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Bachelor Thesis

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Non-Euclidean Geometry

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Contents

Abstract

Inspired by the modern-day version of Euclids parallel postulate, I will be exploring geometries that differ from Euclidean geometry in the sense that they violate this axiom. The two geometries considered are spherical and hyperbolic geometry, and I will show what the straight lines on these two curved surfaces look like. As a consequence of the Lorentz inner product being introduced for modelling hyperbolic space, I will briefly explain how to interpret some of the various angles that arise between different sorts of vectors.

Furthermore, I will dive into inversive geometry, which is the geometry on Euclidean space dealing with reflections in surfaces we can topologically categorize as spheres. These reflections generate a group of transformations known as Möbius transformations. These transformations are conformal, that is, they preserve angles. Eventually, I will show how the group of isometries of hyperbolic space relate to this group of Möbius transformations of Euclidean space. In fact, it turns out that these two groups are isomorphic, leading to a better understanding of the duality of the different geometries.

Introduction

The first part of the thesis will be spent briefly exploring geometries that violate the modern-day version of the parallel postulate of Euclidean geometry

Through a point not lying on a given infinite straight line there is but one infinite straight line parallel to the given line.

The two geometries in question are Spherical Geometry and Hyperbolic Geometry. The model used for spherical geometry is the n-sphere S^n and the model for hyperbolic geometry will be introduced as the *hyperboloid* H^n, each equipped with its own intrinsic metric. The big change in hyperbolic geometry is the introduction of the *Lorentz Inner Product*. The two geometries violate the parallel postulate because one can show the following holds

Through a point x not lying on a line l on the sphere there is no line parallel to l.

Through a point x not lying on a line l on the hyperboloid there are infinitely many lines parallel to l.

where, in the above statements, a line refers to each of the two geometries' own notion of a line, equivalent to the straight lines of Euclidean geometry. In fact, in Chapter 2 and 3, I will be studying these lines, called *Geodesics*. They are the image of a certain type of distance-preserving functions called *Geodesic Lines*. For example, I will prove the following

> Theorem: The geodesics of spherical geometry are the great circles on the sphere.

> Theorem: The geodesics of hyperbolic geometry are the hyperbolic lines, ie. intersections of H^n with Euclidean planes.

Furthermore, the lorentz inner product gives rise to a notion of vectors in $\mathbb{R}^{n+1}$ with different attributes and different angles. These are the *Space-like, Time-like* and *Light-like* vectors. I will shortly look into how to interpret the different angles between these vectors.

In the thesis' second part, I will dive into Inversive Geometry, namely the geometry resulting from transformations of Euclidean spaces in spheres, and see how this relates to hyperbolic geometry. The group of transformations generated by these reflections are called *Möbius Transformations*. The whole purpose of Chapter 4 will be to eventually show the following

> Theorem: The group of isometries of our model for hyperbolic geometry, H^n, is isomorphic to the group of Möbius transformations of the one-point compactification of the Euclidean $(n-1)$-dimensional space.

This will lead to a better understanding of how these geometries relate.

The entire thesis, including the appendix, will be based on chapter 1, 2, 3 and 4 of Ratcliffes book [1], except where otherwise noted. Every now and again I will explicitly refer to the appendix for results or proofs.

Geodesics

First I will give a few general definitions for geometries modelled on metric spaces. Also, I introduce the metric space E^n as $\mathbb{R}^n$ equipped with the usual Euclidean metric.

Definition 1.1 *A geodesic arc in a metric space X is a distance-preserving function $\alpha : [a, b] \to X$, from a closed interval in $\mathbb{R}$ with $a < b$.*

One can show that for a curve $\alpha : [a, b] \to E^n$ to be a geodesic arc, it must precisely be both linear and satisfy $|a'(t)| = 1$ for $t \in [a, b]$.

Definition 1.2 *A geodesic segment from x to y is the image of a geodesic arc $\alpha : [a, b] \to X$ where $\alpha(a) = x$ and $\alpha(b) = y$.*

The geodesic segments of E^n are its lines.

Definition 1.3 *I call a metric space X geodesically convex if and only if for each pair of points $x, y \in X$ that are not alike, there is a unique geodesic segment joining x and y.*

E^n is geodesically convex because any two points x, y, that are not alike, are joined by a unique line segment.

Definition 1.4 *A metric space is geodesically connected if and only if for each pair of points $x, y \in X$ that are not alike, there is a some geodesic segment joining x and y.*

I will show, later on, that in spherical geometry, S^n is geodesically connected but *not* geodesically convex.

Definition 1.5 *A geodesic line in a metric space X is a locally distance preserving function $\lambda : \mathbb{R} \to X$. In other words; for each $x \in \mathbb{R}$ there is a neighbourhood U_x of x on which λ preserves distances.*

Definition 1.6 *A geodesic in a metric space X is the image of a geodesic line.*

In the next two chapters, examples of such geodesic lines and its geodesics will be introduced. In particular, the geodesics of the sphere is what you would intuitively believe it to be, namely the great circles.

Spherical Geometry

The standard model used for spherical geometry is the well-known n-sphere given as

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \,:\, |x| = 1 \right\} \tag{2.0.1}$$

equipped with the intrinsic metric on the sphere, d_s, which I shall now introduce.

2.1 The Spherical Metric

Definition 2.1 *For $x, y \in S^n$ I define the spherical distance between x and y as*

$$d_S(x, y) = \theta(x, y) \tag{2.1.1}$$

where $\theta(x, y)$ is the Euclidean angle between two vectors $x, y \in E^n$. This is the unique real number θ satisfying $x \cdot y = |x|\,|y| \cos\theta$.

And we see that $0 \leq d_S(x, y) \leq \pi$. We call two vectors of S^n *antipodal* if $y = -x$. Thus, the orthogonal transformations of $\mathbb{R}^{n+1}$ obviously preserves spherical distances, because they preserve inner products.

Theorem 2.2 *The spherical distance function d_S is a metric on S^n.*

Proof. It is clearly nonnegative, nondegenerate and symmetric. What is left is to prove the *triangle inequality.* Now, three vectors x, y, z span a vector subspace of dimension at most 3. Because I can freely transform them by an orthogonal transformation by Theorem A.13, I can assume without loss of generality that the three vectors lie in the subspace spanned by e_1, e_2, e_3. Using Theorem A.14, Cauchys inequality, the addition formula for cos and (A.5)

$$\begin{aligned}
\cos(\theta(x,y) + \theta(y,z)) &= \cos\theta(x,y)\cos\theta(y,z) - \sin\theta(x,y)\sin\theta(y,z) = (x \cdot y)(y \cdot z) - |x \times y||y \times z| \\
&\leq (x \cdot y)(y \cdot z) - (x \times y) \cdot (y \times x) = (x \cdot y)(y \cdot z) - ((x \cdot y)(y \cdot z) - (x \cdot z)(y \cdot y)) \\
&= x \cdot z = \cos\theta(x,z)
\end{aligned}$$

And as cos is stricly decreasing on the interval $[0, \pi]$ the above must imply that $\theta(x, z) \leq \theta(x, y) + \theta(y, z)$
□

2.2 Spherical Geodesics

Definition 2.3 *A great circle of S^n is the intersection of S^n with a 2-dimensional vector subspace of $\mathbb{R}^{n+1}$.*

It is clear that two linearly independent vectors x, y on S^n are contained within a unique great circle, namely $S(x, y) = V(x, y) \cap S^n$ where V is the subspace spanned by the two. If, however, they are antipodal, there is a continuum of great circles containing x and y.

Definition 2.4 *Three points $x, y, z \in S^n$ are spherically collinear if and only if a great circle contains all three.*

Lemma 1 *If x, y, z are in S^n and $\theta(x, z) = \theta(x, y) + \theta(y, z)$ then x, y, z are spherically collinear.*

Proof. The three vectors x, y, z span a vector subspace of dimension at most 3. By the calculations in the proof for the preceding theorem

$$(x \times y) \cdot (y \times z) = |x \times y|\,|y \times z|$$

and so they are linearly dependant, meaning $(x \times y) \times (y \times z) = 0$. It now follows from Theorem A.14(3) and A.14(2) that x, y, z are linearly dependant, hence lying on a 2-dimensional vector subspace of $\mathbb{R}^{n+1}$. $\square$

Theorem 2.5 *Let $\alpha : [a, b] \to S^n$ be a curve with $b - a < \pi$. The following are equivalent*

(1) α is a geodesic arc

(2) There are orthogonal vectors $x, y \in S^n$ such that $\alpha(t) = (\cos(t-a))\, x + (\sin(t-a))\, y, \quad t \in [a, b]$

(3) The curve α satisfies $a'' + a = 0$

Proof. Let α be a such curve as described above. Letting α' be the coordinate-wise derivative of α, I see that for A, an orthogonal matrix, I have $(A\alpha)' = A\alpha'$. That means condition (3) holds for α if and only if it holds for $A\alpha$. Thus, I might as well transform α by A when going through the proof.

Suppose (1) holds. let $t \in [a, b]$. Then I have

$$\begin{aligned} \theta(\alpha(a), \alpha(b)) &= |b - a| = b - a = (t - a) + (b - t) \\ &= \theta(\alpha(t), \alpha(a)) + \theta(\alpha(b), \alpha(t)) \end{aligned}$$

meaning that $\alpha(t), \alpha(a)$ and $\alpha(b)$ are spherically collinear. Because $b - a < \pi$ I see that $\theta(\alpha(a), \alpha(b)) < \pi$ and so the points $\alpha(a), \alpha(b)$ are not antipodal. Hence $\alpha(a), \alpha(b)$ lie on a unique great circle S on S^n. And so, the image of α is contained on this great circle by the above as t was arbitrary. I can assume $n = 1$ then. If I apply a rotation of the form

$$\begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}$$

on α, I can rotate $\alpha(a)$ to e_1. But then

$$e_1 \cdot \alpha(t) = \alpha(a) \cdot \alpha(t) = \cos\theta(\alpha(a), \alpha(t)) = \cos(t - a)$$

meaning that $e_2 \cdot \alpha(t) = \pm\sin(t - a)$ for *all* $t \in [a, b]$ seeing as α is continuous and because for all vectors $x = (x_1, x_2)$ on S^1, including $\alpha(t)$, I must have that

$$(e_1 \cdot x)^2 + (e_2 \cdot x)^2 = x_1^2 + x_2^2 = 1$$

If $e_2 \cdot \alpha(t) = -\sin(t - a)$ I can apply a reflection

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

ultimately yielding

$$\alpha(t) = \cos(t - a)e_1 + \sin(t - a)e_2$$

and so (1) implies (2). Assuming (2) holds, for all $s, t \in [a, b]$ with $s \leq t$, I get from the addition formula for cos;

$$\cos\theta(\alpha(t), \alpha(s)) = \alpha(s) \cdot \alpha(t) = \cos(s - a)\cos(t - a) + \sin(s - a)\sin(t - a) \tag{2.3.1}$$

$$= \cos(t - s) \tag{2.3.2}$$

and as $t - s < \pi$, I have $\theta(\alpha(s), \alpha(t)) = t - s$ and so α is a geodesic arc. Thus, (2) and (1) are equivalent.

Straight-forward differentiation shows (2) implies (3). If (3) holds then it is a second order linear differential equation with solution

$$\alpha(t) = \cos(t-a)\alpha(a) + \sin(t-a)\alpha'(a)$$

We know that $\alpha(t) \cdot \alpha(t) = 1$, and differentiating this I get $\alpha(t) \cdot \alpha(t)' = 0$ for all t, including a. Thus, $\alpha(a)$ and $\alpha(a)'$ are in particular orthogonal. As $|\alpha(t)| = |\alpha(t)|^2 = 1$, if I write out

$$|\alpha(t)|^2 = \cos^2(t-a) + \sin^2(t-a)|\alpha'(a)|^2$$

I see that $|\alpha'(a)|$ must be 1, meaning $\alpha'(a) \in S^1$. And so, (3) implies (2). $\square$

In fact, (1) and (2) in Theorem 2.6 are equivalent even if $b - a = \pi$. For (2) to imply (1) even if $b - a = \pi$, notice that the arguments used in (2.3.1), (2.3.2) and just below still hold.

On the other hand, if $\alpha : [a, b] \to S^n$ is a geodesic curve, then I must have $b - a \leq \pi$ because spherical distances go no higher than π. If $b - a = \pi$, then as $\theta(\alpha(a), \alpha(b)) = \pi$ that means $\alpha(a)$ and $\alpha(b)$ are antipodal, and so infinitely many great circles contain the two points. However, the image of α is contained within a unique great circle, because otherwise - as a consequence of Lemma 1 - it would contradict α being a geodesic curve. But that means I can go through the same procedure as in the proof of Theorem 2.5 to 'construct' orthogonal vectors $x, y \in S^n$ such that $\alpha(t) = (\cos(t-a))\, x + (\sin(t-a))\, y$.

Theorem 2.6 *A function $\lambda : \mathbb{R} \to S^n$ is a geodesic line if and only if there exists orthogonal vectors x, y in S^n such that*

$$\lambda(t) = \cos(t)x + \sin(t)y \tag{2.3.3}$$

Proof. If λ satisfies (2.3.3) then we see that λ satisfies $\lambda'' + \lambda = 0$, and so the restriction of λ to any interval $[a, b]$, where $b - a < \pi$, is a geodesic arc by Theorem 2.5. Particularly, it is a distance preserving function on the neighbourhood $(a - \epsilon, b + \epsilon)$ for an appropriate ϵ. Thus, it is a geodesic line. Going in the other direction, for each $r \in \mathbb{R}$ there is an open interval U_r such that λ is a distance preserving function on U_r. Thus, for all $r \in \mathbb{R}$ there is a closed interval $[a_r, b_r]$ on which λ is a geodesic arc, particularly there exists x_r, y_r, orthogonal, such that

$$\lambda(t) = \cos(t-a)x_r + \sin(t-a)y_r, \quad t \in [a_r, b_r] \tag{2.3.4}$$

And so, I take $r, r' \in \mathbb{R}$ and consider the closed interval $[r, r']$. For each $t \in [r, r']$ I can find an open interval U_t on which λ is distance-preserving. As $[r, r']$ is compact and $\bigcup_{t \in [r, r']} U_t$ make up an open cover of $[r, r']$, there is a finite subcover $U_{t_1}, \ldots, U_{t_n}$. As $\mathbb{R}$ is linearly ordered i can assume without loss of generality that $t_1 < \ldots < t_n$. First notice that as these open sets U_{t_i} cover $[r, r']$ there is some path 'along' these open sets from r to r'. This means that there is a subset $U_{t_{j_1}}, \ldots, U_{t_{j_k}}$, appropriately enumerated, of $\bigcup U_{t_i}$ which satisfies $U_{t_{j_i}} \cap U_{t_{j_{i+1}}} \neq \emptyset$ for all $i, i' \in \{1, \ldots, k-1\}$. But then I can choose points p_{j_i} such that $p_{j_i} \in U_{t_{j_i}} \cap U_{t_{j_{i+1}}}$. As this is an intersection of open sets, it is open itself.

Particularly, there is an ϵ_i such that $(p_{j_i} - \epsilon_i, p_{j_i} + \epsilon_i) \subseteq U_{t_{j_i}} \cap U_{t_{j_{i+1}}}$. This all means I can define closed intervals in the following way;

$$I_1 = [r, p_{j_1} + \epsilon_1],\ I_i = [p_{j_{i-1}} - \epsilon_{i-1}, p_{j_i} + \epsilon_i],\ \text{for } i = 2 \ldots k-1 \text{ and } I_{k+1} = [p_{j_k} - \epsilon_k, r']$$

These intervals clearly cover $[r, r']$ and furthermore, any interval has a length less than π as λ is distance-preserving on each U_{t_i}. This all means that for each interval there are orthogonal vectors x_i, y_i for $i = 1, \ldots, k+1$ satisfying (2.3.4).

But as $p_{j_i} \in I_i \cap I_{i+1}$ for all i, then there must exist $s \in I_i$, $s' \in I_{i+1}$ such that

$$\begin{aligned}\lambda(p_{j_i}) &= \cos(s - (p_{j_i} - \epsilon_i))x_i + \sin(s - (p_{j_i} - \epsilon_i))y_i \\ &= \cos(s' - (p_{j_{i+1}} - \epsilon_{i+1}))x_{i+1} + \sin(s' - (p_{j_{i+1}} - \epsilon_{i+1}))y_{i+1}\end{aligned}$$

Now my claim is that the two 2-dimensional vector subspaces spanned by x_i, y_i and x_{i+1}, y_{i+1} respectively are the same. If this was not the case, then they would *only* intersect each other, on S^n, in $\lambda(p_{j_i})$ and $-\lambda(p_{j_i})$. But I see that also the point $p_{j_i} + \epsilon_i/2$ lies in both I_i and I_{i+1}. And so, the two vector subspaces are the same, meaning the four vectors span a two-dimensional subspace. Using this method repeatedly, we see that all x_i, y_i are linearly dependant except for two, say x_1 and y_1, as desired. This means that the vector space they span is 2-dimensional. Thus, $\lambda(r)$ and $\lambda(r')$ lie in the span of x_1 and y_1. As r, r' were arbitrary I have that

$$\lambda(t) = \cos(t)x_1 + \sin(t)y_1$$

□

Corollary 2.7 *The geodesics of S^n are its great circles.*

Theorem 2.8 *S^n is geodesically connected but not geodesically convex.*

Proof. Theorem 2.5 and the above comment shows that S^n is *geodesically connected* because for any two points $x, y \in S^n$, $d_S(x, y) \leq \pi$. Consider the vector $v = x - (x \cdot y)y$. Notice how

$$\begin{aligned}|v^2| &= |x|^2 + (x \cdot y)^2|y|^2 - 2(x \cdot y)^2 \\ &= 1 - \cos^2\theta(x, y)\end{aligned}$$

yielding

$$|v| = \sin\theta(x, y) \tag{2.3.5}$$

Then v, y are orthogonal vectors, and letting $w = v/|v|$, by Theorem 2.5

$$\alpha(t) = \cos(t - a)y + \sin(t - a)w, \quad t \in [a, a + \theta(x, y)]$$

is a geodesic arc. Also, $\alpha(a) = y$ whereas

$$\begin{aligned}\alpha(a + \theta(x, y) &= \cos\theta(x, y)y + \sin\theta(x, y)w \\ &= (x \cdot y)y + w|v| \\ &= x\end{aligned}$$

This all means that there is some geodesic segment where its terminal points are x, y. Hence, S^n is geodesically connected. However, it is not *geodesically convex* because if $x = -y$ there are infinitely many geodesic segments connecting the two points (ie. they are contained in infinitely many great circles) by the preceding corollary. $\square$

Hyperbolic Geometry

In this third chapter I explore hyperbolic geometry and see how it varies from the usual Euclidean geometry.

3.1 Lorentzian n-space and Lorentz transformations

First I will lay out the basics of Lorentzian n-space. I define the *Lorentzian Inner Product* of $x, y \in \mathbb{R}^n$ as

$$x \circ y = -x_1y_1 + x_2y_2 + \ldots + x_ny_n \tag{3.1.1}$$

This is clearly bilinear and symmetric. Nondegeneracy, however, requires an argument. Let $0 \neq x \in \mathbb{R}^n$. If x satisfies $x_1^2 = x_2^2 + \ldots + x_n^2$ then $x \circ x = 0$. However, $x_1 \neq 0$, and so I can choose $y = e_1$ and see that $x \circ y = x_1 \neq 0$. In any other case, I have that $x \circ x \neq 0$. So, by Definition A.1, it is truely an inner product. Notice how it is *not* positive definite. We call the inner product space of $\mathbb{R}^n$ with this inner product the *Lorentzian n-space*, denoted $\mathbb{R}^{1,n-1}$. For simplicity, however, I will continue to speak about $\mathbb{R}^n$ as the vector space of $\mathbb{R}^{1,n-1}$. In addition to this I will define *the Lorentzian norm* of $x \in \mathbb{R}^n$ by the complex number

$$||x|| = (x \circ x)^{\frac{1}{2}} \tag{3.1.2}$$

This is *not* a norm in the usual way, as it can take on both negative and positive imaginary values, as well as positive values. If the norm of a vector x is positive imaginary, the absolute value will be referred to as $|||x|||$. Equivalently I define the *lorentzian distance* between x and y as $d_L(x, y) = ||x - y||$.

Consider the set

$$C^{n-1} = \{x \in \mathbb{R}^n \,:\, ||x|| = 0\} \tag{3.1.3}$$

called the *hypercone*, defined by the equation $x_1^2 = x_2^2 + \ldots + x_n^2$, also called the *light cone.* If $x \in C^{n-1}$ then x is said to be *light-like*, that is positive or negative light-like, respectively, if $x_1 > 0$ or $x_1 < 0$. See Figure 1.

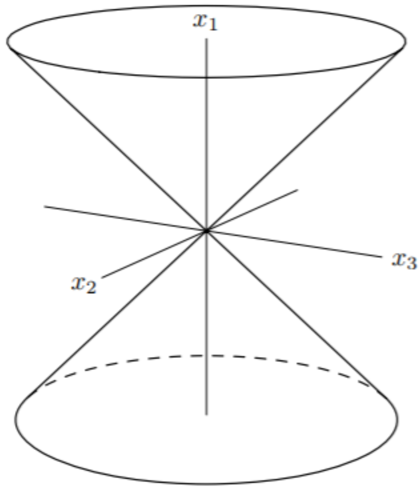

Figure 1: The light cone C^2

If $||x|| > 0$ we call x a *space-like* vector. It is clear that x is space-like if and only if it satisfies $x_1^2 < x_2^2 + \ldots + x_n^2$. The space-like vectors make up the exterior of our light cone. Lastly, if $||x||$ is imaginary, x is called *time-like*, positive and negative respectively if $x_1 > 0$ or $x_1 < 0$. Notice, again, how x is time-like if and only if $x_1^2 > x_2^2 + \ldots + x_n^2$. We also see that if x is time-like, $||x||^2 < 0$. These vectors make up the interior of the light cone.

Theorem 3.1 *The set of positive time-like vectors (or negative, respectively) is a convex subset of* $\mathbb{R}^n$.

Proof. Let x, y be positive (or negative, respectively) time-like vectors. $||tx||^2 = t^2||x||^2 < 0$, and so tx is positive (or negative, respectively) time-like for $t > 0$. Furthermore;

$$\begin{aligned}(x_1 + y_1)^2 &= x_1^2 + 2x_1y_1 + y_1^2 \\ &> (x_2^2 + \ldots + x_n^2) + 2(x_2^2 + \ldots + x_n^2)^{1/2}(y_2^2 + \ldots + y_n2)^{1/2} + (y_2^2 + \ldots + y_n^2) \\ &\geq (x_2 + y_2)^2 + \ldots + (x_n + y_n)^2\end{aligned}$$

where the last inequality follows from Cauchys inequality. Thus, for a $0 < t < 1$, I have $(1-t)x + ty$ is positive (or negative, respectively) time-like. $\square$

Definition 3.2 *A function* $\phi : \mathbb{R}^n \to \mathbb{R}^n$ *is called a Lorentz transformation if and only if it satisfies for all* $x, y \in \mathbb{R}^n$

$$\phi(x) \circ \phi(y) = x \circ y \tag{3.1.4}$$

And I call a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ *Lorentz orthonormal* if and only if $v_1 \circ v_1 = -1$ and $v_i \circ v_j = \delta_{i,j}$.
Example. The standard basis of $\mathbb{R}^n$ is Lorentz orthonormal.

And just like we did with the set of orthogonal matrices, we call the set of real $n \times n$ matrices A whose associated transformation is *Lorentzian* for the *Lorentzian* matrices. Together with matrix multiplication, they form the *Lorentz group* called $O(1, n-1)$.

Theorem 3.3 *A function* $\phi : \mathbb{R}^n \to \mathbb{R}^n$ *is a Lorentz transformation if and only if* ϕ *is linear and* $\{\phi(e_1), \ldots, \phi(e_n)\}$ *is a lorentz orthonormal basis*

Proof. The idea of the proof is as follows. If ϕ is a Lorentz transformation, considerations about $\phi(e_i) \circ \phi(e_j)$ shows that this is a lorentz orthonormal basis and that ϕ is linear. Conversely, if ϕ is linear and the above is a basis, then

$$\begin{aligned}\phi(x) \circ \phi(y) &= \phi\left(\sum_{i=1}^{n} x_i e_i\right) \circ \phi\left(\sum_{j=1}^{n} y_j e_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_i \phi(e_i) \circ \phi(e_j) \\ &= -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = x \circ y\end{aligned}$$

□

Corollary 3.4 . *Let A be a real $n \times n$ matrix. The following are equivalent*

(1) A is Lorentzian

(2) The columns of A form a Lorentz orthonormal basis

(3) $A^t J A = J$ where

$$J = \begin{bmatrix} -1 & \ldots & \ldots & 0 \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

And it clearly shows that this is all somewhat reminiscent of orthogonal matrices. By 3.4(3) we also see that if A is lorentzian, then $\det A = \pm 1$.

Definition 3.5 *A lorentzian matrix is said to be positive (negative, respectively) if and only if A transforms positive time-like vectors into positive time-like vectors (or into negative time-like vectors, respectively).*

Theorem 3.6 *Every Lorentzian matrix is either positive or negative.*

Proof. By Theorem 3.1, the set of positive (negative, respectively) time-like vectors in $\mathbb{R}^n$ are path-connected sets and thus they are particularly connected. In other words, the set of all time-like vectors has two connected components, called T_p and T_n. By the continuity of a lorentzian transformation A, the images $A(T_p)$ and $A(T_n)$ must also be connected. Thus, A maps the set of positive time-like vectors, T_p, entirely into T_p or T_n. □

The group of positive, lorentzian matrices are denoted by $PO(1, n-1)$.

Definition 3.7 *Two vectors $x, y \in \mathbb{R}^n$ are Lorentz orthogonal if and only if $x \circ y = 0$.*

Two space-like vectors can be lorentz orthogonal. As an example, $e_2 \circ e_3 = 0$. However, if either of the two vectors are time-like, neat conditions arise.

Theorem 3.8 . *Let x be a time-like vector, non-zero. If $x \circ y = 0$, then y must be space-like*

Proof. As x is time-like then $x_1^2 > x_2^2 + \ldots + x_n^2$. Hence

$$1 > \left(\sum_{i=2}^{n} x_i^2 \right) / x_1^2 \tag{3.1.5}$$

Because $x \circ y = 0$ then

$$x_1 y_1 = \sum_{i=2}^{n} x_i y_i \quad \Leftrightarrow \quad y_1 = \left(\sum_{i=2}^{n} x_i y_i \right) / x_1 \tag{3.1.6}$$

Notice how the following holds;

$$\left(\sum_{i=2}^{n} x_i^2 \right) \left(\sum_{i=2}^{n} y_i^2 \right) = \sum_{i=2}^{n} \sum_{j=2}^{n} x_i^2 y_j^2 \geq \left(\sum_{i=2}^{n} x_i y_i \right)^2 \tag{3.1.7}$$

And thus, I must have

$$\begin{aligned} ||y||^2 = -y_1^2 + \ldots + y_n^2 &= \sum_{i=2}^{n} y_i^2 - \left(\sum_{i=2}^{n} x_i y_i \right)^2 / x_1^2 \\ &\geq \sum_{i=2}^{n} y_i^2 - \left(\sum_{i=2}^{n} x_i^2 \right) \left(\sum_{i=2}^{n} y_i^2 \right) / x_1^2 \\ &= \sum_{i=2}^{n} y_i^2 \left(1 - \left(\sum_{i=2}^{n} x_i^2 \right) / x_1^2 \right) \geq 0 \end{aligned}$$

Where the last inequality follows from (3.1.5). And so, y is space-like, as it satisfies $y_1^2 < y_2^2 + \ldots + y_n^2$. □

Definition 3.9 *If V is a vector subspace of $\mathbb{R}^n$, V is said to be*

(1) time-like if and only if V contains a time-like vector

(2) space-like if and only if every nonzero vector in V is space-like

(3) light-like otherwise

Now to a theorem which shows that our group $PO(1, n-1)$ behaves well on time-like vector subspaces. It will be very useful going forward to ease other proofs.

Theorem 3.10 . *For every dimension m, the natural action of $PO(1, n-1)$ on the set of m-dimensional time-like vector subspaces of $\mathbb{R}^n$ is transitive.*

Proof By Theorem A.12, I only need to cover the following case. Let V be any m-dimensional, time-like vector subspace. I must show that there exists one W in the set of m-dimensional time-like vector subspaces such that $\exists A \in PO(1, n-1)$ with $A(W) = V$. And so, I let $W = \mathbb{R}^m$, the subspace of $\mathbb{R}^n$ spanned by $e_1, \ldots, e_m$.

Next up, I choose a basis $\{u_1, \ldots, u_m\}$ of V where u_i is time-like, say u_1 without loss of generality. Extend this to a basis $\{u_1, \ldots, u_n\}$ of E^n. Let $w_1 = u_1/|||u_1|||$. It now follows that $w_1 \circ w_1 = ||u_1||^2/|||u_1|||^2 = -1$ as u_1 was time-like. Letting $v_2 = u_2 + (u_2 \circ w_1)w_1$ I have

$$w_1 \circ v_2 = w_1 \circ u_2 + (w_1 \circ w_1)(u_2 \circ w_1) = w_1 \circ u_2 - u_2 \circ w_1 = 0$$

and by Theorem 3.9 I have v_2 is space-like. Letting $w_2 = \frac{v_2}{||v_2||}$, I now define

$$\begin{aligned} v_i &= u_i + (u_i \circ w_1)w_1 - (u_i \circ w_2)w_2 - \ldots - (u_i \circ w_{i-1})w_{i-1} \\ w_i &= \frac{v_i}{||v_i||} \end{aligned}$$

This process results in $\{w_1, \ldots, w_n\}$ being a Lorentz orthonormal basis of $\mathbb{R}^n$ and furthermore, the first m vectors form a basis of V. If I let A be the $n \times n$ matrix whose columns are $w_1, \ldots, w_n$, then by Corollary 3.4 A is lorentzian and furthermore, $A(\mathbb{R}^m) = V$ as the image under A of $\mathbb{R}^m$ is precisely the linear combinations of $w_1, \ldots, w_m$, our basis for V. As $A(e_1) = w_1$, A is also positive as e_1 is a positive time-like vector. □

Theorem 3.11 *Let x, y be positive (negative, respectively) time-like vectors in $\mathbb{R}^n$. Then $x \circ y \leq ||x||\,||y||$ with equality if and only if y and y are linearly dependant.*

Proof. By the preceding theorem there is a positive, lorentz matrix A such that $Ax = te_1$ for $t > 0$. A preserves lorentzian inner products, and so I can replace x with Ax and y with Ay. Thus, assume without loss of generality that $x = x_1e_1$ (in the case where x is a positive time-like vector). As x, y are time-like, their norms are positive imaginary, and thus

$$||x||\,||y|| < 0 \tag{3.1.8}$$

$$\begin{aligned} ||x||^2||y||^2 &= -x_1^2\left(-y_1^2 + y_2^2 + \ldots + y_n^2\right) \\ &= x_1^2y_1^2 - x_1^2(y_2^2 + \ldots + y_n^2) \\ &\leq x_1^2y_1^2 = (x \circ y)^2 \end{aligned}$$

with equality if and only if $y_2^2 + \ldots + y_n^2 = 0$, in other words, if $y = y_1e_1$ ie. if they are linearly dependant. But notice that $x \circ y = -x_1y_1 < 0$ because they are both positive time-like vectors. Thus, from the above calculations, it proves $x \circ y \leq ||x||\,||y||$. □

But Theorem 3.12 now means $\frac{x \circ y}{||x||\,||y||} \geq 1$, using (3.1.8). The above 'induces' a notion of a time-like angle. There is a unique non-negative number $\eta(x, y)$ such that

$$x \circ y = ||x||\,||y|| \cosh \eta(x, y) \tag{3.1.9}$$

Definition 3.12 *Let x,y be positive (negative, respectively) time-like vectors. The number η defined in (3.1.9) is called the time-like angle between x, y.*

3.2 Hyperbolic n-space

To build a model for hyperbolic n-space, I use the 'sphere' of unit imaginary radius defined as

$$F^n = \left\{x \in \mathbb{R}^{n+1} \,:\, ||x||^2 = -1\right\} \tag{3.2.1}$$

But this set is disconnected, as I argued before. It is a hyperboloid of two 'sheets' defined by the equation $x_1^2 - (x_2^2 + \ldots + x_{n+1}^2) = 1$. However, we get around this problem by discarding the negative sheet, meaning the sheet consisting of negative time-like vectors. I define the hyperboloid H^n as the positive sheet of F^n. See Figure 2.

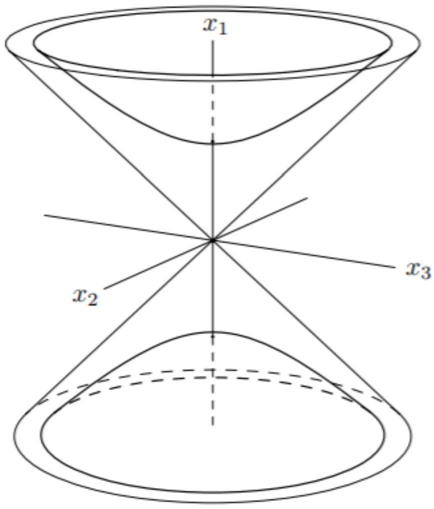

Figure 2: The hyperboloid F^2 inside the light cone C^2

Definition 3.13 *Let x, y be vectors in H^n and let $\eta(x, y)$ be the lorentzian time-like angle between the two. The hyperbolic distance between x and y is*

$$d_H(x, y) = \eta(x, y) \tag{3.2.2}$$

Notice that from (3.1.9) I get $-x \circ y = d_H(x, y)$. To prove that d_H truely is a metric on H^n I first need some results concerning lorentzian cross products.

Let x, y be vectors in $\mathbb{R}^3$ and let J be the matrix from Corollary 3.4.

Definition 3.14 . *The lorentzian cross product of x, y is defined to be*

$$x \otimes y = J(x \times y) \tag{3.2.3}$$

Observing that

$$x \circ (x \otimes y) = x \circ J(x \times y) = x \cdot (x \times y) = 0$$
$$y \circ (x \otimes y) = y \circ J(x \times y) = y \cdot (x \times y) = 0$$

I find $x \otimes y$ is lorentz orthogonal to both x and y.

Lemma 2 . *If x, y are vectors in $\mathbb{R}^3$, then $x \otimes y = J(y) \times J(x)$.*

Proof. Because J is an orientation reversing orthogonal transformation (it changes the sign of the first coordinate), I get

$$J(x \times y) = J(y) \times J(x)$$

by means of the right-hand rule for normal cross products. $\square$

From Theorem A.14 on cross products, the following theorem follows using Lemma 2 to translate it into lorentzian cross products.

Theorem 3.15 *If w, x, y, z are vectors in $\mathbb{R}^3$, the following hold*

(1) $x \otimes y = -y \otimes x$

(2) $(x \otimes y) \circ z = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

(3) $x \otimes (y \otimes z) = (x \circ y)z - (z \circ x)y$

(4) $(x \otimes y) \circ (z \otimes w) = \begin{vmatrix} x \circ w & x \circ z \\ y \circ w & y \circ z \end{vmatrix}$

Corollary 3.16 . *If x, y are positive (negative, respectively) time-like vectors in $\mathbb{R}^3$, then $x \otimes y$ is space-like and $||x \otimes y|| = -||x||\,||y|| \sinh \eta(x, y)$*

Proof. Obviously $x \times y$ is space-like because it is lorentz orthogonal to both x and y, time-like vectors. From Theorem 3.16 (4) I get

$$\begin{aligned} ||x \times y||^2 &= (x \circ y)^2 - ||x||^2||y||^2 \\ &= ||x||^2||y||^2(\cosh^2 \eta(x, y) - 1) \\ &= -\sinh^2 \eta(x, y)||x||^2||y||^2 \end{aligned}$$

□

Corollary 3.17 . *If x, y are space-like vectors in $\mathbb{R}^3$, then*

(1) $|x \circ y| < ||x||\,||y||$ *if and only if* $x \otimes y$ *is time-like*

(2) $|x \circ y| > ||x||\,||y||$ *if and only if* $x \otimes y$ *is space-like*

(3) Equality if and only if $x \otimes y$ *is light-like*

Proof. Using Theorem 3.16 (4) I get

$$||x \otimes y||^2 = (x \circ y)^2 - ||x||^2||y||^2$$

And so, $x \otimes y$ is time-like if and only if $||x \otimes y||^2 < 0$, proving (1). Likewise, $||x \otimes y||^2 > 0$ if and only if $x \otimes y$ is space-like, proving (2). Equality only holds if and only if $x \otimes y$ is light-like. □

Notice how Corollary 3.18 (2) is in opposition with Cauchys inequality concerning the Euclidean inner product. Now, I am all set.

Theorem 3.18 . *The hyperbolic distance function d_H is a metric on H^n.*

Proof. It is non-negative by the considerations leading up to formula (3.1.9). It is also symmetric as $x \circ y = y \circ x$. It is non-degenerate by Theorem 3.12. Now to prove the triangle inequality.

Let $x, y, z \in H^n$. Every positive lorentz transformation act on H^n and it obviously preserves d_H by Definition 3.15. This means I am free to transform x, y, z by a such transformation. Now, x, y, z span a vector subspace of at most 3 dimensions. By Theorem 3.11 I can assume this vector subspace is the span of e_1, e_2 and e_3. From Corollary 3.17 I get

$$||x \otimes y|| = -(-1)\sinh\eta(x,y) = \sinh\eta(x,y) \tag{3.2.4}$$

$$||y \otimes z|| = \sinh\eta(y,z) \tag{3.2.5}$$

because they all have positive imaginary norms. As y is lorentz orthogonal to both of $x \otimes y$ and $y \otimes z$, then y and $(x \otimes y) \otimes (y \otimes z)$ must be linearly dependant, as they lie in a 1-dimensional vector subspace. That must mean the latter of the two is either 0 or time-like because y was time-like. From Corollary 3.18 (1) I find

$$|(x \otimes y) \circ (y \otimes z)| < ||x \otimes y|| \, ||y \otimes z|| \tag{3.2.6}$$

with equality as a possibility if the above mentioned is the zero vector. Put together, using Theorem 3.17 (4), it all yields

$$\begin{aligned}
\cosh(\eta(x,y) + \eta(y,z)) &= \cosh\eta(x,y)\cosh\eta(y,z) + \sinh\eta(x,y)\sinh\eta(y,z) \\
&= \cosh\eta(x,y)\cosh\eta(y,z) + ||x \otimes y|| \, ||y \otimes z|| \\
&\geq \cosh\eta(x,y)\cosh\eta(y,z) + (x \otimes y) \circ (y \otimes z) \\
&= (x \circ y)(y \circ z) + (x \otimes y) \circ (y \otimes z) \\
&= (x \circ y)(y \circ z) + ((x \circ z)(y \circ y) - (y \circ z)(x \circ y)) \\
&= ||y||^2 (x \circ z) = -(x \circ z) = \cosh\eta(x,z)
\end{aligned}$$

and as cosh is strictly increasing on the positive, real axis, $\eta(x,z) \leq \eta(x,y) + \eta(y,z)$. □

Definition 3.19 . *H^n together with d_H is the metric space called the hyperbolic n-space.*

3.3 Hyperbolic Geodesics

Definition 3.20 . *A hyperbolic line of H^n is the intersection of H^n with a 2-dimensional, time-like vector subspace of $\mathbb{R}^{n+1}$.*

For x, y, two time-like vectors with V being the vector subspace they span, $L(x,y) = V(x,y) \cap H^n$ is the unique hyperbolic line containing the two.

Definition 3.21 . *Three points x, y, z of H^n are hyperbolically collinear if and only if there is a hyperbolic line L containing all three points.*

Lemma 3 . *If x, y, z are points of H^n and*

$$\eta(x,z) = \eta(x,y) + \eta(y,z)$$

the three points are hyperbolically collinear.

Proof. Because x, y, z span a time-like vector subspace of $\mathbb{R}^{n+1}$ of dimension 3 at most, we may assume $n = 2$. From the proof of Theorem 3.19 I have

$$(x \otimes y) \circ (y \otimes z) = ||x \otimes y|| \, ||y \otimes z||$$

and by Corollary 3.18, the vector $(x \otimes y) \otimes (y \otimes z)$ is light-like. But by Theorem 3.16 (3), I also get

$$(x \otimes y) \otimes (y \otimes z) = -((x \otimes y) \circ z)y$$

But y was time-like, and so $(x \otimes y) \circ z$ must be 0. Thus, x, y, z are linearly dependant by 3.17 (2), and so they lie on a 2-dimensional time-like vector subspace. In turn, they are hyperbolically collinear. □

Definition 3.22 . *Two vectors x, y in $\mathbb{R}^{n+1}$ are lorentz orthonormal if and only if $||x||^2 = -1$, $x \circ y = 0$ and $||y||^2 = 1$.*

Theorem 3.23 *Let $\alpha : [a, b] \to H^n$ be a curve. The following are equivalent:*

(1) α is a geodesic curve.

(2) There exists lorentz orthonormal vectors x, y in $\mathbb{R}^{n+1}$ such that

$$\alpha(t) = \cosh(t - a)x + \sinh(t - a)y$$

(3) The curve α satisfies $\alpha'' - \alpha = 0$.

Proof. The proof is very similiar to that of Theorem 2.5 using the properties of our lorentz inner-product and hyperbolic distance instead. It is therefore omitted.

Theorem 3.24 . *A function $\lambda : \mathbb{R} \to H^n$ is a geodesic line if and only if there are lorentz orthogonal vectors $x, y \in \mathbb{R}^{n+1}$ such that*

$$\lambda(t) = \cosh(t)x + \sinh(t)y \tag{3.3.1}$$

Proof. Suppose (3.3.1). Then it will most certainly satisfy $\lambda'' - \lambda = 0$, and so the restriction of λ to any interval $[a, b]$ is a geodesic curve by Theorem 3.24. Particularly, we can find neighbourhoods around every $t \in \mathbb{R}$ such that λ is distance-preserving on this neighbourhood. On the other hand, if λ is a geodesic line, by Theorem 3.24 the function satisfies $\lambda'' - \lambda = 0$, and thus it is a second order linear differential equation with solution

$$\lambda(t) = \cosh(t)\lambda(0) + \sinh(t)\lambda'(0) \tag{3.3.2}$$

By similar arguments as those in the proof of Theorem 2.5, $\lambda(0)$ and $\lambda'(0)$ are lorentz orthonormal. □

Corollary 3.25 . *The geodesics of H^n are its hyperbolic lines.*

3.4 Space-like and time-like angles

Definition 3.26 . *Consider the intersection of H^n with an $(m + 1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$. We call this a hyperbolic m-plane.*

A hyperbolic $(n - 1)$-plane of H^n is called a *hyperplane*. Notice that, for $n = 2$ on H^2, the hyperplanes are the hyperbolic 1-planes - the hyperbolic lines. When trying to visualize theorems and results concerning hyperplanes, this should come to mind.

If x is a space-like vector in $\mathbb{R}^{n+1}$, then the lorentzian orthogonal complement of the span of x, denoted $\langle x \rangle^L$, is an n-dimensional time-like vector subspace by Definition 3.10 as contains time-like vectors. That means $P = \langle x \rangle^L \cap H^n$ is a hyperplane. We call this the *hyperplane lorentz orthogonal to x.*

Theorem 3.27 . *Assume x, y to be linearly independent, space-like vectors in $\mathbb{R}^{n+1}$. Then the following are equivalent*

(1) $|x \circ y| < ||x||\,||y||$

(2) The vector subspace V spanned by x, y is space-like.

(3) P and Q, the hyperplanes of H^n lorentz orthogonal to x and y respectively, intersect.

Proof. Omitted. Very similar considerations follow in the proofs of the next few theorems. □

However, the above induces an idea of some sort of angle between two linearly independent, space-like vectors whose span is a space-like vector subspace. By the above theorem, there is some unique real number $\eta(x, y)$ between 0 and π such that

$$x \circ y = ||x||\,||y||\,\cos\eta(x, y) \tag{3.4.1}$$

Definition 3.28 . *The number $\eta(x, y)$ from (3.4.1) is defined to be the lorentzian space-like angle between space-like vectors.*

If I let $\lambda, \mu : \mathbb{R} \to H^n$ be geodesic lines satisfying the condition $\lambda(0) = \mu(0)$ (meaning, visually, that they intersect at a point where both lines bend on H^n), then $\lambda'(0)$ and $\mu'(0)$ span a space-like vector subspace of $\mathbb{R}^{n+1}$, seeing as how $\lambda'(0)$ and $\mu'(0)$ are lorentz orthogonal to $\lambda(0)$ and $\mu(0)$ respectively, and these last two vectors are time-like.

That means I can define the *hyperbolic angle* between λ and μ to be the lorentzian space-like angle between $\lambda'(0)$ and $\mu'(0)$ in the space-like vector subspace.

Definition 3.29 *If P is a hyperplane of H^n and $\lambda : \mathbb{R} \to H^n$ is a geodesic line with $\lambda(0)$ in P (or lies on P, in the case where P is but a geodesic line), then the hyperbolic line $L = \lambda(\mathbb{R})$ is said to be lorentz orthogonal to P if and only if P is the hyperplane of H^n lorentz orthogonal to $\lambda'(0)$.*

Theorem 3.30 . *Assume x, y to be linearly independent, space-like vectors in $\mathbb{R}^{n+1}$. Then the following are equivalent*

(1) $|x \circ y| > ||x||\,||y||$

(2) The vector subspace V spanned by x, y is time-like.

(3) P and Q, the hyperplanes of H^n lorentz orthogonal to x and y respectively, are disjoint and have a common Lorentz orthogonal hyperbolic line.

Proof. Every vector in V, except the scalar multiples of x, are scalar multiples of an element of the form $tx + y$, $t \in \mathbb{R}$. But notice that

$$||tx + y||^2 = t^2||x||^2 + 2t(x \circ y) + ||y||^2 \tag{3.4.2}$$

a quadratic polynomial in t. Its discriminant is $4(x \circ y)^2 - 4||x||^2||y||^2$ and so (3.4.2) takes on negative values (meaning the vector $tx + y$ is time-like) if and only if the discriminant is positive. In other words, if and only if $|x \circ y| > ||x||\,||y||$. Thus, (1) and (2) are equivalent.

Assume (2) holds. Then V^L is space-like. Because $V^L = \langle x\rangle^L \cap \langle y\rangle^L$, $P = \langle x\rangle^L \cap H^n$ and $Q = \langle y\rangle^L \cap H^n$, then $P \cap Q$ must be empty, because otherwise it would imply for some $z \in P \cap Q$ that $z \in V^L$ meaning it is space-like, contradicting $z \in H^n$.

Observe that $N = V \cap H^n$ is a hyperbolic line and that $V \cap \langle x\rangle^L$ is clearly a 1-dimensional subspace of $\mathbb{R}^{n+1}$ as V is spanned by x and y. Also keep in mind that $P \cap N = V \cap \langle x\rangle^L \cap H^n$.

For a vector in $V \cap \langle x\rangle^L$ I get that the solution to the equation

$$(tx + y) \circ x = 0$$

is uniquely given as $t = -\frac{x \circ y}{||x||^2}$. And furthermore, for this chosen t, I get

$$\begin{aligned}
||tx + y||^2 &= \frac{(-x \circ y)^2}{(||x||^2)^2}||x||^2 + 2 \cdot \frac{-x \circ y}{||x||^2}(x \circ y) + ||y||^2 \\
&= -\frac{(x \circ y)^2}{||x||^2} + ||y||^2 \\
&< -||y||^2 + ||y||^2 = 0
\end{aligned}$$

where the last inequality follows from the equivalence of (1) and (2). Thus, $V \cap \langle x\rangle^L$ is time-like. But this must mean that $P \cap N$ is non-empty, and so it must consist of a single point. Likewise for $Q \cap N$. Consider the point

$$u = \frac{-(x \circ y)(x/||x||) + ||x||y}{\pm\sqrt{(x \circ y)^2 - ||x||^2||y||^2}} \tag{3.4.3}$$

where the sign is chosen accordingly such that u is a positive time-like vector. I will check that u satisfies being the only point of $P \cap N$. Clearly $u \in V$ as it is a linear combination of x and y. Furthermore;

$$\begin{aligned}
u \circ x &= \frac{1}{\pm\sqrt{(x \circ y)^2 - ||x||^2||y||^2}}\left(-\frac{(x \circ y)\,||x||^2}{||x||} + ||x||\,y \circ x\right) \\
&= 0
\end{aligned}$$

meaning $u \in \langle x\rangle^L$, and clearly $||u||^2 = -1$. Thus, $u \in V \cap \langle x\rangle^L \cap H^n$ as desired. Likewise, the single point v of $Q \cap N$ is

$$v = \frac{||y||\,x - (x \circ y)(y/||y||)}{\pm\sqrt{(x \circ y)^2 - ||x||^2||y||^2}} \tag{3.4.4}$$

Let $\lambda : \mathbb{R} \to H^n$ be the geodesic line such that $\lambda(0) = u$ and $\lambda(\mathbb{R}) = N$, a choice that makes sense because of Corollary 3.26. Because $\lambda'(0)$ and x both are lorentz orthogonal to u in V, then seeing as V is a two-dimensional subspace, $\lambda'(0)$ *must* be a scalar multiple of x. But as P was the hyperplane lorentz orthogonal to x, P is the hyperplane lorentz orthogonal to $\lambda'(0)$. Thus, N is lorentz orthogonal to P, and the same goes for Q. All in all, (2) $\Rightarrow$ (3).

Assuming (3) holds let N be the common lorentz orthogonal hyperbolic line to P and Q. Then there is a 2-dimensional vector subspace W of $\mathbb{R}^{n+1}$ such that $N = W \cap H^n$. But obviously, $x \in W$ and $y \in W$ by definition. As they are linearly independent, $W = V$, and so V is time-like. $\square$

Notice how, in the above theorem, we see that N, the common hyperbolic line of two disjoint hyperplanes, is uniquely determined. Moreover, if x, y are the space-like vectors lorentz orthogonal to the disjoint hyperplanes, x and y are tangent vectors of N.

And again, if x and y are space-like vectors in $\mathbb{R}^{n+1}$ whose span is a time-like vector subspace, then by Theorem 3.31 there is a unique real positive number $\eta(x,y)$ s.t

$$|x \circ y| = ||x||\,||y||\,\cosh \eta(x,y) \tag{3.4.5}$$

Definition 3.31 . *The number $\eta(x,y)$ given in (3.4.5) is defined to be the time-like angle between space-like vectors.*

How should this 'angle' be interpreted? One can show that if x and y truely span a time-like vector subspace, with hyperplanes P, Q lorentz orthogonal to x and y respectively, then $\eta(x,y)$ is the hyperbolic distance from P to Q measured *along* the hyperbolic line N. I will not prove this, as I will prove something very similar in a few moments.

If x is a space-like vector and y a positive time-like vector in $\mathbb{R}^{n+1}$, then there is a unique nonnegative real number $\eta(x,y)$ so that the following holds

$$|x \circ y| = ||x||\,|||y|||\,\sinh \eta(x,y) \tag{3.4.6}$$

Definition 3.32 . *For a space-like vector x and a positive time-like vector y, the number $\eta(x,y)$ from (3.4.6) is called the Lorentzian time-like angle between x and y.*

And this angle is interpreted in a very similar way as the above. But first, a small lemma.

Lemma 4 *Let y be a point of H^n, and P a hyperplane of H^n. Then there is a unique hyperbolic line N of H^n passing through y and lorentz orthogonal to P.*

Proof. I can let x be a space-like vector lorentz orthogonal to P and let V be the subspace spanned by x and our vector y. Without loss of generality, I can in fact assume x to be a unit vector. Then $N = V \cap H^n$ is obviously a hyperbolic line, and it passes through y. Using again that the vectors in V, except the scalar multiples of x, are scalar multiples of $tx + y$, I get

$$(tx + y) \circ x = 0$$

which has the unique solution $t = -x \circ y$ because $||x||^2 = 1$ just as in the proof of Theorem 3.32. From (3.4.4) the point w given as

$$w = \frac{-(x \circ y)x + y}{\pm\sqrt{(x \circ y)^2 + 1}}$$

is the single point of $P \cap N$. We see that $w \circ x = 0$, $x \circ x = 1$ by assumption and

$$\begin{aligned} w \circ w &= \frac{1}{(x \circ y)^2 + 1}\left(-(x \circ y)^2 - 1\right) \\ &= -1 \end{aligned}$$

meaning that w, x are lorentz *orthonormal* vectors. Thus, by Theorem 3.26, I get

$$\lambda(t) = \cosh(t)w \pm \sinh(t)x$$

and I see $\lambda'(0) = \pm x$. But this means by Definition 3.31 that N is Lorentz orthogonal to P, and this proves the existence of a such hyperbolic line. Supposing N is a hyperbolic line passing through

y and Lorentz orthogonal to P, let $\lambda : \mathbb{R} \to H^n$ be a hyperbolic line such that $\lambda(\mathbb{R}) = N$ and $\lambda(0)$ is in P, which is a fair assumption by Corollary 3.27. But then $\lambda'(0)$ is Lorentz orthogonal to P, meaning $\lambda'(0) = \pm x$ by assumption of x. If W is a 2-dimensional time-like vector subspace such that $N = W \cap H^n$, then because both x and y are are in W, $W = V$, showing N is uniquely determined. □

Theorem 3.33 . *Let x be a space-like vector and y a positive time-like vector in $\mathbb{R}^{n+1}$. Let P be the hyperplane of H^n that is lorentz orthogonal to x. Then the angle $\eta(x, y)$ is in fact the hyperbolic distance from $y/|||y|||$ in H^n to P measured along the hyperbolic line N passing through $y/|||y|||$ lorentz orthogonal to P. Furthermore, $x \circ y < 0$ if and only if x and y are on 'opposite sides' og the hyperplane of $\mathbb{R}^{n+1}$ spanned by P.*

Proof. Notice how the existence of line N as described in the theorem is a consequence of the preceding lemma. Just like in the proof of Theorem 3.32, the single point of $P \cap N$ is

$$u = \frac{-(x \circ y)(x/||x||) + ||x||y}{\pm\sqrt{(x \circ y)^2 - ||x||^2||y||^2}} \tag{3.4.7}$$

letting $v = y/|||y|||$, I have that the distance in question is

$$\begin{aligned}
\cosh d_H(u, v) &= -u \circ v \\
&= \frac{\sqrt{(x \circ y)^2 - ||x||^2||y||^2}}{||x||\,||y||} \\
&= \sqrt{\sinh^2 \eta(x, y) + 1} \\
&= \cosh \eta(x, y)
\end{aligned}$$

showing the first part of the theorem. Furthermore, for the calculation above to hold, the sign of u must be $+$. Going back to (3.4.7) with the plus-sign, I now see that u lies 'in between' x and y in the 2-dimensional subspace V spanned by x and y - which means that the two vectors are on opposite sides of the hyperplane spanned by P - if and only if $-x \circ y$ is positive, ie. if and only if $x \circ y < 0$. □

Inversive geometry

In this chapter, I will study the group of transformations of E^n generated by reflections in hyperplanes and spheres, the latter of which I call inversions. Because it turns out that this group is in fact isomorphic to the group of isometries of H^{n+1}, leading to a deeper understanding of hyperbolic geometry.

4.1 Reflections and Inversions

Letting a be a unit vector in E^n and t a real number I can consider the hyperplane of E^n that is defined in the following way

$$P(a,t) = \{x \in E^n \,:\, a \cdot x = t\}$$

and I notice that every point x in $P(a,t)$ satisfies the equation

$$a \cdot (x - ta) = 0$$

and so, $P(a,t)$ can be thought of as the hyperplane with unit normal a passing through the point ta. Notice how I no longer require a hyperplane to be a subspace, as $t \neq 0$ means $0 \notin P(a,t)$. I now define *the reflection* ρ of E^n in the plane $P(a,t)$ given by the formula

$$\rho(x) = x + 2(t - a \cdot x)a \tag{4.1.1}$$

Theorem 4.1 *If ρ is the reflection of E^n in $P(a,t)$, the following holds*

(1) $\rho(x) = x$ if and only if $x \in P(a,t)$

(2) $\rho(\rho((x)) = x$ for all $x \in E^n$

(3) ρ is a homeomorphism

(4) ρ is an isometry

Proof. (1): Let $\rho(x) = x$, which happens if and only if $x = x + 2(t - a)a$ by (4.1.1), and this happens if and only if $(t - a \cdot x) = 0$. As $a \neq 0$, this again happens if and only if $a \cdot x = t$, meaning $x \in P(a,t)$.

The following calculations prove (2);

$$\begin{aligned}\rho(\rho(x)) &= x + 2(t - a \cdot x)a + 2(t - a \cdot (a + 2(t - a \cdot x)a))a \\ &= x + 2(t - a \cdot x)a + 2(t - a \cdot x - 2(t - a \cdot x))a \\ &= x + a(4(t - a \cdot x) - 4(t - a \cdot x)) = x\end{aligned}$$

And lastly, ρ is distance preserving because

$$\begin{aligned}|\rho(x) - \rho(y)| &= |x - y + a(2(t - a \cdot x) - 2(t - a \cdot y))| \\ &= |x - y + a(2(a \cdot y - a \cdot x))| = |x - y - 2a(a \cdot (x - y))| \\ &= |x - y - 2(x - y)| = |-(x - y)| = |x - y|\end{aligned}$$

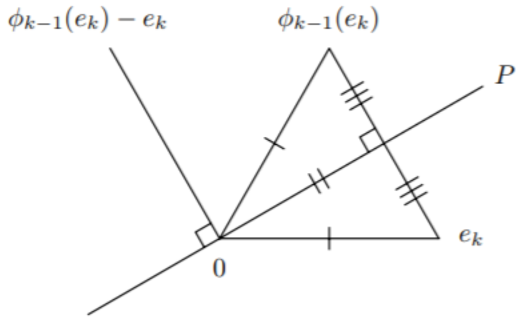

Figure 3: The reflection of e_k, first by ϕ_{k-1} and then ρ_k

as ρ is its own inverse, it is also bijective. Thus, it is an isometry. As ρ is continuous, it is a homeomorphism. □

Theorem 4.2 *Every isometry of E^n is a composition of at most $n+1$ reflections in hyperplanes.*

Proof. Letting $\phi : E^n \to E^n$ be an isometry I will set $v_0 = \phi(0)$. Let ρ_0 be the identity if $v_0 = 0$. If not, let it be the reflection in the plane $P(v_0/|v_0|, |v_0|/2)$, because in that case

$$\begin{aligned}\rho_0(v_0) &= v_0 + 2(|v_0|/2 - v_0/|v_0| \cdot v_0)v_0/|v_0| \\ &= v_0 + 2(-|v_0|/2)v_0/|v_0| \\ &= v_0 - v_0 = 0\end{aligned}$$

In any case, $\rho_0(v_0) = 0$, and so $\rho_0(\phi(0)) = 0$. By Theorem A.8, the function $\phi_0 = \rho_0\phi$ is an orthogonal transformation.

Now supposing that ϕ_{k-1} is an orthogonal transformation of E^n that fixes $e_1, \dots, e_{k-1}$. Letting $v_k = \phi_{k-1}(e_k) - e_k$ and ρ_k be the identity if $v_k = 0$ or the reflection in the plane $P(v_k/|v_k|, 0)$ otherwise, then $\rho_k\phi_{k-1}$ fixes e_k as is shown on Figure 3. Furthermore, for every $j = 1, \dots, k-1$, using that ϕ is an orthogonal transformation fixing each of the basisvectors e_j, I get

$$\begin{aligned}v_k \cdot e_j &= (\phi_{k-1}(e_k) - ek) \cdot e_j \\ &= \phi_{k-1}(e_k) \cdot e_j \\ &= \phi_{k-1}(e_k) \cdot \phi_{k-1}(e_j) \\ &= e_k \cdot e_j = 0\end{aligned}$$

and so e_j is in the plane $P(v_k/|v_k|, 0)$ for each j, meaning ρ_k fixes them. Thus $\phi_k = \rho_k\phi_{k-1}$ fixes $e_1, \dots, e_k$ and by induction there are mappings $\rho_0, \dots, \rho_n$ such that each ϕ_i is either the identity or a reflection and $\rho_n \dots \rho_0\phi$ fixes $0, e_1, \dots, e_n$. But that must mean, by considerations of basises, that $\rho_n \dots \rho_0\phi$ is the identity and so $\phi = \rho_0 \dots \rho_n$, meaning ϕ is a composition of at most $n+1$ reflections.

□

Letting a be a point of E^n and r a positive real number, the *sphere* of E^n of radius r centered at a is defined in the usual way

$$S(a, r) = \{x \in E^n : |x - a| = r\}$$

The *reflection* (or *inversion*) σ of E^n in the sphere $S(a,r)$ is given by

$$\sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a) \tag{4.1.2}$$

Theorem 4.3 *If σ is the reflection of E^n in the sphere $S(a,r)$, then*

(1) $\sigma(x) = x$ if and only if x is in $S(a,r)$

(2) $\sigma(\sigma(x)) = x$ for all $x \neq a$ and

(3) for all $x, y \neq a$

$$|\sigma(x) - \sigma(y)| = \frac{r^2|x-y|}{|x-a|\,|y-a|}$$

(4) σ is a homeomorphism

Proof. For (1), because of formula (4.1.2) I get

$$|\sigma(x) - a|\,|x-a| = r^2$$

meaning that $\sigma(x) = x$ if and only if $|x-a| = r$.

To prove (2), observe how

$$\begin{aligned}
\sigma(\sigma(x)) &= a + \left(\frac{r}{|\sigma(x)-a|}\right)^2 (\sigma(x)-a) \\
&= a + \left(\frac{|x-a|}{r}\right)^2 \left(\frac{r}{|x-a|}\right)^2 (x-a) \\
&= x
\end{aligned}$$

Lastly, to prove (3) I use

$$\begin{aligned}
|\sigma(x) - \sigma(y)| &= r^2 \left| \frac{(x-a)}{|x-a|^2} - \frac{(y-a)}{|y-a|^2} \right| \\
&= r^2 \left[\frac{1}{|x-a|^2} - \frac{2(x-a)\cdot(y-a)}{|x-a|^2\,|y-a|^2} + \frac{1}{|y-a|^2} \right]^{1/2} \\
&= \frac{r^2|x-y|}{|x-a|\,|y-a|}
\end{aligned}$$

As σ is continuous, by (2) it is a homeomorphism. □

Theorem 4.4 *Every reflection of E^n in a hyperplane or a sphere is conformal and reverses orientation.*

Proof. If ρ is the reflection of E^n in $P(a,t)$ then straightforward calculations of its derivate show that

$$\rho(x) = x + 2(t - a\cdot x)a$$

$$\rho'(x) = I - 2A$$

where A is the matrix $(a_i a_j)$, ie. the matrix where the i,j'th entrance is the product of a_i and a_j. By definition of conformality (see Section A.6), it only comes down to $\rho'(x)$. As this is independent of

t, I may assume that $t = 0$. And I can go even further. By Theorem A.13, there is an orthogonal transformation ϕ such that $\phi(a) = e_1$. Consider $\phi\rho\phi^{-1}(x)$. Using the chain rule, I get

$$\det(\phi\rho\phi^{-1})'(x) = \det \rho'(x)$$

because ϕ and ϕ^{-1} are each others inverse. But using that ϕ is linear and preserves inner products I get

$$\begin{aligned}\phi\rho\phi^{-1}(x) &= \phi(\phi^{-1}(x) - 2(a \cdot \phi^{-1}(x))a) = x - 2(a \cdot \phi^{-1}(x))e_1 \\ &= x - 2(\phi(a) \cdot x)e_1 = x - 2(e_1 \cdot x)e_1\end{aligned}$$

and so $\phi\rho\phi^{-1}$ is in fact the reflection in $P(e_1, 0)$. Going forward, I may thus assume that $a = e_1$. But then ρ is orthogonal because

$$\begin{aligned}\rho(x) \cdot \rho(y) &= x \cdot y - 2(a \cdot x)(a \cdot y) + 4(a \cdot x)(a \cdot y)|a|^2 - 2(a \cdot y)(a \cdot x) \\ &= x \cdot y\end{aligned}$$

as $|a|^2 = |e_1|^2 = 1$. Thus $I - 2A$ is an orthogonal matrix, and ρ is conformal.

Furthermore, as $a = e_1$, the matrix $I - 2A$ is in fact the matrix J which was first introduced in Corollary 3.4. And so

$$\det \rho'(x) = \det(I - 2A) = -1$$

which means ρ reverses orientation by Definition A.15.

The case where ρ is the reflection in a sphere $S(a, r)$ follows similarly. □

4.2 Stereographic Projection

Identifying E^n with $E^n \times \{0\}$ in E^{n+1}, then the *stereographic projection* π of E^n onto $S^n - e_{n+1}$ is defined by projecting $x \in E^n$ in towards e_{n+1} until it meets the sphere S^n in the uniquely given point $\pi(x)$, see Figure 4.

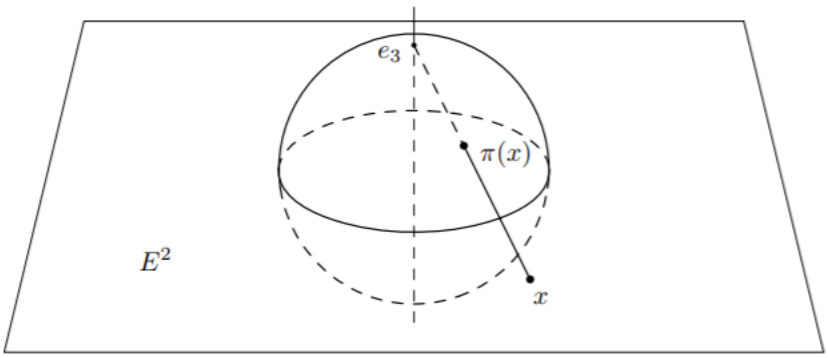

Figure 4: Stereographic projection of the sphere E^2 onto S^2 in E^3

As $\pi(x)$ is on the line passing through x in the direction of $e_{n+1} - x$, there must be some scalar s such that

$$\pi(x) = x + s(e_{n+1} - x)$$

However, the condition that $|\pi(x)|^2 = 1$ means that s has the following value

$$s = \frac{|x|^2 - 1}{|x|^2 + 1}$$

leading to the explicit formula for the projection

$$\pi(x) = \left(\frac{2x_1}{1 + |x|^2}, \dots, \frac{2x_n}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right) \tag{4.2.1}$$

As $|\pi(x)|^2 = 1$ it does indeed map onto $S^n - e_{n+1}$, and it is clear, by looking at the coordinates of the function, that it is both surjective and injective, thus a bijection of E^n onto $S^n - e_{n+1}$.

There is a nice interpretation of stereographic projection in terms of inversion in spheres as explained in section 4.1. Letting σ be the reflection of E^{n+1} in the sphere $S(e_{n+1}, \sqrt{2})$, by (4.1.2) I get

$$\sigma(x) = e_{n+1} + \frac{2(x - e_{n+1})}{|x - e_{n+1}|^2} \tag{4.2.2}$$

and if $x \in E^n$, then because e_{n+1} is linearly independent from x, I get $|x - e_{n+1}|^2 = 1 + |x|^2$, leading to

$$\begin{aligned} \sigma(x) &= e_{n+1} + \frac{2}{1 + |x|^2}(x_1, \dots, x_n, -1) \\ &= \left(\frac{2x_1}{1 + |x|^2}, \dots, \frac{2x_n}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right) \end{aligned}$$

and thus, the restriction of σ to E^n is precisely stereographic projection

$$\pi : E^n \to S^n - e_{n+1}$$

and because σ is its own inverse by Theorem 4.3, we can compute the inverse of π from (4.2.2). Let $y \in S^n - e_{n+1}$. Using that $|y| = 1$

$$\begin{aligned} \sigma(y) &= e_{n+1} + \frac{2(y - e_{n+1})}{|y|^2 - 2y \cdot e_{n+1} + 1} \\ &= e_{n+1} + \frac{1}{1 - y_{n+1}}(y_1, \dots, y_n, y_{n+1} - 1) \\ &= \left(\frac{y_1}{1 - y_{n+1}}, \dots, \frac{y_n}{1 - y_{n+1}}, 0 \right) \end{aligned}$$

and thus, $\pi^{-1}(y) = \left(\frac{y_1}{1 - y_{n-1}}, \dots, \frac{y_n}{1 - y_{n+1}} \right)$.

I now define $\hat{E}^n = E^n \cup \{\infty\}$ where ∞ is a point not in E^{n+1}. This means I can extend π to a bijection $\hat{\pi} : \hat{E}^n \to S^n$ by setting $\hat{\pi}(\infty) = e_{n+1}$ and defining a metric d on $\hat{E}^n$ by the formula

$$d(x, y) = |\hat{\pi}(x) - \hat{\pi}(y)| \tag{4.2.3}$$

and this is clearly a metric by means of the usual Euclidean metric. I will call this the *chordal metric* on $\hat{E}^n$. By definition, the map $\hat{\pi}$ is an isometry from $\hat{E}^n$, with the chordal metric, onto S^n with the Euclidean metric. This means that the metric topology on E^n that is determined by this chordal metric is the same as the Euclidean topology, seeing as π maps E^n homeomorphically onto the open subset $S^n - e_{n+1}$. The metric space $\hat{E}^n$ is compact because it is obtained from E^n by 'adding' a point at infinity, and because π homeomorphism. All of this leads to the theorem

Theorem 4.5 *$\hat{E}^n$ is the one-point compactification of E^n.*

The next theorem shows some properties of the chordal metric which will be useful when I introduce the *cross-ratio* in a bit.

Theorem 4.6 *If x, y are in E^n, then*

(1) $d(x, \infty) = \frac{2}{(1+|x|^2)^{1/2}}$

(2) $d(x, y) = \frac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}}$

Proof. For (1), straight-forward calculation shows

$$\begin{aligned} d(x, y) &= |\hat{\pi}(x) - \hat{\pi}(\infty)| = |\pi(x) - e_{n+1}| \\ &= \left| \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1} - 1 \right) \right| \\ &= \left| \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{-2}{1+|x|^2} \right) \right| \\ &= \frac{2}{1+|x|^2} |(x_1, \ldots, x_n, -1)| = \frac{2}{1+|x|^2} (1+|x|^2)^{1/2} \\ &= \frac{2}{(1+|x|^2)^{1/2}} \end{aligned}$$

Using Theorem 4.3, (2) follows easily

$$\begin{aligned} d(x, y) &= \frac{2|x-y|}{|x - e_{n+1}|\,|y - e_{n+1}|} \\ &= \frac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}} \end{aligned}$$

□

But by the preceding theorem, I will explain why continuity of functions $f : \hat{E}^n \to \hat{E}^n$ behave very well. Because of Theorem 4.6, the distance $d(x, \infty)$ depends solely on $|x|$. Consequently, the open balls around ∞, the balls $B_d(\infty, r)$ where d denotes the chordal metric, is of the form

$$\hat{E}^n - \overline{B(0, s)} \tag{4.2.4}$$

for some $s > 0$. This means that a basis for the topology on $\hat{E}^n$ consists of all open balls $B(x, r)$ in the usual Euclidean sense along with neighbourhoods of the form (4.2.4).

This particularly implies that a function $f : \hat{E}^n \to \hat{E}^n$ is continuous at a point a of $\hat{E}^n$ if and only if $\lim_{x \to a} f(x) = f(a)$ in the Euclidean sense.

Let $P(a, t)$ be a hyperplane in E^n. This leads to the following definition.

Definition 4.7 *An extended hyperplane in $\hat{E}^n$ is given as*

$$\hat{P}(a, t) = P(a, t) \cup \{\infty\}$$

Topologically, we can think of these extended hyperplanes as spheres of lesser dimensions, as shown in the next theorem.

Theorem 4.8 *Let $P(a,t)$ be a hyperplane in E^n. Then $\hat{P}(a,t)$ is homeomorphic to S^{n-1}.*

Proof. Using the translation τ_{-ta} on $P(a,t)$, the resulting $\tau_{-ta}(P(a,t))$ is an $(n-1)$-dimensional vector subspace. A translation is obviously also a homeomorphism. By Theorem A.13 there is an orthogonal transformation ϕ such that $\phi(\tau_{-ta}(P(a,t))) = E^{n-1}$. I notice ϕ is a homeomorphism. By the considerations on continuity following Theorem 4.6, I can extend both τ_{-ta} and ϕ to homeomorphisms $\hat{\tau}_{-ta}$ and $\hat{\phi}$ of $\hat{E}^n$, both mapping ∞ to itself and equal to its 'original' otherwise. Thus, $\hat{P}(a,t)$ is homeomorphic to $\hat{E}^{n-1}$, and because the latter of the two is homeomorphic to S^{n-1}, this finishes the proof because the composition of homeomorphisms is clearly a homeomorphism. □

Letting ρ be the reflection of E^n in $P(a,t)$, I can also extend this reflection to $\hat{\rho} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\rho}(\infty) = \infty$. This shows that $\hat{\rho}(x) = x$ for all $x \in \hat{P}(a,t)$ and that $\hat{\rho}(\hat{\rho}(x))$ is the identity, as desired. This map $\hat{\rho}$ is called the *reflection* of $\hat{E}^n$ in the extended hyperplane.

Theorem 4.9 *Every reflection of $\hat{E}^n$ in an extended hyperplane is a homeomorphism.*

Proof. Letting ρ be the reflection of E^n in a hyperplane, then ρ is continuous as can be seen in formula (4.1.1). From this same formula it is clear that $\lim_{x\to\infty} \rho(x) = \infty$, and thus $\hat{\rho}$ is continuous at ∞ and therefore a continuous function. It is its own inverse, finishing the proof. □

Let σ be the reflection of E^n in $S(a,r)$, a Euclidean sphere. We can also extend σ to a map $\hat{\sigma} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\sigma}(a) = \infty$ and $\hat{\sigma}(\infty) = a$. Then it too satisfies $\hat{\sigma}(x) = x$ for all $x \in S(a,r)$ and $\hat{\sigma}(\hat{\sigma}(x))$ is the identity. We call this the *reflection* of $\hat{E}^n$ in the sphere $S(a,r)$.

Theorem 4.10 *Every reflection of $\hat{E}^n$ in a sphere of E^n is a homeomorphism.*

Proof. Because $\lim_{x\to\infty} \sigma(x) = a$ and $\lim_{x\to a} \sigma(x) = \infty$, from formula 4.1.2, the same arguments as in the proof of Theorem 4.9 applies here. □

Cross Ratio

Letting u, v, x, y be points of $\hat{E}^n$ such that $u \neq v$ and $x \neq y$. The *cross ratio* of these points is defined to be the real number

$$[u,v,x,y] = \frac{d(u,x)d(v,y)}{d(u,v)d(x,y)} \tag{4.2.5}$$

The next theorem follows trivially from Theorem 4.6.

Theorem 4.11 *If $u, v, x, y \in \hat{E}^n$ such that $u \neq v$, $x \neq y$, then*

(1) $[u,v,x,y] = \frac{|u-x\,|v-y|}{|u-v|\,|x-y|}$

(2) $[\infty,v,x,y] = \frac{|v-y|}{|x-y|}$

(3) $[u,\infty,x,y] = \frac{|u-x|}{|x-y|}$

(4) $[u,v,\infty,y] = \frac{|v-y|}{|u-v|}$

(5) $[u,v,x,\infty] = \frac{|u-x|}{|u-v|}$

4.3 Möbius Transformations

By Theorem 4.7, a (topological) sphere Σ of $\hat{E}^n$ is either a Euclidean sphere or an extended hyperplane $\hat{P}(a,t)$.

Definition 4.12 *A Möbius transformation of $\hat{E}^n$ is a finite composition of reflections of $\hat{E}^n$ in spheres.*

We can consider $M(\hat{E}^n)$ as the set of all Möbius transformations of $\hat{E}^n$, and it obviously forms a group under composition. By Theorem 4.2, every isometry of E^n extends to a Möbius transformation of $\hat{E}^n$.

Furthermore, for a positive constant $k > 0$, consider the function $\mu_k : \hat{E}^n \to \hat{E}^n$ given as $\mu_k(x) = kx$. Then this is in fact a Möbius transformation, as it is just the composition of first a reflection in $S(0,1)$ and then a reflection in $S(0,\sqrt{k})$. As can be seen in Theorem A.10, every similiarity of E^n is the composite of first an isometry (namely an orthogonal transformation) and then $\mu_k(x)$, and lastly a translation, another isometry. Thus, every similarity of E^n extends to a Möbius transformation of $\hat{E}^n$.

Lemma 5 *If σ is the reflection of $\hat{E}^n$ in the sphere $S(a,r)$, σ_1 is the reflection in $S(0,1)$ and $\phi : \hat{E}^n \to \hat{E}^n$ a function given as $\phi(x) = a + rx$, then $\sigma = \phi\sigma_1\phi^{-1}$*

Proof.

$$\begin{aligned}
\phi\sigma_1\phi^{-1}(x) &= \phi\sigma_1\left(\frac{x-a}{r}\right) \\
&= \phi\left(\frac{r}{|x-a|^2}(x-a)\right) = a + \frac{r^2(x-a)}{|x-a|^2} \\
&= \sigma(x)
\end{aligned}$$

□

Notice how, in the lemma above, ϕ is in particular a similarity.

Theorem 4.13 *A function $\phi : \hat{E}^n \to \hat{E}^n$ is a Möbius transformation if and only if it preserves cross ratios.*

Proof. Because reflections in hyperplanes are isometries, this case is already covered, as seen in (4.2.5). As ϕ is a Möbius transformation, it is the composition of reflections, and so I may assume that ϕ itself is a reflection. A Euclidean similarity obviously preserves cross ratios, as they merely scale distances. Thus, by Lemma 5, I can assume that $\phi(x)$ is just the reflection in the sphere $S(0,1)$, ie. $\phi(x) = \frac{x}{|x|^2}$. By Theorem 4.3 I then have

$$|\phi(x) - \phi(y)| = \frac{|x-y|}{|x|\,|y|}$$

and by Theorem 4.11 it now clearly follows that

$$[\phi(u), \phi(v), \phi(x), \phi(y)] = [u, v, x, y]$$

if all u, v, x, y are finite and non-zero. The other cases, ie. if one of them is ∞, follows by continuity of ϕ. Thus ϕ preserves cross ratios.

If I assume that ϕ does preserve cross ratios, I want to show that it is a Möbius transformation. Because of that, I can compose ϕ with another Möbius transformation, particularly one that fixes ∞.

Thus, I can assume without loss of generality that $\phi(\infty) = \infty$. Let u, v, x, y be points of E^n such that $u \neq v$, $x \neq y$ and $(u, v) \neq (x, y)$, meaning that either $u \neq x$ or $v \neq y$. The most important is that $x \neq y$, u and v are just chosen accordingly to serve our purpose (and clearly, it can be done).

Assume first that $u \neq x$. Because ϕ preserves cross ratios and $\phi(\infty) = \infty$, I get by Theorem 4.11 that

$$\frac{|\phi(u) - \phi(x)|}{|\phi(x) - \phi(y)|} = \frac{|u - x|}{|x - y|}$$

as well as

$$\frac{|\phi(u) - \phi(x)|}{|\phi(u) - \phi(v)|} = \frac{|u - x|}{|u - v|}$$

But combining these two I get

$$\frac{|\phi(u) - \phi(v)|}{|u - v|} = \frac{|\phi(u) - \phi(x)|}{|u - x|} = \frac{|\phi(x) - \phi(y)|}{|x - y|}$$

and likewise if $v \neq y$

$$\frac{|\phi(u) - \phi(v)|}{|u - v|} = \frac{|\phi(x) - \phi(y)|}{|x - y|}$$

But looking at the above, this means that for all x, y there is some positive constant k such that $k|\phi(x) - \phi(y)| = |x - y|$, and by Definition A.9, ϕ is a similarity. As every similarity of E^n extends to a Möbius transformation, the proof is finished. $\square$

The above proof gives a very useful corollary about precise conditions for a Möbius transformation to be a similarity.

Corollary 4.14 *A Möbius transformation ϕ of $\hat{E}^n$ fixes ∞ if and only if ϕ is a similarity of E^n.*

The Isometric Sphere

Consider a Möbius transformation ϕ of $\hat{E}^n$ that does not fix ∞, and let $a = \phi^{-1}(\infty)$. If σ is the reflection in the sphere $S(a, r)$, then $\phi\sigma(\infty) = \phi(a) = \infty$, and by Corollary 4.14, $\phi\sigma$ is a similarity. That means, by Theorem A.10, that there is some b of E^n, a scalar $k > 0$ and an orthogonal transformation A of E^n such that

$$\phi(x) = b + kA\sigma(x)$$

Supposing that $x, y \in S(a, t)$ for some t. Using Theorem 4.3, then

$$|\phi(x) - \phi(y)| = \frac{kr^2|x - y|}{t^2} = |x - y|$$

where the last equality holds if and only if $t = r\sqrt{k}$. But this means ϕ acts as an isometry on $S(a, r\sqrt{k})$, and this sphere is unique with this property. We call the sphere, $S(a, r\sqrt{k})$, *the isometric sphere of ϕ.*

Theorem 4.15 *Let ϕ be a Möbius transformation of $\hat{E}^n$ with $\phi(\infty) \neq \infty$. Then there is a unique reflection σ in a Euclidean sphere Σ and a unique Euclidean isometry ψ such that $\phi = \psi\sigma$. Moreover, Σ is the isometric sphere of ϕ.*

Proof. Let σ be the reflection in the isometric sphere $S(a, r)$ of ϕ. Then $a = \phi^{-1}(\infty)$ and because $\phi\sigma(\infty) = \infty$, by Corollary 4.14 I see that $\phi\sigma$ is a Euclidean Similarity. If $x, y \in S(a, r)$ I then have

$$|\phi\sigma(x) - \phi\sigma(y)| = |\phi(x) - \phi(y)| = |x - y|$$

and so $\psi = \phi\sigma$ preserves distances for $x, y \in S(a, r)$, but because it is a Euclidean similarity, it does so for all $x, y \in E^n$. Thus ψ is a Euclidean isometry. As σ is its own inverse, I have $\phi = \psi\sigma$.

On the other hand, if σ is a reflection in a sphere $S(a, r)$ and ψ satisfies the above, then $\phi(a) = \infty$ and ϕ acts as an isometry on $S(a, r)$. That must mean $S(a, r)$ is the isometric sphere of ϕ. Thus σ and ψ are both unique, the latter because $\psi = \phi\sigma$. $\square$

Preservation of Spheres

This short subsection deals with the question; let ϕ be a Möbius transformation of $\hat{E}^n$. Does ϕ map spheres of $\hat{E}^n$ to other spheres? To answer this question, and elaborate on it, I first remind myself that the equation defining a sphere $S(a, r)$ and an extended hyperplane $\hat{P}(a, t)$ in $\hat{E}^n$ is, respectively

$$|x|^2 - 2a \cdot x + |a|^2 - r^2 = 0$$

$$-2a \cdot x + 2t = 0$$

I now introduce a common form in which these can be written. I call a vector $(a_0, \ldots, a_{n+1})$ a *coefficient vector* of $\mathbb{R}^{n+2}$. Both of the above can be written in the form

$$a_0|x|^2 - 2a \cdot x + a_{n+1} = 0, \quad |a|^2 > a_0 a_{n+1} \tag{4.3.1}$$

where, in the first case, $a_0 = 1$ and $a_{n+1} = |a|^2 - r^2$ with $a = (a_1, \ldots, a_n)$. Thus, it satisfies the condition $|a|^2 > a_0 a_{n+1}$. Likewise in the second case with $a_0 = 0$ and $a_n + 1 = 2t$. Conversely, any such coefficient vector where $|a|^2 > a_0 a_{n+1}$ and $a = (a_1, \ldots a_n)$ determines a sphere Σ of $\hat{E}^n$ satisfying (4.2.6). If $a \neq 0$ then it is the sphere

$$\Sigma = S\left(\frac{a}{a_0}, \frac{(|a|^2 - a_0 a_{n+1})^{1/2}}{|a_0|}\right)$$

because for every $x \in \Sigma$, I have

$$\begin{aligned} |x - \frac{a}{a_0}|^2 &= \frac{|a|^2 - a_0 a_{n+1}}{|a_0|^2} \\ \Leftrightarrow |x|^2 + \frac{|a|^2}{a_0^2} - \frac{2}{a_0} x \cdot a &= \frac{|a|^2 - a_0 a_{n+1}}{a_0^2} \\ \Leftrightarrow a_0|x|^2 - 2x \cdot a + a_{n+1} &= 0 \end{aligned}$$

Likewise, if $a_0 = 0$ then it is the sphere

$$\Sigma = \hat{P}\left(\frac{a}{|a|}, \frac{a_{n+1}}{2|a|}\right)$$

The coefficient vector for Σ is in either case uniquely determined up to a multiplication by a nonzero scalar.

Theorem 4.16 *Let ϕ be a Möbius Transformation of $\hat{E}^n$. If Σ is a sphere of $\hat{E}^n$, so too is $\phi(\Sigma)$.*

Proof. I may assume that ϕ is a reflection. Let Σ be a sphere of $\hat{E}^n$. Obviously a Euclidean similarity maps spheres to other spheres, and so I may just assume by Lemma 5 that $\phi(x) = \frac{x}{|x|^2}$, meaning ϕ is the reflection in $S(0,1)$. Letting $(a_0, \dots, a_{n+1})$ be a coefficient vector for Σ, then Σ satisfies (4.3.1). But then for $y = \phi(x)$, I get that y satisfies

$$a_0 - 2a \cdot y + a_{n+1}|y|^2 = 0$$

and this is just the equation of another sphere Σ' of $\hat{E}^n$. And so ϕ maps Σ to Σ'. A similar argument shows that ϕ maps Σ' to Σ. And so, $\phi(\Sigma) = \Sigma'$, another sphere. $\square$

The following two theorems will prove equally useful.

Theorem 4.17 *The natural action of $M(\hat{E}^n)$ on the set of spheres of $\hat{E}^n$ is transitive.*

Proof. Letting Σ be a such sphere, it suffices to show that $\exists\, \phi \in M(\hat{E}^n)$ such that $\phi(\Sigma) = \hat{E}^{n-1}$. Because ϕ is a Möbius transformation, a composition of reflections, ϕ has a continuous inverse $\phi^{-1} \in M(\hat{E}^n)$, and so $\phi^{-1}(\hat{E}^{n-1}) = \Sigma$. Because Σ was arbitrary, the sphere $\hat{E}^{n-1}$ serves as my x from Theorem A.12 on transitivity of a group acting on a set.

The group of Euclidean isometries, $I(E^n)$, acts transitively on the set of hyperplanes of E^n. This is shown in the following way. Every hyperplane $P(a,t)$ of the form $ta + W$ for some $(n-1)$-dimensional subspace W, a vector $a \in E^n$ and $t \in \mathbb{R}$. Picking any other hyperplane, on the form $b + V$ for an $(n-1)$-dimensional subspace V, then by Corollary A.6 and Theorem A.13 there is some orthogonal matrix A such that $A(V) = W$. Letting $\phi : E^n \to E^n$ by

$$\phi(x) = (ta - Ab) + Ax$$

then ϕ is also an isometry because it is the composition of a translation and an orthogonal transformation. But I get

$$\phi(b + V) = ta + W$$

And so, if Σ is of the form $\hat{P}(a,t)$ then I am done because by Theorem 4.2 every isometry is in particular a Möbius transformation. Thus I assume that Σ is a Euclidean sphere. However, the group of similarities of E^n, $S(E^n)$, clearly act transitively on the set of Euclidean spheres of E^n. This follows directly from Theorem A.10, because for all such spheres $S(a,r)$ there is a similarity $\phi(x) = a + rAx$ for a suitable orthogonal matrix A such that

$$\phi(S(0,1)) = S(a,r)$$

And so I may assume that $\Sigma = S^{n-1}$. Letting σ be the reflection in the sphere $S(e_n, \sqrt{2})$, then from stereographic projection in section 4.2 I get that $\sigma(S^{n-1}) = \hat{E}^{n-1}$, and so I am done. $\square$

Theorem 4.18 *If ϕ is a Möbius transformation of $\hat{E}^n$ that fixes each point of a sphere Σ of $\hat{E}^n$, then ϕ is either the identity or the reflection in Σ.*

Assume first $\Sigma = \hat{E}^{n-1}$. Then $\phi(\infty) = \infty$, and by Corollary 4.14, ϕ is a similarity. As 0 and e_1 both lie in Σ, then $\phi(0) = 0$ and $\phi(e_1) = e_1$. Consequently, ϕ is a similarity with scale factor $k = 1$ because

$$|\phi(e_1) - \phi(0)| = |e_1 - 0|$$

and because $\phi(0) = 0$, by Theorem A.10, ϕ is an orthogonal transformation.

Because it fixes $e_1, \ldots, e_{n-1}$, the only way its determinant is ± 1 is if $\phi(e_n) = \pm e_n$, as $\phi(e_i)$ for $i = 1, \ldots, n$ are its columns. Thus ϕ is either the identity or the reflection in $P(e_n, 0)$, the latter of which is the reflection in said $\hat{E}^{n-1}$.

If, however, Σ was arbitrary, by the preceding theorem there is a Möbius transformation ψ such that $\psi(\Sigma) = \hat{E}^{n-1}$. Thus $\psi\phi\psi^{-1}$ fixes each point of $\hat{E}^{n-1}$ and so $\psi\phi\psi^{-1}$ is either the identity or a reflection ρ in $\hat{E}^{n-1}$. This means ϕ is either the identity itself or it is $\psi^{-1}\phi\psi$. Letting σ be the reflection in said Σ, then because $\psi\sigma\psi^{-1}$ also fixes $\hat{E}^{n-1}$ and it is not the identity (because σ is not), then $\psi\sigma\psi^{-1} = \rho$. And so

$$\sigma = \psi^{-1}\rho\psi$$

meaning ϕ is either the identity or the reflection σ in Σ. $\square$

4.4 Poincaré Extensions

Identifying E^{n-1} with $E^{n-1} \times \{0\}$ in E^n, a point x of E^{n-1} corresponds uniquely to a point $\tilde{x} = (x, 0)$ in E^n. Let ϕ be a Möbius transformation of $\hat{E}^{n-1}$. I extend this to a Möbius transformation $\tilde{\phi}$ of $\hat{E}^n$ in the following way.

- If ϕ is the reflection in $\hat{P}(a, t)$ in $\hat{E}^{n-1}$, then $\tilde{\phi}$ is the reflection in $\hat{P}(\tilde{a}, t)$ of $\hat{E}^n$.
- If ϕ is the reflection in $S(a, r)$ of $\hat{E}^{n-1}$, then then $\tilde{\phi}$ is the reflection in $S(\tilde{a}, r)$ in $\hat{E}^n$

In either case, by definition of $\tilde{\phi}$, I have for all $x \in E^{n-1}$

$$\tilde{\phi}(x, 0) = (\phi(x), 0) \tag{4.4.1}$$

and so $\tilde{\phi}$ does truely extend ϕ. I now introduce the *upper half-space* U^n as

$$U^n = \{x \in E^n \,:\, x_n > 0\}$$

I will denote the lower half-space of E^n, where $x_n < 0$, as $-U^n$.

Theorem 4.19 *If ϕ is a Möbius transformation of $\hat{E}^{n-1}$, then $\tilde{\phi}$ leaves both $\hat{E}^{n-1}$ and U^n invariant*

Proof. That $\tilde{\phi}$ leaves $\hat{E}^{n-1}$ invariant is clear from its definition and from (4.4.1). Furthermore let $x \in U^n$. If $\tilde{\phi}$ is the reflection in $\hat{P}(\tilde{a}, t)$, then

$$\tilde{\phi}(x) = x + 2(t - \tilde{a} \cdot x)\tilde{a}$$

but as $\tilde{a}_n = 0$, it clearly satisfies that $\left(\tilde{\phi}(x)\right)_n > 0$. If $\tilde{\phi}$ is the reflection in $S(\tilde{a}, r)$, then

$$\tilde{\phi}(x) = \tilde{a} + \left(\frac{r}{|x - \tilde{a}|}\right)^2 (x - \tilde{a})$$

Notice how $|x-\tilde{a}| \neq 0$. But again, as $\tilde{a}_n = 0$, I have that $\left(\tilde{\phi}(x)\right)_n > 0$. $\square$

Let ϕ be an arbitrary Möbius transformation of $\hat{E}^{n-1}$. Then $\phi = \sigma_1 \cdots \sigma_m$, where the dots denote composition. By the preceding arguments, each of these σ_i will leave U^n invariant, and so ϕ will as well. Thus ϕ can be extended to a Möbius transformation $\tilde{\phi}$ of $\hat{E}^{n-1}$ as $\tilde{\phi} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_m$.

Suppose there are two suchs extensions of ϕ, $\tilde{\phi}_1$ and $\tilde{\phi}_2$. Now, for $i = 1, 2$ I have

$$\tilde{\phi}_i = (\phi(x), 0) \quad \forall x \in \hat{E}^{n-1}$$

and as ϕ is its own inverse, it follows that $\tilde{\phi}_1 \tilde{\phi}_2^{-1}$ does in fact fix all $x \in \hat{E}^{n-1}$. As it also leaves U^n invariant, it can not be the reflection in the sphere $\hat{E}^{n-1}$, and so by Theorem 4.18 I have that it is the identity. Thus, $\tilde{\phi}_1 = \tilde{\phi}_2$. So the extension of ϕ is unique. I call this the *Poincaré extension* of ϕ.

Theorem 4.20 *A Möbius transformation ϕ of $\hat{E}^n$ leaves U^n invariant if and only if ϕ is the Poincaré extension of a Möbius transformation of $\hat{E}^{n-1}$.*

Proof. Let ϕ be a such Möbius transformation that leaves U^n invariant. As ϕ is in particular a homeomorphism, it must imply that ϕ also leaves the boundary of U^n invariant. But the boundary is precisely $\hat{E}^{n-1}$. As the restriction of a homeomorpism is also a homeomorphism, then ϕ restricts to a homeomorphism, $\bar{\phi}$, of $\hat{E}^{n-1}$. Because ϕ preserves cross ratios, so does $\bar{\phi}$, and so $\bar{\phi}$ is a Möbius transformation of $\hat{E}^{n-1}$ by Theorem 4.13. Letting $\tilde{\bar{\phi}}$ be its Poincaré extension, then $\tilde{\bar{\phi}}\phi^{-1}$ fixes each point of $\hat{E}^{n-1}$ but leaves U^n invariant. By the arguments made above, $\phi = \tilde{\bar{\phi}}$, a Poincaré extension of a Möbius transformation of $\hat{E}^{n-1}$. The other direction of the proof follows from Theorem 4.19. $\square$

Möbius Transformations of Upper Half-Space

Definition 4.21 *A Möbius transformation of U^n is a Möbius transformation of $\hat{E}^n$ that leaves U^n invariant.*

Letting $M(U^n)$ be the set of such Möbius transformations, then it is a group under composition. Particularly, it is a subgroup of $M(\hat{E}^n)$. This corollary follows immediately from the preceding theorem, and will play an important part later on.

Corollary 4.22 *The group $M(U^n)$ of Möbius transformations of U^n is isomorphic to $M(\hat{E}^{n-1})$.*

Proof. By the arguments above, to each $\phi \in M(\hat{E}^{n-1})$ there is a unique Poincaré extension $\tilde{\phi}$, a Möbius transformation of $\hat{E}^n$, that leaves U^n invariant. It now follows from Theorem 4.20. $\square$

Definition 4.23 *Two spheres Σ and Σ' of $\hat{E}^n$ are orthogonal if and only if they intersect in E^n and at every point of intersection, their normal lines are orthogonal.*

Theorem 4.24 *Two spheres of $\hat{E}^n$ are orthogonal under the following conditions*

(1) The spheres $\hat{P}(a,t)$ and $\hat{P}(b,s)$ are orthogonal if and only if a and b are orthogonal

(2) The spheres $\hat{P}(b,t)$ and $S(a,r)$ are orthogonal if and only if $a \in P(b,s)$

(3) The spheres $S(a,r)$ and $S(b,s)$ are orthogonal if and only if r and s satisfy $|a-b|^2 = r^2 + b^2$

Proof. To prove (1), it is clear that if $\hat{P}(a,t)$ and $\hat{P}(b,s)$ intersect in $\hat{E}^n$ then the two hyperplanes $P(a,t)$ and $P(b,s)$ intersect. These two planes are obviously orthogonal if and only if a and b are orthogonal.

For (2), if those two spheres intersect in E^n, that means $S(a,r)$ and $P(b,t)$ intersect. Obviously, these normal lines at points of intersection are orthogonal if and only if the centre of the circle, a, lies in $P(b,t)$.

To prove (3), consider the following. At each point of intersection x between the two euclidean spheres, the normal lines called u and v have got the equations

$$u = a + t(x-a)$$
$$v = b + t(x-b)$$

for some real t. I can assume this t to be the same by means of reparametrization. These lines are orthogonal if and only if the direction vectors $x-a$ and $x-b$ are orthogonal. Using these calculations

$$\begin{aligned}|a-b|^2 &= |(x-b)-(x-a)|^2 \\ &= |x-b|^2 - 2(x-b)\cdot(x-a) + |x-a|^2 \\ &= s^2 - 2(x-b)\cdot(x-a) + r^2\end{aligned}$$

and so $x-a$ and $x-b$ are orthogonal if and only if $|a-b|^2 = s^2+r^2$. Going in the other direction, suppose $|a-b|^2 = r^2+s^2$. By Figure 5, then there is a right triangle in E^n with vertices a,b,x satisfying the desired.

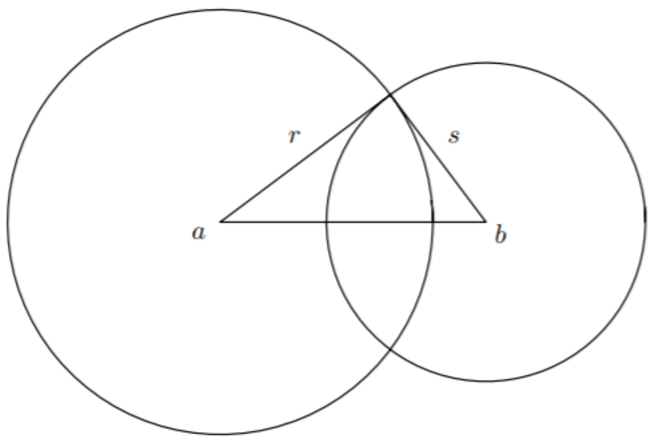

Figure 5: Two orthogonal circles, $S(a,r)$ and $S(b,s)$

□

Remark. Notice that from the proof of Theorem 4.24, I see that two spheres of $\hat{E}^n$ are orthogonal if and only if they are orthogonal at a single point of intersection in E^n.

Theorem 4.25 *A reflection σ of $\hat{E}^n$ in a sphere Σ leaves U^n invariant if and only if $\hat{E}^{n-1}$ is orthogonal to Σ.*

Proof. I let Σ be either $\hat{P}(a,t)$ or $S(a,r)$ where. By Theorem 4.24, $\hat{E}^{n-1}$ and Σ are orthogonal if and only if $a_n = 0$. Because if $\Sigma = \hat{P}(a,t)$, then as $\hat{E}^{n-1}$ can be thought of as the extended hyperplane $\hat{P}(e_n, 0)$ in $\hat{E}^n$, the two spheres are orthogonal if and only if $a_n = 0$ by 4.24 (1). Likewise, if $\Sigma = S(a,r)$, then $a \in \hat{E}^{n-1}$ if and only if $a_n = 0$.

Let $x \in E^n$ and set $y = \sigma(x)$. For all finite values of y I have

$$y_n = \begin{cases} x_n + 2(t - a \cdot x)a_n & \text{if } \Sigma = \hat{P}(a,t) \\ \left(\frac{r}{|x-a|}\right)^2 x_n + \left(1 - \left(\frac{r}{|x-a|}\right)^2\right) a_n & \text{if } \Sigma = S(a,r) \end{cases}$$

Assuming $a_n = 0$, meaning that Σ and $\hat{E}^{n-1}$ are orthogonal, then if $x_n > 0$ that implies $y_n > 0$, and so U^n is left invariant. This is legal because $x \neq a$ as $x_n > 0$.

Conversely, if σ does leave U^n invariant. The same arguments as in the proof of Theorem 4.20 shows σ leaves $\hat{E}^{n-1}$ invariant. Because the reflection in $\hat{E}^{n-1}$ switches U^n to $-U^n$, then I may assume $\Sigma \neq \hat{E}^{n-1}$. Thus $\hat{E}^{n-1} - \Sigma$ is non-empty. Taking an x from this set, assume $y = \sigma(x)$ to be finite. As $x_n = 0 = y_n$, looking at the above expression I get that $a_n = 0$ because neither of the two coefficients of a_n are 0, the latter because $|x - a| \neq r$ as $x \in \Sigma$. □

Möbius Transformations of the unit n-ball

Let σ be the reflection of $\hat{E}^n$ in the sphere $S(e_n, \sqrt{2})$. Thus

$$\sigma(x) = e_n + \frac{2(x - e_n)}{|x - e_n|^2}$$

Furthermore

$$\begin{aligned} |\sigma(x)|^2 &= |e_n|^2 + \frac{4(x - e_n) \cdot e_n}{|x - e_n|^2} + \frac{4|x - e_n|^2}{|x - e_n|^4} \\ &= 1 + \frac{4x_n}{|x - e_n|^2} + \frac{-4}{|x - e_n|^2} + \frac{4}{|x - e_n|^2} \\ &= 1 + \frac{4x_n}{|x - e_n|^2} \end{aligned}$$

And so, if $x \in -U^n$, then $|\sigma(x)|^2 < 1$. Thus σ maps $-U^n$ into the *open unit n-ball*

$$B^n = \{x \in E^n : |x| < 1\}$$

Because σ is a homeomorphism of $\hat{E}^n$, it maps the connected components of $\hat{E}^n - \hat{E}^{n-1}$ homeomorphically onto the connected components of $\hat{E}^n - S^{n-1}$, because the boundary of $-U^n$ is precisely $\hat{E}^{n-1}$ which in turn must be mapped onto the boundary of B^n, which is S^{n-1}. And the other way around.

If ρ is the reflection of $\hat{E}^n$ in $\hat{E}^{n-1}$ then it maps U^n onto $-U^n$. I define

$$\eta = \sigma\rho$$

and I see η maps U^n homeomorphically onto B^n. This Möbius transformation is called the *standard transformation* from U^n to B^n.

Definition 4.26 *A Möbius transformation of the open unit ball B^n is a Möbius transformation of $\hat{E}^n$ that leaves B^n invariant.*

I will now give proof of a theorem similar to Theorem 4.20, but regarding Möbius transformations of B^n instead.

Theorem 4.27 *A reflection σ of $\hat{E}^n$ in a sphere Σ leaves B^n invariant if and only if S^{n-1} and Σ are orthogonal.*

Proof. First, I let η be the standard transformation from U^n to B^n. Letting $\Sigma' = \eta^{-1}(\Sigma)$, then Σ' is a sphere by Theorem 4.16. Let $\sigma' = \eta^{-1}\sigma\eta$. I see that σ' fixes Σ', and thus by Theorem 4.18, it is either the identity or the reflection in Σ'. However, if σ' was to be the identity, then so would $\eta\sigma'\eta^{-1} = \sigma$, but σ is clearly not the identity. And so, $\eta^{-1}\sigma\eta$ is the reflection in Σ'.

Because η maps U^n bijectively to B^n, I have that σ leaves B^n invariant if and only if σ' leaves U^n invariant. However, this latter condition is satisfies if and only if Σ' and $\hat{E}^{n-1}$ are orthogonal by Theorem 4.20. But because η preserves angles by Theorem 4.4, and so Σ' and $\hat{E}^{n-1}$ are orthogonal if and only if $\eta(\Sigma') = \Sigma$ and $\eta(\hat{E}^{n-1}) = S^{n-1}$ are orthogonal. □

4.5 The Conformal Ball Model

I now turn back to the hyperbolic n-space denoted as H^n. I will redefine the Lorentzian Inner Product merely for practical reasons on $\mathbb{R}^{n+1}$. Let $x, y \in \mathbb{R}^{n+1}$

$$x \circ y = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

All the results of Chapter 3 still hold, as this is just a rearrangement of coordinates.

Identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$, I now introduce stereographic projection ζ of B^n onto H^n defined by projecting $x \in B^n$ away from $-e_{n+1}$ until it meets H^n in the unique point $\zeta(x)$. See Figure 6.

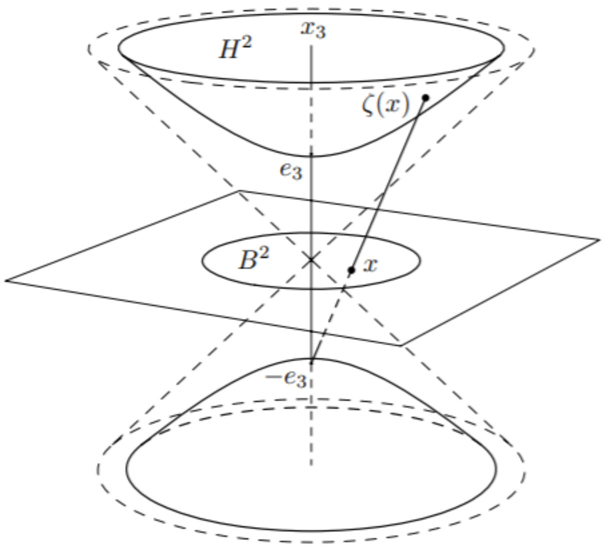

Figure 6: Stereographic projection of B^2 onto H^2

As $\zeta(x)$ lies on the line passing through x in the direction of $x - (-e_{n+1}) = x + e_{n+1}$, there is a scalar s such that

$$\zeta(x) = x + s(x + e_{n+1})$$

But the condition that $||\zeta(x)||^2 = -1$, where the norm $|| \cdot ||$ is now obviously the Lorentzian norm taking complex values, this means

$$s = \frac{1 + |x|^2}{1 - |x|^2}$$

and very much like the stereographic projection of Section 4.2, this gives the formula

$$\zeta(x) = \left(\frac{2x_1}{1 - |x|^2}, \ldots, \frac{2x_n}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right) \tag{4.5.1}$$

and again, this map is clearly both a surjection and is injective, thus a bijection of B^n onto H^n. In fact it is a homemomorphism seeing as its inverse is given by

$$\zeta^{-1}(y) = \left(\frac{y_1}{1 + y_{n+1}}, \ldots, \frac{y_n}{1 + y_{n+1}} \right)$$

another continuous function. Deriving this inverse is done in the exact same way as when finding the inverse π^{-1} of π in Section 4.2.

I can now define a metric d_B, called the *Poincaré metric*, on B^n by the formula

$$d_B(x, y) = d_H(\zeta(x), \zeta(y)), \quad x, y \in B^n$$

and the fact that it is a metric follows because d_H is a metric and ζ is a bijection. I call the metric space (B^n, d_B) the *Conformal Ball Model* of H^n. This is because, by definition of this metric, I see that ζ is in fact an isometry from (B^n, d_B) and onto (H^n, d_H). As $\zeta$is a homeomorphism, and because η is a homeomorpism from U^n to B^n, I see that U^n is homeomorphic to H^n.

Lemma 6 *The metric d_B on B^n satisfies*

$$\cosh d_B(x,y) = 1 + \frac{2 + |x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

Proof. By formula (3.1.9) I get

$$\begin{aligned}
\cosh d_B(x,y) &= \cosh d_H(\zeta(x),\zeta(y)) = -\zeta(x)\circ\zeta(y) \\
&= -\left(x + \frac{1+|x|^2}{1-|x|^2}(x+e_{n+1})\right)\circ\left(y + \frac{1+|y|^2}{1-|y|^2}(y+e_{n+1})\right) \\
&= \frac{(1+|x|^2)(1+|y|^2) - 4x\cdot y}{(1-|x|^2)(1-|y|^2)} \\
&= \frac{(1-|x|^2)(1-|y|^2) + 2(|x|^2+|y|^2) - 4x\cdot y}{(1-|x|^2)(1-|y|^2)} \\
&= 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}
\end{aligned}$$

□

Lemma 7 *If ϕ is a Möbius transformation of B^n and $x, y \in B^n$, then*

$$\frac{|\phi(x)-\phi(y)|^2}{(1-|\phi(x)|^2)(1-|\phi(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

Proof. By Theorem 4.27, ϕ is the reflection in a sphere $S(a,r)$ of E^n orthogonal to S^{n-1}. By Theorem 4.3 I have

$$\frac{|\phi(x)-\phi(y)|}{|x-y|} = \frac{r^2}{|x-a|\,|y-a|}$$

Now, as $S(a,r)$ is orthogonal to S^{n-1}, it follows that $r^2 = |a|^2 - 1$. Furthermore,

$$\phi(x) = a + \frac{r^2}{|x-a|^2}(x-a)$$

which means

$$|\phi(x)|^2 = |a|^2 + \frac{r^4}{|x-a|^2} + \frac{2r^2}{|x-a|^2}a\cdot(x-a)$$

And so, combining all of this I get

$$\begin{aligned}
|\phi(x)|^2 - 1 &= \frac{(|a|^2-1)|x-a|^2 + 2r^2 a\cdot(x-a) + r^4}{|x-a|^2} \\
&= \frac{r^2\left(|x-a|^2 + 2a\cdot(x-a) + |a|^2 - 1\right)}{|x-a|^2} \\
&= \frac{r^2(|x|^2-1)}{|x-a|^2}
\end{aligned}$$

And so, using this twice, I get the result

$$\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2} = \frac{(1-|\phi(x)|^2)(1-|\phi(y)|^2)}{(1-|x|^2)(1-|y|^2)}$$

□

Hyperbolic Translation

Assume $S(a,r)$ is orthogonal to S^{n-1}. Then $r^2 = |a|^2 - 1$, and the reflection σ_a in $S(a,r)$ leaves B^n invariant by Theorem 4.27. Let ρ_a be the reflection in the hyperplane $P(a,0)$. Any such reflection will also leave B^n invariant, just like the composite function $\rho_a\sigma_a$. Let $a^* = \frac{a}{|a|^2}$. Then

$$\rho_a\sigma_a(x) = \frac{(|a|^2-1)}{|x-a|^2}x - \frac{(|x|^2 - 2x\cdot a^* + 1)}{|x-a|^2}a \tag{4.5.2}$$

by (4.1.1) and (4.1.2).

Let $0 \neq b \in B^n$ and $b' = -b^*$. By Theorem 4.24 $S(b', (|b|^2-1)^{1/2})$ is orthogonal to S^{n-1} because

$$((|b|^2-1)^{1/2})^2 + 1^2 = |b'|^2$$

By the above considerations, I can define a Möbius transformation of B^n by

$$\tau_b = \rho_{b'}\sigma_{b'} \tag{4.5.3}$$

Furthermore, in terms of b, by (4.4.3) I get

$$\tau_b(x) = \frac{(1-|b|^2)}{(|b|^2|x|^2 + 2x\cdot b + 1)}x + \frac{(|x|^2 + 2x\cdot b + 1)}{(|b|^2|x|^2 + 2x\cdot b + 1)}b$$

and I notice $\tau_b(0) = b$ for all $b \in B^n$ and that τ_0 is the identity. Now because τ_b is the composite in reflections of a sphere and a hyperplane both orthogonal to the line going through the two points $-\frac{b}{|b|}$ and $\frac{b}{|b|}$ the transformation τ_b can be thought of as a translation along that line. I call this the *hyperbolic translation of* B^n by b.

Theorem 4.28 *Every Möbius transformation of B^n restricts to an isometry of the conformal ball model B^n and every such isometry extends to a unique Möbius transformation of B^n.*

Proof. For a Möbius transformation ϕ of B^n, by Lemma 6 and Lemma 7 I have that

$$\cosh d_B(\phi(x),\phi(y)) = \cosh d_B(x,y)$$

yielding $d_B(\phi(x),\phi(y)) = d_B(x,y)$, so it is an isometry of B^n.

Conversely, assume $\phi : B^n \to B^n$ to be an isometry. Defining $\psi : B^n \to B^n$ by $\psi(x) = \tau^{-1}_{\phi(0)}\phi(x)$, I first see that $\psi(0) = 0$. Furthermore, it is the composite of isometries of B^n by the first part of the theorem, and thus an isometry itself. Let $x, y \in B^n$. Because

$$d_B(\psi(x),\psi(0)) = d_B(\psi(x),0) = d_B(x,0)$$

and Lemma 6, I get

$$\frac{|\psi(x)|^2}{1-|\psi(x)|^2} = \frac{|x|^2}{1-|x|^2}$$

which means $|\psi(x)| = |x|$. Using that ψ is an isometry I also get, from Lemma 7

$$\frac{|\psi(x)-\psi(y)|}{(1-|\psi(x)|^2)(1-|\psi(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

and so $|\psi(x) - \psi(y)| = |x - y|$ because the denominators are equal. It preserves Euclidean distances in B^n. Because of that, it must map each radius of B^n onto a radius of B^n again. Therefore, ψ extends to a function

$$\bar{\psi}(x) : \overline{B^n} \to \overline{B^n}$$

where, for every $x \in S^{n-1}$, the straight line $[0, x)$ from origo in direction of x in B^n is agreed upon by ψ and $\bar{\psi}$, in other words

$$\psi([0, x)) = [0, \bar{\psi}(x))$$

Furthermore, $\bar{\psi}$ is continuous because $\bar{\psi}(x) = 2\psi(x/2)$ for all $x \in \overline{B^n}$, an equivalent way of writing ψ.

But this means $\bar{\psi}$ also preserves Euclidean distances on $\overline{B^n}$ and particularly it preserves inner products. Let $A = \bar{\psi} - \bar{\psi}(0)$. Then $A(0) = 0$ and it preserves Euclidean norms on B^n since

$$|A(x)| = |A(x) - A(0)| = |x - 0| = |x|$$

But this shows $\bar{\psi}$ is in fact the restriction of an orthogonal transformation of E^n, since A satisfies

$$\begin{aligned} 2Ax \cdot Ay &= |Ax|^2 + |Ay|^2 - |Ax - Ay|^2 \\ &= |x|^2 + |y|^2 - |x - y|^2 = 2x \cdot y \end{aligned}$$

and $\bar{\psi}(0) = 0$. Therefore $\tau_{\phi(0)} A$ is a Möbius transformation extending the isometry ϕ, and it is unique because if ϕ_1, ϕ_2 are such two extensions, then they agree on $\overline{B^n}$. Particularly, for all $x \in S^{n-1}$ I have

$$\phi_1 \phi_2^{-1}(x) = x$$

And because it is not the reflection in the sphere $S(0, 1)$ (as it leaves $\overline{B^n}$ invariant), by Theorem 4.18 it must be the identity, hence $\phi_1 = \phi_2$. □

4.6 The Upper Half-Space Model

Letting η be the standard transformation of the upper half-space U^n to B^n, then we recall $\eta = \sigma\rho$ where ρ is the reflection of $\hat{E}^n$ in $\hat{E}^{n-1}$ and σ is the reflection of $\hat{E}^n$ in the sphere $S(e_n, \sqrt{2})$. I now define a metric d_U on U^n, called the *Poincaré metric* on U^n, by the formula

$$d_U(x, y) = d_B(\eta(x), \eta(y))$$

Again, this is clearly a metric. By definition, η is an isometry from (U^n, d_U) to (B^n, d_B). The metric space (U^n, d_U) is called the *upper half-space model* of H^n.

The next theorem follows immediately from Theorem 4.5.2 and how d_U is defined in terms of η, a homeomorphism from U^n to B^n.

Theorem 4.29 *Every Möbius transformation of U^n restricts to an isometry of the upper half-space model U^n, and every such isometry extends uniquely to a Möbius transformation of U^n.*

So where does this all leave us? By the preceding theorem, I can now identify the group of isometries of the upper half-space model, $I(U^n)$, with the group of Möbius transformations of U^n, called $M(U^n)$. The two groups are isomorphic by the theorem.

But the upper half-space model U^n is isometric to the hyperbolic n-space, H^n, by the isometry $\phi : U^n \to H^n$ given as $\phi = \zeta\eta$. This means that the two groups of isometries of the metric spaces, $I(U^n)$ and $I(H^n)$ respectively, are isomorphic. This follows because of the function

$$\begin{array}{rcl} \psi \,:\, I(U^n) & \to & I(H^n) \\ f & \to & \phi f \phi^{-1} \end{array}$$

which is clearly a group homomorphism with inverse

$$\begin{array}{rcl} \psi^{-1} \,:\, I(H^n) & \to & I(U^n) \\ f & \to & \phi^{-1} f \phi \end{array}$$

But as $M(U^n)$ is isomorphic to $M(\hat{E}^{n-1})$ by Corollary 4.22, I get my final result.

Corollary 4.30 *The groups $I(H^n)$ and $M(\hat{E}^{n-1})$ are isomorphic.*

Appendix

A.1 Metric spaces

Definition A.1 *An inner product on a vector space* $V \subseteq \mathbb{R}^n$ *is a function from* $V \times V \to \mathbb{R}$ *denoted by* $(v, w) \to \langle v, w\rangle$ *satisfying for all* $v, w, u \in V$

(1) Bilinearity; $\langle v, w + u\rangle = \langle v, w\rangle + \langle v, u\rangle$ *and* $\langle v + w, u\rangle = \langle v, u\rangle + \langle w, u\rangle$

(2) Symmetry; $\langle v, w\rangle = \langle w, v\rangle$

(3) Nondegeneracy; if $v \neq 0$ *then there is a* $w \neq 0$ *such that* $\langle v, w\rangle \neq 0$

Notice how it is not a demand that the inner product be *positive definite*, meaning $\langle v, v\rangle > 0$ for all $v \neq 0$. Whenever this is the case, however, we can define the *norm* as $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$ as we have seen it before. We can also define a *metric* d_X on a set X in the usual way.

Definition A.2 *A metric on a set* X *is a function* $d : X \times X \to \mathbb{R}$ *such that for all* $x, y, z \in X$*;*

(1) $d(x, y) \geq 0$ *(nonnegativity)*

(2) $d(x, y) = 0$ *if and only if* $x = y$ *(nondegeneracy)*

(3) $d(x, y) = d(y, x)$ *(symmetry)*

(4) $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

This yields the *Metric space* (X, d). As an example of both of these, we have the usual inner product on our Euclidean n-space, E^n, given by $\langle x, y\rangle = x \cdot y$ and its associated metric $d_E(x, y) = |x - y|$.

I will refer to a function $\phi : X \to Y$ between to metric spaces as *distance preserving* if and only if

$$d_Y(\phi(x), \phi(y)) = d_X(x, y), \quad \forall\, x, y \in X$$

Definition A.3 *: An isometry from a metric space* X *to a metric space* Y *is a distance preserving bijection* $\phi : X \to Y$

A translation $\tau_a(x) : \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_a(x) = a + x$ is an isometry, since

$$|\tau_a(x) - \tau_a(y)| = |a + x - (a + y)| = |x - y|$$

and its inverse is $\tau_{-a}(x)$.

A.2 Orthogonal transformations

Definition A.4 *: A function* $\phi : \mathbb{R}^n \to \mathbb{R}^n$ *is an orthogonal transformation if and only if*

$$\phi(x) \cdot \phi(y) = x \cdot y, \quad \forall\, x, y \in \mathbb{R}^n \tag{A.1}$$

and for a real $n \times n$ matrix A I say A is *orthogonal* if the linear transformation associated with it, $A(x) = Ax$ is orthogonal. The set of these matrices together with matrix multiplication form a group called $O(n)$.

We recall the following, central theorem about orthogonal transformations, from which the rest follows.

Theorem A.5 *A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if and only if ϕ is linear and $\{\phi(e_1), \ldots, \phi(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$.*

Proof. Supposing ϕ is an orthogonal transformation, then I have

$$\phi(e_i) \cdot \phi(e_j) = e_i \cdot e_j = \delta_{i,j}$$

Now, to show that $\phi(e_1), \ldots, \phi(e_n)$ are linearly independent, supposing that $\sum_{i=1}^n c_i\phi(e_i) = 0$, upon taking inner products of this equation with $\phi(e_j)$, then by the above I find that all $c_j = 0$ for each j. This means $\{\phi(e_1), \ldots, \phi(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$.

Let any $x = \sum_{i=1}^n x_ie_i \in R^n$. Then there must be coefficients $c_1, \ldots, c_n$ such that $\phi(x) = \sum_{i=1}^n c_i\phi(x_i)$, because it was an orthonormal basis. However, I get that $c_j = \phi(x) \cdot \phi(e_j) = x \cdot e_j = x_j$, meaning that

$$\phi\left(\sum_{i=1}^n x_ie_i\right) = \sum_{i=1}^n x_i\phi(e_i)$$

and so, ϕ is linear.

Suppose instead that ϕ is linear and that $\{\phi(e_1), \ldots, \phi(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$. Then ϕ is orthogonal because

$$\begin{aligned}
\phi(x) \cdot \phi(y) &= \phi\left(\sum_{i=1}^n x_ie_i\right) \cdot \phi\left(\sum_{j=1}^n y_je_j\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n x_iy_j\phi(e_i) \cdot \phi(e_j) \\
&= x \cdot y
\end{aligned}$$

□

Corollary A.6 *Every orthogonal transformation is a Euclidean isometry.*

Proof. Letting ϕ be an orthogonal transformation, then ϕ preserves euclidean norms, as can be seen by

$$|\phi(x)|^2 = \phi(x) \cdot \phi(x) = x \cdot x = |x|^2$$

and so it also preserves distances by use of its linearity

$$|\phi(x) - \phi(y)| = \phi(x - y)| = |x - y|$$

By the details in the proof of Theorem A.4, ϕ is a bijection. Injectivity follows from the linearity and that all $\phi(e_i)$'s are linearly independent. For surjectivity, if $x = \sum_{i=1}^n x_i\phi(e_i)$ is given (because it is a basis), then $\phi\left(\sum_{i=1}^n x_ie_i\right) = x$. Thus, it is an isometry. □

This next theorem follows directly from Theorem A.5.

Theorem A.7 *Letting A be a real $n \times n$ matrix, the following are equivalent*

(1) A is orthogonal

(2) The columns of A form an orthonormal basis of $\mathbb{R}^n$

(3) The rows of A form an orthonormal basis of $\mathbb{R}^n$

Next up, I want to characterize the euclidean isometries.

Theorem A.8 *Let $\phi : E^n \to E^n$ be a function. The following are equivalent*

(1) ϕ is an isometry

(2) ϕ is of the form $\phi(x) = a + Ax$ where A is an orthogonal matrix and $a = \phi(0)$

Proof. Assuming ϕ is an isometry, define $A(x) = \phi(x) - \phi(0)$. Then $A(0) = 0$ and

$$|A(x)| = |A(x)| = |\phi(x) - \phi(0)| = |x - 0| = |x|$$

meaning A preserves norms. Thus, A is orthogonal, because using the law of cosine I get

$$\begin{aligned} 2Ax \cdot Ay &= |Ax|^2 + |Ay|^2 - |Ax - Ay|^2 \\ &= |x|^2 + |y|^2 - |x - y|^2 = 2x \cdot y \end{aligned}$$

This all means that there is an $n \times n$ matrix, which is orthogonal, such that $\phi(x) = Ax + \phi(0)$, meaning (1) implies (2). Conversely, if ϕ is of the form given in (2), then it is the composite of an orthogonal transformation followed by a translation, and so it is an isometry, as it is a composition of isometries. □

A.3 Similarities

Definition A.9 *A similarity from a metric space X to a metric space Y is a bijective function $\phi : X \to Y$ such that $d_X(x, y) = kd_Y(\phi(x), \phi(y))$ for some $k > 0$.*

It is also often referred to as a *change-of-scale*. Notice how the inverse function of a similarity is also a similarity with positive constant $1/k$. The next theorem follows directly from Theorem A.7

Theorem A.10 *Let $\phi : E^n \to E^n$ be a function. The following are equivalent*

(1) ϕ is a similarity

(2) ϕ is of the form $\phi(x) = a + kAx$ for some orthogonal matrix A and a constant $k > 0$, with $a = \phi(0)$.

A.4 Group actions

Recalling the definition of a group G acting on a set X, we have the following definition.

Definition A.11 *: An action of a group G on a set X is transitive if and only if for each x, y in X there is a $g \in G$ such that $gx = y$.*

Notice how the above definition is equivalent to the following;

Theorem A.12 . *The action of a group G on X is transitive if and only if there exists an $x \in X$ such that $G.x = X$*

Proof. Assume G acts transitively. Then for any given $x \in X$, it holds that for all $y \in X$ there exists a $g \in G$ such that $g.x = y$. In other words; $G.x = X$ for all x. In particular, there is at least one x for which it holds. Conversely, for $y_1, y_2 \in X$, pick $g_1, g_2 \in G$ such that $g_1.x = y_1$ and $g_2.x = y_2$. But then I have $g_2 g_1^{-1} y_1 = y_2$. As y_1, y_2 were arbitrary, G acts transitively. □

Theorem A.13 *For every dimension $m > 0$ the natural action of $O(n)$ on our set of m-dimensional vector subspaces of R^n is transitive.*

The idea of the proof is as follows. If V is a vector subspace of m dimensions, choose a basis $\{u_1, \ldots, u_m\}$ of V and extend this to a basis $\{u_1, \ldots, u_n\}$ of E^n. Performing Gram-Schmidt on the vectors $\{u_1, \ldots, u_n\}$ yields an orthonormal basis $\{w_1, \ldots, w_n\}$, where the first m vectors are a basis for V. Thus, the matrix whose columns are $\{w_1, \ldots, w_n\}$, call it A, is an orthogonal matrix and it satisfies $A(\mathbb{R}^m) = V$.

A.5 Cross Products

If x, y are vectors in R^3 then the cross-product of x, y is defined in the usual way

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Theorem A.14 : *If w, x, y, z are vectors in $\mathbb{R}^3$ then the following are equivalent*

(1) $x \times y = -y \times x$

(2) $(x \times y) \cdot z = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

(3) $(x \times y) \times z = (x \cdot z)y - (y \cdot z)x$

(4) $(x \times y) \cdot (z \times w) = \begin{vmatrix} x \cdot z & x \cdot w \\ y \cdot z & y \cdot w \end{vmatrix}$

The proof follows directly from computing the cross products. Notice, however, that the *scalar triple product* of x, y, z given as $(x \times y) \cdot z$ cyclically permutes, ie.

$$(x \times y) \cdot z = (y \times z) \cdot x = (z \times x) \cdot y \tag{A.2}$$

which follows from Theorem A.14(2). For $v, w \neq 0$, I get $|v \times w|^2 = |v|^2 |w|^2 - (v \cdot w)^2$ from Theorem A.14(4). Continuing these calculations;

$$\begin{aligned} |v|^2 |w|^2 - (v \cdot w)^2 &= |v|^2 |w|^2 - |v|^2 |w|^2 \cos^2 \theta(v, w) \\ &= |v|^2 |w|^2 \sin^2 \theta(v, w) \end{aligned}$$

yielding

$$|v \times w| = |v||w| \sin \theta(v, w) \tag{A.3}$$

A.6 Conformal Transformations

For an open subset U of E^n, let $\phi : U \to E^n$ be a differentiable function. Let $\phi'(x)$ be the matrix $\left(\frac{\delta\phi_i}{\delta x_j}(x)\right)$ of partial derivates of ϕ. Then a function ϕ is *conformal* if and only if there is some function $\kappa : U \to \mathbb{R}_+$, called the *scale factor* of ϕ such that $\kappa(x)^{-1}\phi'(x)$ is an orthogonal matrix for each $x \in U$.

Definition A.15 *Let U be an open subset of E^n. Let $\phi' : U \to E^n$ be a differentiable function. Then ϕ is said to preserve (or reverse, respectively) orientation at $x \in U$ if and only if* $\det \phi'(x) > 0$ *(or less than 0, respectively).*

References

[1] John G. Ratcliffe. Foundations of Hyperbolic Manifolds, 1994