FACULTY OF SCIENCE
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Bachelor Thesis in Mathematics

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Automorphisms of G

- with applications to extensions

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Abstract

In this project we will mainly be concerned with trying to understand cohomological obstructions to the extensions of maps and group extensions. We will start out by giving a review of some simplicial homotopy theory. After that we will discuss a structure known as crossed complexes. Then we will give an introduction to various kind of cohomology so we can introduce the concept of a postnikov tower of a Kan complex, and provide two canonical choices for postnikov towers. We will use the coskeleton postnikow tower to present a canonical cocycle which represents the postnikov invariant of a given space. We will then show how to interpret cohomology in terms of certain crossed complexes, and show how to obtain a crossed complex that represents the postnikov invariant of a given space. Finally we will relate it to group extensions. We shall show that for every group there is a canonical obstruction to all group extensions with that kernel being realizable. In addition we will show that this obstruction is the same as the postnikov invariant a certain space.

Resumé

Vi vil i dette projekt primært forsøge at forstå visse kohomologiske obstruktioner til udvidelser af afbildninger og gruppeudvidelser. Vi vil starte med at give en oversigt over noget simpliciel homotopy teori, hvorefter vi giver en introduktion til krydsede komplekser. Vi vil da introducere nogle forskellige former for kohomologi. Dette gør os i stand til at diskutere postnikov tårne af Kan komplekser, og vi vil i den forbindelse give to kanoniske valg af postnikov tårne. Vi vil anvende koskelet postnikov tårnet til at præsentere en kanonisk kocykel som repræsenterer postnikov invarianten af et givet rum. Endelig vil vi relatere til gruppeudvidelser. Vi vil vise at for enhver gruppe kan vi finde en kanonisk obstruktion til at alle gruppeudvidelser med denne kerne er realiserbare. Yderligere vil vi vise at denne obstruktion er den samme som postnikov invarianten af et specielt rum.

Contents

Chapter 1

Simplicial homotopy theory

In this chapter we shall go over some of the necessary background knowledge from simplicial homotopy theory that we will use. This chapter is not intended as a comprehensive reference. We will almost exclusively cover topics that we will be used in this Thesis and leave out many important basic topics. Furthermore we generally will not bother providing proofs and will instead refer to a standard text such as [8]. One exception is the section on $\pi_n(X, *)$ as a pointed homotopy class which as far as I know is not included in any standard reference, despite it being fairly useful.

1.1 Simplicial sets

Definition 1.1.1 (Simplex category)**.** *For every non-negative integer n we define $[n]$ to be the set $\{0, 1, \ldots, n\}$. The simplex category Δ is the category whose objects are all sets of the form $[n]$, and the morphisms $[n] \to [m]$ are the increasing functions $[n] \to [m]$ (increasing is meant weakly, so a constant function is for instance increasing).*

Definition 1.1.2 (Simplicial set)**.** *A simplicial set is a contravariant functor $X : \Delta^{op} \to \mathrm{Set}$. A morphism from a simplicial set X to a simplicial set Y is a natural transformation $X \Rightarrow Y$. Thus the category of simplicial sets* sSet *is simply* $\mathrm{Fun}(\Delta^{op}, \mathrm{Set})$.

We note that this definition allows a straightforward generalization to simplicial objects in an arbitrary category. This obvious generalization is as follows:

Definition 1.1.3 (Simplicial object)**.** *Let a category $\mathcal{C}$ be given. The category of simplicial objects in $\mathcal{C}$ is defined to be* $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ *and denoted $s\mathcal{C}$.*

Let us now be given a simplicial set $X : \Delta^{op} \to \mathrm{Set}$. We then get a collection of sets $X_0, X_1, \ldots$ given by

$$X_i = X([i]) \qquad \text{for } i \in \{0, 1, 2, \ldots\}$$

In addition for every increasing function $\varphi : [n] \to [m]$ we get an induced map $X(\varphi) : X_m \to X_n$. By functoriality we get that for all $i, j, k \in \{0, 1, \ldots\}$ and increasing maps $[i] \xrightarrow{\varphi} [j] \xrightarrow{\psi} [k]$ we have

$$X(1_{[i]}) = 1_{X_i}$$

$$X(\psi \circ \varphi) = X(\varphi) \circ X(\psi)$$

This is exactly functoriality, so alternatively we could say that a simplicial set is a collection of sets $X_0, X_1, \ldots$ and for all increasing functions $\varphi : [n] \to [m]$ an induced map $X_m \to X_n$ satisfying the functoriality conditions. In fact we shall usually think of simplicial sets in this manner.

Definition 1.1.4 (Coface, codegeneracy, face, degeneracy)**.** *Let a non-negative integer n be given. For all $i \in \{0, 1, \ldots, n+1\}$ we define $d^i : [n] \to [n+1]$ and $s^i : [n+1] \to [n]$ by*

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases} \qquad s^i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases}$$

d^i is known as the ith coface map, and s^i is known as the ith codegeneracy map.

Given a simplicial set X the maps $X(d^i) : X_{n+1} \to X_n$ and $X(s^i) : X_n \to X_{n+1}$ are known respectively as the ith face map and the ith degeneracy map.

Theorem 1.1.5 (Cosimplicial identities)**.** *For coface and codegeneracy maps the following identities hold for all i, j where the relevant coface and codegeneracy maps are defined*

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} && \text{if } i < j \\
s^j d^i &= d^i s^{j-1} && \text{if } i < j \\
s^j d^j &= s^j d^{j+1} = 1 && \\
s^j d^i &= d^{i-1} s^j && \text{if } i > j+1 \\
s^j s^i &= s^i s^{j+1} && \text{if } i \leq j
\end{aligned}$$

These identities are collectively known as the cosimplicial identities.

Definition 1.1.6 (Subcomplex)**.** *Let a simplicial set X be given. A subcomplex of X is a simplicial set Y such that*

$$Y_i \subset X_i \qquad \text{for all } i \in \{0, 1, \ldots\}$$

and for all increasing maps $\varphi : [n] \to [m]$ the map $X(\varphi) : X_m \to X_n$ restricts to the map $Y(\varphi) : Y_m \to Y_n$, i.e. the following diagram commutes

$$\begin{tikzcd}
Y_m \arrow[r, hook] \arrow[d, "Y(\varphi)"'] & X_m \arrow[d, "X(\varphi)"] \\
Y_n \arrow[r, hook] & X_n
\end{tikzcd}$$

We shall often write $Y \subset X$ if Y is a subcomplex of X.

Definition 1.1.7 (Standard n-simplex)**.** *For a given non-negative integer n we define the standard n-simplex Δ^n to be the simplicial set $\Delta(-, [n])$, i.e. Δ^n_m is the set of increasing maps $[m] \to [n]$, and given an increasing map $\psi : [m] \to [k]$ we get a map $\Delta^n_\psi : \Delta^n_k \to \Delta^n_m$ which sends an increasing map $\varphi : [k] \to [n]$ to the composition $\varphi \circ \psi : [m] \to [n]$.*

The boundary $\partial\Delta^n$ of the standard n-simplex is defined as the subcomplex of Δ^n for which $(\partial\Delta^n)_k$ is the non-surjective increasing maps $[k] \to [n]$.

For $j \in \{0, 1, \ldots, n\}$ we define the j-th horn Λ^n_j to be the subcomplex of Δ^n for which $(\Lambda^n_j)_k$ is the set of increasing maps $[k] \to [n]$ whose image does not contain $\{0, 1, \ldots, \hat{j}, \ldots, n\}$ where as usual $\hat{j}$ means that j is omitted.

We will often work quite concretely with these simplicial sets so it helps to introduce some helpful notation. A k-simplex of Δ^n is an increasing map $\sigma : [k] \to [n]$. When it is simpler we shall just write

$$\sigma(0)\sigma(1)\cdots\sigma(k)$$

So for instance $d^3 : [4] \to [5]$ is

$$d^3 = 01245$$

There is a bit of ambiguity with this notation as we don't mention the range of the function so for instance $id : [2] \to [2]$ and $d^3 : [2] \to [3]$ are both written 012. When this is an issue we shall be explicit about the range, but in most contexts it is either understood or not relevant. The most common use of this notation is for a X we have its vertices

$$X(0), X(1), \ldots$$

and its 1-simplices

$$X(01), X(12), \ldots$$

Many of our simplicial sets will have a distinguished vertex, but we rarely care about the vertex itself, but rather its generated subcomplex. If a simplicial set has a single distinguished vertex we usually denote it by $*$. An example of this notation would be

$$f(0) = *$$

Here 0 denotes the vertex 0 of Δ^n, but it also denotes its degeneracies s_0*, $s_0 s_0*$, etc.

1.2 Fibrations and Kan complexes

Definition 1.2.1 (Lifting property)**.** *Let maps $f : A \to B$ and $g : X \to Y$ of simplicial sets be given. Suppose that for every pair of maps $\alpha : A \to X$ and $\beta : B \to Y$ such that $g \circ \alpha = \beta \circ f$ there exists a map $F : B \to X$ for which $F \circ f = \alpha$ and $g \circ F = \beta$.*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "F"] & Y \end{tikzcd}$$

In this case we say that $f : A \to B$ has the left lifting property with respect to $g : X \to Y$, or alternatively that $g : X \to Y$ has the right lifting property with respect to $f : A \to B$.

Definition 1.2.2 (Fibration)**.** *A map $f : X \to Y$ of simplicial sets is known as a fibration if it has the right lifting property with respect to all inclusions $\Lambda^n_k \to \Delta^n$.*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Definition 1.2.3 (Kan complex)**.** *Let $*$ denote the terminal object in* sSet*. We say that a simplicial set X is a Kan complex if the unique map $X \to *$ is a fibration.*

Maps of the form $\Lambda^n_k \to \Delta^n$ are not the only maps which all fibrations have the right lifting property with respect to. The following theorem gives us a large class of maps which fibrations have the right lifting property with respect to.

Theorem 1.2.4. *Let $p : X \to Y$ be a fibration. p has the right lifting property with respect to inclusions of the form*

$$(\Lambda^n_k \times B) \cup (\Delta^n \times A) \to \Delta^n \times B \qquad \textit{for } 0 \le k \le n$$

where A is a subcomplex of B.

Proof. See corollary I.4.6 in [8]. $\square$

Definition 1.2.5 (Classifying space)**.** *Let G be a category. We then define the simplicial set BG to be the objects of G in dimension 0, and in dimension i it consists of i composable morphisms. We shall denote an i-simplex by $(g_1, \dots, g_n)$ where $g_1, \dots, g_n$ are morphisms that can be composed.*

1.3 Homotopy

We write $f : (X, A) \to (Y, B)$ when f is a simplicial map $X \to Y$ such that $f(A) \subset B$ where A and B are subcomplexes of X and Y respectively.

Definition 1.3.1 (Simplicial homotopy)**.** *Let K, X be simplicial sets, and let L be a subcomplex of K. Let maps $f, g : K \to X$ be given. A homotopy from f to g is a map $h : \Delta^1 \times K \to X$ such that*

$$\begin{tikzcd} K \arrow[d, "d^1"'] \arrow[dr, "f"] & \\ \Delta^1 \times K \arrow[r, "h"] & X \\ K \arrow[u, "d^0"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. We say that the homotopy is relative to L if there exists a map $\alpha : L \to X$ such that

$$\begin{tikzcd} \Delta^1 \times L \arrow[r, hook] \arrow[d, "pr"'] & \Delta^1 \times K \arrow[d, "h"] \\ L \arrow[r, "\alpha"'] & X \end{tikzcd}$$

commutes. If such a h exists we say that f is homotopic to g relative to L.

If X is a Kan complex then we denote by $[K, X;\ rel\ L]$ the set of simplicial maps $K \to X$ modulo the equivalence relation $f \sim g$ if and only if f is homotopic to g relative to L.

1.4 Homotopy groups

Definition 1.4.1 (Homotopy group)**.** *Let X be a Kan complex with a chosen vertex $* \in X_0$. We define the nth homotopy group $\pi_n(X, *)$ (for $n \geq 0$) to be the set of maps $(\Delta^n, \partial\Delta^n) \to (X, *)$ modulo the equivalence relation defined by letting two such maps be equal if they are homotopic relative to $\partial\Delta^n$.*

Given two elements represented by $f, g : (\Delta^n, \partial\Delta^n) \to (X, *)$ we can construct

$$h : \Lambda_n^{n+1} \to X$$

as $(*, *, \ldots, *, f, -, g)$. We can then extend h to $H : \Delta^{n+1} \to X$ and define the product of $[f]$ and $[g]$ to be $[d_n H]$.

Definition 1.4.2 (Weak equivalence)**.** *A map $f : X \to Y$ of Kan complexes is said to be a weak equivalence if the induced map*

$$f_* : \pi_n(X, *) \to \pi_n(Y, *)$$

is an isomorphism for all $n \geq 0$ and all choices of basepoint.

Proposition 1. *Let X be a Kan complex. A map $f : (\Delta^n, \partial\Delta^n) \to (X, *)$ is homotopic to the constant map relative to $\partial\Delta^n$ if and only if there exists a $(n+1)$-simplex x such that*

$$d_i x = \begin{cases} * & \text{if } i < n+1 \\ f & \text{if } i = n+1 \end{cases}$$

Proof. See lemma I.7.4 of [8]. □

1.5 $\pi_n(X, *)$ as pointed homotopy classes

In the previous section we defined homotopy groups as

$$\pi_n(X, *) = [(\Delta^n, \partial\Delta^n), (X, *) \text{ rel } \partial\Delta^n]$$

However from the homotopy theory of topological spaces we would expect to also be able to define the homotopy group as

$$[(\partial\Delta^{n+1}, 0), (X, *) \text{ rel } 0]$$

This alternative characterization of homotopy groups is not usually included in a standard account of simplicial homotopy theory, but it does hold.

We have a map $\Delta^n \to \partial\Delta^{n+1}$ which just maps onto the 0th face $d_0\Delta^n$. This induces a map

$$\frac{\Delta^n}{\partial\Delta^n} \to \frac{\partial\Delta^{n+1}}{\Lambda_0^{n+1}}$$

The inverse of this composed with the projection $\partial\Delta^{n+1} \to \frac{\partial\Delta^{n+1}}{\Lambda_0^{n+1}}$ gives us a map

$$\pi_n(X,*) = [(\Delta^n, \partial\Delta^n), (X,*) \text{ rel } \partial\Delta^n] \to [(\partial\Delta^{n+1}, 0), (X,*) \text{ rel } *]$$

It is clear that this is well-defined because given a homotopy

$$h : \Delta^1 \times \Delta^n \to X$$

relative to $\partial\Delta^n$ we get a homotopy $\Delta^1 \times d^0\Delta^n \to X$ which we can extend to all of $\partial\Delta^{n+1}$ simply by letting it be $*$ on Λ_0^{n+1}. This is well-defined because h maps $\Delta^1 \times d^0\partial\Delta^n$ to $*$.

Given a map $f : [n] \to [k]$ we say that it contains an element i if i is in the image of f.

Lemma 1.5.1. *Let X be a Kan complex with a vertex $* \in X_0$, and let n, k be integers satisfying $n > k \geq 0$. Let A_k be the subcomplex of $\partial\Delta^{n+1}$ generated by all the k-simplices of $\partial\Delta^{n+1}$ that contains 0.*

$$f : (\partial\Delta^{n+1}, A_k) \to (X,*)$$

is homotopic to a map $(\partial\Delta^{n+1}, A_{k+1}) \to (X,*)$ relative to A_k.

Proof. Let S be the collection of non-degenerate $(k+1)$-simplices of $\partial\Delta^{n+1}$ that contain 0. Let $\omega \in S$. Suppose $\omega : [k+1] \to [n+1]$ is not injective. Then there exists some $i \in [k+1]$ such that $\omega(i) = \omega(i+1)$, so we can write

$$\omega = \omega' \circ s^i$$

for some $\omega' : [k] \to [n+1]$. ω' is then a k-simplex and it contains 0 as

$$0 = \omega(0) = \omega'(s^i(0)) = \omega'(0)$$

But then ω is in A_k which is a contradiction. Thus ω must be injective.

For any j-simplex $\sigma \in \partial\Delta^{n+1}$ with $j < k$, we may construct a k-simplex $\sigma' : [k] \to [n+1]$ by

$$\sigma'(i) = \begin{cases} 0 & \text{if } i < k-j \\ \sigma(i-(k-j)) & \text{if } i \geq k-j \end{cases}$$

σ' is a k-simplex that contains 0, so σ' is in A_k and therefore $\sigma = d_0 \cdots d_0\sigma'$ is in A_k. Thus simplices of dimension less than k are in A_k whether or not they contain 0.

Choose some $\omega \in S$. $f(\omega)$ is a $(k+1)$-simplex of X and therefore can be considered as a map $\Delta^{k+1} \to X$. We claim that for an i-simplex $\varphi \in (\partial\Delta^{k+1})_i$ we have $f(\omega)(\varphi) = *$ if the image of φ contains less than k elements, or if it contains k-elements one of which is 0. We have

$$f(\omega)(\varphi) = \varphi^* f(\omega) = f(\omega \circ \varphi)$$

First suppose the image of φ contains k elements one of which is 0. Let l be the element left out. We can then write

$$\varphi = d_l \circ s \qquad \text{for some } s : [i] \to [k]$$

$\omega \circ d_l : [k] \to [n+1]$ is a k-simplex that contains 0 so it is in A_k, and therefore $\omega \circ d_l \circ s$ is in A_i which implies

$$f(\omega)(\varphi) = f(\omega \circ \varphi) = f(\omega \circ d_l \circ s) = *$$

Now assume instead that the image of φ contains $k-1$ or less elements. We can then find $l_1 < l_2$ in $[k+1]$ that are not in the image of φ. Then we can write φ as

$$\varphi = d_{l_2} \circ d_{l_1} \circ s \qquad \text{for some } s : [i] \to [k-1]$$

$\omega \circ d_{l_2} \circ d_{l_1} : [k-1] \to [n-1]$ is a $(k-1)$-simplex and therefore in A_{k-1}, so as before

$$f(\omega)(\varphi) = *$$

We claim $f(\omega)(\Lambda_0^{k+1}) = *$. Let $\varphi : [i] \to [k+1]$ be an i-simplex of Λ_0^{k+1}. Then the image of φ must either contain $k-1$ or less elements, or it contains k elements one of which is 0. Thus $f(\omega)(\varphi) = *$. Since φ was arbitrary this proves $f(\omega)(\Lambda_0^{k+1}) = *$.

We can therefore define

$$h_\omega : \partial\Delta^1 \times \Delta^{k+1} \cup \Delta^1 \times \Lambda_0^{k+1} \to X$$

by

$$h_\omega(0,t) = * \qquad \text{for } t \in (\Delta^{k+1})_i$$

$$h_\omega(1,t) = f(\omega)(t) \qquad \text{for } t \in (\Delta^{k+1})_i$$

$$h_\omega(s,t) = * \qquad \text{for } s \in (\Delta^1)_i \text{ and } t \in (\Lambda_0^{k+1})_i$$

X is Kan so this can be extended to a map $h'_\omega : \Delta^1 \times \Delta^{k+1} \to X$ by theorem 1.2.4. This is a homotopy from $*$ to $f(\omega)$ relative to Λ_0^{k+1}.

We may now define $H : \Delta^1 \times A_{k+1} \to X$ on i-simplices by

$$H(s,t) = h'_\omega(s,\varphi) \qquad \text{if } t = \varphi^*\omega \text{ for some } \varphi : [i] \to [k+1] \text{ and } \omega \in S$$

$$H(s,t) = * \qquad \text{if } t \text{ is in } A$$

To show that this is well-defined we must show that if $\omega \circ \varphi = \omega' \circ \psi$ for $\varphi, \psi : [i] \to [k+1]$, then

$$h'_\omega(s,\varphi) = h'_{\omega'}(s,\psi)$$

and that $h'_\omega(s,\varphi) = *$ if $\omega \circ \varphi$ is in A.

For the first case suppose $\omega \circ \varphi = \omega' \circ \psi$. ω and ω' are injective so φ is in Λ_0^{k+1} if and only if ψ is. If either is in Λ_0^{k+1} then we get

$$h'_\omega(s,\varphi) = * = h'_{\omega'}(s,\psi)$$

If not, then φ and ψ are either id or d^0. In either case this implies that ω and ω' are equal on $\{1,\dots,k+1\}$, but they are both 0 on 0 so this means $\omega = \omega'$. ω is injective so $\varphi = \psi$.

For the second case suppose $\omega \circ \varphi$ is in A. If $\omega \circ \varphi$ contains 0, then its image must have at most k elements. Hence φ contains 0 and its image contains at most k elements, so φ is in Λ_0^{k+1}. If φ does not contain 0, then its image must have at most $k-1$ elements, but then the image of φ must also contain at most $k-1$ elements. Hence φ is in Λ_0^{k+1}. In either case we get

$$h'_\omega(s,\varphi) = *$$

We can construct a map $H' : \Delta^1 \times A_{k+1} \cup \{1\} \times \partial\Delta^{n+1} \to X$ by

$$H'(s,t) = H(s,t) \qquad \text{if } s \text{ is in } \Delta^1 \text{ and } t \text{ is in } A_{k+1}$$

$$H'(1,t) = f(t) \qquad \text{if } t \text{ is in } \partial\Delta^{n+1}$$

This is well-defined if

$$H(1,t) = f(t) \qquad \text{for } t \in A_{k+1}$$

If $t \in A_k$, then $H(1,t) = *$ and $f(t) = *$. Suppose we can write $t = \omega \circ \varphi$ for some $\varphi : [i] \to [k+1]$ and some $\omega \in S$.

$$H(1,t) = h'_\omega(1,\varphi) = f(\omega)(\varphi) = f(\omega \circ \varphi) = f(t)$$

We can extend H' to a homotopy $\Delta^1 \times \partial\Delta^{n+1} \to X$ by theorem 1.2.4. This is a homotopy from f to a map that is $*$ on A_{k+1} relative to A_k. □

Lemma 1.5.2. *Let K be a Kan complex with vertex $* \in K_0$, and let $n \geq 0$. Every map $f : (\partial\Delta^{n+1}, 0) \to (X, *)$ is homotopic relative to 0 to a map $g : (\partial\Delta^{n+1}, 0) \to (X, *)$ which satisfies $g(\Lambda_0^{n+1}) = *$.*

Proof. In the previous lemma $A_0 = 0$. We can repeatedly apply that lemma to f to get a homotopy to a map $g : (\partial\Delta^{n+1}, 0) \to (X, *)$ relative to 0 which satisfies $g(\Lambda_0^{n+1}) = *$ since $A_n = \Lambda_0^{n+1}$. □

Theorem 1.5.3. *For a Kan complex X the map*

$$\pi_n(X, *) \to [(\partial\Delta^{n+1}, 0), (X, *) \textit{ rel } 0]$$

is a bijection.

Corollary 1.5.4. *Let X be a Kan complex. A map $f : \partial\Delta^n \to X$ is homotopic to a constant map relative to 0 if and only if there exists a map $F : \Delta^n \to X$ such that $F|\partial\Delta^n = f$.*

Proof. Suppose f is homotopic to a constant map relative to 0. Then we get a homotopy

$$h : \Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \to X$$

$$h(0,x) = * \quad h(1,x) = f(x) \qquad h(t,0) = f(0)$$

By theorem 1.2.4 we can extend this to a homotopy $H : \Delta^1 \times \Delta^n \to X$. Then

$$\Delta^n \xrightarrow{d^0} \Delta^1 \times \Delta^n \xrightarrow{H} X$$

is a map which is F on $\partial\Delta^n$.

Conversely suppose we have a map $F : \Delta^n \to X$ such that $F|\partial\Delta^n = f$. By the theorem we can find a homotopy

$$h : \Delta^1 \times \partial\Delta^{n+1} \to X$$

relative to 0 from f to a map $f' : \partial\Delta^{n+1} \to X$ which is $*$ outside Λ_0^{n+1}. By theorem 1.2.4 this extends to a homotopy

$$H : \Delta^1 \times \Delta^{n+1} \to X$$

from F to an extension of f' to Δ^{n+1}. By proposition 1 this means that $[f'] = 0$ as an element of $\pi_n(X, *)$, hence $[f] = 0$. □

1.6 Simplicial abelian group

Let us first recall that a simplicial abelian group is a functor $\Delta^{op} \to \mathrm{Ab}$, and the category of simplicial abelian groups is denoted sAb. The category of chain complexes concentrated in non-negative degrees is denoted Ch_+, from now on we shall use the term chain complex to refer to chain complexes concentrated in non-negative degrees. We will start by defining a functor $N : \mathrm{sAb} \to \mathrm{Ch}_+$. Let a simplicial abelian group A be given. A can be considered a chain complex with boundary operator $\partial : A_n \to A_{n-1}$ defined by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

We call this chain complex the Moore complex .

We let $(NA)_n$ be the abelian subgroup of A_n given by

$$(NA)_n = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1})$$

We give NA the induced boundary operator of the Moore complex, i.e. $\partial : NA_n \to NA_{n-1}$ is $\partial = (-1)^n d_n$. We call NA the normalized chain complex . N gives a functor $N : \mathrm{sAb} \to \mathrm{Ch}_+$.

Let us now define a functor $\Gamma : \mathrm{Ch}_+ \to \mathrm{sAb}$. Let a chain complex $(C_\bullet, \partial_\bullet)$ be given. For $\alpha : [m] \to [n]$ we get a map $\alpha^* : C_n \to C_m$ defined by

$$\alpha^* = \begin{cases} \partial : C_n \to C_{n-1} & \text{if } m = n-1 \text{ and } \alpha = d^n \\ id : C_n \to C_n & \text{if } m = n \text{ and } \alpha = id \\ 0 & \text{otherwise} \end{cases}$$

We define

$$(\Gamma C)_n = \bigoplus_{\sigma : [n] \twoheadrightarrow [k]} C_k$$

For $\theta : [m] \to [n]$ let us define $\theta^* : \Gamma C_n \to \Gamma C_m$ as follows: For a summand C_k indexed by $\sigma : [n] \twoheadrightarrow [k]$ form the composition

$$[m] \xrightarrow{\theta} [n] \xrightarrow{\sigma} [k]$$

and factor as a surjective map $\tau : [m] \to [m']$ followed by an injective map

$\psi : [m'] \to [k]$ (such a factorization is unique).

$$\begin{tikzcd}
 & {[n]} \arrow[dr, "\sigma"] & \\
{[m]} \arrow[ur, "\theta"] \arrow[dr, "\tau"'] & & {[k]} \\
 & {[m']} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Then we have

$$\begin{tikzcd}
C_k \arrow[r, "\psi^*"] \arrow[d, "in_\sigma"'] & C_{m'} \arrow[d, "in_\tau"] \\
\Gamma C_n & \Gamma C_m
\end{tikzcd}$$

Doing this for all summands we get a map $\theta^* : \Gamma C_n \to \Gamma C_m$.

Theorem 1.6.1. *Γ is a left adjoint to N. The unit $\epsilon : 1_{\mathrm{sSet}} \to \Gamma N$ induces isomorphisms $\epsilon_X : X \to \Gamma N X$. The inverse of the unit is given in dimension n as the map*

$$\bigoplus_{[n] \twoheadrightarrow [k]} NA_k \to A_n$$

which on the summand NA_k indexed by $\sigma : [n] \twoheadrightarrow [k]$ is

$$NA_k \xrightarrow{incl} A_k \xrightarrow{\sigma^*} A_n$$

Proof. See proposition III.2.2, corollary III.2.3 (Dold-Kan correspondence), and the preceeding discussion in [8]. $\square$

For a given abelian group A and integer $n \geq 0$ we use $A[n]$ to denote the chain complex that is A in degree n, and 0 in all other degrees. We use $A\langle n+1\rangle$ to denote the chain complex that in dimension $n+2, n+1, n, n-1$ looks like:

$$\cdots \to 0 \to A \xrightarrow{id} A \to 0 \to \cdots$$

Definition 1.6.2 (Eilenberg-MacLane space)**.** *We define the Eilenberg-MacLane space $K(A, n)$ to be*

$$K(A, n) = \Gamma A[n]$$

$$WK(A, n) = \Gamma A\langle n+1\rangle$$

There is a canonical fibration $WK(A, n) \to K(A, n+1)$ with fiber $K(A, n)$ which is induced by the obvious maps

$$1 \to A[n] \to A\langle n\rangle \to A[n+1] \to 1$$

1.7 Simplicial groups

A simplicial group is just a functor $\Delta^{op} \to \mathrm{Grp}$.

Theorem 1.7.1. *All simplicial groups are Kan complexes when considered as simplicial sets.*

Proof. See lemma I.3.4 of [8]. $\square$

Definition 1.7.2. *Let X be a Kan complex. We define $End(X)$ to be the simplicial monoid which in dimension n consists of the simplicial maps $\Delta^n \times X \to X$. The simplicial structure is inherited from the cosimplicial structure of $\Delta^\bullet$. We define the composition of $f, g : \Delta^n \times X \to X$ to be*

$$(fg)(t, x) = f(t, g(t, x)) \qquad \text{for } t \text{ in } \Delta^n \text{ and } x \text{ in } X$$

It is clear that this makes $End(X)$ a simplicial monoid. We let $aut(X)$ be the subcomplex of $End(X)$ consisting of all invertible elements.

Definition 1.7.3. *For a simplicial group G define the simplicial set $\overline{W}G$ to be*

$$\overline{W}G_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0$$

in dimension n. We define the simplicial structure as follows

$$d_n(g_{n-1}, \dots, g_0) = (g_{n-2}, \dots, g_0)$$

$$d_i(g_{n-1}, \dots, g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \dots, d_0(g_{n-i})g_{n-i-1}, g_{n-i-2}, \dots, g_0)$$

$$s_i(g_{n-1}, \dots, g_0) = (s_{i-1}g_{n-1}, s_{i-2}g_{n-2}, \dots, s_0 g_{n-i}, e, g_{n-i-1}, \dots, g_0)$$

$\overline{W}G$ is a connected Kan complex by corollary V.6.8 of [8].

Proposition 2. *For a group G we have*

$$\pi_i(\overline{W}aut(BG)) = \begin{cases} Aut(G) & \text{if } i = 1 \\ ZG & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. A 0-simplex of $aut(BG)$ is an isomorphism $BG \to BG$. Such a map is induced by an automorphism of groups. We claim that two such isomorphisms are homotopic if and only if they differ by an inner automorphism. Let $f, g : G \to G$ be isomorphisms such that

$$f(x) = z^{-1} g(x) z$$

for some $z \in G$. Define the 1-simplex $h : \Delta^1 \times BG \to BG$ by

$$h(00, x) = f(x)$$

$$h(11, x) = g(x)$$

$$h(01, x) = z f(x) = g(x) z$$

We have

$$h(001, (x, y)) = (f(x), g(y) z)$$

$$h(011,(x,y)) = (zf(x), g(y))$$
$$d_1h(001,(x,y)) = g(y)zf(x) = zf(y)f(x) = zf(yx) = h(01, yx)$$
$$d_1h(011,(x,y)) = g(y)zf(x) = zf(y)f(x) = z(fyx) = h(01, yx)$$

so h is well-defined and it is clearly invertible since we could just take the inverses of f and g. Thus $[f] = [g]$.

Now suppose we are given homotopic 0-simplices f, g with a homotopy $h : \delta^1 \times BG \to BG$. Let $z = h(01, e)$. Then

$$h(01, x) = d_1h(001,(x, e)) = h(01, e)h(00, x) = zf(x)$$
$$h(01, x) = d_1h(011,(e, x)) = h(11, x)h(01, e) = g(x)z$$

so f and g differ by an inner automorphism. This shows

$$\pi_0(aut(BG)) = Out(G)$$

Elements of $\pi_1(aut(BG))$ are represented by 1-simplices $f : \Delta^1 \times BG \to BG$ for which $d_ih = 1$. Thus $f(00, x) = f(11, x) = x$ which implies

$$f(01, e)x = f(01, e)f(00, x) = f(11, x)f(01, e) = xf(01, e)$$

Thus $f(01, e) \in ZG$. Suppose two such 1-simplices $f, g : \Delta^1 \times BG \to BG$ are homotopic relative to $\partial\Delta^1$. Then we can find a 2-simplex $h : \Delta^2 \times BG \to BG$ with

$$d_0h = id$$
$$d_1h = f$$
$$d_2h = g$$

Then we get

$$d_1h(012,(e, e)) = f(01, e) = h(12, e)h(01, e) = g(01, e)$$

which implies $f = g$. Thus

$$\pi_1(aut(BG)) = ZG$$

A k-simplex f for $k > 1$ must be trivial if $d_if = id$ for all i. Hence

$$\pi_k(aut(BG)) = 0 \qquad \text{for } k > 1$$

By lemma V.4.1 of [8] we have a fibration

$$WG \to \overline{W}G$$

with fiber G and with WG contractible. Thus from the long exact sequence of homotopy groups we get

$$0 = \pi_{n+1}(WG) \to \pi_{n+1}(\overline{W}G) \to \pi_n(G) \to \pi_n(WG) = 0$$

so $\pi_n(G) = \pi_{n+1}(\overline{W}G)$. The group structure is preserved. □

We will not investigate $\overline{W}aut(BG)$ much more as it would take us too far off course of the project, however it should be noted that it is a very useful space. In particular we can classify fibrations in terms of it. See section 6 of the notes [4] by Edward Curtis for more information. One interesting result is that $aut(BG)$-equivalence classes of principal $aut(BG)$ bundles with base B are in bijection with $[B, \overline{W}aut(BG)]$.

Chapter 2

Crossed complexes

In this chapter we will introduce crossed complexes. Crossed complexes are analogous to chain complexes, but they are non-abelian in low dimensions (≤ 2) which can work very well for topological applications where we often work with fundamental groups in dimension 1. In addition it includes an action of the 1-dimensional part on the rest.

We will start the chapter with a review of groupoids. This includes what it means for a groupoid to be totally disconnected and how groupoids act on each other. Following that we will introduce crossed modules over groupoids which are crossed complexes that are trivial above dimension 2. We will mostly use these as a stepping stone towards crossed complexes, but we will also use crossed modules in a couple of situations.

We will then introduce the notion of a free crossed resolution. We will here discuss a lifting theorem that we will use extensively, and show that lifts are unique up to homotopy. It turns out that all free crossed resolutions are homotopy equivalent, but we will construct a canonical free crossed resolution which can be very useful when working with concrete problems.

Following that we shall introduce the notion of a fundamental crossed complex. To every simplicial set X we associate its fundamental crossed complex which in many ways up to homotopy at least reflect the properties of X. We shall not investige fundamental crossed complexes very deeply. We will revist them in the next chapter when we have introduced the notion of a postnikov invariant and then show that under some assumptions the first non-trivial postnikov invariant of a space can be obtained from its fundamental crossed complex.

The material presented in this chapter is mostly based on the book [2].

2.1 Groupoids

A groupoid is a category G in which all morphisms are isomorphisms. For a groupoid G we let G_0 denote its object set and G_1 denote its morphisms. We get maps $s, t : G_1 \to G_0$ which on a morphism $f : x \to y$ is defined by $sf = x$ and $tf = x$. These maps are called source and target respectively. For objects $p, q \in G_0$ we let $G(p)$ be the automorphism group at p and $G(p, q)$ be the set of isomorphisms $p \to q$. A morphism of groupoids is just a functor.

We call the groupoid G totally disconnected if all morphisms in G are automorphisms. We think of a totally disconnected groupoid as a collection of groups indexed by the object set. It may seem that with such structure one could just as well consider a collection of groups, but usually we have another groupoid which acts on it. A common example of this is that for a Kan pair (X, A) we may form the homotopy group $\pi_n(X, A, *)$ for a chosen basepoint $*$, but we may also form the homotopy groupoid $\pi_n(X, A)$ which is the collection of all such groups for all vertices of A. Then the fundamental groupoid $\pi_f(A)$ acts on $\pi_n(X, A)$ in the sense that for a homotopy class of $p : x \to y$ we get a group morphism

$$p_* : \pi_n(X, A, x) \to \pi_n(X, A, y)$$

Let P, M be given groupoids with M totally disconnected and with equal object sets. We say the P acts on M if for vertices $p, q, r \in P_0$, $x \in M(p)$ and $a \in P(p, q)$ we have an element ${}^a x \in M(q)$ such that for all $b : q \to r$, $a : p \to q$ in P and $x, y \in M(p)$ we have

1. ${}^{ba}x = {}^b({}^a x)$
2. ${}^1 x = x$
3. ${}^a(xy) = {}^a x {}^a y$.

In that case we call M a P-groupoid.

Definition 2.1.1 (Module over groupoid)**.** *We call a totally disconnected P-groupoid M a module over P if all the automorphism groups groups $M(x)$ are abelian.*

For a groupoid P we shall always think of P as acting on its own autmorphism groups by conjugation in the sense that give $a : x \to y$ and $b : x \to x$ we define

$${}^a b = aba^{-1}$$

We call a groupoid morphism $f : M \to N$ a P-groupoid morphism if we have a P-action on both M and N and f preserves this action.

Definition 2.1.2 (P-invariant subgroup)**.** *Let a P-group M be given. A subgroup N of M is called P-invariant if for all $n \in N$ and $p \in P$ we have ${}^p n \in N$.*

Theorem 2.1.3. *Let M be a P-group with a normal subgroup N. The quotient M/N is a P-group with the group action induced by M if and only if N is P-invariant.*

Definition 2.1.4 (Free groupoid)**.** *Let a directed graph R be given. We define G to have the same vertices as R. The morphisms of G are formal compositions of directed edges in R that match up, i.e. sequences of the form*

$$e_n e_{n-1} \cdots e_1$$

where $t(e_i) = s(e_{i+1})$.

2.2 Crossed modules

Definition 2.2.1 ((Pre)crossed module)**.** *A precrossed module is a groupoid P, a totally disconnected P-groupoid M and a P-groupoid morphism $\mu : M \to P$.*

A precrossed module $\mu : M \to P$ is known as a crossed module if

$$^{\mu(m)}m' = mm'm^{-1} \qquad \text{for all } m, m' \in M(x)$$

This identity is known as the Peiffer identity.

A morphism $(\mu : M \to P) \to (\mu' : M' \to P')$ of (pre)crossed modules consists of P-groupoid morphisms $f : M \to M'$ and $g : P \to P'$ such that

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "f"'] & P \arrow[d, "g"] \\ M' \arrow[r, "\mu'"'] & P' \end{tikzcd}$$

commutes.

This gives us the category XMod *of crossed modules as well as the full subcategory* PXMod *of precrossed modules.*

Let us now show how to obtain a crossed module from a precrossed module. Let a totally disconnected P-groupoid M be given. For $m, n \in M(x)$ we define their Peiffer commutator by

$$[[m, n]] = mn^{-1}m^{-1\mu m}n$$

Let $[[M(x), M(x)]]$ be the subgroup of $M(x)$ generated by all the Peiffer commutators.

Lemma 2.2.2. *Let $M(x)$ be a P-group. Then $[[M(x), M(x)]]$ is a P-invariant normal subgroup of M.*

Proof. We note that since μ is equivariant we have

$$p\mu(m) = p\mu(m)p^{-1}p = {}^{p}\mu(m)p = \mu\left({}^{p}m\right)p$$

so

$$\begin{aligned} {}^{p}[[m, n]] &= \left({}^{p}m\right)\left({}^{p}n^{-1}\right)\left({}^{p}m^{-1}\right)\left({}^{p\mu m}n\right) \\ &= \left({}^{p}m\right)\left({}^{p}n^{-1}\right)\left({}^{p}m^{-1}\right)\left({}^{\mu({}^{p}m)p}n\right) \\ &= [[{}^{p}m, {}^{p}n]] \end{aligned}$$

Therefore $[[M, M]]$ is P-invariant.

For all $m, n, n' \in M$ we have

$$\begin{aligned} l[[m, n]]l^{-1} &= lmn^{-1}m^{-1\mu m}nl^{-1} \\ &= lmn^{-1}\left(m^{-1}l^{-1}\left({}^{\mu(lm)}n\right)\left({}^{\mu(lm)}n^{-1}\right)lm\right)m^{-1\mu m}nl^{-1} \\ &= [[lm, n]]\left({}^{\mu(l)\mu(m)}n^{-1}\right)l^{\mu m}nl^{-1} \\ &= [[lm, n]][[l, {}^{\mu m}n]]^{-1} \end{aligned}$$

so $[[M(x), M(x)]]$ is a normal subgroup of $M(x)$. $\square$

Now suppose we are given a precrossed module

$$\mu : M \to P$$

By modding out by the Peiffer commutators we get a precrossed module

$$\mu^{cr} : \frac{M}{[[M, M]]} \to P$$

μ^{cr} maps Peiffer commutators to 1, and therefore satisfies the Peiffer identity. Thus μ^{cr} is actually a crossed module associated to the precrossed module μ.

Definition 2.2.3 (Free crossed module)**.** *Let P be a groupoid and let $f : R \to P$ be a function from a set R. We define*

$$F(f) : M \to P$$

by for $x \in P_0$ letting $M(x)$ be the free group on pairs (p, r) with $r \in R$ and $p : f(r) \to x$. We define the action of P on M by

$$g(p, r) = (gp, r) \qquad \text{for } r \in R,\ p : f(r) \to x,\ q : x \to y$$

Then $F(f)$ is a precrossed module. Its associated crossed module

$$F(f)^{cr} : M^{cr} \to P$$

is known as the free crossed module generated by f.

We note that in this definition we do not require P to be free.

2.3 Crossed complexes

Definition 2.3.1 (Crossed complex)**.** *A crossed complex is a sequence,*

$$\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1$$

where C_1 is a groupoid, C_2 is a totally disconnected C_1-groupoid, C_i is a C_1-module for $i > 2$, and all the arrows are C_1-groupoid homomorphisms, subject to the following axioms

- *$\delta_2 : C_2 \to C_1$ is a crossed module.*
- *$Im(\delta_2)$ acts trivially on C_i for $i > 2$.*
- *$\delta_i \circ \delta_{i+1} = 0$ for $i \geq 2$.*

We shall sometimes denote such a crossed complex by $(C_\bullet, \delta_\bullet)$ or even just C.

A morphism $(f_\bullet) : (C_\bullet, \delta_\bullet) \to (D_\bullet, \delta'_\bullet)$ is a collection of C_1-groupoid morphisms $f_i : C_i \to D_i$ $(i = 1, 2, \ldots)$ for which

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "f_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & D_3 \arrow[r, "\delta'_3"'] & D_2 \arrow[r, "\delta'_2"'] & D_1
\end{tikzcd}$$

commutes.

The resulting category is denoted Crs *(composition is the obvious choice, and it's clear that the axioms for a category are satisfied).*

Definition 2.3.2 (Fundamental groupoid). *Let a crossed complex $(C_\bullet, \delta_\bullet)$ be given. We define its fundamental groupoid to be*

$$\pi_1(C) = \frac{C_1}{Im(\delta_2)} = Coker(\delta_2)$$

For this definition to make sense we must have that $Im(\delta_2)$ is a normal subgroupoid of C_1. This is clear since for $m \in C_2(x)$ and $c \in C_1(x, y)$ we have

$$c\delta_2(m)c^{-1} = {}^c\delta_2(m) = \delta_2\left({}^c m\right) \in Im(\delta_2)$$

Proposition 3. *Let $C_\bullet$ be a crossed complex. The action of C_1 on C_i induces a $\pi_1(C_\bullet)$-module structure on C_i for $i > 2$.*

Proof. For $c \in C_1$ let $[c]$ denote its congruence class in $\pi_1(C_\bullet)$.

It suffices to prove that for $i > 1$, $c, c' \in C_1(x, y)$ and $m \in C_i(x)$ we have

$${}^c m = {}^{c'} m \qquad \text{if } [c] = [c']$$

If $[c] = [c']$, then there exists $z \in Im(C_2)$ for which $c = zc'z^{-1}$. $Im(\delta_2)$ acts trivially on C_i so

$${}^c m = {}^{zc'z^{-1}} m = {}^z({}^{c'} m) = {}^{c'} m$$

□

Definition 2.3.3 (Homology). *Let C be a given crossed complex. We define the nth homology groupoid for $n \geq 2$ to be*

$$H_n(C) = \frac{\ker(\delta_n : C_n \to C_{n-1})}{Im(\delta_{n+1} : C_{n+1} \to C_n)}$$

for a vertex $p \in C_0$ we let $H_n(C, p)$ be the corresponding group of $H_n(C)$. We note that these groupoids are totally disconnected so essentially it is just a collection of homotopy groups.

2.4 Free crossed resolutions

Definition 2.4.1 (Free crossed complex). *A crossed complex*

$$\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1$$

is called free if

- *C_1 is a free groupoid.*
- *$\delta_2 : C_2 \to C_1$ is a free crossed module.*
- *C_n is a free $\pi_1 C$-module.*

Definition 2.4.2 (Crossed resolution). *For a groupoid G a crossed resolution of G is a crossed complex $(C_\bullet, \delta_\bullet)$ and a groupoid homomorphism $\varphi : C_1 \to G$, such that*

$$\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\varphi} G \to 1$$

is exact.

In this case we see that G is isomorphic to πC.

Example 1. *Let a crossed module $M \xrightarrow{\mu} C$ be given. Define,*

$$A = ker(\mu) \qquad G = \frac{C}{Im(\mu)}$$

Then

$$0 \to A \to M \xrightarrow{\mu} C \to G \to 1$$

is an exact sequence. The action on C on M restricts to an action of C on A. For $a, b \in A$ we have

$$a = {}^{\mu b}a = bab^{-1}$$

so A is abelian. Thus the action gives A the structure of a C-module. In addition by the Peiffer identity we get

$$^{\mu(m)}a = mam^{-1} = mm^{-1}a = a \qquad \textit{for all } m \in M,\, a \in A$$

so

$$0 \to A \to M \xrightarrow{\mu} C \to G \to 1$$

is a crossed resolution of G.

In the following theorems we shall construct lifts by mapping out of free structures. From the proof it is clear that all the choices of where to send the generators are arbitrary so if we want some elements send a specific place, then we can.

Theorem 2.4.3. *Let $(C_\bullet, \delta_\bullet)$ be a free crossed resolution of G with $\varphi : C_1 \to G$, and let $(D_\bullet, \delta'_\bullet)$ be a crossed resolution of H with $\psi : D_1 \to H$. Given a map $f_0 : G \to H$ we can construct a morphism $f_\bullet : C_\bullet \to D_\bullet$.*

Proof. C_1 is a free groupoid and therefore freely generated by some subgraph $S_1 \subset C_1$. For $s : x \to y$ in S_1 choose $f'(s) \in D_1(x, y)$ such that $\psi(f'(s)) = f_0(\varphi(s))$ which is possible since ψ is surjective. By the universal property of free groups this induces a map $f_1 : C_1 \to D_1$ such that

$$\begin{tikzcd} C_1 \arrow[r, "\varphi"] \arrow[d, dashed, "f_1"'] & G \arrow[d, "f_0"] \\ D_1 \arrow[r, "\psi"'] & H \end{tikzcd}$$

commutes.

For $s \in C_2$ we have

$$\psi(f_1(\delta_2(s))) = f_0(\varphi(\delta_2(s)) = f_0(1) = 1$$

so $f_1(\delta_2(s)) \in ker(\psi) = \delta'_2(D_2)$. Thus we can choose $f'_2(s) \in D_2$ for which $\delta'_2(f'_2(s)) = f_1(\delta_2(s))$. By the universal property of free C_1-crossed modules we get a C_1-groupoid morphism $f_2 : C_2 \to D_2$ such that

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d, dashed, "f_1"'] & C_1 \arrow[r, "\varphi"] \arrow[d, "f_1"] & G \arrow[d, "f_0"] \\ D_2 \arrow[r, "\delta'_2"'] & D_1 \arrow[r, "\psi"'] & H \end{tikzcd}$$

commutes.

Now suppose $f_k, f_{k-1}, \dots, f_1$ have been defined such that

$$\begin{tikzcd} C_k \arrow[r, "\delta_k"] \arrow[d, "f_k"'] & \cdots \arrow[r, "\delta_2"] & C_1 \arrow[r, "\varphi"] \arrow[d, "f_1"'] & G \arrow[d, "f_0"] \\ D_k \arrow[r, "\delta'_k"] & \cdots \arrow[r, "\delta'_2"] & D_1 \arrow[r, "\psi"'] & H \end{tikzcd}$$

as before we may then define a map $f_{k+1} : C_{k+1} \to D_{k+1}$ such that

$$\begin{tikzcd} C_{k+1} \arrow[r, "\delta_{k+1}"] \arrow[d, dashed, "f_{k+1}"'] & C_k \arrow[r, "\delta_k"] \arrow[d, "f_k"'] & \cdots \arrow[r, "\delta_2"] & C_1 \arrow[r, "\varphi"] \arrow[d, "f_1"'] & G \arrow[d, "f_0"] \\ D_{k+1} \arrow[r, "\delta'_{k+1}"'] & D_k \arrow[r, "\delta'_k"] & \cdots \arrow[r, "\delta'_2"] & D_1 \arrow[r, "\psi"'] & H \end{tikzcd}$$

commutes by the freeness of C_{k+1} and the exactness at D_k of the bottom row. $\square$

Theorem 2.4.4. *Let F be a free crossed resolution of a groupoid G, and let E be a crossed resolution of the groupoid H. Let two lifts $f, f' : F \to E$ be given of the map $g : G \to H$. Then there exists a sequence of maps $h_i : F_i \to E_{i+1}$ for $i \geq 0$ (we write $F_0 = G$) such that*

$$\delta_{n+2}(h_{n+1}(x)) = f_{n+1}(x)f'_{n+1}(x)^{-1}h_n(\delta_{n+1}x)^{-1} \qquad \textit{for all } n \geq 0,\ x \in F_n$$

$h_0 : G \to E_1$ maps $x \in G$ to the identity of the vertex $g(x)$.

Proof.

$$\psi(f_1(x)f'_1(x)^{-1}) = 1$$

for all $x \in X$ so we can choose $h_1(x) \in E_2$ such that

$$\delta_2(h_1(x)) = f_1(x)f'_1(x)^{-1}$$

Since F_1 is free this gives us a map $h_1 : F_1 \to E_2$ that satisfies this. For all $x \in F_2$ we have

$$\delta_2(f_2(x)f'_2(x)^{-1}h_1(\delta_2 x)^{-1}) = \delta_2(f_2(x)f'_2(x)^{-1})f'_1(\delta_2 x)f_1(\delta_2 x)^{-1} = 1$$

so as before we can form $h_2 : F_2 \to E_3$ such that

$$\delta_3(h_2(x)) = f_2(x)f'_2(x)^{-1}h_1(\delta_2 x)^{-1}$$

for all $x \in F_2$. Now suppose we have constructed all h_i up to h_n. Then

$$\begin{aligned} &\delta_{n+1}(f_{n+1}(x)f'_{n+1}(x)^{-1}h_n(\delta_{n+1}x)^{-1}) \\ &= f_n(\delta_{n+1}x)f'_n(\delta_{n+1}x)^{-1}h_{n-1}(\delta_n\delta_{n+1}x)f'_n(\delta_{n+1}x)f_n(\delta_{n+1}x)^{-1} \\ &= 0 \end{aligned}$$

where in the last step we used that E_n is abelian for $n > 2$ and $\delta_n\delta_{n+1} = 0$. Hence we can construct $h_{n+1} : F_{n+1} \to E_{n+2}$ such that

$$h_{n+1}(x) = f_{n+1}(x)f'_{n+1}(x)^{-1}h_n(\delta_{n+1}x)^{-1}$$

for all $x \in F_{n+1}$. By induction this finishes the proof. $\square$

Remark 1. *The map h just constructed is a homotopy from f to f'. We will not bother to define homotopies in general for crossed complexes, as we will not use any deep homotopical results about crossed complexes. We will however note that an immediate corollary of this theorem is that free resolutions of G are homotopy equivalent. The interested reader can see part 2 of [2] which has a comprehensive introduction to the homotopy category of crossed complexes.*

2.5 The standard free resolution

Definition 2.5.1 (Standard free resolution)**.** *Given a group G the standard free resolution $F(G)$ is defined by letting $F(G)_1$ be the free group on the set of symbols $[g]$ for $g \in G$. $F(G)_2$ is the free G-group on the set of symbols $[g_1, g_2]$ for $g_1, g_2 \in G$. $F(G)_n$ ($n > 2$) is the free G-module on the set of symbols $[g_1, \ldots, g_n]$ for $g_1, \ldots, g_n \in G$. The differentials are defined by*

$$\varphi : F(G)_1 \to G \qquad [g] \mapsto g$$

$$\delta_2([g_1, g_2]) = [g_1][g_2][g_1g_2]^{-1}$$

$$\delta_3([g_1, g_2, g_3]) = g_1[g_2, g_3][g_1, g_2g_3][g_1g_2, g_3]^{-1}[g_1, g_2]^{-1}$$

$$\begin{aligned}\delta_n([g_1, \ldots, g_n]) = g_1[g_2, \ldots, g_n] + (-1)^n[g_1, \ldots, g_{n-1}] \\ + \sum_{i=1}^{n-1}(-1)^i[g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_n]\end{aligned}$$

Let us verify that the standard free resolution is actually a crossed resolution.

$$(\varphi \circ \delta_2)([g_1, g_2]) = g_1g_2(g_1g_2)^{-1} = 1$$

$$\begin{aligned}&(\delta_2 \circ \delta_3)([g_1, g_2, g_3]) \\ &= [g_1]\delta_2[g_2, g_3][g_1]^{-1}\delta_2[g_1, g_2g_3]\delta_2[g_1g_2, g_3]^{-1}\delta_2[g_1, g_2]^{-1} \\ &= [g_1][g_2][g_3][g_2g_3]^{-1}[g_1]^{-1}[g_1][g_2g_3][g_1g_2g_3]^{-1}\delta_2[g_1g_2, g_3]^{-1}\delta_2[g_1, g_2]^{-1} \\ &= [g_1][g_2][g_3][g_1g_2g_3]^{-1}[g_1g_2g_3][g_3]^{-1}[g_1g_2]^{-1}\delta_2[g_1, g_2]^{-1} \\ &= [g_1][g_2][g_1g_2]^{-1}\delta_2[g_1, g_2]^{-1} \\ &= [g_1][g_2][g_1g_2]^{-1}[g_1g_2][g_2]^{-1}[g_1]^{-1} = 1\end{aligned}$$

By the Peiffer identity we then get

$$d_3([x, y, z])[w, t]d_3([x, y, z])^{-1} = d_2(d_3([x, y, z]))[w, t] = [w, t]$$

which shows that the image of d_3 lies in the center of $F(G)_3$.

To show that $\delta_3 \circ \delta_4 = 1$ we shall use a trick from the article [7] of Eilenberg and MacLane. For $x, y, z, w \in G$ define

$$J = {}^{[x][y]}[z, w]{}^{[x]}[y, zw][x, yzw]$$

We now calculate J in two different ways. Let

$$H = [x, y][xy, z][xyz, w]$$

From the definition of δ_3 we get

$$\delta_3([a,b,c])[a,b][ab,c] = {}^{[a]}[b,c][a,bc]$$

Applying this and the fact that $\delta_3([a,b,c])$ is in the center of $F(G)_3$ we get

$$\begin{aligned}
J &= {}^{[x][y]}[z,w]{}^{[x]}[y,zw][x,yzw] \\
&= {}^{[x]}({}^{[y]}[z,t][y,zw])[x,yzw] \\
&= {}^{[x]}(\delta_3([y,z,w])[y,z][yz,w])[x,yzw] \\
&= {}^{[x]}\delta_3([y,z,w]){}^{[x]}[y,z]{}^{[x]}[yz,w][x,yzw] \\
&= {}^{[x]}\delta_3([y,z,w]){}^{[x]}[y,z]\delta_3([x,yz,w])[x,yz][xyz,w] \\
&= {}^{[x]}\delta_3([y,z,w])\delta_3([x,yz,w]){}^{[x]}[y,z][x,yz][xyz,w] \\
&= {}^{[x]}\delta_3([y,z,w])\delta_3([x,yz,w])\delta_3([x,y,z])H
\end{aligned}$$

We note also that δ_2 may be written as

$$[x][y] = d_2([x,y])[x,y]$$

and that one of the axioms of a crossed module state

$${}^{\delta_2([x,y])}[z,w] = [x,y][z,w][x,y]^{-1}$$

Thus we may calculate

$$\begin{aligned}
J &= {}^{[x][y]}[z,w]{}^{[x]}[y,zw][x,yzw] \\
&= {}^{\delta_2([x,y])[xy]}[z,w]{}^{[x]}[y,zw][x,yzw] \\
&= [x,y]{}^{[xy]}[z,w][x,y]^{-1}{}^{[x]}[y,zw][x,yzw] \\
&= [x,y]{}^{[xy]}[z,w][x,y]^{-1}\delta_3([x,y,zw])[x,y][xy,zw] \\
&= \delta_3([x,y,zw])[x,y]{}^{[xy]}[z,w][xy,zw] \\
&= \delta_3([x,y,zw])[x,y]\delta_3([xy,z,w])[xy,z][xyz,w] \\
&= \delta_3([x,y,zw])\delta_3([xy,z,w])H
\end{aligned}$$

Setting the two expressions of J equal to each other and cancelling out H we then get

$$\begin{aligned}
0 &= {}^{[x]}\delta_3([y,z,w]) - \delta_3([xy,z,w]) + \delta_3([x,yz,w]) - \delta_3([x,y,zw]) + \delta_3([x,y,z]) \\
&= \delta_3(\delta_4([x,y,z,w]))
\end{aligned}$$

Since $x,y,z,w \in G$ were arbitrary and elements of the form $[x,y,z,w]$ generate $F(G)_4$ we get $\delta_3 \circ \delta_4 = 0$. Finally we need to show $\delta_n \circ \delta_{n+1} = 0$ for $n > 3$. When $n > 3$ this is the exact same formula as the bar resolution in homological algebra, so we will not reproduce one here. Compared to the mess of the non-abelian, low-dimensional behavior the higher dimensions are fairly easy to deal with.

2.6 Fundamental crossed complex

Definition 2.6.1 (Fundamental crossed complex)**.** *Let a simplicial set X be given. We define the fundamental crossed complex ΠX as follows:*

1. *Let* $(\Pi X)_0 = X_0$.
2. *Let* $(\Pi X)_1$ *be the free groupoid generated by the directed graph* X_1.
3. *Let* $(\Pi X)_2 \to (\Pi X)_1$ *be the free crossed module generated by the function* $\delta_2 : X_2 \to (\Pi X)_1$ *defined by,*

$$\delta_2(x) = [d_2(x)][d_0(x)][d_1(x)]^{-1} \qquad \text{for all } x \in X_2$$

4. *For* $k > 2$ *let* $(\Pi X)_k$ *be the free* $(\Pi X)_1$*-module generated by* X_k. *The boundary* $(\Pi X)_k \to (\Pi X)_{k-1}$ *is the map induced by* $\delta_k : X_k \to (\Pi X)_{k-1}$ *defined by*

$$\delta_k(x) = \begin{cases} {}^{[p^*x]}[d_0x][d_2x][d_1x]^{-1}[d_3x]^{-1} & \text{if } k = 3 \\ {}^{[p^*x]}[d_0x] + \sum_{i=1}^{k}(-1)^i[d_kx] & \text{if } k > 3 \end{cases}$$

   where $p : [1] \to [k]$ *is the inclusion.*

This defines a functor $\Pi : \mathrm{sSet} \to \mathrm{Crs}$.

The proof that we actually have $\delta_n \circ \delta_{n+1} = 0$ goes as for free crossed resolutions.

Proposition 4. *Let* X *be a Kan complex. There is a natural isomorphism of groupoids*

$$\theta : \pi_n(X) \to H_n(\Pi X)$$

for $n \geq 2$.

Proof. This is proposition 2.6 in [3]. □

Chapter 3

Cohomology and invariants

In this chapter we shall discuss various kinds of cohomology. We introduce group cohomology as well as the equivariant cohomology of simplicial sets. We also show that certain crossed complexes can be used to determine cohomology. Our coverage of group cohomology and equivariant cohomology are only covered because we need them. We will not discuss them more deeply than necessary and most proofs will be omitted in favor of references. The main objective of this chapter is to define postnikov invariants, construct a canonical postnikov invariant and show how to get a postnikov invariant from the fundamental crossed complex of a simplicial set.

3.1 Group cohomology

Let us first recall from homological algebra that a resolution of a R-module M is an exact sequence

$$\cdots \to L_2 \to L_1 \to L_0 \to M \to 0$$

of R-modules. For a group G we now construct a canonical resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Definition 3.1.1 (Bar resolution)**.** *For $i \geq 0$ we define B_i to be the free $\mathbb{Z}G$-module with basis all n-tuples $[g_1|g_2|\cdots|g_n]$ with $g_1, \dots, g_n \in G$ (this notation just refers to the n-tuple $(g_1, \dots, g_n)$). In particular we note that B_0 is free on the one element $[]$. We define $d_i : B_n \to B_{n-1}$ by*

$$d_i([g_1|\cdots|g_n]) = \begin{cases} g_1[g_2|\cdots|g_n] & \text{if } i = 0 \\ [g_1|\cdots|g_i g_{i+1}|\cdots|g_n] & \text{if } i \in \{1, 2, \dots, n-1\} \\ [g_1|\cdots|g_{n-1}] & \text{if } i = n \end{cases}$$

on basis elements. Now let $d : B_n \to B_{n-1}$ be defined by

$$d = \sum_{i=0}^{n} (-1)^i d_i$$

Define $\epsilon : B_0 \to \mathbb{Z}$ by $[] \mapsto 1$ on basis elements. The bar resolution of G is defined to be

$$\dots \xrightarrow{d} B_2 \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Proposition 5. *The bar resolution is actually a resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$.*

Definition 3.1.2 (Group cohomology)**.** *Let G be a group and let M be a G-module. We define the cohomology of G with coefficients in M to be the cohomology of the cochain complex*

$$0 \to Hom(B_0, M) \to Hom(B_1, M) \to Hom(B_2, M) \to \cdots$$

where $Hom(B_i, M)$ denotes the G-module morphisms $B_i \to M$. We shall denote cocycles by $Z^n(G, M)$, coboundaries by $B^n(G, M)$ and the cohomology group by $H^n(G, M)$.

3.2 Cohomology of cosimplicial abelian group

We recall that a simplicial abelian group is a contravariant functor $\Delta^{op} \to \mathrm{Ab}$. Dually we may define a cosimplicial abelian group as follows.

Definition 3.2.1 (Cosimplicial abelian group)**.** *A cosimplicial abelian group is a functor $\Delta \to \mathrm{Ab}$. The category of cosimplicial abelian groups is the functor category $\mathrm{Fun}(\Delta, \mathrm{Ab})$ and this is denoted cAb.*

We recall that for a simplicial object X we define face maps $X(d^i) : X_n \to X_{n-1}$. Similarly for a cosimplicial abelian group we define the coface maps $d^i = X(d^i)$ and the codegeneracy maps $s^i = X(s^i)$. We note that there is some ambiguity in this notation, but context will always make clear whether d^i and s^i refers to $d^i : [n-1] \to [n]$ and $s^i : [n] \to [n-1]$ or $d^i : X^{n-1} \to X^n$ and $s^i : X^n \to X^{n-1}$. In addition for a simplicial object X we adopted the notation $X_n = X([n])$, similarly for a cosimplicial abelian group X we write $X^n = X([n])$.

Of course if X is a cosimplicial abelian group we may define $d : X_{n-1} \to X_n$ by

$$d = \sum_{i=0}^{n} (-1)^i d^i$$

We then get a map $D = d \circ d : X_{n-2} \to X_n$ which is:

$$D = \sum_{i=0}^{n} (-1)^i d^i \circ \left(\sum_{j=0}^{n-1} (-1)^j d^j \right) = \sum_{(i,j) \in S} (-1)^{i+j} d^i d^j$$

where S is the cartesian product $[n] \times [n-1]$. We write S as the disjoint union of

$$A = \{(i,j) \in S | j < i\} \qquad B = \{(i,j) \in S | j \geq i\}$$

We note that $(i,j) \mapsto (j, i-1)$ defines a bijection $A \to B$ with inverse given by $(i,j) \mapsto (j+1, i)$. This combined with the cosimplicial identity (1.1.5)

$d^i d^j = d^j d^{i-1}$ for $i > j$ gives us that

$$\begin{aligned}
D &= \sum_{(i,j)\in S} (-1)^{i+j} d^i d^j \\
&= \sum_{(i,j)\in A} (-1)^{i+j} d^i d^j + \sum_{(i,j)\in B} (-1)^{i+j} d^i d^j \\
&= \sum_{(i,j)\in A} (-1)^{i+j} d^j d^{i-1} + \sum_{(i,j)\in B} (-1)^{i+j} d^i d^j \\
&= \sum_{(i,j)\in B} (-1)^{(j+1)+i} d^i d^{(j+1)-1} + \sum_{(i,j)\in B} (-1)^{i+j} d^i d^j \\
&= -\sum_{(i,j)\in B} (-1)^{j+i} d^i d^j + \sum_{(i,j)\in B} (-1)^{i+j} d^i d^j = 0
\end{aligned}$$

Thus,

$$\cdots \to 0 \to X^0 \xrightarrow{d} X^1 \xrightarrow{d} X^2 \xrightarrow{d} \cdots$$

is a cochain complex.

Definition 3.2.2 (Cohomology)**.** *The cohomology the cosimplicial abelian group X is defined to be the cohomology of the cochain complex*

$$\cdots \to 0 \to X^0 \xrightarrow{d} X^1 \xrightarrow{d} X^2 \xrightarrow{d} \cdots$$

3.3 Construction of homotopy colimits

A common simplicial object is the homotopy colimit of a functor $F : I \to \mathrm{sSet}$ for some index category I. It is a homotopy-invariant version of the usual colimit object. We will not deal with the abstract development of this theory, but simply present a common way to construct it.

Definition 3.3.1 (Homotopy colimit)**.** *Let us be given categories $I, \mathcal{C}$ and a functor $F : I \to \mathcal{C}$ where $\mathcal{C}$ is either* sSet *or* Ab*. We then form the simplicial set* $\mathrm{hocolim}_I F$ *which in degree n is the disjoint union*

$$(\operatorname*{hocolim}_I F)_n = \coprod_{\sigma:\Delta^n \to BI} F(\sigma(0))$$

For an increasing map $\theta : [m] \to [n]$ we define

$$\theta^* : (\operatorname*{hocolim}_I F)_n \to (\operatorname*{hocolim}_I F)_m$$

to be the map such that for all $\sigma : \Delta^n \to BI$ the diagram

$$\begin{tikzcd}
F(\sigma(0)) \arrow[r, "F(\sigma(0,\theta(0)))"] \arrow[d, "i_\sigma"'] & F(\sigma(\theta(0))) \arrow[d, "i_{\sigma\circ\theta}"] \\
(\mathrm{hocolim}_I F)_n \arrow[r, "\theta^*"'] & (\mathrm{hocolim}_I F)_m
\end{tikzcd}$$

commutes where i_σ denotes the obvious inclusion

$$F(\sigma(0)) \to \coprod_{\tau:\Delta^n \to BI} F(\tau(0))$$

3.4 Equivariant cohomology

For a fixed groupoid Γ we have the category sSet $\downarrow B\Gamma$ of simplicial sets over $B\Gamma$ whose objects are pairs (X, φ) where X is a simplicial set and φ is a map $X \to B\Gamma$. A morphism from $\varphi : X \to B\Gamma$ to $\psi : Y \to B\Gamma$ is a map $f : X \to Y$ such that $\psi \circ f = \varphi$. In particular for a given simplicial set X with fundamental groupoid Γ we get a canonical map $X \to B\Gamma$.

Definition 3.4.1 (Local coefficient system)**.** *Let X be a Kan complex. A local coefficient system on X is a functor $\pi_f(X) \to \mathrm{Ab}$.*

For a vertex $v \in X_0$ we usually denote $X(v)$ by X_v. We note that our previous construction of homotopy colimits also works for local coefficient systems.

Definition 3.4.2. *For a Kan complex X and a local coefficient system A on X we define $C^n_\Gamma(X, A)$ to be the set of maps $\alpha : X_n \to (\mathrm{hocolim}_\Gamma A)_n$ such that*

$$\begin{tikzcd} X_n \arrow[rr, "\alpha"] \arrow[dr] & & (\mathrm{hocolim}_\Gamma A)_n \arrow[dl] \\ & (B\Gamma)_n & \end{tikzcd}$$

commutes.

Lemma 3.4.3. *For a groupoid Γ, let $\varphi : X \to B\Gamma$ and $\psi : Y \to B\Gamma$ be spaces over $B\Gamma$, and let A be a functor $\Gamma \to \mathrm{Ab}$. Let an increasing map $\theta : [m] \to [n]$ be given, and let $f : Y_n \to X_m$ be a function such that*

$$\begin{tikzcd} Y_n \arrow[r, "f"] \arrow[d, "\psi"'] & X_m \arrow[d, "\varphi"] \\ B\Gamma_n \arrow[r, "\theta^*"'] & B\Gamma_m \end{tikzcd}$$

commutes. Then for every function $\alpha : X_m \to (\mathrm{hocolim}_\Gamma A)_m$ over $B\Gamma_m$, there exists a function $f^*(\alpha) : Y_n \to (\mathrm{hocolim}_\Gamma A)_n$ over $B\Gamma_n$ for which

$$\begin{tikzcd} Y_n \arrow[r, "f^*(\alpha)"] \arrow[d, "f"'] & (\mathrm{hocolim}_\Gamma A)_n \arrow[d, "\theta^*"] \\ X_m \arrow[r, "\alpha"'] & (\mathrm{hocolim}_\Gamma A)_m \end{tikzcd}$$

commutes.

Proof. Let $y \in Y_n$ be given. We have a map

$$\psi(y)(0) \xrightarrow{\psi(y)(0(\theta(0)))} \psi(y)(\theta(0))$$

in Γ. Γ is a groupoid so this map is an isomorphism and therefore we get a map

$$A_{\psi(y)(\theta(0))} \xrightarrow{A(\psi(y)(0(\theta(0)))^{-1})} A_{\psi(y)(0)}$$

Let $f^*(\alpha)(y)$ be the image of $\alpha(f(y)) \in A_{\varphi(f(y))}$ under this isomorphism. $\square$

From this we can give $C^n_\Gamma(X, A)$ the structure of a cosimplicial abelian group.

Definition 3.4.4 (Cohomology)**.** *For a Kan complex X and local coefficient system A on X we define the cohomology $H^*(X, A)$ of X with coefficients in A as the cohomology of $C^*_\Gamma(X, A)$.*

Theorem 3.4.5. *Let G be a group and A a G-module. Then A can be considered as a local coefficient system on BG, and we get a natural isomorphism*

$$H^*(G, A) \cong H^*(BG, A)$$

Proof. A G-module is just a functor $G \to \mathrm{Ab}$, while a local coefficient system on BG is a functor $\pi_f(BG) \to \mathrm{Ab}$, but $\pi_f(BG) = G$ so a G-module can be considered as a local coefficient system.

Consider a cocycle $f \in Z^n(G, A)$. This may be considered as a function $f : G^n \to A$ satisfying the cocycle condition. This obviously determines a map

$$BG_n \to \operatorname*{hocolim}_G A$$

over BG_n. From the cosimplicial structure on $C^n_\Gamma(X, A)$ it is clear that f is a cocycle in $C^n_\Gamma(X, A)$. $\square$

We use $[,]_\Gamma$ to denote the homotopy class over $B\Gamma$.

Theorem 3.4.6. *There is an isomorphism*

$$[X, \operatorname*{hocolim}_\Gamma K(A, n)]_\Gamma \cong H^n_\Gamma(X, A)$$

which is given by sending a cocycle $f : X_n \to \mathrm{hocolim}_\Gamma A_n$ to the map $f' : X \to$ $\mathrm{hocolim}_\Gamma K(A, n)$ over $B\Gamma$. Every m-simplex $x \in X_m$ can be written uniquely as $x = \theta^* x'$ where $x' \in X_k$ is non-degenerate and $\theta : [m] \twoheadrightarrow [k]$ is surjective. We then define $f'(x)$ by

$$f'(\theta^* x) = \begin{cases} (\theta, f(x')) & \text{if } k = n \\ (\theta, 0) & \text{if } k \neq n \end{cases}$$

where we note that a m-simplex of $K(A, n)$ is a map $[m] \twoheadrightarrow [k]$ and an element of $A[n]_k$ which is 0 unless $n = k$, in which case it is A.

Proof. See section VI.4 of [8]. $\square$

3.5 Postnikov towers

Definition 3.5.1 (Postnikov tower)**.** *Let X be a connected Kan complex. A Postnikov tower of X is a tower*

$$\cdots \xrightarrow{q_2} X_2 \xrightarrow{q_1} X_1 \xrightarrow{q_0} X_0$$

as well as maps $i_n : X \to X_n$ for all $n \geq 0$. This data must satisfy the following axioms,

1. $\pi_i(X_n) = 0$ *for* $i > n$.

2. For $i \leq n$, i_n induces an isomorphism $(i_n)_* : \pi_i(X) \to \pi_i(X_n)$.

3. $q_{n+1} \circ i_{n+1} = i_n$ for all $n \geq 0$.

We may note that we can form the limit of such a tower and get a map $X \to \lim X_n$. Usually we want this to be a weak equivalence so the postnikov tower gives us a way to build X up to homotopy.

There are several ways to construct Postnikov towers for an arbitrary connected Kan complex X. Consider two q-simplices $x, y : \Delta^q \to X$ of X. We write $x \sim_n y$ if and only if $x|sk_n\Delta^q = y|sk_n\Delta^q$. Then $\sim_n$ is an equivalence relation, and it is compatible with the simplicial structure. Thus for all $n \geq 0$ we get a simplicial complex $X(n) = X/\sim_n$. We then have obvious maps

$$p_n : X(n) \to X(n-1) \qquad i_n : X \to X(n)$$

Let us confirm that this is a postnikov tower. First suppose we are given two elements of $\pi_i(X(n))$ represented by $f, g : \Delta^i \to X(n)$ where $i > n$. $d_i f = * = d_i g$ for all i so f and g agree on their $(i-1)$-skeleton, and therefore they also agree on their n-skeleton which shows $[f] = [g]$. Thus $\pi_i(X(n)) = 0$ for $i > n$. Now suppose $i \leq n$. Clearly $(i_n)_* : \pi_i(X) \to \pi_i(X(n))$ is an isomorphism.

Definition 3.5.2 (Moore-Postnikov tower)**.** *The tower $\{X(n)\}$ just defined is known as the Moore-Postnikov tower of X.*

In the article [5] by W.G. Dwyer and D.M. Kan an alternative Postnikov tower known as the coskeleton tower is presented. Let X be an arbitrary connected Kan complex. The n-coskeleton of X is the simplicial set $cosk_n X$ whose k-simplices are maps

$$sk_n\Delta^k \to X$$

with the obvious simplicial structure induced by the cosimplicial structure of $\Delta^\bullet$. Define $Q_n X = cosk_{n+1}X$. We then get canonical maps

$$q_n : Q_n X \to Q_{n-1}X \qquad i_n : X \to Q_n X$$

Let us now show that this is a Postnikov tower. First let $k > n$ and let $f, g : sk_{n+1}\Delta^k \to X$ represent arbitrary elements in $\pi_k(Q_n X)$. If $k > n+1$, then f, g are both constantly $*$ so $[f] = [g]$ and $\pi_k(Q_n X) = 0$. Suppose therefore $k = n+1$. Define

$$h : sk_{n+1}\Delta^{k+1} = \partial\Delta^{k+1} \to X$$

by

$$d_i h = \begin{cases} * & \text{if } i = 0, 1, \ldots, k-1 \\ f & \text{if } i = k \\ g & \text{if } i = k+1 \end{cases}$$

which is well-defined since $d_j * = * = d_j f = d_j g$. h is a $(k+1)$-simplex in $Q_n X$ so by the definition of addition in $\pi_k(Q_n X)$ we get $[f] = [g]$. Thus we have $\pi_k(Q_n X) = 0$ for $k > n$. Now suppose $k \leq n$. We have an induced map

$$\pi_k(X) \to \pi_k(Q_n X)$$

A k-simplex of Q_nX is a map $sk_{n+1}\Delta^k = \Delta^k \to X$ so the map is surjective. All non-degenerate simplices of $\Delta^k \times \Delta^1$ have dimension $\leq k+1 \leq n+1$ so every homotopy

$$\Delta^{k+1} \times \Delta^1 \to Q_nX$$

comes from a homotopy

$$\Delta^{k+1} \times \Delta^1 \to X$$

and therefore $\pi_k(Q_nX) \to \pi_k(X)$ is also injective. We have therefore now computed

$$\pi_k(Q_nX) = \begin{cases} \pi_k(X) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

In addition the argument shows that the induced map

$$(i_n)_* : \pi_k(X) \to \pi_k(Q_nX)$$

is an isomorphism for $k \leq n$.

Definition 3.5.3 (Coskeleton tower)**.** *For a connected Kan complex X the constructed tower*

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_3"] \\
 & Q_2X \arrow[d, "q_2"] \\
 & Q_1X \arrow[d, "q_1"] \\
X \arrow[r, "i_0"'] \arrow[ur] \arrow[uur] \arrow[uuur] & Q_0X
\end{tikzcd}$$

with $Q_kX = cosk_{k+1}X$, is a Postnikov tower of X.

3.6 Postnikov invariants

Definition 3.6.1 (Postnikov invariant)**.** *Let X be a given connected Kan complex. Choose a Postnikov tower $\{X_n\}$ for X. The nth Postnikov invariant of X is a cohomology class $k_n \in H^{n+1}_\Gamma(X_{n-1}, \pi_n(X))$ represented by a map*

$$k_n : X_{n-1} \to \operatorname*{hocolim}_\Gamma K(\pi_n(X), n+1)$$

such that we have a homotopy pullback

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "q_n"'] & \operatorname{hocolim}_\Gamma WK(\pi_n(X), n+1) \arrow[d] \\
X_{n-1} \arrow[r, "k_n"'] & \operatorname{hocolim}_\Gamma K(\pi_n(X), n+1)
\end{tikzcd}$$

over $B\Gamma$.

3.7 An explicit construction of postnikov invariants

We will now present an explicit construction of postnikov invariants. This construction was given in the article [5] by W.G. Dwyer and D.M. Kan, but it is essentially a simplicial analog of the k^3 invariant for topological space described in the article [6] by Eilenberg and MacLane, with the obvious generalization to higher dimensions.

For a connected Kan complex X ($\Gamma = \pi_f(X)$) and integer $n \geq 2$ we define a map $k_n : Q_{n-1}X \to \operatorname{hocolim}_\Gamma K(\pi_n X, n+1)$ over $B\Gamma$. Such a map is completely determined by where it sends $(n+1)$-simplices. k_n sends the $(n+1)$-simplex $\sigma : \partial\Delta^n \to X$ to its equivalence class in $\pi_n(X)$.

The following theorem is presented in [5] without proof.

Theorem 3.7.1. *The cocycle*

$$k_n : Q_{n-1}X \to \operatorname*{hocolim}_{\Gamma} K(\pi_n X, n+1)$$

represents the Postnikov invariant of X.

Proof. The map

$$p : WK(\pi_n, n) \to K(\pi_n, n+1)$$

is a fibration so by lemma VI.4.2 of [8] the induced map

$$p_* : \operatorname*{hocolim}_{\Gamma} WK(\pi_n, n) \to \operatorname*{hocolim}_{\Gamma} K(\pi_n, n+1)$$

is a fibration over $B\Gamma$.

Form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{hocolim}_\Gamma WK(\pi_n(X), n) \arrow[d, "p_*"] \\ Q_{n-1} \arrow[r, "k_n"'] & \operatorname{hocolim}_\Gamma K(\pi_n(X), n+1) \end{tikzcd}$$

Let $\omega : Q_n X \to \operatorname{hocolim}_\Gamma WK(\pi_n, n)$ be the 0 map. Then

$$\begin{tikzcd} Q_n X \arrow[r, "\omega"] \arrow[d, "q"] & \operatorname{hocolim}_\Gamma WK(\pi_n, n) \arrow[d, "p_*"] \\ Q_{n-1}X \arrow[r, "k_n"'] & \operatorname{hocolim}_\Gamma K(\pi_n, n+1) \end{tikzcd}$$

commutes. The only dimension in which it is not trivial is dimension $n+1$, but there we start out with a map $x : \Delta^{n+1} \to X$ and $q(x) : \partial\Delta^{n+1} \to X$ is the restriction of x and therefore represents 0 in $\pi_n(X)$ as it can be extended. This shows that both ways to go from top left to bottom right gives 0 in dimension

$n+1$. By the universal property we get a map $f : Q_nX \to P$ such that

$$\begin{tikzcd}
Q_nX \arrow[dr, dashed, "f"] \arrow[drrr, "\omega"] \arrow[ddr, "q"'] & & & \\
& P \arrow[rr, "\beta"] \arrow[d, "\alpha"] & & \operatorname{hocolim}_\Gamma WK(\pi_n(X), n) \arrow[d, "p_*"] \\
& Q_{n-1} \arrow[rr, "k_n"'] & & \operatorname{hocolim}_\Gamma K(\pi_n(X), n+1)
\end{tikzcd}$$

commutes.

We wish to show that f is a weak equivalence. The k-simplices of P for $k < n$ consists a map $x : \Delta^n \to X$ and an element $\sigma \in B\Gamma_k$ and they must be compatible in the sense that $x(ij) = \sigma(ij)$ for all $0 \leq i \leq j \leq k$. We claim that a k-simplex (x, σ) which is $*$ on $\partial\Delta^k$ represents 0 in $\pi_k(P)$ only if $[x] = 0$ in $\pi_k(X)$. So suppose (x, σ) represents 0 in $\pi_k(P)$. Then we can find a $(k+1)$-simplex $y : \Delta^{n+1} \to P$ with

$$d_i y = \begin{cases} * & \text{if } i < k+1 \\ (x, \sigma) & \text{if } i = k+1 \end{cases}$$

Composing with

$$P \xrightarrow{\alpha} Q_{n-1}X$$

we get a $(k+1)$-simplex in $Q_{n-1}X$, and therefore in X as $k < n$, with

$$\alpha(d_i y) = \begin{cases} * & \text{if } i < k+1 \\ x & \text{if } i = k+1 \end{cases}$$

which shows that $[x] = 0$ if $[(x, \sigma)] = 0$. This implies that

$$f_* : \pi_k(Q_nX) \to \pi_k(P)$$

is injective for $k < n$ and it is clearly surjective for $k < n$ since σ is determined by x. Thus f_* is an isomorphism for $k < n$.

The n-simplices of P consists of a map $x : \Delta^n \to X$, $\sigma \in B\Gamma_n$, and $w \in \pi_n$, and these must be compatible in the sense that $\sigma(ij) = x(ij)$ for $0 \leq i \leq j \leq n$. If the n-simplex is $*$ on $\partial\Delta^n$, then $\sigma = *$. Let $(x, *, 0)$ and $(*, *, w')$ represent elements of $\pi_n(P)$. We claim that $[x, *, w)] = [(*, *, w')]$ if and only if $[x] + w = w'$. First suppose $[(x, *, w)] = [(*, *, w')]$. Then there exists a $(n+1)$-simplex $m \in P_{n+1}$ with

$$d_i m = \begin{cases} * & \text{if } i < n \\ (x, *, w) & \text{if } i = n \\ (*, *, w') & \text{if } i = n+1 \end{cases}$$

Such a $(n+1)$-simplex consists of a map $F : \partial\Delta^n \to X$, for every surjective map $\tau : [n+1] \twoheadrightarrow [n]$ an element $w_\tau \in \pi_n$ and a group $z \in \pi_n$ associated to the identity $[n+1] \to [n+1]$. This must be compatible in the sense that $z = [F]$. We now wish to calculate $d_i m$ in terms of these elements. Every $\tau : [n+1] \twoheadrightarrow [n]$ is of the form s^j for some j. We have

$$s^j \circ d^i = id \qquad \text{if and only if } j = i \text{ or } j = i+1$$

By the simplicial structure of $WK(\pi_n, n)$ we therefore get

$$d_{n+1}m = (d_{n+1}F, *, z + w_n)$$

$$d_i m = (d_i F, *, w_{i-1} + w_i) \qquad \text{for } 0 < i < n+1$$

$$d_0 m = (d_0 F, *, w_0)$$

From which we get

$$* = d_{n+1}F \qquad x = d_n F$$

$$w' = z + w_n \qquad w = w_{n-1} + w_n$$

$$w_{i-1} + w_i = 0 \qquad \text{for } 0 < i < n$$

$$w_0 = 0$$

$$d_i F = * \qquad \text{for } i < n$$

By the definition of homotopy groups in terms of maps $\partial\Delta^{n+1} \to X$ we get $[x] = [F] = z$. Clearly $w_i = 0$ for $i < n$ so

$$w' = z + w_n = [x] + w$$

Thus $[(x, *, w)] = [(*, *, w')]$ if $[x] + w = w'$. Now conversely suppose $[x] + w = w'$. Then we can form the $(n+1)$-simplex $F : \partial\Delta^n \to X$, $w_0, w_1, \dots, w_n, z$ by

$$w_i = 0 \qquad \text{for } i < n$$

$$w_n = w \quad z = w' - w_n$$

and let $F = s_n x$. Then it satisfies all the constraints we just wrote and this is sufficient for it to define a $(n+1)$-simplex. Thus $[(x, *, w)] = [(*, *, w')]$ if $[x] + w = w'$.

In particular this proves that $\pi_n(P) \cong \pi_n(X) \cong \pi_n(Q_n X)$. And the map

$$f_* : \pi_n(Q_n X) \to \pi_n(P)$$

sends an element represented by $x : \Delta^n \to X$ to $[(x, *, 0)]$. Thus f_* induces an isomorphism in dimension n.

Finally let us show that $\pi_k(P) = 0$ for $k > n$. There only exists one k-simplex $x : \Delta^k \to x$ for which $x(\partial\Delta^k)$ when $k > n+1$, so we only need to consider the case $k = n+1$. Let $x : \Delta^{n+1} \to P$ represent an element of $\pi_{n+1}(P)$, so in particular it is $*$ on $\partial\Delta^{n+1}$. There is only one such $(n+1)$-simplex in $Q_{n-1}X$ so we get that $\pi_{n+1}(P) \cong \pi_{n+1}(F)$ where F is the fiber of the fibration

$$p_* : \operatorname*{hocolim}_{\Gamma} WK(\pi_n, n) \to \operatorname*{hocolim}_{\Gamma} K(\pi_n, n+1)$$

The fiber is $K(\pi_n, n)$. Thus

$$\pi_{n+1}(P) \cong \pi_{n+1}(K(\pi_n, n)) = 0$$

We have now shown that f_* is a weak equivalence.

By example XI.4.1.(iv) of [1] the homotopy pullback and the pullback are weakly equivalent if one of the legs is a fibration. Hence Q_nX is weakly equivalent to the homotopy pullback of our pullback diagram, but that implies that it is a homotopy pullback of the diagram.

$$\begin{tikzcd}
Q_n \arrow[r] \arrow[d, "q"'] & \mathrm{hocolim}_\Gamma WK(\pi_n(X), n) \arrow[d, "p_*"] \\
Q_{n-1} \arrow[r, "k_n"'] & \mathrm{hocolim}_\Gamma K(\pi_n(X), n+1)
\end{tikzcd}$$

□

3.8 Homotopy addition theorem

Let us recall a well-known useful homotopy addition lemma.

Lemma 3.8.1. *Let (X, A) be a pair of topological spaces with basepoint $*$. Let a continuous map*

$$f : (|\partial\Delta^{n+1}|, |sk_{n-1}\Delta^{n+1}|, 0) \to (X, A, *)$$

be given. Define

$$f_i = f \circ |d^i| : (|\Delta^n|, |\partial\Delta^n|, *) \to (X, A, f(d^i(0)))$$

for $i = 0, 1, \ldots, n, n+1$. We note $d^i(0) = 0$ for $i > 0$ so

$$\alpha_i = [f_i] \in \pi_n(X, A, *) \qquad \text{for } i > 0$$

Define

$$p = f \circ |i_{01}| : |\Delta^1| \to A$$

where $i_{01} : \Delta^1 \to \partial\Delta^{n+1}$ is the map induced by 01. Since $d^0(0) = 1$ and from the action of the fundamental groupoid $\pi_f(A)$ of A on the groupoid $\pi_n(X, A)$ we get

$$\alpha_0 = {}^{[p]}[f_0] \in \pi_n(X, A, *)$$

Let

$$j_* : \pi_n(X, *) \to \pi_n(X, A, *)$$

be the canonical map induced by inclusion $(X, *, *) \to (X, A, *)$. Then

$$j_*([f]) = \begin{cases} \alpha_2\alpha_0\alpha_1^{-1} & \text{if } n = 1 \\ \alpha_0\alpha_2\alpha_1^{-1}\alpha_3^{-1} & \text{if } n = 2 \\ \sum_{i=0}^{n+1}(-1)^i\alpha_i & \text{if } n > 2 \end{cases}$$

Proof. This is theorem 6.1 in [9]. □

3.9 $OpExt^n(G, M)$

Definition 3.9.1 (Crossed n-fold extension)**.** *Let G be a group and M a G-module. A crossed n-fold extension of M by G is a crossed resolution E of G of the form*

$$\cdots \to 0 \to M \xrightarrow{\delta_{n+1}} E_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} E_1 \xrightarrow{\varphi} G \to 1$$

A morphism of crossed n-fold extensions of M by G is a morphism of crossed complexes which is the identity on G and M. If there exists a zig-zag of morphisms from a crossed n-fold extension E of M by G to another crossed n-fold extension E', then we define E and E' to be equivalent. The resulting set of equivalence classes of crossed n-fold M by G extensions is denoted $OpExt^n(G, M)$.

Theorem 3.9.2. *There is a natural bijection*

$$H^{n+1}(G, M) \cong OpExt^n(G, M)$$

Proof. Let a crossed n-fold extension E of M by G be given. By theorem 2.4.3 we can construct a lift as in the following diagram

$$\begin{tikzcd}
F_{n+2}(G) \arrow[r, "\delta_{n+2}"] \arrow[d, "f_{n+2}"] & F_{n+1}(G) \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & F_n(G) \arrow[r, "\delta_n"] \arrow[d, "f_n"] & \cdots \arrow[r, "\delta_2"] & F_1(G) \arrow[r, "\epsilon"] \arrow[d, "f_1"] & G \arrow[d, "id"] \\
0 \arrow[r] & M \arrow[r, "\delta_{n+1}"] & E_n \arrow[r, "\delta_n"] & \cdots \arrow[r, "\delta_2"] & E_1 \arrow[r, "\varphi"] & G
\end{tikzcd}$$

That the leftmost square commutes means exactly that f_{n+1} decides a cocycle $Z^{n+1}(G, M)$. If we we had chosen another lift f' then by theorem 2.4.4 we can find $h_n : F_n(G) \to E_{n+1}(G)$ such that

$$h_n(\delta_{n+1}x) = f_{n+1}(x)f'_{n+1}(x)^{-1}$$

which means that the cocycles f_{n+1} and f'_{n+1} differ by a coboundary and therefore represent the same cohomology class in $H^{n+1}(G, M)$.

It is clear that if we are given a morphism $E \to E'$ of crossed n-fold extensions of M by G, then by composition we can extend our lift of E to E'. Thus equivalent crossed n-fold extensions give equivalent cohomology classes. We have now defined a map

$$OpExt^n(G, M) \to H^{n+1}(G, M)$$

Let a cocycle $f : G^{n+1} \to M$ represent an element of $H^{n+1}(G, M)$. Since f is a cocycle it determines a map $f_{n+1} : F_{n+1}(G) \to M$ for which $f\delta_{n+2} = 0$. Now define

$$E_i = \begin{cases} F_i(G) & \text{if } i < n \\ (F_n(G) \times M)/D & \text{if } i = n \end{cases}$$

where D is the submodule of $F_n(G) \times M$ generated by elements of the form $(\delta_{n+1}x, f_{n+1}x)$ for all $x \in F_{n+1}(G)$. $\delta_{n+1} : M \to E_n$ is the inclusion $x \mapsto (0, x)$ followed by the projection that mods out D. The map $\delta_n : E_n \to E_{n-1}$ is just induced by $\delta_n : F_n \to F_{n-1}$. For $k < n$ $\delta_k : E_k \to E_{k-1}$ is just $\delta_k : F_k \to F_{k-1}$.

To show that this is a crossed n-fold extension of M by G we must show that $\delta_{n+1} : M \to E_n$ is injective and that $Im(\delta_{n+1}) = ker(\delta_n)$. Suppose there exists $x \in F_{n+1}(G)$ such that $(0, m) = (\delta_{n+1}x, f_{n+1}x)$. Then $x \in ker(\delta_{n+1})$ so there must exist $x' \in F_{n+2}(G)$ for which $x = \delta_{n+2}(x')$, but then

$$m = f_{n+1}\delta_{n+2}x' = 0$$

This proves that δ_{n+1} is injective. It is clear that $\delta_n \circ \delta_{n+1} = 0$. Suppose $(f, m) \in F_n(G) \times M$ represents an element of $ker(\delta_n)$. Then $\delta_n f = 0$ so there exists $f' \in F_{n+1}(G)$ such that $f = \delta_{n+1}f'$, but this implies that (f, m) equals $(0, m - f_{n+1}(f'))$ modulo D. We have therefore shown $Im(\delta_{n+1}) = ker(\delta_n)$. Thus we have constructed a crossed n-fold extension of M by G. We may now define $f_i : F_i(G) \to E_i$ by

$$f_i = id \qquad \text{if } i \in \{1, 2, \ldots, n-1\}$$

and let $f_n : F_n(G) \to E_n$ be the inclusion $x \mapsto (x, 0)$ followed by the projection $F_n(G) \times M \twoheadrightarrow E_n$. It is clear that

$$\begin{tikzcd}
F_{n+2}(G) \arrow[r, "\delta_{n+2}"] \arrow[d, "f_{n+2}"] & F_{n+1}(G) \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & F_n(G) \arrow[r, "\delta_n"] \arrow[d, "f_n"] & \cdots \arrow[r, "\delta_2"] & F_1(G) \arrow[r, "\epsilon"] \arrow[d, "f_1"] & G \arrow[d, "id"] \\
0 \arrow[r] & M \arrow[r, "\delta_{n+1}"] & E_n \arrow[r, "\delta_n"] & \cdots \arrow[r, "\delta_2"] & E_1 \arrow[r, "\varphi"] & G
\end{tikzcd}$$

commutes. Suppose we are given a cohomologous cocycle $f'_{n+1} : F_{n+1}(G) \to M$. Then there exists some $g : F_n(G) \to M$ such that $f'_{n+1} - f_{n+1} = g_n \circ \delta_{n+1}$. f'_{n+1} then determines another crossed n-fold extension E' of M by G. However the only part of the construction that differs is that what we called D before will now be D' which is generated by elements of the form $(\delta_{n+1}x, f'_{n+1}x)$ for all $x \in F_{n+1}(G)$. We get an isomorphism $\theta : D \to D'$ by

$$\theta(y, z) = (y, z + g_n(y))$$

This is an isomorphism because

$$\theta(\delta_{n+1}x, f_{n+1}x) = (\delta_{n+1}x, f_{n+1}x + f_n(\delta_{n+1}x)) = (\delta_{n+1}x, f'_{n+1}x)$$

and we can obviously construct an inverse in the same manner. This isomorphism gives us a morphism $E_n \to E'_n$ and therefore a morphism $E \to E'$ of crossed n-fold extension of M by G. We have now shown that this construction gives a map

$$H^{n+1}(G, M) \to OpExt^n(G, M)$$

which is obviously inverse to our previous map. This finishes the proof. $\square$

3.10 The first non-trivial postnikov invariant

Theorem 3.10.1. *The cokernel of* $\delta_2 : \Pi X_2 \to \Pi X_1$ *is* $\pi_f(X)$. *Suppose* $\pi_i(X) = 0$ *for* $1 < i < n$, *then*

$$0 \to \frac{\ker(\delta_n)}{Im(\delta_{n+1})} \to \frac{\Pi X_n}{Im(\delta_{n+1})} \to \Pi X_{n-1} \to \cdots \Pi X_1 \to \frac{\Pi X_1}{Im(\delta_2)} \to 1$$

is a crossed n-fold resolution of $\pi_f(X)$ by $\pi_n(X)$. This crossed resolution is denoted

$$\operatorname{cosk}^n \operatorname{cotr}_n \Pi X$$

If there exists a subcomplex A of the CW complex $|X|$ such that $|sk_{n-1}X| \subset A$ and $\pi_n(A, *) = 0$, then the associated cohomology class in $H^{n+1}(\pi_1, \pi_n)$ is the first non-trivial Postnikov invariant of X in the sense that it corresponds to that postnikov invariant via the isomorphism

$$H^{n+1}(\pi_1, \pi_n) \cong H^{n+1}_\Gamma(Q_{n-1}, \pi_n)$$

In particular if $n = 2$, then we can let A be the 1-skeleton of $|X|$ which is a topological graph and therefore has a contractible universal covering space. Hence $\pi_2(A, *) = 0$.

Proof. Let us first calculate the cokernel of δ_2, $G = \frac{\Pi X_1}{Im(\delta_2)}$. We can map every 1-simplex of X to their equivalence class in $\pi_f(X)$. This gives a map

$$\Pi X_1 \to \pi_f(X)$$

Let a 2-simplex x be given. Then $[d_2 x][d_0 x] = [d_1 x]$ by the definition of composition in $\pi_f(X)$. Thus $\Pi X_1 \to \pi_f(X)$ factors through G which gives us a map

$$\theta : G \to \pi_f(X)$$

We claim that θ is a groupoid isomorphism. Clearly θ is surjective. Let us be given 1-simplices x, y such that $d_0 x = d_1 y$. Then since X is Kan we get a 2-simplex z with $d_0 z = y$, $d_2 z = x$. Thus $[x][y] = [d_2 z][d_0 z] = [d_1 z]$ in G. Addition is defined similarily in $\pi_f(X)$ so θ is a groupoid morphism. If a 1-simplex mapped to 0 in $\pi_f(X)$, then there exists a 2-simplex z with $d_2 z = x$ and $d_1 z = *$, $d_0 z = *$ which implies $[x] = 0$ in G. Thus θ is injective. We have now shown that θ is an isomorphism.

By proposition 4 we get that

$$\frac{\ker(\delta_n)}{Im(\delta_{n+1})} \cong \pi_n(X)$$

Similarily we get exactness at ΠX_i for $1 < i < n$ since $\pi_i(X) = 0$. Exactness at $\frac{\Pi X_n}{Im(\delta_{n+1})}$ is obvious.

We have now shown that the crossed complex is a crossed n-fold extension of π_f by π_n.

$Q_{n-1}X$ is a Kan complex and therefore the canonical weak equivalence

$$\alpha : Q_{n-1}X \to B\pi_1$$

is a homotopy equivalence. Let $\beta : B\pi \to Q_{n-1}X$ be a homotopy inverse of α. On k-simplices for $k \leq n$ this map in dimension k sends a sequence $(g_1, \ldots, g_k)$ of elements of π_1 to some $f_k(g_1, \ldots, g_k) : \Delta^k \to X$ which is just a k-simplex of X. Thus this determines a map

$$f_k : F_k(G) \to \Pi X_k$$

Since β is simplicial we get

$$d_0 f_{k+1}(g_1, \ldots, g_{k+1}) = f_k(g_2, g_3, \ldots, g_{k+1})$$

$$d_{k+1}f_{k+1}(g_1,\ldots,g_{k+1}) = f_k(g_1,\ldots,g_k)$$
$$d_i f_{k+1}(g_1,\ldots,g_{k+1}) = f_k(g_2,\ldots,g_ig_{i+1},\ldots,g_{k+1}) \qquad \text{for } 0<i<k+1$$

From this we immediately see that

$$\begin{tikzcd} F_{k+1}G \arrow[r, "\delta_{k+1}"] \arrow[d, "f_{k+1}"'] & F_kG \arrow[d, "f_k"] \\ \Pi X_{k+1} \arrow[r, "\delta_{k+1}"'] & \Pi X_k \end{tikzcd}$$

commutes for $k = 1,2,\ldots,n-1$.

On $n+1$ simplices $\beta : B\pi_1 \to Q_{n-1}X$ sends a sequence $(g_1,\ldots,g_{n+1})$ to a map $F = \beta(g_1,\ldots,g_{n+1}) : \partial\Delta^{n+1} \to X$. F represents an element $f_{n+1}(g_1,\ldots,g_{n+1})$ of $\pi_n(X,*)$. Taking the geometric realization we get an element $|F| : |\partial\Delta^{n+1}| \to |X|$ which represents an element of $\pi_n(|X|,*)$. By the homotopy addition theorem 3.8.1 the map

$$j_* : \pi_n(|X|,*) \to \pi_n(|X|,A)$$

sends $[|F|]$ to

$$j_*[|F|] = \begin{cases} \alpha_2\alpha_0\alpha_1^{-1} & \text{if } n=1 \\ \alpha_0\alpha_2\alpha_1^{-1}\alpha_3^{-1} & \text{if } n=2 \\ \sum_{i=0}^{n+1}(-1)^i\alpha_i & \text{if } n>2 \end{cases}$$

where $\alpha_i = [|d_iF|] \in \pi_n(|X|,A)$ for $i>0$, and

$$\alpha_0 = {}^{[F\circ i_{01}]}[|d_0F|]$$

where $i_{01} : \Delta^1 \to \partial\Delta^{n+1}$ sends 0 and 1 to 0 and 1.

Let a n-simplex $x : \Delta^n \to X$ be given. Then

$$|x(\partial\Delta^n)| \subset |sk_{n-1}X| \subset A$$

so it determines an element of $\pi_n(|X|,A)$. This gives us a map $q : \Pi X_n \to \pi_n(|X|,A)$. Now consider an $(n+1)$-simplex $x : \Delta^{n+1} \to X$. If we restrict to $\partial\Delta^{n+1}$ and take geometric realization we get an element of $\pi_n(|X|,*)$, but it represents 0 as it can be extended. We then have

$$j_*[x|\partial\Delta^{n+1}] = q(\delta_{n+1}x)$$

since the formula for $\delta_{n+1} : \Pi X_{n+1} \to \Pi X_n$ is the same as for the homotopy addition theorem. We therefore have that $q(Im(\delta_{n+1})) = 0$ so we get

$$q' : \frac{\Pi X_n}{Im(\delta_{n+1})} \to \pi_n(|X|,A)$$

The following diagram is then commutative

$$\begin{tikzcd} 0 \arrow[r] & \frac{ker(\delta_n)}{Im(\delta_{n+1})} \arrow[r, "i"] \arrow[d, "\cong"'] & \frac{\Pi X_n}{Im(\delta_{n+1})} \arrow[d, "q'"] \\ 0 \arrow[r] & \pi_n(|X|,*) \arrow[r, "j_*"'] & \pi_n(|X|,A) \end{tikzcd}$$

j_* is injective since $\pi_n(A,*) = 0$. We have $j_*[|F|] = q'(f_n(\delta_{n+1}F))$ as they are both given by the homotopy addition formula. However since i is injective, $\frac{ker(\delta_{n+1})}{Im(\delta_n)} \to \pi_n(|X|,*)$ is an isomorphism, and j_* is injective so $i(f_{n+1}F) = f_n(\delta_{n+1}F)$. Thus

$$\begin{tikzcd}
F_{n+1}G \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"'] & F_nG \arrow[d, "f_n"] \\
\frac{ker(\delta_n)}{Im(\delta_{n+1})} \arrow[r, "i"] & \frac{\Pi X_n}{Im(\delta_{n+1})}
\end{tikzcd}$$

commutes.

Finally we need to show $f_{n+1}(\delta_{n+2}(g_1, \ldots, g_{n+2})) = 0$. However

$$i(f_{n+1}(\delta_{n+2}(g_1, \ldots, g_{n+2}))) = f_n(\delta_{n+1}\delta_{n+2}(g_1, \ldots, g_{n+2})) = 0$$

and because i is injective we are done. □

The technical requirement that $|X|$ contains a certain subcomplex seems a bit out of place. Indeed I suspect that the proposition holds without any such requirement, but have been unable to show it. However we will mainly use the case $n = 2$ so for the purposes of this thesis the requirement is not too detrimental.

Chapter 4

Group extensions

In this chapter we will cover group extensions. In particular we will introduce an equivalence relation on group extensions. We will then construct an obstruction to a certain group extension being realizable. Should a group extension be realizable we will classify all group extensions of that kind in terms of 2-dimensional group cohomology.

In addition we will investigate for what kernels no obstruction ever occurs. This in particular is the case for abelian kernels, but some non-abelian kernels have vanishing universal obstruction as well. We will also show how to obtain this universal obstruction as a postnikov invariant. Most of this chapter is based on the paper [7] by Eilenberg and MacLane. We do some things differently because we have the machinery of crossed complexes, but overall it is the same techniques.

4.1 Group extensions

Definition 4.1.1 (Group extension)**.** *Let N, Q be groups. A group extension of Q by N is a short exact sequence*

$$1 \to N \to E \to Q \to 1$$

We say that group extensions of Q by N

$$1 \to N \to E \to Q \to 1 \quad \text{and} \quad 1 \to N \to E' \to Q \to 1$$

are equivalent if there exists a group homomorphism $f : E \to E'$ such that

$$\begin{tikzcd} & & E \arrow[dd, "f"] \arrow[dr] & & \\ 1 \arrow[r] & N \arrow[ur] \arrow[dr] & & Q \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

commutes. We let $Ext(Q, N)$ be the collection of equivalence classes of group extensions of Q by N.

4.2 Group extensions with abelian kernel

Proposition 6. *A group extension η:*

$$1 \to N \xrightarrow{i} E \xrightarrow{p} Q \to 1$$

induces a canonical group homomorphism $\psi : Q \to Out(N)$ such that

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "c"'] & Q \arrow[d, "\psi"] \\ Aut(N) \arrow[r, two heads] & Out(N) \end{tikzcd}$$

commutes, where $c : E \to Aut(N)$ is defined by $c(e)(n) = ene^{-1}$.

We let $Ext_\psi(Q, N)$ denote the subset of $Ext(Q, N)$ consisting of group extensions that induce ψ.

Proof. Let us first note that the definition of c makes sense because N is the kernel of p and therefore normal in E, so $ene^{-1} \in N$.

p is surjective so we can choose a section $s : Q \to E$ of p, and we may choose it such that $s(1) = 1$. We call such a section normalized. Now define $\psi : Q \to Out(N)$ by

$$\psi = Q \xrightarrow{s} E \xrightarrow{c} Aut(N) \twoheadrightarrow Out(N)$$

The automorphism $c(s(q))c(s(q'))c(s(qq'))^{-1}$ is conjugation by $s(q)s(q')s(qq')^{-1}$, but

$$p(s(q)s(q')s(qq')^{-1}) = 1$$

so $s(q)s(q')s(qq')^{-1} \in iN$ which proves

$$\psi(q)\psi(q') = \psi(qq')$$

so ψ is a group homomorphism.

Choose another section $s' : Q \to E$ of p and define

$$\psi' = Q \xrightarrow{s'} E \xrightarrow{c} Aut(N) \twoheadrightarrow Out(N)$$

We have $p(s'(q)^{-1}s(q)) = q^{-1}q = 1$ so $s'(q)^{-1}s(q) \in ker(p) = i(N)$. Thus,

$$s'(q)^{-1}s(q)i(n)s(q)^{-1}s'(q) = i(n')i(n)i(n')^{-1} = c_{i(n')}(n)$$

for some $n' \in N$. This proves $c(s(q))$ and $c(s(q'))$ differ by an element of $Inn(N)$ and consequently $\psi = \psi'$.

For an arbitrary $e \in E$ we may choose a section $s_e : Q \to E$ of p such that $s_e(p(e)) = e$. By the preceeding paragraph we have

$$\psi = Q \xrightarrow{s_e} E \xrightarrow{c} Aut(N) \twoheadrightarrow Out(N)$$

Therefore

$$\psi(p(e)) = [c(e)]$$

where $[-] : Aut(N) \to Out(N)$ is the canonical projection. Thus the diagram in the proposition commutes. ☐

We note however that if N is abelian, then $Out(N) = Aut(N)$ so ψ actually gives us an action of Q on N, i.e. N is a Q-module.

Theorem 4.2.1. *Let $\psi : Q \to Aut(A)$ be a given group homomorphism that induces a Q-module structure on A. Then we have a bijection*

$$Ext_\psi(Q, A) \cong H^2(Q, A)$$

Proof. Let a group extension

$$\eta : \qquad 0 \to A \xrightarrow{i} E \xrightarrow{p} Q \to 1$$

in $Ext_\psi(Q, A)$ be given. Choose a normalized section $s : Q \to E$ of p.

$$0 \to A \to E \to Q \to 1$$

is a crossed 1-fold resolution of Q by A. Thus it determines a cohomology class in $H^2(Q, A)$. The equivalence of crossed 1-fold extensions is precisely the same as the equivalence relation of Q by A group extensions. Hence we have a well-defined map

$$Ext_\psi(Q, A) \to H^2(Q, A)$$

In addition every 1-fold extension gives us an element of $Ext_\psi(Q, A)$. Thus we have

$$Ext_\psi(Q, A) = OpExt^1(Q, A) \cong H^2(Q, A)$$

□

4.3 Obstruction class

Definition 4.3.1. *Let two group extensions*

$$1 \to N \xrightarrow{i} E_1 \xrightarrow{p} Q \to 1 \quad 1 \to N' \xrightarrow{i'} E_2 \xrightarrow{p'} Q \to 1$$

be given where the center of N and N' are the same. Let $E_1 \times_Q E_2$ be the pullback of

$$\begin{tikzcd} & E_2 \arrow[d, "p'"] \\ E_1 \arrow[r, "p"'] & Q \end{tikzcd}$$

and let S be the normal subgroup of $E_1 \times_Q E_2$ consisting of elements of the form (n, n^{-1}) for $n \in ZN = ZN'$. Then we get an induced surjection

$$\frac{E_1 \times_Q E_2}{S} \twoheadrightarrow Q$$

which sends (e_1, e_2) to $p(e_1) = p'(e_2)$. This surjection has kernel $N \times N'/S$. The extension

$$1 \to \frac{N \times N'}{S} \to \frac{E_1 \times_Q E_2}{S} \to Q \to 1$$

is known as the product of the given extensions.

Let a group extension

$$\eta: \qquad 1 \to N \xrightarrow{i} E \xrightarrow{p} Q \to 1$$

be given.

The automorphism crossed module $N \xrightarrow{c} Aut(N)$ gives rise the crossed resolution

$$0 \to ZN \to N \xrightarrow{c} Aut(N) \twoheadrightarrow Out(N) \to 1$$

By theorem 2.4.3 and 2.4.4 we note that we can lift any group homomorphism $\psi : Q \to Out(N)$ as follows

$$\begin{tikzcd}
F_4(Q) \arrow[r, "\delta_4"] \arrow[d] & F_3(Q) \arrow[r, "\delta_3"] \arrow[d, "f_3"] & F_2(Q) \arrow[r, "\delta_2"] \arrow[d, "f_2"] & F_1(Q) \arrow[r, "\epsilon"] \arrow[d, "f_1"] & Q \arrow[d, "\psi"] \\
0 \arrow[r] & ZN \arrow[r] & N \arrow[r] & Aut(N) \arrow[r] & Out(N)
\end{tikzcd}$$

It is clear from this that f_3 determines a cocycle in $Z^3(Q, ZN)$. By theorem 2.4.4 any other lift gives a cocycle that differs by a coboundary, and every cocycle in this cohomology class is induced by some lift. We denote the induced cohomology class by $\theta_\psi \in H^3(Q, ZN)$.

Proposition 7. *The set $Ext_\psi(Q, N)$ is empty if and only if $\theta_\psi \neq 0$.*

Proof. Suppose there exists an extension

$$1 \to N \xrightarrow{i} E \xrightarrow{p} Q \to 1$$

which induces ψ. Choose a normalized section $s : Q \to E$ of P. We can then define $f_1 : F_1(Q) \to Aut(N)$ to send $q \in Q$ to conjugation by $s(q)$. Define $f_2 : F_2(Q) \to N$ by sending (q, q') to

$$f_2(q, q') = s(q)s(q')s(qq')^{-1}$$

Finally define $f_3 : F_3(Q) \to ZN$ by

$$f_3(q, q', q'') = f_2(\delta_3(q, q', q''))$$

We note

$$\begin{aligned}
{}^{s(q)}f_2(q', q'')f_2(q, q'q'') &= s(q)s(q')s(q'')s(q'q'')^{-1}s(q)^{-1}s(q)s(q'q'')s(qq'q'')^{-1} \\
&= s(q)s(q')s(q'')s(qq'q'')^{-1}
\end{aligned}$$

$$\begin{aligned}
f_2(q, q')f_2(qq', q'') &= s(q)s(q')s(qq')^{-1}s(qq')s(q'')s(qq'q'')^{-1} \\
&= s(q)s(q')s(q'')s(qq'q'')^{-1}
\end{aligned}$$

so $f_3(q, q', q'') = 1$ and therefore it represents 0 in $H^3(Q, ZN)$.

Now suppose $\theta_\psi = 0$. When choosing a lift of ψ we can choose f_1, f_2, f_3 such that $f_1(1) = 1$ and $f_2(1, g) = f_2(g, 1) = 1$ and $f_3 = 0$. Let $E = Q \times N$ with group structure defined by

$$(a, x)(b, y) = (af_1(x)(b)f_2(x, y), xy)$$

Clearly $(1,1)$ is an identity element, and assocativity follows from

$$f_2(\delta_3(x,y,z)) = 0$$

Thus E is a group and we clearly have a group extension

$$1 \to N \to E \to Q \to 1$$

which induces ψ via the section $s : Q \to E$, $q \mapsto (q,1)$.

We have now shown that there exists a group extension inducing ψ if and only if $\theta_\psi = 0$. $\square$

Proposition 8. *Let a group extension* η_1

$$1 \to N \xrightarrow{i} E_1 \xrightarrow{p_1} Q \to 1$$

inducing $\psi : Q \to Out(N)$ *be given. Multiplication then gives a bijection*

$$Ext(Q, ZN) \to Ext_\psi(Q, N)$$

Proof. Let a group extension η_2

$$1 \to N \xrightarrow{i} E_2 \xrightarrow{p_2} Q \to 1$$

be a group extension in $Ext_\psi(Q,N)$. We may then choose normalized sections $s_i : Q \to E_i$ of p_i such that conjugation by $s_1(q)$ and conjugation by $s_2(q)$ gives the same automorphism $\alpha(q) \in Aut(N)$. By the lifting theorem we get

$$\begin{tikzcd}
F_4(Q) \arrow[r, "\delta_4"] \arrow[d, "l_i"] & F_3(Q) \arrow[r, "\delta_3"] \arrow[d, "g_i"] & F_2(Q) \arrow[r, "\delta_2"] \arrow[d, "f_i"] & F_1(Q) \arrow[r, "\epsilon"] \arrow[d, "\alpha"] & Q \arrow[d, "\psi"] \\
0 \arrow[r] & ZN \arrow[r] & N \arrow[r] & Aut(N) \arrow[r] & Out(N)
\end{tikzcd}$$

for $i = 1,2$. By the previous proposition the maps l_i represent 0 in $H^3(Q, ZN)$ so we can choose lifts in such a way that $f_3^i = 0$. We note

$$c(f_2^1(x,y)f_2^2(x,y)^{-1}) = f_1(x)f_1(y)f_1(xy)^{-1}f_1(xy)f_1(y)^{-1}f_1(x)^{-1} = 1$$

By 2.4.4 we get $h : F_1(Q) \to N$ and $d : F_2(Q) \to ZN$ such that

$$c(h(x)) = \alpha(x)\alpha(x)^{-1} = 1$$

$$d(x,y) = g_2(x,y)g_1(x,y)^{-1}h(\delta_2(x,y))$$

$$d(\delta_3(x,y,z)) = l_1(x,y,z)l_2(x,y,z)^{-1} = 0$$

We can choose $h(x) = 0$. In that case

$$d(x,y) = g_2(x,y)g_1(x,y)^{-1} \in ZN$$

Now define a group structure on $D = ZN \times Q$ by

$$(a,x)(b,y) = (a + {}^x b + d(x,y), xy)$$

Then $(0,1)$ is an identity element. We get associativity as follows

$$\begin{aligned}((a,x)(b,y))(c,z) &= (a + {}^x b + d(x,y), xy)(c,z) \\ &= (a + {}^x b + d(x,y) + {}^{xy} c + d(xy,z), xyz)\end{aligned}$$

$$\begin{aligned}(a,x)((b,y)(c,z)) &= (a,x)(b + {}^y c + d(y,z), yz) \\ &= (a + {}^x b + {}^{xy} c + {}^x d(y,z) + d(x,yz), xyz)\end{aligned}$$

Thus for associativity to hold we must have

$$d(x,y) + {}^{xy} c + d(xy,z) = {}^{xy} c + d(xy,z)$$

which holds since d is a cocycle.

Define $ZN \to D$ by $n \mapsto (n,0)$. And define $D \xrightarrow{p'} Q$ by $(0,q) \mapsto q$. We then have a group extension

$$1 \to ZN \to D \xrightarrow{p'} Q \to 1$$

Choose a section $s' : Q \to D$ of p'. Multiplication in D is then given by

$$ns'(q)n's(q') = n^q n' d(q,q')s'(qq')$$

Multiplication in E_i is given by

$$ns_i(q)n's_i(q') = n^q n' g_i(q,q')s_i(qq')$$

Thus multiplication in the product $D \otimes E_1$ with section $s : Q \to D \otimes E_1$ is given by

$$ns(q)n's(q') = n^q n' d(q,q')g_1(q,q')s(qq')$$

But $d(q,q')g_1(q,q') = g_2(q,q')$ so multiplication in $D \otimes E_1$ and in E_2 is the same. Therefore they are equivalent.

Let us now be given two extensions

$$1 \to ZN \to D_i \to Q \to 1$$

with chosen sections $r_i : Q \to D_i$ and let $d_i : Q^2 \to ZN$ be cocycles determined by them. Then

$$r_i(q)r_i(q') = d_i(q,q')r_i(qq')$$

We now form the products $E_1 = E \otimes D_1$ and $E_2 = E \otimes D_2$ with sections $s_i : Q \to E \otimes D_i$. Suppose these two extensions are equivalent. Then we have an isomorphism $\tau : E_1 \to E_2$. We have

$$\tau(s_1(q)) = b(q)s_2(q) \qquad \text{for some } b(q) \in ZN$$

because $p_2\tau = p_1$. We apply τ to

$$s_1(q)n = {}^q n s_1(q)$$

and get

$$b(q)s_2(q)n = {}^q n b(q)s_2(q)$$

Inserting $s_2(q)n = {}^q n s_2(q)$ we get

$$b(q)^q n = {}^q n b(q)$$

which implies that $b(q) \in ZN$ since conjugation by q is an automorphism. Now apply τ to

$$s_1(q)s_1(q') = d_1(q,q')f(q,q')s_1(qq')$$

to get

$$b(q)s_2(q)b(q')s_2(q') = d_1(q,q')f(q,q')b(qq')s_2(qq')$$

but since $b(q)$ is in the center of N and

$$s_2(q)s_2(q') = d_2(q,q')f(q,q')s_2(qq')$$

we get

$$b(q) + {}^q b(q') = d_1(q,q') - d_2(q,q') + b(qq')$$

But this means that d_1 and d_2 are cohomologous so D_1 and D_2 are equivalent.

□

4.4 The universal obstruction class

We have now seen how to every map $Q \to Out(N)$ we can associate a canonical obstruction to the existence of group extensions of Q by N that induce our given map. This is a pleasant result, but it is still a bit cumbersome that we have to deal with what maps $Q \to Out(N)$ are induced. Ideally we would like something like for abelian kernels where we simply get a bijection of equivalence classes of extensions with the second cohomology group. It turns out that for a kernel N we actually have an obstruction to such a correspondence.

Let $\theta_N \in H^3(Out(N), ZN)$ be the canonical obstruction associated with the identity $Out(N) \to Out(N)$. In fact given a map $\psi : Q \to Out(N)$ we have an induced map

$$H^3(Out(N), ZN) \xrightarrow{\psi^*} H^3(Q, ZN)$$

and we have $\psi^*\theta_N = \theta_\psi$. This is clear since given a lift of $Out(N) \to Out(N)$ we can further extend it as in the following diagram

$$\begin{tikzcd}
F_4(Q) \arrow[r] \arrow[d, dashed] & F_3(Q) \arrow[r] \arrow[d, dashed] & F_2(Q) \arrow[r] \arrow[d, dashed] & F_1(Q) \arrow[r] \arrow[d, dashed] & Q \arrow[d, "\psi"] \\
F_4(Out(N)) \arrow[r] \arrow[d] & F_3(Out(N)) \arrow[r] \arrow[d] & F_2(Out(N)) \arrow[r] \arrow[d] & F_1(Out(N)) \arrow[r] \arrow[d] & Out(N) \arrow[d, "1"] \\
0 \arrow[r] & ZN \arrow[r] & N \arrow[r] & Aut(N) \arrow[r] & Out(N)
\end{tikzcd}$$

where $F_iQ \to F_iOut(N)$ is just induced by $\psi : Q \to Out(N)$.

Theorem 4.4.1. *If the obstruction θ_N vanishes, then θ_ψ vanishes for all $\psi : Q \to Out(N)$.*

Proof. If θ_N vanishes, then

$$\theta_\psi = \varphi^*\theta_N = \varphi^*0 = 0$$

□

4.5 Equivalence of invariants of $Baut(BG)$

Theorem 4.5.1. *The crossed 2-fold extension*

$$\mathrm{cosk}^2\mathrm{cotr}_2\Pi\overline{W}aut(BN)$$

is equivalent to

$$0 \to ZN \to N \to Aut(N) \to Out(N) \to 1$$

Proof. Let $X = \overline{W}aut(BN)$.

A 1-simplex of $\overline{W}aut(BN)$ is just an automorphism of N so we get a map

$$f_1 : \Pi X_1 \to \mathrm{Aut}(N)$$

A 2-simplex consists of invertible maps $g_1 : \Delta^1 \times BG \to BG$ and $g_0 : BG \to BG$. For every such 2-simplex associate $g_1(01,e)^{-1} \in N$. This gives a map

$$f_2 : \Pi X_2 \to N$$

For a two simplex (g_1, g_0) we have

$$\begin{aligned} f_1(\delta_2(g_1,g_0)) &= f_1([d_1g_1][g_0][d_0(g_1)g_0]^{-1}) \\ &= (d_1g_1)(g_0)(g_0)^{-1}(d_0(g_1))^{-1} \\ &= (d_1g_1)(d_0g_1)^{-1} = (c(g(01,e)^{-1}d_0g_1)(d_0g_1)^{-1} = c(g(01,e)^{-1}) \end{aligned}$$

$$c(f_2(g_1,g_0)) = c(g_1(01,e)^{-1})$$

Thus

$$\begin{tikzcd} \Pi X_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & \Pi X_1 \arrow[d, "f_1"] \\ N \arrow[r, "c"'] & Aut(N) \end{tikzcd}$$

commutes.

One can similarily see that the image of δ_3 is in the kernel of f_2. Thus we can mod out by $Im(\delta_3)$ to get a diagram

$$\begin{tikzcd} ZN \arrow[r] \arrow[dr] & \frac{\Pi X_2}{Im(\delta_3)} \arrow[r, "\delta_2"] \arrow[d, "f_2"] & \Pi X_1 \arrow[r] \arrow[d, "f_1"] & Out(N) \\ & N \arrow[r, "c"'] & Aut(N) \arrow[ur] & \end{tikzcd}$$

where $ZN \to \frac{\Pi X_2}{Im(\delta_3}$ sends $z \in ZN$ to the 2-simplex that is the identity on the boundary and $g_1(01,e) = z^{-1}$.

This commutes so $\mathrm{cosk}^2\mathrm{cotr}_2\Pi X$ is equivalent to the automorphism crossed extension. □

Corollary 4.5.2. *The non-trivial Postnikov invariant of $\overline{W}aut(BN)$ is the canonical obstruction $\theta_N \in H^3(Out(N), ZN)$.*

Proof. The cohomology class associated to the automorphism crossed extension is clearly θ_N, and by theorem 3.10.1 the cohomology class associated to

$$\mathrm{cosk}^2\mathrm{cotr}_2\Pi\overline{W}aut(BN)$$

is the first non-trivial Postnikov invariant of $\overline{W}aut(BN)$. □

4.6 Examples

Let D_{16} be the dihedral group of order 16. We write it

$$D_{16} = \langle a, b | a^8 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$$

Let $\alpha \in Aut(D_{16})$ be the automorphism defined by

$$\alpha(a) = a^5 \qquad \alpha(b) = ab$$

This makes sense because

$$\alpha(a^8) = a^{40} = 1 \quad \alpha(b^2) = abab = aa^{-1} = 1$$

$$\alpha(abab) = a^5aba^5ab = a^6ba^6b = a^6bba^{-6} = 1$$

It is an automorphism as the map $a \mapsto a^5$, $b \mapsto a^3b$ defines an inverse.

We have

$$\alpha(\alpha(a)) = a^{25} = a \qquad \alpha(\alpha(b)) = a^6b$$

Therefore α^2 is conjugation by a^3 or a^{-1}, so it represents 0 in $Out(D_{16})$. We can therefore define

$$\psi : C_2 \to Out(D_{16}), \qquad x \mapsto [\alpha]$$

where x denotes the generate of the cyclic group C_2 of order 2.

Assume there exists an

$$1 \to D_{16} \xrightarrow{i} E \xrightarrow{p} C_2 \to 1$$

inducing ψ. Let $s : C_2 \to E$ be a normalized section such that conugation by $s(x)$ is α. $s(e)^2 \in D_{16}$ because $\alpha^2 \in Inn(D_{16})$. Conjugation by $s(e)^2$ must then be the same as conjugation by a^3 or a^{-1}. Hence we have $[a^3, s(e)] = 1$ or $[a^{-1}, s(e)] = 1$. However

$$[a^3, s(e)] = a^3s(e)a^5s(e)^{-1} = a^3c(s(e))(a^5) = a^3\alpha(a^5) = a^4 \neq 1$$

$$[a^7, s(e)] = a^7s(e)a^1s(e)^{-1} = a^7c(s(e))(a^1) = a^7\alpha(a^1) = a^4 \neq 1$$

This is a contradiction. Thus $Ext_\psi(C_2, D_{16})$ is empty. We must therefore have that the canonical obstruction $\theta_\psi \neq 0$, but then the universal obstruction cannot vanish either so $\theta_N \neq 0$. This then also implies that the postnikov invariant of $\overline{W}aut(BD_{16})$ is non-trivial.

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