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Localizations in higher algebra and algebraic K-theory

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Abstract

There is a notion of commutative monoid and group in an ∞-category with finite products, which generalizes the usual notion of commutative monoid or group object in an ordinary category (in particular, the usual notion of commutative monoid or group). The tensor product on abelian groups is uniquely determined by the requirement that the free abelian group functor $\mathbf{Set} \to \mathbf{Grp}$ carries products to tensor products. In analogy with this, we show that for any presentable ∞-category $\mathcal{C}$, the ∞-categories of commutative monoids and groups admit unique symmetric monoidal structures making the free functors from $\mathcal{C}$ symmetric monoidal. Since it is in general very difficult to specify a symmetric monoidal structure on an ∞-category, we employ an indirect approach, due to Gepner-Groth-Nikolaus [4], using the universal properties of the ∞-categories of monoids and groups. The key result is that each of these constructions is a localization of the ∞-category Pr^{L} of presentable ∞-categories, and that this localization is compatible with the symmetric monoidal structure on Pr^{L} in a certain sense. As a consequence, we obtain a functor from $\mathbb{E}_\infty$ rig spaces to $\mathbb{E}_\infty$ ring spectra. We discuss applications of this machinery to algebraic K-theory.

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1 Introduction

Suppose we are given a commutative monoid M. Then we can build an abelian group, the *group completion* of M, by formally inverting every element of M. This defines a functor $\mathbf{CMon} \to \mathbf{Ab}$ which is left adjoint to the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{CMon}$. Now, suppose we are given a multiplication map on our monoid M. In other words, M has the structure of a *rig* (a "ring without negatives"). Then we would like to give its group completion the structure of a ring. Of course, one can do this explicitly, but there is also a more conceptual way of approaching this problem.

A multiplication, both in commutative monoids and abelian groups, is the same thing as a map

$$M \otimes M \to M$$

for certain symmetric-monoidal structures $(\mathbf{CMon}, \otimes)$, $(\mathbf{Ab}, \otimes)$. These can be characterized by the property that the free functors $\mathbf{Set} \to \mathbf{CMon}$ and $\mathbf{Set} \to \mathbf{Ab}$ send products of sets to tensor products of monoids/groups. Since group completion takes free monoids to free groups, it follows that group completion admits a symmetric monoidal structure. This implies it takes algebra objects in $\mathbf{CMon}$ (rigs) to algebra objects in $\mathbf{Ab}$ (rings).

This is hardly the most efficient way of proving this theorem. However, it has the advantage of generalizing, with some work, to a more general setting. There are "homotopy-theoretic" generalizations of the categories of commutative monoids and abelian groups, which we will call $\mathrm{CMon}(\mathcal{S})$ and $\mathrm{CGrp}(\mathcal{S})$. These are ∞-categories which are important, for instance, in algebraic K-theory. Endowing these things with a symmetric-monoidal structure is an important problem. It is not feasible to simply write down a tensor product, and a symmetric monoidal structure on the free functors and the group completion, as one can do in the classical case. Hence, we employ the approach sketched above, of showing that there is a unique symmetric monoidal structure determined by the requirement that the free functor is monoidal. To do this, we make use of a theory of *localizations* in symmetric-monoidal ∞-categories.

We will prove the following statement:

Theorem A. Let $\mathcal{C} \to \mathcal{D}$ be a symmetric monoidal left adjoint functor between presentably symmetric monoidal ∞-categories. Then this extends to a diagram of symmetric monoidal functors between presentably symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "F"] & \mathcal{C}_* \arrow[r] \arrow[d, "(F)_*"] & \mathrm{CMon}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{CMon}(F)"] & \mathrm{CGrp}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{CGrp}(F)"] & \mathrm{Sp}(\mathcal{C}) \arrow[d, "\mathrm{Sp}(F)"] \\
\mathcal{D} \arrow[r] & \mathcal{D}_* \arrow[r] & \mathrm{CMon}(\mathcal{D}) \arrow[r] & \mathrm{CGrp}(\mathcal{D}) \arrow[r] & \mathrm{Sp}(\mathcal{D})
\end{tikzcd}$$

in an essentially unique way. In particular, each of the categories above admits canonical symmetric monoidal structure, uniquely determined by the requirement that the free functors be symmetric monoidal.

As a direct corollary of this, we get the following:

Theorem B. The K-theory functor

$$\mathcal{C}\mathrm{at}^{\otimes} \to \mathrm{Sp}$$

admits a canonical lax symmetric monoidal structure.

(Here $\mathcal{C}\mathrm{at}^{\otimes}$ denotes the ∞-category of symmetric monoidal categories).

In particular, this implies that for a commutative ring R, $K(R)$ has the structure of an $\mathbb{E}_\infty$-ring spectrum, with the multiplication coming from $\otimes$.

∞-categories

In this project, we work in the language of *∞-categories*. This is a formalism for doing "abstract homotopy theory", introduced by Boardmann-Vogt [2] under the name *Weak Kan complex*, and developed heavily by Joyal (who named them *quasicategories*) in [5], and Lurie in [8], who used the term ∞-categories.

Definition 1.1. An ∞-category is a simplicial set $\mathcal{C}$ so that for each diagram of this form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

where $0 < i < n$, there is an extension as indicated by the dashed arrow.

To motivate this definition, note for instance that a simplicial set is the nerve of an ordinary category if and only if each such diagram admits a *unique* filler.

Given an ∞-category $\mathcal{C}$, one should interpret the 0-simplices as the objects, and the 1-simplices as the morphisms. A 2-simplex of the form

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

should be interpreted as a homotopy from $g \circ f$ to h. The higher simplices should similarly be interpreted as homotopies between homotopies (between homotopies ...).

Note that, in an ∞-category, there is no one composite between to morphisms $X \to Y \to Z$. Rather, there is a choice of composites given by the fillers for the corresponding inner horn (and by assumption there is always at least one such composite). It can be shown that composites are essentially unique in a strong sense (the map $\mathrm{Map}_{sSet}(\Delta^2, \mathcal{C}) \to \mathrm{Map}_{sSet}(\Lambda^2_1, \mathcal{C})$ is a trivial Kan fibration - in fact this condition is equivalent to asking that $\mathcal{C}$ be a ∞-category).

A functor between ∞-categories cannot just be defined by giving a value on objects and morphisms and asking that it preserves the composition - indeed, the equation $F(f \circ g) = F(f) \circ F(g)$ makes no sense in an ∞-category. Rather, we must ask that for each 2-simplex as above, we get a corresponding one in the codomain, and similarly for the higher simplices. In other words, a functor between ∞-categories is just a map of simplicial sets.

It turns out that one can develop most of the machinery of ordinary category theory in this setting. However, ∞-categories also function as an alternative for model categories. For instance, every model category has an "underlying" ∞-category, and limits and colimits in this ∞-category (which can be defined as initial/terminal objects of a "slice ∞-category ", analogously to the definitions in ordinary category theory) correspond to homotopy limits and colimits in the model category.

In fact, in the setting of the main result of this project, *presentable ∞-categories*, the correspondence is extremely strong: every presentable ∞-category is the underlying ∞-category of a (left proper, combinatorial) model category, and any colimit-preserving functor "lifts" to a left Quillen functor between the corresponding model categories.

We will elaborate slightly on two of the more technical aspects of this theory.

First of all, one often needs to work with functors into $\mathcal{C}\mathrm{at}_\infty$, the ∞-category of ∞-categories. However, it can be quite hard to write down such a functor, since one needs to specify an infinite amount of coherence data. To approach this problem, Lurie introduced the *straightening equivalence*, which allows us to identify functors $\mathcal{K} \to \mathcal{C}\mathrm{at}_\infty$ with certain maps of simplicial sets $\mathcal{E} \to \mathcal{K}$, the so-called *coCartesian fibrations*. We will not go into a study of the straightening equivalence here - for the details, see [8, Chapter 3].

Theorem 1.2 (Lurie)**.** Let $\mathcal{K}$ be an ∞-category. Then there is an equivalence of ∞-categories

$$(\mathcal{C}\mathrm{at}_\infty)^{\mathrm{coCart}}_{/\mathcal{K}} \simeq \mathrm{Fun}(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty),$$

the *straightening equivalence*. Here the left-hand side refers to the full subcategory of the slice $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{K}}$ spanned by those functors $\mathcal{E} \to \mathcal{K}$ which are *coCartesian fibrations*.

Second, we will need to use the notion of Kan extension for ∞-categories. Again, we will not give an account of the theory, but note that it mostly follows the same form as the version for ordinary categories. The notable difference is that the Kan extensions considered herein are analogous to what is usually called a *pointwise* Kan extension. The theory is developed in [8, Section 4.3].

Lastly, we note the following proposition, which is crucial for the project.

Theorem 1.3 ([6, Theorem 5.14])**.** Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the set of objects. Then the restriction functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$ is conservative.

Overview of the project

In section 1, we describe the theory of localizations in ∞-categories, which plays a key role.

In section 2, we state some results about presentable ∞-categories, the type which the main theorem is concerned with.

In section 3, we describe the algebraic ∞-categories which are the focus of the project: the ∞-categories of pointed objects, commutative monoids and commutative groups. We prove a number of basic results about these categories.

In section 4, we give an abbreviated account of symmetric monoidal ∞-categories, which are the ∞-categorical analogue of symmetric monoidal categories. Importantly, we describe a symmetric monoidal structure on Pr^{L} so that commutative algebras in this structure are precisely the presentable closed symmetric monoidal categories.

In section 5, we begin the path towards the main theorem, by proving that each of the assignments discussed in section 3 can be viewed as a localization of Pr^{L}, and we describe these localizations.

In section 6, we briefly describe how a similar construction works for the category of *spectrum objects in* $\mathcal{C}$. We also give a short description of the "free" functors between the various categories we have introduced.

In sections 7, we describe how these localizations, because they are compatible with the symmetric monoidal structure in a certain sense, induce localizations of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, which give the canonical symmetric monoidal structures we are interested in.

In section 8, we describe how this relates to algebraic K-theory, by giving a description of the K-theory spectrum in the language of this project, and showing how we obtain a multiplicative structure essentially for free. We also discuss how these results compare to classical work.

Conventions

We write $h\mathcal{C}$ for the homotopy category of $\mathcal{C}$, if $\mathcal{C}$ is an ∞-category.

We write $\mathrm{Map}(X, Y)$ or $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for the mapping spaces in an ∞-category or simplicial category, in contrast $\mathrm{Hom}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for the Hom-sets of an ordinary category.

We denote by $\mathrm{Fun}^{\mathrm{L}}$ (resp. $\mathrm{Fun}^{\mathrm{R}}$) the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve colimits (resp. limits) in each variable.

We write $\mathcal{S}$ for the ∞-category of spaces, $\mathcal{C}\mathrm{at}_\infty$ for the ∞-category of ∞-categories.

A *space* for us will always mean an object of the ∞-category of spaces, or more concretely a Kan complex (although in most cases it could equivalently be a CW-complex).

We generally use $\mathcal{C}, \mathcal{D}$ for ∞-categories, while we use $\mathbf{C}, \mathbf{D}$ for ordinary categories.

$\widehat{\mathcal{C}\mathrm{at}}_\infty$ refers to the category of not necessarily small ∞-categories. In general, we use a hat to denote any collection of large objects.

If $\mathcal{C}$ is an ∞-category, or an ordinary category, we denote by $\mathcal{C}^{\simeq}$ the maximal subgroupoid, i.e the subcategory spanned by all isomorphisms or equivalences in $\mathcal{C}$. Recall that in the case of ∞-categories, this is the largest Kan complex contained in $\mathcal{C}$.

We generally use the symbol I for the unit object in a symmetric monoidal (∞-)category, $*$ for terminal objects, and 1 for identities.

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2 Localizations of ∞-categories

For ordinary categories, there is a very useful theory of *localizations*. This goes back at least to work of Adams in [1] and Bousfield in [3].

Since the papers [1] and [3] are good references to the ordinary theory of localizations, we will not recall it here, but it is very useful to keep it in mind as we proceed, since the ∞-categorical version is essentially the same.

Since things are in general much more complicated in the world of ∞-categories, one might fear that a good theory of localizations would also be very technical. However, it turns out that one can develop the theory in a completely analogous way. This is done by Lurie in [8, Section 5.2.7]. We will cite that section heavily, although we lay out the material in a slightly different way. We also prove a theorem about how the local equivalences determine a localization which is absent from [8], although again it is a very easy consequence of the theorems there.

We start with a number of definitions. The relationships between these will be exactly as in the ordinary theory.

Definition 2.1. Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then L is called a *localization* if it admits a right adjoint which is fully faithful.

Definition 2.2. Let $L : \mathcal{C} \to \mathcal{C}$ be an endofunctor of the ∞-category $\mathcal{C}$, and let $\eta : 1_{\mathcal{C}} \to L$ be a natural transformation. We say that η exhibits L as an *idempotent functor* if for each $X \in \mathcal{C}$, there exists an equivalence $\eta_{LX} \simeq L(\eta_X) : LX \to LLX$, and both η_{LX} and $L(\eta_X)$ are localizations.

Definition 2.3. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory of the ∞-category $\mathcal{C}$, stable under equivalence We say that a map $X \to \tilde{X}$, where $X \in \mathcal{C}, \tilde{X} \in \mathcal{C}_0$ is a *$\mathcal{C}_0$-localization of* X, or a localization *relative to* $\mathcal{C}_0$ if the induced map

$$\mathrm{Map}(\tilde{X}, Y) \to \mathrm{Map}(X, Y)$$

is an equivalence for each $Y \in \mathcal{C}_0$.

Definition 2.4. Let W be a collection of morphisms in the ∞-category $\mathcal{C}$, which has the 2-out-of-3 property and contains all equivalences. We say that an object X is W-local if, for each map $f : A \to B$ in W, the induced map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is an equivalence. We will denote by $\mathcal{C}_W$ the full subcategory of W-local objects.

Remark 2.5. In this case, $\mathcal{C}_W$ is stable under equivalence.

Definition 2.6. If $L : \mathcal{C} \to \mathcal{C}$ is an idempotent functor, we say that a morphism $f : X \to Y$ in $\mathcal{C}$ is a *local equivalence for* L if Lf is an equivalence. If the choice of L is understood, we frequently just say that f is a local equivalence.

We now discuss the relationship between these things

Proposition 2.7 ([8, Prop 5.2.7.4])**.** Suppose $L : \mathcal{C} \to \mathcal{C}$ is an endofunctor on the ∞-category $\mathcal{C}$, and let $\eta : 1_{\mathcal{C}} \to L$ be a natural transformation. Then the following are all equivalent:

(1) L is left adjoint to the inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$ of the essential image, with unit $\eta : 1_{\mathcal{C}} \to L$.

(2) $\eta_{LX}, L(\eta_X) : LX \to LLX$ are equivalences for each $X \in \mathcal{C}$.

(3) η exhibits L as an idempotent functor.

Remark 2.8. It is clear that, in the first case, the functor $L : \mathcal{C} \to L\mathcal{C}$ is a localization. On the other hand, if $L : \mathcal{C} \to \mathcal{D}$ is a localization, we can identify $\mathcal{D}$ with a full subcategory of $\mathcal{C}$ via the right adjoint, which will bring us back to the situation of (1). Hence this theorem describes a sort of equivalence between localizations and idempotent functors. We will often abuse notation slightly and refer to a functor $L : \mathcal{C} \to \mathcal{C}$ as a localization, if $L : \mathcal{C} \to L\mathcal{C}$ is one.

Proposition 2.9 ([8, Prop. 5.2.7.8])**.** Suppose $\mathcal{C}_0 \subseteq \mathcal{C}$ is a full subcategory stable under equivalence. Then the following are equivalent:

(1) Each object in $\mathcal{C}$ admits a $\mathcal{C}_0$-localization.

(2) The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint.

Proposition 2.10. Let $L : \mathcal{C} \to \mathcal{C}$ be a functor. Let $\eta : 1_{\mathcal{C}} \to L$ be a natural transformation which exhibits L as an idempotent functor, and let W be the collection of local equivalences. Then a map $f : C \to D$ is an $L\mathcal{C}$-localization if and only if $D \in L\mathcal{C}$ and f is in W. In particular, each $\eta_X : X \to LX$ is an $L\mathcal{C}$-localization. Furthermore, $L\mathcal{C}$ is precisely the collection of W-local objects.

The proof is somewhat long, and does not really contain any technical insight. The above proposition is true essentially for the same reason that it is true for ordinary categories.

Proof. First we show $L\mathcal{C} = \mathcal{C}_W$. For the first inclusion, we must show that $X \in L\mathcal{C}$ is W-local, i.e. that if $A \to B \in W$, the map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is an equivalence. By commutativity of this naturality diagram

$$\begin{tikzcd} \mathrm{Map}(B, X) \arrow[r] \arrow[d] & \mathrm{Map}(A, X) \arrow[d] \\ \mathrm{Map}(LB, X) \arrow[r, "\sim"] & \mathrm{Map}(LA, X) \end{tikzcd}$$

it suffices to show this for the maps

$$\mathrm{Map}(LA, X) \to \mathrm{Map}(A, X)$$

But this is precisely the natural equivalence defining the adjunction, so it certainly must be an equivalence.

Suppose X is W-local, i.e for each $f : A \to B \in W$, the map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is an equivalence. Note that $\eta_X : X \to LX$ is in W. Hence

$$\mathrm{Map}(LX, X) \to \mathrm{Map}(X, X)$$

is an equivalence. It follows that there is some (essentially unique) map $g : LX \to X$ so that $g\eta_X \simeq 1_X$. Since LX is W-local, the map

$$\mathrm{Map}(LX, LX) \to \mathrm{Map}(X, LX)$$

given by precomposition with η is an equivalence. Clearly the identity and ηg both map to $\eta g \eta \simeq \eta$. Hence $\eta g \simeq 1_{LX}$, which shows that g is an equivalence. Hence X is really in the essential image of L, as desired.

Now for the first part of the statement. First, consider $f : X \to Y$. Suppose $f \in W$, and that $Y \in L\mathcal{C}$. We must show that

$$\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is an equivalence, for each $Z \in L\mathcal{C}$. By the above, $Z \in \mathcal{C}_W$, and the result follows immediately.

Now suppose $f : X \to Y$ is an $L\mathcal{C}$-localization. Then by definition $Y \in L\mathcal{C}$. We must show that $f \in W$, in other words, that Lf is an equivalence. By definition, we have an equivalence

$$\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

whenever $Z \in L\mathcal{C}$. In particular, we have an equivalence

$$\mathrm{Map}(Y, LX) \to \mathrm{Map}(X, LX)$$

This defines a map $g : Y \to LX$ with the property that $gf \simeq \eta_X$. By naturality, we have the following commuting diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"] & Y \arrow[d, "\eta_Y"] \\ LX \arrow[r, "Lf"] & LY \end{tikzcd}$$

In other words $\eta_Y f \simeq Lf\eta_X \simeq Lfgf$. Since f is an $L\mathcal{C}$ localization, $Lfg \simeq \eta_Y$, i.e $Lf(g\eta_Y^{-1}) \simeq 1_{LX}$. Now consider $g\eta_Y^{-1}Lf : LX \to LX$. We wish to show that it equals 1_{LX}. It is enough to observe that this is true after precomposition with η_X, since this is an $L\mathcal{C}$-localization. We have shown that Lf is an equivalence, as desired. This completes the proof. ☐

Warning 2.11. If $L : \mathcal{C} \to \mathcal{C}$ is a localization functor, then each object X has an $L\mathcal{C}$-localization $X \to LX$. If the objects and morphisms of $\mathcal{C}$ are (∞-)categories and functors (e.g. Pr^L, the main example of interest), this can cause some confusion, since the localization map $X \to LX$ could reasonably be called a "localization functor", but is not one in the sense of definition 2.1.

Proposition 2.12 ([8, Prop. 5.2.7.12])**.** Let $L : \mathcal{C} \to \mathcal{C}$ be an idempotent functor. Let $\mathcal{D}$ be another ∞-category. Then composition with L induces a functor

$$\mathrm{Fun}(L\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

This functor is fully faithful, and its essential image consists of those functors F which send local equivalences to equivalences.

This is a slightly modified version of [8, Prop. 5.2.7.12].

The preceding propositions describe how the localization $L : \mathcal{C} \to \mathcal{C}$, the full subcategory of local objects, and the local equivalences all determine one another. We will use this correspondence frequently. Proposition 2.12 essentially states that $L\mathcal{C}$ is the ∞-category obtained from $\mathcal{C}$ by formally inverting the local equivalences.

The following propositions will be convenient later:

Proposition 2.13. Suppose $L : \mathcal{C} \to \mathcal{C}$ is a localization, and let $\mathcal{D}$ be any ∞-category. Then L induces a functor $L_* : \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$. This functor is a localization, with essential image consisting of those functors with image contained in $L\mathcal{C}$

Proof. It is clear that the essential image of L_* contains only functors with essential image contained in $L\mathcal{C}$. Conversely, if a functor F has essential image contained in $L\mathcal{C}$, the natural transformation $\eta F : F \to LF$ is an equivalence at each object, hence a natural equivalence.

To see that L_* is a localization, we will construct a natural transformation $1 \to L_*$ in the category

$$\mathrm{Fun}(\mathrm{Fun}(\mathcal{C},\mathcal{D}),\mathrm{Fun}(\mathcal{C},\mathcal{D})).$$

This is the same thing as a map $\mathrm{Fun}(\mathcal{D},\mathcal{C}) \times \Delta^1 \times \mathcal{D} \to \mathcal{C}$. We are given a map $\eta : \mathcal{C} \times \Delta^1 \to \mathcal{C}$. We can use the canonical map $e : \mathrm{Fun}(\mathcal{D},\mathcal{C}) \times \mathcal{D} \to \mathcal{C}$, composing it with η, to build a map of the correct type. Undwinding the definitions, it is not hard to see that it will have the right properties. □

Proposition 2.14. Suppose $L : \mathcal{C} \to \mathcal{C}$ is a localization, and $\mathcal{D} \subseteq \mathcal{C}$ is a full subcategory such that $L\mathcal{D} \subseteq \mathcal{D}$. Then the restriction $L|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a localization.

Proof. Let η be a natural transformation exhibiting L as idempotent. Then it is immediate that the restriction of η to $L|_{\mathcal{D}}$ will exhibit it as idempotent, and we are done. □

Corollary 2.15. Suppose we have inclusions $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{E}$ of ∞-categories, and that furthermore both $\mathcal{C} \hookrightarrow \mathcal{E}$ and $\mathcal{D} \hookrightarrow \mathcal{E}$ admit left adjoints. Then also $\mathcal{C} \hookrightarrow \mathcal{D}$ admits a left adjoint, and these three left adjoints can be chosen to fit into a commutative triangle like this

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, shift left] \arrow[r, phantom, "\top"] \arrow[dr, hook, shift left] \arrow[dr, phantom, "\top"] & \mathcal{D} \arrow[l, shift left] \arrow[d, hook, shift left] \arrow[d, phantom, "\dashv"] \\
& \mathcal{E} \arrow[u, shift left] \arrow[ul, shift left]
\end{tikzcd}$$

3 Presentable ∞-categories

The overall idea of this project is to study certain localizations in ∞-categories of ∞-categories. Localizations can be identified with certain full subcategories, which are essentially the same thing as properties of the objects (i.e. of ∞-categories). We will be particularly interested in certain properties, which we will discuss momentarily, for instance *preadditivity*. However, it is not quite true that the subcategory of preadditive ∞-categories, as a subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$, (the ∞-category of large ∞-categories), is a localization.

For this to be true, we need to work with the somewhat more technical notion of *presentable* ∞-categories.

Warning 3.1. This section is not very enlightening. If one simply remembers that presentable ∞-categories have all limits and colimits, and satisfy the adjoint functor theorem, it will probably be safe to ignore this section. (The only exception is when we have to show that the categories we construct are actually presentable)

For a comprehensive reference on the material in this section, see [8, Section 5.4 & 5.5]

Definition 3.2. Let κ be a regular cardinal. An ∞-category is κ-*accessible* if it is locally small, admits κ-filtered colimits, the subcategory of κ-compact objects $\mathcal{C}^\kappa$ is essentially small, and generates $\mathcal{C}$ under κ-filtered colimits. We say that $\mathcal{C}$ is accessible if it is κ-accessible for some κ.

Remark 3.3. For the most part, the definitions of κ-accessible, κ-compact, κ-filtered, etc, can be ignored. We will need them only briefly to show that certain ∞-categories are presentable.

Definition 3.4. An ∞-category is *presentable* if it is accessible and admits all (small) colimits.

We will need the following lemma.

Lemma 3.5. Let $\mathcal{C}$ be κ-accessible. Then a map $f : X \to Y$ in $\mathcal{C}$ is an equivalence if and only if each map $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is an equivalence, for $A \in \mathcal{C}^\kappa$.

Proof. The "only if" direction is trivial. Now assume that $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is an equivalence whenever A is κ-compact.

Observe that $X \to Y$ is an equivalence if and only if each induced map $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is an equivalence, for $A \in \mathcal{C}$.

Now let any A be given. Then by assumption we can write $A = \mathrm{colim}_{i \in I} A_i$, where I is κ-filtered and $A_i \in \mathcal{C}^\kappa$. Then $\mathrm{Map}(A, -)(f) = \mathrm{Map}(\mathrm{colim}_{i \in I} A_i, -)(f) = \lim_{i \in I^{op}} (\mathrm{Map}(A_i, -)(f))$ By assumption, each $\mathrm{Map}(A_i, -)(f)$ is an equivalence, hence this limit is an equivalence. This concludes the proof. □

Remark 3.6. This proof only uses the assumption that $\mathcal{C}$ is generated by the collection $\mathcal{C}^\kappa$ under colimits.

Proposition 3.7 ([8, Corollary 5.5.2.4])**.** Let $\mathcal{C}$ be a presentable ∞-category. Then it has all limits.

Proposition 3.8 ([8, Corollary 5.5.2.9], The adjoint functor theorem for ∞-categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.

(1) F admits a right adjoint if and only if F preserves all colimits.

(2) F admits a left adjoint if and only if F preserves all limits, and κ-filtered colimits for some regular cardinal κ.

Remark 3.9. A functor which preserves κ-filtered colimits is called κ-accessible. A functor which is κ-accessible for some κ is called accessible.

The ∞-category Pr^L

The category of presentable ∞-categories in which we will work is the following

Definition 3.10. Pr^L is the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by the presentable ∞-categories and colimit-preserving functors. We also define a category Pr^L of presentable ∞-categories and limit-preserving, accessible functors in an analogous way.

Definition 3.11. If $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, we denote by $\mathrm{Fun}^{Ra}(\mathcal{C}, \mathcal{D})$ the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the limit-preserving and accessible functors.

Corollary 3.12 (of the adjoint functor theorem)**.** There is a categorical equivalence $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{op}$ mapping each category to itself, and each functor to a right adjoint.

We will also need a few ways of building new presentable categories.

Proposition 3.13. Suppose $\mathcal{C}, \mathcal{D}$ are presentable ∞-categories, and K is a small simplicial set.

(1) $\mathrm{Fun}(K, \mathcal{C})$ is presentable.

(2) $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable.

Proposition 3.14. Let $\mathcal{C}$ be a presentable ∞-category, and let W be a (small) set of morphisms in $\mathcal{C}$. Then $\mathcal{C}_W \hookrightarrow \mathcal{C}$ admits a left adjoint, and $\mathcal{C}_W$ is presentable.

Remark 3.15. In this situation, we say that $\mathcal{C}_W$ is a presentable localization of $\mathcal{C}$.

4 Pointed objects, commutative monoids and groups

We now introduce the ∞-categories of "algebraic objects" that we will work with: commutative groups and monoids. We will begin by introducing the category of *pointed objects* in an ∞-category $\mathcal{C}$. There are two reasons for this. First, it is sometimes necessary to work with e.g. pointed spaces, and it can be convenient to understand how they fit into the story of this project. More importantly however, the category of pointed objects will serve as a simple example of the theory we are developing. For this reason, some of the proofs have been made more complicated than strictly necessary, in order to emphasize the parallels with the case of monoids and groups.

Pointed objects

Definition 4.1. Let $\mathcal{C}$ be an ∞-category with a terminal object. Define $\mathcal{C}_*$ as the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms beginning at a terminal object.

Proposition 4.2. Suppose $\mathcal{C}$ is presentable. Then also $\mathcal{C}_*$ is presentable, and the forgetful functor

$$\mathcal{C}_* \to \mathcal{C}$$

admits a left adjoint.

Proof. First, we establish presentability. Since $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is presentable, it suffices to exhibit a set of maps W, so that $\mathcal{C}_*$ is precisely the subcategory of local objects. Consider functor $ev_0 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ given by evaluating at 0. Since it preserves limits and colimits, it admits a left adjoint, F.

Now by construction,

$$\mathrm{Map}_{\mathcal{C}}(X, f(0)) \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1,\mathcal{C})}(FX, f)$$

An object $f \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ is in $\mathcal{C}_*$ if and only if this space is contractible for each X, which is the same as requiring that the map

$$\mathrm{Map}(FX, f) \to \mathrm{Map}(F\emptyset, f)$$

is a homotopy equivalence, where $\emptyset$ is an initial object of $\mathcal{C}$ (which implies that $F\emptyset$ is initial as well). By accessibility, there is a small subset of $\mathcal{C}^\kappa$ so that is suffices to check this for X in this subset. Taking the maps $F\emptyset \to FX$, for each of these X, as our set W, it follows that the W-local objects are precisely the objects of $\mathcal{C}_*$.

The left adjoint can be constructed as the composite

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}_*$$

of the left adjoint of the "evaluation at 1" functor and the localization. $\square$

Remark 4.3. This proposition could certainly have been proved more simply. For instance, the left adjoint is given by $X \mapsto (* \hookrightarrow X \sqcup *)$, and it's not hard to check directly that this is an adjunction. The proof was made more complicated than necessary to emphasize the analogy with the corresponding proofs in the case of commutative monoids and groups.

Proposition 4.4. Let $\mathcal{C}, \mathcal{D}$ be complete ∞-categories. Then $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D}_*) \simeq \mathrm{Fun}^R(\mathcal{C}, \mathcal{D})_*$. Moreover, the two obvious functors into $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$ are identified under this identification.

Proof. Both categories can be identified with the subcategory of $\mathrm{Fun}(\Delta^1 \times \mathcal{C}, \mathcal{D})$ spanned by the functors which preserve limits in the second variable, and so that $F(0, x)$ is a terminal object for all x. $\square$

The category of finite pointed sets

Before we introduce commutative monoids and groups, we must make a brief digression to introduce the category of finite pointed sets. This category will also play a role in the next section, when we discuss symmetric monoidal ∞-categories. We will use the following conventions:

Definition 4.5. The (ordinary) category Fin_* has objects (X, x), where X is a finite set and $x \in X$. A morphism $(X, x) \to (Y, y)$ is a map $f : X \to Y$ with the property that $f(x) = y$.

It is instructive to think of a map $(X, x) \to (Y, y)$ in Fin_* as a *partially defined* function $X \setminus \{x\} \to Y \setminus \{y\}$.

Definition 4.6. We denote the element $(\{*, 1, 2, \ldots, n\}, *)$ of Fin_* by $\langle n \rangle$. We let $\langle n \rangle^\circ = \{1, 2, \ldots, n\} \subseteq \langle n \rangle$.

Remark 4.7. Since the collection of $\langle n \rangle$, with $n = 0, 1 \ldots$ is a skeleton of Fin_*, we often just work with these sets.

Definition 4.8. We say a map $f : (X, *) \to (Y, *)$ in Fin_* is *inert* if $f^{-1}(\{y\})$ is a singleton for each $y \neq *$ in Y. We say a map $f : (X, *) \to (Y, *)$ in Fin_* is *active* if $f^{-1}(\{*\}) = \{*\}$. We denote by $\mathrm{Fin}_*^{\mathrm{int}}$ the subcategory of inert maps, and by $\mathrm{Fin}_*^{\mathrm{act}}$ the subcategory of active maps.

Remark 4.9. $\mathrm{Fin}_*^{\mathrm{act}}$ is isomorphic to the category of finite sets Fin.

Definition 4.10. For each i, we denote by ρ^i the unique inert map $\langle n \rangle \to \langle 1 \rangle$ so that $\rho^i(i) = 1$. (This notation is slightly abusive, since it also depends on n, but we supress this).

Definition 4.11. We denote the unique active map $\langle n \rangle \to \langle 1 \rangle$ by m.

Commutative monoids and groups

Definition 4.12. Let $\mathcal{C}$ be an ∞-category with finite products. Let $A : N(\mathrm{Fin}_*) \to \mathcal{C}$ be a functor. If each of the maps

$$A(\langle n \rangle) \overset{\prod A(\rho^i)}{\to} A(\langle 1 \rangle)^n$$

is an equivalence, we say that A is a *monoid in $\mathcal{C}$*. We refer to $A(\langle 1 \rangle)$ as the *underlying object* of A. We often abuse notation by denoting this by A as well.

If A is a commutative monoid, we get a map $\mu : A \times A \simeq A(\langle 2 \rangle) \overset{A(m)}{\to} A$. Consider the map $s : A \times A \to A \times A$, given by projection on the first coordinate and μ on the second coordinate. We call this the *shear map*. If it is an equivalence, we say that A is a *commutative group* (in $\mathcal{C}$).

We denote by $\mathrm{CMon}(\mathcal{C}), \mathrm{CGrp}(\mathcal{C})$ the full subcategories of $\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C})$ spanned by respectively the commutative monoids and groups.

These notions go back to Segal [10], who developed them in the case of spaces and categories. (Although he did not work in ∞-categories, Segal's definitions of Γ*-space* is equivalent to our notion of commutative monoid in $\mathcal{S}$, in a suitable sense.)

Remark 4.13. We will often refer to the condition that a functor $N(\mathrm{Fin}_*) \to \mathcal{C}$ is a commutative monoid as the *Segal condition*.

Remark 4.14. Given a commutative monoid A, the map μ defined above makes $A(\langle 1 \rangle)$ into a commutative monoid object of $h\mathcal{C}$. It is a commutative group if and only if this monoid object is a group object.

Proposition 4.15. Suppose $\mathcal{C}$ is a presentable ∞-category. Then also $\mathrm{CMon}(\mathcal{C}), \mathrm{CGrp}(\mathcal{C})$ are presentable.

Proof. Since both $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C})$ are full subcategories of the presentable category $\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C})$, it will suffice to exhibit some set of morphisms so that they are precisely the local objects.

First look at the evaluation functors $ev_{\langle n \rangle} : \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C}) \to \mathcal{C}$. Since they preserve limits and colimits, they admit left adjoints, $F_{\langle n \rangle} : \mathcal{C} \to \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C})$.

By definition, a functor $M : \mathrm{N}(\mathrm{Fin}_*) \to \mathcal{C}$ is a commutative monoid if and only if each map $M(\langle n \rangle) \to \prod M(\langle 1 \rangle)$ is an equivalence. This is equivalent to requiring that, for each $C \in \mathcal{C}$, the map

$$\mathrm{Map}_{\mathcal{C}}(C, M(\langle n \rangle)) \to \mathrm{Map}_{\mathcal{C}}(C, \prod M(\langle 1 \rangle)) \tag{1}$$

is an equivalence of spaces. By accessibility of $\mathcal{C}$, using lemma 3.5, there is some regular cardinal κ so that it suffices to ask that 1 be an equivalence for objects C in $\mathcal{C}^{\kappa}$. Using the adjoint property of F and the the universal property of coproducts, we see that 1 is induced by a map

$$\phi_{n,C} : \coprod_{n} F_{\langle 1 \rangle} C \to F_{\langle n \rangle} C$$

Hence if we let S be the set of $\phi_{n,C}$, with n running through $\mathbb{N} \cup \{0\}$, and C running through representatives for the equivalence classes of $\mathcal{C}^{\kappa}$, the commutative monoids are precisely the S-local objects.

A commutative group is a commutative monoid with the additional requirement that the shear map

$$M(\langle 1 \rangle) \times M(\langle 1 \rangle) \to M(\langle 1 \rangle) \times M(\langle 1 \rangle)$$

is an equivalence. This is the same as requiring that the map

$$M(\langle 2 \rangle) \to M(\langle 1 \rangle) \times M(\langle 1 \rangle)$$

given by $M(\rho^1)$ on the first coordinate and $M(m)$ on the second coordinate. is an equivalence. We can translate this requirement into being local with respect to a set of morphisms in the same way as above. □

Corollary 4.16. Suppose $\mathcal{C}$ is presentable. Then the forgetful functors $\mathrm{CGrp}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ each admit a left adjoint.

Proof. The desired functor $\mathcal{C} \to \mathrm{CMon}(\mathcal{C})$ can be obtained as the composite

$$\mathcal{C} \to \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C}) \to \mathrm{CMon}(\mathcal{C})$$

of the left adjoint to evaluation at $\langle 1 \rangle$, and the localization. The functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C})$ can be constructed in a similar way. □

Proposition 4.17. Let $\mathcal{C}$ be a complete ∞-category. Limits in $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C})$ are calculated pointwise. In particular, the forgetful functors $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}, \mathrm{CGrp}(\mathcal{C}) \to \mathcal{C}$ preserve limits.

Proof. This is known for the functor category $\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C})$. It then suffices to observe that each subcategory $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C})$ is closed under taking limits, which is again simple because limits are calculated pointwise. □

Lemma 4.18. Let $\mathcal{C}, \mathcal{D}$ be complete ∞-categories. Then we have canonical equivalences

$$\mathrm{CMon}(\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^R(\mathcal{C}, \mathrm{CMon}(\mathcal{D})) \tag{2}$$

$$\mathrm{CGrp}(\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^R(\mathcal{C}, \mathrm{CGrp}(\mathcal{D})) \tag{3}$$

Moreover, under this equivalence, the forgetful functor $\mathrm{CMon}(\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$ is identified with the post-composition functor $\mathrm{Fun}^R(\mathcal{C}, \mathrm{CMon}(\mathcal{D})) \to \mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$, and analogously for groups.

Proof. We have a fully faithful inclusion

$$\begin{aligned}\mathrm{CMon}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D})) &\subseteq \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*),\mathrm{Fun}(\mathcal{C},\mathcal{D})) && (4)\\ &\simeq \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*)\times\mathcal{C},\mathcal{D}) && (5)\end{aligned}$$

with essential image precisely those functors F so that each functor $F(-,C)$ is a commutative monoid, and each $F(\langle n\rangle,-)$ preserves limits. We also have a fully faithful inclusion

$$\begin{aligned}\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathrm{Mon}(\mathcal{D})) &\subseteq \mathrm{Fun}(\mathcal{C},\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*)\mathcal{D})) && (6)\\ &\simeq \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*)\times\mathcal{C},\mathcal{D}) && (7)\end{aligned}$$

which has precisely the same essential image. This proves the case of commutative monoids. The case of groups is entirely analogous.

The claimed identification is immediate, since both functors are the restriction of the functor $\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*)\times\mathcal{C},\mathcal{D})\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$ given by holding the first parameter constant at $\langle 1\rangle$. □

5 Symmetric monoidal ∞-categories

The main aim of this project is to construct canonical symmetric monoidal structures on certain ∞-categories. In this section, we give a very abbreviated account of the basic theory of symmetric monoidal ∞-categories. For a thorough discussion of such things, see [7, Chapter 2].

Foundations

Definition 5.1. A *symmetric monoidal ∞-category* is a monoid in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. It is small if the underlying category is small, which is the same as requiring it to be a monoid in $\mathcal{C}\mathrm{at}_\infty$. The ∞-category of (small) symmetric monoidal ∞-categories are defined in the obvious way:

$$\widehat{\mathcal{C}\mathrm{at}}_\infty^{\otimes} = \mathrm{CMon}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$$

$$\mathcal{C}\mathrm{at}_\infty^{\otimes} = \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$$

A map in $\mathcal{C}\mathrm{at}_\infty^{\otimes}$ (or $\widehat{\mathcal{C}\mathrm{at}}_\infty^{\otimes}$) is called a *symmetric monoidal functor*.

If $\mathcal{C}:\mathrm{N}(\mathrm{Fin}_*)\to\widehat{\mathcal{C}\mathrm{at}}_\infty$ is a symmetric monoidal ∞-category, we refer to $\mathcal{C}(\langle 1\rangle)$ as the *underlying ∞-category*. As usual, we will frequently abuse notation and use the same symbol for the symmetric monoidal ∞-category and the underlying ∞-category, or say that $\mathcal{C}=\mathcal{C}(\langle 1\rangle)$ is a symmetric monoidal ∞-category.

This definition makes a lot of intuitive sense. If we replace ∞-categories with topological (or simplicial) categories, we also recover essentially Segal's original definition of Γ-category.

Remark 5.2. It can also be shown if $\mathcal{C}$ is a symmetric monoidal icat, then the functor

$$\otimes:\mathcal{C}\times\mathcal{C}\simeq\mathcal{C}(\langle 2\rangle)\overset{\mathcal{C}(m)}{\to}\mathcal{C}$$

and the object

$$I:1\simeq\mathcal{C}(\langle 0\rangle)\overset{m}{\to}\mathcal{C}$$

equip $h\mathcal{C}$ with the structure of a symmetric monoidal category. For instance, the commutator natural isomorphism $A\otimes B\to B\otimes A$ is constructed by applying $\mathcal{C}$ to the commutative diagram

$$\begin{tikzcd}\langle 2\rangle \arrow[r, "\tau"] \arrow[dr, "m"'] & \langle 2\rangle \arrow[d, "m"] \\ & \langle 1\rangle\end{tikzcd}$$

where τ swaps 1 and 2.

In particular, a symmetric monoidal ∞-category whose underlying category is the nerve of an ordinary category gives rise to a symmetric monoidal structure (in the usual sense) on that category. This goes in the other direction as well.

However, this definition can be hard to use in practice. The basic reason for this is that it's quite hard to write down a functor into $\mathcal{C}\mathrm{at}_\infty$. We can approach this problem via the straigtening equivalence (see theorem 1.2).

This leads to the following alternative approach to symmetric monoidal ∞-categories.

Definition 5.3. An *unstraight symmetric monoidal ∞-category* consist of a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$, which satisfies the following condition. Each map $f : \langle n \rangle \to \langle m \rangle$ in Fin_* induces a functor $\bar{f} : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle m \rangle}$ by straightening. In particular, we get a functor

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{\prod \bar{\rho}^i} \prod \mathcal{C}^{\otimes}_{\langle 1 \rangle}$$

This functor should be a categorical equivalence for all n.

Definition 5.4. We say that a map $f : X \to Y$ in a symmetric monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ is *inert* if $p(f)$ is inert and f is p-coCartesian.

Example 5.5. In the symmetric monoidal ∞-category $\mathrm{N}(\mathrm{Fin}_*)$, the two definitions of the word "inert" agree.

Definition 5.6. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ and $q : \mathcal{D}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ be unstraight symmetric monoidal ∞-categories. Then an *unstraight symmetric monoidal functor* $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ which carries p-coCartesian morphisms to q-coCartesian morphisms, and so that

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "F"] \arrow[dr, "p"'] & \mathcal{D}^{\otimes} \arrow[d, "q"] \\ & \mathrm{N}(\mathrm{Fin}_*) \end{tikzcd}$$

commutes. If we relax this condition, requiring instead only that F carries inert morphisms to inert morphisms, we say that F is a *lax* unstraight symmetric monoidal functor.

We denote by $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the symmetric monoidal functors.

Remark 5.7. The term "unstraight symmetric monoidal ∞-category " is our own - the standard in the literature is to use definition 5.3 as the definition of symmetric monoidal ∞-category. We chose to define symmetric monoidal ∞-categories in terms of monoids because it seemed more intuitive, and to emphasize the relation with Segal's Γ-categories.

Proposition 5.8. The full subcategory of $(\mathcal{C}\mathrm{at}_\infty)^{\mathrm{coCart}}_{/\mathrm{N}(\mathrm{Fin}_*)}$ spanned by the unstraight symmetric monoidal ∞-categories is categorically equivalent to $\mathcal{C}\mathrm{at}^{\otimes}_{\infty}$. Moreover, the mapping space between two symmetric monoidal ∞-categories is equivalent to $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})^{\simeq}$.

Proof. The first statement is essentially by definition: an object of $(\mathcal{C}\mathrm{at}_\infty)^{\mathrm{coCart}}_{/\mathrm{N}(\mathrm{Fin}_*)}$ is an unstraight symmetric monoidal ∞-category if and only if it goes to a symmetric monoidal ∞-category under the equivalence

$$(\mathcal{C}\mathrm{at}_\infty)^{\mathrm{coCart}}_{/\mathrm{N}(\mathrm{Fin}_*)} \simeq \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \mathcal{C}\mathrm{at}_\infty)$$

For the other statement, we identify the mapping spaces $\mathrm{Map}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ in $(\mathcal{C}\mathrm{at}_\infty)^{\mathrm{coCart}}_{/\mathrm{N}(\mathrm{Fin}_*)}$ with the subspace of $\mathrm{Fun}_{(}\, \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{\simeq}$ spanned by those functors so that this diagram commutes

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r] \arrow[dr] & \mathcal{D}^{\otimes} \arrow[d] \\ & \mathrm{N}(\mathrm{Fin}_*) \end{tikzcd}$$

and which furthermore preserve coCartesian edges. This is precisely $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$. □

Remark 5.9. This propositions shows that there is really no difference between symmetric monoidal categories and unstraight symmetric monoidal categories. Hence we omit the term "unstraight" from now on. Since the unstraight version is genrally much easier to work with, all our symmetric-monoidal categories will be represented as such from now on, unless we note otherwise.

Remark 5.10. Notably, it is not so easy to define lax symmetric monoidal functors using the description of symmetric monoidal categories as monoids. This is another advantage of the description as coCartesian fibrations.

Remark 5.11. If $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category, we denote the fiber over $\langle n \rangle$ by $\mathcal{C}^{\otimes}_{\langle n \rangle}$.

Definition 5.12. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, we have a canonical equivalence of ∞-categories

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \left(\mathcal{C}^{\otimes}_{\langle 1 \rangle}\right)^n$$

We denote the object in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ identified with the n-tuple $(C_1, C_2, \dots, C_n)$ by $C_1 \oplus C_2 \cdots \oplus C_n$, or by $\bigoplus_{i=1}^n C_i$

Remark 5.13. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the active maps $m : \langle n \rangle \to \langle 1 \rangle$, along with the indentification $\mathcal{C}_{\langle n \rangle} \simeq (\mathcal{C}_{\langle 1 \rangle})^n$, give rise to functors

$$\bigotimes : \mathcal{C}^n \simeq \mathcal{C}_{\langle n \rangle} \to \mathcal{C}$$

$$I : \Delta^0 \simeq \mathcal{C}_{\langle 0 \rangle} \to \mathcal{C}$$

Proposition 5.14. Suppose $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category, let $X = \bigoplus_{i \in \langle n \rangle^\circ} X_i, Y = \bigoplus_{j \in \langle m \rangle^\circ}$ be objects, and $f : \langle n \rangle \to \langle m \rangle$ a map. Then the fiber over f of the map

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(X, Y) \to \mathrm{Map}_{\mathrm{N}(\mathrm{Fin}_*)}(\langle n \rangle, \langle m \rangle)$$

is the space

$$\prod_{j \in \langle m \rangle^\circ} \mathrm{Map}_{\mathcal{C}}\left(\bigotimes_{f(i)=j} X_i, Y_j\right)$$

Proof. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category. Let $f : \langle n \rangle \to \langle m \rangle$ be a map, and $\bigoplus X_i$ be an object over $\langle n \rangle$ in $\mathcal{C}^{\otimes}$. f lifts to a functor $\otimes_f : C^{\otimes}_{\langle n \rangle} \to C^{\otimes}_{\langle m \rangle}$.

Essentially by definition of this functor, there is a coCartesian lift of f, $\bar{f} : \bigoplus X_i \to \prod_f (\bigoplus X_i)$. Moreover, by postcomposing f with ρ^i and using functoriality of straightening, we can find a categorical equivalence

$$\bigotimes_f \left(\bigoplus X_i\right) \simeq \bigoplus_{j \in \langle m \rangle} \bigotimes_{f(i)=j} X_i$$

lying over the identity $\langle m \rangle \to \langle m \rangle$. Certainly the composition of these morphisms is a p-coCartesian map

$$\bigoplus X_i \to \bigoplus_{j \in \langle m \rangle} \bigotimes_{f(i)=j} X_i$$

Now by [8, Prop 2.4.4.2], we see that the fiber we are interested in can be written as

$$\mathrm{Map}_{\mathcal{C}^{\otimes}_{\langle m\rangle}}\left(\bigoplus_{j\in\langle m\rangle}\bigotimes_{f(i)=j} X_i, \bigoplus Y_j\right)$$

and now the result follows because of the categorical equivalence $\mathcal{C}^{\otimes}_{\langle m\rangle} \to \mathcal{C}^m$. □

Lemma 5.15. Let $F : \bigoplus X_i \to \bigoplus Y_j$ be a map in $\mathcal{C}^{\otimes}$ lying over f. We can identify this with a family of maps $\bigotimes_{f(i)=j} X_i \to Y_j$ for each j. F is a coCartesian lift of f if and only if each of these maps is an equivalence.

Proof. This is immediate using the above and [8, p. 2.4.4.3] □

Commutative algebra objects

In [7], Lurie treats the notion of symmetric monoidal ∞-category as a special case of a more general theory of ∞-operads. Apart from being a generalization of symmetric monoidal ∞-categories, one could also think of an ∞-operad as a "type of algebra". For instance, there is an ∞-operad, $\mathbb{E}_1$, which classifies (in a suitable sense) associative algebras. We will not need to work with any sort of algebra except commutative algebras, and so we will not need this theory. We mention it only because many of our results will hold for more general algebra objects, and this is an important part of the theory. For instance, it is important to know that the suspension spectrum of an $\mathbb{E}_1$-space is an $\mathbb{E}_1$-ring spectrum, which will indeed follow from the fact that Σ^{∞}_{+} is symmetric monoidal.

Definition 5.16. Let $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category. A *commutative algebra object of* $\mathcal{C}$, or a *commutative algebra in* $\mathcal{C}$, is a section $A : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ of the structure map $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$, with the additional property that it carries inert morphisms to inert morphisms. We let $\mathrm{CAlg}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes})$ spanned by the commutative algebras.

Remark 5.17. A commutative algebra is the same thing as a lax symmetric monoidal functor $N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$, i.e. $* \to \mathcal{C}$.

Remark 5.18. Let $A : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ be a commutative algebra. Then one can define a commutative algebra object in $h\mathcal{C}$ (with the symmetric monoidal structure of remark 5.2), with underlying object $A(\langle 1\rangle) = A$. To get the multiplication, first note that, since $A(\rho^i)$ is an inert map, this gives an isomorphism $A(\langle 2\rangle) \simeq A \oplus A$. Then $A(\langle m\rangle) : A(\langle 2\rangle) \to A(\langle 1\rangle)$ corresponds to a map $A \otimes A \to A$, which is the multiplication. The other structure maps (and the commutativity relation) can be constructed similarly.

Proposition 5.19. Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax symmetric monoidal functor. Then F induces a functor

$$\mathrm{Fun}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{D}^{\otimes})$$

Which restricts to a functor

$$\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$$

Proof. It suffices to check that, if $A : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ is a commutative algebra, then so is $AF : N(\mathrm{Fin}_*) \to \mathcal{D}^{\otimes}$. For this, it is enough to observe that F carries inert morphisms to inert morphisms. □

Corollary 5.20. The assignment $\mathcal{C} \mapsto \mathrm{CAlg}(\mathcal{C})$ defines a functor $\mathcal{C}\mathrm{at}^{\otimes}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$ (or $\widehat{\mathcal{C}\mathrm{at}}^{\otimes}_{\infty} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$)

Lemma 5.21. The forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is conservative.

Proof. Let $A, B : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ be commutative algebras, and let $\eta : A \to B$ be a natural transformation. Suppose further that $\eta_{\langle 1 \rangle} : A \to B$ is an equivalence. We must show that η is a natural equivalence. It suffices to show that each map $\eta_{\langle n \rangle} : A(\langle n \rangle) \to B(\langle n \rangle)$ is an equivalence. Naturality of η means that for each $i \in \langle n \rangle^{\circ}$, we have a commutative diagram

$$\begin{tikzcd} A(\langle n \rangle) \arrow[r, "\eta_{\langle n \rangle}"] \arrow[d, "A(\rho^i)"] & B(\langle n \rangle) \arrow[d, "B(\rho^i)"] \\ A(\langle 1 \rangle) \arrow[r, "\eta_{\langle 1 \rangle}"] & B(\langle 1 \rangle) \end{tikzcd}$$

in $\mathcal{C}^{\otimes}$. Moreover, the vertical maps are inert. This implies that, under the identification $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n$, if we write $A(\langle n \rangle) = \bigoplus A_j$, $B(\langle n \rangle) = \bigoplus B_j$, we have a commutative diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & B_i \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

where the vertical and bottom maps are equivalences. This implies that $\eta_{\langle n \rangle}$ is (identified with) a product of equivalences in $\mathcal{C}^n$, which finishes the proof. □

Cartesian monoidal structure

If $\mathcal{C}$ is an (ordinary) category with finite products, we can choose a product for every pair of objects, and a terminal object, to give $\mathcal{C}$ the structure of a symmetric monoidal category. We would like to mimic this procedure for symmetric monoidal ∞-categories. This construction embodies the difference between the ordinary notion of symmetric monoidal category and symmetric monoidal ∞-categories: We must specify a much more complicated system of data, but we do not have to make the ad hoc choice of products as above.

The construction of the Cartesian monoidal structure is due to Lurie, in [7]. However, the ideas in the construction go back to [10]. Lurie develops a very general theory of Cartesian structures, but we will only need a small amount of the theory.

Our main interest in the Cartesian monoidal structure will be the fact that symmetric monoidal categories can themselves be described as the commutative algebra objects in a certain symmetric monoidal category.

Definition 5.22. We define an (ordinary) category, $\Gamma^{\times}$, in the following way:

- The objects of $\Gamma^{\times}$ are pairs $(\langle n \rangle, S)$, where $\langle n \rangle \in \mathrm{Fin}_*$, and $S \subset \langle n \rangle^{\circ}$.
- A map $(\langle n \rangle, S) \to (\langle n' \rangle, S')$ is a map $\alpha : \langle n \rangle \to \langle n' \rangle$ with the property that $\alpha^{-1}(S') \subset S$.

Remark 5.23. There is an obvious forgetful functor $\Gamma^{\times} \to \mathrm{Fin}_*$. It has a canonical section $s(\langle n \rangle) = (\langle n \rangle, \langle n \rangle^{\circ})$.

Definition 5.24. Let $\mathcal{C}$ be an ∞-category. Then we define a simplicial set over $N(\mathrm{Fin}_*)$, $\tilde{\mathcal{C}}^{\times}$ by the following universal property: For each map $K \to N(\mathrm{Fin}_*)$, there is a bijection

$$\mathrm{Hom}_{N(\mathrm{Fin}_*)}(K, \tilde{\mathcal{C}}^{\times}) \to \mathrm{Hom}_{sSet}(K \times_{N(\mathrm{Fin}_*)} N(\Gamma^{\times}), \mathcal{C})$$

Take $\langle n \rangle \in N(\mathrm{Fin}_*)$, and consider the fiber $\tilde{\mathcal{C}}^{\times}_{\langle n \rangle}$ An n-simplex of this set is, by the above, an element of

$$\mathrm{Hom}_{sSet}(\Delta^n \times_{N(\mathrm{Fin}_*)} N(\Gamma^{\times}), \mathcal{C}),$$

where the map $\Delta^n \to N(\mathrm{Fin}_*)$ is constant at $\langle n \rangle$. Then this is the same as

$$\mathrm{Hom}(\Delta^n \times N(P^{op}), \mathcal{C}) = \mathrm{Map}(N(P^{op}), \mathcal{C})_n$$

Where P is the poset of subsets of $\langle n\rangle^\circ$.

In other words, we can identify the fiber $\tilde{\mathcal{C}}^\times_{\langle n\rangle}$ with the simplicial set of maps from $N(P^{op})$ to $\mathcal{C}$. Under this identification, define $\mathcal{C}^\times$ as the simplicial subset spanned by those maps f where, for each $S \subset \langle n\rangle^\circ$, the maps $f(S) \to f(\{j\})$ exhibit $f(S)$ as the product $\prod_{j\in S} f(\{j\})$.

Lastly, composition with $s : N(\mathrm{Fin}_*) \to N(\Gamma^\times)$ induces a map

$$\mathrm{Hom}(K \times_{N(\mathrm{Fin}_*)} N(\Gamma^\times), \mathcal{C}) \to \mathrm{Hom}(K \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*), \mathcal{C}) = \mathrm{Hom}(K, \mathcal{C}),$$

which gives a canonical functor $\pi : \tilde{C}^\times \to \mathcal{C}$.

The two following propositions are not especially important for this project, and we omit the proofs. We list them here to give the reader an idea of the properties that the Cartesian symmetric monoidal structure is supposed to have.

Proposition 5.25. Suppose $\mathcal{C}$ has finite products.

(1) $\mathcal{C}^\times \to N(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C}$.

(2) The composite $\mathcal{C}^n \simeq \mathcal{C}^\times_{\langle n\rangle} \to \mathcal{C}$ maps

$$C_1 \oplus C_2 \cdots \oplus C_n \mapsto C_1 \times C_2 \cdots \times C_n$$

These statements are cases of [7, Prop. 2.4.2.5].

Proposition 5.26. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories with finite products. Then the forgetful functor $\mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, with image consisting precisely of the product-preserving functors.

This is a special case of [7, Cor. 2.4.1.8].

We think of the objects of $\mathrm{CMon}(\mathcal{C})$ as "commutative monoid objects in $\mathcal{C}$, up to coherent homotopy". There is another natural interpretation of this idea, namely a commutative algebra in $\mathcal{C}$, with the Cartesian monoidal structure. The following proposition shows that these interpretations agree.

Proposition 5.27. Let $\mathcal{C}$ be an ∞-category with finite products. Then post-composition with $\pi : \mathcal{C}^\times \to \mathcal{C}$ gives a functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C})$. This functor is fully faithful, with essential image $\mathrm{CMon}(\mathcal{C})$. In particular, $\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C})$.

Proof. By definition, $\mathrm{CAlg}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{C}^\times)$ spanned by the functors that take inert maps to inert maps. By definition of $\mathcal{C}^\times$, this is the same as the full subcategory of

$$\mathrm{Fun}(N(\mathrm{Fin}_*) \times_{N(\mathrm{Fin}_*)} N(\Gamma^\times), \mathcal{C}) = \mathrm{Fun}(N(\Gamma^\times), \mathcal{C})$$

spanned by functors satisfying certain conditions.

(1) In order to land in $\mathcal{C}^\times \subseteq \tilde{\mathcal{C}}^\times$, for each subset $S \subset \langle n\rangle^\circ$, we must have

$$F(\langle n\rangle, S) \to \prod_{j\in S} F(\langle n\rangle, \{j\})$$

an equivalence.

(2) To correspond to an algebra, for each inert map $f : \langle n\rangle \to \langle m\rangle$, and every subset $S \subseteq \langle m\rangle^\circ$, the induced map $F(\langle n\rangle, f^{-1}S) \to F(\langle m\rangle, S)$ must be an equivalence in $\mathcal{C}$.

The described functor into $\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C})$, under the above indentification, corresponds to precomposing with the canonical section $s : N(\mathrm{Fin}_*) \to N(\Gamma^\times)$. In other words, it maps $F : N(\Gamma^\times) \to \mathcal{C}$ into the functor $\langle n \rangle \mapsto F(\langle n \rangle, \langle n \rangle^\circ)$.

The condition that this functor defines a monoid is then that the map

$$F(\langle n \rangle, \langle n \rangle^\circ) \to \prod_{j \in \langle n \rangle} F(\langle 1 \rangle, \langle 1 \rangle^\circ)$$

is an equivalence. The map $F(\langle n \rangle, \langle n \rangle^\circ) \overset{F(\rho^j)}{\to} F(\langle 1 \rangle, \langle 1 \rangle^\circ)$ can be factorized as

$$F(\langle n \rangle, \langle n \rangle^\circ) \to F(\langle n \rangle, \{j\}) \to F(\langle 1 \rangle, \langle 1 \rangle^\circ)$$

The second condition on F implies that the latter map is an equivalence. The first condition implies that the product of the first map for each j is an equivalence - this implies the desired result.

We have described a functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{C})$. Observe that $s : \mathrm{Fin}_* \to \Gamma^\times$ is actually a full subcategory inclusion. In fact, it admits a left adjoint L, sending $(\langle n \rangle, S)$ to $S \cup \{*\}$. (It is not hard to check that this is really a left adjoint). By proposition 2.12, we see that

$$\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}) \to \mathrm{Fun}(\Gamma^\times, \mathcal{C})$$

is fully faithful, with essential image those functors that carry local equivalences to equivalences. Using 2 out of 3 for equivalences, we see that this is equivalent to (2) above. Hence $\mathrm{CAlg}(\mathcal{C})$ is contained in the essential image. Observing that the Segal condition on the left-hand side implies (1), we conclude that this restricts to a fully faithful functor

$$\mathrm{CMon}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}).$$

To see essential surjectivity, it suffices to remark that by the above, every algebra is isomorphic to some functor of the form $\Gamma^\times \xrightarrow{L} N(\mathrm{Fin}_*) \xrightarrow{M} \mathcal{C}$, and it is clear to see that in this setting, (1) also implies the Segal condition for M.

It only remains to observe that precomposition with s is a one-sided inverse to this equivalence, hence a proper inverse. □

The following corollary will be of key importance:

Corollary 5.28. Consider $\mathcal{C}\mathrm{at}_\infty$. Since it admits finite products, we may equip it with the cartesian symmetric monoidal structure. Then $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathcal{C}\mathrm{at}_\infty^\otimes$. Similarly, $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \simeq \widehat{\mathcal{C}\mathrm{at}}_\infty^\otimes$. Moreover, these equivalences preserve the forgetful functors to $\mathcal{C}\mathrm{at}_\infty$.

Proof. $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) = \mathcal{C}\mathrm{at}_\infty^\otimes$ □

Proposition 5.29. The equivalence $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ given by $\mathcal{C} \mapsto \mathcal{C}^{op}$ extends to an equivalence $\mathcal{C}\mathrm{at}_\infty^\otimes \to \mathcal{C}\mathrm{at}_\infty^\otimes$.

Proof. We wish to define a functor $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$ given by postcomposition with $(-)^{op}$. It suffices to observe that $(-)^{op}$ preserves products. □

Definition 5.30. A symmetric monoidal category $\mathcal{C}$ is called *coCartesian* if the unit $I_\mathcal{C}$ is initial, and the canonical map $A \sqcup B \simeq (A \otimes I) \sqcup (I \otimes B) \to A \otimes B$ is an equivalence. Here the latter map is defined on the first summand as 1_A tensored with the unique map from I to B, and analogously on the second summand.

By combining this with the construction of Cartesian symmetric monoidal structures above, we obtain the following corollary

Corollary 5.31. Suppose $\mathcal{C}$ is a category with finite coproducts. Then $\mathcal{C}$ admits a coCartesian symmetric monoidal structure.

Corollary 5.32. Suppose $\mathcal{C}, \mathcal{D}$ are ∞-categories which admit finite coproducts, and $F : \mathcal{C} \to \mathcal{D}$ preserves them. Then if $\mathcal{C}, \mathcal{D}$ are equipped with the coCartesian symmetric monoidal structure, there is a (essentially) unique symmetric monoidal structure on F.

We will need the following proposition:

Proposition 5.33. Suppose $\mathcal{C}^{\otimes}$ is a coCartesian symmetric monoidal category. Then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

This is a very special case of [7, Prop. 2.4.3.8].

Tensor product of presentable ∞-categories

We will describe a symmetric monoidal structure on Pr^{L}. In some sense this is the most important symmetric monoidal category in this project - we will construct the symmetric monoidal structures of the main theorem by analyzing it. It was developed by Lurie in [7, Section 4.8.1]. Since this is not the primary concern of this project, we will just state the definition and the basic results we need, omitting the proofs.

Definition 5.34. Consider the Cartesian symmetric monoidal structure on $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. We define $\mathrm{Pr}^{\mathrm{L},\otimes} \subseteq \widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\times}$ in the following way:

- An object $\bigoplus_i \mathcal{C}_i \in (\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\times})_{\langle n\rangle}$ is in $\mathrm{Pr}^{\mathrm{L},\otimes}$ if each $\mathcal{C}_i$ is a presentable ∞-category.
- Let $f : \bigoplus_{i\in\langle n\rangle^{\circ}} \mathcal{C}_i \to \bigoplus_{j\in\langle m\rangle^{\circ}} \mathcal{D}_j$ be a map lying over $\bar{f} : \langle n\rangle \to \langle m\rangle$. Then this is equivalently a collection of functors $\prod_{\bar{f}(i)=j} \mathcal{C}_i \to \mathcal{D}_j$, for each j. The map f is in $\mathrm{Pr}^{\mathrm{L},\otimes}$ if each of these functors preserves colimits in each variable. By this we mean that, for each of these functors $F : \prod_i \mathcal{C}_i \to \mathcal{D}$, for each k, for each collection of objects $\{C_i \in \mathcal{C}_i\}_{k\neq i}$, the functor $\mathcal{C}_k \to \mathcal{D}$ given by
$$X \mapsto F(C_1, \dots C_{k-1}, X, C_{k+1}, \dots)$$
preserves colimits.

We note that by restriction of the canonical map $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\times} \to \mathrm{N}(\mathrm{Fin}_*)$ to $\mathrm{Pr}^{\mathrm{L},\otimes}$, we obtain a canonical map $\mathrm{Pr}^{\mathrm{L},\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$

Lurie spends a lot of work developing a fairly general theory, of which we again only need a few pieces

Proposition 5.35 (Lurie)**.**

(1) The obvious map to $\mathrm{N}(\mathrm{Fin}_*)$ makes $\mathrm{Pr}^{\mathrm{L},\otimes}$ into a symmetric monoidal ∞-category with underlying ∞-category Pr^{L}.

(2) The tensor product $\mathcal{C} \otimes \mathcal{D}$ of two presentable ∞-categories receives a functor from $\mathcal{C} \times \mathcal{D}$, which preserves colimits in each variable. Furthermore
$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$
is fully faithful for every presentable ∞-category $\mathcal{E}$, with essential image consisting of those functors that preserve colimits in each variable.

(3) There is a natural equivalence $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{Ra}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$

(4) Composing with the inclusion gives a functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_{\infty})$. This is fully faithful, with image precisely those symmetric monoidal ∞-categories which are presentable, and where the tensor product preserves in each variable.

This type of symmetric monoidal ∞-category is important enough (and awkward enough to write) that we will introduce a term for it.

Definition 5.36. A *presentably symmetric monoidal ∞-category* is a presentable symmetric monoidal ∞-category with the property that the tensor product preserves colimits in each variable. If $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C}$, we will also refer to $\mathcal{C}^{\otimes}$ as a *presentable symmetric monoidal structure* on $\mathcal{C}$.

6 The universal properties of $\mathcal{C}_*, \mathrm{CMon}(\mathcal{C})$, and $\mathrm{CGrp}(\mathcal{C})$

In this section, we will show that each of the assignments described in the title is a localization of Pr^{L}.

Pointed ∞-categories

Definition 6.1. An ∞-category $\mathcal{C}$ is *pointed* if it has a terminal object, an initial object, and they are isomorphic. Such an object is called a zero object, and denoted 0.

Remark 6.2. If $\mathcal{C}$ has an initial object $\emptyset$ and a terminal object $*$, there is always a unique map $\emptyset \to *$. The condition is then that this map is an equivalence.

Remark 6.3. If X, Y are objects of a pointed ∞-category, there is an essentially unique map $0 : X \to Y$ which factors over 0.

Proposition 6.4. Suppose $\mathcal{C}$ is pointed. Then $\mathcal{C}_* \to \mathcal{C}$ is an equivalence.

Proof. Use the identification $\mathcal{C}_* \simeq \mathcal{C}_{*/}$ and the fact that $*$ is also initial, since $\mathcal{C}$ is pointed. ☐

Proposition 6.5. If $\mathcal{C}$ is a complete ∞-category, then $\mathcal{C}_*$ is pointed.

Proof. Since limits in functor categories are pointwise, $* \to *$ is terminal in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Hence it is also terminal in the full subcategory $\mathcal{C}_*$. We must show that it is also initial. By definition, an initial object of $\mathcal{C}_{*/} \simeq \mathcal{C}_*$ is the same thing as a colimit diagram for the constant diagram $* : \Delta^0 \to *$. But clearly the map $* \to *$ is such a diagram (see e.g. [8, Lemma 4.3.2.3]). ☐

Proposition 6.6. Suppose $\mathcal{C}$ and $\mathcal{D}$ are complete, and either $\mathcal{C}$ or $\mathcal{D}$ is pointed. Then $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ is pointed.

Proof. The second case is clear since

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D}_*) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})_*$$

For the first case, let $\mathrm{const}_* : \mathcal{C} \to \mathcal{D}$ be the constant functor at $* \in \mathcal{D}$. It is terminal in $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$; we must show that it is initial. Let $\{0\} \subseteq \mathcal{C}$ be the full subcategory spanned by the zero object. Consider the functor $\{0\} \to \mathcal{D}$ sending 0 to $*$. It is easy to verify that const_* is the left Kan extension of this functor. This implies that for any functor $F : \mathcal{C} \to \mathcal{D}$,

$$\mathrm{Map}(\mathrm{const}_*, F) \simeq \mathrm{Map}_{\mathcal{D}}(*, F(0))$$

If F is limit-preserving, it maps 0 to $*$, so this space is contractible. This finishes the proof. ☐

Corollary 6.7. Suppose $\mathcal{C}$ is presentable. Then the functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint. This is a localization of $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}$ relative to the pointed ∞-categories.

Proof. Via the anti-equivalence, it suffices to show that the functor

$$\mathrm{Fun}^{\mathrm{R}a}(\mathcal{D},\mathcal{C}_*) \to \mathrm{Fun}^{\mathrm{R}a}(\mathcal{D},\mathcal{C})$$

is an equivalence whenever $\mathcal{D}$ is pointed.

First we consider

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C}_*) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})$$

This functor can be identified with the composition.

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C}_*) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})_* \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})$$

which is an equivalence since $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})$ is pointed. Now we show that this restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{R}a}(\mathcal{D},\mathcal{C}_*) \to \mathrm{Fun}^{\mathrm{R}a}(\mathcal{D},\mathcal{C})$$

First, it's clear that composition with $U : \mathcal{C}_* \to \mathcal{C}$ sends accessible functors to accessible functors, since it is itself accessible (being a right adjoint).

Second, suppose $F : \mathcal{D} \to \mathcal{C}$ is an accessible functor. Then we have $F \simeq U \circ G$ for some $G : \mathcal{D} \to \mathcal{C}_*$, and we must show that G is also accessible. Let κ be such that F is κ-accesible, And let $d : K \to \mathcal{D}$ be a κ-filtered diagram. We must show that the map $\operatorname{colim} G \circ d \to G(\operatorname{colim} d)$ is an equivalence. Since U is conservative, it will suffice to show that the map $U(\operatorname{colim} G \circ d) \to UG(\operatorname{colim} d)$ is an equivalence. Since U is accessible, we can identify this with the map $\operatorname{colim}(U \circ G) \circ d \to (U \circ D)(\operatorname{colim} d)$. Since $U \circ G \simeq F$, which is κ-accessible, this is an equivalence. □

Remark 6.8. The style of the proof above, where we mostly work in Pr^{R} rather than Pr^{L}, is typical of this section. It is due to the fact that the localizations we work with here are perhap more naturally thought of as "colocalizations" of the subcategory of $\mathcal{C}\mathrm{at}_\infty$ given by complete categories and limit-preserving functors. We are essentially making them into localizations by restricting to Pr^{R} and using the anti-equivalence between this and Pr^{L}.

Corollary 6.9. There is a functor $(-)_* : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ with essential image the subcategory of pointed ∞-categories. It is left adjoint to the inclusion. Furthermore, its value on the object $\mathcal{C}$ is isomorphic to the category $\mathcal{C}_*$ defined above (justifying our notation), and the unit $\mathcal{C} \to \mathcal{C}_*$ is identified under this equivalence with $(-)_+$.

Preadditive ∞-categories

Definition 6.10. Let $\mathcal{C}$ be a pointed ∞-category. Suppose $\mathcal{C}$ admits finite products and coproducts. Then we can define a map $A \sqcup B \to A \times B$ given by $(1_A, 0) \sqcup (0, 1_B)$ If this map is an equivalence for all A, B, we say that $\mathcal{C}$ is *preadditive*. In this case we denote such an object by $A \oplus B$.

Warning 6.11. There is some disagreement in the literature on this term - for instance, Lurie calls categories of this type *semiadditive*. We use the terminology of [4].

Proposition 6.12. Suppose $\mathcal{C}$ is a category with finite products. Then the following are equivalent:

(1) $\mathcal{C}$ is preadditive.

(2) The Cartesian symmetric monoidal structure on $\mathcal{C}$ is coCartesian.

Proof. Note that $\mathcal{C}$ being preadditive is equivalent to the statement

(1) The terminal object $*$ is initial.

(2) The canonical map $X \sqcup Y \simeq (X \times *) \sqcup (* \times Y) \to X \times Y$ is an equivalence.

Using the definition of the Cartesian monoidal structure, and the definition of coCartesian, the equivalence is now immediate. ☐

Proposition 6.13. Suppose $\mathcal{C}$ is preadditive. Then the forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

Proof. By proposition 6.12 and proposition 5.27. ☐

Lemma 6.14. Let $\mathcal{C}$ be an ∞-category with finite products. Suppose the terminal object $*$ is initial, and that for each object $X \in \mathcal{C}$ there exists a map $\delta^X : X \times X \to X$ with the following properties:

(1) Let $* \to X$ be a map (unique up to homotopy). Then the composition $X \simeq X \times * \to X \times X \xrightarrow{\delta^X} X$ is homotopic to the identity.

(2) For each map $X \to Y$, there is a commutative diagram

$$\begin{tikzcd} X \times X \arrow[r, "f \times f"] \arrow[d, "\delta^X"] & Y \times Y \arrow[d, "\delta^Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

(3) For $X, Y \in \mathcal{C}$, there is a commutative diagram

$$\begin{tikzcd} (X \times X) \times (Y \times Y) \arrow[rr, "\sim"] \arrow[dr, "\delta^X \times \delta^Y"] & & (X \times Y) \times (X \times Y) \arrow[dl, "\delta^{X \times Y}"] \\ & X \times Y & \end{tikzcd}$$

Then $\mathcal{C}$ is preadditive.

Remark 6.15. This is the (3) $\Rightarrow$ (1) case of [7, Prop. 2.4.3.19], in the case of a Cartesian symmetric monoidal structure.

Proof. We must show that the two maps

$$i_X : X \simeq X \times * \to X \times Y \leftarrow * \times Y \simeq Y : i_Y$$

exhibit $X \times Y$ as the coproduct of X and Y. This is equivalent to the statement that the map

$$\varphi : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, Z) \times \mathrm{Map}(Y, Z)$$

is a homotopy equivalence for each Z. We claim that the following map

$$\psi : \mathrm{Map}(X, Z) \times \mathrm{Map}(Y, Z) \xrightarrow{\times} \mathrm{Map}(X \times Y, Z \times Z) \xrightarrow{\delta^Z \circ -} \mathrm{Map}(X \times Y, Z)$$

is an inverse.

First consider the composite $\varphi \circ \psi$. To see that this is homotopic to the identity, it suffices to verify that this diagram commutes:

$$\begin{tikzcd} \mathrm{Map}(X, Z) \times \mathrm{Map}(Y, Z) \arrow[r, "\times"] \arrow[d, "p_1"] & \mathrm{Map}(X \times Y, Z \times Z) \arrow[r, "- \circ i_X"] & \mathrm{Map}(X, Z \times Z) \arrow[d, "\delta^Z \circ -"] \\ \mathrm{Map}(X, Z) \arrow[rr, "="] & & \mathrm{Map}(X, Z) \end{tikzcd}$$

To see this, consider this diagram:

$$\begin{tikzcd}
\mathrm{Map}(X,Z)\times\mathrm{Map}(Y,Z) \arrow[r, "\times"] \arrow[d] \arrow[dd, bend right=60, "p_1"'] & \mathrm{Map}(X\times Y, Z\times Z) \arrow[d] & \\
\mathrm{Map}(X,Z)\times\mathrm{Map}(1,Z) \arrow[r, "\times"] \arrow[d, "p_1"] & \mathrm{Map}(X\times 1, Z\times Z) & \\
\mathrm{Map}(X,Z) \arrow[r, "i_Z\circ -"] & \mathrm{Map}(X, Z\times Z) \arrow[u, "\simeq"] \arrow[r, "\delta^Z\circ -"] & \mathrm{Map}(X,Z)
\end{tikzcd}$$

Each cell is seen to commute because of standard properties of the product. The composition along the top is the map which we wish to compare to p_1. The desired homotopy exists because the composition along the bottom is homotopic to the identity, by assumption.

Now consider $\psi\circ\varphi$. By condition (2), this can be identified with the map induced by

$$X\times Y \to (X\times 1)\times(Y\times 1)\to (X\times Y)\times(X\times Y)\overset{\delta^{X\times Y}}{\to} X\times Y$$

Which is homotopic to the identity by (1) and (3). $\square$

Proposition 6.16. For any ∞-category $\mathcal{C}$ with finite products, CMon($\mathcal{C}$) is preadditive.

Remark 6.17. As a consequence of this, the converse of proposition 6.13 is also true.

Proof. First we will show that CMon($\mathcal{C}$) is pointed. This means that the commutative monoid given by the functor $N(\mathrm{Fin}_*)\to\mathcal{C}$ constant at $*$ is initial. We denote this commutative monoid by const_*. We observe that const_* is left Kan extended from $\{\langle 0\rangle\}\subseteq\mathcal{C}$. This is fairly obvious, since each category of maps $\{\langle 0\rangle\}_{/\langle n\rangle}$ is just a point. Hence

$$\mathrm{Map}_{\mathrm{CMon}(\mathcal{C})}(\mathrm{const}_*, A)\simeq\mathrm{Map}_{\mathcal{C}}(*, A(\langle 0\rangle))\simeq\mathrm{Map}_{\mathcal{C}}(*,*)\simeq *$$

This shows that CMon($\mathcal{C}$) is pointed.

Let $\vee:\mathrm{Fin}_*\times\mathrm{Fin}_*\to\mathrm{Fin}_*$ be the coproduct functor. By composing with the diagonal, we get a functor

$$\bullet\vee\bullet:\mathrm{Fin}_*\to\mathrm{Fin}_*,$$

Which induces a functor $C_\bullet\mapsto C_{\bullet\vee\bullet}$ on CMon($\mathcal{C}$). The natural transformations $p_1,p_2:\langle n\rangle\vee\langle n\rangle\to\langle n\rangle$ given by mapping the second or first summand to $*$ induce natural transformations

$$\pi_1,\pi_2: C_{\bullet\vee\bullet}\to C_\bullet$$

It is a consequence of the Segal condition that these maps exhibit $C_{\bullet\vee\bullet}$ as the product of $C_\bullet$ with itself.

The natural transformation $\langle n\rangle\vee\langle n\rangle\to\langle n\rangle$ given by the identity on both summands induces a natural transformation

$$\delta: C_{\bullet\vee\bullet}\to C_\bullet$$

which is called the fold map.

We now claim that, putting $\delta^X=\delta_X$ for each $X\in\mathcal{C}$, we get a collection of maps satisfying the hypotheses of lemma 6.14

To see (1), observe that the map $C_\bullet\to C_{\bullet\vee\bullet}$ is the one induced by the inclusion of the first wedge summand. The relation then follows from the corresponding relation in Fin_* The diagram in (2) is just a naturality diagram, so clearly commutes. To see (3), observe this diagram:

$$\begin{tikzcd}
C_{\bullet\vee\bullet}\times D_{\bullet\vee\bullet} \arrow[r] \arrow[d, "\pi_1"] & (C\times D)_{\bullet\vee\bullet} \arrow[r, "\Delta"] \arrow[d, "(\pi_1)_{\bullet\vee\bullet}"] & C_\bullet\times D_\bullet \arrow[d, "\pi_1"] \\
C_{\bullet\vee\bullet} \arrow[r, "="] & C_{\bullet\vee\bullet} \arrow[r, "\Delta"] & C_\bullet
\end{tikzcd}$$

The left square commutes because intertwining products commutes with projections, while the right square commutes by (2). This implies (3). □

Proposition 6.18. Let $\mathcal{C}, \mathcal{D}$ be complete ∞-categories. Suppose either $\mathcal{C}$ or $\mathcal{D}$ is preadditive. Then also $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ is preadditive.

Proof. First, if $\mathcal{D}$ is preadditive, $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is stable under both finite products and coproducts. Hence finite coproducts and products in $\mathrm{Fun}^{\mathrm{R}}$ are calculated pointwise, and by preadditivity of $\mathcal{D}$, they agree.

Now suppose $\mathcal{C}$ is preadditive. Then it is pointed, so by proposition 6.6, $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ is pointed. Let $(-)^2 : \mathcal{C} \to \mathcal{C}$ be the functor $X \mapsto X \times X$. There is a natural transformation $(-)^2 \to 1_{\mathcal{C}}$ given by the fold map $X \times X \simeq X \sqcup X \to X$. This induces a natural transformation $\delta^H : H \times H \to H$ for any $H : \mathcal{C} \to \mathcal{D}$. We claim this data satisfied the hypotheses of lemma 6.14. This is not hard to verify - each diagram can be constructed from an analogous commutative diagram in $\mathcal{C}$. □

Corollary 6.19. Let $\mathcal{C}$ be a presentable ∞-category. Then the forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint. This is a localization of $\mathcal{C}$ in Pr^{L} relative to the preadditive categories.

Proof. Analogous to the proof of corollary 6.7 □

Additive ∞-categories

Definition 6.20. If $A \in \mathcal{C}$ is an object in a preadditive category, we define the *shear map*

$$s : A \oplus A \to A \oplus A$$

as the projection $p_1 : A \times A \to A$ on the first factor and the fold map $A \sqcup A \to A$ on the second factor.

Definition 6.21. An *additive* ∞-category is a preadditive ∞-category in which each shear map is an equivalence.

Proposition 6.22. Suppose $\mathcal{C}$ is an ∞-category with finite products. Then $\mathrm{CGrp}(\mathcal{C})$ is additive.

Proof. Since $\mathrm{CGrp}(\mathcal{C}) \subseteq \mathrm{CMon}(\mathcal{C})$ is closed under products, and hence under coproducts (by preadditivity of CMon), it is preadditive. Now take $A \in \mathrm{CGrp}(\mathcal{C})$. Observe that the shear map $s : A \times A \to A \times A$ is given on underlying objects by the analogous map coming from the commutative monoid structure: $s_{\langle 1 \rangle} : A(\langle 1 \rangle)^2 \to A(\langle 1 \rangle)^2$ is given by composition on the first coordinate and projection on the second. It is clear that, if $A(\langle 1 \rangle)$ is a group object, this is an equivalence. Since the forgetful functor is conservative, so is the original shear map. This is all we needed to show. □

Proposition 6.23. Suppose $\mathcal{C}$ is an additive ∞-category. Then the forgetful functor $\mathrm{CGrp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

Proof. We know that $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence. Hence the statement is equivalent to the claim that all objects in $\mathrm{CMon}(\mathcal{C})$ are also groups, which we now prove. Let $X \in \mathcal{C}$. Then there is a unique commutative monoid $\bar{X}$ with underlying object X. The multiplication $X \times X \to X$ is given by the fold map $X \coprod X \to X$. Since $\mathcal{C}$ is additive, the shear map $X \times X \to X \times X$ is an equivalence. Hence so is $\bar{X} \times \bar{X} \to \bar{X} \times \bar{X}$, finishing the proof. □

Proposition 6.24. Let $\mathcal{C}, \mathcal{D}$ be complete ∞-categories. Suppose either $\mathcal{C}$ or $\mathcal{D}$ is additive. Then also $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ is additive.

Proof. As the proof of proposition 6.18. In the case that $\mathcal{C}$ is additive, we must check that the shear map $F \times F \to F \times F$ is an equivalence for each functor $F : \mathcal{C} \to \mathcal{D}$. Both the fold map $\delta^F : F \times F \to F$ and the projection $p_1 : F \times F \to F$, arise from applying F to the corresponding natural transformations $X \times X \to X$ in $\mathcal{C}$. Hence the shear map is objectwise given by the map $F(s) : FX \times FX \to FX \times FX$, where $s : X \times X \to X \times X$ is the shear map in $\mathcal{C}$. Since $\mathcal{C}$ is additive, this is an equivalence, finishing the proof. □

Proposition 6.25. Let $\mathcal{C}$ be presentable. Then $\mathrm{CGrp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, which is a localization of $\mathcal{C}$ relative to the additive categories.

Proof. As corollary 6.7 and corollary 6.19 □

As a consequence of the theory described here, the assignments $\mathcal{C} \mapsto \mathrm{CMon}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{CGrp}(\mathcal{C})$ extend to functors $\mathrm{Pr}^L \to \mathrm{Pr}^L$. We may choose these functors to be the opposites of the functors $\mathrm{Pr}^R \to \mathrm{Pr}^R$ sending a functor $\mathcal{C} \to \mathcal{D}$ to the functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{D})$ given by post-composition. In other words, given a colimit-preserving functor $\mathcal{C} \to \mathcal{D}$, we take its adjoint, construct the functor $\mathrm{CMon}(\mathcal{D}) \to \mathrm{CMon}(\mathcal{C})$, then take the adjoint of this.

If given a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves colimits and finite products, we may instead define a functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{D})$ by just post-composing with F.

Proposition 6.26. Suppose given a left adjoint, finite product-preserving functor $L : \mathcal{C} \to \mathcal{D}$. Suppose further that finite products in $\mathcal{C}$ and $\mathcal{D}$ preserve countable colimits. Then the two functors described above are equivalent.

Proof. Let R be the right adjoint of L. Then postcomposing with L or R gives an adjunction

$$L_*\colon \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}) \rightleftarrows \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{D}) : R_*$$

Both of these functors restrict to functors

$$\underline{L} : \mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{D})$$

$$\underline{R} : \mathrm{CMon}(\mathcal{D}) \to \mathrm{CMon}(\mathcal{D})$$

Since adjoints are unique up to equivalence, it suffices to prove that this is an adjunction. Let $\eta : 1_{\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C})} \to R_* L_*$.be a unit for the adjunction $L_* | R_*$, in the sense of [8, Definition 5.2.2.7]. Then we can simply restrict η to get a natural transformation $\eta : 1_{\mathrm{CMon}(\mathcal{C})} \to \underline{RL}$, which will also be a unit. This proves that the two functors are adjoint, finishing the proof. □

Remark 6.27. By exactly the same argument, the two possible functors $\mathrm{CGrp}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{D})$ are also equivalent.

7 Sp($\mathcal{C}$), and the free functors

Stable ∞-categories and spectrum objects

We will briefly discuss the relationship between $\mathcal{C}_*$, $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C})$, and the category $\mathrm{Sp}(\mathcal{C})$ of *spectrum objects in* $\mathcal{C}$. We will not discuss $\mathrm{Sp}(\mathcal{C})$ itself in any great detail. When $\mathcal{C} = \mathcal{S}$, it is called *the infinity-category of spectra*, and is equivalent to the underlying ∞-category of the model category of symmetric spectra ([7, Example 4.1.8.6]). Moreover the symmetric-monoidal structure induced on it by applying the construction of this project is the same as the one induced by the (strict) symmetric monoidal structure on symmetric spectra. Hence we can regard $(\mathrm{Sp}(\mathcal{S}), \otimes)$ as a suitable ∞-categorical version of the smash product of spectra.

Definition 7.1. A *stable* ∞-category $\mathcal{C}$ is a pointed ∞-category which admits finite limits and colimits, and with the property that any square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pullback if and only if it is a pushout.

Remark 7.2. Compare the fact that homotopy Cartesian and homotopy coCartesian squares agree in e.g. the model category of symmetric spectra.

Remark 7.3. This is different from, but equivalent to, Lurie's definition of stable ∞-category. See [7, Prop. 1.1.3.4].

Remark 7.4. It is not difficult to verify that for $\mathcal{C}$ an ∞-category,

$$\mathcal{C} \text{ stable} \Rightarrow \mathcal{C} \text{ additive} \Rightarrow \mathcal{C} \text{ preadditive} \Rightarrow \mathcal{C} \text{ pointed}$$

The only nontrivial case is to see that stable implies additive. This follows from Corollary 1.4.2.17 and Remark 1.1.3.5 in [7].

Proposition 7.5 ([7, Cor. 1.4.4.5])**.** Let $\mathcal{C}$ be presentable. Then it admits a localization relative to the stable ∞-categories (a *stabilization*). This is the functor $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$, which is left adjoint to the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$.

Free functors

As a consequence of the above, for each presentable ∞-category $\mathcal{C}$, there is a canonically determined sequence of colimit-preserving functors

$$\mathcal{C} \to \mathcal{C}_* \to \mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}),$$

each of which is a localization of the domain relative to the successive classes of ∞-categories. The first three functors are left adjoint to the respective forgetful functors.

Proposition 7.6. Let $\mathcal{C}$ be a presentable ∞-category. Suppose the product in $\mathcal{C}$ preserves countable colimits in each variable separately. Then the free monoid functor $\mathcal{C} \to \mathrm{CMon}(\mathcal{C})$ takes $X \in \mathcal{C}$ to the functor

$$\langle n \rangle \mapsto \mathrm{colim}_{\langle a \rangle \to \langle n \rangle \in (\mathrm{Fin}_*^{\mathrm{int}})_{/\langle n \rangle}} X^{\langle a \rangle^\circ}$$

In particular, the composition $\mathcal{C} \to \mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is given by the formula $X \mapsto \sqcup_n (X^n)_{\Sigma_n}$, where Σ_n acts on X^n by permutation.

Proof. It suffices to prove that the given functor has the correct universal property. To see this, observe that the formula above is precisely the formula for the left Kan extension to $N(\mathrm{Fin}_*)$ of the functor $F : \mathrm{Fin}_*^{\mathrm{int}} \to \mathcal{C}$ given by $\langle n \rangle \mapsto X^{\langle n \rangle^\circ}$. By the universal property of left kan extension, it suffices to check that

$$\mathrm{Map}_{\mathrm{Fun}(N(\mathrm{Fin}_*)^{\mathrm{int}},\mathcal{C})}(F, G) \simeq \mathrm{Map}_{\mathcal{C}}(X, G(\langle 1 \rangle))$$

whenever G is the restriction of $N(\mathrm{Fin}_*)^{\mathrm{int}}$ of a monoid. To see this, observe that

$$\mathrm{Fun}^{\mathrm{Seg}}(N(\mathrm{Fin}_*^{\mathrm{int}}), \mathcal{C}) \simeq \mathcal{C},$$

with the inverse given by right Kan extension.

Using the fact that the product distributes over colimits, one can easily check the Segal condition.

The formula follows in the following way: since every map in Fin_* factors uniquely as an inert followed by an active map, the collection of active maps $\langle a \rangle \to \langle 1 \rangle$ is cofinal in the index category. This allows us to replace the colimit with one indexed by active maps $\langle a \rangle \to \langle 1 \rangle$, which is equivalently just the category of isomorphisms in Fin. This clearly gives the correct value. ∎

Remark 7.7. The left adjoint $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C})$ is called group completion. In can be identified with Segal's group completion functor for Γ-spaces.

Remark 7.8. The functor $\mathrm{CGrp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ is fully faithful. Its essential image is the subcategory of *connective spectrum objects*. This is shown in [7, p. 5.2.6].

8 Localizations of symmetric monoidal ∞-categories

After much preparation, we are finally approaching the heart of the matter, namely the interplay between localizations and symmetric monoidal ∞-categories.

Symmetric monoidal and smashing localizations

Definition 8.1. Suppose $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category, and that $L : \mathcal{C} \to \mathcal{C}$ is a localization of the underlying ∞-category. Then we say that L is *compatible with the symmetric monoidal structure* if, whenever $X \to Y$ is a local equivalence, and Z is any object, the map $X \otimes Z \to Y \otimes Z$ is also a local equivalence.

Remark 8.2. Definition 8.1 is equivalent to the following condition: given a collection of local equivalences $X_i \to Y_i$, the map $\bigotimes_i X_i \to \bigotimes_i Y_i$ is a local equivalence. This is the definition of compatibility used in [7].

The following proposition describes how compatible localizations extend to the symmetric monoidal structure. In many ways, this is really the key proposition of the project.

Proposition 8.3. Suppose $p : \mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category, and that $L : \mathcal{C} \to \mathcal{C}$ is a localization of the underlying ∞-category, which is compatible with the symmetric monoidal structure. Let $(L\mathcal{C})^\otimes$ be the subcategory of $\mathcal{C}^\otimes$ spanned by the objects of the form

$$LC_1 \oplus \cdots \oplus LC_n.$$

Then the following hold:

(1) The restriction $p : (L\mathcal{C})^\otimes \to N(\mathrm{Fin}_*)$ exhibits $(L\mathcal{C})^\otimes$ as a symmetric monoidal ∞-category with underlying ∞-category $L\mathcal{C}$

(2) The inclusion $(L\mathcal{C})^\otimes \subseteq \mathcal{C}^\otimes$ admits a left adjoint, $L^\otimes$. Furthermore, $L^\otimes|_{\mathcal{C}} \simeq L$, and the unit η of the adjunction can be chosen so that $p(\eta) = 1$.

(3) $L^\otimes$ is a symmetric monoidal functor.

(4) The inclusion $(L\mathcal{C})^\otimes \subseteq \mathcal{C}^\otimes$ is a lax symmetric monoidal functor.

We will need the following lemma, which we do not prove.

Lemma 8.4 ([7, Lemma 2.2.1.11])**.** Suppose $p : \mathcal{C} \to \mathcal{D}$ is a coCartesian fibration, and that $L : \mathcal{C} \to \mathcal{C}$ is a localization functor. Suppose further that $pL \simeq p$. Then

(1) L carries p-coCartesian morphisms of $\mathcal{C}$ to p-coCartesian morphisms of $L\mathcal{C}$.

(2) The functor $L\mathcal{C} \to \mathcal{D}$ is a coCartesian fibration

Proof of proposition 8.3. We will begin by constructing the functor $L^\otimes$ of (2). For each object $\bigoplus_{i\in\langle n\rangle^\circ} X_i \in \mathcal{C}^\otimes$, consider the map $\bigoplus_{i\in\langle n\rangle^\circ} X_i \to \bigoplus_{i\in\langle n\rangle^\circ} L(X_i)$. We claim that this is a localization of $\bigoplus_{i\in\langle n\rangle^\circ} X_i$ relative to $(L\mathcal{C})^\otimes$.

To see this, let $\bigoplus_{j\in\langle m\rangle^\circ} LY_j$ be an arbitrary object of $L(\mathcal{C})^{\otimes}$. Consider the map

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(\bigoplus_{i\in\langle n\rangle^\circ} LX_i, \bigoplus_{j\in\langle n\rangle^\circ} LX_j) \to \mathrm{Map}_{\mathcal{C}^{\otimes}}(\bigoplus_{i\in\langle n\rangle^\circ} X_i, \bigoplus_{j\in\langle n\rangle^\circ} LX_j)$$

We know that the mapping space

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(\bigoplus_{i\in\langle n\rangle^\circ} LX_i, \bigoplus_{j\in\langle n\rangle^\circ} LX_j)$$

can be written as the disjoint union over $f : \langle n\rangle \to \langle m\rangle$ of $\prod_{j\in\langle m\rangle} \mathrm{Map}_{\mathcal{C}}(\otimes_{f(i)=j} LX_i, LX_j)$, similarly for the other mapping spaces. Moreover, since the map we are composing with lies over the identity on $\langle n\rangle$, it preserves this decomposition. Hence it suffices to observe that on each component it is a homotopy equivalence, which is true since each map $\otimes X_i \to \otimes LX_i$ is a localization in $\mathcal{C}$ relative to $L\mathcal{C}$, since L is compatible with $\otimes$.

The above implies that the inclusion $(L\mathcal{C})^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ admits a left adjoint. The restriction of this left adjoint to $\mathcal{C}$ must be a left adjoint to the inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$, hence be equivalent to L. It is also clear that constructed in this way, η goes to the identity on $N(\mathrm{Fin}_*)$.

(2) being done, we move on to (1), and (3), which are now easy consequences of lemma 8.4. To see (4), note that map $F : \bigoplus_i X_i \to \bigoplus_j Y_j$ lying over $f : \langle n\rangle \to \langle m\rangle$, which can be identified with a collection of maps $\otimes_{f(i)=j} X_i \to Y_j$, is a coCartesian lift of f if and only if each of these maps is an equivalence. If f is inert, each of these tensor products is unary. By this, it is clear that the inclusion preserves inert maps. $\square$

Remark 8.5. The inclusion is almost, but not quite, fully symmetric monoidal. It does preserve tensor products, except nullary ones, i.e. the unit object. Instead, the unit object in $(L\mathcal{C})^{\otimes}$ is the localization of the unit in $\mathcal{C}$, which is not necessarily the unit itself.

Corollary 8.6. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $L : \mathcal{C} \to \mathcal{C}$ be a compatible localization. Then L induces a functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(L\mathcal{C})$, which is left adjoint to the inclusion.

Proof. It will be sufficient to observe that $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(L\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ is a localization. The above can be identified with the functor $\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes}) \to \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes})$ given by post-composition with $L^{\otimes}$, restricted to the full subcategory spanned by the commutative algebras (and with restricted image). Then applying proposition 2.13 and proposition 2.14 gives us the result. $\square$

Corollary 8.7. A map in $\mathrm{CAlg}(\mathcal{C})$ is a local equivalence if and only if the underlying map in $\mathcal{C}$ is.

Proof. Follows from the fact that a map of algebras is an equivalence if and only if the underlying map is, and the fact that the localization is just L on the underlying objects. $\square$

Corollary 8.8. Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object with underlying object $A_{\langle 1\rangle}$. Then there is a map $A \to B$ in $\mathrm{CAlg}(\mathcal{C})$ with underlying map $\eta_{A_{\langle 1\rangle}} : A_{\langle 1\rangle} \to LA_{\langle 1\rangle}$, namely the localization of A. Furthermore, if $A \to B'$ is another algebra morphism with the same underlying map (i.e. another algebra structure on $LA_{\langle 1\rangle}$ making it a homomorphism), there is a (essentially) unique equivalence $B \to B'$ so that

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & B' \end{tikzcd}$$

commutes. The underlying endomorphism of $B_{\langle 1\rangle}$ is the identity.

Proof. By corollary 8.7, any such map of algebras is a localization of A. This implies the unique existence of the equivalences $B \to B'$. The underlying map $LA_{\langle 1\rangle} \to LA_{\langle 1\rangle}$ is unique with the property that

$$\begin{tikzcd} A_{\langle 1\rangle} \arrow[r] \arrow[dr] & LA_{\langle 1\rangle} \arrow[d] \\ & LA_{\langle 1\rangle} \end{tikzcd}$$

commutes, hence must be the identity. $\square$

This corollary says that, given a commutative algebra structure on the object X, there is a unique structure on LX making η_X a homomorphism.

We now define smashing localizations.

Definition 8.9. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *smashing localization* of $\mathcal{C}$ is a functor $L : \mathcal{C} \to \mathcal{C}$ of the form $L(X) = X \otimes E$ for some fixed object E, which is a localization functor.

Smashing localizations can be identified with so-called *idempotent objects*, which we now describe.

Definition 8.10. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Let $e : I \to E$ be a morphism in $\mathcal{C}$. Then we say that e exhibits E as an idempotent object if the map

$$e \otimes 1_E : I \otimes E \to E \otimes E$$

is an equivalence.

Remark 8.11. It is a consequence of this definition that the analogous map $E \otimes I \to E \otimes E$ is an equivalence, since it is equivalent to the composite $E \otimes I \to I \otimes E \to E \otimes E$.

Proposition 8.12. Let $e : I \to E$ be a map in a symmetric monoidal ∞-category $\mathcal{C}$. Then e induces a natural transformation $(-) \otimes e : 1_{\mathcal{C}} \simeq (-) \otimes I \to (-) \otimes E$.

Then $(-) \otimes e$ exhibits $(-) \otimes E$ as an idempotent object if and only if e exhibits E as an idempotent functor.

Proof. Suppose first e exhibits E as idempotent. Then we must check that the two maps $X \otimes E \to X \otimes E \otimes E$ are both equivalences. One is given by

$$((-) \otimes e)_{X \otimes E} = 1_{X \otimes E} \otimes e = 1_X \otimes 1_E \otimes e,$$

which is an equivalence. The other is given by

$$((-) \otimes e)_X \otimes 1_E = 1_X \otimes e \otimes 1_E,$$

which is also an equivalence.

Now suppose $(-) \otimes e$ exhibits $(-) \otimes E$ as an idempotent functor. Then in particular

$$((-) \otimes e)_{1 \otimes E} : I \otimes E \simeq I \otimes E \otimes I \overset{1_I \otimes 1_E \otimes e}{\to} 1 \otimes E \otimes E$$

is an ismorphism. But this means $1_I \otimes (1_E \otimes e)$ is an equivalence, but then certainly so is $1_E \otimes e$, which is the desired property for e. $\square$

Remark 8.13. In fact it is a consequence of the second half of this proof that, given any smashing localization, taking $\lambda_1 : I \to I \otimes E \simeq E$ will exhibit E as an idempotent object. So $(-) \otimes E$ is idempotent if and only if E is idempotent. However, there may be idempotent structures on $(-) \otimes E$ which do not arise from idempotent structures on E by this construction.

As an example, consider $N(\mathrm{Vect}_{\mathbb{R}})$, the nerve of the ordinary category of vector spaces, with direct sum as the symmetric monoidal structure. Then $\mathbb{R}^0 = \{0\}$ is the tensor unit, and admits a unique idempotent structure. However, the corresponding localization is just the identity functor $1_{N(\mathrm{Vect}_{\mathbb{R}})}$ Any natural automorphism will exhibit this as an idempotent functor, and there are many nontrivial such, e.g. $\alpha_V : V \to V$ given by $v \mapsto -v$.

For this project, we are mainly interested in showing that a localization is compatible with the symmetric monoidal structure, and for this purpose, there is no issue.

Proposition 8.14. Let $L = (-) \otimes E : \mathcal{C} \to \mathcal{C}$ be a smashing localization on the symmetric monoidal ∞-category $\mathcal{C}$. Then it is compatible with the symmetric monoidal structure.

Proof. We must check that if $f : X \to Y$ is a local equivalence, then also $f \otimes 1_Z : X \otimes Z \to Y \otimes Z$ is a local equivalence. This means that, if $X \otimes E \to Y \otimes E$ is an equivalence, then also $X \otimes Z \otimes E \to Y \otimes Z \otimes E$ should be one. This is immediate by swapping the tensor factors. $\square$

$(-)_*$, CMon, and CGrp as smashing localizations

Proposition 8.15. Each of the localizations $\mathcal{C} \mapsto \mathcal{C}_*, \mathcal{C} \mapsto \mathrm{CMon}(\mathcal{C}), \mathcal{C} \mapsto \mathrm{CGrp}(\mathcal{C}), \mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ is smashing.

Proof. Let L be one of these localizations. In each case, we have

$$\begin{aligned} L(\mathcal{C}) &\simeq L(\mathcal{C} \otimes \mathcal{S}) && (8)\\ &\simeq L(\mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{S})) && (9)\\ &\simeq \mathrm{Fun}^R(\mathcal{C}^{op}, L\mathcal{S}) && (10)\\ &\simeq \mathcal{C} \otimes L(\mathcal{S}) && (11) \end{aligned}$$

The third equivalence is the only nontrivial part of this, and follows from lemma 4.18 in the case of CMon and CGrp. For $(-)_*$, it is proposition 4.4. For Sp, it follows from the description of $\mathrm{Sp}(\mathcal{C})$ as a colimit - see [7, Example 4.8.1.23] for details. In fact, by chasing through the definitions, one can observe that the previously defined maps $\mathcal{C} \to L\mathcal{C}$ (left adjoints to the forgetful functors) are actually identified with the map $\mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes L\mathcal{S}$. $\square$

As a direct consequence of this theorem, we achieve all our goals.

Corollary 8.16. Let L be one of the localizations above. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Then there is a unique presentable symmetric monoidal structure on $L\mathcal{C}$ with the same property so that the functor $\mathcal{C} \to L\mathcal{C}$ is symmetric monoidal.

Proof. Since L is smashing, it is compatible with the symmetric monoidal structure on Pr^L. Now just apply corollary 8.8 above, recalling that presentable symmetric monoidal structures can be identifies with commutative algebra objects of Pr^L. $\square$

Corollary 8.17. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Then the sequence of functors

$$\mathcal{C} \to \mathcal{C}_* \to \mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$$

refines to a sequence of symmetric monoidal left adjoint functors, using the symmetric monoidal structure of corollary 8.16

Proof. Follows from the above using also corollary 2.15. $\square$

Corollary 8.18 (Theorem A)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal, colimit-preserving functor of presentably symmetric monoidal ∞-categories. Then there is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"] & \mathcal{C}_* \arrow[r] \arrow[d, "(F)_*"] & \mathrm{CMon}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{CMon}(F)"] & \mathrm{CGrp}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{CGrp}(F)"] & \mathrm{Sp}(\mathcal{C}) \arrow[d, "\mathrm{Sp}(F)"] \\ \mathcal{D} \arrow[r] & \mathcal{D}_* \arrow[r] & \mathrm{CMon}(\mathcal{D}) \arrow[r] & \mathrm{CGrp}(\mathcal{D}) \arrow[r] & \mathrm{Sp}(\mathcal{D}) \end{tikzcd}$$

of symmetric monoidal left adjoint functors using the structures above. Moreover, this diagram is determined up to equivalence.

Note that the group completion functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C})$ is actually a localization itself. This means that we can also apply the above theory in this case.

Proposition 8.19. The localization

$$L : \mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C}) \hookrightarrow \mathrm{CMon}(\mathcal{C})$$

is compatible with the symmetric monoidal structure.

Proof. Suppose $X \to Y$ is a local equivalence in $\mathrm{CMon}(\mathcal{C})$, i.e $LX \to LY$ is an equivalence. Then $L(X \otimes Z) \to L(Y \otimes Z)$ can be identified with the map $LX \otimes LZ \to LY \otimes LZ$, since L is symmetric monoidal. Here the tensor products are computed in $\mathrm{CGrp}(\mathcal{C})$. But all the same this map is clearly an equivalence, finishing the proof. □

Corollary 8.20. Suppose R is an object of $\mathrm{CAlg}(\mathrm{CMon}(\mathcal{C}))$ (an "$\mathbb{E}_\infty$ semiring in $\mathcal{C}$"). Then there is a unique way to give $LR \in \mathrm{CGrp}(\mathcal{C})$ the structure of an object of $\mathrm{CAlg}(\mathrm{CGrp}(\mathcal{C}))$ (an "$\mathbb{E}_\infty$ ring in $\mathcal{C}$").

Since understanding the interplay between the group completion functor and multiplicative structures is a classically important problem (see Section 9), this corollary is very powerful. For instance, it says that the multiplicative structue on $K(R)$ is uniquely determined by the multiplicative structure on the commutative monoid in $\mathcal{S}$ which is constructed as an intermediate step.

9 Algebraic K-theory

Suppose we are given an (essentially small) ordinary category $\mathbf{C}$ with a symmetric monoidal structure $\oplus$. Then we can extract an abelian group in the following way:

(1) Pass to the set of isomorphism classes $\pi_0(\mathbf{C})$

(2) Equip it with the structure of a commutative monoid via $\oplus$.

(3) Take the group completion of this monoid.

The resulting functor is Grothendieck's $K_0(\mathbf{C})$, the *0th algebraic K-group* of $\mathbf{C}$. An important idea, which goes back to Quillen (developed by Segal, [10]), is that a more powerful version of this invariant can be developed by passing instead to the classifying space of $\mathbf{C}^{\simeq}$ in step 1. To make steps 2 and 3 work in this setting, one needs a suitably general notion of commutative monoid and group, and it is for this purpose that Segal introduced Γ-spaces.

If $\mathbf{C}$ has another symmetric monoidal structure $\otimes$, which distributes over $\oplus$ in a suitable sense, $K_0(\mathbf{C})$ becomes a ring. The most important example of this is if R is a commutative ring. The category of finitely generated projective R-modules $\mathbf{R}-\mathbf{Mod}^{\mathrm{FP}}$ with $\oplus$ the direct sum and $\otimes$ the tensor product is an example of this. This gives a ring structure to $K_0(R) = K_0(\mathbf{R}-\mathbf{Mod}^{\mathrm{FP}})$. Right since [10], it has been an important problem to extend this phenomenon to $K(R)$. In fact, this was an important motivation for developing a good notion of ring spectrum. Segal sketched in [10] how to get a commutative algebra object in $h\mathrm{Sp}$ (the stable homotopy category). There have been a large number of refinements of this result, notably May in [9]. In that paper, May develops a way of passing from "bipermutative categories" to ring spectra (in a certain sense). Here the notion of bipermutative category is a way of formalizing the notion that $\otimes$ distributes over $\oplus$.

However, this approach has certain limitations

(1) The notion of bipermutative category is unecessarily restrictive. It requires that certain diagrams commute strictly, when it it sufficient that they commute up to coherent homotopy in a certain sense.

(2) Moreover, the notion of bipermutative category is very *ad hoc*. It is not clear how to relax it, for instance, to obtain a way of assigning "noncommutative ring spectra" to categories where $\otimes$ is not symmetric.

The methods of this project can solve both of these issues. However we will not really remark on (2), as doing so will require more machinery from [7]

Definition 9.1. The *algebraic* K*-theory functor* is defined as the following composite

$$\mathsf{K} : \mathcal{C}\mathrm{at}_\infty^\otimes = \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) \overset{\mathrm{CMon}(-)^\simeq}{\to} \mathrm{CMon}(\mathcal{S}) \to \mathrm{Sp}$$

Remark 9.2. This also defines algebraic K-theory for symmetric monoidal ordinary categories. This definition also extends the approach of [9] and [10] by allowing ∞-categories as input.

Proposition 9.3. K admits a canonical lax symmetric monoidal structure.

Remark 9.4. This implies Theorem B.

Proof. We have already equipped the functor $\mathrm{CMon}(\mathcal{S}) \to \mathrm{Sp}$ with a symmetric monoidal structure (which is in particular a lax symmetric monoidal structure). On the other hand, the functor $(-)^\simeq : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ can be defined as the right adjoint of the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$. This inclusion preserves products, so that it gives a symmetric monoidal functor $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$. This gives a symmetric monoidal structure on the corresponding functor $\mathrm{CMon}(\mathcal{S}) \to \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$, which we have seen is left adjoint to the functor $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{CMon}(\mathcal{S})$. Hence this right adjoint receives a lax symmetric monoidal structure, finishing the proof. $\square$

Definition 9.5. A *commutative rig ∞-category* is an object of $\mathrm{CAlg}(\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty))$. We name this category $\mathtt{CRig}(\mathcal{C}\mathrm{at}_\infty)$ A *commutative ring spectrum* is an object of $\mathrm{CAlg}(\mathrm{Sp}))$. We name this category $\mathtt{CRingSp}$

Corollary 9.6. K induces a functor

$$\mathsf{K} : \mathrm{CRig}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{CRingSp}$$

$\mathrm{CRig}(\mathcal{C}\mathrm{at}_\infty)$ is our refinement of May's bipermutative categories. It does have the disadvantage of being less explicit, so it is harder to simply write down examples. However, we can at least construct them in the following way:

Proposition 9.7. Suppose $\mathcal{C}$ is a symmetric monoidal ∞-category with coproducts, and that the tensor product preserves coproducts in each variable separately. Then $\mathcal{C}$ is canonically an element of $\mathrm{CRig}(\mathcal{C}\mathrm{at}_\infty) = \mathrm{CAlg}(\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty))$

Proof. Let $\mathcal{C}\mathrm{at}_\infty^\Sigma$ be the ∞-category of ∞-categories with coproducts and coproduct-preserving functors. It can be shown that $\mathcal{C}\mathrm{at}_\infty^\Sigma$ admits a symmetric monoidal structure with the following property: There is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$, which is initial among functors that preserve coproducts in each variable. With this structure, an object in $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty^\Sigma)$ is precisely a symmetric monoidal ∞-category as in the statement of the proposition. This is [7, p. 4.8.1.4], with $\mathcal{K}$ the collection of discrete simplicial sets.[1]

First, we claim that $\mathcal{C}\mathrm{at}_\infty^\Sigma$ is preadditive. First of all, it's presentable, so in particular it has finite products and coproducts. It suffices to check that for $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty^\Sigma$, their product in $\mathrm{h}\mathcal{C}\mathrm{at}_\infty^\Sigma$, which is just their product in $\mathrm{h}\mathcal{C}\mathrm{at}_\infty$, satisfies the universal property of the coproduct in $\mathrm{h}\mathcal{C}\mathrm{at}_\infty^\Sigma$. Let $f : \mathcal{C} \to \mathcal{E}, \mathcal{D} \to \mathcal{E}$ be coproduct-preserving functors. Then they extend to the functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E} \times \mathcal{E} \overset{\sqcup}{\to} \mathcal{E}$, and this is unique up to homotopy since $(c, d) \simeq (c, \emptyset) \sqcup (\emptyset, d)$ for any $(c, d) \in \mathcal{C} \times \mathcal{D}$.

Now observe that by [7, Remark 4.8.1.9], the inclusion $\mathcal{C}\mathrm{at}_\infty^\Sigma \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ admits a left adjoint, which is symmetric monoidal. Hence it induces a symmetric monoidal functor $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty^\Sigma) \simeq \mathcal{C}\mathrm{at}_\infty^\Sigma$. The right adjoint of this functor then has a canonical lax symmetric monoidal structure.

Thus it induces a functor $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty^\Sigma) \to \mathrm{CAlg}(\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty))$ (which preserves the underlying category), which is what we wanted. $\square$

[1] Note that Lurie's notation $\mathcal{C}\mathrm{at}_\infty^\otimes$ for the Cartesian symmetric monoidal structure on $\mathcal{C}\mathrm{at}_\infty$ conflicts with his use of the same symbol for the ∞-category of symmetric monoidal ∞-categories

Corollary 9.8. Let R be a commutative ring. Then $K(R)$ is canonically an object of CRingSp, with multiplication coming from $\otimes$.

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