\tau \cdot \sigma)_X = \tau_X \circ \sigma_X$. Since the big diagram commutes we see that composition is associative. And the identity transformation is just the component wise identity $id_T(X) = id_{TX}$.

Example A.17: The category of *simplicial sets* is a functor category for the functors, $X : \Delta^{op} \to \mathbf{Set}$, where Δ^{op} is the opposite category of Δ [A.5],[A.7].

Example A.18: The functor category is sometimes called the category of *diagrams in $\mathcal{B}$ with the shape of $\mathcal{C}$*. Let for example $\mathcal{C}$ be $\{X \to Y\}$, the functors $R : \mathcal{C} \to \mathcal{B}$ are the morphisms $f : RX \to RY$ in $\mathcal{B}$, so the objects in $\mathcal{B}^{\mathcal{C}}$ are those morphisms f. And the morphisms $\mathcal{B}^{\mathcal{C}}$ are the natural transformations $\tau : f \to g$ given by the commutative diagrams;

$$\begin{array}{ccc} RX & \xrightarrow{\tau_X} & SX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ RY & \xrightarrow{\tau_Y} & SY \end{array}$$

This special case of a functor category is called the *category of morphisms* and is denoted $\mathbf{Mor}(\mathcal{C})$.

Definition A.19: An object X in a category $\mathcal{C}$ is a *retract* of an object A if there exists morphisms $i : X \to A$ and $r : A \to X$ such that $r \circ i = 1_X$. A morphism f is a *retract* of a morphism g if it is as an object in the category $\mathbf{Mor}(\mathcal{C})$.

To make the meaning of the retract of a morphism more clear, we look at the following diagram,

$$\begin{array}{ccccc} X & \xrightarrow{i} & A & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i'} & B & \xrightarrow{r'} & Y \end{array}$$

which commutes when f is a retract of g, with the compositions, ri, $r'i'$ the relative identity maps.

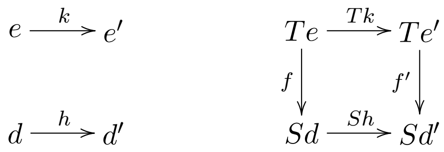
Now we examine another type of category, the comma category denoted $S \downarrow T$, we will use this construction later on in chapter 4.

Definition A.20: Given categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and functors, T and S in the following configuration;

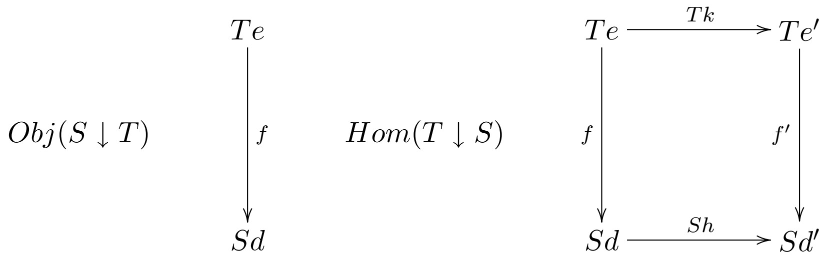
$$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$$

the *comma category* $T \downarrow S$ has as it's objects $\langle e, d, f\rangle$ the triples with $e \in Obj(\mathcal{E})$, $d \in Obj(\mathcal{D})$ and $f : Te \to Sd$. Furthermore $T \downarrow S$ has as morphisms, the pairs $\langle k, h\rangle : \langle d, e, f\rangle \to$

$\langle d', e', f' \rangle$, where $k : e \to e'$ in $\mathcal{E}$ and $h : d \to d'$ in $\mathcal{D}$ such that;

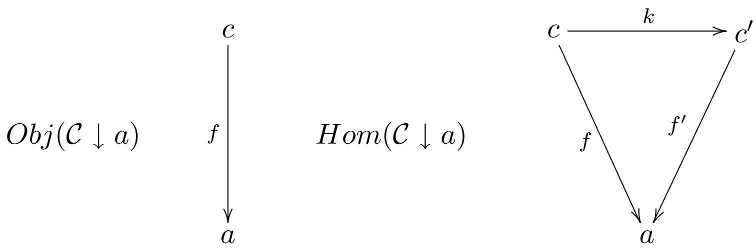


commutes. So we can look at this category as;



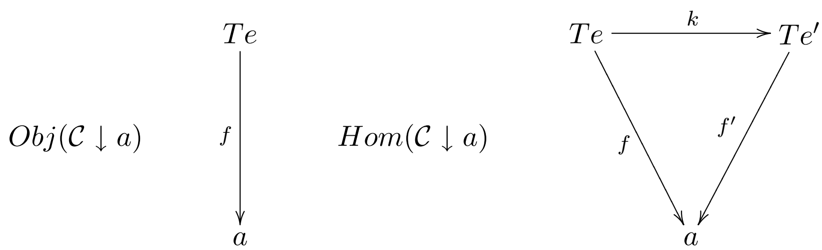
Next we will look at two special comma categories, firstly the category of *objects over a*, denoted $\mathcal{C} \downarrow a$ and secondly the category of T*-objects over a*, denoted $T \downarrow a$.

Example A.21: Let $S \downarrow T$ be a comma category as in definition [A.17] and let $\mathcal{E} = \mathcal{C}$, $\mathcal{D} = *$. Furthermore let the functors be; $T = id_{\mathcal{C}}$ and $S : * \mapsto a$. An object in $S \downarrow T$, $\langle *, c, f \rangle$ can be displayed as a map f from c to a, we just write $\langle c, f \rangle$. The morphism $\langle k, h \rangle$ will be determined by k, since the only choice for h is i_* and we will only write k. We can visualize this category as follows;



This comma category is the category of *objects over a* or also the *over category.*

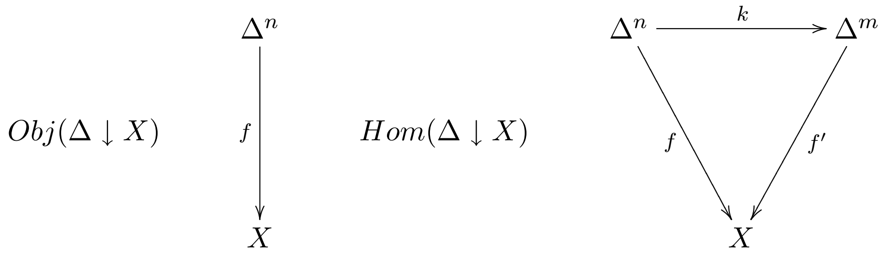
Example A.22: Let $S \downarrow T$ be a comma category as in definition 1.0.17. and let $\mathcal{D} = *$. Furthermore let $S : * \mapsto a$. An object in $S \downarrow T$, $\langle e, f \rangle$ can be displayed as a map f from Te to a. Morphisms in this comma category will be induced by morphisms $k : e \to e'$ in $\mathcal{E}$. We can visualize this category as follows;



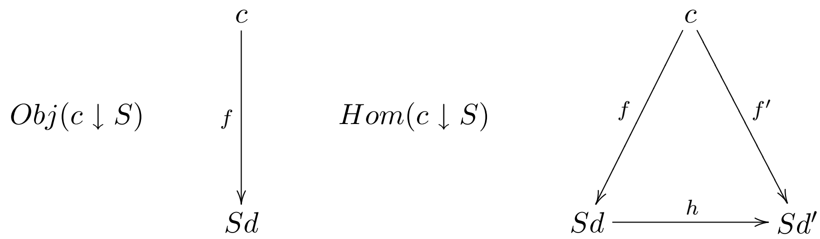
This comma category is the category of T*-objects over* a, notation $T \downarrow a$.

Similarly we can construct the category of *objects under* b and S*-objects under* b.

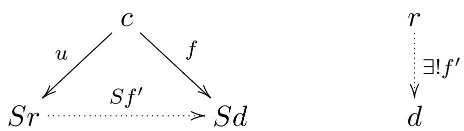
Example A.23: The *simplex category* of a simplicial set [A.19] X is the category of $Hom_{\Delta}(-, [n])$-objects over X, and is denoted by $\Delta \downarrow X$. This category can be visualized as follows;



Definition A.24: Let $S : \mathcal{D} \to \mathcal{C}$ be a functor, c an object of $\mathcal{C}$. An *universal arrow* from c to S is an initial object [A.8] $\langle r, u \rangle$ in $c \downarrow S$, the category of S-objects under c. The category is;



So for every object $\langle d, f \rangle$ in $c \downarrow S$, there is a unique morphism $f' : r \to d$ such that $Sf' \circ u = f$. So we have a commutative diagram for each morphism $f : c \to Sd$ like this;



With the use of universal arrows we will now introduce limits and colimits. These are very general concepts, and after the definition we will give the most commonly met examples of both colimits and limits. To introduce the concept we need to introduce a new functor first, namely the *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$, where $\mathcal{J}$ is called the index category, for reasons that will become clear later. The index category is usually small and often finite. Recall from definition [A.18] that the objects in $\mathcal{C}^{\mathcal{J}}$ are functors $F : \mathcal{J} \to \mathcal{C}$. The diagonal functor sends each object $c \in \mathcal{C}$ to the constant functor Δ_c which sends each object $i \in J$ to c and each morphism $i \xrightarrow{f} j$ to i_c. Furthermore the diagonal functor sends morphisms in $\mathcal{C}$ to constant natural transformations in $\mathcal{C}^{\mathcal{J}}$.

$$
\begin{array}{llll}
\Delta: & \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathcal{J}} \\
 & c & \mapsto & \Delta_c \\
 & & & \Delta_c(d) = c, \;\; \Delta_c(d \xrightarrow{f} d') = id_c \\
 & \left(c \xrightarrow{f} c'\right) & \mapsto & \Delta_f : \Delta_c \to \Delta_{c'}
\end{array}
$$

Now since a functor $F : \mathcal{J} \to \mathcal{C}$ is an object in $\mathcal{C}$ and Δ is a functor from $\mathcal{C}$, we can define a universal arrow from F to Δ as in definition [A.24].

Definition A.25: The *colimit* of a functor $F : \mathcal{J} \to \mathcal{C}$ is the universal arrow from F to Δ.

This definition in my opinion is too abstract to be able to really visualize something. So we will look at it in a bit more detail. A universal arrow $\langle r, u\rangle$ from F to Δ consists of an object $r \in \mathcal{C}$ and a morphism $u : F \to \Delta_r$ which is a natural transformation in the functor category $\mathcal{C}^{\mathcal{J}}$. The universal arrow has the property that for every other object $\langle d, f\rangle$ in $F \downarrow \Delta$, there is a unique morphism from r to d such that the following diagram commutes;

$$
\begin{array}{ccccc}
 & & F & & \\
 & \swarrow^{u} & & \searrow^{f} & \\
\Delta_r & & \xrightarrow{\Delta'_f} & & \Delta_d
\end{array}
\qquad\qquad
\begin{array}{c}
r \\
\Big\downarrow {\scriptstyle \exists ! f'} \\
d
\end{array}
$$

Where u, v and $\Delta_{f'}$ are natural transformations. This is still rather abstract, so we will look at the most common examples of colimits now.

Example A.26: Let $\mathcal{J}$ be a category with three objects and two morphisms (besides the identity morphisms) as follows $\{a \leftarrow b \to c\}$. The colimit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called the *pushout*. The functor F can be depicted as;

$$F(a) \leftarrow F(b) \to F(c)$$

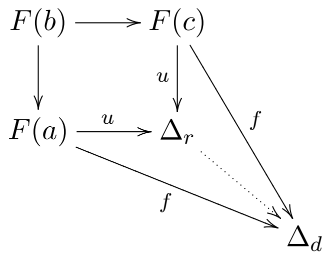
a diagram with the shape of $\mathcal{J}$ in $\mathcal{C}$. Now a universal arrow $\langle r, u\rangle$ can be visualized as follows;

$$
\begin{array}{ccccc}
F(a) & \longleftarrow & F(b) & \longrightarrow & F(c) \\
{\scriptstyle u}\Big\downarrow & & {\scriptstyle u}\Big\downarrow & & {\scriptstyle u}\Big\downarrow \\
\Delta_r & = & \Delta_r & = & \Delta_r
\end{array}
$$

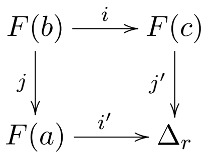
Where all the squares commute. Since the lower row are just identifications we usually write;

$$
\begin{array}{ccc}
F(b) & \longrightarrow & F(c) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle u} \\
F(a) & \xrightarrow{u} & \Delta_r
\end{array}
$$

We don't write the arrow from $F(b)$ to Δ_r since it's just the composite. Now the universality of $\langle r, u\rangle$ means that whenever there is another functor Δ_d and a natural transformation f, from F to it, then there is a natural transformation from Δ_r to Δ_d making the following diagram commute;

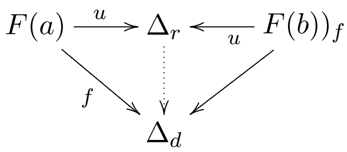


Definition A.27: In the pushout diagram from above;



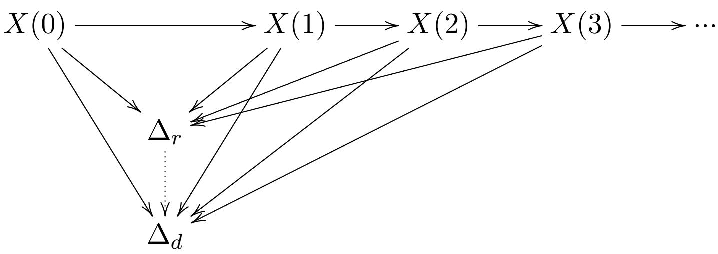
we call the map i' the *cobase change* of i (along j) and similarly j' the cobase change of j (along i).

Example A.28: When $\mathcal{J} = \{a, b\}$ a set with objects and no non-identity morphisms the colimit is called the *coproduct* and can be visualized by the following diagram;



For example in **sSet** the coproduct is just the disjunct union of sets and in **Top** it is the disjoint union of spaces.

Example A.29: When $\mathcal{J} = 0 \to 1 \to 2 \to 3 \to ...$ the colimit is called the sequential colimit and can be depicted as;

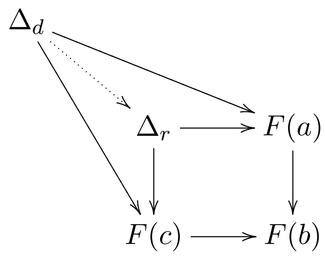


If the arrows are inclusions in **Top** then the colimit can be interpreted as an increasing union of the $X(n)$.

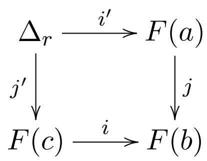
Definition A.30: The dual of a colimit is the *limit* of a functor. Let $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ again be the diagonal functor and $F : \mathcal{J} \to \mathcal{C}$ a functor, the limit of F is a universal arrow $\langle r, u \rangle$ from Δ to F.

We will look at the two most common examples, the duals of the first two colimits from above, namely the pullback and product.

Example A.31: Let $\mathcal{J} = \{a \to b \leftarrow c\}$, the limit of F is now called the *pullback* of F and looks like;

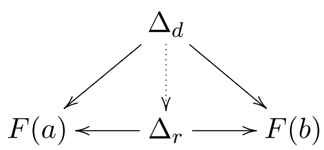


Definition A.32: In the pullback diagram from above;



we call the map i' the *base change* of i (along j) and similarly j' the base change of j (along i).

Example A.33: Let $\mathcal{J} = a, b$, the limit of F is the product;



The product in **sSet** and **Top** is what is usually called the 'direct product' or the 'Cartesian product'.

B Homotopy relations

In this appendix we proof some properties of the homotopy relations we defined in section two. The concept of left and right homotopy are dual, so we will only discuss right homotopy and path objects extensively. This appendix follows section four from [DS95] really closely. In their article Dwyer and Spalinski have a real good outline and to change this would not improve it. We do try to give the proofs in a bit more detailed way, so that they are easier to read for a reader new to the subject. This appendix does not contain any examples, so might be a bit abstract. That is why we decided not to obtain it in the article itself.

In this paragraph $\mathcal{C}$ will always be a model category. We start out by given a property of maps in a model category without proving them, for a proof we refer to [DS95][pg. 16-17].

Proposition B.1:
(i) The class of fibrations is stable under cobase change [A.27].
(ii) The class of acyclic fibrations is stable under cobase change.
And again dually;
(iii) The class of cofibrations is stable under base change[A.32].
(iv) The class of acyclic cofibrations is stable under base change.

Definition B.2: A *path object* for $X \in \mathcal{C}$ is an object X^I of C together with a diagram

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

which factors the diagonal map $(id_X, id_X) : X \to X \times X$. So the following diagram commutes;

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X^I \\ & \searrow^{id_X + id_X} & \downarrow p \\ & & X \times X \end{array}$$

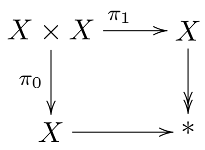
Now a path object is called a *good path object* when p is a fibration and a *very good path object* if in addition the map from X to X^I is a (necessarily acyclic) cofibration.

By axiom **MC5** there exists at least one very good path object for every X, since the map $(id_X, id_X) : X \to X \times X$ can be factored (as all maps can) as a fibration composed with an acyclic cofibration.

Recall from the definition of a product [A.33] that there are maps, call them π_0 and π_1, from $X \times X$ to X (the form the universal arrow). Now denote the two maps $X^I \to X$ by $p_0 = \pi_0 p$ and $p_1 = \pi_1 p$.

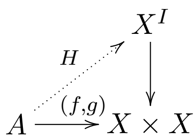
Lemma B.3: If X is fibrant and X^I is a good path object for X, then the maps p_0, p_1 are acyclic fibrations.

Proof: It is enough to check this just for one of the maps, p_0, since they are defined symmetrically. The identity map on X factors as $X \xrightarrow{\sim} X^I \xrightarrow{p_0} X$. The identity map is a weak equivalence, so p_0 is a weak equivalence as well by axiom **MC2**. To show that p_0 is a fibration we will show that both p and π_0 are fibrations, then since p_0 is just the composition of those two maps, it will itself be a fibration. The morphism p is a fibration by definition of a good cylinder object. To check that π_0 is a fibration we look at the following diagram,



the product $X \times X$ is the defined to be this pullback. The map on the right is a fibration because X is a fibrant object. Now it follows that π_0 is a fibration as well, since the class of fibrations is stable under base change [A.32] and we finished the proof.

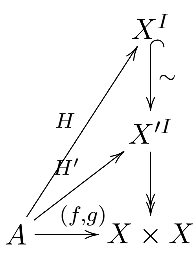
Definition B.4: Two maps $f, g : A \to X$ are said to be *right homotopic* if the product map $(f, g) : A \to X \times X$ can be lifted to a map $H : A \to X^I$ as illustrated in the diagram;



Notation; $f \overset{r}{\sim} g$. The map H is called a *right homotopy* from f to g. The right homotopy is said to be *good* respectively *very good* if the path object is. Note that if two maps are right homotopic, $f \overset{r}{\sim} g$, then f is a weak equivalence if and only if g is, because the maps p_0, p_1 are weak equivalences, so if $f = p_0 H$ is a weak equivalence then H is a weak equivalence and so $g = p_1 H$ is a weak equivalence.

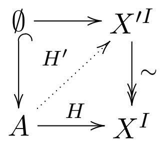
Lemma B.5: If $f \overset{r}{\sim} g : A \to X$, then there exists a good right homotopy from f to g.

Proof: We apply axiom **MC5**(ii) to the map $X^I \to X \times X$ to get a factorization $X^I \overset{\sim}{\hookrightarrow} X'^I \twoheadrightarrow X \times X$, so we can replace X^I by X'^I, and H by H';



Lemma B.6: If $f \overset{r}{\sim} g : A \to X$ and in addition A is cofibrant, then there exists a very good right homotopy from f to g.

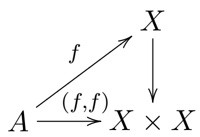
Proof: We now choose a good left homotopy $H : A \to X^I$ from f to g. We want to construct a very good path object X'^I, we again use **MC5**, but now to factor the map $X \xrightarrow{\sim}$ as $X \overset{\sim}{\hookrightarrow} X'^I \twoheadrightarrow X^I$. By **MC2** the second map is a weak equivalence so indeed we find a factorization $X \overset{\sim}{\hookrightarrow} X'^I \twoheadrightarrow X \times X$. Now we have to find the right homotopy, we look at the following diagram;



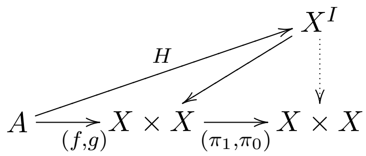
the lift H' exists by axiom **MC4**

Lemma B.7: If X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on $hom_{\mathcal{C}}(A, X)$.

Proof: (i) $f \overset{r}{\sim} f$? Note that X is a cylinder object for itself $X \xrightarrow{\sim} X \to X \times X$, where the first map is the identity. Now to get a right homotopy from f to itself we can just use f itself.



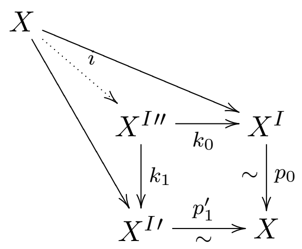
(ii) $f \overset{r}{\sim} g \Rightarrow g \overset{r}{\sim} f$? So we have a right homotopy H. Now we can use the map (π_1, π_0) : $X \times X \to X \times X$ to switch factors, so $(g, f) = (f, g)(\pi_1, \pi_0)$.



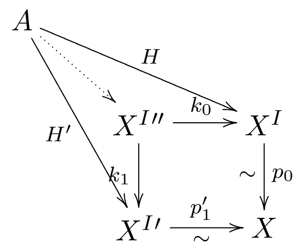
So H is a right homotopy from g to f too, with a slightly different path object.
(iii) $f \overset{r}{\sim} g$, $g \overset{r}{\sim} h \Rightarrow f \overset{r}{\sim} h$? Let $H : A \to X^I$ be a good right homotopy from f to g and $H' : A \to X^{I'}$ a good right homotopy from g to h. We want to find a right homotopy from f to h, to do this we construct $X^{I''}$ as the pullback of the following diagram;

$$X^{I'} \underset{\sim}{\xrightarrow{p_1'}} X \underset{\sim}{\xleftarrow{p_0}} X^I$$

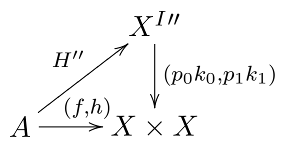
The maps p_1' and p_0 are weak equivalences by lemma [B.3]. Now by the universal property of colimits we find a map i from X to $X^{I''}$ that makes the following diagram commute,



this means that we found a factorization of (id_X, id_X) namely $(p_0k_0, p_1'k_1)i$, to show that $X^{I''}$ is a path object we need i to be a weak equivalence. By **MC2** it is enough to show that the map $X \to X^I$ and k_0 are weak equivalences. The first is, since X^I is a path object. And k_0 is a weak equivalence because it is the base change of p_1'. So indeed $X^{I''}$ is a path object. Now we can construct H'' that gives the wanted homotopy;



We see that his map H'' makes the following diagram commute;



since $p_0k_0H'' = p_0 = f$ and $p_1'k_1H'' = p_1'H' = h$.

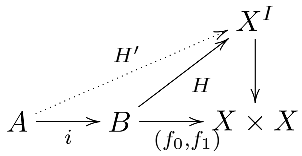
Let $\pi^r(A, X)$ denote the set of equivalence classes of $hom_{\mathcal{C}}(A, X)$ under the equivalence relation generated by right homotopy.

Lemma B.8: If X is fibrant and $i : A \to B$ is an acyclic cofibration, then composition with i induces a bijection;

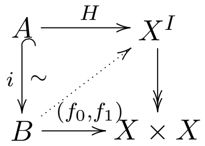
$$i^* : \pi^r(B, X) \to \pi^r(A, X)$$

Proof: First of all we have to check that this map is well defined. So when $f \overset{r}{\sim} g : B \to X$

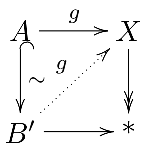
then we want $fi \overset{r}{\sim} gi$ too. We find this homotopy just by composing with i;



So indeed the map i^* is well defined. To see that it's an injective map we have to check that if $f_0 i \overset{r}{\sim} f_1 i$ then $f_0 \overset{r}{\sim} f_1$. Choose a good homotopy H from $f_0 i$ to $f_1 i$. We can find a lift in the following diagram to give the wanted homotopy;



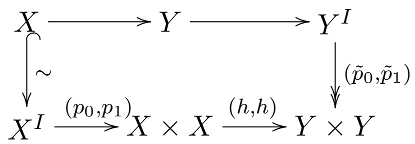
So i^* is injective. Now to show that i^* is surjective we pick a map $g : A \to X$ and try to find a map $g' : B \to X$ such that $g'i = g$. The map g' is the lift in the following diagram;



where the right map is a fibration since X is fibrant.

Lemma B.9: Suppose that A is cofibrant, that f and g are right homotopic maps from A to X, and that $h : X \to Y$. Then $hf \overset{r}{\sim} hg$.

Proof: First we choose a good path object for Y, namely $Y \xrightarrow{\sim} Y^I \twoheadrightarrow Y \times Y$, then we consider the following diagram and a very good right homotopy H from f to g. There is a lift k in the following diagram by **MC4**;



The composition $kH : A \to Y^I$ gives the wanted homotopy, since $\tilde{p}_0 kH = hp_0 H = hf$ and $\tilde{p}_1 kH = hp_1 H = hg$.

Lemma B.10: If A is cofibrant then the composition in $\mathcal{C}$ induces a map

$$\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y)$$

Proof: Note that elements that represent the same class in $\pi^r(A, X)$ need not be directly related by a right homotopy, but since the equivalence relation is generated by the rightt homotopies it still is enough to check the following. For $f \overset{r}{\sim} g : A \to X$ and $h \overset{r}{\sim} k$ the elements hf and kg are in the same right homtopy class class. By the previous lemma we find $hf \overset{r}{\sim} hg$. And by composing the homotopy between h and k with g we find $hg \overset{r}{\sim} hk$.

We can repeat this whole section dually to define a concept of left homotopy by defining cylinder objects. We will just give the definition for a cylinder object and left homotopy, then all the duals of the above statements hold. In their article, Dwyer and Spalinski describe the left homotopy extensively instead of right homotopy, so if one would really insist on reading this, we refer to [DS95].

Definition B.11: A *cylinder object* for A is an object $A \wedge I$ of $\mathcal{C}$ together with a diagram;

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the map $id_A + id_A : A \coprod A \to A$. Now a cylinder object is called a *good cylinder object* when i is a cofibration and a *very good cylinder object* if in addition the map from $A \wedge I$ to A is a (necessarily acyclic) fibration.

By axiom **MC5** there exists at least one very good cylinder object for every A, since the map $(id_A + id_A) : A \coprod A \to A$ can be factored (as all maps can) as an acyclic fibration composed with a cofibration.

Recall from the definition of a coproduct [A.28] that there are maps, call them ι_0 and ι_1, from A to $A \coprod A$ (they form the universal arrow in the definition of coproduct). Now denote the two maps $A \to A \wedge I$ by $i_0 = i\iota_0$ and $i_1 = i\iota_1$.

Definition B.12: Two maps $f, g : A \to X$ are said to be *left homotopic* if the sum map $f + g : A \coprod A \to X$ can be extended to a map $H : A \wedge I \to X$ as illustrated in the diagram;

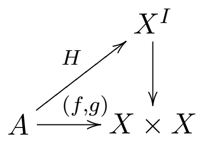
$$\begin{array}{ccc} A \coprod A & \xrightarrow{f+g} & X \\ \downarrow i & \overset{H}{\nearrow} & \\ A \wedge I & & \end{array}$$

Notation; $f \overset{l}{\sim} g$. The map H is called a *left homotopy* from f to g. The left homotopy is said to be *good* respectively *very good* if the cylinder object is.

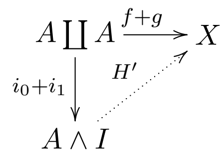
Now that we have defined left and right homotopy on the basis of our model category axioms we will continue by looking at the relationships between them. We will show that the two notions coincide when A is cofibrant and X is fibrant. Furthermore we will observe that in the situation where both A and X are fibrant and cofibrant, a map $f : A \to X$ will be a weak equivalence if and only if it has a homotopy inverse (e.g. a map $g : X \to A$ such that the compositions are homotopic to the relative identities). We will use this observation in the following chapter on the homotopy category, when we want a condition for being isomorphic in this category. And as mentioned in the introduction, we would like to get homotopy type as the isomorphism type.

Lemma B.13: Let $f, g : A \to X$ be morphisms in $\mathcal{C}$;
(i) If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{r}{\sim} g$, and dually
(ii) If A is cofibrant and $f \overset{r}{\sim} g$, then $f \overset{r}{\sim} g$.

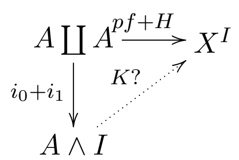
Proof: Since the two statement are each others duals it suffices to just proof one of them, we will only proof the first statement here. We can choose a good right homotopy from f to g;



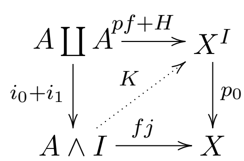
And we would like to get a map $H' : A \wedge I \to X$ making the following diagram commute



So we want $H'i_0 = f$ and $H'i_1 = g$, note that both f and g both can be written as compositions of p_1 and another function. Namely $f = id_X f = p_1 p f =$ and $g = p_1 H$, so it would be enough to find an extension K in the following diagram, and then let $H' = p_1 K$;



We can find this by making the diagram a bit bigger;



Now the lift K exists since p_0 is an acyclic fibration by [B.3] and $A \coprod A \to A \wedge I$ is a cofibration, because we can choose $A \wedge I$ to be a good cylinder object.

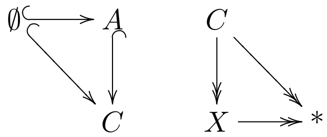
Definition B.14: When A is cofibrant and X fibrant, we will call the identical relation of right and left homotopic maps simply *homotopic maps*. We will use the symbol $\sim$ to denote homotopic maps, and the set of equivalence classes with respect to this relation will be denoted $\pi(A, X)$.

Lemma B.15: When we are in the situation that A and X are both fibrant and cofibrant in $\mathcal{C}$, then a map $f : A \to X$ is a weak equivalence if and only if it has a homotopy inverse.

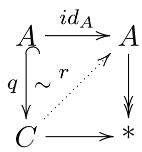
Proof: "⇒" We can factor our map f as we did before as follows (**MC5** and **MC2**);

$$A \overset{\sim}{\hookrightarrow} C \overset{\sim}{\twoheadrightarrow} X$$

Note that C is a cofibrant and fibrant object too by the following diagrams;



There exists a lift r in the following diagram;



this is a left inverse for q, e.g. $rq = id_A$. Now we can use lemma [B.8] on the acyclic cofibration q and the fibrant object C to get a bijection;

$$q^* \pi^r(C, C) \to \pi^r(A, C)$$

Since C and A are both fibrant and cofibrant, the right homotopy sets are actually just the homotopy sets. Now $q^*([qr]) = [qrq] = [q] = [id_C q] = q^*([id_C])$ and thus we find $qr \sim id_C$. A completely dual argument shows that there is a morphism $s : X \to C$ with $ps = id_X$ and $sp \sim id_C$. Now the composite gives the wanted homotopy inverse $g = rs$;

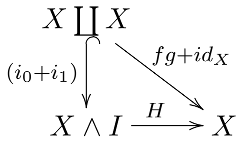
$$fg = pqrs \sim pid_C s = ps = id_X$$
$$gf = rspq \sim rid_C q = rq = id_A$$

"⇐" Now suppose that f has a homotopy inverse, we want to show that f is a weak equivalence. By **MC5** we can again factor f as;

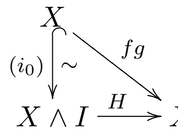
$$A \overset{\sim}{\hookrightarrow} C \twoheadrightarrow X$$

And showing that f is a weak equivalence becomes the same as showing that p is, since weak equivalences are stable under composition. We will construct a map s such that p is a retract

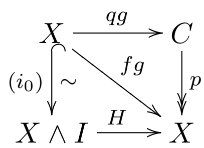
of sp and will show that sp is a weak equivalence, so p is as well.
Let g be the homotopy inverse of f and let $H : X \wedge I \to C$ be a good homotopy between fg and id_X. So recall from the definition of a left homotopy that we have a diagram;



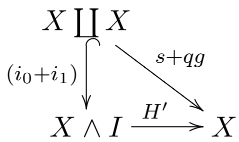
We can reduce this diagram to;



Where i_0 is an acyclic cofibration by the dual of lemma [B.3] with X cofibrant. Now since $fg = pqg$ we can extend this reduced diagram as follows;



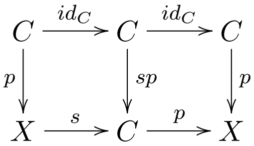
By **MC4** we can find a lift $H' : X \wedge I \to C$. Now define $s = H'i_0$, in this way H' becomes a homotopy from s to qg.



So we have $s \sim qg$. Now that we have defined s we want to show that sp is a weak equivalence. To do this we observe that the map q is a weak equivalence and thus by the first part of the proof has a homotopy inverse, call it r. Now since $f = pq$ we find $fr = pqr \sim p$, so $p \sim fr$. Now from lemma [B.10] and its dual it follows that;

$$sp \sim qgp \sim qgfr \sim qr \sim id_C$$

The identity id_C on C is a weak equivalence and this implies that sp is, see [B.4]. Now we are left to show that p is a retract of sp, which follows from the following commutative diagram;



So to summarize this chapter, we have defined what a model category is, a category where we have three types of maps, weak equivalences, fibrations and cofibrations. From the axioms of a model category we have build a notion of homotopy relations in this abstract setting. Now we are ready to define a homotopy category.

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