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The Dold-Kan Correspondence and Derived Functors of Non-additive Functors

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Abstract

In classical homological algebra one defines the derived functor of an additive covariant functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$. Our goal is to generalize this such that F need not be additive. In order to do this we introduce the ordinal number category Δ, the category of simplicial objects $s\mathscr{C}$ induced by the category $\mathscr{C}$, and we define the functors $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$ and $\Gamma : \mathbf{Ch}_+^\Lambda \to s\mathbf{Mod}_\Lambda$ which form an equivalence of categories called the Dold-Kan correspondence. We will use these functors to give a new definition of the derived functor of F which does not require F to be additive, and which coincides with the classical definition if F is additive. In the end we give some examples in which we apply the left derived functor of a non-additive functor.

Resumé

I klassisk homologisk algebra definerer man den differentierede funktor af en additiv kovariant funktor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$. Vores mål er at generalisere dette, således at F ikke behøver at være additiv. For at gøre dette introducerer vi ordinaltal-kategorien Δ, kategorien af simplicielle objekter $s\mathscr{C}$ induceret af en kategori $\mathscr{C}$, og vi definerer funktorerne $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$ og $\Gamma : \mathbf{Ch}_+^\Lambda \to s\mathbf{Mod}_\Lambda$, hvilke udgør en ækvivalens af kategorier kaldet Dold-Kan korrespondancen. Vi vil benytte disse funktorer til at give en ny definition af den differentierede funktor af F, som ikke kræver, at F er additiv, og som stemmer overens med den klassiske definition, hvis F er additiv. Til sidst giver vi nogle eksempler, hvor vi anvender den venstre differentierede funktor af en ikke-additiv funktor.

Contents

1 Introduction and Homological Algebra

1.1 Introduction

This project is inspired by the article [2], written by Albrecht Dold and Dieter Puppe in 1958. In the article they define a projective simplicial resolution of a Λ-module (Λ being a ring) and use the functor $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$, which we define in Section 2.3, to define derived functors of non-additive functors. We will go about this in a different way. We define the functor $\Gamma : \mathbf{Ch}_+^\Lambda \to s\mathbf{Mod}_\Lambda$ which together with N forms an equivalence of categories, the so called Dold-Kan correspondence (see Section 2.3), and which preserve homotopy (see Section 3.1). By using the fact that homotopy in $s\mathbf{Mod}_\Lambda$ does not depend on additivity, when applying a non-additive functor to a simplicial Λ-module, we still preserve homotopy. These facts combined allow us to define the derived functor of a non-additive functor.

The first main result of this project is Theorem 2.3.4, the Dold-Kan correspondence. In [4] they give a sketch of the proof and in [8] they call it an easy consequence. Section 2 is dedicated to defining this theorem, and giving a detailed proof.

The other main result of this project is Theorem 3.2.4 which shows that Definition 3.2.3 is a generalization of the classical derived functors studied in classical homological algebra. Section 3 is dedicated to showing that the Dold-Kan correspondence preserves homotopy, and applying this in order to give a generalized definition of derived functors. In the end we will give some examples of how to use the left derived functor of a non-additive functor.

1.2 Chain Complexes and Classical Derived Functors

Classical homological algebra deals with chain complexes of modules and derived functors of additive functors. We start out by recalling the relevant definitions and important theorems which can be found in [6]. In the following Λ will denote a unital ring (with non-zero unit), and we will be working over (left or right) Λ-modules.

Definition 1.2.1. A *graded* Λ*-module* $C_\bullet$ a collection $(C_n)_{n\in\mathbb{Z}}$ of Λ-modules. A *map* f *of degree* d between two graded modules $C_\bullet$ and $D_\bullet$, written $f : C_\bullet \to D_\bullet$, is a collection of homomorphisms $(f_n)_{n\in\mathbb{Z}}$ such that $f_n : C_n \to D_{n+d}$. If f has degree d we write $|f| = d$.

A *chain complex* $C_\bullet$ is a graded Λ-module together with a map $\partial : C_\bullet \to C_\bullet$ of degree -1 called the *differential*, such that $\partial_{n+1}\partial_n = 0$. We call $f : C_\bullet \to D_\bullet$ a *morphism* if $|f| = 0$ and f commutes with differentials, i.e.

if $f_n \partial_{n+1} = \partial_n f_n$. Note that here ∂_{n+1} is a differential in $C_\bullet$ and ∂_n is a differential in $D_\bullet$.

The category $\mathbf{Ch}^\Lambda$ is the category in which chain complexes of Λ-modules are the objects and the morphisms are morphisms between chain complexes. Furthermore if $C_\bullet$ is a chain complex where $C_n = 0$ for $n < 0$, we call $C_\bullet$ a *non-negative chain complex.* We let $\mathbf{Ch}_+^\Lambda$ denote the subcategory of $\mathbf{Ch}^\Lambda$ in which the objects are non-negative chain complexes of Λ-modules.

With these basic definitions in mind we can define what homology is. From now on $C_\bullet$ and $D_\bullet$ are chain complexes unless other is notet.

Definition 1.2.2. The *n'th homology module* of $C_\bullet$ is the module $H_n(C_\bullet) = \ker \partial_n / \mathrm{Im} \partial_{n+1}$ and $H(C_\bullet)$ is the graded Λ-module $(H_n(C_\bullet))_{n \in \mathbb{Z}}$. If $f : C_\bullet \to D_\bullet$ is a morphism of chain complexes let $H(f) = (H_n(f))_{n\in\mathbb{Z}} : H(C_\bullet) \to H(D_\bullet)$ be the induced map between graded Λ-modules of degree 0. This makes $H(-)$ into a functor. Moreover f is called a *quasi-isomorphism*, written $f : A_\bullet \xrightarrow{\sim} B_\bullet$, if $H_n(f) : H_n(A_\bullet) \to H_n(B_\bullet)$ is an isomorphism for every n.

Definition 1.2.3. Let $f, g : C_\bullet \to D_\bullet$ be morphisms. We say that f and g are *homotopic*, written $f \simeq g$, if there exists a map $\Sigma : C_\bullet \to D_\bullet$ of degree $+1$ such that

$$f_n - g_n = \partial_{n+1} \Sigma_n + \Sigma_{n-1} \partial_n.$$

Moreover we call Σ a *homotopy* from f to g and write $\Sigma : f \simeq g$.

Two chain complexes $C_\bullet, D_\bullet$ are said to be *homotopy equivalent* if there exist morphisms $f : C_\bullet \to D_\bullet$, $g : D_\bullet \to C_\bullet$ such that $gf \simeq id_{C_\bullet}$ and $fg \simeq id_{D_\bullet}$. Moreover the morphism f (and g) is called a *homotopy equivalence.*

It is now time to recall some important theorems from homological algebra. These all play an important part in defining derived functors.

Theorem 1.2.4. *Let $f, g : C_\bullet \to D_\bullet$ be morphisms. If $f \simeq g$ then $H(f) = H(g) : H(C_\bullet) \to H(D_\bullet)$.*

Corollary 1.2.5. *If $C_\bullet$ and $D_\bullet$ are homotopy equivalent and $f : C_\bullet \to D_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Theorem 1.2.6. *The homotopy relation "$\simeq$" is an equivalence relation in $\mathbf{Ch}^\Lambda$.*

Before stating the next theorem recall that if $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ is a (covariant) functor of modules and $C_\bullet$ is a chain complex, then there is an induced chain complex $FC_\bullet$ given by the collection of modules (FC_n) and the differentials $F\partial_n : FC_n \to FC_{n-1}$. Moreover if $f : C_\bullet \to D_\bullet$ is a morphism then $F(f)$ is the induced morphism where $F(f)_n = F(f_n) : FC_n \to FD_n$.

Theorem 1.2.7. *If $f \simeq g : C_\bullet \to D_\bullet$ and if $F : \boldsymbol{Mod}_\Lambda \to \boldsymbol{Mod}_{\Lambda'}$ is an additive functor, then $H(Ff) = H(Fg) : H(FC_\bullet) \to H(FD_\bullet)$*

Definition 1.2.8. Let $C_\bullet$ be a non-negative chain complex. Then $C_\bullet$ is called *projective* if C_n is projective for all $n \geq 0$, and $C_\bullet$ is called *acyclic* if $H_n(C_\bullet) = 0$ for $n \geq 1$. A projective and acyclic complex $P_\bullet$ is called a *projective resolution* of a Λ-module A if there exists an isomorphism $H_0(P_\bullet) \cong A$. Similarly we define a *free resolution.*

Theorem 1.2.9. *Let A be a Λ-module. Then A has a projective resolution. Moreover two rojective resolutions of A are homotopy equivalent.*

The next theorem is a generalization of [6] Theorem IV.4.1. A proof of this can be found in [7] Lemma 2.3.6.

Theorem 1.2.10. *Let $P_\bullet$ be a non-negative projective chain complex. Then every diagram*

$$\begin{array}{ccc} & & C_\bullet \\ & \overset{f}{\nearrow} & \downarrow{\scriptstyle \pi}\ \sim \\ P_\bullet & \xrightarrow{g} & D_\bullet \end{array}$$

where $C_\bullet$ and $D_\bullet$ are chain complexes, π is a surjective quasi-isomorphism and g is a morphism, there exists a morphism $f : P_\bullet \to C_\bullet$ such that $\pi f = g$. Furthermore f is unique up to homotopy.

Definition 1.2.11. Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be a functor. Then F is said to be *additive* if for any Λ-modules A and B and any homomorphisms $\varphi, \psi : A \to B$ then $F(\varphi + \psi) = F\varphi + F\psi$.

Theorem 1.2.12. *Let $F : \boldsymbol{Mod}_\Lambda \to \boldsymbol{Mod}_{\Lambda'}$ be a covariant functor. Then F is additive if and only if for any Λ-modules $A_1, \ldots, A_n$ the homomorphism $\langle F\iota_{A_i} \rangle_{i=1}^n : \bigoplus_{i=1}^n FA_i \to F(\bigoplus_{i=1}^n A_i)$ is an isomorphism, where ι_{A_j} is the inclusion map $A_j \to \bigoplus_{i=1}^n A_i$.*

Definition 1.2.13 (Classical Left Derived Functors)**.** Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be an additive (covariant) functor and define the n'th *left derived functor* of F, denoted L_nF, in the following way: let A be a Λ-module and $P_\bullet$ a projective resolution of A. Then let $L_nF(A) = H_nF(P_\bullet)$. Let $\varphi \in \mathrm{Hom}_\Lambda(A, B)$ and $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively. By considering A and B as chain complexes $A_\bullet$ and $B_\bullet$ where $A_0 = A$ and $A_n = 0$ for $n \neq 0$ and correspondingly for $B_\bullet$, Theorem 1.2.10 implies the existence of a morphism $f_\varphi : P_\bullet \to Q_\bullet$ which is unique up to homotopy. Then let $L_nF(\varphi) = H_nF(f_\varphi)$. This makes $L_nF : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ into a (covariant) functor.

Note that due to the above theorems $L_nF(A)$ does not depend (up to isomorphism) on the choice of projective resolution $P_\bullet$, and $L_nF(\varphi)$ does not depend (up to isomorphism) on the choice of projective resolutions $P_\bullet$ and $Q_\bullet$ or of the choice of $f_\varphi : P_\bullet \to Q_\bullet$.

Every definition and theorem above can be dualized. This gives rise to the n'th *right derived functor* $R^nF : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ of an additive functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ which is the dual definition of the left derived functor.

One of our goals in this project is to generalize the definition of left (and right) derived functors such that it is not necessary to assume that F is additive. But in order to do this we must introduce the Dold-Kan correspondence and find some nice properties of this correspondence.

2 The Dold-Kan Correspondence

2.1 The Ordinal Number Category

The Dold-Kan correspondence is an equivalence of categories between non-negative chain complexes and simplicial objects. In order to understand the Dold-Kan correspondence one must understand simplicial objects, and in order to understand simplicial objects one must understand the ordinal number category. In this section we define the ordinal number category and define the morphisms cofaces and codegeneracies. We formulate the cosimplicial identities and prove that every morphism has a unique factorization of an injective and a surjective morphism, called the epi-monic factorization.

Definition 2.1.1. The *ordinal number category* Δ is the category in which the objects are the totally ordered sets $[n] = (\{0, 1, \ldots, n\}, \leq)$ for any non-negative integer n, and the morphisms are the weakly order-preserving maps $\varphi : [n] \to [m]$. Moreover the *cofaces* are the morphisms $d^i : [n-1] \to [n]$, $0 \leq i \leq n$ given by

$$d^i(k) = \begin{cases} k & \text{for } k < i \\ k+1 & \text{for } k \geq i \end{cases}$$

and the *codegeneracies* are the morphisms $s^i : [n+1] \to [n]$, $0 \leq i \leq n$ given by

$$s^i(k) = \begin{cases} k & \text{for } k \leq i \\ k-1 & \text{for } k > i \end{cases}$$

Remark 2.1.2. Note that the coface $d^i : [n-1] \to [n]$ is the injective morphism which "skips" i in $[n]$, and the codegeneracy $s^i : [n+1] \to [n]$ is the surjective morphism where $s^i(i) = s^i(i+1) = i$. Furthermore for any injective morphism $\varphi : [n] \rightarrowtail [m]$ where $\varphi(k) = i_k$ it can easily be verified that $\varphi = d^m d^{m-1} \cdots d^{i_n+1} d^{i_n-1} \cdots d^{i_0+1} d^{i_0-1} \cdots d^1 d^0$ if $n < m$ and $\varphi = id$ if $n = m$. Thus any injective morphism which is not the identity is a composite of cofaces.

Theorem 2.1.3. *Any morphism $\varphi : [n] \to [m]$ can be written as a composite of cofaces and codegeneracies.*

Proof. We prove this by induction on n. Any morphism $\varphi : [0] \to [m]$ is injective for any m and thus a composite of cofaces or the identity by Remark 2.1.2. Assume that the assertion is true for n and let $\varphi : [n+1] \to [m]$. If φ is injective the result follows from Remark 2.1.2. If φ is not injective there exists a $k \in [n+1]$ such that $\varphi(k) = \varphi(k+1)$. Define the morphism $\varphi' : [n] \to [m]$ by $\varphi'(i) = \varphi(i)$ for $i \leq k$ and $\varphi'(i) = \varphi(i+1)$ for $i > k$. Then $\varphi = \varphi' s^k$ and since φ' can be written as a composite of cofaces and codegeneracies, so can φ. □

Due to this theorem whenever we look at something related to the morphisms of Δ it suffices to look at the cofaces and codegeneracies. This will be very useful later on. The following lemma, the cosimplicial identities, will be used often.

Lemma 2.1.4 (The Cosimplicial Identities)**.** *In the ordinal number category Δ the following identities called the* cosimplicial identities *hold:*

$$\begin{array}{ll} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = id = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{array}$$

Proof. We only prove the first identity. The rest are proved in a similar facion.

Let $i < j$ and $k \in [n]$. Then

$$d^j d^i(k) = d^j \left(\left\{ \begin{array}{ll} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{array} \right\} \right) = \left\{ \begin{array}{ll} k & \text{if } k < i \\ k+1 & \text{if } i \leq k, k+1 < j \\ k+2 & \text{if } j \leq k+1 \end{array} \right.$$

and

$$d^i d^{j-1}(k) = d^i \left(\left\{ \begin{array}{ll} k & \text{if } k < j-1 \\ k+1 & \text{if } k \geq j-1 \end{array} \right\} \right) = \left\{ \begin{array}{ll} k & \text{if } k < i \\ k+1 & \text{if } i \leq k, k+1 < j \\ k+2 & \text{if } j \leq k+1 \end{array} \right.$$

Hence $d^j d^i = d^i d^{j-1}$ if $i < j$. □

The following theorem gives a unique way of writing a surjective morphism as a composite of codegeneracies. This will be used frequently later on.

Theorem 2.1.5. *Any surjective morphism $\sigma : [n] \twoheadrightarrow [m]$ where $\sigma \neq id$ can be written as a composite of codegeneracies $\sigma = s^{j_1} s^{j_2} \cdots s^{j_{n-m}}$ with $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$. Furthermore this form is unique for every surjective morphism.*

Proof. We prove this by induction on n. For $n = 1$ the only surjective morphism (which is not id) is s^0. Assume that the assertion is true for some n and let $m < n+1$ and a surjective map $\sigma : [n+1] \twoheadrightarrow [m]$ be given. Let $j \in [n+1]$ be the least element where $\sigma(j) = \sigma(j+1)$. Then there exists a surjective morphism $\sigma' : [n] \twoheadrightarrow [m]$ such that $\sigma = \sigma' s^j$. If $\sigma' = id$ we are done. Hence we can assume that $m < n$. Then by assumption there exist (unique) $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$ such that $\sigma' = s^{j_1} \cdots s^{j_{n-m}}$. Assume that $j > j_{n-m}$. Then

$$s^{j_{n-m}} s^j (j_{n-m} + 1) = s^{j_{n-m}} (j_{n-m} + 1) = s^{j_{n-m}} (j_{n-m}) = s^{j_{n-m}} s^j (j_{n-m})$$

and thus $\sigma(j_{n-m}) = \sigma(j_{n-m} + 1)$, which contadicts the minimality of j. Hence $j \leq j_{n-m}$.

Next assume that $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$, $m \geq i_1 \geq \cdots \geq i_{n-m} \geq 0$ such that $\sigma = s^{j_1} \cdots s^{j_{n-m}} = s^{i_1} \cdots s^{i_{n-m}}$. Assume that $j_{n-m} = i_{n-m}, \ldots j_{k+1} = i_{k+1}, j_k < i_k$ for some k. Then $s^{j_k} \cdots s^{j_{n-m}}(j_k + n - m - k) = j_k$ and $s^{i_k} \cdots s^{i_{n-m}}\,(j_k + n - m - k) = s^{i_k}(j_k + 1) = j_k + 1$. Since s^j fixes j_k for $j \geq j_k$ and fixes $j_k + 1$ for $j \geq i_k$ we get that $s^{j_1} \cdots s^{j_{n-m}}(j_k + n - m - k) = j_k$ and $s^{i_1} \cdots s^{i_{n-m}}(j_k + n - m - k) = j_k + 1$ which is a contradiction. Hence the composite of codegeneracies is unique. □

Another important fact in the ordinal number category is that every morphism has a unique factorization of an injective and a surjective morphism. This is the epi-monic factorization.

Theorem 2.1.6 (The Epi-monic Factorization)**.** *Every morphism $\varphi : [n] \to [m]$ has a unique factorization $\varphi = \mu\sigma$ where μ is injective and σ is surjective. This factorization is called* the epi-monic factorization *of φ.*

Proof. Theorem 2.1.3 and the cosimplicial identities imply the existence of such a factorization. Let $\mu_i : [k_i] \to [m]$ be injective and $\sigma_i : [n] \to [k_i]$ be surjective for $i = 1, 2$, such that $\varphi = \mu_1\sigma_1 = \mu_2\sigma_2$. Since σ_i maps onto $[k_i]$ and μ_i maps to $k_i + 1$ distinct elements in $[m]$, φ must map to $k_1 + 1 = k_2 + 1$ distinct elements, and since μ_1, μ_2 are injective with equal image $\mu_1 = \mu_2 := \mu$. For $j \in [n]$ we get $\mu\sigma_1(j) = \mu\sigma_2(j)$ which implies that $\sigma_1(j) = \sigma_2(j)$ since μ is injective. Hence $\sigma_1 = \sigma_2$. □

Remark 2.1.7. We can use the first cosimplicial identity to reorder any composition of cofaces to be of the form in Remark 2.1.2 and the last cosimplicial identity to reorder any composition of codegeneracies to be of the form in Theorem 2.1.5. The proof of this is easily verified and is therefor omitted. This will be useful in the next section when showing that something is in fact a simplicial object.

2.2 Simplicial Objects

In this section we want to apply our knowledge of the ordinal number category to define simplicial objects in a category $\mathscr{C}$, which are a generalization of non-negative chain complexes, and define the category $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. Then we will introduce the simplicial identities and define what homotopy is on a simplicial category.

Definition 2.2.1. A *simplicial object* in a category $\mathscr{C}$ is a contravariant functor A from Δ to $\mathscr{C}$. The category $s\mathscr{C}$ is the category of simplicial objects in $\mathscr{C}$ with morphisms being the natural transformations between the simplicial objects. Moreover we call $A_n := A([n])$ the *n-simplex*, $d_j := A(d^j)$ the *faces* and $s_j := A(s^j)$ the *degeneracies* of A. In general if φ is a morphism in Δ we denote $A(\varphi)$ by φ^*.

Lemma 2.2.2 (The Simplicial Identities)**.** *For any simplicial object A the* *following identities called the* simplicial identities *hold:*

$$\begin{array}{ll} d_i d_j = d_{j-1} d_i & \textit{if } i < j \\ d_i s_j = s_{j-1} d_i & \textit{if } i < j \\ d_j s_j = id = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \textit{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \textit{if } i \leq j \end{array}$$

Proof. Follows from Lemma 2.1.4. □

Remark 2.2.3. Note that the simplicial identities together with Theorem 2.1.6 and Remark 2.1.7 imply that in order to check that a collection of n-simplices with faces and degeneracies is in fact a simplicial object, it is enough to check that the faces and degeneracies respect the simplicial identities. Furthermore when checking that a map $f : A \to B$ is in fact a morphism of simplicial objects, one should check that $f = (f_n : A_n \to B_n)_{n \geq 0}$ commutes with faces and degeneracies, i.e. that $f_n d_i = d_i f_{n+1}$ and $f_{n+1} s_i = s_i f_n$ for every n.

Just as in the category of chain complexes we can define homotopy in a simplicial category. But the definition of homotopy in the category of chain complexes uses the additivity of the category which we do not have. Therefor we must go about this in a different way.

Definition 2.2.4. Let $\mathscr{C}$ be some category and let $f, g : A \to B$ be morphisms in $s\mathscr{C}$. We say that f *is homotopic to* g, written $f \simeq g$, if there exist morphisms $h_i^n : A_n \to B_{n+1}$, $0 \leq i \leq n$ in $\mathscr{C}$ such that

$$\begin{array}{rcll} d_0 h_0 & = & f_n & \\ d_{n+1} h_n & = & g_n & \\ d_i h_j & = & h_{j-1} d_i & \text{if } i < j \\ d_{j+1} h_{j+1} & = & d_{j+1} h_j & \\ d_i h_j & = & h_j d_{i-1} & \text{if } i > j+1 \\ s_i h_j & = & h_{j+1} s_i & \text{if } i \leq j \\ s_i h_j & = & h_j s_{i-1} & \text{if } i > j. \end{array}$$

Furthermore we call h a *homotopy* from f to g and write $h : f \simeq g$.

That this definition of homotopy and the definition of homotopy on chain complexes are the same, will be proven in Section 3.1.

Remark 2.2.5. Let **Set** denote the category of sets and $\Delta[n]$ denote the n'th standard simplex, i.e. the functor $\mathrm{Hom}_\Delta(-, [n]) : \Delta \to \mathbf{Set}$. Homotopy in

s**Set** is defined such that $f, g : X \to Y$ are homotopic, $f \simeq g$, if the diagram

$$\begin{array}{ccc} X \times \Delta[0] \cong X & & \\ {\scriptstyle id\times d^1}\downarrow & \searrow^{f} & \\ X \times \Delta[1] & \xrightarrow{h} & Y \\ {\scriptstyle id\times d^0}\uparrow & \nearrow_{g} & \\ X \times \Delta[0] \cong X & & \end{array}$$

commutes for some morphism h (see [4] Section I.6). It turns out that this definition is equivalent to Definition 2.2.4. The advantage of our definition is that it works on any category $s\mathscr{C}$.

Note that in contrary to homotopy on chain complexes, the homotopy relation "$\simeq$" on an arbitrary category $s\mathscr{C}$ need not be an equivalence relation. But it can be shown that "$\simeq$" is an equivalence relation if the target of the morphisms are Kan Complexes (see [8], §6). Since simplicial modules are Kan Complexes this implies that "$\simeq$" is an equivalence relation on the category of simplicial modules. In Theorem 3.1.4 we go about this in another way, and show that "$\simeq$" is an equivalence relation on the category of simplicial modules using the Dold-Kan correspondence.

Definition 2.2.6. Let $\mathscr{C}$ be some category and let A, B be objects in $s\mathscr{C}$. We say that A and B are *homotopic equivalent* if there exist morphisms $f : A \to B$ and $g : B \to A$ such that $fg \simeq id_A$ and $gf \simeq id_B$.

2.3 The Dold-Kan Correspondence

As mentioned in the beginning of section 2.2, simplicial objects are a generalization of non-negative chain complex. This is because of the Dold-Kan correspondence which gives an equivalence between the category of non-negative chain complexes and the category of simplicial modules. In this section we define the functors $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$ and $\Gamma : \mathbf{Ch}_+^\Lambda \to s\mathbf{Mod}_\Lambda$ and show that these induce an equivalence of categories, the Dold-Kan correspondence.

In the following Λ will denote a unital ring (with non-zero unit), $\mathbf{Mod}_\Lambda$ will denote the category of (left or right) Λ-modules and $\mathbf{Ch}_+^\Lambda$ will denote the category of non-negative chain complexes of Λ-modules. Furthermore we call the objects in $s\mathbf{Mod}_\Lambda$ for simplicial Λ-modules.

Definition 2.3.1. Let the functor $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$ be defined in the following way: for a simplicial Λ-module A let

$$N(A)_n = \bigcap_{i=0}^{n-1} \ker d_i \subseteq A_n.$$

with differentials $\partial_n = (-1)^n d_n : N(A)_n \to N(A)_{n-1}$. For a morphism $f : A \to B$ we let $N(f)_n = f_n : N(A)_n \to N(B)_n$, i.e. f_n restricted to $N(A)_n$. The non-negative chain complex $N(A)$ is called the *normalized complex* and is denoted NA.

It might not be clear that N is a functor but this is easily shown. By the simplicial identities $d_i d_n = d_{n-1} d_i$ and thus $\partial_n : NA_n \to NA_{n-1}$ is a well-defined and $\partial\partial = 0$. Hence NA is a non-negative chain complex of Λ-modules. Let $f : A \to B$ be a morphism in $s\mathbf{Mod}_\Lambda$. Then for any $x \in NA_n$ and $i < n$ we get $d_i f_n(x) = f_{n-1} d_i(x) = 0$ and hence $N(f)_n$ is well-defined. Furthermore

$$\partial f_n(x) = (-1)^n d_n f_n(x) = (-1)^n f_{n-1} d_n(x) = f_{n-1}\partial(x)$$

and hence Nf is a morphism. If $f : A \to B$ and $g : B \to C$ then clearly $N(gf) = N(g)N(f)$. Moreover $N(id) = id$ and thus $N : s\mathbf{Mod}_\Lambda \to \mathbf{Ch}_+^\Lambda$ is a functor.

Now define the *Moore complex* $A_\bullet$ of any simplicial Λ-module A as the chain complex of Λ-modules $A_\bullet : \cdots \to A_2 \to A_1 \to A_0$ with differentials

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}$$

That $\partial\partial = 0$ follows from the simplicial identities. Let

$$DA_n := \sum_{i=0}^{n-1} \mathrm{Im}(s_i) \subseteq A_n.$$

Note that for $x \in A_{n-1}$ the simplicial identities imply that

$$\begin{aligned} \partial s_j(x) &= \sum_{i=0}^{n} (-1)^i d_i s_j(x) \\ &= \sum_{i=0}^{j-1} (-1)^i s_{j+1} d_i(x) + \sum_{i=j+2}^{n} (-1)^i s_j d_{i+1}(x) \in DA_{n-1} \end{aligned}$$

and thus we get a chain complex $A_\bullet/DA : \cdots \to A_1/DA_1 \to A_0/DA_0$ where the differentials are the induced homomorphisms $\partial : A_n/DA_n \to A_{n-1}/DA_{n-1}$. The following theorem shows that this chain complex is isomorphic to the normalized complex NA and thus we need not distinguish between these.

Theorem 2.3.2. *For any simplicial* Λ*-module* A *the composite*

$$NA \hookrightarrow A_\bullet \xrightarrow{\pi} A_\bullet/DA$$

(where π *is the canonical projection) is an isomorphism of chain complexes.*

Proof. Let

$$N_jA_n := \bigcap_{i=0}^{j} \ker(d_i) \subseteq A_n, \qquad D_jA_n := \sum_{i=0}^{j} \mathrm{Im}(s_i) \subseteq A_n$$

and let ϕ_j denote the composite $N_jA_n \hookrightarrow A_n \xrightarrow{\pi} A_n/D_jA_n$. We wish to show that ϕ_j is an isomorphism by induction on j and n. Let $x \in A_n$. Then $[x - s_0d_0(x)] = [x] \in A_n/\mathrm{Im}(s_0)$ and $x - d_0s_0(x) \in \ker(d_0)$ since $d_0(x - s_0d_0(x)) = d_0(x) - d_0s_0d_0(x) = 0$ and thus $\phi_0(x - s_0d_0(x)) = [x]$. Hence ϕ_0 is surjective. Let $x \in \ker(\phi_0)$. Then there exists $y \in A_{n-1}$ such that $s_0(y) = x$ and we get that $0 = d_0(x) = d_0s_0(y) = y$. Hence $x = s_0(y) = 0$ and thus ϕ_0 is injective.

Given $n > j$, assume that $\phi_k : N_kA_m \to A_m/D_kA_m$ is an isomorphism for every $k < j$ where $k \leq m \leq n$. Consider the diagrams

$$\begin{array}{ccccc} N_{j-1}A_n & \hookrightarrow & A_n & \xrightarrow{\pi} & A_n/D_{j-1}A_n \\ \uparrow & & \| & & \downarrow \pi \\ N_jA_n & \hookrightarrow & A_n & \xrightarrow{\pi} & A_n/D_jA_n \end{array} \qquad \begin{array}{ccc} N_{j-1}A_n & \xrightarrow[\phi_{j-1}]{\cong} & A_n/D_{j-1}A_n \\ \uparrow & & \downarrow \pi \\ N_jA_n & \xrightarrow{\phi_j} & A_n/D_jA_n \end{array}$$

Since both squares in the first diagram commute, so does the second diagram, since this is the composite square in the first diagram. Let $x \in A_n$. Due to the second diagram above there exists $y \in N_{j-1}A_n$ such that $\pi\phi_{j-1}(y) = [x] \in A_n/D_jA_n$. As before $y - s_jd_j(y) \in N_jA_n$ and $\phi_j(y - s_jd_j(y)) = [y - s_jd_j(y)] = [x]$. Hence ϕ_j is surjective.

It remains to show that ϕ_j is injective. For $x \in N_{j-1}A_{n-1}$ we get $d_is_j(x) = s_{j-1}d_i(x) = 0$ for $i < j$ and thus $s_j : N_{j-1}A_{n-1} \to N_{j-1}A_n$ is well-defined. Furthermore $s_js_i = s_is_{j-1}$ for $i < j$ and thus $s_j : A_{n-1}/D_{j-1}A_{n-1} \to$

$A_n/D_{j-1}A_n$ is well-defined. Hence we get the following diagram

$$\begin{array}{ccccc} N_{j-1}A_{n-1} & \xrightarrow{s_j} & N_{j-1}A_n & \hookleftarrow & N_jA_n \\ \cong\downarrow \phi_{j-1} & & \cong\downarrow \phi_{j-1} & & \downarrow \phi_j \\ A_{n-1}/D_{j-1}A_{n-1} & \xrightarrow{s_j} & A_n/D_{j-1}A_n & \xrightarrow{\pi} & A_n/D_jA_n \end{array}$$

which has commutative squares. Let $x \in A_n$ such that $\pi([x]) = [0]$. Then there exist $x_0, \ldots, x_j \in A_{n-1}$ such that $x = \sum_{i=0}^{j} s_i(x_i)$. Hence

$$s_j([x_j]) = [s_j(x_j)] = \left[\sum_{i=0}^{j} s_i(x_i)\right] = [x] \in A_n/D_{j-1}A_n$$

and thus $\ker\pi \subseteq s_j(A_{n-1}/D_{j-1}A_{n-1})$. Now let $x \in N_jA_n$ such that $\phi_j(x) = 0$. Using that the squares in the above diagram commute, that the ϕ_{j-1} are isomorphisms and that $\ker\pi \subseteq s_j(A_{n-1}/D_{j-1}A_{n-1})$ we can find a $y \in N_{j-1}A_{n-1}$ such that $s_j(y) = x$. Hence $0 = d_j(x) = d_js_j(y) = y$ and thus $x = s_j(y) = 0$.

Hence ϕ_n is an isomorphism and since $(-1)^n d_n\big|_{NA_n} = \sum(-1)^i d_i\big|_{NA_n}$ it follows that ϕ_n commutes with differentials for every n. Hence the composite $NA \hookrightarrow A_\bullet \xrightarrow{\pi} A_\bullet/DA$ is an isomorphism. □

Definition 2.3.3. Let $\Gamma : \mathbf{Ch}_+^\Lambda \to s\mathbf{Mod}_\Lambda$ be the functor defined in the following way: let $C_\bullet$ be a non-negative chain complex of Λ-modules. Define

$$\Gamma(C_\bullet)_n := \bigoplus_{\sigma:[n]\twoheadrightarrow[m]} C_m.$$

The face $d_i : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)_{n-1}$ is defined in the following way: let $\sigma : [n] \twoheadrightarrow [m]$ and $\mu\sigma_0$ be the epi-monic factorization of σd^i. On the coordinate corresponding to σ we define

$$d_i(x) = \begin{cases} \iota_{\sigma_0}(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^j, j < m \\ (-1)^m \iota_{\sigma_0}\partial(x) & \text{if } \mu = d^m \end{cases}$$

where ι_{σ_0} is the inclusion map into the coordinate corresponding to σ_0. The degeneracy $s_i : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)_{n+1}$ is defined on the coordinate corresponding to σ by

$$s_j(x) = \iota_{\sigma s^j}(x).$$

For a morphism of chain complexes $f = (f_n) : C_\bullet \to D_\bullet$ we define $\Gamma(f)$ by $\Gamma(f)_n := \langle \iota_\sigma f_m \rangle_{\sigma:[n]\twoheadrightarrow[m]}$.

From this point on we let $\iota_\sigma : C_m \to \Gamma(C_\bullet)_n = \bigoplus C_k$ for $\sigma : [n] \twoheadrightarrow [m]$ denote the inclusion map into the coordinate corresponding to σ. Note that it is not at all clear why Γ is a functor. In order to make sure that Γ is indeed a functor we need to show that $\Gamma(C_\bullet)$ is in fact a simplicial Λ-module, i.e. by Remark 2.2.3 to show that the faces and degeneracies respect the simplicial identities, that $\Gamma(f)$ commutes with the faces and degeneracies, that $\Gamma(gf) = \Gamma(g)\Gamma(f)$ and that $\Gamma(id) = id$.

We will only show that the first simplicial identity holds. The rest is more or less similar (and easier) to prove. Given $\sigma : [n] \twoheadrightarrow [m]$ and $i < j$, let $\mu_1\sigma_1 = \sigma d^j$, $\mu_2\sigma_2 = \sigma_1 d^i$, $\mu_3\sigma_3 = \sigma d^i$ and $\mu_4\sigma_4 = \sigma_3 d^{j-1}$ be the epi-monic factorizations. We wish to show that $d_i d_j$ and $d_{j-1}d_i$ are equal on the coordinate corresponding to σ. Note that $\mu_1\mu_2 = \mu_3\mu_4$ and $\sigma_2 = \sigma_4$ due to the uniqueness of the epi-monic factorization.

If $\mu_1\mu_2 = id$ then $\mu_1 = id$ and $\mu_2 = id$ because of the uniqueness of the epi-monic factorization. Similarly $\mu_3 = id$ and $\mu_4 = id$ and thus $d_i d_j(x) = \iota_{\sigma_2}(x)$ and $d_{j-1}d_i(x) = \iota_{\sigma_4}(x) = \iota_{\sigma_2}(x)$.

If $\mu_1\mu_2 = d^k$ for some $k \leq m$ then either $\mu_1 = id$ and $\mu_2 = d^k$ or $\mu_1 = d^k$ and $\mu_2 = id$ by the uniqueness of the epi-monic factorization. But the same holds for μ_3 and μ_4 and thus if $k < m$ then $d_i d_j(x) = d_{j-1}d_i(x) = 0$, and if $k = m$ then $d_i d_j(x) = d_{j-1}d_i(x) = (-1)^m \iota_{\sigma_2}\partial(x)$.

Assume that $\mu_1\mu_2 : [m-2] \to [m]$. Then $\mu_1 = d^k$ and $\mu_2 = d^l$ for some k, l. The only μ_1 and μ_2 for which $d_i d_j(x)$ is not immediately 0 (by the definition of the faces) is if $\mu_1 = d^m$ and $\mu_2 = d^{m-1}$. But then

$$d_i d_j(x) = (-1)^m d_i \iota_{\sigma_1}\partial(x) = -\iota_{\sigma_2}\partial\partial(x) = 0.$$

Similarly we get that $d_{j-1}d_i(x) = 0$ and thus $d_i d_j = d_{j-1}d_i$.

Hence $\Gamma(C_\bullet)$ is a simplicial Λ-module. We will now show that $\Gamma(f)$ commutes with the faces and degeneracies. On the coordinate corresponding to $\sigma : [n] \twoheadrightarrow [m]$ we get that

$$\Gamma(f)_{n+1}s_i(x) = \Gamma(f)_{n+1}\iota_{\sigma s^i}(x) = \iota_{\sigma s^i} f_m(x) = s_i \iota_\sigma f_m(x) = s_i \Gamma(f)_n(x)$$

and hence $\Gamma(f)$ commutes with degeneracies. Let $\mu\sigma_0$ be the epi-monic factorization of σd^i. On the coordinate corresponding to σ we get

$$\begin{aligned}
\Gamma(f)_{n-1} d_i(x) &= \begin{cases} \Gamma(f)_{n-1}\iota_{\sigma_0}(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^k, k < m \\ (-1)^m \Gamma(f)_{n-1}\iota_{\sigma_0}\partial(x) & \text{if } \mu = d^m \end{cases} \\
&= \begin{cases} \iota_{\sigma_0} f_m(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^k, k < m \\ (-1)^m \iota_{\sigma_0}\partial f_m(x) & \text{if } \mu = d^m \end{cases} \\
&= d_i \iota_\sigma f_m(x) \\
&= d_i \Gamma(f)_n(x).
\end{aligned}$$

Hence $\Gamma(f)$ commutes with faces. Let $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$. Then $\Gamma(gf)_n = \langle \iota_\sigma g_m f_m \rangle = \langle \iota_\sigma g_m \rangle \langle \iota_\sigma f_m \rangle = \Gamma(g)_n \Gamma(f)_n$. Since $\Gamma(id)_n = \langle \iota_\sigma \rangle = id$ it follows that Γ is a functor from $\mathbf{Ch}_+^\Lambda$ to $s\mathbf{Mod}_\Lambda$. We now have enough definitions to state the Dold-Kan correspondence.

Theorem 2.3.4 (The Dold-Kan Correspondence)**.** *The functors N and Γ form an equivalence of the categories $\boldsymbol{Ch}_+^\Lambda$ and $s\boldsymbol{Mod}_\Lambda$.*

The proof of this equivalence is rather long and complicated and thus the rest of this section is devoted to proving this theorem.

Theorem 2.3.5. *Any non-negative chain complex $C_\bullet$ of Λ-modules is isomorphic to $N\Gamma(C_\bullet)$ in $\boldsymbol{Ch}_+^\Lambda$.*

Proof. Let $C_\bullet$ be a non-negative chain complex of Λ-modules. We wish to show that $\Gamma(C_\bullet)_\bullet / D\Gamma(C_\bullet)$ is isomorphic to $C_\bullet$ (here $\Gamma(C_\bullet)_\bullet$ denotes the Moore complex of $\Gamma(C_\bullet)$). First note that for any n

$$D\Gamma(C_\bullet)_n = \sum_{i=0}^{n-1} \operatorname{Im}(s_i) = \sum_{i=0}^{n-1} \operatorname{Im}\left(\langle \iota_{\sigma s^i} \rangle_{\sigma:[n-1]\twoheadrightarrow[m]}\right)$$

For any surjective morphism $\sigma : [n] \twoheadrightarrow [m]$ with $m \neq n$, Theorem 2.1.5 implies that there exist $\sigma_0 : [n-1] \twoheadrightarrow [m]$ and $i \in \{0, \dots, m\}$ such that $\sigma = \sigma_0 s^i$. Hence $\bigoplus_{\sigma:[n]\twoheadrightarrow[m], m\neq n} C_m \subseteq D\Gamma(C_\bullet)_n$ and since $\sigma s^i \neq id$ for any $\sigma : [n-1] \twoheadrightarrow [m]$ and $i \in \{0, \dots, m\}$ we get $D\Gamma(C_\bullet)_n \subseteq \bigoplus_{\sigma:[n]\twoheadrightarrow[m], m\neq n} C_m$. Hence

$$\Gamma(C_\bullet)_n / D\Gamma(C_\bullet)_n = \frac{\bigoplus_{\sigma:[n]\twoheadrightarrow[m]} C_m}{\bigoplus_{\sigma:[n]\twoheadrightarrow[m], m\neq n} C_m} \cong C_n.$$

It remains to show that the diagram

$$\begin{array}{ccc} \Gamma(C_\bullet)_n/D\Gamma(C_\bullet)_n & \xrightarrow{\partial'} & \Gamma(C_\bullet)_{n-1}/D\Gamma(C_\bullet)_{n-1} \\ \downarrow \cong & & \downarrow \cong \\ C_n & \xrightarrow{\partial} & C_{n-1} \end{array}$$

commutes. Consider the Moore complex $\Gamma(C_\bullet)_\bullet$ which has differentials of the form $\sum_{i=0}^{n}(-1)^i d_i : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)_{n-1}$. Let $x \in C_n$ and $\tilde{x} \in \Gamma(C_\bullet)_n$ be the element which is x on the coordinate corresponding to id and zero everywhere else. Then by the construction of d_i we get

$$\sum_{i=0}^{n}(-1)^i d_i(\tilde{x}) = (-1)^n(-1)^n \iota_{id}\partial(x) = \iota_{id}\partial(x)$$

which implies that the above diagram commutes. Hence by Theorem 2.3.2 we get that

$$N\Gamma(C_\bullet) \cong \Gamma(C_\bullet)_\bullet/D\Gamma(C_\bullet) \cong C_\bullet, \text{ in } \mathbf{Ch}_+^{\Lambda}.$$

□

Corollary 2.3.6. *The functors $N\Gamma$ and $I_{\mathbf{Ch}_+^{\Lambda}}$ are naturally isomorphic.*

Proof. For every non-negative chain complex $C_\bullet$ let $\phi^{C_\bullet}$ be the composite of

$$N\Gamma(C_\bullet) \hookrightarrow \Gamma(C_\bullet)_\bullet \xrightarrow{\pi} \Gamma(C_\bullet)_\bullet/D\Gamma(C_\bullet) \xrightarrow{\cong} C_\bullet.$$

Theorem 2.3.2 and the proof of Theorem 2.3.5 imply that $\phi^{C_\bullet}$ is an isomorphism. Let $f : C_\bullet \to D_\bullet$ be a morphism. If the squares in the diagram

$$\begin{array}{ccccccc} N\Gamma(C_\bullet) & \hookrightarrow & \Gamma(C_\bullet)_\bullet & \xrightarrow{\pi} & \Gamma(C_\bullet)_\bullet/D\Gamma(C_\bullet) & \xrightarrow{\cong} & C_\bullet \\ {\scriptstyle N\Gamma(f)}\downarrow & & & & {\scriptstyle \widetilde{\Gamma(f)}}\downarrow & & \downarrow{\scriptstyle f} \\ N\Gamma(D_\bullet) & \hookrightarrow & \Gamma(D_\bullet)_\bullet & \xrightarrow{\pi} & \Gamma(D_\bullet)_\bullet/D\Gamma(D_\bullet) & \xrightarrow{\cong} & D_\bullet \end{array}$$

commute, then ϕ is a natural isomorphism of $N\Gamma$ and $I_{\mathbf{Ch}_+^{\Lambda}}$. Here $\Gamma(f)$ is viewed as a morphism between the Moore complexes, and $\widetilde{\Gamma(f)}$ is the induced morphism between quotients, which is well-defined since $\Gamma(f)$ commutes with degeneracies. For $x \in N\Gamma(C_\bullet)_n$ we get that

$$\widetilde{\Gamma(f)}_n\pi(x) = \widetilde{\Gamma(f)}_n([x]) = [\Gamma(f)_n(x)] = \pi(\Gamma(f)_n(x)) = \pi N\Gamma(f)_n(x)$$

where the last equality follows since $N\Gamma(f)_n = \Gamma(f)_n\big|_{N\Gamma(C_\bullet)_n}$. Hence the first square commutes. Let $(x_\sigma) \in \Gamma(C_\bullet)_n$. Then $\widetilde{\Gamma(f)}_n([(x_\sigma)]) = [\langle \iota_\sigma f_m \rangle (x_\sigma)]$ which by the proof of Theorem 2.3.5 is just $id^* f_n(x_{id}) = f_n(x_{id})$ when mapped to D_n. Since $[(x_\sigma)]$ is mapped to x_{id} in C_n it follows that the second square commutes. Hence $N\Gamma$ and $I_{\mathbf{Ch}_+^\Lambda}$ are naturally isomorphic. □

Our next goal is to prove that $\Gamma N(A) \cong A$ for any simplicial Λ-module A. In order to do this we will require some lemmas about the ordinal number category.

Lemma 2.3.7. *If $\sigma : [n] \twoheadrightarrow [m]$ and $d^k : [n-1] \to [n]$ such that the epi-monic factorization of σd^k is $d^m \sigma_0$ for some surjective morphism σ_0, then $k = n$.*

Proof. It is clear if $\sigma = id$. Assume that $k \le m < n$ and write $\sigma = s^{j_1} \cdots s^{j_{n-m}}$ uniquely with $m \ge j_1 \ge \cdots \ge j_{n-m} \ge 0$ by Theorem 2.1.5. Due to the uniqueness of the epi-monic factorization, the cosimplicial identities imply that $k = m$ and $j_i > k = m$ for $i = 1, \dots, n-m$ which is a contradiction. Now assume that $m < k \le n$. Then by the cosimplicial identities

$$\sigma d^k = s^{j_1} \cdots s^{j_{n-m}} d^k = s^{j_1} \cdots s^{j_{n-m-1}} d^{k-1} s^{j_{n-m}}.$$

If $k-1 = m$ we get the same contradiction as above. Hence $k-1 > m$. By doing this recursively we get that $k - (n-m-1) > m$ which implies that $k = n$. □

Remark 2.3.8. Note that this lemma gives us another way to define the faces $d_i : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)_{n-1}$ for $i = 0, \dots, n-1$: on the coordinate corresponing to $\sigma : [n] \twoheadrightarrow [m]$ we get

$$d_i(x) = \begin{cases} \iota_{\sigma d^i}(x) & \text{if } \sigma d^i \text{ is surjective} \\ 0 & \text{otherwise} \end{cases}$$

Furthermore if $\sigma : [n] \twoheadrightarrow [m]$ and $n > m$ write $\sigma = s^{j_1} \cdots s^{j_{n-m}}$ uniquely with $m \ge j_1 \ge \cdots \ge j_{n-m} \ge 0$. Then by the cosimplicial identities

$$\sigma d^n = s^{j_1} \cdots s^{j_{n-m}} d^n = s^{j_1} d^{m+1} s^{j_2} \cdots s^{j_{n-m}}.$$

Hence σd^n is either surjective or $\sigma d^n = d^m \sigma'$ where $\sigma' = s^{j_1} \cdots s^{j_{n-m}} : [n-1] \twoheadrightarrow [m-1]$. This allows us to define the face $d_n : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)_{n-1}$ on the coordinate corresponding to σ as

$$d_n(x) = \begin{cases} \iota_{\sigma d^i}(x) & \text{if } \sigma d^i \text{ is surjective} \\ (-1)^m \iota_{\sigma'} \partial(x) & \text{otherwise} \end{cases}$$

Consider the set of surjective morphisms $\sigma : [n] \twoheadrightarrow [m]$ with $n > m$. Theorem 2.1.5 induces a total order $\preceq$ on this set by

$$s^0 s^0 s^0 \cdots s^0 \preceq s^1 s^0 s^0 \cdots s^0 \preceq s^1 s^1 s^0 \cdots s^0 \preceq \cdots \preceq s^m \cdots s^m.$$

We will need this total order to formulate the next lemma and also for proving Theorem 2.3.10 below.

Lemma 2.3.9. *Let $n > m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$, and let $\mu = d^{j_{n-m}} \cdots d^{j_1} : [m] \to [n]$. If $\sigma : [n] \twoheadrightarrow [m]$ such that $\sigma\mu = id$ then $\sigma \preceq s^{j_1} \cdots s^{j_{n-m}}$.*

Proof. Let $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$ and $m \geq i_1 \geq \cdots \geq i_{n-m} \geq 0$ such that at least one $j_k \neq i_k$ and $s^{j_1} \cdots s^{j_{n-m}} \preceq s^{i_1} \cdots s^{i_{n-m}}$. If $i_1 = j_1$ then the cosimplicial identities imply that

$$\begin{aligned} s^{i_1} \cdots s^{i_{n-m}} d^{j_{n-m}} \cdots d^{j_1} &= s^{i_2} \cdots s^{i_{n-m}} s^{i_1+n-m-1} d^{j_1+n-m-1} d^{j_{n-m}} \cdots d^{j_2} \\ &= s^{i_2} \cdots s^{i_{n-m}} d^{j_{n-m}} \cdots d^{j_2} \end{aligned}$$

which gives us the exact same problem for surjective morphisms from $[n-1]$ onto $[m]$. Hence we will without loss of generality assume that $i_1 > j_1$.

Note that for $0 \leq k \leq n-m-1$ we get that $i_1 + n - m - 1 - k \geq i_1 > j_1 \geq \cdots \geq j_{n-m}$. Hence the cosimplicial identities imply that

$$\begin{aligned} s^{i_1} \cdots s^{i_{n-m}} d^{j_{n-m}} \cdots d^{j_1} &= s^{i_2} \cdots s^{i_{n-m}} s^{i_1+n-m-1} d^{j_{n-m}} \cdots d^{j_1} \\ &= s^{i_2} \cdots s^{i_{n-m}} d^{j_{n-m}} \cdots d^{j_1} s^{i_1-1} \end{aligned}$$

which can never be the identity morphism on $[m]$ since $i_1 - 1$ and i_1 are mapped to the same element. □

We now have enough tools in order to prove the following theorem which is the second part of the Dold-Kan correspondence.

Theorem 2.3.10. *Any simplicial Λ-module A is isomorphic to $\Gamma N(A)$ in s**Mod**$_\Lambda$.*

Proof. Given $A \in s\mathbf{Mod}_\Lambda$, let $\psi_n : \bigoplus_{[n]\twoheadrightarrow[m]} NA_m \to A_n$ be given on the summand corresponding to σ by $\psi_n(x) = \sigma^*(x)$ (i.e. $\psi_n = \langle \sigma^*|_{NA_m} \rangle_{\sigma:[n]\twoheadrightarrow[m]}$). First we wish to show that ψ_n is an isomorphism by induction on n. Observe that $NA_0 = A_0$. Since the only (surjective) map $[0] \to [0]$ is id, we get that $\psi_0 = id : A_0 \to A_0$ which is an isomorphism. Assume that ψ_k is an isomorphism for $k < n$. The diagram

$$\begin{array}{ccc} \bigoplus NA_m & \xrightarrow[\cong]{\psi_{n-1}} & A_{n-1} \\ \downarrow{\scriptstyle s_i} & & \downarrow{\scriptstyle s_i} \\ \bigoplus NA_m & \xrightarrow{\psi_n} & A_n \end{array}$$

commutes for all $i = 0, \ldots, n-1$ since

$$s_i\psi_{n-1}(x) = s_i\langle\sigma^*\rangle(x) = \langle s_i\sigma^*\rangle(x) = \langle(\sigma s^i)^*\rangle(x) = \langle\sigma^*\rangle\langle\iota_{\sigma s^i}\rangle(x) = \psi_n s_i(x).$$

Hence it follows that $DA_n \subseteq \mathrm{Im}(\psi_n)$. Let $x \in A_n$. By Theorem 2.3.2 let $y \in NA_n$ such that $\phi_n(y) = [y] = [x] \in A_n/DA_n$ and let $z \in DA_n$ such that $x = y + z$. Then there exists $z_0 \in \bigoplus NA_m$ such that $\psi_n(z_0) = z$ and thus

$$\psi_n(\iota_{id}(y) + z_0) = id^*(y) + \psi_n(z_0) = y + z = x.$$

Hence ψ_n is surjective.

We wish to show that the diagram

$$\begin{array}{ccc} \bigoplus NA_m & \xrightarrow{\psi_{k-1}} & A_{k-1} \\ {\scriptstyle d_i}\uparrow & & \uparrow{\scriptstyle d_i} \\ \bigoplus NA_m & \xrightarrow{\psi_k} & A_k \end{array}$$

commutes for every $i = 0, \ldots, k$. Let $\sigma : [k] \twoheadrightarrow [m]$, $0 \leq i \leq k$ be given and let $\mu\sigma_0$ be the epi-monic factorization of σd^i. Then on the coordinate corresponding to σ we get

$$\begin{aligned} \psi_{k-1}d_i(x) &= \begin{cases} \psi_{k-1}\iota_{\sigma_0}(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^l, l < m \\ (-1)^m\psi_{k-1}\iota_{\sigma_0}((-1)^m d_m(x)) & \text{if } \mu = d^m \end{cases} \\ &= \begin{cases} \sigma_0^*(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^l, l < m \\ \sigma_0^* d_m(x) & \text{if } \mu = d^m \end{cases} \\ &= \sigma_0^*\mu^*(x) = (\mu\sigma_0)^*(x) = (\sigma d^i)^*(x) = d_i\sigma^*(x) = d_i\psi_k(x) \end{aligned}$$

where we used that $x \in NA_m$. Hence the above diagram commutes.

Let $(x_\sigma) \in \bigoplus NA_m$ such that $\psi_n((x_\sigma)) = 0$, and let $m < n$. We wish to prove that $x_\sigma = 0$ for every $\sigma : [n] \twoheadrightarrow [m]$ by induction on σ using the total order $\preceq$. Remark 2.3.8 implies that the coordinate in $d_0 \cdots d_0((x_\sigma))$ corresponding to id is the sum of all x_σ where $d^0 \cdots d^0$ is a section for σ.[1] Now Lemma 2.3.9 implies that the only surjective morphism which has

[1] Note that in [4] Proposition III.2.2 they choose a section μ for a surjective morphism σ_0 and say that the coordinate in $\mu^*((x_\sigma))$ corresponding to id is x_{σ_0}. But this is only the case if $\sigma = s^0 \cdots s^0$ or $\sigma = s^m \cdots s^m$ and we choose the section $\mu = d^0 \cdots d^0$ or $\mu = d^{m+1} \cdots d^{m+1}$ respectively. Otherwise the coordinate corresponding to id in $\mu^*((x_\sigma))$ will be the sum of all x_σ for which μ is a section of σ.

$d^0 \cdots d^0$ as a section is $s^0 \cdots s^0$. Hence $d_0 \cdots d_0((x_\sigma))_{id} = x_{s^0 \ldots s^0}$. Since ψ_k commutes with faces we get that

$$\psi_m d_0 \cdots d_0((x_\sigma)) = d_0 \cdots d_0 \psi_n((x_\sigma)) = 0$$

and since ψ_m is an isomorphism $x_{s^0 \ldots s^0} = 0$.

Given $\sigma_0 : [n] \twoheadrightarrow [m]$ assume that $x_\sigma = 0$ for every $\sigma \preceq \sigma_0$ where $\sigma \neq \sigma_0$. Write $\sigma_0 = s^{j_1} \cdots s^{j_{n-m}}$ with $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$. Again Remark 2.3.8 implies that the coordinate in $d_{j_1} \cdots d_{j_{n-m}}((x_\sigma))$ corresponding to id is the sum of all x_σ where $d^{j_{m-n}} \cdots d^{j_1}$ is a section for σ. But Lemma 2.3.9 implies that if $\sigma d^{j_{m-n}} \cdots d^{j_1} = id$ then $\sigma \preceq \sigma_0$ and thus by our hypothesis the coordinate in $d_{j_1} \cdots d_{j_{n-m}}((x_\sigma))$ corresponding to id is x_{σ_0}. As before

$$\psi_m d_{j_1} \cdots d_{j_{n-m}}((x_\sigma)) = d_{j_1} \cdots d_{j_{n-m}} \psi_n((x_\sigma)) = 0$$

and since ψ_m is an isomorphism $x_{\sigma_0} = 0$. Hence $x_\sigma = 0$ for every $\sigma \neq id$. Finally we get that $0 = \psi_n((x_\sigma)) = \sum \sigma^*(x_\sigma) = x_{id}$ and hence ψ_n is injective and thus an isomorphism.

Since ψ_k commutes with faces and degeneracies, (ψ_n) is an isomorphism in $s\mathbf{Mod}_\Lambda$ and thus $\Gamma N(A) \cong A$ in $s\mathbf{Mod}_\Lambda$. □

Corollary 2.3.11. *The functors ΓN and $I_{s\mathbf{Mod}_\Lambda}$ are naturally isomorphic.*

Proof. For any simplicial Λ-module A let ψ^A be the isomorphism defined in the proof of Theorem 2.3.10. It remains to show that the diagram

$$\begin{array}{ccc} \Gamma N(A) & \xrightarrow{\psi^A} & A \\ {\scriptstyle \Gamma N(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \Gamma N(B) & \xrightarrow{\psi^B} & B \end{array}$$

commutes for any $A, B \in s\mathbf{Mod}_\Lambda$ and any $f : A \to B$. But this follows easily since

$$\psi_n^B \Gamma N(f)_n = \psi_n^B \langle \iota_\sigma f_m\big|_{NA_m} \rangle = \langle \sigma^* f_m\big|_{NA_m} \rangle$$

and

$$f_n \psi_n^A = f_n \langle \sigma^*\big|_{NA_m} \rangle = \langle f_n \sigma^*\big|_{NA_m} \rangle = \langle \sigma^* f_m\big|_{NA_m} \rangle$$

where the last equality follows since f_k commutes with degeneracies (and faces) for any k. Hence ΓN and $I_{s\mathbf{Mod}_\Lambda}$ are naturally isomorphic. □

With this last corollary we can finally give a proof of the Dold-Kan correspondence.

Proof of the Dold-Kan Correspondence. The equivalence is a direct consequence of Corollary 2.3.6 and Corollary 2.3.11. □

3 Derived Functors of Non-additive Functors

3.1 The Dold-Kan Correspondence and Homotopy

When defining the classical derived functors, homotopy played a big part. The same is the case for us. In this section we prove that the functors N and Γ which induce the Dold-Kan correspondence preserve homotopy. Furthermore we use this to show that the homotopy relation "$\simeq$" is an equivalence relation on $s\mathbf{Mod}_\Lambda$.

Theorem 3.1.1. *The functor N preserves homotopy, i.e. if $f, g : A \to B$ and $f \simeq g$ then $N(f) \simeq N(g)$.*

Proof. Let $f, g : A \to B$ in $s\mathbf{Mod}_\Lambda$ such that $f \simeq g$, and let h be a homotopy from f to g. Define $\Sigma' = (\Sigma'_n)$ by

$$\Sigma'_n = \sum_{i=0}^{n} (-1)^i h_i : A_n \to B_{n+1}.$$

Let $A_\bullet$ and $B_\bullet$ be the Moore complexes of A and B respectively, and $f, g : A_\bullet \to B_\bullet$ the induced morphisms. By the definition of h we get

$$\begin{aligned}
\partial_{n+1}\Sigma'_n &= \sum_{i=0}^{n+1}\sum_{j=0}^{n}(-1)^{i+j} d_i h_j \\
&= f_n - g_n + \sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^{i+j} h_{j-1} d_i + \sum_{j=0}^{n-1}\sum_{i=j+2}^{n+1}(-1)^{i+j} h_j d_{i-1} \\
&= f_n - g_n - \sum_{j=0}^{n-1}(-1)^j h_j \sum_{i=0}^{n}(-1)^i d_i \\
&= f_n - g_n - \Sigma'_{n-1}\partial_n.
\end{aligned}$$

Hence $f_n - g_n = \partial_{n+1}\Sigma'_n + \Sigma_{n-1}\partial_n$ and thus $\Sigma' : f \simeq g$. Note that for $0 \leq j \leq n-1$

$$\Sigma'_n s_j = \sum_{i=0}^{n}(-1)^i h_i s_j = \sum_{i=0}^{j}(-1)^i s_{j+1} h_i + \sum_{i=j+1}^{n}(-1)^i s_j h_{i-1}$$

and therefor $\Sigma'_n(DA_n) \subseteq DB_{n+1}$. Hence the induced maps $\Sigma_n : A_n/DA_n \to B_{n+1}/DB_{n+1}$ are well-defined. Furthermore since f and g commute with degeneracies, the induced morphisms $\widetilde{f}, \widetilde{g} : A_\bullet/DA \to B_\bullet/DB$ are well-defined and clearly $\Sigma : \widetilde{f} \simeq \widetilde{g}$.

Let $\phi^A : NA \to A_\bullet/DA$ and $\phi^B : NB \to B_\bullet/DB$ be the isomorphisms from Theorem 2.3.2. We wish to show that $(\phi^B)^{-1}\Sigma\,\phi^A : N(f) \simeq N(g)$. We get that

$$\begin{aligned} & \partial_{n+1}(\phi^B_{n+1})^{-1}\Sigma_n\phi^A_n + (\phi^B_n)^{-1}\Sigma_{n-1}\phi^A_{n-1}\partial_n \\ = \; & (\phi^B_n)^{-1}(\partial_{n+1}\Sigma_n + \Sigma_{n-1}\partial_n)\phi^A_n \\ = \; & (\phi^B_n)^{-1}\widetilde{f_n}\phi^A_n - (\phi^B_n)^{-1}\widetilde{g_n}\phi^A_n. \end{aligned}$$

Since $N(f)_n = f_n\big|_{NA_n}$ the squares in the diagram

$$\begin{array}{ccccc} NA_n & \hookrightarrow & A_n & \xrightarrow{\pi} & A_n/DA_n \\ {\scriptstyle N(f)_n}\downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \widetilde{f_n}} \\ NB_n & \hookrightarrow & B_n & \xrightarrow{\pi} & B_n/DB_n \end{array}$$

commute, and hence $\widetilde{f_n}\phi^A_n = \phi^B_n N(f)_n$. Similarly $\widetilde{g_n}\phi^A_n = \phi^B_n N(g)_n$, and hence

$$(\phi^B_n)^{-1}\widetilde{f_n}\phi^A_n - (\phi^B_n)^{-1}\widetilde{g_n}\phi^A_n = N(f)_n - N(g)_n.$$

This gives us the homotopy $(\phi^B)^{-1}\Sigma\,\phi^A : N(f) \simeq N(g)$. □

Theorem 3.1.2. *The functor* Γ *preserves homotopy, i.e. if* $f, g : C_\bullet \to D_\bullet$ *and* $f \simeq g$ *then* $\Gamma(f) \simeq \Gamma(g)$.

Proof. Let $f, g : C_\bullet \to D_\bullet$ in $\mathbf{Ch}^\Lambda_+$ such that $f \simeq g$, and let Σ be a homotopy from f to g. We define $h^n_j : \Gamma(C_\bullet)_n \to \Gamma(D_\bullet)_{n+1}$ in the following way: let $\sigma : [n] \twoheadrightarrow [m]$. If $\sigma = id$ put $k = 0$ and $\tilde{\sigma} = id_{[n+1]}$. If not write $\sigma = s^{j_1}\cdots s^{j_{n-m}}$ with $m \geq j_1 \geq \cdots \geq j_{n-m} \geq 0$. Then let $0 \leq k \leq n-m$ be given such that $j_k + n - m - k + 1 > j$ and $j_{k+1} + n - m - k \leq j$ and put $\tilde{\sigma} = s^{j_1+1}\cdots s^{j_k+1}s^{j_{k+1}}\cdots s^{j_{n-m}} : [n+1] \twoheadrightarrow [m+1]$. Now we define h^n_j on the coordinate corresponding to σ by

$$h^n_j(x) = \begin{cases} \iota_{s^m\tilde{\sigma}}(f_m(x) - \Sigma_{m-1}\partial(x)) + (-1)^m\iota_{\tilde{\sigma}}\Sigma_m(x) & \text{if } k = n-j \\ \iota_{s^{m-1}\tilde{\sigma}}f_m(x) - \iota_{s^m\tilde{\sigma}}\Sigma_{m-1}\partial(x) & \text{if } k = n-j-1 \\ \iota_{s^{j-n+m+k}\tilde{\sigma}}f_m(x) & \text{if } k < n-j-1 \end{cases}$$

This will give us a homotopy $h : \Gamma(f) \simeq \Gamma(g)$. We will first prove that $d_{n+1}h^n_n = \Gamma(g)_n$ and then that $d_ih_j = h_{j-1}d_i$ if $i < j$. The rest are left for the reader to do on a cold and lonely night.

Let $\sigma : [n] \twoheadrightarrow [m]$. If $\sigma = id$ let $\tilde{\sigma} = id_{[n+1]}$. If not, write $\sigma = s^{j_1}\cdots s^{j_{n-m}}$. Since $j_1 + n - m \leq n$ let $\tilde{\sigma} = s^{j_1}\cdots s^{j_{n-m}} : [n+1] \twoheadrightarrow [m+1]$. The cosimplicial identities imply that $s^m\tilde{\sigma}d^{n+1} = \sigma s^n d^{n+1} = \sigma$ and $\tilde{\sigma}d^{n+1} = d^{m+1}\sigma$ and hence on the coordinate corresponding to σ we get

$$\begin{aligned} d_{n+1}h_n(x) &= d_{n+1}(\iota_{s^m\tilde{\sigma}}(f_m(x) - \Sigma_{m-1}\partial(x)) + (-1)^m \iota_{\tilde{\sigma}}\Sigma_m(x)) \\ &= \iota_\sigma(f_m(x) - \Sigma_{m-1}\partial(x) + (-1)^{m+m+1}\partial\Sigma_m(x)) \\ &= \iota_\sigma g_m(x) \\ &= \Gamma(g)_n(x). \end{aligned}$$

Hence $d_{n+1}h_n = \Gamma(g)_n$.

Now let $i < j \leq n$. We wish to show that $d_i h^n_j = h^{n-1}_{j-1}d_i$. On the coordinate corresponding to id we get

$$h_j(x) = \begin{cases} \iota_{s^n}(f_n(x) - \Sigma_{n-1}\partial(x)) + (-1)^n \iota_{id}\Sigma_n(x) & \text{if } j = n \\ \iota_{s^{n-1}}f_n(x) - \iota_{s^n}\Sigma_{n-1}\partial(x) & \text{if } j = n-1 \\ \iota_{s^j}f_n(x) & \text{if } j < n-1 \end{cases}$$

By the cosimplicial identities $s^j d^i = d^i s^{j-1}$ and $s^n d^i = d^i s^{n-1}$. Hence $d_i h_j(x) = 0$ and since $d_i(x) = 0$ we get that $d_i h_j(x) = h_{j-1}d_i(x) = 0$ on the coordinate corresponding to id.

Let $\sigma : [n] \twoheadrightarrow [m]$ with $n > m$ and write $\sigma = s^{j_1} \cdots s^{j_{n-m}}$. Define k and $\tilde{\sigma}$ as in the construction of h_j. First note that since $j_1 + n - m - 1 > \cdots > j_k + n - m - k \geq j > i$ the cosimplicial identities imply that

$$\begin{aligned} \sigma d^i &= s^{j_1} \cdots s^{j_{n-m}} d^i \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} s^{j_k+n-m-k} \cdots s^{j_1+n-m-1} d^i \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} d^i s^{j_k+n-m-k-1} \cdots s^{j_1+n-m-2} \end{aligned}$$

and that

$$\begin{aligned} \tilde{\sigma} d^i &= s^{j_1+1} \cdots s^{j_k+1} s^{j_{k+1}} \cdots s^{j_{n-m}} d^i \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} s^{j_k+n-m-k+1} \cdots s^{j_1+n-m} d^i \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} d^i s^{j_k+n-m-k} \cdots s^{j_1+n-m-1}. \end{aligned}$$

Hence by the uniqueness of the epi-monic factorization σd^i is surjective if and only if $\tilde{\sigma} d^i$ is surjective. Now note that since $j - n + m + k \geq j_{k+1} \geq \cdots \geq j_{n-m}$ we get

$$\begin{aligned} s^{j-n+m+k}\tilde{\sigma} d^i &= s^{j-n+m+k} s^{j_{k+1}} \cdots s^{j_{n-m}} d^i s^{j_k+n-m-k} \cdots s^{j_1+n-m-1} \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} s^j d^i s^{j_k+n-m-k} \cdots s^{j_1+n-m-1} \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} d^i s^{j-1} s^{j_k+n-m-k} \cdots s^{j_1+n-m-1} \\ &= s^{j_{k+1}} \cdots s^{j_{n-m}} d^i s^{j_k+n-m-k-1} \cdots s^{j_1+n-m-2} s^{j-1} \\ &= \sigma d^i s^{j-1}. \end{aligned}$$

Again by the uniqueness of the epi-monic factorization $s^{j-n+m+k}\tilde{\sigma}d^i$ is surjective if and only if σd^i is surjective. Similarly we also get $s^{j+1-n+m+k}\tilde{\sigma}d^i$ is surjective if and only if σd^i is surjective.

All these facts combined together with Remark 2.3.8 imply, that if σd^i is not surjective then on the coordinate corresponding to σ we get $d_i h_j(x) = 0$ and $d_i(x) = 0$ and hence $d_i h_j(x) = h_{j-1} d_i(x)$.

If σd^i is surjective, then by what we showed above

$$d_i h_j(x) = \begin{cases} \iota_{s^m \tilde{\sigma} d^i}(f_m(x) - \Sigma\partial(x)) + (-1)^m \iota_{\tilde{\sigma} d^i}\Sigma(x) & \text{if } k = n-j \\ \iota_{s^{m-1}\tilde{\sigma} d^i} f_m(x) - \iota_{s^m \tilde{\sigma} d^i}\Sigma\partial(x) & \text{if } k = n-j-1 \\ \iota_{s^{j-n+m+k}\tilde{\sigma} d^i} f_m(x) & \text{if } k < n-j-1 \end{cases}$$

on the coordinate corresponding to σ. Furthermore if σd^i is surjective, then $s^{j_{k+1}} \cdots s^{j_{n-m}} d^i$ is surjective and by the cosimplicial identities can be written as $s^{i_{k+1}} \cdots s^{i_{n-m-1}}$ with $j_{k+1} \geq i_{k+1} \geq \cdots \geq i_{n-m-1}$. Hence $\sigma d^i = s^{j_1} \cdots s^{j_k} s^{i_{k+1}} \cdots s^{i_{n-m-1}}$ with $m \geq j_1 \geq \cdots \geq j_k \geq i_{k+1} \geq \cdots \geq i_{n-m-1} \geq 0$. Note that $j_k + (n-1) - m - k + 1 > j - 1$ and $i_{k+1} + (n-1) - m - k \leq j - 1$ since $i_{k+1} \leq j_{k+1}$. By Remark 2.3.8 $d_i(x) = \iota_{\sigma d^i}(x)$ and since $\tilde{\sigma} d^i = s^{j_1+1} \cdots s^{j_k+1} s^{i_{k+1}} \cdots s^{i_{n-m-1}}$ we get that

$$h_{j-1} d_i(x) = \begin{cases} \iota_{s^m \tilde{\sigma} d^i}(f_m(x) - \Sigma\partial(x)) + (-1)^m \iota_{\tilde{\sigma} d^i}\Sigma(x) & k = n-j \\ \iota_{s^{m-1}\tilde{\sigma} d^i} f_m(x) - \iota_{s^m \tilde{\sigma} d^i}\Sigma\partial(x) & k = n-j-1 \\ \iota_{s^{(j-1)-(n-1)+m+k}\tilde{\sigma} d^i} f_m(x) & k < n-j-1 \end{cases}$$

on the coordinate corresponding to σ and hence $d_i h_j(x) = h_{j-1} d_i(x)$ when σd^i is surjective. Hence $d_i h_j = h_{j-1} d_i$ whenever $i < j$.

□

Hence Γ and N preserve homotopy when passing between non-negative chain complexes and simplicial modules. This is an important property which amongst other things allow us to generalize the notion of derived functors which we will do in the following section. But first let us apply it to give a short proof of the homotopy relation on morphisms between simplicial modules beeing an equivalence relation.

Lemma 3.1.3. *Let $f, g : A \to B$ in $s\boldsymbol{Mod}_\Lambda$. If $\Gamma N(f) \simeq \Gamma N(g)$ then $f \simeq g$.*

Proof. Let $h : \Gamma N(f) \simeq \Gamma N(g)$ and let $\psi^A : \Gamma N(A) \xrightarrow{\cong} A$, $\psi^B : \Gamma N(B) \xrightarrow{\cong} B$ be the isomorphisms from Corollary 2.3.11. Now $f = \psi^B \Gamma N(f)(\psi^A)^{-1}$ and $g = \psi^B \Gamma N(g)(\psi^A)^{-1}$ and since ψ^A and ψ^B commute with faces and degeneracies it easily follows that $\psi^B h(\psi^A)^{-1} : f \simeq g$. □

Corollary 3.1.4. *The homotopy relation "$\simeq$" is an equivalence relation in* $s\boldsymbol{Mod}_\Lambda$.

Proof. First recall that the homotopy relation in $\mathbf{Ch}_+^\Lambda$ is an equivalence relation. Let $f, g, h : A \to B$ be morphisms in $s\mathbf{Mod}_\Lambda$. Then $N(f) \simeq N(f)$ and hence $\Gamma N(f) \simeq \Gamma N(f)$. By Lemma 3.1.3, $f \simeq f$ and hence "$\simeq$" is reflexive.

Assume that $f \simeq g$. Then $N(f) \simeq N(g)$ and since this relation is symmetric it follow that $N(g) \simeq N(f)$. Hence $\Gamma N(g) \simeq \Gamma N(f)$ and by Lemma 3.1.3 it follows that $g \simeq f$. Hence "$\simeq$" is symmetric.

Finally assume that $f \simeq g$ and $g \simeq h$. Then $N(f) \simeq N(g)$ and $N(g) \simeq N(h)$ which implies that $N(f) \simeq N(h)$. Thus $\Gamma N(f) \simeq \Gamma N(h)$ and by Lemma 3.1.3 it follows that $f \simeq h$. Hence "$\simeq$" is transitive and thus an equivalence relation. □

3.2 Derived Functors of Non-additive Functors

In Section 1.2 we defined the classical left derived functor of an additive covariant functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$. In this section we give a new definition of left derived functors of a functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ which does not require F to be additive, and then show that if F is indeed additive, this definition coincides with the defintion of the classical left derived functor.

Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be a (covariant) functor and let $f : A \to B$ be a morphism in $s\mathbf{Mod}_\Lambda$. We define FA to be the simplicial Λ'-module where the n-simplices are $F(A_n)$ and the faces and degeneracies are Fd_i and Fs_i respectively. Moreover we define the morphism $Ff : FA \to FB$ to be the morphism where $(Ff)_n = Ff_n : (FA)_n \to (FB)_n$. This makes $F : s\mathbf{Mod}_\Lambda \to s\mathbf{Mod}_{\Lambda'}$ into a functor. Note that if $f, g : A \to B$ and $h : f \simeq g$ then $Fh : Ff \simeq Fg$. This is clear since e.g. $Fd_iFh_j = F(d_ih_j) = F(h_{j-1}d_i) = Fh_{j-1}Fd_i$ for $i < j$. Hence the functor $F : s\mathbf{Mod}_\Lambda \to s\mathbf{Mod}_{\Lambda'}$ preserves homotopy, and thus the covariant functor $NF\Gamma : \mathbf{Ch}_+^\Lambda \to \mathbf{Ch}_+^{\Lambda'}$ preserves homotopy due to Theorem 3.1.1 and Theorem 3.1.2.

We wish to use this to generalize the notion of left derived functors. Again let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be a functor, let A be a Λ-module and $P_\bullet$ be a projective resolution of A. Define for $n \geq 0$

$$L_n^{P_\bullet} F(A) := H_n(NF\Gamma P_\bullet),$$

i.e. the n'th homology module of $NF\Gamma P_\bullet$. The following theorem states that it does not matter which projective resolution of A we choose.

Theorem 3.2.1. *Let $F : \boldsymbol{Mod}_\Lambda \to \boldsymbol{Mod}_{\Lambda'}$ be a functor, A a Λ-module and $P_\bullet, Q_\bullet$ projective resolutions of A. Then $L_n^{P_\bullet} F(A) \cong L_n^{Q_\bullet} F(A)$ for every n.*

Proof. From section 1.2 we know that $P_\bullet$ and $Q_\bullet$ are homotopic equivalent and thus $NF\Gamma P_\bullet$ and $NF\Gamma Q_\bullet$ are homotopic equivalent since $NF\Gamma$ preserves homotopy and maps identity morphisms to identity morphisms. Again from section 1.2 we know that any homotopy equivalence $f : NF\Gamma P_\bullet \to NF\Gamma Q_\bullet$ is a quasi-isomorphism and thus $H_n(f) : L_n^{P_\bullet} F(A) \to L_n^{Q_\bullet} F(A)$ is an isomorphism. □

Now let A, B be Λ-modules and $\varphi \in \mathrm{Hom}_\Lambda(A, B)$. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively. By Theorem 1.2.10 there exists a morphism $f_\varphi : P_\bullet \to Q_\bullet$, which is unique up to homotopy, such that the diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\sim} & A \\ {\scriptstyle f_\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Q_\bullet & \xrightarrow{\sim} & B \end{array}$$

commutes. Here we think of A and B as beeing chain complexes where $A_0 = A$, $A_n = 0$ for $n \neq 0$ and similarly for B. Now for $n \geq 0$ we define

$$L_n^{f_\varphi} F(\varphi) := H_n(NF\Gamma f_\varphi) : L_n^{P_\bullet} F(A) \to L_n^{Q_\bullet} F(B).$$

The following theorem shows that this definition does not depend on the projective resolutions $P_\bullet$ and $Q_\bullet$ or of the choice of f_φ.

Theorem 3.2.2. *Let $F : \boldsymbol{Mod}_\Lambda \to \boldsymbol{Mod}_{\Lambda'}$ be a functor, A and B be Λ-modules, $\varphi : A \to B$ be a homomorphism, $P_\bullet^1, P_\bullet^2$ be projective resolutions of A and $Q_\bullet^1, Q_\bullet^2$ be projective resolutions of B, and $f_\varphi^i : P_\bullet^i \to Q_\bullet^i$ be some morphisms induced by Theorem 1.2.10 for $i = 1, 2$. Then for every $n \geq 0$ the diagram*

$$\begin{array}{ccc} L_n^{P_\bullet^1} F(A) & \xrightarrow{\cong} & L_n^{P_\bullet^2} F(A) \\ {\scriptstyle L_n^{f_\varphi^1} F(\varphi)}\downarrow & & \downarrow{\scriptstyle L_n^{f_\varphi^2} F(\varphi)} \\ L_n^{Q_\bullet^1} F(B) & \xrightarrow{\cong} & L_n^{Q_\bullet^2} F(B) \end{array}$$

commutes for some isomorphisms.

Proof. By Theorem 1.2.10 there exist morphisms $g_p^1 : P_\bullet^1 \to P_\bullet^2$ and $g_p^2 : P_\bullet^2 \to P_\bullet^1$ such that the diagram

$$\begin{array}{ccccc} & & P_\bullet^2 & & \\ & \overset{g_p^1}{\nearrow} & \sim \Big\downarrow {\scriptstyle \psi_p^2} & \overset{g_p^2}{\searrow} & \\ P_\bullet^1 & \xrightarrow[\psi_p^1]{\sim} & A & \xleftarrow[\psi_p^1]{\sim} & P_\bullet^1 \end{array}$$

commutes. Furthermore Theorem 1.2.10 implies that $g_p^2 g_p^1 \simeq id_{P_\bullet^1}$. Similarly $g_p^1 g_p^2 \simeq id_{P_\bullet^2}$ and hence g_p^1 and g_p^2 are homotopy equivalences of $P_\bullet^1$ and $P_\bullet^2$. Similarly we get morphisms $g_q^1 : Q_\bullet^1 \to Q_\bullet^2$ and $g_q^2 : Q_\bullet^2 \to Q_\bullet^1$ such that a similar diagram commutes and such that g_q^1 and g_q^2 are homotopy equivalences of $Q_\bullet^1$ and $Q_\bullet^2$. Hence we get a commutative diagram

$$\begin{array}{ccccc} & & \overset{g_p^1}{\longrightarrow} & & \\ P_\bullet^1 & \xrightarrow[\sim]{} & A & \xleftarrow[\sim]{} & P_\bullet^2 \\ {\scriptstyle f_\varphi^1}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle f_\varphi^2} \\ Q_\bullet^1 & \xrightarrow{\sim} & B & \xleftarrow{\sim} & Q_\bullet^2 \\ & & \underset{g_q^2}{\longleftarrow} & & \end{array}$$

and again by Theorem 1.2.10 we get that $f_\varphi^1 \simeq g_q^2 f_\varphi^2 g_p^1$. Since $NF\Gamma$ preserves homotopy $NF\Gamma f_\varphi^1 \simeq NF\Gamma(g_q^2 f_\varphi^2 g_p^1)$ and also $NF\Gamma g_p^1$ and $NF\Gamma g_q^2$ are homotopy equivalences and thus quasi-isomorphisms. Hence $H_n(NF\Gamma(f_\varphi^1)) = H_n(NF\Gamma(g_q^2 f_\varphi^2 g_p^1))$ and thus we get a commutative diagram

$$\begin{array}{ccc} L_n^{P_\bullet^1} F(A) & \xrightarrow[\cong]{H_n(NF\Gamma g_p^1)} & L_n^{P_\bullet^2} F(A) \\ {\scriptstyle L_n^{f_\varphi^1} F(\varphi)}\Big\downarrow & & \Big\downarrow{\scriptstyle L_n^{f_\varphi^2} F(\varphi)} \\ L_n^{Q_\bullet^1} F(B) & \xrightarrow[\cong]{(H_n(NF\Gamma g_q^2))^{-1}} & L_n^{Q_\bullet^2} F(B) \end{array}$$

□

We can now give the definition of a left derived functor.

Definition 3.2.3 (Left Derived Functor)**.** Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be a covariant functor and define the n'th *left derived functor* of F, $L_n F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ as follows: let A be a Λ-module and let $P_\bullet$ be a projective resolution of A. Then let $L_n F(A) = L_n^{P_\bullet} F(A)$. Let $\varphi : A \to B$ be a homomorphism and let $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively. Choose a morphism $f_\varphi : P_\bullet \to Q_\bullet$ by Theorem 1.2.10. Then let $L_n F(\varphi) = L_n^{f_\varphi} F(\varphi)$.

By Theorem 3.2.1 and Theorem 3.2.2, $L_n F$ is uniquely determined up to isomorphism, just as the classical definition of derived functors. Our next goal is to show that if the functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ is additive then the Defintion 3.2.3 coincides with the definition of the classical left derived functor.

Theorem 3.2.4. *Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be an additive covariant functor. Then Definition 1.2.13 and Definition 3.2.3 are equivalent.*

In order to prove this we will require some lemmas.

Lemma 3.2.5. *Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be an additive covariant functor. Then $NF\Gamma C_\bullet \cong FC_\bullet$ for any non-negative chain complex.*

Proof. We will first show that $F\Gamma C_\bullet \cong \Gamma F C_\bullet$. Recall that since F is additive, the morphism of modules

$$(\Gamma F C_\bullet)_n = \bigoplus\nolimits_{[n]\twoheadrightarrow[m]} FC_m \xrightarrow[\cong]{\langle F\iota_\sigma\rangle} F\left(\bigoplus\nolimits_{[n]\twoheadrightarrow[m]} C_m\right) = (F\Gamma C_\bullet)_n$$

is an isomorphism for every n. Hence it remains to show that this isomorphism commutes with faces and degeneracies for every n. Let $\sigma : [n] \twoheadrightarrow [m]$ and let $\mu\sigma_0$ be the epi-monic factorization of σd^i. Then on the coordinate corresponding to σ we get

$$\langle F\iota_{\sigma'}\rangle d_i(x) = \begin{cases} (F\iota_{\sigma_0})(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^k, k < m \\ (-1)^m (F\iota_{\sigma_0})(F\partial)(x) & \text{if } \mu = d^m \end{cases}$$

where we used that the differentials in $FC_\bullet$ are $(F\partial_n)_{n\geq 0}$. Again on the coordinate corresponding to σ we get

$$Fd_i\langle F\iota_{\sigma'}\rangle(x) = F(d_i\iota_\sigma)(x) = \begin{cases} (F\iota_{\sigma_0})(x) & \text{if } \mu = id \\ 0 & \text{if } \mu = d^k, k < m \\ F((-1)^m \iota_{\sigma_0}\partial)(x) & \text{if } \mu = d^m \end{cases}$$

which is clearly equal to $\langle F\iota_{\sigma'}\rangle d_i(x)$. Hence $\langle F\iota_\sigma\rangle$ commutes with faces. Again let $\sigma : [n] \twoheadrightarrow [m]$. On the coordinate corresponding to σ we get

$$\langle F\iota_{\sigma'}\rangle s_i(x) = \langle F\iota_{\sigma'}\rangle \iota_{\sigma s^i}(x) = (F\iota_{\sigma s^i})(x) = F(s_i\iota_\sigma)(x) = Fs_i\langle F\iota_{\sigma'}\rangle(x).$$

Hence $\langle F\iota_\sigma\rangle$ commutes with degeneracies and thus $F\Gamma C_\bullet \cong \Gamma F C_\bullet$. Now the Dold-Kan correspondence implies that $NF\Gamma C_\bullet \cong N\Gamma F C_\bullet \cong FC_\bullet$. □

Lemma 3.2.6. *Let $F : \boldsymbol{Mod}_\Lambda \to \boldsymbol{Mod}_{\Lambda'}$ be an additive covariant functor and let $f : C_\bullet \to D_\bullet$ be a morphism. Then the diagram*

$$\begin{array}{ccc} FC_\bullet & \xrightarrow{\cong} & NF\Gamma C_\bullet \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle NF\Gamma f} \\ FD_\bullet & \xrightarrow{\cong} & NF\Gamma D_\bullet \end{array}$$

commutes for some isomorphisms.

Proof. First we will show that the diagram

$$\begin{array}{ccc} (\Gamma FC_\bullet)_n & \underset{\cong}{\xrightarrow{\langle F\iota_\sigma\rangle}} & (F\Gamma C_\bullet)_n \\ {\scriptstyle (\Gamma Ff)_n}\downarrow & & \downarrow{\scriptstyle (F\Gamma f)_n} \\ (\Gamma FD_\bullet)_n & \underset{\cong}{\xrightarrow{\langle F\iota_\sigma\rangle}} & (F\Gamma D_\bullet)_n \end{array}$$

is commutative for any n. Let $\sigma : [n] \twoheadrightarrow [m]$. On the coordinate corresponding to σ we get

$$\begin{aligned} \langle F\iota_{\sigma'}\rangle(\Gamma Ff)_n(x) &= \langle F\iota_{\sigma'}\rangle\iota_\sigma Ff_m(x) = F(\iota_\sigma f_m)(x) = F(\Gamma(f)_n\iota_\sigma)(x) \\ &= (F\Gamma f)_n\langle F\iota_{\sigma'}\rangle(x). \end{aligned}$$

Consider the diagram

$$\begin{array}{ccccc} FC_\bullet & \xrightarrow{\cong} & N\Gamma FC_\bullet & \xrightarrow{\cong} & NF\Gamma C_\bullet \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle N\Gamma Ff} & & \downarrow{\scriptstyle NF\Gamma f} \\ FD_\bullet & \xrightarrow{\cong} & N\Gamma FD_\bullet & \xrightarrow{\cong} & NF\Gamma D_\bullet \end{array}$$

By what we just proved the second square commutes, and due to the Dold-Kan correspondence the first square commutes. Hence the composite square commutes. □

With this lemma we now have enough tools to prove Theorem 3.2.4.

Proof of Theorem 3.2.4. Let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be an additive covariant functor, A be a Λ-module and $P_\bullet$ be a projective resolution of A. By Lemma 3.2.5, $FP_\bullet \cong NF\Gamma P_\bullet$ and hence $H_n(FP_\bullet) \cong L_nF(A)$ for every n.

Let $\varphi : A \to B$ be a homomorphism between modules, let $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively and let $f_\varphi : P_\bullet \to Q_\bullet$ be a morphism induced by Theorem 1.2.10. Then Lemma 3.2.6 implies that the diagram

$$\begin{array}{ccc} H_n(FP_\bullet) & \xrightarrow{\cong} & L_nF(A) \\ {\scriptstyle H_n(Ff_\varphi)}\downarrow & & \downarrow{\scriptstyle L_nF(\varphi)} \\ H_n(FQ_\bullet) & \xrightarrow{\cong} & L_nF(B) \end{array}$$

commutes for every n. Hence Definition 1.2.13 and Definition 3.2.3 are equivalent. □

Remark 3.2.7 (Right Derived Functor). In order to generalize the *right derived functor* one must go about this in a different way, which we will shortly sketch. We define *cosimplicial objects* in a category $\mathscr{C}$ to be the covariant functors between Δ and $\mathscr{C}$. Then one can define functors N and Γ which form an equivalence of the category of non-negative cochain complexes and the category of cosimplicial modules, i.e. another version of the Dold-Kan correspondence. Again one can show that N and Γ preserve homotopy. Now let $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ be a covariant functor. We define the n'th *right derived functor*, $R^nF : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$ in the following way: let A be a Λ-module and $I_\bullet$ be an *injective resolution* of A (i.e. the dual of a projective resolution). Then $R^nF(A) := H^n(NF\Gamma I_\bullet)$, i.e. the n'th cohomology module of the non-negative cochain complex $NF\Gamma I_\bullet$. Let $\varphi : A \to B$ be a homomorphism and let $I_\bullet$ and $J_\bullet$ be injective resolutions of A and B respectively. By the dual of Theorem 1.2.10 there is an induced morphism $f_\varphi : I_\bullet \to J_\bullet$ which is unique up to homotopy. Then we define $R^nF(\varphi) := H^n(NF\Gamma f_\varphi)$.

Just as for the left derived functor, this definition does not depend on the choice of injective resolution $I_\bullet$ or of the choice of the induced morphism f_φ. One can then show that if F is an additive functor this definition coincides with the classical definition of the right derived functor of an additive functor.

3.3 Applications and Examples

Every theorem in Section IV.5 in [6] can be fitted such that it applies to Definition 3.2.3, by adding to the theorem that the functor must be additive. Note that we do not need to change Proposition 5.2, 5.5 and 5.6 since they recuire our functor F to be left (or right) exact, which implies that it is additive. In this section we generalize Proposition IV.5.3 in [6], and by introducing the symmetric power and the symmetric algebra functors, we give some examples of how to apply the left derived functor of a non-additive functor.

We start out by proving a generalization of Proposition IV.5.3 in [6]

Theorem 3.3.1. *Let $F : \textbf{\textit{Mod}}_\Lambda \to \textbf{\textit{Mod}}_{\Lambda'}$ be a functor and P a projective Λ-module. Then $L_0F(P) = FP$ and $L_nF(P) = 0$ for $n \geq 1$.*

Proof. Let $P_\bullet$ be the non-negative chain complex where $P_0 = P$ and $P_n = 0$ for $n \neq 0$. This is a projective resolution of P. Note that $\Gamma P_\bullet$ is the simplicial module where $(\Gamma P_\bullet)_n = P$ for every n and every face and degeneracy is id_P. Hence $F\Gamma P_\bullet$ is the simplicial module where $(F\Gamma P_\bullet)_n = FP$ and every face and degeneracy is id_{FP} and since $\ker id_{FP} = 0$

$$NF\Gamma P_\bullet : \cdots \to 0 \to 0 \to FP$$

Hence $L_0F(P) = FP$ and $L_nF(P) = 0$ for $n \geq 1$. □

The next theorem is a direct consequence of Proposition IV.5.4 in [6].

Theorem 3.3.2. *Let $F : \textbf{\textit{Mod}}_\Lambda \to \textbf{\textit{Mod}}_{\Lambda'}$ be an additive functor. Then L_nF is additive for every n.*

One may ask if L_nF is additive even though F is not additive. This is not the case in general. In order to give an example of this not being true we define the symmetric power and the symmetric algebra of a module.

Definition 3.3.3 (Symmetric Power and Symmetric Algebra)**.** For a Λ-module A and $n \geq 1$ let $A^{\otimes n}$ denote the n'th *tensor power* of A, i.e. $A \otimes_\Lambda \cdots \otimes_\Lambda A$ where there are n factors. Now define the equivalence relation "$\sim$" on $A^{\otimes n}$ by $a_1 \otimes \cdots \otimes a_n \sim b_1 \otimes \cdots \otimes b_n$ if and only if there exists a permutation $\sigma \in S_n$ such that $a_1 \otimes \cdots \otimes a_n = b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$. We define the functor $S^n : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_\Lambda$ for $n \geq 1$ such that $S^n(A) = A^{\otimes n}/\sim$ and for a homomorphism $\varphi : A \to B$ let $S^n(\varphi)$ be the induced homomorphism $\varphi \otimes \cdots \otimes \varphi : S^n(A) \to S^n(B)$. Furthermore we denote $S^0(A) = \Lambda$ and $S^0(\varphi) = id_\Lambda$. We call $S^n(A)$ the n'th *symmetric power* of A.

Now define the functor $S : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_\Lambda$ by $S(A) = \bigoplus_{n\geq 0} S^n(A)$ and for a homomorphism $\varphi : A \to B$ let $S(\varphi) = \bigoplus_{n\geq 0} S^n(\varphi)$. We call $S(A)$ the *symmetric algebra* of A.

Note that S^n is not additive for $n \geq 2$. E.g. consider the Λ-modules Λ and Λ^2. If S^n was additive then $S^n(\Lambda^2) \cong S^n(\Lambda)^2$ but $S^n(\Lambda) \cong \Lambda$ and $S^n(\Lambda^2) \cong \Lambda^{n+1}$ and thus S^n is not additve. Since Λ and Λ^2 are free modules Theorem 3.3.1 implies that L_0S^n is not additive for $n \geq 2$. Hence L_nF is generally non-additive for a non-additive functor $F : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_{\Lambda'}$.

In the first example we wish to calculate $L_0S(\mathbb{Z}/p)$ and $L_1S(\mathbb{Z}/p)$ where $\mathbb{Z}/p$ is a $\mathbb{Z}/p^2$-module and p is a prime number. This is a nice example of how to apply the left derived functor of a non-additive functor in general. Note that if $S : \mathbf{Mod}_\Lambda \to \mathbf{Mod}_\Lambda$ then there is a canonical isomorphism $S(\bigoplus_{i=1}^n \Lambda) \cong \Lambda[x_1, \dots, x_n]$ by mapping each basis element to a variable.

Example 3.3.4. Let $\Lambda := \mathbb{Z}/p^2$ for some prime number p. We wish to calculate $L_0S(\mathbb{Z}/p)$ and $L_1S(\mathbb{Z}/p)$. Note that $P_\bullet : \cdots \to \Lambda \xrightarrow{\cdot p} \Lambda \xrightarrow{\cdot p} \Lambda$ is a projective resolution of $\mathbb{Z}/p$. Now we get that $(S\Gamma P_\bullet)_0 = S(\Lambda) \cong \Lambda[x]$ and that $(S\Gamma P_\bullet)_1 = S(\Lambda_{s^0} \oplus \Lambda_{id}) \cong \Lambda[x, y]$ where x corresponds to s^0 and y to id. Since $Ss_0(x^n) = x^n$ for $n \geq 0$ we get that

$$(NS\Gamma P_\bullet)_1 = \frac{\Lambda[x, y]}{\mathrm{Im} Ss_0} = \frac{\Lambda[x, y]}{\Lambda[x]} = y\Lambda[x, y].$$

Note that the quotient above is not zero since $\Lambda[x, y]$ is a Λ-module and not an algebra. Now since $Sd_0(y) = 0$, $Sd_1(x) = x$, $Sd_1(y) = -px$ we get that

$$\partial_1(x^n y^m) = (Sd_0 - Sd_1)(x^n y^m) = -(-p)^m x^{n+m}$$

where we used that $m \geq 1$. Hence $\mathrm{Im}\partial_1 = px\Lambda[x]$ and thus

$$L_0S(\mathbb{Z}/p) = \frac{\Lambda[x]}{px\Lambda[x]} \cong \Lambda \oplus x(\mathbb{Z}/p)[x] \cong S(\mathbb{Z}/p)$$

Our next goal is to find $\ker \partial_1$. We consider the polynomials which map to polynomials of the form ax^{n+1}. These are the polynomials of the form $y \sum_{i=0}^n a_i x^{n-i} y^i$. We get that

$$\partial_1 \left(y \sum_{i=0}^n a_i x^{n-i} y^i \right) = p \sum_{i=0}^n a_i (-p)^i x^{n+1} = pa_0 x^{n+1}$$

and thus $y \sum_{i=0}^n a_i x^{n-i} y^i \in \ker \partial_1$ if and only if $a_0 \in \mathbb{Z}/p$. Hence $\ker \partial_1 \cong (y^2\Lambda[x, y]) \oplus \bigoplus_{n \geq 1} \mathbb{Z}/p$ by the isomorphism φ given by

$$\varphi(ax^n y^m) = \begin{cases} (ax^n y^m, 0) & \text{if } m > 1 \\ (0, \iota_n a) & \text{if } m = 1 \end{cases}$$

Now $(S\Gamma P_\bullet)_2 = S(\Lambda_{s^0 s^0} \oplus \Lambda_{s^0} \oplus \Lambda_{s^1} \oplus \Lambda_{id}) \cong \Lambda[x, y, z, w]$ where x corresponds $s^0 s^0$, y to s^0, z to s^1 and w to id. We get that $Ss_0(x^n y^m) = x^n y^m$ and $Ss_1(x^n y^m) = x^n z^m$. Hence $\mathrm{Im} Ss_0 + \mathrm{Im} Ss_1 = \Lambda[x, y] + \Lambda[x, z]$ and thus

$$(NS\Gamma P_\bullet)_2 = \frac{\Lambda[x, y, z, w]}{\Lambda[x, y] + \Lambda[x, z]} \cong w\Lambda[x, y, z, w] \oplus yz\Lambda[x, y, z].$$

Now for $m \geq 1$

$$\partial_2(x^k y^l z^n w^m) = (Sd_0 - Sd_1 + Sd_2)(x^k y^l z^n w^m) = (-1)^l p^{l+m} x^{k+l} y^{n+m}$$

since $Sd_0 w = Sd_1 w = 0$ and thus $\partial_2(w\Lambda[x,y,z,w]) = py\Lambda[x,y]$. Now for $l, n \geq 1$ we get

$$\partial_2(x^k y^l z^n) = -x^k y^{l+n} + (-p)^l x^{k+l} y^n$$

where we used that $Sd_0 z = 0$, $Sd_1 y = Sd_1 z = y$, $Sd_2 y = -px$ and $Sd^2 z = y$. It can now be verified that that

$$\mathrm{Im}\partial_2 = y((y - px)\Lambda[x,y] + p\Lambda[x,y]) = \ker \partial_1$$

and thus $L_0 S(\mathbb{Z}/p) = 0$.

Our next goal is to give a generalized form of how to calculate $L_0 S^2(A)$ for any Λ-module A.

Example 3.3.5. Let A be a Λ-module and $P_\bullet : \cdots P_2 \to P_1 \to P_0$ be a projective resolution of A and denote the differentials ∂', as not to confuse these with the differentials ∂ in $NS^2\Gamma P_\bullet$. Then $(\Gamma P_\bullet)_0 = P_0$ and thus $(NS^2\Gamma P_\bullet)_0 = S^2(P_0)$. Note that since $(S^2 d_0 - S^2 d_1)(\mathrm{Im} S^2 s_0) = 0$ we get that $\mathrm{Im}\partial_1 = \mathrm{Im}(S^2 d_0 - S^2 d_1)$ by Theorem 2.3.2, where $S^2 d_0 - S^2 d_1 : (S^2\Gamma P_\bullet)_1 = ((P_{1,id} \oplus P_{0,s^0})^{\otimes 2}/\sim) \to (S^2\Gamma P_\bullet)_0 = S^2(P_0)$. Here we indexed the modules in $(\Gamma P_\bullet)_1$ by the surjective morphism $[1] \twoheadrightarrow [m]$ to which they correspond. Let $(a_1, a_2) \otimes (b_1, b_2) \in (P_{1,id} \oplus P_{0,s^0})^{\otimes 2}$. Then

$$\begin{aligned}
& (S^2 d_0 - S^2 d_1)[(a_1, a_2) \otimes (b_1, b_2)] \\
= \; & [d_0(a_1, a_2) \otimes d_0(b_1, b_2)] - [d_1(a_1, a_2) \otimes d_1(b_1, b_2)] \\
= \; & [a_2 \otimes b_2] - [(a_2 - \partial'(a_1)) \otimes (b_2 - \partial'(b_1))] \\
= \; & [(a_2 - \partial'(a_1) \otimes \partial'(b_1)] + [b_2 \otimes \partial'(a_1)]
\end{aligned}$$

Clearly $\mathrm{Im}(S^2 d_0 - S^2 d_1) = (P_0 \otimes_\Lambda \mathrm{Im}\partial'_1)/\sim$ and hence

$$L_0 S^2(A) = \frac{S^2 P_0}{\mathrm{Im}(S^2 d_0 - S^2 d_1)} = \frac{(P_0 \otimes_\Lambda P_0)/\sim}{(P_0 \otimes_\Lambda \mathrm{Im}\partial'_1)/\sim}$$

Example 3.3.5 gives us an easy way of calculating $L_0 S^2(A)$ for some Λ-module A. Our final example shows us how we can use this in a simple matter.

Example 3.3.6. Let k be a field and $\Lambda := k[x, y]$. We will use Example 3.3.5 to show that $L_0S^2(k) = k$ and $L_0S^2(k^2) = k^3$, where $k := \Lambda/(x\Lambda + y\Lambda)$. Let $P_\bullet : \cdots \to 0 \to \Lambda \to \Lambda^2 \to \Lambda$ be the projective resolution of k where $\partial_1(a, b) = ax + bx$. Then

$$L_0S^2(k) = \frac{(\Lambda \otimes_\Lambda \Lambda)/\sim}{(\Lambda \otimes_\Lambda (x\Lambda + y\Lambda))/\sim} \cong \frac{\Lambda}{x\Lambda + y\Lambda} = k.$$

Let $Q_\bullet = P_\bullet \oplus P_\bullet : \cdots \to 0 \to \Lambda^2 \to \Lambda^4 \to \Lambda^2$ which is a projective resolution of k^2. Using the canonical isomorphism $(\Lambda^2 \otimes_\Lambda \Lambda^2)/\sim\, \cong \Lambda^3$ one can easily verify that $(\Lambda^2 \otimes_\Lambda (x\Lambda + y\Lambda))/\sim\, \cong (x\Lambda + y\Lambda)^3$. Hence

$$L_0S^2(k^2) = \frac{(\Lambda^2 \otimes_\Lambda \Lambda^2)/\sim}{(\Lambda^2 \otimes_\Lambda (x\Lambda + y\Lambda)^2)/\sim} \cong \frac{\Lambda^3}{(x\Lambda + y\Lambda)^3} = k^3$$

Let $\Delta : A \to A^2$ be given by $\Delta(a) = (a, a)$ for any Λ-module A. We wish to show that for $\Delta : k \to k^2$ we have $L_0S^2\Delta = \{id, 2id, id\}$. The diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\sim} & k \\ {\scriptstyle \Delta_\bullet}\downarrow & & \downarrow{\scriptstyle \Delta} \\ Q_\bullet & \xrightarrow{\sim} & k^2 \end{array}$$

is commutative, where $\Delta_\bullet$ is the morphism which is $\Delta : P_n \to P_n \oplus P_n = Q_n$ in degree n. Now

$$(S^2\Gamma\Delta)_0 = \Delta \otimes \Delta : S^2(\Lambda) = \Lambda \to S^2(\Lambda^2) = \Lambda^3$$

We now get that $(\Delta \otimes \Delta)(a) = (a, 2a, a)$ and thus $L_0S^2\Delta : k \to k^3$ is given by $L_0S^2\Delta(a) = (a, 2a, a)$ which is that $L_0S^2\Delta = \{id, 2id, id\}$.

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