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Master project in Mathematics

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Quasi-categories
- basic constructions

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0.1 Abstract

The project consists of two chapters and an appendix. In the first chapter define quasi-categories, and perform basic constructions with these, some of which are motivated by ordinary category theory. Among these, we shall build under-categories and colimits. Also we briefly review some different notions of fibrations needed for the next chapter, and introduce a pair of adjoint functors called straightening and unstraightening.

The second chapter is devoted to a certain application of the tools developed in chapter 1. However we will have to apply auxillary results since developing all we need is outside the scope of this project. There will be some loose discussion of the results we apply and those we obtain. We will try to relate the work of the chapter to a theorem in topology, which states that in certain cases (not too restrictive), there is a weak equivalence $G^{ad}/\!\!/G \simeq LBG$ for a simplicial group G. A classical proof of this is given in the appendix.

Contents

0.2 Introduction

In this project we shall introduce some vocabulary of quasi-categories which can be taken as a model of ∞-categories. Also referred to as $(\infty, 1)$-categories in analogue to (n, r)-categories, these structures are often thought of as higher categories for which all j-morphisms are equivalences, for $j > 1$.

In this project we shall define them as a special class of simplicial sets, and work through basic constructions from there. we will for the most part leave interpretations to the reader. In chapter 1 we will cover basic constructions rather explicitly.

Some results needed for the project, are not build from scratch, but quoted from the main reference [Lur09]. I will try to justify some of them with reference to chapter 1 and classic results outside the theory of quasi-categories. Results from model category theory will just be stated, partly because there is not room to develop the theory as thouroughly as the rest of chapter 1, and partly due to the authors inexperience in working with this subject.

The reader will be expected to know of basic constructions in category theory, and to have basic knowledge of algebraic topology, and in particular of simplicial sets. Having read [Hat02] or [Bre93], and chapter 1 of [GJ99] will be more than enough.

Quasi-categories

1.1 Definition

As stated in the introduction, our choice of definition is a certain class of simplicial sets. I will not try to motivate this choice of definition, but simply note that this gives a familiar frame to work in, and has other (in some respects) nice implications. Among these is the comforting fact that we need not worry about set theoretic issues that might show up in our constructions, when working inside this class. However, we will need to leave this frame at some points along the way, and when we do so, I will implicitly assume that the classes we encounter are sufficiently small for set theory to work out. Of course this is a limitation, but aside from this remark I will refer the reader to [Lur09] with regards to this.

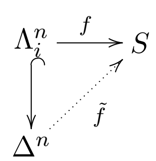
Now we state definitions

Definition 1.1.1 (Horns). *Let Δ^n be the standard n-simplex in simplicial sets. The* i'th horn *on Δ^n is the simplicial set $\Lambda_i^n \subset \Delta^n$, where the non-degenerate k-simplicies are given by*

$$(\Lambda_i^n)_k = \begin{cases} (\Delta^n)_k & \text{for } k < n-1 \\ (\Delta^n)_k - d_i(\Delta^n) & \text{for } k = n-1 \\ \emptyset & \text{for } k \geq n \end{cases} \qquad 0 \leq i \leq n$$

and structure maps induced from those on Δ^n. We say that Λ_i^n is an inner horn *if $0 < i < n$.*

Definition 1.1.2 (Quasi-categories). *A* quasi-category *is a simplicial set S which the following extension property*



for $0 < i < n$. I.e. that every map f from an inner horn to S, can be lifted to a map $\tilde{f}$ from the corresponding simplex.

We say that a quasi-category is a simplicial set that has *inner horn fillers*. It may be noted that in particular Kan complexes are quasi-categories, and that these will play a central role later in this project.

1.2 Colimits and limits

In this section we shall define what a colimit in a quasi-category is. We shall only deal with colimits, and note whenever dualizing statements will result in a limit construction.

First we shall recall a construction in simplicial sets and in categories.

1.2.1 Join

Definition 1.2.1 (Join in sSet). *Let X, Y be simplicial sets. We define the* join *of X and Y, to be the simplicial set $X \star Y$, with n-simplicies*

$$(X \star Y)_n = \coprod_{i+j=n-1} X_i \times Y_j, \qquad -1 \leq i, j \leq n$$

with the convention that $Z_{-1} = *$ for a simplicial set Z, and similar that a (-1)-simplex $\sigma_{-1} = *$. The structure maps

$$d_i \colon (X \star Y)_n \to (X \star Y)_{n-1}$$
$$s_i \colon (X \star Y)_n \to (X \star Y)_{n+1}$$

are given by

$$d_k(x_i, y_j) = \begin{cases} (d_k^X(x_i), y_j) & \text{for } k \leq i \\ (x_i, d_{k-j-1}^Y(y_j)) & \text{for } k > i \end{cases}$$

$$s_k(x_i, y_j) = \begin{cases} (s_k^X(x_i), y_j) & \text{for } k \leq i \\ (x_i, s_{k-j-1}^Y(y_j)) & \text{for } k > i \end{cases}$$

Since the structure maps are given coordinate-wise by those from X and Y, simplicial identities are seen to hold.

We can think of the join $X \star Y$ as $X \sqcup Y$ with a 1-simplex σ added for every pair of 0-simplices $(x, y) \in X \times Y$, satisfying the conditions $d_0(\sigma_{(x,y)}) = y$ and $d_1(\sigma_{(x,y)}) = x$.

Note that in particular we have an inclusion of the disjoint union $X \sqcup Y \hookrightarrow X \star Y$.

Definition 1.2.2 (Join in Cat). *Let $\mathcal{C}, \mathcal{D}$ be categories. We define the* join *of $\mathcal{C}$ and $\mathcal{D}$, to be the category $\mathcal{C} \star \mathcal{D}$ with $ob(\mathcal{C} \star \mathcal{D}) = ob\mathcal{C} \sqcup ob\mathcal{D}$, and morphisms*

$$Mor_{\mathcal{C} \star \mathcal{D}}(a, b) = \begin{cases} Mor_{\mathcal{C}}(a, b) & a, b \in ob\mathcal{C} \\ Mor_{\mathcal{D}}(a, b) & a, b \in ob\mathcal{D} \\ * & a \in ob\mathcal{C}, b \in ob\mathcal{D} \\ \emptyset & a \in ob\mathcal{D}, b \in ob\mathcal{C} \end{cases}$$

The coinciding terminology and notation for the two notions of join defined so far is not coincidential. As the following proposition states the nerve functor preserves the join construction.

Proposition 1.2.3. *If $\mathcal{C}, \mathcal{D}$ are categories, then there is an isomorphism of simplicial sets*

$$N(\mathcal{C} \star_{Cat} \mathcal{D}) \simeq N(\mathcal{C}) \star_{sSet} N(\mathcal{D})$$

Proof. Level-wise we have

$$(N(\mathcal{C}) \star_{\text{sSet}} N(\mathcal{D}))_n = \coprod_{i+j=n-1} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

Define a map $\phi\colon N(\mathcal{C}) \star_{\text{sSet}} N(\mathcal{D}) \to N(\mathcal{C} \star_{\text{Cat}} \mathcal{D})$ by

$$\phi(c_i, d_j) = c_i \star d_j$$

where $c_i \star d_j$ is the functor $[n] \to \mathcal{C} \star_{\text{Cat}} \mathcal{D}$ such that $c_i \star d_j|_{[0,\dots,i]} = c_i$, $c_i \star d_j|_{[i+1,\dots,n]} = d_j$, where $[a, \dots, b]$ denotes the full subcategory spanned by the objects $a, \dots, b$. We note that $c_i \star d_j$ is defined on the morphism $i \le i+1$ by $c_i \star d_j(i \le i+1) = c_i \star d_j(i) \to c_i \star d_j(i+1)$, which is unique, since source and target of this morphism are objects of respectively $\mathcal{C}$ and $\mathcal{D}$. Also note that we still allow for the cases $j = -1$ or $i = -1$, in which $c_n \star d_{-1} = c_n$ and $c_{-1} \star d_n = d_n$.

This is a simplicial map as

$$\phi(s_k(c_i, d_j)) = \phi(s_k(c_i), d_j) = s_k(c_i) \star d_j = s_k(c_i \star d_j) = s_k(\phi(c_i, d_j)), \quad k \le i$$

and similar for $k > i$, and for the face maps.

Further it is a bijection as we shall see. For a given functor $[n] \xrightarrow{F} \mathcal{C} \star_{\text{Cat}} \mathcal{D}$ there are three cases to consider. Either F maps objects entirely to one of $\mathcal{C}$ or $\mathcal{D}$, or there is a unique object $i \in [n]$ such that $s(F(i \le i+1)) \in \mathcal{C}$ and $t(F(i \le i+1)) \in \mathcal{D}$. Thus either $F = F \star F_{-1} = \phi(F, F_{-1})$, or $F = F_{-1} \star F = \phi(F_{-1}, F)$; or $F = F|_{[0,\dots,i]} \star F|_{[i+1,\dots,n]} = \phi(F|_{[0,\dots,i]}, F|_{[i+1,\dots,n]})$. Hence ϕ is surjective.

Suppose now that $\phi(c_i, d_j) = \phi(c'_k, d'_l)$. In particular

$$c_i \star d_j(i \le i+1) = c'_k \star d'_l(i \le i+1) = c'_k \star d'_l(k \le k+1)$$

which is unique, so $i = k$ and $j = l$. Then

$$c_i = c_i \star d_j|_{[0,\dots,i]} = c'_i \star d'_j|_{[0,\dots,i]} = c'_i$$

and similar for the d's. Hence ϕ is injective, and so ϕ is an isomorphism of simplicial sets. □

Taking linearly ordered sets as categories, we observe that $[n+1] = [n] \star [0]$. Thus $\Delta^{n-1} \star \Delta^0 = N([n]) \star N([0]) = N([n+1]) = \Delta^n$, and more generally $\Delta^n \star \Delta^m = \Delta^{n+m+1}$.

1.2.2 Under-quasi-categories

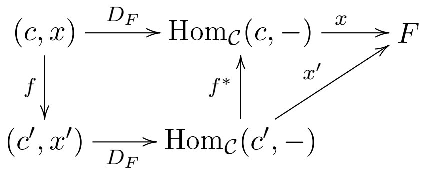
Before we proceed with defining a colimit for quasi-categories, we shall need the following technical proposition also found in [ML98].

Proposition 1.2.4. *Let $\mathcal{E}$ be a concrete monoidal category, and let $\mathcal{C}$ be a category enriched over $\mathcal{E}$. Any functor $F\colon \mathcal{C} \to \mathcal{E}$, is a colimit of a diagram of representables $Hom_{\mathcal{C}}(d, -)$ for $d \in ob\mathcal{C}$.*

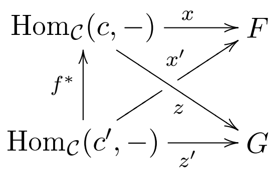
Proof. Consider the category of elements $1 \downarrow F$ given as follows. Objects are pairs (c, x) for $c \in \text{ob}\mathcal{C}$ and $x \in \text{ob}F(c)$, and morphisms

$$\text{Mor}_{1\downarrow F}((c, x), (c', x')) = \{f \in \text{Mor}_{\mathcal{C}}(c, c') \mid F(f)(x) = x'\}$$

Now let $D_F\colon 1 \downarrow F \to \mathcal{E}^{\mathcal{C}}$ be the functor given by sending an object (c, x) to the functor $\text{Hom}_{\mathcal{C}}(c, -)$, and a morphism $f\colon c \to c'$ to the natural transformation $f^*\colon \text{Hom}_{\mathcal{C}}(c', -) \to \text{Hom}_{\mathcal{C}}(c, -)$. We now have the commutative diagram



where the natural transformations x, x' are given by the Yoneda lemma, which states that there is a natural isomorphism $\text{Hom}_{\mathcal{E}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(c, -), F) \simeq F(c)$. To show that F is a colimit, let $G\colon \mathcal{C} \to \mathcal{E}$ be any other functor with natural transformations z, z'



such that the diagram commutes. Again by the Yoneda lemma, z, z' are given as elements of respectively $G(c)$ and $G(c')$. Hence we can construct a natural transformation $\theta\colon F \to G$ by setting the component at c, $\theta_c\colon F(c) \to G(c)$ to be given by $\theta_c(x) = z$. This determines θ as the diagrams above are for any object in $1 \downarrow F$.

It is easy to verify that θ unique, and naturality follows from naturality of the Yoneda isomorphism.

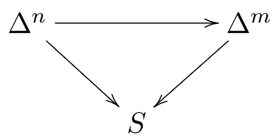
Thus F is the colimit of the diagram D_F, those representables for which the natural transformations between them are induced by morphisms in $1 \downarrow F$. □

This proposition immediatly gives us the corollary

Corollary 1.2.5. *Any simplicial set S is a colimit of its simplicies $\Delta^n \to S$.*

Proof. This is the case of $S\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ from the lemma above, since $\Delta^n := \mathrm{Hom}_{\Delta^{\mathrm{op}}}([n], -)$.

In this case, the Yoneda lemma lets us identify the category of elements with the category with objects morphisms from representables Δ^n to S, and morphisms commuting triangles



By this description it is obvious that the colimit of the diagram D_S in sSet, is S. □

Proposition 1.2.6. *Let S, K be simplicial sets, and let $p\colon K \to S$ be a simplicial map. There exist a simplicial set $S_{p/}$ such that*

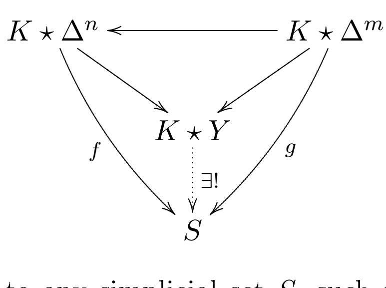
$$Hom_{sSet}(Y, S_{p/}) \simeq Hom_{sSet,p}(K \star Y, S)$$

for any simplicial set Y. Here the subscript p indicates that we restrict the Hom-set to those morphisms $f\colon K \star Y \to S$ such that $f|_K = p$.

Proof. Define $(S_{p/})_n$ to be the set $\mathrm{Hom}_{\mathrm{sSet},p}(K \star \Delta^n, S)$. In the case that Y is a simplex Δ^n,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{Hom}_{\mathrm{sSet},p}(K \star \Delta^-, S)) &\simeq \mathrm{Hom}_{\mathrm{sSet},p}(K \star \Delta^-, S)_n \\ &= \mathrm{Hom}_{\mathrm{sSet},p}(K \star \Delta^n, S)\end{aligned}$$

Applying lemma 1.2.5, we see that in the general case it is enough to check that $\mathrm{colim}_{1\downarrow Y}((K \star \Delta^n) \simeq K \star Y$. First note that given any morphism $K \star \Delta^n \to K \star \Delta^m$ in the image of the diagram D_Y from above, there are maps to $K \star Y$ induced by inclusions of simplicies in Y such that the upper triangle of the following diagram commutes



Then given maps f, g to any simplicial set S, such that the outer triangle commutes, we should produce a unique map as in the diagram, making the whole diagram commute. Level-wise we have $(K \star Y)_n = \coprod_{i+j=n-1} K_i \times Y_j$

for $-1 \leq i, j \leq n$, so it is enough to define a map $K_i \times Y_j \to S_n$ for any i, j in this range. We have $Y = (\mathrm{colim}_{1\downarrow Y} D_Y)$, so in particular we have that

$$\begin{aligned} Y_j &= (\mathrm{colim}_{(l,y)\in 1\downarrow Y} \Delta^l)_j \\ &= \mathrm{colim}_{(l,y)\in 1\downarrow Y} (\Delta^l)_j \end{aligned}$$

and in sets we know that

$$K_i \times \mathrm{colim}_{(l,y)\in 1\downarrow Y} (\Delta^l)_j = \mathrm{colim}_{(l,y)\in 1\downarrow Y} K_i \times (\Delta^l)_j$$

So we just have to produce maps $K_i \times (\Delta^l)_j$ for all such simplicies in the image of the diagram D_Y. But $K_i \times (\Delta^l)_j$ are exactly the n-simplicies of $K \star \Delta^l$, when i, j run through $-1, \dots n$, so the maps similar to f or g in the above diagram, from the appropriate source in the image of D_Y, will work. Remember that this diagram was for any morphism in the image of D_Y. □

Definition 1.2.7. *Let Q be a quasi-category, K be a simplicial set, and $p\colon K \to Q$ a simplicial map. We define the* under-quasi-category*, to be the simplicial set $Q_{p/}$.*

Dually we can define over-quasi-categories, $Q_{/p}$ by replacing $K \star Y$ with $Y \star K$ in the previous proposition and definition. As the choice of name suggests, these simplicial sets are indeed quasi-categories. To show this we need a bit more machinery, which will be introduced in section 1.3.

1.2.3 Colimit for quasi-categories

Definition 1.2.8 (Initial object)**.** *Let Q be a simplicial set, and let $q \in Q$ be a vertex. We say that q is an* initial object *of Q if q is an initial object in hQ.*

For this definition to make sense we should say what $\mathrm{h}S$ is, for a simplicial set S. Define therefor $\mathrm{h}S = \mathrm{h}\mathfrak{C}[S]$, the homotopy category for the simplicial category $\mathfrak{C}[S]$ defined in section 1.4.1. This homotopy category is obtained by taking π_0 of all morphism spaces (once defined, that is).

Definition 1.2.9 (Strongly initial object)**.** *Let S be a simplicial set, and $x \in S$ a vertex. We say that x is* strongly initial *if the projection $S_{x/} \to S$ is a trivial fibration of simplicial sets.*

In this definition we identify $x \in S$ with the inclusion $x\colon \Delta^0 \hookrightarrow S$ of the vertex x into S. We will define the notion of (trivial) fibrations in section 1.3.

Definition 1.2.10 (Colimit for quasi-categories)**.** *Let Q be a quasi-category, and $p\colon K \to Q$ a simplicial map. The* colimit *of p is then a strongly initial object of $Q_{p/}$.*

Note that this is not the definition given in [Lur09](definition 1.2.13.4), but due to [Lur09] corollary 1.2.12.5, we can take this as a definition, when working with quasi-categories.

Dually we may define a *final object* of Q to be a final object of $\mathrm{h}Q$, a *strongly final* object, as a vertex of $y \in S$ for which the projection $S_{/y} \to S$ is a trivial fibration, and a *limit* for p to be a strongly final object of $Q_{/p}$.

1.3 Fibrations

Here we shall briefly introduce some basic notions of fibrations of simplicial sets. The types that will be of use to us are the trivial fibrations and the right fibrations. A few other are listed for comparison.

Definition 1.3.1 (Fibrations). *Let $p\colon S \to X$ be a map of simplicial sets. We say that p is a*

a) trivial *fibration if*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & S \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

there exists an extension as indicated, such that the diagram commutes.

b) Kan fibration *if for $0 \leq i \leq n$*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & X \end{array} \tag{1.1}$$

there exist an extension as indicated, such that the diagram commutes.

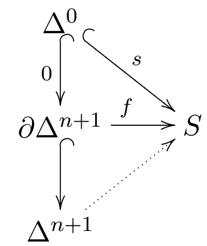
c) left fibration *if the extension in* (1.1) *exists for $0 \leq i < n$*

d) right fibration *if the extension in* (1.1) *exists for $0 < i \leq n$*

e) inner fibration *if the extension in* (1.1) *exists for $0 < i < n$*

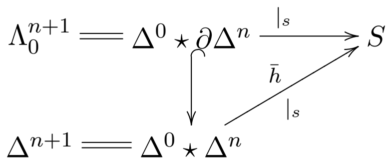
Note that $a) \Rightarrow b) \Rightarrow c), d) \Rightarrow e)$, that $c) \wedge d) \Rightarrow b)$. Also, knowing what a trivial fibration is, we may now rephrase our definition of a colimit in quasi-categories.

Lemma 1.3.2. *A vertex s of a simplicial set S is strongly initial if and only if for any map f as in the diagram*

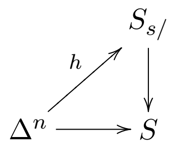


there exists an extension as indicated, such that the diagram commutes. Here 0 denotes the inclusion of the zero'th vertex, and s denotes the inclusion of the vertex s.

Proof. Suppose $s \in S$ is strongly initial. Then in particular we get a commutative diagram



On the other hand we also know that the adjoint map with respect to the join, h makes the following diagram commute



which gives us the last face of Λ_0^{n+1} above, since an n-simplex in $(S_{p/})_n = \mathrm{Hom}_s(\Delta^0 \star \Delta^n, S)$ is an n-simplex in S lying under s, the image of the 0'th vertex in Λ_0^{n+1}. So we get a map $\Delta^{n+1} \to S$, with the 0'th vertex mapping to s, given that we have both a map from the Λ_0^{n+1}, which maps the 0'th vertex to s, and a map from an n-simplex to the (missing) face opposite s in this horn. This is evidently a lift, given a map from the boundary as we wanted.

The other implication is just trailing this argument backwards. □

The notions of fibrations are introduced partly as a tool to check that the definitions given in the previous section can be justified, and partly for the next chapter where they will play a central role. First we set out to prove the following

Proposition 1.3.3. *Let Q be a quasi-category, K a simplicial set, and $p\colon K \to Q$ a simplicial map. The simplicial set $Q_{p/}$ is a quasi-category.*

Following [Lur09], we can obtain this as a corollary of the following

Proposition 1.3.4. *Let*

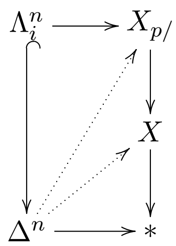
$$K_0 \subseteq K \xrightarrow{p} X \xrightarrow{q} S$$

be a diagram of simplicial sets, with q an inner fibration. Let $r = q \circ p$, $p_0 = p|_{K_0}$, and $r_0 = r|_{K_0}$. The induced map

$$X_{p/} \longrightarrow X_{p_0/} \times_{S_{r_0/}} S_{r/}$$

is a left fibration.

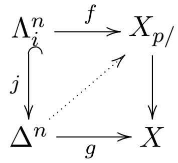
proof of 1.3.3. In the terminology of 1.3.4, choose $K_0 = \emptyset$ and $S = *$. Then we get a left fibration $X_{p/} \to X \times_* * = X$, and when X is a quasi-category, we get the following lifts for $0 < i < n$



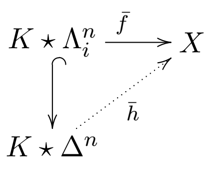
The bottom one from X being a quasi-category, and the top one from $X_{p/} \to X$ being a left fibration, and thus in particular an inner fibration. Thus $X_{p/}$ is a quasi-category. □

We will not prove proposition 1.3.4 in general here, but only in the case needed for proposition 1.3.3.

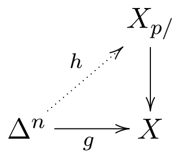
Proof. As established above we should verify that the following lift exists for $0 \leq i < n$



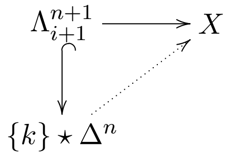
The given data can be reinterpreted by the adjoint relation for joins as follows. We are given a map $\bar{f}\colon K \star \Lambda^n_i \xrightarrow{|_p} X$ restricting to p on K, and a map $g\colon \Delta^n \to X$ such that $\bar{f}|_{\Lambda^n_i} = g \circ j$. Following this interpretation, we want a map $\bar{h}$ such that the following diagram commutes



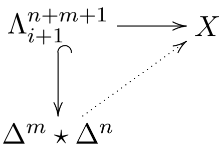
and such that the adjoint map h makes the following diagram commute



To illuminate the task at hand, suppose that K is just a single vertex k, in which case the given data is just that of a map $\Lambda^{n+1}_{i+1} \to X$ (where g supplies the i'th face), such that k is the 0'th simplex of the horn. Then the lift exists in



since $0 < i+1 < n+1$, $\{k\} \star \Delta^n$ is a Δ^{n+1}, and X is a quasi-category. Completely similarly if $K = \Delta^m$ for any $m \geq 0$, the data given is that of a map $\Lambda^{n+m+1}_{i+1} \to X$ such that K is the first m-simplex in this horn. Then the lift exists in



since $0 < i+1 < m+n+1$, $\Delta^m \star \Delta^n$ is a Δ^{m+n+1}, and X is a quasi-category. It is obvious that the adjoint map to this lift satisfies the diagram it has to.

Since the join preserves colimits (cf. proof of proposition 1.2.6), it exists for any simplicial set K. □

As an example of a fibration we may consider

Example 1.3.5 Let X, Y be based Kan complexes. The evaluation at the base point $ev\colon \mathrm{Map}(X, Y) \to Y$ is a Kan fibration, in particular it is a left and right fibration. This is a classical result, but will not show it here since it is easiest done using other tools than developed here.

1.4 Straightening and unstraightening

In this section we will define a pair of functors St_S, Un_S associated to a simplicial set S, called respectively the straightening and unstraightening functor. It can be shown that this is a Quillen adjunction, and even a Quillen equivalence, but we shall not do that here, as we will steer clear of model category theory.

First step in this is to define simplicial categories, and a way of passing between these and simplicial sets.

1.4.1 Simplicial categories

Definition 1.4.1 (Simplicial categories)**.** *Let $\mathcal{S}$ be a category. We say that $\mathcal{S}$ is a* simplicial category *if the set of morphisms $Mor_{\mathcal{S}}(a, b)$ comes equipped with a simplicial structure for all objects $a, b \in ob\mathcal{S}$.*

We shall define a functor $\mathfrak{C}\colon \text{sSet} \to \text{sCat}$. First we define $\hat{\mathfrak{C}}$ on the full subcategory generated by the representables $\Delta^n \in \text{sSet}$, denoted here by $\hat{\Delta}$.

There is a canonical identification between the representable simplicial set Δ^n and the category $[n]$, for any n, given by the nerve and τ functors. Due to this we shall divert slightly from the definition in [Lur09], and treat Δ^n as a linearly ordered set.

Definition 1.4.2. *We define $\hat{\mathfrak{C}}[\Delta^n]$ to be the simplicial category with objects $ob[n]$, and morphisms*

$$Mor_{\hat{\mathfrak{C}}[\Delta^n]}(i, j) = \begin{cases} \emptyset & i > j \\ N(P_{i,j}) & i \leq j \end{cases}$$

where $P_{i,j}$ is the poset $\{\Delta^m \subset \Delta^n \mid i, j \in \Delta^m, \forall k \in \Delta^m : i \leq k \leq j\}$ ordered by inclusion of subsets. Finally if $i \leq j \leq k$, then the composition

$$Mor_{\hat{\mathfrak{C}}[\Delta^n]}(i, j) \times Mor_{\hat{\mathfrak{C}}[\Delta^n]}(j, k) \to Mor_{\hat{\mathfrak{C}}[\Delta^n]}(i, k)$$

is induced by the the map of posets

$$\begin{aligned} P_{i,j} \times P_{j,k} &\to P_{i,k} \\ (\Delta^p, \Delta^q) &\mapsto \Delta^p \cup \Delta^q \end{aligned}$$

Note that this also defines identities, as there is only a single 0-simplex of morphisms $i \leq i$, and so we may think of $\hat{\mathfrak{C}}$ as sending the vertex $i \subset \Delta^n$ to the object $i \in \text{ob}[n]$. In order to get a functor $\hat{\mathfrak{C}}$ from this we should show that $\hat{\mathfrak{C}}[\Delta^n]$ is functorial, i.e. simplicial maps are taken to simplicial functors.

Definition 1.4.3. *If $f \in Hom_{\hat{\Delta}}(\Delta^n, \Delta^m)$, we define $\hat{\mathfrak{C}}[f](i) = f(i) \in ob\hat{\mathfrak{C}}[\Delta^m]$ on objects, and on morphisms we define*

$$\hat{\mathfrak{C}}[f](i \leq j) = Mor_{\hat{\mathfrak{C}}[\Delta^n]}(i, j) \xrightarrow{N(\bar{f})} Mor_{\hat{\mathfrak{C}}[\Delta^m]}(f(i), f(j))$$

where $\bar{f}\colon P_{i,j} \to P_{f(i),f(j)}$ given by $\Delta^n \mapsto f(\Delta^n)$.

We now verify that

Proposition 1.4.4. *The assignment $\Delta^n \mapsto \hat{\mathfrak{C}}[\Delta^n]$ is functorial.*

Proof. This amounts to checking that $\hat{\mathfrak{C}}[f]$ is a functor of simplicial categories. This is more or less obvious from the definiton, however we here check that $\hat{\mathfrak{C}}$ respects composition in f, i.e. that for $\Delta^n \xrightarrow{g} \Delta^m \xrightarrow{f} \Delta^l$, the simplicial functors $\hat{\mathfrak{C}}[f] \circ \hat{\mathfrak{C}}[g]$ and $\hat{\mathfrak{C}}[f \circ g]$ are naturally isomorphic. As the nerve is functorial, we have

$$N(\overline{f \circ g}) = N(\bar{f} \circ \bar{g}) = N(\bar{f}) \circ N(\bar{g})$$

So the identity morphisms provide a natural isomophism of simplicial functors. □

Now we proceed to extend the functor $\hat{\mathfrak{C}}$, to the following

Proposition 1.4.5 (The functor $\mathfrak{C}$). *Let S be a simplicial set. The assignment*

$$S \mapsto \mathfrak{C}[S] := \mathrm{colim}_{1 \downarrow S}(\hat{\mathfrak{C}} \circ D_S)$$

defines a functor $\mathfrak{C}$: sSet $\to$ sCat, which agrees with the functor $\hat{\mathfrak{C}}$ previously defined on $\hat{\Delta}$.

Note that we follow the usual convention regarding colimits in categories, i.e. that we pass to simplicial sets by the nerve, then form the colimit and finally apply the adjoint to the nerve τ, to get back to categories. In our case the nerve takes us to bisimplicial sets (which also has colimits), remembering the structure on morphisms, and the adjoint τ is then back to sCat. So the formula for $\mathfrak{C}[S]$ actually reads

$$\mathfrak{C}[S] = \tau\left(\mathrm{colim}_{1 \downarrow S}(N(\hat{\mathfrak{C}} \circ D_S))\right)$$

For the proof we shall need a few lemmas.

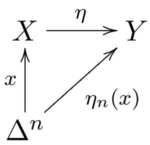
Lemma 1.4.6. *The construction $1 \downarrow F$ depends functorially on F. I.e. if $F, G\colon \mathcal{C} \to \mathcal{E}$ are two functors, then a natural transformation $\eta\colon F \to G$, induces a functor $E\colon 1 \downarrow F \to 1 \downarrow G$.*

Proof. Let $\eta\colon F \to G$ be such a natural transformation. Now define a functor $E\colon 1 \downarrow F \to 1 \downarrow G$ on objects by $E(c, x) = (c, \eta_c(x))$. Now let $f\colon (c, x) \to (c', x')$ be a morphism of $1 \downarrow F$, and define $E(f) = f$. This is indeed a morphism of $1 \downarrow G$, since $G(f)(\eta_c(x)) = \eta_{c'}(F(f)(x)) = \eta_{c'}(x')$ by naturality of η, and using that f is a morphism of $1 \downarrow F$.

It is easily seen that identities are preserved, and compositions in η are respected. □

Lemma 1.4.7. *Let $\eta\colon X \Rightarrow Y$ be a natural transformation of simplicial sets, inducing a functor $E\colon 1 \downarrow X \to 1 \downarrow Y$. Then there exists a natural isomorphism $\bar{\eta}\colon D_X \Rightarrow D_Y \circ E$ of functors $1 \downarrow X \to$ sSet.*

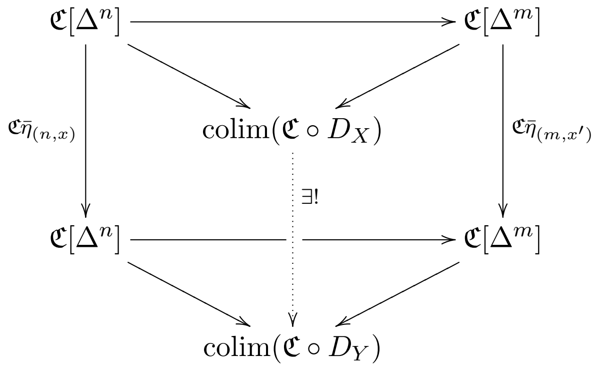
Proof. First note that for $x \in X(n)$ the triangle



with the natural transformations x and $\eta_n(x)$ given by the Yoneda lemma, commutes. This is since these transformations are given by where they map $id_n \in \mathrm{Hom}_{\Delta^{op}}(n, n)$.

Now, define the natural transformation $\bar{\eta}$, by letting the components $\bar{\eta}_{(n,x)} \colon \mathrm{Hom}_{\Delta^{op}}(n, -) \to \mathrm{Hom}_{\Delta^{op}}(n, -)$ be the identity map. This is of course natural. □

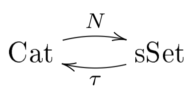
proof of proposition 1.4.5. By lemmas 1.4.6 and 1.4.7, a map of simplicial sets $\eta \colon X \to Y$ gives us a natural transformation $\bar{\eta} \colon D_X \Rightarrow D_Y \circ E$. We now verify that this induces a map $\mathrm{colim}(\mathfrak{C} \circ D_X) \to \mathrm{colim}(\mathfrak{C} \circ D_Y)$. Consider the following diagram



Here the top triangle is a cocone for $\mathrm{colim}(\mathfrak{C} \circ D_X)$, and the bottom triangle is a cocone for $\mathrm{colim}(\mathfrak{C} \circ D_X)$. By the universal property of $\mathrm{colim}(\mathfrak{C} \circ D_X)$, there exists a unique map as indicated, depending only on $\bar{\eta}$.

This construction is easily seen to respect composition in η, and indentities. Finally we check that this is actually an extension of the $\mathfrak{C}$ defined on $\hat{\Delta}$, i.e. that for $\Delta^n \in \hat{\Delta}$ we have $\mathfrak{C}[\Delta^n] = \mathrm{colim}(\mathfrak{C} \circ D_{\Delta^n})$. But this follows from the fact that Δ^n is terminal in D_{Δ^n}. □

As mentioned above there is an adjunction



and by uniqueness (up to natural isomorphism) of adjoints, and the result 1.2.5, the following bijection characterizes the nerve functor completely

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, N(\mathcal{C})) \simeq \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$$

Now, the functor $\mathfrak{C}$ is defined as a colimit, so it commutes with colimits (in the sence described above). By Freyd's adjoint functor theorem it then has a right adjoint. This, together with the above characterization of the nerve, motivates the next definition

Definition 1.4.8 (Coherent nerve)**.** *Let $\mathcal{S}$ be a simplicial category. We define the* coherent nerve *of $\mathcal{S}$ to be the simplicial set $N(\mathcal{S})$, such that*

$$Hom_{sSet}(\Delta^n, N(\mathcal{S})) \simeq Hom_{sCat}(\mathfrak{C}[\Delta^n], \mathcal{S})$$

From this point we shall always mean coherent nerve whenever we say nerve, or write $N(-)$. Finally we will end this section with a short remark.

Remark 1.4.9 The category sSet can, in a natural way be thought of as a simplicial category. Indeed the set of simplicial maps between to simplicial sets $\mathrm{Hom}_{\mathrm{sSet}}(S,T)$, carries a simplical structure. This is just the usual structure on $\mathrm{Map}(S,T)$, where

$$\mathrm{Map}(S,T)_n = \mathrm{Hom}_{\mathrm{sSet}}(S \times \Delta^n, T)$$

and the structure maps are induced by those on $\Delta^{\cdot}$, thought of as a cosimplicial space.

Next step is to define the two functors. We do not need the full machinery developed in [Lur09] here, so we will simplify matters slightly. Notably we will only deal with the case where ϕ is the identity (with the notation of [Lur09], cf. the introduction of section 2.2.1 in this reference).

1.4.2 Straightening

Let S be a simplical set, and let X be an object in sSet/S. Denote by $X^{\triangleright}$ the *right cone* $X \star \Delta^0$, and let v denote the cone point in $X^{\triangleright}$. Now consider the simplicial category

$$\mathcal{M}_X = \mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} \mathfrak{C}[S]^{op}$$

Note that we may as also obtain this by taking the similar pushout in sSet, and then apply $\mathfrak{C}$ to the pushout.

To the simplicial set X, we can assign a simplicial functor

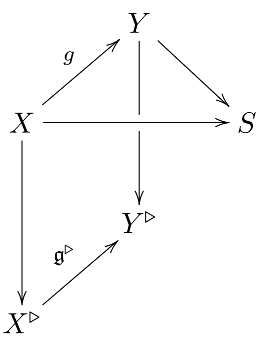
$$St_S(X)\colon \mathfrak{C}[S]^{op} \to \mathrm{sSet}$$

which is defined on objects by $St_S(X)(s) = \mathrm{Map}_{\mathcal{M}}(s,v)$. On a morphism $f \in \mathrm{Map}_{\mathfrak{C}[S]^{op}}(s,s')$ it is the natural transformation $St_S(X)(f)\colon \mathrm{Map}_{\mathcal{M}_X}(s,v) \to \mathrm{Map}_{\mathcal{M}_X}(s',v)$ given by pre-composing with f.

Of course we should check that this is functorial in X.

Proposition 1.4.10. *Let $g\colon X \to Y$ be a map of simplicial sets over S. This induces a functor of simplicial categories $\mathcal{M}_g\colon \mathcal{M}_X \to \mathcal{M}_Y$. Further, for any $s \in S$ we get a map of morphism spaces $m_g^s\colon Map_{\mathcal{M}_X}(s, v_X) \to Map_{\mathcal{M}_Y}(s, v_Y)$.*

Proof. Let $g^{\triangleright}\colon X^{\triangleright} \to Y^{\triangleright}$ be the unique extension of g to the cones on X and Y. Then we have a commutative diagram in simplicial sets as follows



(1.2)

which induce a morphism of the colimits. Then applying $\mathfrak{C}$ yields the map $\mathcal{M}_g\colon \mathcal{M}_X \to \mathcal{M}_Y$. For $s \in S$ we then have that $\mathcal{M}_g(s) = s$ since the upper triangle of (1.2) commutes. Also $\mathcal{M}_g(v_X) = v_Y$ where v_X, v_Y are the respective cone points, since $\mathfrak{C}[g^{\triangleright}](v_X) = v_Y$. Thus $\mathcal{M}_g$, being a functor of simplicial categories, in particular gives a map of the morphisms spaces, $m_g^s\colon \mathrm{Map}_{\mathcal{M}_X}(s, v_X) \to \mathrm{Map}_{\mathcal{M}_Y}(s, v_Y)$ as we wanted. □

By this, we may define $St_S(g)_s = m_g^s$, and conclude that $St_S\colon \mathrm{sSet}/S \to \mathrm{sSet}^{\mathfrak{C}[S]^{op}}$ is a functor. This functor commutes with colimits, since both $\mathcal{M}_-$ and $\mathrm{Map}(-, v)$ does so, and therefor it has an adjoint.

1.4.3 Unstraightening

Define $Un_S\colon \mathrm{sSet}^{\mathfrak{C}[S]^{op}} \to \mathrm{sSet}/S$, to be the right adjoint of St_S. We may think of unstraightening as something related to the Grothendieck construction, but this is not very precise.

Recall the classic Grothendieck construction for a functor from an ordinary category into the category of small categories, $F\colon \mathcal{D} \to \mathrm{Cat}$. This construction is again a category which we denote by $\mathcal{D} \int F$, and which is given as follows. The objects of $\mathcal{D} \int F$ are pairs (d, x) with $d \in \mathcal{D}$ and $x \in F(d)$. The morphisms are given by

$$\begin{aligned}&\mathrm{Mor}_{\mathcal{D} \int F}((d, x), (d', x')) \\ &\qquad = \{(\alpha, \beta) \mid \alpha \in \mathrm{Mor}_{\mathcal{D}}(d, d'),\, \beta \in \mathrm{Mor}_{F(d')}(F(\alpha)(x), x')\}\end{aligned}$$

and given

$$\begin{aligned}&(\alpha, \beta) \in \mathrm{Mor}_{\mathcal{D} \int F}((k, x), (k', x')) \quad \text{and} \\ &(\alpha', \beta') \in \mathrm{Mor}_{\mathcal{D} \int F}((k', x'), (k'', x''))\end{aligned}$$

we define composition in $\mathcal{D} \int F$ by

$$(\alpha', \beta') \circ (\alpha, \beta) = (\alpha' \circ \alpha, \beta' \circ F(\alpha')(\beta))$$

Also recall that the Grothendieck construction is functorial in F, and that there is a forgetful functor $\mathcal{D} \int F \to \mathcal{D}$. This should been seen as an analogue to the unstraightening giving us a simplcial set over another, and we may think of unstraightening as a appropriately weakened version of the Grothendieck construction.

1.4.4 Straightening over a point

Let us try to understand the simplest case of these two functors. Let $S = *$ be a single point, and consider sSet$/*$ which is just sSet since $*$ is terminal in sSet. Then the explicit formula for straightening over S applied to this single object of $\mathfrak{C}[S]$ reads

$$St_S(X)(*) = \mathrm{Mor}_{\mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} \mathfrak{C}[S]}(*, v)$$

The functor $St_S(X) \colon \mathfrak{C}[S] \to$ sSet may then be identified with this simplicial set, since there is only the identity morphism. We now claim that this simplicial set is the same as the geometric realization of X with respect to the cosimplicial object $Q^\bullet$ of sSet, which we define as follows

$$Q^n = \mathrm{Mor}_{\mathfrak{C}[(\Delta^n)^{\triangleright}] \coprod_{\mathfrak{C}[\Delta^n]} \mathfrak{C}[*]}(*, v)$$

Taking the geometric realization with respect to $Q^\bullet$ (formally a coend construction), can be written

$$|X|_{Q^\bullet} = \coprod_{n \geq 0} X_n \times Q^n / \sim$$

where we identify in the usual way.

Evidently, the two constructions are the same in the case of X being a simplex. Since they preserve colimits, they are equal for all simplicial sets. The functor $| - |_{Q^\bullet}$ is thus naturally seen as being from sSet to itself, and preserving colimits has an adjoint. This adjoint of is then a special case of unstraightening, and will be denoted by $\mathrm{Sing}_{Q^\bullet}$. This functor is given by the formula

$$\mathrm{Sing}_{Q^\bullet}(X)_n = \mathrm{Hom}_{\mathrm{sSet}}(Q^n, X)$$

but we shall not elaborate on this, since we will only have to use its properties as an adjoint.

We now have most of our notation in place for chapter 2, and by that we finish this chapter.

An application in topology

In this chapter we shall have to apply a few results that we do not have room to develop in this project. We will motivate these with reference to the results from the previous chapter, and they can all be found in the later parts of our main reference [Lur09], where they are developed as a natural extension of what we have done in chapter 1.

2.1 A Quillen equivalence

We record the results we need here.

Theorem 2.1.1. *Given a simplicial set Y,*

$$St_Y\colon sSet/Y \rightleftarrows sSet^{\mathfrak{C}[Y]^{op}}\colon Un_Y$$

is a Quillen equivalence, where $sSet/Y$ is given the contravariant model structure, and $sSet^{\mathfrak{C}[Y]^{op}}$ is given the projective model structure.

Both model structures are described in [Lur09], and the theorem follows directly from [Lur09] proposition 2.2.1.2. We note that the fibrant objects in these structures are respectively right fibrations, and functors taking values in Kan copmlexes (fibrant objects of the codomain). More generally the projective model structure is defined by letting weak equivalences and fibrations be given pointwise.

We note that $|-|_{Q^\bullet}, \mathrm{Sing}_{Q^\bullet}$ also is a Quillen equivalence.

Proposition 2.1.2. *Consider a Quillen equivalence $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$. Then the following holds*

a) Let $\alpha\colon C \xrightarrow{\simeq} C'$ be a weak equivalence in $\mathcal{C}$, with C cofibrant. Then $F(\alpha)\colon F(C) \xrightarrow{\simeq} F(C')$ is a weak equivalence, and $F(C)$ is cofibrant.

b) Let $\beta\colon D \xrightarrow{\simeq} D'$ be a weak equivalence in $\mathcal{D}$ with D' fibrant. Then $G(\beta)\colon G(D) \xrightarrow{\simeq} G(D')$ is a weak equivalence, and $G(D')$ is fibrant.

2.2 Straightening evaluated on an object

In this section, we will establish

Proposition 2.2.1. *Let $f\colon X \to S$ be a right fibration and $s \in S$ a vertex. There is a weak equivalence $|X_s|_{Q^\bullet} \xrightarrow{\simeq} St_S(f)(s)$*

Proof. Choose a fibrant replacement for $St_S(f)$ to get a weak equivalence $St_S(f) \xrightarrow{\simeq} \overline{St_S(f)}$, where $\overline{St_S(f)}$ denotes the fibrant object. Then the adjoint map which factors trough the unit for the adjunction is a weak equivalence

$$\begin{array}{ccc} f & \xrightarrow{\qquad\simeq\qquad} & Un_S(\overline{St_S(f)}) \\ & {\scriptstyle\text{unit}}\searrow \quad \nearrow & \\ & Un_S \circ St_S(f) & \end{array}$$

Also by 2.1.2 b) the right map in the diagram is a weak equivalence to a fibrant object, and by assumption f is a right fibration. Thus [Lur09] proposition 2.2.3.13 tells us that we get a weak equivalence of fibres

$$\begin{array}{ccc} X_s & \xrightarrow{\qquad\simeq\qquad} & (Un_S(\overline{St_S(f)}))|_s \\ & \searrow \quad \nearrow & \\ & (Un_S \circ St_S(f))|_s & \end{array}$$

factoring as the diagram suggests. [Lur09] remark 2.2.2.11 gives an isomorphism $Un_S(f)|_s \simeq \mathrm{Sing}_{Q^\bullet} f(s)$, so we get the following diagram

$$\begin{array}{ccccc} |X_s|_{Q^\bullet} & \longrightarrow & |\mathrm{Sing}_{Q^\bullet} St_S(f)(s)|_{Q^\bullet} & \longrightarrow & |\mathrm{Sing}_{Q^\bullet} \overline{St_S(f)}(s)|_{Q^\bullet} \\ & {\scriptstyle\simeq}\searrow & \downarrow & & \downarrow{\scriptstyle\simeq} \\ & & St_S(f)(s) & \xrightarrow[\simeq]{} & \overline{St_S(f)}(s) \end{array}$$

The vertical maps are given by the counit for the $\mathrm{Sing}_{Q^\bullet}, |-|_{Q^\bullet}$ adjunction, and the bottom map is the fibrant replacement we began with taken pointwise. The composite map in the top is still a weak equivalence by 2.1.2 a), the right map is a weak eequivalence since $\overline{St_S(f)}(s)$ is fibrant, and the bottom map is by definition a weak equivalence. Thus by the 2 out of 3 property for weak equivalences, first the composite over the left counit map from $|X_s|_{Q^\bullet}$ to $\overline{St_S(f)}(s)$ is a weak equivalence, and hence the left diagonal map is too. □

Now let X, Y be a Kan complexes such that $X_0 = Y_0 = *$, and consider the evaluation map $ev\colon \mathrm{Map}(X, Y) \to Y$. We saw in 1.3.5 that ev is a fibrant object of sSet$/Y$, and by 2.2.1 there is a weak equivalence $|\mathrm{Map}_*(X, Y)|_{Q^\bullet} \xrightarrow{\simeq} St_Y(ev)(*)$. By [Lur09] proposition 2.2.2.7 there is also a weak equivalence $|\mathrm{Map}_*(X, Y)|_{Q^\bullet} \xrightarrow{\simeq} \mathrm{Map}_*(X, Y)$, and we have then established

Proposition 2.2.2. *There is a zig-zag of weak equivalences such that*

$$Map_*(X,Y) \simeq St_Y(ev)(*)$$

2.3 Unstraightening as colimit

Since straightening and unstraightening is a Quillen equivalence, we know that $f \simeq Un_S(St_S(f))$ for a right fibration $f\colon X \to S$. In the previous section we succesfully identified $St_Y(f)$. The objective of this section is then to recognize $Un_Y(F)$ as something we know, for a fibrant functor $F\colon \mathfrak{C}[Y]^{op} \to \mathrm{sSet}$ of simplicial categories. In the model structure we work with, F being fibrant simply means that it takes values in Kan complexes.

Unwinding the definitions, one interpretation is given in [Lur09] proposition 3.3.4.6. We will not show it here, but the proposition states that we may identify the unstraightening functor with taking the colimit for quasi-categories constructed in chapter 1. Further this colimit may be identified with the ordinary homotopy colimit. We wont show this either, but we can motivate it as follows.

Thomasson's theorem relates hocolim to the ordinary Grothendieck construction

Theorem 2.3.1 (Thomasson's theorem)**.** *Let $\mathcal{K}$ be a small category, $F\colon \mathcal{K} \to Cat$ a functor, and consider the composition*

$$\mathcal{K} \xrightarrow{F} Cat \xrightarrow{N} sSet$$

where N denotes the regular nerve. Now there exists a natural weak equivalence

$$\mathrm{hocolim}_{\mathcal{K}} NF \xrightarrow{\sim} N\left(\mathcal{K} \int F\right)$$

The previous discussion about unstraightening suggest that the right hand side of this weak equivalence is close to the unstraightening functor.

To further justify thinking of unstraightening as a homotopy colimit, we work through an example suggesting that homotopy colimit and our quasi-colimit are the same.

Example 2.3.2 Consider a simplicial group G and the simplicial set NG, the coherent nerve of G. We will begin to show that NG is the colimit of the simplicial map $p\colon NG \to \mathrm{Kan}$, given by mapping everything to the trivial Kan complex, which is a point. Here 'Kan' is to be understood as the coherent nerve applied to the simplicial category of Kan copmlexes as sitting inside the simplicial category sSet.

Note that classically BG is the homotopy colimit of the trivial functor $G \to \mathrm{Top}$ mapping everything to a point.

We should check that NG is a strongly initial object of $\mathrm{Kan}_{p/}$. Following 1.3.2 we check that for every $n \geq 0$, there is a lift in the following diagram, when the 0'th vertex of is mapped to NG.

$$\begin{array}{ccc} \partial\Delta^{n+1} & \longrightarrow & \mathrm{Kan}_{p/} \\ \downarrow & \nearrow & \\ \Delta^{n+1} & & \end{array}$$

By definition this is equivalent to a lift in

$$\begin{array}{ccc} NG \star \partial\Delta^{n+1} & \longrightarrow & \mathrm{Kan} \\ \downarrow & \nearrow & \\ NG \star \Delta^{n+1} & & \end{array}$$

with the same condition on the 0'th vertex of $\partial\Delta^n$. We analyse the case $n = 0$.

In the top left of the diagram we have two distinct right cones on NG, with the first having the cone point mapping to NG, and the second having the cone point mapping to some element K in Kan.

To get the lift we will first establish that the data of the map on the second cone, is that of a map $NG \to K$. An (non-degenerate) $(n+1)$-simplex in $NG \star \Delta^0$, is of the form (g_n, v_0), where v_0 is the vertex of Δ^0, and g_n is an n-simplex of NG, and is thus determined by g_n. The map gives an n-simplex $\tau_n \in \mathrm{Map}(*, K)_n$, for every such g_n, but this is exactly the data of an n-simplex in K. Thus we have produced a 1-simplex δ_1 in Kan, with boundary NG and K, which is the same as a 0-simplex in $\mathrm{Map}_{\mathrm{Kan}}(NG, K)$ the space of morphisms from NG to K in the simplicial category Kan. Note that this is true also for $K = NG$, where the we just get g_n back.

For a k-simplex $g_k \in NG_k$ we should now produce a $(k+2)$-simplex σ_{k+2} of Kan, which agrees with τ_k, g_k and δ_1 on the boundary. Now such a σ_{k+2} corresponds to an element of $\mathrm{Hom}_{\mathrm{sCat}}(\mathfrak{C}[\Delta^{k+2}], \mathrm{Kan})$ where we have specified how to map objects, and certain morphisms.

By construction of δ_1, we get a degenerate $(k+1)$-simplex in the space $\mathrm{Map}_{\mathrm{Kan}}(*, K)$, from the arrow τ_k to the composite g_k followed by δ_1, since (with slight abuse of notation) $\delta_1(g_k) = \tau_k$. By inspection, this is a $(k+2)$-simplex in Kan satisfying our boundary conditions.

This is seen to be a lift one simplex at a time in the diagram as we wanted. The cases for $n > 0$ becomes very cumbersome to check, so we will leave it at the case $n = 0$ for now.

It can be made more precise that these two colimit notions are in fact the same, but we shall not do that here.

2.4 Action of monoid on "fibre"

In this final section we shall summarize what we have done in this chapter, and state this as a theorem. From this we see a connection to an analogous theorem, for which a classical proof is supplied in the appendix for comparison.

Let X, Y be as in the previous section. The space $\mathrm{Map}_{\mathfrak{C}[Y]^{op}}(*,*)$ is a monoid, with composition given by composition of morphisms, and the identity morphism id_*, as the unit element.

This monoid acts on $\mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*,v)$, in the following way

$$\begin{array}{c} \mathrm{Map}_{\mathfrak{C}[Y]^{op}}(*,*) \times \mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*,v) \\ \big\downarrow \\ \mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*,*) \times \mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*,v) \\ \big\downarrow \circ \\ \mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*,v) \end{array}$$

First include, and then use the composition available in the simplicial category $\mathcal{M}_{\mathrm{Map}(X,Y)}$.

Applying the unstraightening functor corresponds to taking the homotopy colimit of the functor giving this action, and since we started out with a fibrant object, we get back an object in sSet/Y, weakly equivalent to ev. This may be stated as

Theorem 2.4.1. *Let X, Y be pointed Kan complexes with $X_0 = Y_0 = *$, and consider the fibration*

$$ev\colon Map(X,Y) \to Y$$

given by the evaluation map. There is a weak equivalence

$$Map(X,Y) \simeq Map_{\mathcal{M}_{Map(X,Y)}}(*,v) /\!\!/ Map_{\mathfrak{C}[Y]^{op}}(*,*)$$

We may think of this monoid as a weak version of ΩY, or maybe the Moore loops $\Omega_m Y$. This is the space of based maps from compact intervals of the non-negative real line of the form $[0,a]$, to Y, with the condition that $f(0) = f(a)$ for any $f \in \Omega_m Y$. This is also a monoid, by concatenation of paths and the constant path $0 \mapsto *$ as the unit element. The advantage over the regular loop space is that with the Moore loops, we do not have to reparametrize in order to get a monoid.

Consider the set of 0-simplicies for this monoid $\mathrm{Map}_{\mathfrak{C}[Y]^{op}}(*,*)_0$. This can be described as sequences of composable arrows in Y, i.e. the set of non-degenerate maps

$$\Delta^1 \times_{\Delta^0} \Delta^1 \times_{\Delta^0} \cdots \times_{\Delta^0} \Delta^1 \to Y$$

together with the degeneracy on a single 1-simplex into Y. This is itself a monoid in the obvious way, and we see that there is an inclusion of the 0-simplicies of the ordinary loopspace $(\Omega Y)_0 = \{\Delta^1 \to Y\} \subset \mathrm{Map}_{\mathfrak{C}[Y]^{op}}(*,*)_0$. The higher simplicies for both monoids are in some sense determined by the 0-simplicies. In ΩY we declare 0-simplicies to be equal if they bound a 1-simplex, where as in $\mathrm{Map}_{\mathfrak{C}[Y]^{op}}(*,*)$ the 1-simplicies can be thought of as homotopies between the bounding 0-simplicies. 2-simplicies are then homotopies between the bounding homotopies, and so on. Thus compared to the ordinary model for loop spaces we weaken the condition that simplicies bounding a higher simplex are equal, to that they agree up to homotopies which are determined up to homotopies, and so on.

Further we recall that $\mathrm{Map}_{\mathcal{M}_{\mathrm{Map}(X,Y)}}(*, v) \simeq \mathrm{Map}_*(X, Y)$, justifying the title of the section, even though the action does not necessarily transport. Then with this in mind, the special case where $X = S^1$ and $Y = BG$, theorem 2.4.1 looks a lot like the following

Theorem 2.4.2. *Let G be a simplicial group which is a Kan complex. There is a weak equivalence*

$$\mathit{Map}(S^1, BG) \simeq \mathit{Map}_*(S^1, BG)/\!\!/\Omega BG$$

We refer to the appendix for a proof of this theorem.

2.5 Final remark

There is a good chance of generalizing this much further, since investegating the evaluation map is just a worked example of the theory in [Lur09]. Any other right fibration might work just as well, and actually we don't even need right fibrations, but may look at the more general class of (co)cartesian fibrations. This will also get us from Kan complexes to quasi-categories in general. However, this is how far we got.

Appendix

A.1 A classical proof

Theorem A.1.1. *Let G be a simplicial group which is Kan. Then there is a weak homotopy equivalence*

$$G^{ad}/\!\!/G \simeq LBG$$

between the homotopy orbit space of G acting on itself by conjugation, and the free loop space on the classifying space of G.

Before we prove this we shall make some observations, and state a few lemmas. Throughout this appendix, G is a group as above.

We begin by noting that $G^{ad}/\!\!/G = EG \times_G G$ with the diagonal action on the right of both factors, and the adjoint action on the G-factor. Further we have that $LBG = \mathrm{Map}(S^1, BG)$, which has trivial G-action, and we denote by $p\colon EG \to BG$ the canonical fibration.

Lemma A.1.2. *The map $EG \times_G G \to BG$ induced by projecting off G, is a fibration with fibre G.*

The proof is omitted, but we refer to [May75] theorem 8.2, from which it can be shown that the map is actually a fibre bundle.

Lemma A.1.3. *Let $p\colon E \to B$ a be fibration, with E contractible and fibre F. Then there exist a weak homotopy equivalence $\Omega B \xrightarrow{\simeq} F$.*

Proof. By the Puppe sequence for fibrations we get

$$* \simeq \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \simeq *$$

Thus $\Omega B \to F$ is a weak homotopy equivalence by the long exact sequence for fibrations. □

Note that in particular we get a weak equivalence $G \simeq \Omega BG$.

Proof of theorem A.1.1. First choose the standard model for BG, the realization of the simplicial set with a single 0-simplex, and G worth of 1-simplicies. Choose the corresponding model for EG.

Let $\delta_e \colon I \to EG$ denote the path from $e \in EG$ to the basepoint of EG, which is the realization of the corresponding 1-simplex. Further let $g \in G$, and let $\gamma_g \in \Omega BG$ be the image of g under the weak equivalence from lemma A.1.3.

Define a map $q \colon EG \times G \to LBG$ by letting $q(e, g) = p(\delta_e) \cdot \gamma_g \cdot \overline{p(\delta_e)}$, where $\cdot$ denotes concatenation of paths, and the bar denotes inverse path. This is seen to be a well-defined loop in BG. We note that this is invariant for the G-action since for any $h \in G$

$$\begin{aligned} q(h.e, h.g) &= p(\delta_{eh}) \cdot \gamma_{hgh^{-1}} \cdot \overline{p(\delta_{eh})} \\ &= p(h\delta_e \cdot \delta_h) \cdot \overline{\gamma_h} \cdot \gamma_g \cdot \gamma_h \cdot \overline{p(h\delta_e \cdot \delta_h)} \\ &= p(\delta_e) \cdot \gamma_g \cdot \overline{p(\delta_e)} \\ &= q(e, g) \end{aligned}$$

Where we use that $\delta_{eh} = h\delta_e \cdot \delta_h$, and that $p(\delta_h) = \gamma_h$. Hence we get a map $\bar{q} \colon EG \times_G G \to LBG$. Now we claim that this is a weak homotopy equivalence.

Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\simeq} & \Omega BG \\ \downarrow & & \downarrow \\ EG \times_G G & \xrightarrow{\bar{q}} & LBG \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ BG & = & BG \end{array}$$

The bottom square obviously commutes, and the top square commutes since the loop given by $\bar{q}$ is constructed using exactly the top map. By lemma A.1.2 the left column is a fibration sequence, and the right column is clearly also a fibration sequence. Lemma A.1.3 gives the topmost weak equivalence, and by the five lemma we conclude that also $\bar{q}$ is a weak homotopy equivalence. $\square$

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