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Bachelor Thesis in Mathematics.
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Introductory Knot Theory

The Knot Group and The Jones Polynomial

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July 5, 2011

Abstract

In this thesis, basic knot theory is introduced, along with concepts from topology, algebra and algebraic topology, as they relate to knot theory. In the first chapter, basic definitions concerning knots are presented. In the second chapter, the fundamental group is applied as a method of distinguishing knots. In particular the torus knots are classified using the fundamental group, and a general algorithm for computing the group of certain well-behaved knots is shown to be successful. In the third chapter, the Jones polynomial is developed by considering diagrams of oriented links, and shown to be unaffected by certain changes in these diagrams. Finally, the relative strengths and weaknesses of the Jones polynomial and the knot group are discussed.

Resumé

I det følgende præsenteres indledende knudeteori, samt topologi, algebra og algebraisk topologi, i det omfang de relaterer til knudeteori. I første kapitel anvendes fundamentalgruppen som en metode til at skelne knuder fra hinanden. Specielt bliver torus knuderne klassificeret ved denne metode, og en generel algoritme til at beregne fundamentalgruppen af visse tilstrækkeligt pæne knuder præsenteres og vises successfuld. I tredje kapitel udvikles Jones polynomiet ved at betragte diagrammer af orienterede knuder og vises at være uændret under visse operationer på disse diagrammer. Til sidst diskuteres fundamentalgruppens og Jones polynomiets indbyrdes stærke og svage sider.

Contents

Introduction

This thesis deals with introductory knot theory. The principal goal will be to answer the question: In how many different ways can S^1 be embedded in S^3? As it turns out there is no complete answer to this question yet, however a partial solution is given using two different knot invariants, namely the knot group and the Jones polynomial. The knots and links discussed in this thesis will always be embeddings of S^1 in either $\mathbb{R}^3$ or S^3.

The thesis is divided into three chapters. In the first chapter, general knots are introduced and discussed. Two different definitions of knots and links are given, and corresponding to these are two different defintions of knot equivalence. The reason for this, is a technical advantage in the developement of the two invariants. In the second chapter, the fundamental group is discussed in the context of knot theory, in particular it is shown how the fundamental group can be used to distinguish knots by applying it to the complement of a knot. The so-called torus knots are classified and an algorithm for computing the knot group of certain well-behaved knots is proven successful. The main emphasis is on this chapter and, as a result, the notions introduced are dealt with in greather detail. In the third chapter, it is shown how a certain polynomial, called the Jones polynomial, can be associated to oriented links and knots in such a way that equivalent knots are assigned identical polynomials. During the course of the thesis the differences between the two invariants will be discussed, and their relative strength and weaknesses compared.

The definitions given in the first chapter are inspired by the books [Rol76], [Hat02] and [Lic97]. Unless otherwise specified, the second chapter is based on chapter 3 of [Rol76], and the third chapter is based on [Lic97], chapters 1 and 3.

Acknowledgements

I would like to thank my advisor Jesper Michael Møller. He has been a great help and made the experience pleasant. I would also like to thank Karen Bakke Haga and Rasmus K. Rendsvig for correcting a large number of spelling errors, bad formulations and making sure that the thesis is readable in general, as well as Kristian Knudsen Olesen for numerous useful discussions and comments during the writing process. Any errors that remain is entirely the fault of the author.

Chapter 1

Introducing Knots

1.1 Notation and Terminology

Throughout, the standard unit ball of $\mathbb{R}^n$ will be denoted by $D^n = \{x \in \mathbb{R}^n | \ \|x\| \leq 1\}$ and $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n | \|x\| = 1\}$. The symbol $\cong$ means that two spaces are homeomorphic or that two groups are isomorphic, depending on the context. The symbol $\simeq$ will denote that two spaces have the same homotopy type.

Given two groups G and H the free product of G and H will be denoted $G * H$. If $C_1, C_2, ...$ are infinite cyclic groups, such that x_i generates C_i, then $G = C_1 * C_2 * \cdots$ is said to be the free group on the generators $x_1, x_2, ...$ If G is the trivial group, this will be denoted $G = 0$

Regardless of which equivalence relation between knot (or links) that is used, the equivalence class of a knot (or link) K will be called the knot type (or link type) of K.

1.2 Knots and Equivalence

The definitions in this section can be found in [Rol76, Chapter 1] unless otherwise specified

Definition 1.2.1. A subspace $K \subset X$ of a topological space is said to be a *knot* in X if it is homeomorphic to S^n, for some $n \in \mathbb{N}$. A subspace $L \subset X$ is said to be a *link* of k components in X if it is homeomorphic to the disjoint union $S^{n_1} \sqcup \cdots \sqcup S^{n_k}$ for some $n_1, ..., n_k \in \mathbb{N}$.

In the following, the space X will always be taken to be either $\mathbb{R}^3$ or S^3. Since S^1 is compact and both $\mathbb{R}^3$ and S^3 are Hausdorff, any injective, continuous map $f : S^1 \to S^3$ will be an embedding. Sometimes the embedding itself will be referred to as the knot or link, rather than the image of the embedding as stated in the definition. No confusion should arise from this abuse of terminology.

Definition 1.2.2 (Simple equivalence)**.** Two knots or links $K, K' \subset X$ are said to be equivalent if there exists a homeomorphism $f : X \to X$ such that $f(K) = K'$.

This definition of equivalence of knots relates the knots regarded as sets, and thus there is not given any orientation of the knots or links in question, or the space X. Equivalent knots will be regarded as the same knot, and therefore a formulation like "The knots K and K' are distinct" means that the knots K and K' are not equivalent. The notion of equivalence defined above is referred to as simple equivalence, although this is not standard terminology. To avoid possible pathological examples when considering knots, one often distinguishes between wild and tame knots.

Definition 1.2.3. A knot in $\mathbb{R}^3$ is said to be *polygonal* if it is the union of a finite number of closed, straight-line segments. A knot is said to be *tame* if it is equivalent to a polygonal knot.[1]

In this definition 'straight' is in the linear structure of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty \cong S^3$, or in the linear structure of the simplexes that make up S^3 in a triangulation as defined in the next section. A knot is *wild* if it is not tame.

1.3 Another View of Knots

1.3.1 Simplexes

The concepts and definitions in this section are introduced in a semi-formal matter and will not be dealt with in depth, since they will not play a significant role in the following chapters. They are necessary and since they play a central role in the notion of equivalence they are certainly not unimportant, however too much emphasis on the technical details of the constructions will distract from the principal goal, namely studying knots and invariants of these. In the interest of completeness, some of the basic definitions are given. The definitions can be found in [Hat02, chapter 2]

Definition 1.3.1. Let $v_0, ..., v_n \in \mathbb{R}^m$ be points such that the difference vectors $v_1 - v_0, ..., v_n - v_0$ are linearly independent. The points $v_0, ..., v_n$ are said to be vertices in the n-simplex $[v_0, ..., v_n]$ defined by:

$$[v_0, ..., v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; \sum_{i=0}^{n} t_i = 1 \wedge t_i \geq 0 \text{ for all } i \right\}.$$

The standard n-simplex Δ^n has as vertices the endpoints of the standard unit vectors in $\mathbb{R}^{n+1}$ i.e.

$$\Delta^n = \left\{ (t_0, ..., t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \wedge t_i \geq 0 \right\}.$$

[1] [CF77, p. 5]

Any two n-simplexes are canonically homeomorphic via the map

$$\sum_{i=0}^{n} t_i v_i \mapsto \sum_{i=0}^{n} t_i w_i.$$

Hence any n-simplex can be identified with the standard n-simplex. If a vertex is deleted from $[v_0, ...v_n]$, one obtains a $(n-1)$-simplex which is called a face of $[v_0, ..., v_n]$. If the vertex v_i is deleted, the resulting face will be called the i'th face of the simplex and this will be denoted $[v_0, ..., \hat{v_i}, ..., v_n]$.

In general the faces of a simplex $[v_0, ..., v_n]$ are all simplexes of the form

$$[v_{i_1}, ..., v_{i_m}]$$

where $0 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq m < n$.

As a particular instance of the discussion above, it should be noted that faces with same dimension of any two simplexes can be identified.

Definition 1.3.2. Let $\Delta_\alpha^{n_\alpha}$ for $\alpha \in A$ be a collection of simplexes where n_α depends on α and let $\mathcal{F}_1, .., \mathcal{F}_q$ be collections of some of the faces of the $\Delta_\alpha^{n_\alpha}$ such that the faces of each $\mathcal{F}_i$ have the same dimension. The quotient space of the disjoint union $\sqcup_{\alpha \in A} \Delta_\alpha^{n_\alpha}$ obtained by identifying all of the faces in each $\mathcal{F}_i$ with a simplex, is called a Δ-complex.

Definition 1.3.3. A simplicial complex is a Δ-complex, where each simplex is uniquely determined by its vertices.

Intuitively a simplicial complex is obtained by taking a number of simplexes and 'gluing' them together along their edges, in such a way that the endpoints are glued to endpoints, lines to lines, triangles to triangles and so on.

The notion of triangulation may be defined as follows:

Definition 1.3.4. A triangulation of the space X, consists of a simplicial complex S along with a homeomorphism

$$f : S \to X$$

Using the notion of triangulation, a piece-wise linear condition on continuous functions from S^3 to itself can be defined thusly:

Definition 1.3.5. Given a triangulation of S^3, the function $f : S^3 \to S^3$ is said to be piece-wise linear if simplexes are mapped to simplexes in a linear way, i.e. if f maps vertices to vertices and $f(\sum_{i=0}^{j} t_i v_i) = \sum_{i=0}^{n} t_i f(v_i)$. It is orientation preserving, if each of the restrictions $f|_{\Delta^n}$ to a simplex is orientation preserving.

1.3.2 Knots and Links

Using the concepts introduced, new definitions for knots, links and equivalence between them will be given, all of which can be found in [Lic97, chapter 1].

Definition 1.3.6. A link L of m components in S^3 is a subspace that consists of m disjoint piece-wise linear, simple closed curves. A knot is a link of one component.

A simple closed curve is any space that is homeomorphic with S^1. The piece-wise linear condition means that each component is made up of a finite number of straight line segments, straight being in the linear structure of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty \cong S^3$ or in the simplexes that make up S^3 in a triangulation. Thus in this definition wild knots and links are excluded permanently and we restrict our attention to tame links. Thus definition 1.3.6 can be seen as a combination of definition 1.2.1 and definition 1.2.3.

The piece-wise linear condition will not be given much emphasis in the chapters to come, and in practice the knots and links will be drawn rounded. Given this new definition of links and knots, a new definition of equivalence is required.

Definition 1.3.7 (Oriented equivalence)**.** Two knots $K, K' \subset S^3$ is said to be equivalent if there exists an orientation preserving, piece-wise linear homeomorphism $f : S^3 \to S^3$ such that $f(K) = K'$.

That this definition is not equivalent to the defintion of simple equivalence should be obvious. The equivalence defined above is usually referred to as oriented equivalence.

If $\rho : S^3 \to S^3$ is an orientation reversing homeomorphism and $L \subset S^3$ a link, the link $\rho(L)$ will be denoted $\overline{L}$. Up to equivalence of $\overline{L}$, the choice of ρ is immaterial. Thus, in the following chapters, the standard orientation-reversing homeomorphism will be $(x, y, z) \mapsto (x, y, -z)$, when S^n is considered as $\mathbb{R}^n \cup \infty$.

Chapter 2

The Knot Group

Before the knot group is explored, a few general results is stated though not proven, and the terminology adopted in this chapter will be introduced. Note that in this chapter the notion of simple equivalence is used, unless otherwise specified, and knots are defined as in definition 1.2.1.

2.1 General Results

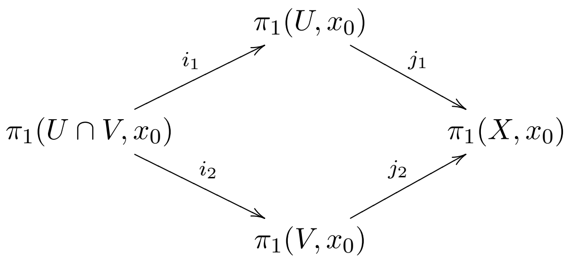
Theorem 2.1.1. *Let $f : X \to Y$ be a homotopy equivalence with $f(x_0) = y_0$. Then f induces an isomorphism*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

Proof. The reader is referred to [Mun00, p. 264] for a proof. □

This theorem will be used quite frequently, usually without reference. The homotopy equivalence will often be a homeomorphism or a deformation retract, or a sequence of these.

Theorem 2.1.2 (van Kampen)**.** *Let U and V be open, path-connected subsets of X such that $U \cup V = X$ and $U \cap V$ is path-connected. Assume that $x_0 \in U \cap V$. Consider the following diagram:*



where all the indicated homomorphisms are induced by inclusion. Let

$$h : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi(X, x_0)$$

be the homomorphism of the free product that extends j_1, j_2. Then h is surjective and its kernel is the least normal subgroup containing all elements represented by words on the form $i_1(g)^{-1}i_2(g)$ with $g \in \pi_1(U \cap V, x_0)$.

Proof. A proof can be found in [Mun00, Theorem 70.2 p. 420] □

The requirement that the sets U and V are open often complicates computations and therefore these will be replaced by closed sets A and B, such that there is a deformation retract of U onto A and of V onto B. This consideration will be carried out in silence in the applications to follow as it causes no difficulty. Another formulation of this theorem is very useful in the computation of knot groups.

Theorem 2.1.3 (van Kampen)**.** *Assume the hypothesis of Theorem 2.1.2 and that there are presentations:*

$$\begin{aligned}\pi_1(U, x_0) &= (x_1, x_2, ...|r_1, r_2, ...);\\ \pi_1(V, x_0) &= (y_1, y_2, ...|s_1, s_2, ...);\\ \pi_1(U \cap V, x_0) &= (z_1, z_2, ...|t_1, t_2, ...).\end{aligned}$$

Then the fundamental group of X has presentation:

$$\pi_1(X, x_0) \cong (x_1, ..., y_1, ...|r_1, ..., s_1, ..., i_1(z_1) = i_2(z_1), ...).$$

The notation $G = (x_1, x_2, ...|r_1, r_2, ...)$ means that G has a presentation with generators $x_1, x_2, ...$ and the relations $r_1, r_2, ...$ This means that G is generated by the elements $x_1, x_2, ...$ and that the relations $r_1, r_2, ...$ generate all relations that holds between the generators, in the sense that every relation can be deduced from the r_i's using the standard algebraic operations. For a brief introduction to group presentations see appendix A

Lemma 2.1.4. *Let $p \in S^n$ be $(0, ..., 0, 1) \in \mathbb{R}^{n+1}$. Then the stereographic projection $f : (S^n - p) \to \mathbb{R}^n$ given by*

$$f(x_1, ..., x_n, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n)$$

is a homeomorphism.

Proof. The stereographic projection is obviously continuous. One shows that it is bijective by checking that $g : \mathbb{R}^n \to (S^n - p)$ given by

$$g(y) = g(y_1, ..., y_n) = (t(y)y_1, ..., t(y)y_n, 1 - t(y)),$$

where $t(y) = \frac{2}{1+\|y\|^2}$, is a left and right inverse of f.[1] □

Theorem 2.1.5. *Let X, X' be locally compact Hausdorff spaces and Y, Y' such that*

1. *X is a subspace of Y and X' is a subspace of Y'*

[1]This is inspired by [Mun00, p. 369]

2. *The sets $Y - X$ and $Y' - X'$ both consist of a single point*

3. *Y, Y' are compact Hausdorff spaces*

If $f : X \to X'$ is a homeomorphism, then f extends to a unique homeomorphism $h : Y \to Y'$.

Proof. Let $f : X \to X'$ be a homeomorphism, $Y - X = \{p\}$, $Y' - X' = \{q\}$ and define $h : Y \to Y'$ by $h|_X = f$ and $h(p) = q$. It will be shown that if $U \subset Y$ is open then $h(U) \subset Y'$ is open. The desired result will then follow from symmetry.

Let $U \subset Y$ be open and assume that $p \notin U$. Then $U \subset X$ and therefore U is open in X, hence $h(U) = f(U)$ is open in X'. It follows that $f(U)$ is open in Y'.

Assume that $p \in U$ and consider $C = Y - U$. Since C is closed in Y it is compact in Y. Since $C \subset X$ it is also compact in X. Hence $h(C) = f(C)$ is compact in X'. Since X' is Hausdorff it follows that $f(C)$ is closed in X' and hence is closed in Y'. Therefore $Y' - h(C) = h(U)$ is open in Y'.[2] □

This result in conjunction with lemma 2.1.4 can be used to prove:

Corollary 2.1.6. *Let $\mathbb{R}^n \cup \infty$ denote the one-point compactification of $\mathbb{R}^n$. Then $S^n \cong (\mathbb{R}^n \cup \infty)$.*

2.2 The Basics

Definition 2.2.1. Let $K \subset X$ be a knot in X. The fundamental group $\pi_1(X - K, x_0)$, for some $x_0 \in (X - K)$ is called the knot group of K.

Sometimes reference to the basepoint x_0 is omitted to ease notation. As all spaces considered in this thesis are path-connected, different basepoints will give rise to isomorphic knot groups. Note that it follows from theorem 2.1.1 that any homeomorphism $f : X \to X$ with $f(K) = K'$, orientation preserving or otherwise, will induce an isomorphism $f_* : \pi_1(X - K) \to \pi_1(X - K')$, which explains why simple equivalence is adopted in this chapter, rather than oriented equivalence.

It should be obvious that the knot group is a knot invariant, but it will be proven regardless.

Theorem 2.2.2. *Let $K, K' \subset X$ be equivalent knots in X. Then*

$$\pi_1(X - K) \cong \pi_1(X - K').$$

Proof. By definition there exists a homeomorphism $f : X \to X$ such that $f(K) = K'$ and thus $f|_{X-K} : X - K \to X - K'$ is a homeomorphism. It follows that $\pi_1(X - K) \cong \pi_1(X - K')$. □

[2]This is a mildly modified version of the proof of [Mun00, Theorem 29.1, p. 183]

It has been proven that the complement of a knot is a complete invariant[3] although the proof of this is outside the scope of this thesis. In contrast the knot group is not a complete invariant, as demonstrated in example 3D10 and 8E15 in [Rol76].

Definition 2.2.3. *The trivial knot* is defined to be the image under the embedding $(x, y) \mapsto (x, y, 0)$.

In the following proof the notation $X \sqcup_f Y$ is used. This is the quotient space obtained from $X \sqcup Y$ by considering a map $f : A \to Y$, where $A \subset X$, and identifying a and $f(a)$.

Lemma 2.2.4. *The knot group of the trivial knot $K \subset \mathbb{R}^3$ is $\mathbb{Z}$.*

Proof. It is clear that $\mathbb{R}^3 - S^1$ deformation retracts onto an closed ball centered at the origin with a small circle removed from the interior. Furthermore, it is visually clear that this deformation retracts onto $S^2 \sqcup_f D^1$ where $f : \partial D^1 \to S^2$ is given by $f(-1) = (-1, 0, 0)$ and $f(1) = (1, 0, 0)$. Consider then $g : \partial D^1 \to S^2$ given by $g(-1) = g(1) = (1, 0, 0)$. Since f is homotopic to g it follows that[4] $S^2 \sqcup_f D^1 \simeq S^2 \sqcup_g D^1 \cong S^2 \vee S^1$. Since $\pi_1(S_2) = 0$ an easy application of van Kampen's theorem yields that

$$\pi_1(S^2 \vee S^1, x_0) \cong \pi_1(S^2, x_0) * \pi_1(S^1, x_0) \cong \pi_1(S^1, x_0) = \mathbb{Z},$$

where $x_0 = (1, 0, 0)$ is the basepoint. □

Theorem 2.2.5. *If B is any bounded subset of $\mathbb{R}^n$, such that $\mathbb{R}^n - B$ is path-connected, and $n \geq 3$, then the inclusion, under the identification $S^n \cong \mathbb{R}^n \cup \infty$, induces an isomorphism:*

$$i_* : \pi_1(\mathbb{R}^n - B) \to \pi_1(S^n - B).$$

Proof. Consider $S^n \cong \mathbb{R}^n \cup \infty$ and $\mathbb{R}^n \subset S^n$. Choose a neighbourhood U of ∞ in S^n such that $U \cap B = \emptyset$ and $U \cong \mathbb{R}^n$. Such a neighbourhood exists, since B is a bounded subset of $\mathbb{R}^n$.

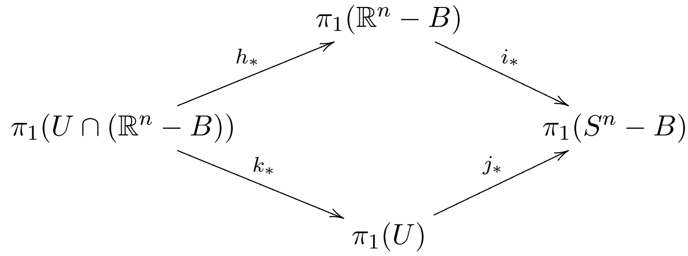
Let $B_1 \subset \mathbb{R}^n$ be a closed ball centered at the origin and radius M, such that B_1 contains B. Consider the map $f = \varphi \circ \psi \circ \varphi^{-1}$, where φ is the extension of the stereographic projection to S^n and $\psi : S^n \to S^n$ is given by $\psi(x_1, ..., x_n, x_{n+1}) = (x_1, ..., x_n, -x_{n+1})$. Then f maps B_1 homeomorphically to $\{x \in \mathbb{R}^n | \|x\| < 1/M\}$ which is homeomorphic to $\mathbb{R}^n$. It follows that

$$U \cap (\mathbb{R}^n - B) = U - \infty \cong \mathbb{R}^n - \mathbf{0} \simeq S^{n-1}$$

[3] [GL89,]

[4] [Hat02, p. 13]

and thus $(\mathbb{R}^n - B) \cap U$ is simply connected. Consider the diagram



where each of the homomorphisms are induced by inclusion. Then van Kampen's theorem yields that

$$f : \pi_1(U) * \pi_1(\mathbb{R}^n - B) \to \pi_1(S^n - B),$$

where f is the homomorphism that extends i_* and j_*, is surjective and that $\ker(f)$ is the least normal subgroup containing all elements of the form $h_*(u)k_*(u)^{-1}$. Since $\pi_1(U \cap (\mathbb{R}^n - B)) = 0$, it holds that h_*, k_* are both the trivial homomorphism and thus $\ker(f) = 0$. Hence f is an isomorphism. Furthermore, since $\pi_1(U)$ is trivial it follows that j_* is trivial and therefore $f = i_*$. □

The above theorem applies to the desired cases, i.e. where B is a knot in $\mathbb{R}^3$, although this is not easy to prove. Since every knot is a compact subset of $\mathbb{R}^n$ it is closed and bounded. The difficult part is showing that $\mathbb{R}^n - K$ is path-connected for any knot K. This is a consequence of the Jordan-Brouwer theorem and is beyond the scope of this thesis, but a proof can be found in [Hat02] p. 169-170.

The key conclusion is that knots can be considered as embedded in either $\mathbb{R}^3$ or S^3 as needed, without any difficulty.

2.3 Torus Knots

The knots considered in this section are the so-called torus knots. They are the only special class of knots considered, and they are studied because they can be completely classified and their knot groups are relatively simple.

Torus knots are of the form $K : S^1 \to S^1 \times S^1$ given by $e^{i\theta} \mapsto (e^{2p\pi i\theta}, e^{2q\pi i\theta})$ for $0 \leq \theta \leq 1$ and S^1 suitably identified with a subset of $\mathbb{C}$, followed by the standard embedding of $S^1 \times S^1$ in $\mathbb{R}^3$. Maps of this type does not always give rise to a knot, but the following theorem gives a classification of the values of p and q, for which the described map is indeed a knot.

Theorem 2.3.1. *The map $f : S^1 \to S^1 \times S^1$ given by $e^{i\theta} \mapsto (e^{2p\pi i\theta}, e^{2q\pi i\theta})$ is an embedding if, and only if, $gcd(p, q) = 1$.*

Proof. This will be done by proving that the map described is injective if, and only if, $\gcd(p,q) = 1$. Since S^1 is compact and $S^1 \times S^1$ is Hausdorff the result will follow.

Assume that $\gcd(p,q) = 1$. Choose $\theta, \theta' \in [0,1)$ such that

$$(e^{2p\pi i\theta}, e^{2q\pi i\theta}) = (e^{2p\pi i\theta'}, e^{2q\pi i\theta'}),$$

and let $\theta \geq \theta'$. It follows that $p(\theta - \theta') \in \mathbb{Z}$ and that $q(\theta - \theta') \in \mathbb{Z}$ and hence $\theta - \theta' = \frac{s}{t}$, where s and t can be chosen such that $\gcd(s,t) = 1$. It then follows that $t \mid p$ and $t \mid q$. Therefore the assumption implies that $t = 1$ and thus $(\theta - \theta') = s \in \mathbb{Z}$. Since $\theta \geq \theta'$ and $\theta, \theta' \in [0,1)$, it holds that $\theta - \theta' = 0$ and therefore the map is injective.

Assume that $\gcd(p,q) = d > 1$. Then $\frac{1}{d} \in (0,1)$ and $\frac{p}{d}, \frac{q}{d} \in \mathbb{Z}$ and thus

$$(e^{2p\pi i}, e^{2q\pi i}) = (e^{(2p\pi i)/d}, e^{(2q\pi i)/d})$$

therefore the map is not injective. □

Knots of the form $e^{i\theta} \mapsto (e^{2p\pi i\theta}, e^{2q\pi i\theta})$ with $\gcd(p,q) = 1$ are denoted $T_{p,q}$. The knot group of any torus knot will now be found, but first a proposition which will be quite useful.

Lemma 2.3.2. *S^3 is homeomorphic to $D^2 \times S^1 \cup S^1 \times D^2$.*

Proof. The homeomorphism comes from considering $S^3 = \partial D^4$ with $D^4 \subset \mathbb{R}^4$. First it will be proven that $D^4 \cong D^2 \times D^2$.

To do this, consider both $D^2 \times D^2$ and D^4 as having coordinates in $\mathbb{R}^2 \times \mathbb{R}^2$ and the map $\varphi : D^2 \times D^2 \to D^4$ given by

$$(x,y) \mapsto \begin{cases} \frac{\max\{\|x\|,\|y\|\}}{\|(x,y)\|}(x,y) & \text{if } (x,y) \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Consider first the restriction of φ to $D^2 \times D^2 - \mathbf{0}$ denoted $\hat{\varphi}$. It is clear that $\max\{\|x\|,\|y\|\}$ defines a norm on $\mathbb{R}^4$ and it induces the same topology on $\mathbb{R}^4$ as $\|(x,y)\|$, since

$$\max\{\|x\|,\|y\|\} \leq \|(x,y)\| \leq \sqrt{2}\max\{\|x\|,\|y\|\}.$$

It follows that $\max\{\|x\|,\|y\|\}$ is continuous with respect to the topology induced by $\|(x,y)\|$, i.e. the standard topology on $\mathbb{R}^4$, hence $\hat{\varphi}$ is continuous. Furthermore, it follows easily that $\|\varphi(x,y)\| \leq 1$ and therefore φ is a well-defined continuous map. To see that it is bijective, consider $\psi : D^4 \to D^2 \times D^2$ given by

$$\begin{cases} \psi(x,y) = \frac{\|(x,y)\|}{\max\{\|x\|,\|y\|\}}(x,y) & \text{if } (x,y) \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

and let $\hat{\psi}$ be the restriction to $D^4 - \mathbf{0}$. It is not difficult to show that $\hat{\psi} = (\hat{\varphi})^{-1}$, but the computations will be omitted on account of their lack in aesthetics. Since $\hat{\psi}$ is also continuous by the same arguments as before, $\hat{\varphi}$ is a homeomorphism. It follows from theorem 2.1.5 that φ is a homeomorphism and $\varphi^{-1} = \psi$.

From the discussion above the desired conclusion follows as

$$\begin{aligned}
\partial D^4 &\cong \partial(D^2 \times D^2) \\
&= \overline{D^2 \times D^2} - \mathrm{int}(D^2 \times D^2) \\
&= (D^2 - \mathrm{int}D^2) \times D^2 \cup D^2 \times (D^2 - \mathrm{int}D^2) \\
&= (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2).
\end{aligned}$$

□

Theorem 2.3.3. *Let $G_{p,q} = (x, y | x^p = y^q)$. Then $\pi_1(T_{p,q}) \cong G_{p,q}$.*

Proof. Consider $T_{p,q}$ as embedded in $S^1 \times D^2 \cup D^2 \times S^1$, since it was shown above that $S^3 \cong S^1 \times D^2 \cup D^2 \times S^1$, and let $T_1 = S^1 \times D^2$, let $T_2 = D^2 \times S^1$ and $T = T_1 \cap T_2 = S^1 \times S^1$. Choose an arbitrary basepoint $x_0 \in (T - T_{p,q})$.

Since $T_1 \simeq S^1$ and $T_2 \simeq S^1$ it follows that $\pi_1(T_1) \cong \mathbb{Z}$ and hence $\pi_1(T_1)$ has presentation $(x|-)$. Similarly, $\pi_1(T_2)$ has presentation $(y|-)$. It is visually clear that $(T - T_{p,q}) \cong R$, where $R \subseteq \mathbb{R}^2$ is an annulus. Therefore $T - T_{p,q} \simeq S^1$ and thus $\pi_1(T - T_{p,q})$ has presentation $(z|-)$. An application of van Kampens theorem yields that

$$\pi_1(S^3 - T_{p,q}) \cong (x, y | i_*(z) = j_*(z)),$$

where i_*, j_* are the homomorphisms induced by the inclusions $i : T \to T_1$ and $j : T \to T_2$.

It is clear that $z = \langle p, q \rangle \in \pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$ and since any loop in D^2 is nullhomotopic it follows that $i_*(z) = p$. Similarly it holds that $j_*(z) = q$ and thus it follows that $\pi_1(S^3 - T_{p,q})$ has presentation $(x, y | x^p = y^q)$. □

It obviously holds that if $p = 1$ or $q = 1$, this is the trivial knot. Moreover, the type of the knot is unaffected by changing the sign of either p or q or interchanging p and q. Otherwise the knots $T_{p,q}$ are distinct, as will be proven in the following series of propositions.

Lemma 2.3.4. *Let G and H be groups. All torsion elements in $G * H$ are conjugates of torsion elements in G or H.*

Proof. This will be proven by induction on the length of the word. The statement is trivial for words of length 0 and 1. Therefore, assume that $w = x_1 \cdots x_n \in G * H$ is a torsion element of length $n > 1$ and that the statement holds for all words of length $k < n$. It must hold that $x_1 = x_n^{-1}$, since otherwise $x_n x_1 \neq 1$ and therefore w^m can not be reduced to the empty word for any $m \geq 1$, hence w is not torsion. Therefore w will have the form $aw'a^{-1}$. But then w' will have length $n-2$, hence $w' = v'g(v')^{-1}$ for some torsion element g in either G or H. Thus $w = vgv^{-1}$ where $v = av'$ and therefore the statement holds. □

Theorem 2.3.5. *Let $1 < p < q$. Then $G_{p,q}$ determines the pair (p,q) uniquely.*

Proof. $G_{p,q}/\langle x^p \rangle \cong C_p * C_q$: Since x^p commutes with both generators of $G_{p,q}$ it holds that $\langle x^p \rangle \subset \mathrm{Cent}(G_{p,q})$ and hence it is a normal subgroup of $G_{p,q}$. Consider then the homomorphism $\varphi : G_{p,q} \to C_p * C_q$ given by $\varphi(x) = a$ and $\varphi(y) = b$, where a and b generate C_p and C_q respectively. It is clear that φ is surjective and $\ker(\varphi) = \langle x^p \rangle$ and thus φ induces an isomorphism $f : G_{p,q}/\langle x^p \rangle \to C_p * C_q$.

The abelianization of $C_p * C_q$ is $C_p \times C_q$. Consider the homomorphism $\psi : C_p * C_q \to C_p \times C_q$ given by $\psi(x) = x$ and $\psi(y) = y$. Since $C_p \times C_q$ is abelian, it follows that $[C_p * C_q, C_p * C_q] \subset \ker(\psi)$.
For the reversed inclusion consider an element $x^{n_1}y^{m_1} \cdots x^{n_k}y^{m_k} \in \ker(\psi)$. This obviously holds if, and only if, $p \mid \left(\sum_{i=1}^{k} n_i\right)$ and $q \mid \left(\sum_{i=1}^{k} m_i\right)$. Define $a_j \in C_p * C_q$ to be the element

$$a_j = (y^{\sum_{i=1}^{j} m_{i-1}} x^{\sum_{i=1}^{j} n_i}) y^{m_j}$$

and $m_0 = 0$. It is not difficult to see that

$$x^{n_1}y^{m_1} \cdots x^{n_k}y^{m_k} = [a_1, a_1] \cdots [a_k, a_k],$$

but the computations are a bit long and not very informative, and are therefore omitted. It follows that $x^{n_1}y^{m_1} \cdots x^{n_k}y^{m_k} \in [C_p * C_q, C_p * C_q]$. Hence ψ induces an isomorphism $g : C_p * C_q / [C_p * C_q, C_p * C_q] \to C_p \times C_q$. Since $|C_p \times C_q| = pq$ the product pq is uniquely determined by the group $G_{p,q}$.

It follows from lemma 2.3.4 and the assumption that $p < q$, that the maximal order of a torsion element in $C_p * C_q$ is q. It has therefore been shown that the product pq and q is uniquely determined by $G_{p,q}$, hence the pair (p,q) is uniquely determined. □

This concludes the section on torus knots since the above theorem, along with observations made earlier, gives a complete classification of the torus knots. It should be noted in passing that the above theorem also proves the existence of non-trivial knots in $\mathbb{R}^3$.

2.4 The Wirtinger Presentation

In this section, an algorithm for computing the fundamental group of any tame knot will be presented and proven successful, as well as applied to a selection of prime knots.

First let the basepoint of the knot group be $x = (0, 0, 1)$. The basepoint x should be thought of as "the eye of the beholder". To ease calculations, an image of the knot K in question is needed. To do this, consider K as a knot in $\mathbb{R}^3$, and take the image to be a finite number of disjoint arcs $\alpha_1, ..., \alpha_n$ in the xy-plane, such that each α_i is connected to α_{i+1} $(mod\ n)$, by an undercrossing arc β_i, dipping down ε below the plane. Futhermore, give each α_i an orientation, compatible with the ordering of the subscripts. Finally, let x_i be an arrow crossing underneath α_i from right to left, relative to the orientation that is chosen. This will represent an element of the knot group as the homotopy class of the loop consisiting of a straight line from x to the tail of x_i, then going along to the head and finally a straight line from the head of x_i back to x. The situation is exemplified in figure 2.1. Note that when drawing a knot, the knot is pictured as seen from above, namely from the basepoint.

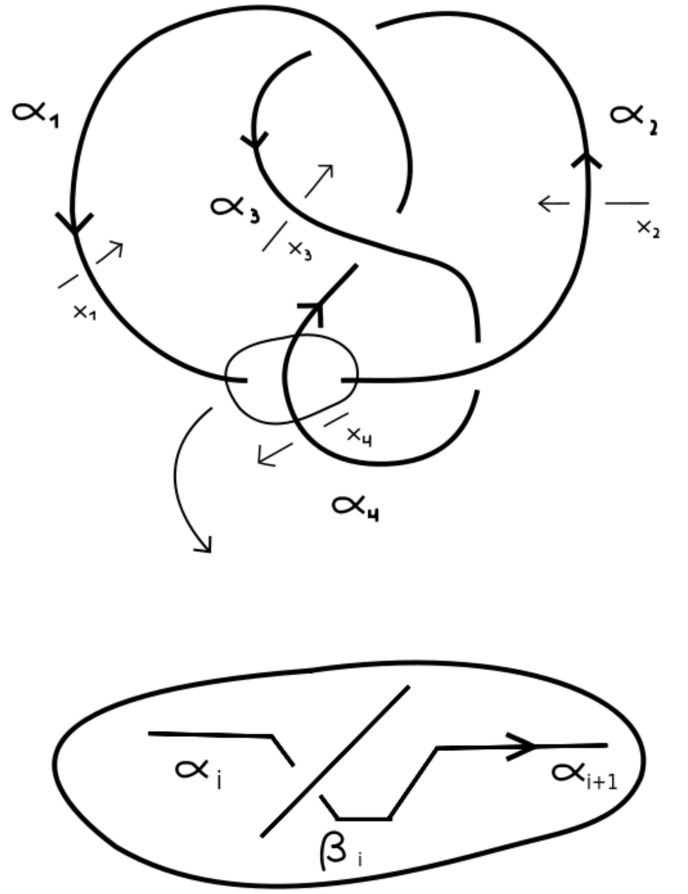


Figure 2.1: An example of the construction used in the Wirtinger presentation

Since the x_i's represents loops in $\mathbb{R}^3$ we can form products of them in the standard way. Consider the case where α_k crosses over β_i from right to left. In this case a relation must obviously hold between the x_i's, as demonstrated in figure 2.2 namely $r_i:\ x_k x_i = x_{i+1} x_k$. In the event that α_k crosses from left to right, the relation r_i will be $x_k x_{i+1} = x_i x_k$. It will be shown that this constitutes a complete set of relations for $\pi_1(\mathbb{R}^3 - K, x)$

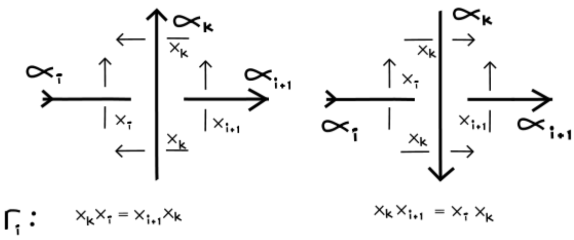


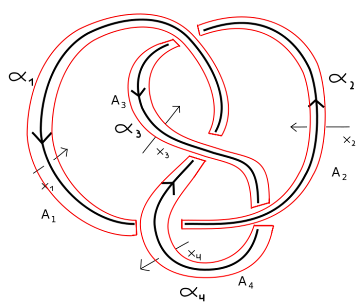
Figure 2.2: Relations between generators

Theorem 2.4.1. *The group* $\pi_1(\mathbb{R}^3 - K)$ *is generated by the* x_i*'s and has presentation*

$$\pi_1(\mathbb{R}^3 - K) = (x_1, ..., x_k | r_1, ..., r_n)$$

Furhtermore, one of the relations $r_1, ..., r_n$ *may be omitted and the above remains true.*

Proof. In order to apply van Kampen's theorem, $\mathbb{R}^3$ will be dissected into $2n+2$ pieces in the following way. Let $A = \{(x,y,z) \in \mathbb{R}^3 | z \geq -\varepsilon\} - K$. Subdivide further by letting A_i be an open neighbourhood of α_i, such that A_i contains none of the other arcs α_j, along with an arc from the top to x. The neighbourhood is chosen as an open, solid box whose bottom touches the lower boundary of A, with sides following the curve of α_i. The situation is exemplified in figure 2.4



Let $A' = \overline{A - (A_1 \cup \cdots \cup A_n)}$. Next let B_i be a solid rectangular box whose top fits on the lower boundary of A and contains β_i, then remove

β_i and adjoin an arc running from the top to x missing K. Choose the B_i's so they are disjoint. Finally let C be the closure of everything below $A \cup B_1 \cdots \cup B_n$.

First $\pi_1(A, x)$ is computed. This will be done by adjoining the A_i's to A' one at a time. Clearly both A' and $A' \cap A_1$ are simply connected, whereas A_1 is not. It is clear that A_1 is homeomorphic to a closed ball, with a diameter removed and a arc adjoined to the boundary, i.e. $(D^3 \cup L) - M$, where $L = \{(x, y, z) | (x = z = 0) \wedge (1 \leq y \leq 2)\}$ and $M = \{(x, y, z) | (x = y = 0) \wedge (-1 \leq z \leq 1)\}$.

Futhermore, $(D^3 \cup L) - M \simeq S^1$ via the homotopy $g_t(x, y, z) = (x, y, (1 - t)z)$ followed by $h_t(x, y) = (1 - t)(x, y) + t\frac{(x,y)}{||(x,y)||}$. Tracing through these homotopy equivalences x gets send to $(0, 1) \in \mathbb{R}^2$, and since $\pi_1(S^1)$ is infinite cyclic it follows from van Kampen that $\pi_1(A' \cup A_1, x)$ has presentation $(x_1|-)$. Similarly, $\pi_1(A' \cup A_1 \cup A_2, x)$ has presentation $(x_1, x_2|-)$ and so on until it is achieved that

$$\pi_1(A, x) = (x_1, ..., x_n|-)$$

Next, the effect of adjoining B_1 to A is examined. It is clear that B_1 is simply-connected and $B_1 \cap A$ is a rectangle with β_1 removed so $\pi_1(A \cap B_1, x)$ is infinite cyclic with generator y.

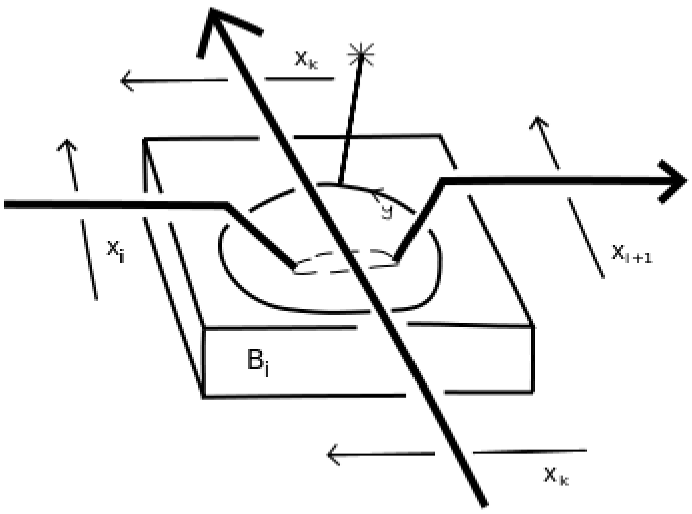


Figure 2.3:

As is clear from the picture (fig 2.3), the inclusion of y in A equals the word $x_1^{-1}x_k^{-1}x_2x_k$. Thus it follows from van Kampen's theorem that

$$\pi_1(A \cup B_1, x) = (x_1, ..., x_n | x_1^{-1}x_k^{-1}x_2x_k = 1),$$

which is equivalent to

$$\pi_1(A \cup B_1, x) = (x_1, ..., x_n | r_1).$$

This obviously only holds if the relation is of the first type in figure 2.2. However, the other type of relation follows in a completely similar way. Similar computations show that

$$\pi_1(A \cup B_1 \cup \cdots \cup B_n, x) = (x_1, ..., x_n | r_1, ..., r_n)$$

Finally since both C and $C \cap (A \cup B_1 \cup \cdots \cup B_n) \cong \mathbb{R}^2$ are simply connected, adjoining C has no effect on the fundamental group.

This completes the first part of the proof, leaving only the fact that one of the relations, say r_n, can be omitted. To see this, consider the knot as embedded in S^3, and let $A' = A \cup \infty$ and $C' = C \cup B_n \cup \infty$. Computations go exactly as before, except that $(A' \cup B_1 \cup \cdots \cup B_{n-1}) \cap C'$ is homeomorphic to a 2-sphere minus an arc, and is therefore simply connected. □

Theorem 2.4.1 gives a relatively simple way of obtaining the knot group of any tame knot, and as such is widely applicable. This will now be demonstrated by applying the algorithm on a selection of knots.

Example 2.4.1. The knot group of the trefoil $T_{2,3}$, as pictured in figure 2.4, has presentation $(x, y | xyx = yxy)$.

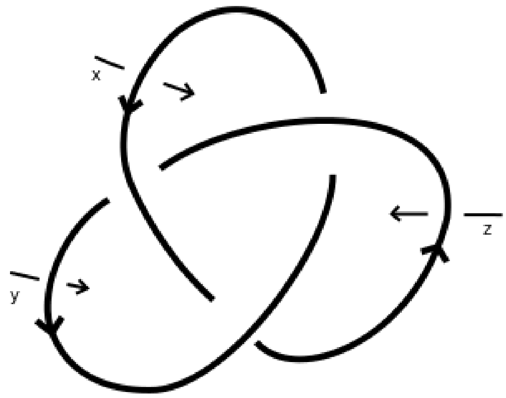


Figure 2.4: The Trefoil

A simple application of Theorem 1.3.1 yields that

$$\pi_1(\mathbb{R}^3 - T_{2,3}) = (x, y, z | xz = yx, zy = yx)$$

Note that there are only two relations in the presentation, as the theorem states that one of the crossings can be ignored in computations. Furthermore, the second relation can be transformed to $z = yxy^{-1}$, and through a substitution, one obtains the equivalent presentation

$$\pi_1(\mathbb{R}^3 - T_{2,3}) = (x, y | xyx = yxy).$$

As suggested by the notation $T_{2,3}$, the trefoil is a torus knot, which should be clear. As was proven in the previous section, it therefore has knot group

$$\pi_1(\mathbb{R}^3 - T_{2,3}) = (a, b | a^3 = b^2)$$

It follows that the two presentations must be equivalent, although this is not exactly obvious.

Let $G = (x, y | xyx = yxy)$ and $H = (a, b | a^3 = b^2)$, and consider the homomorphims $f : G \to H$ given by $f(x) = a^{-1}b, f(y) = b^{-1}a^2$ and $g : H \to G$ given by $g(a) = xy, g(b) = xyx$. To show that these homomorphism are well-defined, it must be shown that $f(xyxy^{-1}x^{-1}y^{-1}) = e_H$ and similarly $g(a^3b^{-2}) = e_G$. It should be obvious that $g(a^3b^{-2}) = e_G$. That f

is well-defined follows from the computations:

$$\begin{aligned}f(xyxy^{-1}x^{-1}y^{-1}) &= a^{-1}bb^{-1}a^2a^{-1}ba^{-2}bb^{-1}aa^{-2}b \\ &= ba^{-3}b \\ &= e_H\end{aligned}$$

where the last equality follows from the relation $a^3 = b^2$. Hence the homomorphisms are well-defined. The computations that show $f = g^{-1}$ are not difficult, but rather tedious and are therefore omitted.

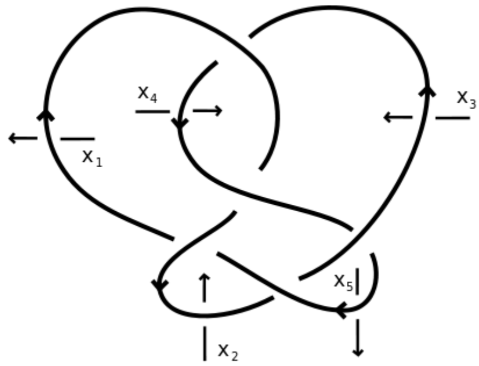


Figure 2.5: Example 2.4.2

Example 2.4.2. Let K be the knot pictured in figure 2.5.

It follows directly from theorem 2.4.1 that the knot group of K is

$$(x_1, x_2, x_3, x_4, x_5 | x_1x_3 = x_4x_1, x_4x_1 = x_2x_4, x_5x_3 = x_3x_4, x_5x_2 = x_3x_5)$$

This presentation can be changed by noting that the above relations hold if, and only if,

$$\begin{aligned}x_3 &= x_5x_2x_5^{-1}, \\ x_4 &= x_3^{-1}x_5x_3.\end{aligned}$$

A substitution yields that

$$x_4 = x_5x_2^{-1}x_5x_2x_5^{-1}.$$

Similarly, noting that

$$x_1 = x_4^{-1} x_2 x_4$$

and substituting where it is appropriate, one obtains the equivalent presentation

$$(x_2, x_5 | x_2 x_5 x_2^{-1} x_5 = x_5 x_2^{-1} x_5^{-1} x_2 x_5^{-1} x_2 x_5 x_2^{-1} x_5 x_2).$$

Example 2.4.3. In this example the knot group of the knot K_1 pictured in figure 2.6, is computed.

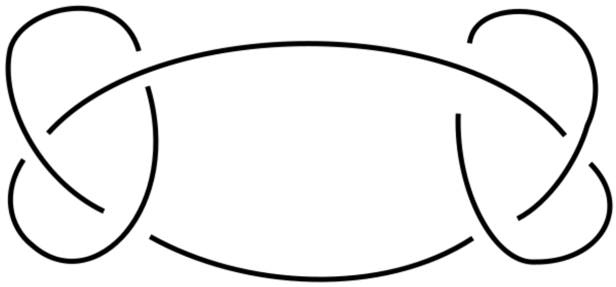


Figure 2.6: The knot K_1

This can be done by applying theorem 2.4.1 directly, or one can take a shortcut by considering:

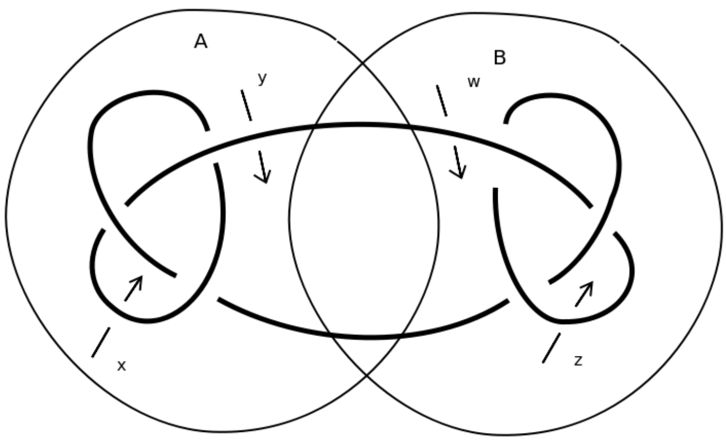


Figure 2.7: The knot K_1

Figure 2.7 is to be understood as follows: The circles A and B represents open balls in $\mathbb{R}^3$ containing the parts of K_1 that are indicated. It is clear that $\mathbb{R}^3 - K_1$ deformation retracts onto $(A - K_1) \cup (B - K_1)$, and hence $\pi_1(\mathbb{R}^3 - K_1, x_0) \cong \pi_1((A - K_1) \cup (B - K_1), x_0)$, when it is assumed that $x_0 \in (A \cap B)$. The knot group of K_1 can then be computed by noting that $A - K_1$ obviously has the same homotopy type as $A - T_{2,3}$ and similarly $B - K_1$ has the same homotopy type as $B - T_{2,-3}$. As proven in example 2.4.1 it holds that

$$\pi_1(A - K_1, x_0) = (x, y | xyx, yxy)$$
$$\pi_1(B - K_1, x_0) = (z, w | zwz = wzw)$$

It has been used, as remarked upon earlier, that $T_{2,3}$ and $T_{2,-3}$ are equivalent. Using Van Kampens theorem we obtain:

$$\pi_1(\mathbb{R}^3 - K_1, x_0) \cong ((A \cup B) - K_1, x_0) = (x, y, z | xyx = yxy, zyz = yzy)$$

Very similar computations show that the knot K_2, pictured in figure 2.8, has knot group

$$\pi_1(\mathbb{R}^3 - K_2) = (x, y, z | xyx = yxy, yzy = zyz).$$

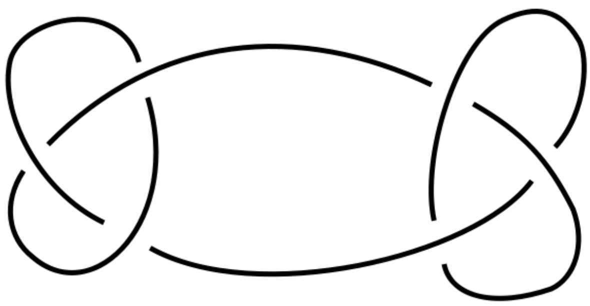


Figure 2.8: The knot K_2

It is easy to see that the knot groups of K_1 and K_2 in example 2.4.3 are isomorphic. As it turns out K_1 and K_2 are not equivalent. It is proven in examples 3D10 and 8E15 in [Rol76].

The above examples should serve to demonstrate that group presentations are in general difficult to handle. Although the two presentations in example 2.4.1 are relatively simple, the isomorphism between them does not seem to be an obvious or canonical choice. Similarly, the knots $T_{2,3}$ and K

from example 2.4.2 are not equivalent, and it is therefore not clear whether the two presentations describe isomorphic groups or not. On the face of it, they are rather similar, since they both have two generators and a single relation. However, it is a non-trivial problem to determine whether they are isomorphic or not, since it can be proven that there exists no algoritm for this task.

As a result, although the knot groups of given knots are quite easily calculated, the results are often very difficult to compare. It is therefore natural to ask whether a knot invariant exists, for which the end results are more readily compared and indeed there does. The Jones polynomial is one such invariant, which will be explored in the next chapter.

Chapter 3

The Jones Polynomial

In this chapter another knot invariant will be derived and applied to a few knots. The approach is rather different from the last chapter, and a selection of basic tools needs to be introduced. The key idea is that given diagrams, much like in the previous chapter, of two equivalent links the diagrams must somehow be related. This idea is formalized in the Reidemeister moves.

In this chapter the notion of oriented equivalence is used, and the reason for this will be clear later on. The definition of links adopted is definition 1.3.5.

3.1 Diagrams and Reidemeister Moves

Reidemeister moves requires a diagram of the knot or link in question, much like the picture of a knot discussed in the previous section. Recall that a link of m components is defined as m piece-wise linear, simple closed curves.

A link $L \subset \mathbb{R}^3$ is said to be in general position with respect to the standard projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, if:

1. Each line segment in L is projected to a line segment in $\mathbb{R}^2$
2. The projection of two segments intersect in at most one point and for disjoint line segments this is not an endpoint.
3. Each point belongs to the projection of at most two line segments.

It is easy to construct examples of links that are not in general position. However it can be proven, that for every link L in $\mathbb{R}^3$ there exists a link L' such that L is equivalent to L', and L' is in general position with respect to p. This result will not be proven here.

Hence given a link L, a diagram D of L will be the image under the projection p if it is in general position, otherwise it will be the image of a link L', that is equivalent to L and is in general position, along with 'over-and-under' information at each crossing in the diagram. 'Over-and-under' information refers to the relative values of the z-values in the coordinates of the points being projected to a crossing. This information will be indicated by breaks in the under-passing segment.

Recall that given a link L, the reflection of L, being the image of L under an orientation-reversing homeomorphism, is denoted $\overline{L}$. If D is a diagram of L, and $\overline{D}$ is obtained from D by changing all the unders and overs, then it is not difficult to see that $\overline{D}$ is a diagram of $\overline{L}$. If L is given an orientation and D is a diagram of L, this will amount to drawing an arrow in the diagram D.

Given such a diagram each Reidemeister move replaces a configuration of arcs with another configuration of arcs. There are three types of Reidemeister moves, all of which are illustrated in figure 3.1. Note that all moves illustrated works both ways. For instance a type I move either inserts or removes a kink in a diagram. The importance of Reidemeister moves are demonstrated by the following theorem, that will not be proven.

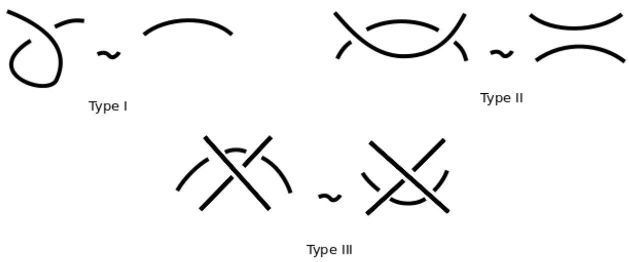


Figure 3.1: Reidemeister moves

Theorem 3.1.1. *Let L and L' be links in $\mathbb{R}^3$, let D be any diagram of L and D' any diagram of L'. Then L and L' are equivalent links if and only if D is related to D' through a series of Reidemeister moves and an orientation-preserving homeomorphism of the plane.*

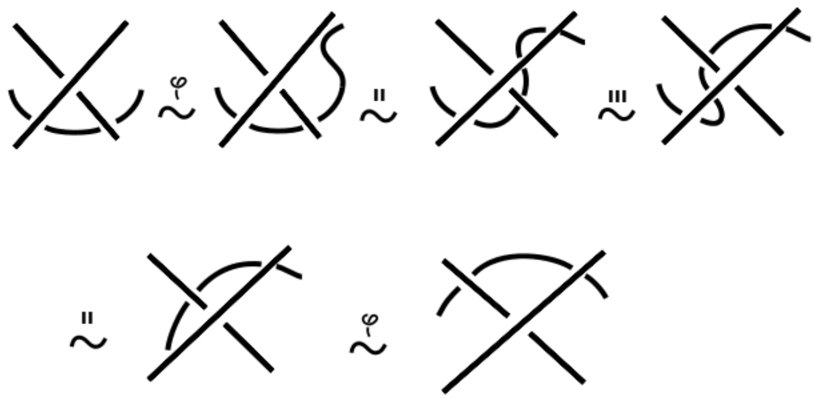
This was proven by Kurt Reidemeister in *Elementare Begründung der Knotentheorie* published in 1926.

There is a number of moves which are a direct consequence of the Reidemeister moves pictured above. Indeed there is an obvious variant of the type I move and similarly three obvious variants of the type III move.

Example 3.1.1. In this example a common variation of a type I move is shown to be a consequence of the Reidemeister moves:

where $\overset{\varphi}{\sim}$ denotes that the two diagrams are related through an orientation-preserving homeomorphism of the plane and $\overset{i}{\sim}$ denotes that they are related through a type i Reidemeister move, where $i \in \{I, II, III\}$.

Below a common variation of a type III move is shown to be a consequence of the Reidemeister moves:



Other obvious moves on knot diagrams can be shown to be a consequence of the Reidemeister moves, using very similar techniques. As a consequence the Reidemeister moves can be thought of as a generating set for the 'legal' moves that can be performed on link diagrams.

The above discussion will prove quite useful in developing the Jones polynomial, since it is defined by various operations on the diagram of a link. In particular, it follows that any function from link diagrams to some other space, will be an invariant if it is unchanged by the Reidemeister moves.

3.2 The Kauffman Bracket

In this section the Kaufmann bracket polynomial will be defined, as a step towards defining the Jones polynomial. It will simply be referred to as the bracket polynomial. The Laurent polynomials with integer coefficients in some indeterminate A will be denoted $\mathbb{Z}[A, A^{-1}]$.

Definition 3.2.1. The bracket polynomial is a function:

$$\begin{aligned} \langle \cdot \rangle : \{\text{unoriented link diagrams}\} &\to \mathbb{Z}[A, A^{-1}] \\ D &\mapsto \langle D \rangle \end{aligned}$$

characterized by:

(i) $\left\langle \bigcirc \right\rangle = 1$;

(ii) $\left\langle D \sqcup \bigcirc \right\rangle = (-A^{-2} - A^{2})\langle D \rangle$;

(iii) $\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle$.

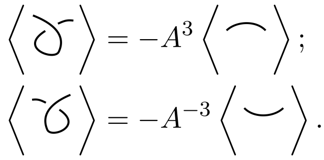
This definition is to be understood in the following way. In (i) the diagram represents the trivial knot with no crossings. In (ii) the left hand side is meant to represent a diagram D along with a simple closed curve, with no crossings between the two. In (iii) the diagrams represent the same link except in a small disc around the crossing, where they differ in the way indicated. Any orientation-preserving homeomorphism of the plane must preserve all crossing information, hence the bracket polynomial is unchanged by such an homeomorphism.

It will now be verified that the bracket polynomial is indeed a Laurent polynomial. This will be done by induction on the number of crossings. Assume therefore that D is a diagram with k components and no crossings. Then $k-1$ applications of (ii) and one application of (i) yields that $\langle D \rangle = (A^{-2} - A^{-2})^{k-1}$. Next assume that D is a diagram with n crossings. Then, by (iii), $\langle D \rangle$ can be expressed as a linear sum of 2^n diagrams with no crossings, and hence the conclusion follows.

In the process above one must apply (iii) in some order, and it must therefore be shown that the order does not affect the outcome. This can be done noting that any permutation of $\{1, ..., n\}$ can be expressed as a product of adjacent transpositions, and that transposing adjacing crossings does not affect the outcome. The calculations are not difficult, but not very informative either, and will therefore be omitted.

Next, the effects of Reidemeister moves on the bracket polynomial are examined. As will be clear, the bracket polynomial is not a link invariant, however the results will provide a clue as to how it can be modified into an invariant.

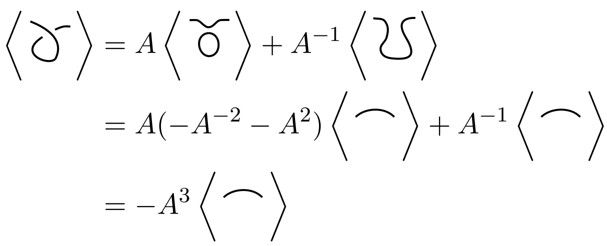
Lemma 3.2.2. *If a diagram is changed by a type I move its bracket polynomial changes in the following way:*



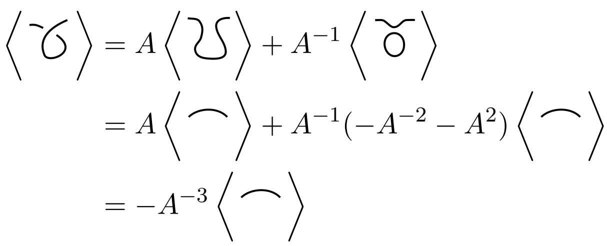
(3.1)

(3.2)

Proof. First (3.1) is proved:

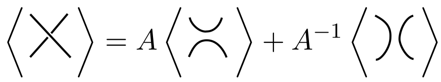


Very similar computations show that (3.2) is also true:



□

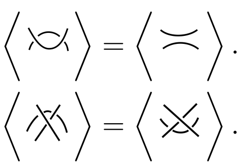
Notice that (iii) in definition 3.2.1, rotated by $\frac{\pi}{2}$, gives



and hence for any diagram D it holds that $\langle\overline{D}\rangle = \overline{\langle D\rangle}$, where $\overline{D}$ is the reflection of the diagram, as defined earlier and $\overline{\langle D\rangle}$ denotes the operation of interchanging A and A^{-1}.

Next the effects on the bracket polynomial, when performing a type II or type III move is investigated:

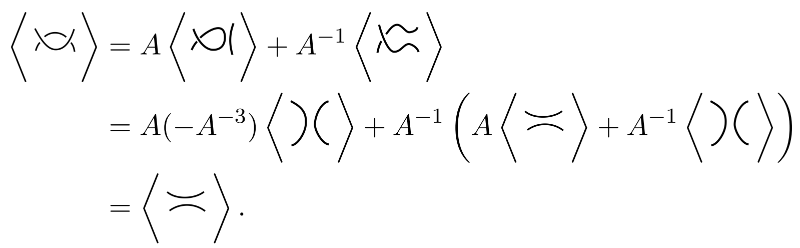
Lemma 3.2.3. *If a diagram D is changed by a type II or type III move, $\langle D\rangle$ is unchanged.*



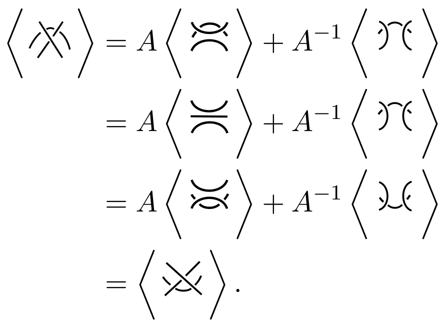
(3.3)

(3.4)

Proof. (3.3) is proven by this simple manipulation:



For the proof of (3.4) notice that (3.3) is used twice in the process.



∎

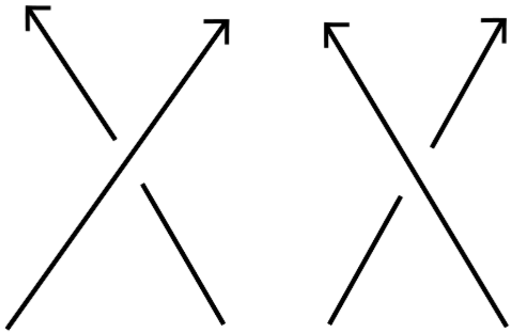
It is clear from this section, that the bracket polynomial is not a link invariant, since it is affected by a type I move. However, it is quite close. In the next section the *writhe* of an oriented link diagram is defined, and it is shown how this can be used to define an invariant which is closely related to the Kauffman bracket.

3.3 Developing The Jones Polynomial

In this section the Jones polynomial will be defined, using the bracket polynomial developed in the previous section, but first a new concept needs to be introduced. Since oriented links are involved, fix an orientation of $\mathbb{R}^3$. When working in S^3 matters are more complicated, but it can be shown that every S^n is a connected, orientable manifold, and any such manifold has exactly two distinct orientations. A proof of this can be found in [GH81, part III, section 22]. Any one of these orientations can be chosen, so we simply fix one of them as the given orientation.

Note that in $\mathbb{R}^3$ or S^3 for that matter, the orientation of a link L amounts to drawing an arrow in the diagram D of L.

Definition 3.3.1. The writhe of an oriented link diagram $\omega(D)$ is the sum of the signs of the crossings, where each crossing has sign $+1$ or -1 as defined (by convention) below.



The positive crossing is called a right-hand crossing and the negative a left-hand crossing.

Theorem 3.3.2. *Let D be a diagram of an oriented link. Then the expression*

$$(-A)^{-3\omega(D)}\langle D\rangle$$

is unchanged under the Reidemeister moves.

Proof. As proven in the previous section the bracket polynomial is unaffected by Reidemeister moves of type II and III. Furthermore since a type II move replaces crossings of opposite signs with no crossings (regardless of which orientation is chosen), $\omega(D)$ is also unchanged by a type II move. Similarly $\omega(D)$ is unchanged by a type III move and hence the claim is true for Reidemeister moves of type II and III.

Consider a diagram D and let D' be the diagram after a kink has been inserted using a type I move. Then $\omega(D') = \omega(D) + 1$ regardless of the orientation. Hence:

$$\begin{aligned}(-A)^{-3\omega(D')}\langle D'\rangle &= (-A)^{-3(\omega(D)+1)}\langle D'\rangle \\ &= (-A)^{-3\omega(D)}(-A)^{-3}(-A)^{3}\langle D\rangle \\ &= (-A)^{-3\omega(D)}\langle D\rangle.\end{aligned}$$

□

With this result in hand the Jones polynomial may be defined as follows:

Definition 3.3.3. The Jones polynomial $V(L)$ of a oriented link L is the laurent polynomial in $t^{1/2}$, with integer coefficients, defined by:

$$V(L) = \left((-A)^{-3\omega(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}}$$

where D is any oriented diagram of L.

It follows from theorem 3.3.2 that $V(L)$ is well-defined and that it is an invariant. Note that if the orientation of all components in a link are changed, the writhe is unchanged. Thus for a knot K it follows that $V(K)$ does not depend on the orientation.

Example 3.3.1. The Jones polynomial of the knot in figure 3.2, called the left-hand trefoil, will now be computed. First the bracket polynomial is

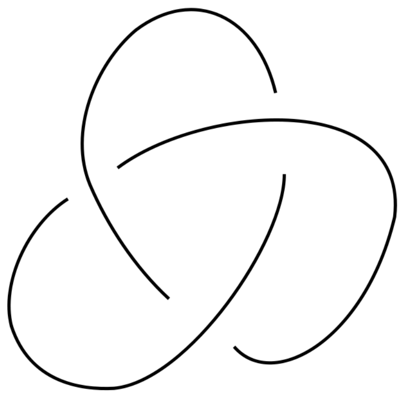
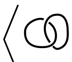


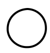
Figure 3.2: The left-hand trefoil

computed:

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

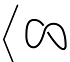
The two diagrams on the right-hand side will be treated seperately:

$\Big\langle$  $\Big\rangle = -A^3 \Big\langle$  $\Big\rangle$

$= (-A^3)(-A^3) \Big\langle$  $\Big\rangle$

$= A^6,$

and next:

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$= A(-A^3) + A^{-1}(-A^{-3})$

$= -A^4 - A^{-4}.$

Hence it holds that

$$\left\langle \right.$$  $$\left.\right\rangle = A \cdot A^6 + A^{-1}(-A^4 - A^{-4})$$
$$= A^7 - A^3 - A^{-5}.$$

Since all the crossings are left-hand crossings, $\omega(K) = -3$. Therefore

$$\begin{aligned} V(K) &= \left((-A)^{-3\omega(K)}\langle K\rangle\right)_{t^{1/2}=A^{-2}} \\ &= \left(-A^9(A^7 - A^3 - A^{-5})\right)_{t^{1/2}=A^{-2}} \\ &= \left(-A^{16} + A^{12} + A^4\right)_{t^{1/2}=A^{-2}} \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

The last result concerning the Jones polynomial will now be proven.

Theorem 3.3.4. *Let L be an oriented link. Then*

$$V(\overline{L}) = \overline{V(L)},$$

where $\overline{V(L)}$ denotes that $t^{1/2}$ and $t^{-1/2}$ have been interchanged.

Proof. Let D be a diagram of L. As remarked earlier $\langle\overline{D}\rangle = \overline{\langle D\rangle}$. Since $\omega(\overline{D}) = -\omega(D)$ it follows that

$$\begin{aligned} V\left(\overline{L}\right) &= \left((-A)^{3\omega(D)}\langle\overline{D}\rangle\right)_{t^{1/2}=A^{-2}} \\ &= \left((-A)^{3\omega(D)}\overline{\langle D\rangle}\right)_{t^{1/2}=A^{-2}} \\ &= \overline{V(L)}. \end{aligned}$$

□

3.4 Examples

In this section a selection of examples are treated.

Example 3.4.1. The left-hand trefoil is not equivalent to its reflection, called the right-hand trefoil. This is a straightforward application of theorem 3.3.4, since $\overline{-t^{-4} + t^{-3} + t^{-1}} = -t^4 + t^3 + t \neq -t^{-4} + t^{-3} + t^{-1}$.

A more general result could be stated by defining a polynomial $p \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$ to be *symmetric* if it can be written as follows:

$$p = \sum_{i=1}^{n} a_n t^{-i/2} + a_0 + \sum_{i=1}^{n} a_n t^{i/2}$$

and *asymmetric* otherwise.

Corollary 3.4.1. *Let L be a oriented link. If $V(L)$ is asymmetric then L and $\overline{L}$ are not equivalent.*

Proof. Trivial $\square$

Example 3.4.2. As a final application of the teory presented in this chapter consider the knot K_1 shown in figure 3.3

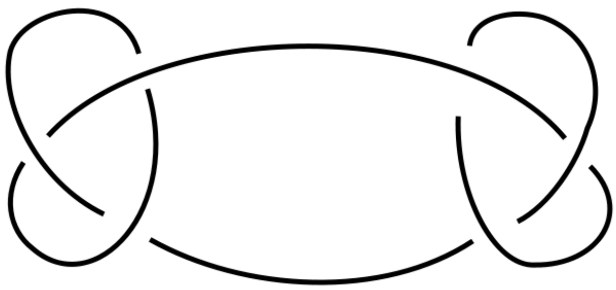


Figure 3.3: The knot K

It is not difficult to show that

$$V(K) = -t^{-3} + t^{-2} + t + 3 + t + t^2 - t^3.$$

Since $V(K)$ is a symmetric polynomial, K may be equivalent to its reflection Next, consider the knot K' pictured below

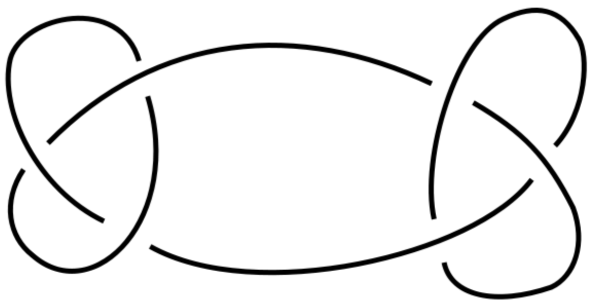


Figure 3.4: The knot K'

This has Jones polynomial

$$V(K') = t^{-8} - 2t^{-7} + t^{-6} - 2t^{-5} + t^{-4} + t^{-2}$$

Since $V(K) \neq V(K')$ the two knots are distinct. Furthermore, since $V(K)$ is symmetric, it follows that $\overline{V(K)} = V(K) \neq V(K')$, hence the reflection of K is not equivalent to K' either.

Chapter 4

The Jones Polynomial vs. The Knot Group

Which of the invariants introduced is the 'best'? This question is not easy to answer. As mentioned at the end of chapter 2, the Jones polynomial has the advantage of attaching polynomials to knots, which are much easier to compare to one another, than the group presentations produced by the algorithm in theorem 2.4.1. On the other hand, theorem 2.4.1 is easier to apply, at least to knots with relativly few crossings. Moreover, the knot group made the classification of torus knots possible, a result which would be difficult to prove using the Jones polynomial.

The selection of examples seems to illustrate that the Jones polynomial is a stronger invariant, since it has been able to distinguish more knots than the knot group. However, it is possible to construct examples of distinct knots, which the Jones polynomial fails to distinguish, and the knot group is succesful (Lick side 29). Therefore, the answer to this question is not simple, however their relative strengths and weaknesses may be summarized in the following way:

The knot group of a tame knot is very ease to calculate using theorem 2.4.1. However the groups produced by application of this theorem are often quite difficult to compare.

The Jones polynomial is a bit more difficult to calculate, however the results of the calculation are easily compared. It is able to distinguish between a knot and its mirror image, something the knot group is not capable of.

Appendix A

Group Presentations

This appendix is based on definitions and remarks made in [Mun00, section 69] and [Rol76, appendix A]. Informally a group presentation of a group G is a set of generators for the group along with at set of relations, such that every relation between the generators that are 'true' in G can be derived from these relations. This intuitive idea is worth holding on to, although it obviously is not optimal for a formalization of the concept.

To introduce this rigorously, let $\{x_1, ...\}$ be a (possibly infinte) set of generators and let $F(x_1, ...)$ denote the free group on the generators $x_1, x_2, ...$. Then there is a canonical homomorphism

$$\varphi : F \to G$$

defined by setting $\varphi(x_i) = x_i$ and expanding. This is obviously surjective and hence $F/N \cong G$, where N is the least normal subgroup containing $\ker(\varphi)$.

This subgroup N is meant to represent the total sum of all relations holding between the generators of G. This can be stated more precisely by letting $\{r_1, ...\}$ be a set of reduced words in the letters $x_1, ...$ such that the r_i and their conjugates generate N. These words are called the relators of G and the relations mentioned in the beginning are obtained by considering $\varphi(r_i) = r_i = 1$.

Now we introduce some notation. In the situation described above, i.e. $x_1, ...$ generate G and $r_1, ...$ and their conjugates generate N we write:

$$G = (x_1, ...|r_1, ...)$$

Any word $w \in N$ is said to be a consequence of the relators $r_1, ...$. In practice one often replaces the more formally correct relators with the more intuitive relations, as demonstrated throughout the thesis. In this case any relation in G is said to be a consequence of the relations $r_1, ...$ if it can be deduced from them, using standard algebraic operations.

Now we consider when two presentations descirbe the same group, i.e. when two presentations are isomorphic. The notation is the same as adopted above.

Lemma A.0.2. *Let G, H be groups and G have presentation $(x_1, ...|r_i, ...)$, and $f : \{x_1, ..., \} \to H$ be any function. If $f(r_i) = 1$ for each relator r_i, then f induces a unique homomorphism $\varphi : G \to H$.*

Proof. This is just a reformulation of a well-known result from algebra. Since any function $f : \{x_1, ..., \} \to H$ extends to a unique homomorphism $g : F \to H$, where F is the free group on the generators $x_1, ...$ (follows from [Mun00, lemma 69.1, . 421]) and $f(r_i) = 1$ for all the relators, it holds that $g(N) = \{1\}$ and hence there exists precisely one homomorphism $\varphi : G \to H$ such that $f = \varphi \circ \kappa$, where κ is the standard projection. □

Theorem A.0.3. *Let $(x_1, ..., |r_1, ...)$ and $(x'_1, ...|r'_1, ...)$ be group presentations. Let f be a function that assigns a word in $x'_1, ...$ to each x_i and g be a function that to each x'_i assigns a word in $x_1, ...$. If*

1. *$f(r_i)$ is a consequence of $r'_1, ...$ for each relator r_i and $g(r'_i)$ is a consequence of $r_1, ...$*
2. *$(f \circ g)(x'_i) = x'_i$ is a consequence of $r'_1, ...$ and $(g \circ f)(x_i) = x_i$ is a consequence of $r_i, ...$*

then the presentations describe isomorphic groups.

Proof. Let N_1 be the least normal subgroup of $F(x_1, ...)$ containing $r_1, ...,$ and N_2 be the least normal subgroup of $F(x'_1, ...)$ containing $r'_1, ...$ By lemma 69.1 in [Mun00] f and g extend to unique homomorphisms

$$f_* : F(x_1, ...) \to F(x'_1, ...)$$
$$g_* : F(x'_1, ...) \to F(x_1, ...)$$

Condition 1, along with Lemma 68.9 p. 420 in [Mun00] implies that $f_*(N_1) \subseteq N_2$ and $g_*(N_2) \subseteq N_1$, hence they both induce homomorphisms

$$\varphi : F(x_1, ...)/N_1 \to F(x'_1, ...)/N_2$$
$$\psi : F(x'_1, ...)/N_2 \to F(x_1, ...)/N_1$$

Condition 2 then implies that these induced homomorphism are each others inverses. The conclusion follows from a change in notation. □

In general it is difficult to determine whether two group presentations describe isomorphic groups. It can be proven that there exists no general algorithm to determine whether two group presentations describe isomorphic groups, or even an algorithm determining whether a group presentation describes the trivial group.

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