



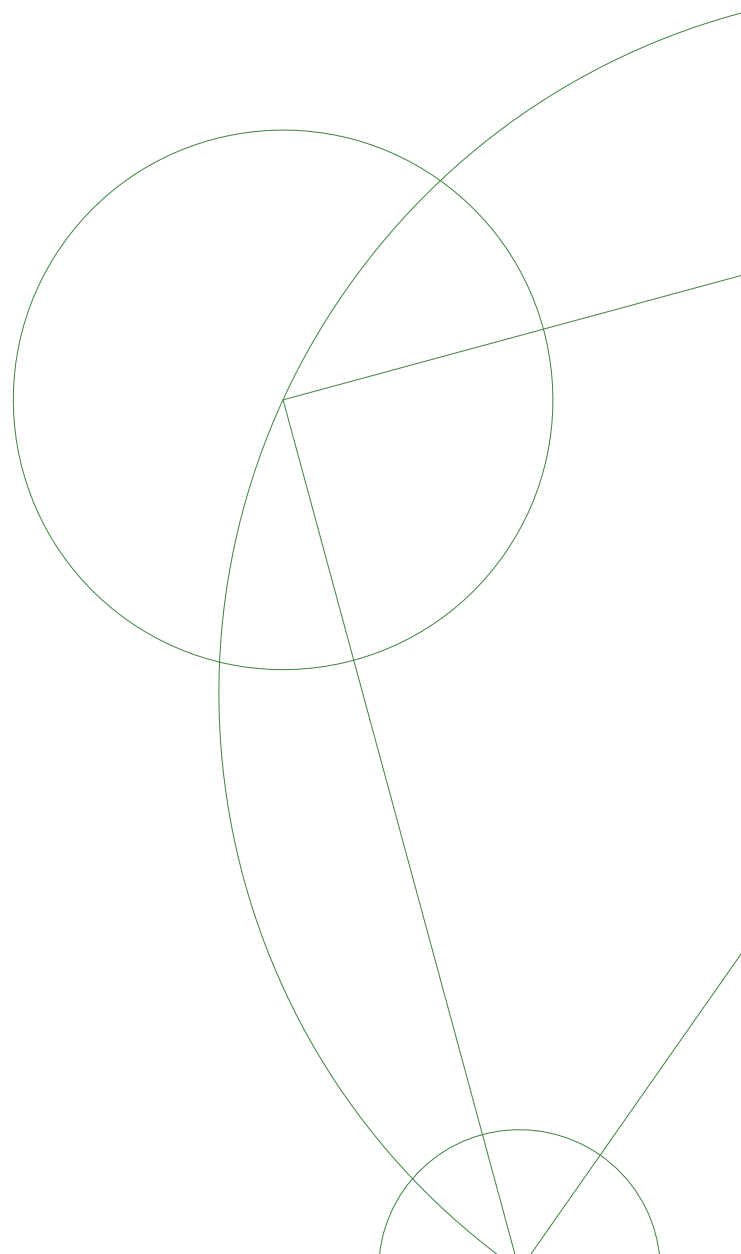
Bachelor Thesis in Mathematics

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Vietoris–Rips complexes of point sets in Euclidean space

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Submitted: 2015-06-04



Abstract

A Vietoris–Rips complex \mathcal{R} is the clique complex of a proximity graph. This graph can arise from a finite set of points in Euclidean n -space, and then there is a natural projection onto its n -dimensional shadow; $p : \mathcal{R} \rightarrow \mathcal{S}$.

An article by Chambers et al.[Cha+10] demonstrates that p induces an isomorphism of fundamental groups for $n = 1, 2$, while the induced homomorphism fails to be surjective for $n \geq 4$. In the case of $n = 3$, neither injectivity nor surjectivity are known. The existing proof of surjectivity for $n = 2$ proceeds by lifting paths from \mathcal{S} to \mathcal{R} .

In this report we propose a new proof of surjectivity for $n = 2$. We proceed by induction on the vertex set, thus we examine the geometry of the shadow in the neighborhood of a single vertex. At this vertex we can make decompositions of \mathcal{R} and \mathcal{S} to apply the Seifert–van Kampen theorem. This yields a diagram of fundamental groups, from which surjectivity follows by a category-theoretic argument.

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Introduction

Our main result is a new proof of the following theorem.

Theorem 2.9. [Cha+10][Part of Theorem 3.1] Let X be a finite set of points in \mathbb{R}^2 . Let $\mathcal{R} = \mathcal{R}(X)$ be the Rips complex of X and let \mathcal{S} be its image under the projection map $p : \mathcal{R} \rightarrow \mathbb{R}^2$. Then the induced homomorphism $\pi_1 p : \pi_1 \mathcal{R} \rightarrow \pi_1 \mathcal{S}$ is surjective. \square

We proceed by induction on the vertex set of $\mathcal{R}(X)$. This strategy immediately provides a new proof of [Cha+10, Proposition 5.2], which states that the Vietoris–Rips complex and its shadow are homotopy equivalent for a set of points in \mathbb{R}^1 . The argument is detailed in Proposition 2.2.

We now sketch the proof of Theorem 2.9. First, we decompose the spaces \mathcal{R} and \mathcal{S} at each vertex, so as to apply the Seifert–van Kampen theorem. In order to verify path-connectedness conditions, we inspect the geometry of the shadow in the neighborhood of a vertex. We state this as our first original result in Proposition 2.4, the proof of which is combinatorial in nature and uses geometric properties of \mathbb{R}^2 .

Second, by functoriality of π_1 and the Seifert–van Kampen theorem, we obtain a cube-like diagram of fundamental groups. In Section 2.3.2, we introduce the categorical tools that make up most of the remaining arguments. To the best of our knowledge, the statement of Proposition 2.5 is new in its given form, although its proof is elementary. Corollary 2.6 is our main tool for the proof of Theorem 2.9, and it is interesting in its own right.

It is noteworthy that only the first part of the proof relies on geometric considerations. Moreover, by induction, it suffices to consider the geometry within a small neighborhood of a vertex, which simplifies matters for a set of points in \mathbb{R}^3 , the open case. This could lead to the discovery of a counterexample in \mathbb{R}^3 , or extend our proof of surjectivity to this case. Indeed, we believe that by virtue of its abstract nature, the second part of the argument will remain sound, and not increase in complexity.

Chapter 1

Topological notions

In this chapter we introduce the needed tools from algebraic topology. Proofs are omitted and can be found in standard texts such as [Hat02] and [Rot88]. For simplicial complexes we refer to [Koz08]. Categorical language is borrowed from [Mac98] and [Lei14].

Unless otherwise stated, maps between topological spaces are assumed to be continuous. In this chapter, X, Y, Z are topological spaces, and I is the real unit interval $[0, 1] \subset \mathbb{R}$.

1.1. Paths and homotopy

Definition. A *path* in X is a map $f : I \rightarrow X$. The points $f(0)$ and $f(1)$ are called its endpoints, and we say that f is a path from $f(0)$ to $f(1)$. \lrcorner

The *reverse path* of f is denoted f^{-1} and defined as $t \mapsto f(1 - t)$, which is a path from $f(1)$ to $f(0)$. If $f(0) = f(1)$, then f is called a *loop*. The *constant path* at $x_0 \in X$ is given by $c_{x_0} : t \mapsto x_0$.

Two paths suitably matched at one endpoint can be concatenated as follows:

Definition. Given paths f, g with $f(1) = g(0)$, we define their *path product*

$$f \cdot g = \begin{cases} f(2s), & s \in [0, \frac{1}{2}] \\ g(2s - 1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

\lrcorner

The path product is continuous. If we think of $f(t)$ as encoding the position of a continuously moving point at time $t \in [0, 1]$, then the path product corresponds to traveling successively through f and g , now at twice the speed.

We will consider two paths equivalent whenever they can be continuously deformed into one another, in a sense that will now be made precise, starting from slightly more general definitions.

Definition. Let f_0, f_1 be maps $X \rightarrow Y$. A *homotopy* between f_0 and f_1 is a map $F : X \times I \rightarrow Y$ such that, for all $x \in X$,

$$(i) F(x, 0) = f_0(x),$$

$$(ii) F(x, 1) = f_1(x).$$

We then say that f_0 and f_1 are *homotopic*, denoted $f_0 \simeq f_1$. ┘

Intuitively, this means that f_0 can be continuously deformed into f_1 via the induced intermediate functions $f_t : x \mapsto F(x, t)$. The above can be generalized further.

Definition. Suppose $f_0, f_1 : X \rightarrow Y$ agree on some subspace $A \subseteq X$, i.e. $f_0|_A = f_1|_A$. If a homotopy $F : f_0 \simeq f_1$ satisfies that

$$f_t(a) = f_0(a) = f_1(a)$$

for every $a \in A$ and $t \in I$, then we say that f_0 and f_1 are *homotopic relative to A* , denoted $f_0 \simeq f_1 \text{ rel } A$. ┘

We think of homotopy rel A as a deformation that leaves A unchanged.

Proposition 1.1. For fixed $A \subseteq X$, homotopy rel A is an equivalence relation on the set of maps $X \rightarrow Y$. ┘

We will primarily be concerned with paths.

Definition. Two paths $f_0, f_1 : I \rightarrow X$ are *path homotopic* if they are homotopic as maps, relative to $\{0, 1\} \subset I$. In particular, homotopic paths must have matching endpoints. ┘

In the above situation, we will sometimes abuse the language by saying that the paths are homotopic relative to their endpoints, and simply write $f_0 \simeq f_1$. The class of a path f under the corresponding equivalence relation will be denoted $[f]$.

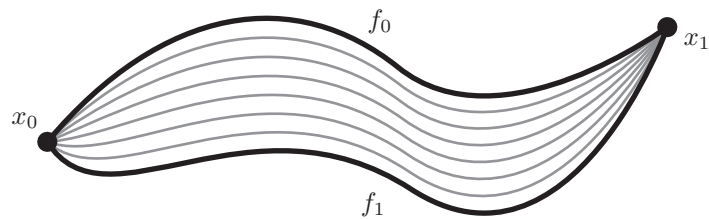


Figure 1.1: Path homotopy.

In some cases, it is easy to exhibit a path homotopy:

Example 1.2. Let f_0, f_1 be two paths in a convex subspace $U \subseteq \mathbb{R}^n$. Since U is convex, it contains every segment linking $f_0(s)$ to $f_1(s)$. Hence we can define the so-called *linear homotopy*

$$F(s, t) = (1 - t)f_0(s) + tf_1(s).$$

This map is continuous since addition and multiplication are continuous operations on \mathbb{R}^n , and it is clearly a homotopy $f_0 \simeq f_1$. Thus there is only one homotopy class of paths for each pair of endpoints. \lrcorner

1.2. The fundamental group

Definition. Given a fixed basepoint $x_0 \in X$, we denote by $\pi_1(X, x_0)$ the set of path classes of loops at x_0 . \lrcorner

A reason for considering loops at a fixed basepoint is that their concatenation is always well-defined. Together with path reversal, this gives rise to a group structure on $\pi_1(X, x_0)$, called the *fundamental group* of X (at basepoint x_0).

Proposition 1.3. The set $\pi_1(X, x_0)$ is a group, under the operation $[f][g] = [f \cdot g]$, with identity $[c_{x_0}]$, and inverses $[f]^{-1} = [f^{-1}]$. \lrcorner

Our current construction of the fundamental group depends on a choice of basepoint. This choice turns out to be irrelevant, under the assumption that the space is path-connected.

Proposition 1.4. Let $x_0, x_1 \in X$ belong to the same path-component of X . Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. \lrcorner

By the above, we will use the abbreviation $\pi_1(X)$ for the fundamental group of a path-connected space X . In general however, we must consider pairs (X, x_0) consisting of a space and a basepoint. Such pairs are called *based topological spaces* and form the category \mathbf{Top}_* , where morphisms are basepoint-preserving continuous maps.

The fundamental group will allow us to translate topological properties to algebraic properties. This process applies to maps as well.

Proposition 1.5. Let $\varphi : X \rightarrow Y$ with $\varphi(x_0) = y_0$. The set map

$$\pi_1\varphi : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

defined by

$$\pi_1\varphi : [f] \mapsto [\varphi f]$$

is a well-defined group homomorphism, called the *induced homomorphism*. \lrcorner

Remark 1.6. The assignment $\pi_1 : \varphi \mapsto \pi_1\varphi$ respects identity and composition, in the sense that $\pi_1 \text{id}_X = \text{id}_{\pi_1(X)}$ and $\pi_1(\varphi\psi) = (\pi_1\varphi)(\pi_1\psi)$, hence defines a functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$. \lrcorner

The fundamental group is a homotopy invariant, in the following sense:

Definition. Two based topological spaces $(X, x_0), (Y, y_0)$ are said to be *homotopy equivalent* if there exists a pair of basepoint-preserving maps $f : X \rightarrow Y, g : Y \rightarrow X$ such that $gf \simeq \text{id}_X$ and $fg \simeq \text{id}_Y$, relative to basepoints. We then write $(X, x_0) \simeq (Y, y_0)$, and the maps f, g are called *homotopy equivalences*. \lrcorner

This defines an equivalence relation, and equivalent spaces are said to have same *homotopy type*. A space that is equivalent to a one-point space is said to be *contractible*. We say that a space is *simply connected* if it is path-connected and has trivial fundamental group. In particular, every contractible space is simply connected.

Proposition 1.7. Let $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$ be homotopy equivalent spaces. Then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. \lrcorner

In particular, the fundamental groups of homeomorphic spaces are isomorphic.

Computing fundamental groups is a difficult task in general. On occasion, one may appeal to geometric intuition, as in the classic examples below.

Example 1.8. The fundamental group of the unit circle is isomorphic to the group of integers, i.e. $\pi_1(S^1) \cong \mathbb{Z}$. Letting $S^1 = \{e^{2\pi it} \in \mathbb{C} \mid t \in \mathbb{R}\}$, it can be shown that the homotopy class of the loop $t \mapsto e^{2\pi it}$ is a generator for $\pi_1(S^1)$, of infinite order. \lrcorner

Example 1.9. For $n \geq 2$, $\pi_1(S^n) \cong 0$. In particular, this shows that the converse to Proposition 1.7 fails: the 2-sphere has trivial fundamental group, like the one-point space, but is non-contractible. \lrcorner

1.2.1. The Seifert–van Kampen theorem

A powerful tool for computing fundamental groups is the Seifert–van Kampen theorem, which expresses the fundamental group of a space in terms of the fundamental groups of nice subspaces.

To introduce the theorem, recall this fact from general topology: let A, B be open subspaces covering X , i.e. $X = A \cup B$. Then the following diagram, with inclusion maps, is a pushout square in \mathbf{Top} :

$$\begin{array}{ccc} A \cap B & \xhookrightarrow{i} & A \\ i' \downarrow & & \downarrow j \\ B & \xhookrightarrow{j'} & X. \end{array}$$

That is, for any Y and any pair of maps $f : A \rightarrow Y$, $g : B \rightarrow Y$ such that $f|_{A \cap B} = g|_{A \cap B}$, there exists a unique $h : X \rightarrow Y$ such that $h|_A = f$ and $h|_B = g$.

The next theorem asserts that under certain conditions, this pushout carries over to fundamental groups.

Theorem 1.10. (*Seifert–van Kampen*). Let $(X, x_0) \in \mathbf{Top}_*$. Let A, B be open subspaces of X such that $X = A \cup B$. Suppose that $A \cap B$ is path-connected and contains x_0 . Then the following diagram, with homomorphisms induced by inclusion maps, is a pushout square in **Grp**:

$$\begin{array}{ccc} \pi_1(A \cap B, x_0) & \xrightarrow{\pi_1 i} & \pi_1(A, x_0) \\ \pi_1 i' \downarrow & & \downarrow \pi_1 j \\ \pi_1(B, x_0) & \xrightarrow{\pi_1 j'} & \pi_1(X, x_0). \end{array}$$

More generally, suppose X is the union of a finite family $\{A_\alpha\}$ of path-connected open subspaces such that every possible intersection contains the basepoint, and is path-connected. This yields a diagram consisting of all inclusion maps induced by intersections. Then $\pi_1(X, x_0)$ is the colimit of the image of this diagram under the functor π_1 . \lrcorner

We now examine some consequences of the Seifert–van Kampen theorem.

Example 1.11. In the setup of Theorem 1.10, assume that X is path-connected and $A \cap B$ is simply connected. In particular, $\pi_1(A \cap B) = 1$ is the trivial group, which is initial in **Grp**. Then $\pi_1(X) \cong \pi_1(A) * \pi_1(B)$, where $*$ denotes the coproduct in **Grp** (known as the *free product*). \lrcorner

A classic application of the Seifert–van Kampen theorem is to compute the fundamental group of a wedge of circles.

Definition. Let (X_α, x_α) be a family of based topological spaces. Let \sim be the equivalence relation generated by identification of all basepoints. The space

$$\bigvee_{\alpha} X_i = \bigsqcup_{\alpha} X_i / \sim$$

with basepoint $[x_0]_{\sim}$ is called the *wedge sum* of the family and is the coproduct of this family in \mathbf{Top}_* . \lrcorner

The Seifert–van Kampen theorem then asserts that under the right conditions, the functor π_1 takes coproducts to coproducts. In particular, we have the following:

Example 1.12. Letting $X_\alpha \simeq S^1$ for $\alpha \in \{1, \dots, n\}$, we have

$$\pi_1 \left(\bigvee^n S^1 \right) \cong F_n,$$

where F_n denotes the *free group* on n generators, which is isomorphic to the free product of n copies of \mathbb{Z} in **Grp**. \lrcorner

1.3. Simplicial complexes

1.3.1. Abstract simplicial complexes

Let V be a finite set.

Definition. An *abstract simplicial complex* is a collection Δ of subsets of V such that if $\tau \in \Delta$ and $\sigma \subseteq \tau$, then $\sigma \in \Delta$. \lrcorner

$V = \text{Vert}(\Delta)$ is called the *vertex set* of Δ , and elements of Δ are called *simplices* or *faces*. A *subcomplex* of Δ is a subcollection of Δ that is also an abstract simplicial complex. There is a notion of map between abstract simplicial complexes:

Definition. Let Δ_1, Δ_2 be abstract simplicial complexes. A *simplicial map* $\varphi : \Delta_1 \rightarrow \Delta_2$ is a map $\varphi : \text{Vert}(\Delta_1) \rightarrow \text{Vert}(\Delta_2)$ such that $\sigma \in \Delta_1 \Rightarrow \varphi\sigma \in \Delta_2$, in the sense that if $\{v_0, \dots, v_k\} \in \Delta_1$, then $\{\varphi(v_0), \dots, \varphi(v_k)\} \in \Delta_2$. \lrcorner

Abstract simplicial complexes together with simplicial maps form a category, denoted **ASC**. An isomorphism in this category is a bijective simplicial map whose inverse is simplicial.

The naming choices in the definitions below will make sense in view of Section 1.3.2, where we provide a way to construct topological spaces from abstract simplicial complexes.

Definition. Let $\sigma \in \Delta$ be a simplex and let $\text{card } \sigma$ be its cardinality as a set. The *dimension* of σ is defined as $\dim \sigma = \text{card } \sigma - 1$, and the dimension of Δ is then defined as $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. \lrcorner

Definition. The k -skeleton of an abstract simplicial complex Δ is the subcomplex of Δ consisting of all simplices $\sigma \in \Delta$ such that $\dim \sigma \leq k$. It is denoted $\Delta^{(k)}$. \lrcorner

1.3.2. Geometric realization

Let Δ be an abstract simplicial complex and let σ be a simplex of Δ , and suppose $\dim \sigma = n$, i.e. $\text{card } \sigma = n + 1$.

Consider the linear topological space \mathbb{R}^σ , which is isomorphic to \mathbb{R}^{n+1} , and has elements of σ as basis. We define Δ^σ to be the convex hull of σ inside \mathbb{R}^σ . In this way, Δ^σ is homeomorphic to the convex hull of the standard basis in \mathbb{R}^{n+1} . In general, a space with this property is called an n -simplex.

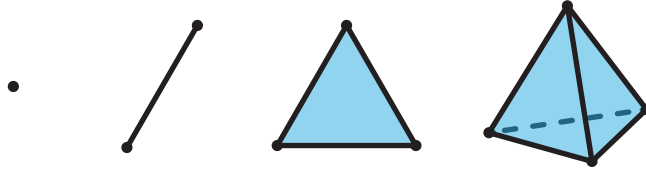


Figure 1.2: n -simplices for $n = 0, 1, 2, 3$.

These spaces will be our building blocks for constructing a topological space from the combinatorial data of Δ . Note that whenever we have a subcomplex $\tau \subseteq \sigma$, there is an inclusion of bases which induces a natural inclusion map $f_{\tau\sigma} : \Delta^\tau \hookrightarrow \Delta^\sigma$. We define

$$\Omega = \bigsqcup_{\sigma \in \Delta} \Delta^\sigma,$$

equipped with the coproduct topology, and let \sim denote the equivalence relation on Ω generated by the prescription

$$y \sim x \text{ whenever } y = f_{\tau\sigma}(x) \text{ for some } f_{\tau\sigma}.$$

Definition. In the above notation, the *geometric realization* of Δ is defined to be the topological space

$$|\Delta| = \Omega / \sim$$

equipped with the quotient topology. ┘

The assignment $|\cdot| : \mathbf{ASC} \rightarrow \mathbf{Top}$ is a functor. In particular, isomorphic abstract simplicial complexes have homeomorphic geometric realizations. From now, we will freely speak of topological properties Δ when meaning $|\Delta|$. In the same spirit, we use the term *complex* for either an abstract simplicial complex or its geometric realization.

We will use the following remark in Chapter 2:

Remark 1.13. In general, subcomplexes of Δ are closed subspaces. However, one may replace each subcomplex by an open neighborhood, chosen to be sufficiently small to preserve homotopy type. This allows us to apply the Seifert–van Kampen. More precisely, let Δ_1, Δ_2 be path-connected subcomplexes of Δ , such that $\Delta_1 \cup \Delta_2 = \Delta$. To avoid basepoint considerations, suppose Δ is path-connected. Then the Seifert–van Kampen theorem yields the following pushout diagram of topological spaces:

$$\begin{array}{ccc} \pi_1(\Delta_1 \cap \Delta_2) & \longrightarrow & \pi_1(\Delta_1) \\ \downarrow & & \downarrow \\ \pi_1(\Delta_2) & \longrightarrow & \pi_1(\Delta) \end{array}$$

┘

1.3.3. Piecewise linear paths

Suppose Δ is a path-connected abstract simplicial complex. Paths in $|\Delta|$ can be described combinatorially in the following way:

Definition. Let $f : I \rightarrow |\Delta|$ be a path within the 1-skeleton of Δ , i.e. $f(I) \subseteq |\Delta^{(1)}|$, such that there exists a sequence of vertices $v_0, \dots, v_k \in \text{Vert}(\Delta)$ and a subdivision of I :

$$0 = a_0 < \dots < a_k = 1$$

such that, for $i \in \{0, \dots, k\}$,

$$(i) \quad f(a_i) = v_i,$$

$$(ii) \quad f((1-t)a_i + ta_{i+1}) = (1-t)v_i + tv_{i+1} \text{ for all } t \in I.$$

Such a map f is called a *piecewise linear path*. In particular, endpoints of a piecewise linear path are vertices of Δ . \lrcorner

As with regular paths, one can define piecewise linear loops, the constant path, and path products. Piecewise linear paths are said to be homotopic if they are path homotopic in the usual sense.

Let us see that a sequence of vertices uniquely determines a piecewise linear path, up to homotopy. Suppose f, g are piecewise linear paths through vertices v_0, \dots, v_k , with two respective subdivisions of I given by a_0, \dots, a_k and b_0, \dots, b_k . The map $h : I \rightarrow I$ defined by $h((1-t)a_i + ta_i) = (1-t)b_i + tb_i$ is a reparametrization such that $fh = g$, hence $f \simeq g$.

The geometric realization of an abstract simplicial complex belongs to the class of so-called CW-complexes. We shall make use of the result below, which is derived from the cellular approximation theorem for CW-complexes.

Proposition 1.14. Let $f : I \rightarrow |\Delta|$ be a path in a simplicial complex, with endpoints in $|\Delta^{(1)}|$. Then f is homotopic to some piecewise linear path \tilde{f} such that $\tilde{f}(I) \subseteq |\Delta^{(1)}|$, relative to endpoints. \lrcorner

1.3.4. Deletion, Star and Link

Let Δ be an abstract simplicial complex, and let $\tau \in \Delta$. We define some subcomplexes of Δ that will provide a useful decomposition of Δ .

Definition. The *deletion* of τ is the abstract simplicial complex defined by

$$\text{dl}_\Delta(\tau) = \{\sigma \in \Delta \mid \sigma \not\supseteq \tau\}.$$

The *star* of τ is the abstract simplicial complex defined by

$$\text{st}_\Delta(\tau) = \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta\}.$$

The *link* of τ is the abstract simplicial complex defined by

$$\text{lk}_\Delta(\tau) = \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta \text{ and } \sigma \cap \tau = \emptyset\}.$$

┘

Proposition 1.15. Let v be a single vertex in Δ . Then we have the following decomposition: $\Delta = \text{dl}_\Delta(v) \cup \text{st}_\Delta(v)$, with intersection $\text{dl}_\Delta(v) \cap \text{st}_\Delta(v) = \text{lk}_\Delta(v)$. This can be depicted in the diagram below, which is a pushout square in **ASC**:

$$\begin{array}{ccc} \text{lk}_\Delta(v) & \hookrightarrow & \text{st}_\Delta(v) \\ \downarrow & & \downarrow \\ \text{dl}_\Delta(v) & \hookrightarrow & \Delta, \end{array}$$

whose image under the geometric realization functor is a pushout in **Top**. ┘

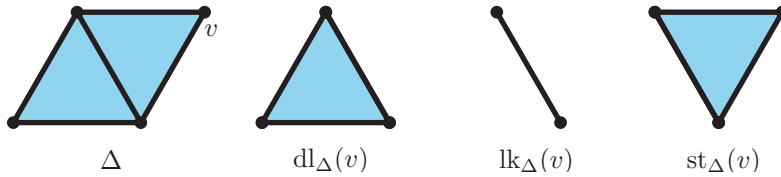


Figure 1.3: A simple instance of Proposition 1.15.

Chapter 2

Vietoris–Rips complexes of point sets in \mathbb{R}^n

We now introduce a class of complexes that will be our main interest, known as *Vietoris–Rips* complexes (Rips complexes for brevity).

Definition. Let (M, d) be a metric space and $X \subseteq M$ a set of distinct points of M . Let $r \geq 0$. The *Rips complex* $\mathcal{R}_r(X)$ is the complex obtained by letting $\sigma \subseteq 2^X$ be a simplex of $\mathcal{R}_r(X)$ whenever σ has diameter at most r . \lrcorner

Elements of X are called vertices, and r is called the *radius* of \mathcal{R}_r . From now on, we will assume $r = 1$ and use the shorthand notation $\mathcal{R} = \mathcal{R}(X) = \mathcal{R}_1(X)$. Moreover, we restrict our attention to $M = \mathbb{R}^n$.

The above definition uniquely determines the complex $\mathcal{R}_r(X)$. Alternatively, \mathcal{R} can be constructed in two steps as follows: first, construct the *proximity graph* of X by joining $u, v \in X$ with an edge whenever $d(u, v) \leq 1$. Then define \mathcal{R} as the largest complex with this graph as 1-skeleton.

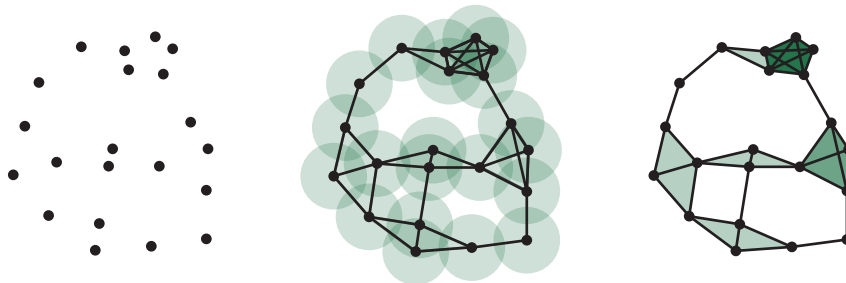


Figure 2.1: A set of planar points, its proximity graph, and its 4-dimensional Rips complex.

2.1. The Shadow of a Rips complex

Suppose $X \subset \mathbb{R}^n$. In general, the dimension of $\mathcal{R}(X)$ may be larger than n . However, there is a natural projection map $p : \mathcal{R} \rightarrow \mathbb{R}^n$, that maps each simplex affinely onto the convex hull of its vertices. We define $\mathcal{S} = p(\mathcal{R})$, occasionally denoted $\mathcal{S}(X)$. Equivalently, we have

$$\mathcal{S}(X) = \bigcup_{\substack{S \subset X \\ \text{diam}(S) \leq 1}} \text{conv}(S)$$

In fact, as described in [Cha+10], in the above it suffices to take the union over S with $\text{card}(S) \leq n + 1$ instead of all $S \subset X$.

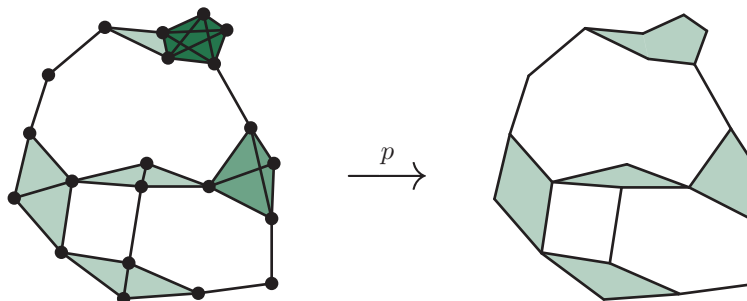


Figure 2.2: A Rips complex and its shadow. Darker colors are meant to represent higher dimensional simplices.

In the 2-dimensional case, we define a decomposition of the shadow. First, let a *shadow vertex* be any point that either lies in $X = p(\mathcal{R}^{(0)})$ or is a transverse intersection of images of Rips edges under the projection p . Let $\mathcal{S}^{(0)}$ be the set of such shadow vertices. Next, define a *shadow edge* to be the closure of any path-component of $p(\mathcal{R}^{(1)}) - \mathcal{S}^{(0)}$, and let $\mathcal{S}^{(1)}$ be the set of such shadow edges. Finally, define a *shadow face* to be the closure of any component of $p(\mathcal{R}^{(2)}) - \mathcal{S}^{(1)}$. The shadow of any Rips complex has the homotopy type of a wedge of circles.

Our main concern will be to compare the fundamental groups of the complexes \mathcal{R} and \mathcal{S} . By Proposition 1.14, we can describe paths combinatorially as sequences of Rips or shadow edges. Such paths will be called Rips paths and shadow paths. Let us establish some notation. Vertices will typically be labeled A, B, C, u, v . Simplices of \mathcal{R} will be denoted e.g. $[ABC]$, and $\langle ABC \rangle$ denotes the subcomplex of \mathcal{R} generated by vertices A, B, C . Shadow edges will be denoted e.g. AB , and we write $|AB|$ for the Euclidean length of such an edge.

The article [Cha+10] examines the relation between the homotopy types of \mathcal{R} and \mathcal{S} . One of the first steps is the examination of path-components.

Proposition 2.1. [Cha+10, Proposition 5.1] Let \mathcal{R} be a Rips complex over a set of points in \mathbb{R}^n . The map $p : \mathcal{R} \rightarrow \mathcal{S}$ induces a bijection $\pi_0 p : \pi_0 \mathcal{R} \rightarrow \pi_0 \mathcal{S}$. \lrcorner

2.2. The Rips complex for $n = 1$

Proposition 2.2. [Cha+10, Proposition 5.2] Let $X \subset \mathbb{R}$ be a finite set of distinct points, such that $\mathcal{R}(X)$ is path-connected. Then $\mathcal{R}(X)$ and $\mathcal{S}(X)$ are both contractible, and so $p : \mathcal{R} \rightarrow \mathcal{S}$ is a homotopy equivalence. \lrcorner

The proof below is different from the original, in that we use induction and the decomposition described in Proposition 1.15. This serves as a preview of the strategy adopted in proving our main result (Theorem 2.9).

Proof of Proposition 2.2. By Proposition 2.1, $\mathcal{S}(X)$ is connected, hence is an interval of \mathbb{R} , thus it is contractible. We show that $\mathcal{R}(X)$ is contractible by induction. When $\text{card}(X) = 1$, it is clear. Let $v = \max\{X\}$ and suppose $\mathcal{R}(X-v)$ is contractible. Let $X_v = X \cap [v-1, v]$ and $X' = X_v - v$. Then we have the decomposition

$$\mathcal{R}(X) = \mathcal{R}(X-v) \cup \mathcal{R}(X_v),$$

with intersection

$$\mathcal{R}(X-v) \cap \mathcal{R}(X_v) = \mathcal{R}(X').$$

Note that this is precisely the decomposition from Proposition 1.15. Both X_v and X' have diameter at most 1, hence the spaces $\mathcal{R}(X_v)$ and $\mathcal{R}(X')$ are contractible. By the induction hypothesis, the space $\mathcal{R}(X-v)$ is contractible. Thus, $\mathcal{R}(X)$ is the union of contractible spaces whose intersection is contractible, hence it is contractible. \square

2.3. Technical lemmas

2.3.1. Geometric lemmas in \mathbb{R}^2

From here onwards assume that X is a finite subset of \mathbb{R}^2 , and $\mathcal{R}(X)$ is path-connected. This section collects preliminary results to be used in section 2.4. We will make extensive use of the following geometric observation:

Proposition 2.3. [Cha+10, Proposition 2.1] Let $\mathcal{R} = \langle ABCD \rangle$ be a Rips complex containing Rips edges $[AB]$ and $[CD]$, such that AB and CD intersect in \mathcal{S} . Then \mathcal{R} is a cone. \lrcorner

Proof. Let x be a point of intersection of AB , CD . Up to relabeling we can assume that A is the vertex with minimal distance to x , so that in particular $|Ax| \leq |Cx|$. This together with the triangle inequality yields

$$|AD| \leq |Ax| + |xD| \leq |Cx| + |xD| = |CD| \leq 1,$$

hence $[AD]$ is a Rips edge. Similarly $[AC]$ is a Rips edge, so that \mathcal{R} is a cone with apex A . \square

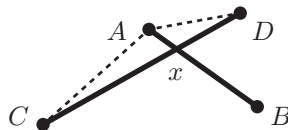


Figure 2.3: The situation in Proposition 2.3.

Let us fix a vertex $v \in X$, and consider the closed ball $\mathcal{B} = \mathcal{B}(v, 1)$ of radius 1, centered at v . We denote $X_v = X \cap \mathcal{B}$ and $X' = X_v - v$. The next proposition is conceived for a later application of the Seifert–van Kampen theorem.

Proposition 2.4. Let $\mathcal{R}(Y)$ be a path component of $\mathcal{R}(X')$. For every $x \in \mathcal{S}(Y \cup v) \cap \mathcal{S}(X - v)$, there is a path from x to some point of $\mathcal{S}(Y)$, such that this path is entirely contained in $\mathcal{S}(Y \cup v) \cap \mathcal{S}(X - v)$. In particular, $\mathcal{S}(Y \cup v) \cap \mathcal{S}(X - v)$ is path-connected. \lrcorner

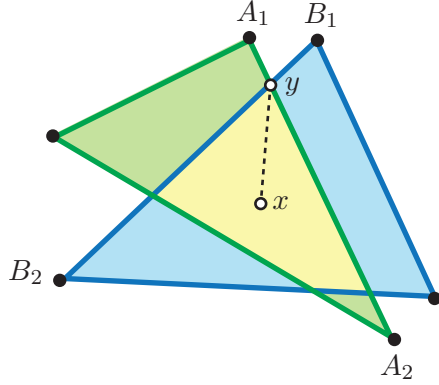
Proof. By Proposition 2.1, $\mathcal{S}(Y)$ is a path component of $\mathcal{S}(X')$. Recall that the shadow can be expressed as

$$\mathcal{S}(X) = \bigcup_{\substack{i,j,k \in X \\ \text{diam}(\{i,j,k\}) \leq 1}} \text{conv}(\{i,j,k\})$$

Therefore, we can assume that x belongs to the intersection of two solid triangles in the plane, say $T_a \subseteq \mathcal{S}(X_v)$ and $T_b \subseteq \mathcal{S}(X - v)$.

We first argue that if either triangle is properly contained inside the other, then $x \in \mathcal{S}(Y)$, from which the conclusion follows. On the one hand, suppose $T_b \subsetneq T_a$. Then $T_b \subseteq \mathcal{S}(Y \cup v)$, but $v \notin T_b$, hence $T_b \subseteq \mathcal{S}(Y)$. On the other hand, suppose $T_a \subsetneq T_b$. Then $v \notin T_a$, hence $T_a \subseteq \mathcal{S}(Y)$.

Now assume that neither proper containment holds. Then there exists a point $y \in T_a \cap T_b$, such that y is the intersection of an edge $A_1A_2 \subseteq T_a$ and an edge $B_1B_2 \subseteq T_b$, with $A_1, A_2 \in Y \cup v$ and $B_1, B_2 \in X - v$. Moreover, $T_a \cap T_b$ is a convex subset of \mathbb{R}^2 , hence there exists a path from x to y , entirely contained in $T_a \cap T_b$:



Thus, without loss of generality, we may assume that $x = y$. We now examine cases depending on the position of the vertices A_1, A_2, B_1, B_2 , relative to \mathcal{B} .

If neither A_1 nor A_2 equals v , then $x \in A_1A_2 \subseteq \mathcal{S}(Y)$. Thus assume $A_2 = v$. Suppose $B_1, B_2 \in \mathcal{B}$. Since B_1, B_2 must both lie in the same path component of $\mathcal{S}(Y \cup v)$, and the intersection $B_1B_2 \cap A_1A_2$ is nonempty, it follows that both B_1 and B_2 lie in $\mathcal{S}(Y)$. But then $x \in B_1B_2 \subseteq \mathcal{S}(Y)$.

Now, up to relabeling, two cases remain:

- (i) $B_1 \notin \mathcal{B}$ and $B_2 \in \mathcal{B}$,
- (ii) $B_1, B_2 \notin \mathcal{B}$.

In the first case, $\langle A_1vB_1B_2 \rangle$ is a cone by Proposition 2.3. By assumption, $|B_1v| > 1$, hence neither B_1 nor v is an apex, so $[A_1B_2] \in \mathcal{R}(X)$. In particular, B_2 is in the same path component as A_1 , i.e. $B_2 \in \mathcal{S}(Y)$. Moreover, $[B_2v] \in \mathcal{R}(X)$ since $B_2 \in \mathcal{B}$. Thus the subsegment xB_2 is contained in the triangle $A_1B_2v \subseteq \mathcal{S}(Y \cup v)$. Since $xB_2 \subseteq B_1B_2 \subseteq \mathcal{S}(X-v)$, it follows that the linear path from x to $B_2 \in \mathcal{S}(Y)$ is entirely contained within $\mathcal{S}(X-v) \cap \mathcal{S}(Y \cup v)$, as desired.

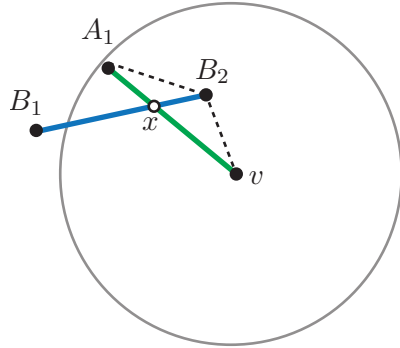


Figure 2.4: Case (i) of Proposition 2.4.

In the second case, $\langle A_1vB_1B_2 \rangle$ is a cone by Proposition 2.3. But by assumption, neither B_1 nor B_2 nor v can be an apex, hence A_1 is the apex. Then the subsegment xA_1 is contained in the triangle $A_1B_1B_2 \subseteq \mathcal{S}(X-v)$. Since $xA_1 \subseteq A_1v \subseteq \mathcal{S}(Y \cup v)$, it follows that the linear path from x to $A_1 \in \mathcal{S}(Y)$ is entirely contained within $\mathcal{S}(X-v) \cap \mathcal{S}(Y \cup v)$, as desired.

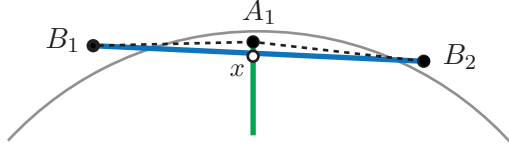


Figure 2.5: Case (ii) of Proposition 2.4.

□

2.3.2. Categorical tools

Let \mathbf{C} be a category. Let C_0, C_1, \dots, C_k be a collection of objects in \mathbf{C} , together with a collection of morphisms $c_i : C_0 \rightarrow C_i$ for $i \in \{1, \dots, k\}$. This can be represented by the following diagram:

$$\begin{array}{ccc} & C_0 & \\ c_1 \swarrow & & \searrow c_k \\ C_1 & \dots & C_k. \end{array} \quad (2.1)$$

Suppose we have another similar diagram,

$$\begin{array}{ccc} & C'_0 & \\ c'_1 \swarrow & & \searrow c'_k \\ C'_1 & \dots & C'_k. \end{array} \quad (2.2)$$

and a natural transformation between the diagrams 2.1 and 2.2, i.e. a collection of morphisms $p_i : C_i \rightarrow C'_i$, such that for each $i \in \{1, \dots, k\}$, the following diagram commutes:

$$\begin{array}{ccc} C_0 & \xrightarrow{c'_i} & C_i \\ \downarrow p_0 & & \downarrow p_i \\ C'_0 & \xrightarrow{c'_i} & C'_i. \end{array} \quad (2.3)$$

We say that the “vertical” morphisms $\{p_i\}_1^k$ allows us to “compare” the diagrams 2.1 and 2.2. Now suppose these two diagrams admit colimits. Let C be the colimit of the diagram 2.1, and denote d_i the coprojections into the colimit, $d_i : C_i \rightarrow C$, making the following diagram commute:

$$\begin{array}{ccccc}
& & C_0 & & \\
& c_1 \swarrow & & \searrow c_k & \\
C_1 & & \dots & & C_k \\
& d_1 \swarrow & & \searrow d_k & \\
& & C & &
\end{array} \tag{2.4}$$

We use corresponding notation for the diagram 2.2. Then by universal property of the colimit C , the collection of composed morphisms $\{c'_i p_i\}_i^k$ induce a unique morphism $p : C \rightarrow C'$ such that the following diagram commutes, for every $i, j \in \{1, \dots, k\}$:

$$\begin{array}{ccccc}
& & C_0 & \xrightarrow{c_j} & C_j \\
& c_i \swarrow & \downarrow & \searrow d_j & \\
C_i & \xrightarrow{d_i} & C & & \\
\downarrow p_i & \downarrow p_0 & \downarrow p & & \downarrow p_j \\
& & C'_0 & \xrightarrow{c'_j} & C'_j \\
& c'_i \swarrow & & \searrow d'_j & \\
C'_i & \xrightarrow{d'_i} & C' & &
\end{array}$$

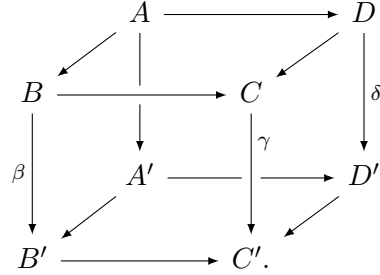
Proposition 2.5. In the above situation, suppose that p_i is an epimorphism for each $i \in \{1, \dots, k\}$. Then p is an epimorphism. \square

Proof. Let $D \in \mathbf{C}$ be any object and let $f, g : C' \rightarrow D$ be a pair of morphisms such that $fp = gp$. Precomposing with d_i yields $fpd_i = gpd_i$, so that $gd'_i p_i = fd'_i p_i$ by commutativity of the diagram. Then, since p_i is an epimorphism, we have

$$fd'_i = gd'_i. \tag{2.5}$$

Denote $e'_i = gd'_i$. Clearly $e'_i c'_i = e'_j c'_j$ for every i, j , hence by the universal property of the colimit C' , there is a unique $h : C' \rightarrow D$ with the property that $hd'_i = e'_i$ for every i . But by definition of e'_i and by Eq. (2.5), both f and g satisfy this property, hence $f = g$ by uniqueness of h , and we conclude that p is an epimorphism. \square

Next we state Proposition 2.5 for $k = 2$, which will be useful for applications of the Seifert–van Kampen theorem. We call a commutative diagram of the following form a *cube*:



For short, we refer to *faces* of the cube: front, back, top, bottom, left and right.

Corollary 2.6. Suppose the top and bottom faces of the above cube are pushout squares. If β and δ are epimorphisms, then γ is an epimorphism. \lrcorner

Remark 2.7. We will primarily be concerned with groups. Recall that in the category **Grp**, a homomorphism is an epimorphism if and only if it is surjective as a set map. \lrcorner

2.3.3. π_1 -surjectivity under point gluings

Let $f : A \rightarrow B$ be a map between path-connected topological spaces. For $i \in \{1, \dots, k\}$, let $a_i \in A$ be distinct, and $b_i = f(a_i) \in B$ be distinct. Denote \sim (resp. \sim') the equivalence relation on A (resp. B) generated by identifying every a_i (resp. b_i). Taking quotient maps induces a unique \tilde{f} such that the following diagram commutes:

$$\begin{array}{ccc}
A & \longrightarrow & A/\sim \\
f \downarrow & & \downarrow \tilde{f} \\
B & \longrightarrow & B/\sim'.
\end{array}$$

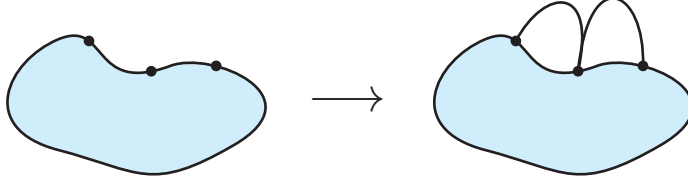
Proposition 2.8. In the above situation, if $\pi_1 f$ is surjective, then $\pi_1 \tilde{f}$ is surjective. \lrcorner

Proof. Let $0 = e_1 < \dots < e_k = 1$ be a partition of the unit interval I . Define attaching maps $\alpha : e_i \mapsto a_i$ and $\beta : e_i \mapsto b_i$, and define adjunction spaces by the following pushout squares:

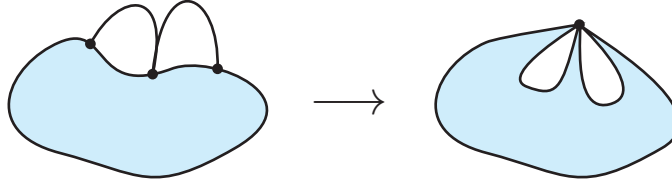
$$\begin{array}{ccc}
\{e_i\}_1^k & \xrightarrow{\alpha} & A \\
\downarrow & & \downarrow \\
I & \longrightarrow & A \sqcup_{\alpha} I,
\end{array}
\qquad
\begin{array}{ccc}
\{e_i\}_1^k & \xrightarrow{\beta} & B \\
\downarrow & & \downarrow \\
I & \longrightarrow & B \sqcup_{\beta} I.
\end{array}
\tag{2.6}$$

We denote γ (resp. γ') the image of I in $A \sqcup_{\alpha} I$ (resp. $B \sqcup_{\beta} I$), and so we write $A \cup \gamma = A \sqcup_{\alpha} I$ and $B \cup \gamma' = B \sqcup_{\beta} I$. The identity map $id : I \rightarrow I$

and the map f yield a canonical map $f \cup id : A \cup \gamma \rightarrow B \cup \gamma'$. Concretely, these spaces are obtained by attaching a 1-cell connecting every a_i (resp. b_i):



Note that γ, γ' are contractible, so contracting γ, γ' yields homotopy equivalences $A \cup \gamma \simeq A/\sim$ and $B \cup \gamma \simeq B/\sim'$, as illustrated below:



We then have the following commutative diagram:

$$\begin{array}{ccc} A \cup \gamma & \xrightarrow{\simeq} & A/\sim \\ f \cup id \downarrow & & \downarrow \tilde{f} \\ B \cup \gamma' & \xrightarrow{\simeq} & B/\sim'. \end{array}$$

We now use [NR93, Lemma 1.8]. The maps

$$(f \cup id)|_A = f, (f \cup id)|_\gamma = id, (f \cup id)|_{A \cap \gamma} = f \alpha$$

are all 1-connected, since A, B are path-connected and $\pi_1 f$ is surjective, and therefore $\pi_1 \tilde{f}$ is surjective. □

2.4. Surjectivity of $\pi_1 p$ in \mathbb{R}^2

A theorem by Chambers et al. [Cha+10, Theorem 3.1] states that the fundamental groups $\pi_1 \mathcal{R}$ and $\pi_1 \mathcal{S}$ are isomorphic. In particular, the fundamental group $\pi_1 \mathcal{R}$ is free, since \mathcal{S} is a wedge of circles. In the rest of this section, we prove the following weaker result:

Theorem 2.9. [Cha+10][Part of Theorem 3.1] Let X be a finite set of points in \mathbb{R}^2 . Let $\mathcal{R} = \mathcal{R}(X)$ be the Rips complex of X and let \mathcal{S} be its image under the projection map $p : \mathcal{R} \rightarrow \mathbb{R}^2$. Then the induced homomorphism $\pi_1 p : \pi_1 \mathcal{R} \rightarrow \pi_1 \mathcal{S}$ is surjective. ┘

Proof. We proceed by induction on the vertex set. Fix $v \in X$ and suppose the induced homomorphism $\pi_1 \mathcal{R}(X-v) \rightarrow \pi_1 \mathcal{S}(X-v)$ is surjective. We successively examine two cases:

- (i) $\mathcal{R}(X-v)$ is path-connected,
- (ii) $\mathcal{R}(X-v)$ has several path-components.

(i)

Suppose $\mathcal{R}(X-v)$ is path-connected. Let $\mathcal{R}(X'_1), \dots, \mathcal{R}(X'_k)$ denote the path components of $\mathcal{R}(X')$. Let $i \in \{1, \dots, k\}$ and define

$$\mathcal{R}_i = \mathcal{R}(X'_i \cup v) \cup \mathcal{R}(X-v).$$

Note that $\mathcal{R}(X'_i \cup v) \cap \mathcal{R}(X-v) = \mathcal{R}((X'_i \cup v) \cap (X-v)) = \mathcal{R}(X'_i)$. Thus we have the following pushout square of topological spaces:

$$\begin{array}{ccc} \mathcal{R}(X'_i) & \hookrightarrow & \mathcal{R}(X'_i \cup v) \\ \downarrow & & \downarrow \\ \mathcal{R}(X-v) & \hookrightarrow & \mathcal{R}_i. \end{array}$$

Correspondingly, in the shadow we have the following pushout square:

$$\begin{array}{ccc} \mathcal{S}(X-v) \cap \mathcal{S}(X'_i \cup v) & \hookrightarrow & \mathcal{S}(X'_i \cup v) \\ \downarrow & & \downarrow \\ \mathcal{S}(X-v) & \hookrightarrow & \mathcal{S}_i. \end{array}$$

By taking restrictions of $p : \mathcal{R} \rightarrow \mathcal{S}$ to subspaces, we thus obtain the following commutative cube in **Top**:

$$\begin{array}{ccccc} & & \mathcal{R}(X'_i) & \hookrightarrow & \mathcal{R}(X'_i \cup v) \\ & \swarrow & \downarrow & & \downarrow \\ \mathcal{R}(X-v) & \hookrightarrow & \mathcal{R}_i & & \mathcal{R}(X'_i \cup v) \\ \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{S}(X'_i) & & \mathcal{S}(X'_i \cup v) \\ & & \downarrow & & \downarrow \\ & & \mathcal{S}(X-v) \cap \mathcal{S}(X'_i \cup v) & \hookrightarrow & \mathcal{S}(X'_i \cup v) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}(X-v) & \hookrightarrow & \mathcal{S}_i & & \mathcal{S}(X'_i \cup v) \end{array}$$

Let us verify that the top and bottom faces satisfy the conditions of the Seifert–van Kampen theorem, where we use Remark 1.13.

The spaces $\mathcal{R}(X'_i \cup v)$ and $\mathcal{S}(X'_i \cup v)$ are cones with apex v , hence they are contractible, and in particular they are path connected. $\mathcal{R}(X-v)$ and $\mathcal{R}(X'_i)$ are path connected by assumption, hence so is \mathcal{R}_i . The projection map is continuous, hence the images of these spaces are also path connected. Finally, Proposition 2.4 ensures that the space $\mathcal{S}(X-v) \cap \mathcal{S}(X'_i \cup v)$ is path connected.

Thus, by the Seifert–van Kampen theorem, and functoriality of π_1 , we obtain the following commutative cube in **Grp**:

$$\begin{array}{ccccc}
& & \pi_1 \mathcal{R}(X'_i) & \xrightarrow{\quad} & 1 \\
& \swarrow & \downarrow & & \downarrow \\
\pi_1 \mathcal{R}(X-v) & \xrightarrow{\quad} & \pi_1 \mathcal{R}_i & & 1 \\
\downarrow & & \downarrow & \xrightarrow{\pi_1 p_i} & \downarrow \\
& & \pi_1(\mathcal{S}(X-v) \cap \mathcal{S}(X'_i \cup v)) & \xrightarrow{\quad} & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1 \mathcal{S}(X-v) & \xrightarrow{\quad} & \pi_1 \mathcal{S}_i & & 1
\end{array}$$

Here, 1 denotes the trivial group. By the induction hypothesis, the map $\pi_1 \mathcal{R}(X-v) \rightarrow \pi_1 \mathcal{S}(X-v)$ is surjective, hence by Corollary 2.6, the map

$$\pi_1 p_i : \pi_1 \mathcal{R}_i \rightarrow \pi_1 \mathcal{S}_i \quad (2.7)$$

is surjective. Now, for each i , there are inclusion maps $\mathcal{R}(X-v) \hookrightarrow \mathcal{R}_i$ and $\mathcal{S}(X-v) \hookrightarrow \mathcal{S}_i$. Hence we have the following two diagrams:

$$\begin{array}{ccc}
& \mathcal{R}(X-v) & \\
\swarrow & & \searrow \\
\mathcal{R}_1 & \dots & \mathcal{R}_k,
\end{array}
\quad
\begin{array}{ccc}
& \mathcal{S}(X-v) & \\
\swarrow & & \searrow \\
\mathcal{S}_1 & \dots & \mathcal{S}_k.
\end{array}
\quad (2.8)$$

Letting $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{S}}$ denote the respective colimits of these diagrams, we obtain:

$$\begin{array}{ccc}
& \mathcal{R}(X-v) & \\
\swarrow & & \searrow \\
\mathcal{R}_1 & \dots & \mathcal{R}_k \\
\searrow d_1 & & \swarrow d_k \\
& \tilde{\mathcal{R}} &
\end{array}
\quad
\begin{array}{ccc}
& \mathcal{S}(X-v) & \\
\swarrow & & \searrow \\
\mathcal{S}_1 & \dots & \mathcal{S}_k \\
\searrow d'_1 & & \swarrow d'_k \\
& \tilde{\mathcal{S}} &
\end{array}
\quad (2.9)$$

where d_i, d'_i denote the coprojections into the colimit. Now, the two previous diagrams in 2.8 can be compared by restrictions of the projection map, $p_i = p|_{\mathcal{R}_i}$. Note that for each i , there are inclusion maps such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}_i & \hookrightarrow & \mathcal{R} \\ p_i \downarrow & & \downarrow p \\ \mathcal{S}_i & \hookrightarrow & \mathcal{S}. \end{array} \quad (2.10)$$

Thus there exists unique maps q, q', \tilde{p} , such that the following diagram commutes, for every i, j :

$$\begin{array}{ccccc} & \mathcal{R}(X-v) & \longrightarrow & \mathcal{R}_j & \\ & \swarrow & & \searrow d_j & \\ \mathcal{R}_i & \xrightarrow{d_i} & \tilde{\mathcal{R}} & \xrightarrow{q} & \mathcal{R} \\ & \downarrow & \downarrow p & \downarrow p_j & \downarrow p \\ p_i \downarrow & \mathcal{S}(X-v) & \longrightarrow & \mathcal{S}_j & \\ & \swarrow & & \searrow d'_j & \\ \mathcal{S}_i & \xrightarrow{d'_i} & \tilde{\mathcal{S}} & \xrightarrow{q'} & \mathcal{S}. \end{array} \quad (2.11)$$

Let us now verify that the two diagrams in 2.9 satisfy conditions for the Seifert–van Kampen theorem. By definition, $\tilde{\mathcal{R}} = \bigcup \mathcal{R}_k$. Within the space $\tilde{\mathcal{R}}$, each pairwise intersection $\mathcal{R}_i \cap \mathcal{R}_j$ equals $\mathcal{R}(X-v)$, which is path connected, hence every possible intersection is connected. Analogous arguments apply in the shadow. Therefore, the Seifert–van Kampen theorem applies.

By functoriality of π_1 , we then obtain a diagram satisfying the conditions for Proposition 2.5, since every vertical homomorphism $\pi_1 p_i : \pi_1 \mathcal{R}_i \rightarrow \pi_1 \mathcal{S}_i$ is surjective, by Eq. (2.7). Therefore, we conclude that

$$\pi_1 \tilde{p} : \pi_1 \tilde{\mathcal{R}} \rightarrow \pi_1 \tilde{\mathcal{S}} \quad (2.12)$$

is surjective.

Let us denote $v_i = d_i$ and $v'_i = d'_i$ the images of v under the coprojections from diagrams 2.9. By definition of q from diagram 2.11, we have $q(v_i) = v$ and $q(x) = x$ for all $x \neq v_i$, for all i . That is, q simply identifies the different images of v . Let \sim denote the equivalence relation generated by this identification. Then we have

$$\mathcal{R} = \tilde{\mathcal{R}} / \sim.$$

Similarly, letting \sim' denote the identification of images v'_i under q' , we have:

$$\mathcal{S} = \tilde{\mathcal{S}} / \sim'.$$

Finally, by Proposition 2.8, we conclude that the homomorphism $\pi_1 p : \pi_1 \mathcal{R} \rightarrow \pi_1 \mathcal{S}$ is surjective.

(ii)

Now suppose $\mathcal{R}(X-v)$ is not path-connected. Denote $\mathcal{R}(X^1-v), \dots, \mathcal{R}(X^l-v)$ its path-components. We then have the decompositions $\mathcal{R}(X) = \bigcup_{i=1}^l \mathcal{R}(X^i)$ and $\mathcal{S}(X) = \bigcup_{i=1}^l \mathcal{S}(X^i)$. Let us verify that these decompositions satisfy the conditions for the Seifert–van Kampen theorem.

Note that each space $\mathcal{R}(X^i)$ is path-connected, since we assumed $\mathcal{R}(X)$ to be path-connected. By continuity of p , each space $\mathcal{S}(X^i)$ is path-connected. Thus the intersections $\bigcap_{i=1}^l \mathcal{R}(X^i)$ and $\bigcap_{i=1}^l \mathcal{S}(X^i)$ are intersections of path-connected spaces all containing v , hence they are path-connected, and the Seifert–van Kampen theorem applies.

By the previous case, the restriction of p to each $\mathcal{R}(X^i)$ induces a surjective homomorphism $\pi_1 \mathcal{R}(X^i) \rightarrow \pi_1 \mathcal{S}(X^i)$. It follows from Proposition 2.5 that $\pi_1 p : \mathcal{R}(X) \rightarrow \mathcal{S}(X)$ is surjective. □

2.5. The Rips complex for $n \geq 3$

An example by Chambers et al. [Cha+10, Proposition 5.3] demonstrates that Theorem 2.9 is false for $n \geq 4$. Specifically, our Proposition 2.4, which relied on examining the intersection of triangles, fails in this instance. Extending this proposition to the case $n = 3$ would require examining the intersection of tetrahedra. As Proposition 2.4 relied on the geometric observation that intersecting edges generate a cone (Proposition 2.3), a first step is to extend this observation. We speculate that this could be a statement of the following form: given a Rips triangle $[XYZ]$ and a Rips edge $[AB]$ whose images intersect in the shadow (see Fig. 2.6), the generated Rips complex $\langle ABXYZ \rangle$ is a cone.

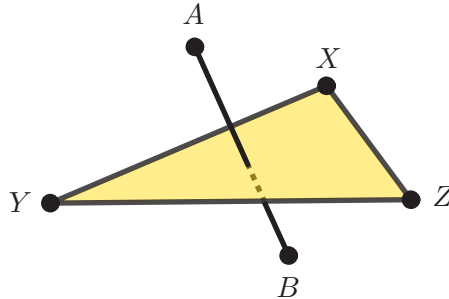


Figure 2.6

Bibliography

- [Cha+10] Erin W Chambers, Vin De Silva, Jeff Erickson, and Robert Ghrist. “Vietoris–Rips complexes of planar point sets”. In: *Discrete and Computational Geometry* 44.1 (2010), pp. 75–90.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.
- [Koz08] Dmitry Kozlov. *Combinatorial algebraic topology*. Vol. 21. Algorithms and Computation in Mathematics. Springer, Berlin, 2008, pp. xx+389. ISBN: 978-3-540-71961-8.
- [Lei14] Tom Leinster. *Basic category theory*. Vol. 143. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2014, pp. viii+183. ISBN: 978-1-107-04424-1.
- [Mac98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314. ISBN: 0-387-98403-8.
- [NR93] “Brick’s Quasi Simple Filtrations and 3-Manifolds”. In: *Geometric Group Theory*. Ed. by Graham A. Niblo and Martin A. Roller. Cambridge University Press, 1993, pp. 188–203.
- [Rot88] Joseph J. Rotman. *An introduction to algebraic topology*. Vol. 119. Graduate Texts in Mathematics. Springer-Verlag, New York, 1988, pp. xiv+433. ISBN: 0-387-96678-1.