Department of Mathematical Sciences
UNIVERSITY OF COPENHAGEN

Anders Jess Pedersen, Magnus Baunsgaard Kristensen

Straightening and Unstraightening

Advisor
Lars Hesselholt

January 25, 2018

Contents

1 Introduction

In this paper we study a classical construction, the Grothendieck construction, and its higher categorical analogue, Straightening and Unstraightening. The first of these is a procedure for producing a fibered category from the data of a pseudofunctor. Those familiar with the theory of pseudofunctors will know that this is an involved process, simply because the data is so extensive. The purpose of the Grothendieck construction is then to repackage this data into the easier to understand theory of fibered categories. The main result of the first part is that this repackaging is lossless, in the sense that the Grothendieck construction is an equivalence of bicategories. This is shown in section 2. As we shall see, a large part of this section will be devoted to setting up the machinery; even the definition of "equivalence of bicategories" requires a great deal of work.

In the second part of the paper, we introduce many notions of higher category theory, so that we may formulate theorems in it's language. This will consist of a preliminary discussion of ∞-categories, including many unproven facts, an extensive discussion of the several model structures which come up, and finally an outline of the proof of the analogue of the Grothendieck construction equivalence. In doing this, we shall see that even in this new setting, it is quite hard to work with simplicial functors, which are the analogue of pseudofunctors, and relatively easy to work with the relevant over category. Having presented the theoretical framework, we will give some simple examples of what this equivalence actually does to a specific slice category, so that we may gain some insight into the machinery of Straightening and Unstraightening.

1.1 Universes

Now we will present the usual foundations for the category theoretical framework, namely that of universes. The important part of this exposition is not to understand the details, but rather to understand that there are foundational issues, but that these can be resolved.

The primary goal of classical category theory is to talk about categories like the category of R-modules, for some ring R. One could naively try to form this category by simply taking all R-modules. This runs into the problem that the free R-module generated by a set x is an R-module. So for every set we get a module, but the collection of all sets is not a set, so the collection of R-modules cannot be a set. We remedy this by restricting our attention to only R-modules of cardinality bounded by some $\mathcal{U}$. Usually we will call this cardinality the universe, and we might call sets contained in $\mathcal{U}$ “small”. Usually one would call these sets “$\mathcal{U}$-small”, but since the $\mathcal{U}$ is fixed throughout this paper we will suppress it in the notation.

This approach is due to Grothendieck, and people will often say "Grothendieck universe" instead of just universe. There are some set theoretic restrictions to be aware of in the choice of universe; more specifically we require that $\mathcal{U}$ is the collection of all sets less than some strongly inaccessible cardinal:

Definition 1.1
Let α be an uncountable cardinal. We say that α is strongly inaccessible if it cannot be obtained by the usual set theoretic operations from strictly smaller cardinals. Explicitly, we can say that α is not the sum of less than α cardinals each smaller than α, and $2^\beta < \alpha$ for all $\beta < \alpha$.

This approach is a slight modernization of Grothendiecks construction. The original list of properties was the following:

Definition 1.2
We say that a set $\mathcal{U}$ is a universe if

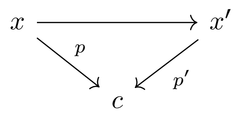
- It is transitive, in the sense that the membership relation is transitive when restricted to $\mathcal{U}$;
- for any two sets $x, y \in \mathcal{U}$ we also have their unordered pair $\{x, y\} \in \mathcal{U}$;
- $\mathcal{U}$ is closed under the powerset operation;
- and finally, if $I \in \mathcal{U}$ and $\{x_i\}_{i\in I}$ is a family of sets in $\mathcal{U}$, the union $\cup_{i\in I} x_i \in \mathcal{U}$.

This list of properties will seem familiar to anyone that has encountered the ZFC axioms in their pure form. Indeed, it might be informally translated to "a universe is a set which is a model of ZFC". Of course, this is only true under the assumption that one already has a model of ZFC which contains a cardinal which satisfies the conditions of definition 1.1, which is why these approaches can be thought of as equivalent. To those not familiar with the particulars of set theory, this treatment might seem superfluous or mystic. For the benefit of those readers, we remark that one can safely ignore these set theoretic issues in ones everyday life. Nevertheless, it is important that these things are possible, so that the constructions of our mathematics make sense.

2 The Grothendieck Construction

We start by stating the familiar definition of a over category.

Definition 2.1
Let $\mathscr{C}$ be a category, and $c \in \mathscr{C}$ an object. The objects of $\mathscr{C}_{/c}$ is the collection of pairs (x, p) where x is an object of $\mathscr{C}$ and $p : x \to c$ is a map in $\mathscr{C}$, and a morphism of pairs $(x, p) \to (x', p')$ is a commuting triangle



This category is called the over category. One can define the dual notion of under category by considering pairs (x, i) where i is a morphism $c \to x$ and the morphisms are the obvious dual of the above.

Examples of such categories are naturally ubiquitous. One trivial example is when the category has a terminal object. The over category over this object will clearly be isomorphic to the original category (the condition that morphisms be compatible with the the only possible projection is no condition at all). These cases are not very interesting at all, but by either choosing a base object with some desirable structure, or considering certain sub-categories of the over categories we can create interesting structures.

2.1 Fibered Categories

The case we will be considering in this chapter is the following, where $\mathscr{C}$ is the 2-category of small categories Cat.

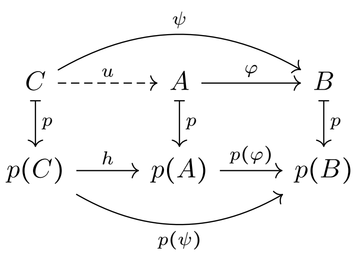
We first introduce some terminology which was developed by Grothendieck in his work on descent theory [AG04].

Definition 2.2
Consider the category $\mathsf{Cat}_{/\mathscr{E}}$ where $\mathscr{E}$ is some category. Elements of this category are called $\mathscr{E}$-categories, or epsilon categories.

It is worth making this definition slightly more explicit. Since we are in Cat to begin with, the morphisms are functors; which is to say that an object $(\mathscr{C}, p) \in \mathsf{Cat}_{/\mathscr{E}}$ consists of a category and a privileged functor $p : \mathscr{C} \to \mathscr{E}$. We will call this functor the projection functor, or simply the projection.

We will be considering a further refinement of this notion, namely that of fibered categories. Defining this notion will amount to restricting the possible morphisms. In the following many diagrams will contain arrows of different types, we will denote the privileged functor $p : \mathscr{C} \to \mathscr{E}$, in diagrams as $\mapsto$, to indicate the passing from $\mathscr{C}$ to $\mathscr{E}$.

Definition 2.3
Let $\mathscr{C}$ be an $\mathscr{E}$-category. An arrow $\varphi : A \to B \in \mathscr{C}$ is called Cartesian if for any arrow $\psi : C \to B \in \mathscr{C}$ and any arrow $h : p(C) \to p(A) \in \mathscr{E}$ with $p(\varphi) \circ h = p(\psi)$, there exists a unique arrow $u : C \to A \in C$ such that $p(u) = h$ and $\varphi \circ u = \psi$. Diagrammatically



Remark 2.4
We will say that $f : A \to B$ is Cartesian over a functor $F : \mathscr{C} \to \mathscr{D}$, when it is Cartesian over $F(f) : F(A) \to F(B)$.

We will see later that his is indeed the correct restriction for our purposes, but first, we will show that it is a valid restriction on the morphisms.

Definition 2.5
A fibered category over $\mathscr{E}$ is a $\mathscr{E}$-category, $\mathscr{C}$, such that given an arrow $f : A \to B \in \mathscr{E}$ and object $T \in \mathscr{C}$ such that $p(T) = B$, there exists Cartesian arrow $\varphi : W \to T$ with $p(\varphi) = f$.

The collection of categories fibered over a common base, is itself a category in the following sense.

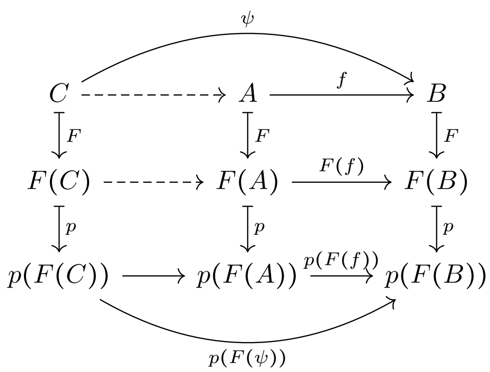
Definition 2.6
Let $\mathscr{C}$ and $\mathscr{D}$ be fibered categories over $\mathscr{E}$, with projections p and q respectively, then a morphism of fibered categories $F : \mathscr{C} \to \mathscr{D}$ is a functor such that

1. F is base preserving, $q \circ F = p$.
2. F sends Cartesian arrows to Cartesian arrows, where Cartesian is interpreted as Cartesian over the respective projections.

This notion of morphism makes the collection of fibered categories over $\mathscr{E}$ into a category of its own denoted $\mathsf{Fib}(\mathscr{E})$. The fact that this is a category is a direct consequence of the following lemma.

Proposition 2.7
Let $p : \mathscr{D} \to \mathscr{E}$ be the associated projection to the $\mathscr{E}$-category $\mathscr{D}$ and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let $f : A \to B \in \mathscr{C}$. If f is Cartesian over F and $F(f)$ is Cartesian over p, then f is Cartesian over $p \circ F$.

Proof. Consider the diagram



Because $F(f)$ is Cartesian over $p(F(f))$ we can fill in the lower dashed arrow, such that the lower two squares commute, and there exists a map $F(C) \to F(B)$ which projects to $p(F(\psi))$ and equal to the composite of the dashed arrow and $F(f)$. Now since f is Cartesian over $F(f)$ we can fill in the upper dashed arrow, such that the upper two squares commutes. Hence f is Cartesian over $p \circ F$. □

The language of fibered categories and projections suggest that we might be able to speak of fibers over some category and projection, and indeed we can.

Definition 2.8
Let $\mathscr{C}$ be a fibered category over $\mathscr{E}$. Given an object $\alpha \in \mathscr{E}$, the fiber $p^{-1}(\alpha)$ of $\mathscr{C}$ over α is the subcategory of $\mathscr{C}$ whose objects are the objects $A \in \mathscr{C}$ such that $p(A) = \alpha$, and whose arrows are arrows $f \in \mathscr{C}$ with $p(f) = \mathrm{id}_\alpha$.

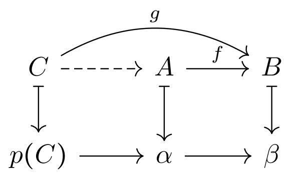
Note that the notation $p^{-1}(\alpha)$ is abuse of notation, since this is not the same as the set theoretic notion of preimage.

Before moving on, we should also remark that this is not all the structure. As we have seen, this category has essentially been obtained by iterated refinement of the category of categories. This hints at the fact that there might be more structure, analogous to the 2-categorical structure on Cat mediated by natural transformations. Indeed, it is a bicategory, which we have yet to define. We therefore leave out the details of checking that this is the case, other than remarking that the 2-morphisms in this case are simply the natural transformations of functors, with the caveat that the fibers of the natural transformations should stay in $p^{-1}(x)$ for some x.

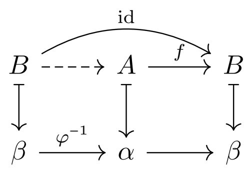
The tentative goal of this section is to construct an object called a pseudofunctor from a fibered category, and to do this we will need the following proposition.

Proposition 2.9
Let $p : \mathscr{C} \to \mathscr{E}$ be the projection of the $\mathscr{E}$-category $\mathscr{C}$. Let $\varphi : \alpha \to \beta$ be an isomorphism in $\mathscr{E}$. An arrow $f : A \to B \in \mathscr{C}$ such that $p(f) = \varphi$, is an isomorphism if and only if f it is Cartesian.

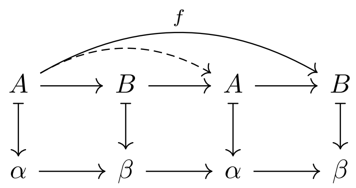
Proof. First we assume that f is an isomorphism. In this case, we can produce the desired lift in



simply by postcomposing by the inverse of f. Assuming now that f is Cartesian, we will in particular get a lift



Which produces a right inverse to f, but we also have a lift of



Clearly the compostion of the solid arrows $A \to B \to A$ constitute a possible lift, but so is the identity on A. Hence by uniqueness the two are equal and we have produced a left inverse to f, hence it must be an isomorphism. ☐

As we remarked with the 2-morphisms, we will have to require that the morphisms of fibered categories are compatible with the projections. But in this case we are in luck; they in fact respect the structural maps automatically.

Remark 2.10
A morphism of fibered categories $F : \mathscr{C} \to \mathscr{D}$, sends $p^{-1}(\alpha)$ to $q^{-1}(\alpha)$. This means that there is a restriction functor, restricting both domain and codomain to the fiber over a specific element, $F_\alpha : p^{-1}(\alpha) \to q^{-1}(\alpha)$.

It will sometimes be convenient to have a fixed system of Cartesian arrows in a fibered category.

Definition 2.11
A cleavage of a fibered category $\mathscr{C}$ over $\mathscr{E}$ consist of a class K of Cartesian arrows such that for each arrow $\varphi : \alpha \to \beta \in \mathscr{E}$ and each object $B \in p^{-1}(\beta)$, there exist a unique arrow $f : A \to B$ in K such that $p(f) = \varphi$. Diagrammatically

$$\begin{array}{ccc} A & \overset{f}{\dashrightarrow} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \alpha & \underset{\varphi}{\longrightarrow} & \beta \end{array}$$

Where the solid arrows are the assumptions and the dashed arrow and A are simultaneously induced.

Note that in this definition $p^{-1}(\beta)$ can be empty, in which case the statement is tautological. In fact, one can always assume that fibered category admits a cleavage by the following:

Proposition 2.12
Every fibered category $\mathscr{C}$ has a cleavage.

Proof. Because $\mathscr{C}$ is fibered we have for each diagram

$$\begin{array}{ccc} A & & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \alpha & \underset{\varphi}{\longrightarrow} & \beta \end{array}$$

some Cartesian map $f : A \to B$ for which $f \mapsto \varphi$. There are potentially many such arrows. Using axiom of choice we pick for each such lifting problem a Cartesian f, and collect them into K. ☐

We now consider a construction which will play a crucial role in the proof of the main theorem of this section.

Construction 2.13
Let $\mathscr{C}$ be a fibered category of $\mathscr{E}$ with cleavage. All maps in the following will come from the cleavage of $\mathscr{C}$. Let $\varphi : \alpha \to \beta \in \mathscr{E}$. For each $B \in p^{-1}(\beta)$, we choose a pullback $\varphi_B : \varphi^\bullet B \to B$. Define $\varphi^\bullet : p^{-1}(\beta) \to p^{-1}(\alpha)$ by sending each object $B \in p^{-1}(\beta)$ to $\varphi^\bullet B$, and each Cartesian arrow $f : A \to B$ to the unique arrow $\varphi^\bullet f : \varphi^\bullet A \to \varphi^\bullet B$. We now construct $\varphi^\bullet f$. Consider the diagram

$$\begin{array}{ccccc} \varphi^\bullet B & \overset{u}{\dashrightarrow} & A & \overset{f}{\longrightarrow} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \alpha & \overset{h}{\longrightarrow} & \beta & \longrightarrow & \beta \end{array} \qquad (\varphi_B : \varphi^\bullet B \to B)$$

where u is induced because f is Cartesian. Next consider

$$\begin{array}{ccccc} \varphi^\bullet A & \overset{u'}{\dashrightarrow} & \varphi^\bullet B & \overset{f}{\longrightarrow} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \alpha & \overset{h}{\longrightarrow} & \alpha & \longrightarrow & \beta \end{array} \qquad (f \circ \varphi_A : \varphi^\bullet A \to B)$$

We have constructed the desired map by setting $u' = \varphi^\bullet f$. Note that the maps makes the following diagram commute

$$\begin{array}{ccc} \varphi^\bullet A & \overset{\varphi_A}{\longrightarrow} & A \\ \downarrow{\scriptstyle \varphi^\bullet f} & & \downarrow{\scriptstyle f} \\ \varphi^\bullet B & \overset{\varphi_B}{\longrightarrow} & B \end{array}$$

Now using this construction we can associate to each object $\alpha \in \mathscr{E}$ a category $p^{-1}(\alpha)$, and to each arrow $\varphi : \alpha \to \beta$ we associate a functor $\varphi^{\bullet} : p^{-1}(\beta) \to p^{-1}(\alpha)$.

Remark 2.14
Now note that $\mathrm{id}_{\alpha}^{\bullet} : p^{-1}(\alpha) \to p^{-1}(\alpha)$, are not necessarily identities, because the objects of $p^{-1}(A)$ might be scrambled by $\mathrm{id}_{\alpha}^{\bullet}$. Which shows that the above correspondence is not a functor, because it does not preserve identities. But when $\alpha \in \mathscr{C}$ and $A \in p^{-1}(\alpha)$, we have that $E_{\alpha}(A) : \mathrm{id}_{\alpha}^{\bullet}(A) \to A$ is an isomorphism, via proposition 2.9, which become the components of an isomorphism of functors $E_{\alpha} : \mathrm{id}_{\alpha}^{\bullet} \simeq \mathrm{id}_{p^{-1}(\alpha)}$. An analogous problem arise with composition. Suppose we have two arrows $\varphi : \alpha \to \beta$ and $\psi : \beta \to \gamma$ in $\mathscr{E}$, and an object $C \in p^{-1}(\gamma)$. Then $\varphi^{\bullet}\psi^{\bullet}C$ is a pullback of C to α, but there is no reason this should coincide with $(\psi\varphi)^{\bullet}C$. But again there is a canonical isomorphism $\alpha_{\varphi,\psi}(C) : \varphi^{\bullet}\psi^{\bullet}C \cong (\psi\varphi)^{\bullet}C$ in $p^{-1}(\alpha)$, because they both are pullbacks, and again these are the components of the isomorphism of functors $\alpha_{\varphi,\psi} : \varphi^{\bullet}\psi^{\bullet} \cong (\varphi\psi)^{\bullet}$.

The primary purpose of the next section is to work around the fact that this does not define a functor.

2.2 Bicategories

We saw in the previous section that choosing a fibered category over $\mathscr{E}$ with cleavage, we almost obtain a functor $\mathscr{E} \to \mathsf{Cat}$, up to the problems mentioned above. We obtain something more general namely a pseudo-functor, which is a consequence of the fact that Cat is a bicategory. This motivates the following discussion. This section is based on an unpublished book written by Lars Hesselholt and Ib Madsen [LH].

Definition 2.15
A bicategory, $\mathscr{C}$, consists of

- a set of objects $\mathrm{ob}\,\mathscr{C}$;
- for each pair of objects $c_0, c_1 \in \mathrm{ob}\,\mathscr{C}$ a category $\mathrm{Hom}_{\mathscr{C}}(c_0, c_1)$, where the objects of the category are called 1-morphisms and the morphisms are called 2-morphisms;
- for each triple of objects c_0, c_1, c_2, a composition functor

$$\mathrm{Hom}_{\mathscr{C}}(c_0, c_1) \times \mathrm{Hom}_{\mathscr{C}}(c_1, c_2) \to \mathrm{Hom}\,\mathscr{C}(c_0, c_2)$$

- for each object c, a 1-morphism id_c;
- an associativity constraint, in the form of, for each triple f_1, f_2, f_3 of composable morphisms, an invertible 2-morphism $(f_3 \circ f_2) \circ f_1 \Rightarrow f_3 \circ (f_2 \circ f_1)$;
- left and right identity constraints, in the form of invertible 2-morphisms $\mathrm{id}_c \circ f \Rightarrow f$ and $f \circ \mathrm{id}_c \Rightarrow f$.

Furthermore, these objects and morphisms are subject to some constraints on the interplay between the different 2-morphisms:

- For any four composable morphisms, f_1, f_2, f_3 and f_4, the two different ways to obtain $f_4 \circ (f_3 \circ (f_2 \circ f_1))$ from $(f_4 \circ (f_3 \circ f_2)) \circ f_1$ agree, i.e, the following commutes:

$$\begin{array}{ccccc}
(f_4 \circ f_3) \circ (f_2 \circ f_1) & & \Longrightarrow & & f_4 \circ (f_3 \circ (f_2 \circ f_1)) \\
\Uparrow & & & & \Uparrow \\
((f_4 \circ f_3) \circ f_2) \circ f_1 & \Longrightarrow & (f_4 \circ (f_3 \circ f_2)) \circ f_1 & \Longrightarrow & f_4 \circ ((f_3 \circ f_2) \circ f_1)
\end{array}$$

- For every pair of composable morphisms, f_1, f_2, the two ways to get from $f_2 \circ (\mathrm{id} \circ f_1)$ to $f_2 \circ f_1$ agree, i.e, the following commutes:

$$\begin{array}{ccc} f_2 \circ (\mathrm{id} \circ f_1) & \Longrightarrow & (f_2 \circ \mathrm{id}) \circ f_1 \\ & \searrow \quad \swarrow & \\ & f_2 \circ f_1 & \end{array}$$

As with usual categories, we have a notion of structure preserving map into bicategories.

Definition 2.16
Let $\mathscr{C}$ and $\mathscr{D}$ be bicategories. A pseudofunctor, $F{:}\mathscr{C} \to \mathscr{D}$, is

- An assignment of an object $F(c) \in \mathscr{D}$ to each object $c \in \mathscr{C}$;
- a functor $\mathrm{Hom}_{\mathscr{C}}(c_0, c_1) \to \mathrm{Hom}_{\mathscr{D}}(F(c_0), F(c_1))$ for each pair of objects $c_0, c_1 \in \mathscr{C}$ (as usual we denote the image of f by $F(f)$;
- an invertible 2-morphisms $r_F : id_{F(c)} \Rightarrow F(id_c)$ and $l_F : F(id_c) \Rightarrow id_{F(c)}$;
- an invertible 2-morphism $\alpha_{f,g}{:}F(g) \circ F(f) \Rightarrow F(g \circ f)$, called the compositor morphism.

These collections of 2-morphisms have to satisfy two compatibility criteria, namely that for composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3$$

the different ways to get from $(F(f_1) \circ F(f_2)) \circ F(f_3)$ to $F(f_3 \circ (f_2 \circ f_1))$ agree, i.e., the following commutes

$$\begin{array}{ccc} F(f_3) \circ (F(f_2) \circ F(f_1)) & \xRightarrow{\alpha_{f_1,f_2}} & F(f_3) \circ F(f_2 \circ f_1) \\ \Uparrow & & \Downarrow \alpha_{(f_2 \circ f_1),f_3} \\ (F(f_3) \circ F(f_2)) \circ F(f_1) & & F(f_3 \circ (f_2 \circ f_1)) \\ \Downarrow \alpha_{f_1,f_2} & & \Uparrow \\ F(f_3 \circ f_2) \circ F(f_1) & \xRightarrow{\alpha_{f_1,(f_3 \circ f_2)}} & F((f_3 \circ f_2) \circ f_1) \end{array}$$

where the unlabeled arrows are the associativity constraints. The other criterion is that the two ways to cancel identities, both from the right and from the left, agree. In other words, for each morphism $f_0 : c_0 \to c_1$ in $\mathscr{C}$ the following commutes:

$$\begin{array}{ccc} \mathrm{id}_{F(c_0)} \circ F(f_0) & \Longrightarrow & F(f_0) \\ \Downarrow & & \Uparrow \\ F(\mathrm{id}_{c_0}) \circ F(f_0) & \xRightarrow{\alpha_{f_0,\mathrm{id}}} & F(\mathrm{id}_{c_0} \circ f_0) \end{array} \qquad \begin{array}{ccc} F(f_0) \circ \mathrm{id}_{F(c_0)} & \Longrightarrow & F(f_0) \\ \Downarrow & & \Uparrow \\ F(f_0) \circ F(\mathrm{id}_{c_0}) & \xRightarrow{\alpha_{\mathrm{id},f_0}} & F(f_0 \circ \mathrm{id}_{c_0}) \end{array}$$

Note that in this definition it is enough to require the existence of either the collection r_F or the collection l_F, since either would be invertible and hence induce the other. The reason we require the existence of both is simply to have a canonical notation.

In the following we will instead use the restricted definition where only the target of the pseudofunctor is a bicategory. One obtains this restricted definition by simply removing the constraints from the codomain category.

Theorem 2.17
Every fibered category $\mathscr{C}$ over $\mathscr{E}$, defines a pseudo-functor $\mathscr{E} \to \mathsf{Cat}$.

Proof. Via construction 2.13 and remark 2.14 remark we see that we get the data of a pseudofunctor, hence we only need to show that the coherence diagrams commute. Consider composable arrows $\varphi : \alpha \to \beta$, $\psi : \beta \to \gamma$ and $\eta : \gamma \to \xi$ and an object $D \in p^{-1}(\xi)$, we check that the following diagram commute

$$\begin{array}{ccc} \varphi^\bullet(\psi^\bullet(\eta^\bullet D)) & \xrightarrow{\alpha_{\varphi,\psi}(\eta^\bullet D)} & ((\psi\varphi)^\bullet(\eta^\bullet D)) \\ \Big\downarrow{\scriptstyle \varphi^\bullet\alpha_{\psi,\eta}(D)} & & \Big\downarrow{\scriptstyle \alpha_{\psi\varphi,\eta}(D)} \\ \varphi^\bullet((\eta\psi)^\bullet D) & \xrightarrow{\alpha_{\varphi,\eta\psi}(D)} & (\eta\psi\varphi)^\bullet D \end{array}$$

Now both $\varphi^\bullet\psi^\bullet\eta^\bullet D$ and $(\eta\psi\varphi)^\bullet D$ are pullbacks of D, hence both project onto ξ. Consider the identity on ξ in $\mathscr{E}$, because we fixed a cleavage on $\mathscr{C}$ we obtain a unique Cartesian arrow between $\varphi^\bullet\psi^\bullet\eta^\bullet D$ and $(\eta\psi\varphi)^\bullet D$ which project onto id_ξ. Diagrammatically

$$\begin{array}{ccc} \varphi^\bullet(\psi^\bullet(\eta^\bullet D)) & \xrightarrow{!\exists} & (\eta\psi\varphi)^\bullet D \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ \xi & \xrightarrow{\mathrm{id}_\xi} & \xi \end{array}$$

At this point we note that both the compositions $\alpha_{\psi\varphi,\eta}(D) \circ \alpha_{\varphi,\psi}(\eta^\bullet D)$ and $\alpha_{\varphi,\eta\psi}(D) \circ \varphi^\bullet\alpha_{\psi,\eta}(D)$ are such arrows, hence they must be the same. Now consider the map $\varphi : \alpha \to \beta$, and an object of $B \in p^{-1}(\beta)$. We wish to construct maps $\mathrm{id}_\alpha^\bullet(\varphi^\bullet B) \to \varphi^\bullet B$ and $\varphi^\bullet(\mathrm{id}_\beta^\bullet B) \to \varphi^\bullet B$. We show the existence of the first, because the other one is similiar. By assumptions $\epsilon_\alpha : \mathrm{id}_\alpha^\bullet \to \mathrm{id}_{p^{-1}(\alpha)}$ is an isomorphism, hence we have the following diagram

$$\begin{array}{ccc} \mathrm{id}_\alpha^\bullet\,\varphi^\bullet B & \longrightarrow & \mathrm{id}_{p^{-1}(\alpha)}\,\varphi^\bullet B \\ \Big\downarrow & & \Big\| \\ (\varphi \circ \mathrm{id}_\alpha)^\bullet B & =\!=\!= & \varphi^\bullet B \end{array}$$

Where the vertical map is an isomorphism by the same argument as above, and the two isomorphisms are the same up to 2-morphism by the above, hence the desired morphism exists. □

In order to construct something like a equivalence of categories we need the categorical structure on both the domain and codomain. The same thing is going to be the case here.

Definition 2.18
Let $\mathscr{E}$ be a category. The collection of contravariant pseudofunctors will be denoted $[\mathscr{E}^{op}, \mathsf{Cat}]$. In fact this collection is a bicategory, in the following sense. The objects are pseudofunctors $\mathscr{E}^{op} \to \mathsf{Cat}$. Given $F, G \in [\mathscr{E}^{op}, \mathsf{Cat}]$ the morphisms are

$$c \mapsto u_c : F(c) \to G(c)$$

together with a 2-morphism u_f for each $f : c \to d$, such that the following coherence diagram commute

$$\begin{array}{ccc} F(c) & \xrightarrow{u_c} & G(c) \\ {\scriptstyle F(f)}\Big\downarrow & \overset{u_f}{\nearrow\!\!\!\nearrow} & \Big\downarrow{\scriptstyle G(f)} \\ F(d) & \xrightarrow{u_d} & G(d) \end{array}$$

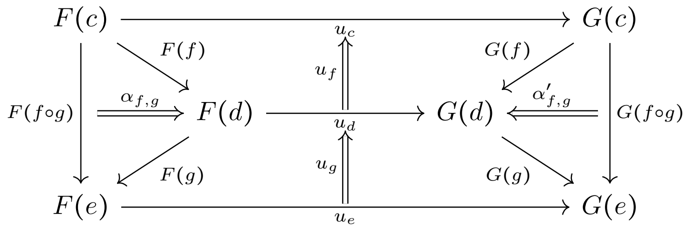
furthermore given $g : d \to e$ we obtain the following diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{u_c} & G(c) \\ {\scriptstyle F(f)}\Big\downarrow & \overset{u_f}{\nearrow\!\!\!\nearrow} & \Big\downarrow{\scriptstyle G(f)} \\ F(d) & \xrightarrow{u_d} & G(d) \\ {\scriptstyle F(g)}\Big\downarrow & \overset{u_g}{\nearrow\!\!\!\nearrow} & \Big\downarrow{\scriptstyle G(g)} \\ F(e) & \xrightarrow{u_e} & G(e) \end{array}$$

and we want to compose the two coherence morphisms we write $u_f \circ u_g$ for the following composition

$$u_e \circ F(g) \circ F(f) \overset{u_g \circ \mathrm{id}_{F(f)}}{\Longrightarrow} G(g) \circ u_d \circ F(f) \overset{\mathrm{id}_{G(g)} \circ u_f}{\Longrightarrow} G(g) \circ G(f) \circ u_c$$

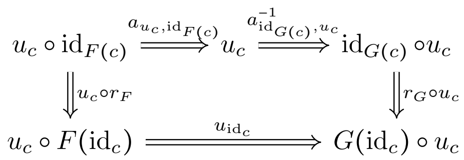
which is subject to the following coherence diagram, where $\alpha_{f,g}$ is the to F associated coherence morphism and $\alpha'_{f,g}$ the one associated to G,



Together with a invertible 2-morphism $u_{f\circ g} : F(f \circ g) \Rightarrow G(f \circ g)$. Which has to satisfy:

$$a_{f,g} \circ (u_f \circ u_g) \circ a'^{-1}_{f,g} = u_{f\circ g}$$

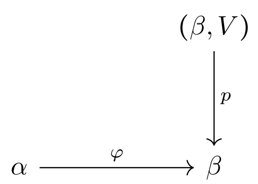
Furthermore for every object $c \in \mathscr{E}$, the diagram of 2-morphisms in Cat must commute



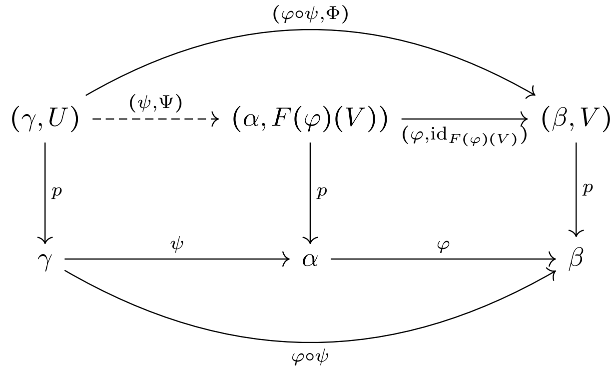
Now we wish to go the other direction, i.e. take a pseudofunctor and produce a fibered category. But first we will consider the case of a functor $F\colon \mathscr{E}^{op} \to \mathsf{Cat}$. This can also be considered as a motivating example for the definition of the Grothendieck construciton.

Example 2.19
In this case, we can consider for each object of $\mathscr{E}$ the image under F, and try to glue these together into a fibered category over $\mathscr{E}$. To do this, we consider pairs of objects in $\alpha \in \mathscr{E}$ and $U \in F(\alpha)$. To get morphisms between such pairs $(\alpha, U), (\beta, V)$, we would like to have a morphism on each factor, and in particular $U \to V$ But U and V are not objects of the same category, so we start with a morphisms $\varphi : \alpha \to \beta$. From this, we can obtain a morphism on the second factor by moving U to $F(\beta)$ by $F(\varphi)$. A morphism between pairs $(\alpha, U), (\beta, V)$ will then be a pair (φ, f), of a morphism $\varphi : \alpha \to \beta$ together with another morphism $f : U \to F(\varphi)(V)$. The identities are then just $(\mathrm{id}, \mathrm{id})$, and we can define composition coordinatewise so that this becomes a category. In the following, we will denote this category by $\int F$, for reasons that will become clear later.

This construction of the category comes with a natural choice of functor into $\mathscr{E}$, namely projection onto the first coordinate. We claim that this makes $\int F$ a category fibered over $\mathscr{E}$. Given a diagram like this



We wish to produce a Cartesian arrow into (β, V), and the only obvious candidate is $(\varphi, \mathrm{id}_{(F(\varphi)(V))}) : (\alpha, F(\varphi)(V)) \to (\beta, V)$, hence we show that this is Cartesian. Consider the diagram



Where $\Phi : U \to F(\varphi \circ \psi)(V)$. $\Psi : U \to F(\psi)(F(\varphi)(V))$, but note by functoriality $F(\varphi \circ \psi) = F(\psi)(F(\varphi))$, hence we can pick Ψ to be Φ hence $(\varphi, \mathrm{id}_{F(\varphi)(V)})$ is Cartesian.

Definition 2.20
Let $\mathscr{E}$ be a category and let $F : \mathscr{E}^{op} \to \mathsf{Cat}$ be a contravariant pseudofunctor. We define the Grothendieck construction $\int F$ of F as follows: its objects are pairs (α, A) where $\alpha \in \mathscr{E}$ and $A \in F(\alpha)$ and morphisms $(\varphi, f) : (\alpha, A) \to (\beta, B)$ are pairs $\varphi : \alpha \to \beta \in \mathscr{E}$ and $f : F(\varphi)(\alpha) \to B \in F(\beta)$.

Remark 2.21
The composite of (φ, f) and $(\psi, g) : (\beta, B) \to (\gamma, C)$ is given as $(\psi \circ \varphi, h)$ where h is the composite

$$A \xrightarrow{f} F(\varphi)(\beta) \xrightarrow{F(\psi)} F(\varphi)(F(\psi)(C)) \xRightarrow{a_{\varphi,\psi}} F(\varphi \circ \psi)(C)$$

This composition is associative. It turns out, as shown in *theorem* 2.23, that $\int F$ is a category. Here we clearly see the difference between using the Grothendieck construction on an ordinary functor versus an pseudofunctor, which we have alluded to in the past couple of pages.

Remark 2.22
The projection $p : \int \mathscr{C} \to \mathscr{E}$ is given by $(\alpha, A) \mapsto \alpha$, and it is functorial, therefore $\int \mathscr{C} \in \mathsf{Cat}_{\mathscr{E}/}$.

Theorem 2.23
There is an equivalence of bicategories

$$\int : [\mathscr{E}^{op}, \mathsf{Cat}] \to \mathsf{Fib}(\mathscr{E})$$

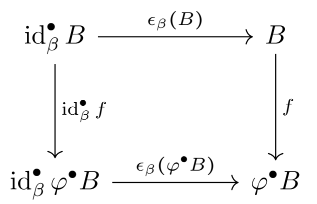
Where $[\mathscr{E}^{op}, \mathsf{Cat}]$ *is the* 2*-category of pseudofunctors* $\mathscr{E}^{op} \to \mathsf{Cat}$.

Proof. To prove that we have an equivalence of 2-categories, we need to produce a pseudo-inverse to $\int$, i.e., a pseudofunctor going in the other direction, and a pseudonatural transformation from each composition to the identities on the respective 2-categories.

Consider a pseudofunctor $F : \mathscr{E}^{op} \to \mathsf{Cat}$. We check that the Grothendieck construction of F in fact gives a fibered category over $\mathscr{E}$. As claimed above the composition is associative. Given an object $(\beta, B) \in \int F$, we have the isomorphism $\epsilon_\beta(B) : \mathrm{id}_\beta^\bullet B \to B$, therefore we can define the identity to be $\mathrm{id}_{(\beta,B)} : (\beta, B) \to (\beta, B)$ as $(\epsilon_\beta(B)^{-1}, \mathrm{id}_\beta)$. To check that this acts as the neutral element with respect to the defined composition, take an arrow $(\varphi, f) : (\beta, B) \to (\gamma, C)$. Writing out the composition yields

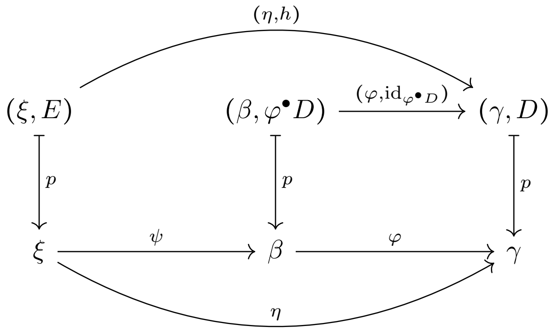
$$(\varphi, f) \circ (\epsilon_\beta(B)^{-1}, \mathrm{id}_\beta) = (\varphi, \beta_{\mathrm{id}_\beta,\varphi}(C) \circ \mathrm{id}_\beta^\bullet f \circ \epsilon_\beta(B)^{-1}).$$

but by definiton $\beta_{\mathrm{id}_\beta,\varphi}(C) = \epsilon_\beta(\varphi^\bullet B)$, while the diagram



commutes, because ϵ_β is a natural transformation. This implies that $\beta_{\mathrm{id}_\beta,\varphi}(C) \circ \mathrm{id}_\beta^\bullet f \circ \epsilon_\beta(B)^{-1} = f$, hence the proposed identity is in fact the right identity. A similiar argument gives that it is also a left inverse. Hence the Grothendieck construction applied to a functor is a category. Consider the projection $\int F \to \mathscr{E}$. We show that this projection makes $\int F$ into a fibered category over $\mathscr{E}$.

Take an arrow $\varphi : \beta \to \gamma$ in $\mathscr{E}$, and an object $(\gamma, D) \in \int F$. Now we wish to construct a map into (γ, D) which is Cartesian; we claim $(\varphi, \mathrm{id}_{\varphi^\bullet D}) : (\beta, \varphi^\bullet D) \to (\gamma, D)$ does the job. Suppose we are given the following diagram



We need to show the existence of a unique arrow $(\psi, g) : (\xi, E) \to (\beta, \varphi^\bullet D)$. Note that if it exist, it must be true that the following composition

$$(\varphi, \mathrm{id}_{\varphi^\bullet D}) \circ (\psi, g) = (\psi \circ \varphi, \alpha_{\psi,\varphi}(D) \circ \varphi^\bullet \, \mathrm{id}_{\varphi^\bullet D} \circ g) = (\psi \circ \varphi, \alpha_{\psi,\varphi}(D) \circ g),$$

must equal (η, h), hence we can only define $(\psi, g) = (\eta, \alpha_{\psi,\varphi}(D)^{-1} \circ h)$. Hence $\int F$ is a fibered category over $\mathscr{E}$, and this also gives us a cleavage, because there was only one possible map.

Note that for all objects $\alpha \in \mathscr{E}$ there is a functor $p^{-1}(\alpha) \to F(\alpha)$ sending an object (α, A) to α and an arrow (φ, f) to φ. This is an isomorphism of categories. The cleavage constructed above gives, for every arrow $\varphi : \alpha \to \beta$, functors $\varphi^\bullet : p^{-1}(\beta) \to p^{-1}(\alpha)$. Identifying each $p^{-1}(\beta)$ with $F(\alpha)$ via the isomorphism above, then these functors correspond to $f^\bullet : F(\beta) \to F(\alpha)$. Therefore starting with a pseudofunctor we construct a fibered category with cleavage, and considering the associated pseudofunctor these two are isomorphic in appropriate sense, and also vice versa. □

3 Facts about ∞-category

For an excellent introduction to higher category theory one should read the first chapter of [Lur09]; here we give a bare-bones description of the setting which is developed there by Lurie. In the previous section we saw that the classical categry theory can be extended to a theory of bicategories. Bicategories are quite complex to define and work with, so one might have gone in a slightly different direction, and merely defined a 2-category as a category with morphisms between the 1-morphisms, and call these 2-morphisms. This describes the several well known examples, for example Cat is a 2-category in this sense. Another example is the category of topological spaces. Here we can take the 2-morphisms to be homotopies. But now we notice that a homotopy is itself a continuous map, hence a 1-morphism of Top, and therefore it makes sense to speak of homotopies between homotopies. It's a fundamental property of homotopies are invertible, and so to describe this scenario it is natural to require that n-morphisms are invertible in some sense for $n > 1$. We take this to be our informal definition of $(\infty, 1)$-category (where the 1 signifies the invertibility condition). We will throughout the text write ∞-category as shorthand for the model of $(\infty, 1)$-category which we define now. For our formal definition we adopt the viewpoint of [Lur09], and define $(\infty, 1)$-category as follows:

Definition 3.1
An ∞-category (or quasi-category) is a weak Kan complex, i.e, a simplicial set S, which has all inner horn fillers . More explicitly, any map $f_0 : \Lambda^n_i \to S$ for $0 < i < n$ extends to an n-simplex $f : \Delta^n \to S$.

Example 3.2
Kan complexes are ∞-categories, because they have all horn fillers.

Example 3.3
The usual nerve of a small category is also an ∞-category, and the inner horn fillers are even unique.

One important thing missing in this definition is the uniqueness condition usually imposed in such situations. Informally, this is exactly the thing that separates this theory from ordinary category theory. Of course, there still has to be some structure, and we replace the usual unique composition by a contractible space of choices of such.

The usual nerve is inadequate when working with simplicial categories since it will forget all the simplicial structure. To fix this, one instead defines what is called the homotopy coherent nerve. This construction will somehow retain this information, and one very important attribute is that it is part of an adjunction. We now review the first step towards constructing the adjoint to the homotopy coherent nerve.

Definition 3.4
Let J be a finite nonempty linearly ordered set. We define the simplicial category $\mathfrak{C}[\Delta^J]$.

- Its objects are the elements of J.
- For $i, j \in J, then$

  $$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \begin{cases} N(P_{i,j}) & \text{if } i \leq j, \\ \varnothing & \text{if } j < i. \end{cases}$$

  where is the poset $P_{i,j} = \{I \subset J : i, j \in J \text{ and } k \in I \text{ such that } i \leq k \leq j\}$ and N is the ordinary nerve.
- If $i_0 \leq i_1 \leq \cdots \leq i_n$ then the compoition

  $$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i_0, i_n) \times \cdots \times \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i_{n-1}, i_n) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i_0, i_n)$$

  is induced by the map of partially ordered sets

  $$\begin{aligned} P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} &\to P_{i_0,i_n} \\ (I_1, ..., I_n) &\mapsto I_1 \cup ... \cup I_n. \end{aligned}$$

Furthermore let $f : J \to J'$ be a monotone map between linearly ordered sets. The simplicial functor $\mathfrak{C}[f] : \mathfrak{C}[\Delta^J] \to \mathfrak{C}[\Delta^{J'}]$ is defined as

- For each $i \in \mathfrak{C}[\Delta^J]$, $\mathfrak{C}[f](i) = f(i) \in \mathfrak{C}[\Delta^{J'}]$.
- If $i \leq j$ in J, then the map $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j))$ induced by f is the nerve of the map

$$P_{i,k} \to P_{f(i),f(j)},$$
$$I \mapsto f(I).$$

Remark 3.5
This construction defines a functor from $\mathfrak{C} : \Delta \to \mathsf{Cat}_\Delta$, given by $\Delta^n \mapsto \mathfrak{C}[\Delta^n]$, and because Cat_Δ admits all colimits, the functor $\mathfrak{C}$, extends uniquely to a functor $\mathsf{sSet} \to \mathsf{Cat}_\Delta$. Note that the simplicial structure on sets of morphisms comes from the fact that we applied the nerve during the construction to get these sets.

Now we are ready for the homotopical nerve construction.

Definition 3.6
Let $\mathscr{C} \in \mathsf{Cat}_\Delta$. We define the simplicial nerve $N(\mathscr{C})$ as the simplicial set described by the formula

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, N(\mathscr{C})) = \mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathscr{C}). \tag{1}$$

N defines a functor

$$N : \mathsf{Cat}_\Delta \to \mathsf{sSet}.$$

The following remark will be used often, and will turn out to be a crucial component of the main proof.

Remark 3.7
Note that (1) extends via colimits to an adjunction, and one of the very important facts of this theory is that the pair $\mathfrak{C}$ and N form an Quillen equivalence. In the following section we will make this statement precise.

4 Model structures

4.1 The Quillen-Kan model structure on simplicial sets

In the following we will introduce a number of model structure on a number of categories. For a full discussion of model categories see [Qui67]. The ultimate goal is to understand the projective model structure on the category of simplicial functors from a small category $\mathscr{C}$ to the category of simplicial sets sSet and the covariant model structure on the category of simplicial sets over a fixed simplicial set.

This section is dedicated to setting the stage for understanding the ∞-categorical analogue of the Grothendieck construction, where the $\mathsf{sSet}_{/X}$ will play the role of fibered categories and $\mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet})$, where this is the simplicially enriched functors, will play the role of pseudo-functors. We start by recalling the definition of a model category, and some basic facts relating to these.

Definition 4.1
A model category $\mathscr{C}$ is a (co)complete category equipped with three distinquished classes of maps which are, weak equivalences W, cofibrations Cof and fibrations Fib. W must have the two-out-of-three property, and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ must be weak factorization systems on $\mathscr{C}$.

Many of these definitions are included for completeness, even though familiarity is assumed. In particular we will assume familiarity with the Quillen-Kan model structure on sSet.

Theorem 4.2 (Quillen-Kan Model structure.)
The category sSet, together with the following three classes of morphisms, is a model category.

- *The class of the weak homotopy equivalences, W, which will play the role of weak equivalences;*
- *the class of level-wise injections(i.e, the monomorphisms),* Cof*, which will play the role of cofibrations;*
- *the class of Kan-fibrations (i.e. maps with the right lifting property with respect to horn-inclusions),* Fib*, which will play the role of the fibrations.*

The following easily checked facts will be important in later on.

Lemma 4.3
The following holds true for sSet equipped with the Quillen-Kan model structure,

- *The fibrant objects are the Kan-complexes, i.e. simplicial sets which has all horn fillers.*
- *All simplicial sets are cofibrant.*
- *A morphism of fibrant simplicial sets is a weak equivalence if it induces a isomorphisms of the all the associated homotopy groups.*

It turns out that many of the relevant model structures have additional structure, namely that they are combinatorial. The definition of this builds on the concepts of presentable categories, and weakly saturated classes. A detailed discussion of presentablility of categories is out of scope for this project, so the following definition will be slightly vague. In particular, many set-theoretic technicalities are completely ignored, and replaced by the informal condition that things be "small". This is also the case for another concept, namely that of accessible subcategories, where we will let these concepts be vague and informal to avoid a technical discussion of cardinality issues.

Definition 4.4
We say a category $\mathscr{C}$ is presentable if it satisfies the following conditions

1. $\mathscr{C}$ is cocomplete.
2. There exists a set S of objects of $\mathscr{C}$ such that every object of $\mathscr{C}$ can be obtained as the colimit of a small diagram in S.

3. Every object of $\mathscr{C}$ is small. (Its corepresentable functor preserve filtered colimits.)
4. For any pair of objects $X, Y \in \mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is small.

This definition serves two purposes. First, the main challenges in producing model structures with certain weak equivalences are often set theoretic. In these cases it will play a central role in the proofs that the relevant weak saturations indeed form model structures, that the underlying category is presentable. The second utility derived from this definition is that it makes it almost trivial to apply the adjoint functor theorem. The main criterion in the adjoint functor theorem is that the functor satisfies what is called the solution set condition. In cases where we refer to the adjoint functor theorem, one can usually check that the functor involved satisfies these conditions essentially only invoking the presentability of the target category.

One example of a presentable category which we will use many times is that of sSet.

Lemma 4.5
The category sSet *is presentable.*

This lemma is important, since it will play a central role when we show that the Quillen-Kan model structure is combinatorial, left proper and right proper. We will not prove this fact. It can be seen in [JA94].

Definition 4.6
Let $\mathscr{C}$ be a cocomplete category and let S be a class of morphisms of $\mathscr{C}$. We say that S is weakly saturated if it has the following properties

1. (Closure under pushout) Given a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{g} & W \end{array}$$

   such that $f \in S$, then $g \in S$.
2. (Closure under transfinite composition) Let $C \in \mathscr{C}$, let α be an ordinal, and let $\{D_\beta\}_{\beta<\alpha}$ be a system of objects in the under category $\mathscr{C}_{C/}$. For $\beta \leq \alpha$ we let $D_{<\beta}$ be a colimit of the system $\{D_\gamma\}_{\gamma<\beta}$ in $\mathscr{C}_{C/}$. Suppose that for each $\beta < \alpha$, the natural map $D_{<\beta} \to D_\beta$ is in S. Then the induced map $C \to D_{<\alpha}$ belongs to S.
3. (Closure under retracts) Given a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow f & & \downarrow g & & \downarrow f \\ Y & \longrightarrow & Y' & \longrightarrow & Y. \end{array}$$

   Then if g belongs to S, so does f.

In practice, it turns out that many classes which are described by having certain lifting properties are weakly saturated. For example, in a model category, the cofibration and fibration classes are each weakly saturated.

There are many ways in which a model category can be nice, here we review two:

Definition 4.7
We say a model category $\mathscr{C}$ is combinatorial if it satisfies the following conditions,

1. $\mathscr{C}$ is presentable.
2. There exists a set I of *generating cofibrations* such that Cof is the smallest weakly saturated class of morphisms containing I.

3. There exists a set J of *generating trivial cofibrations* such that $\mathrm{Cof} \cap W$ is the smallest weakly saturated class of morphisms containing J.

Note that the second and third conditions are what is usually called cofibrantly generated.

Definition 4.8
A model structure $\mathscr{C}$ is left proper if, for any pushout square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ A' & \xrightarrow{i'} & B' \end{array}$$

in which $i \in \mathrm{Cof}$ and $j \in W$, then $j' \in W$. Dually, a model structure is right proper if for each pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{p'} & B \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ A' & \xrightarrow{p} & B' \end{array}$$

where p is a fibration and j' is a weak equivalence, j is a weak equivalence.

The following proposition lets us assert via lemma 4.3 that sSet with the Quillen-Kan model structure is left proper.

Proposition 4.9
Let $\mathscr{C}$ be a model category in which every object is cofibrant. Then the model structure on $\mathscr{C}$ is left proper.

The proof can be found in [Lur09].

Definition 4.10
Let $\mathscr{C}$ be a presentable category. A class of maps P in $\mathscr{C}$ is perfect if it satisfies the following conditions

1. Every isomorphism belongs to P.
2. It satisfies the two-out-of-three property.
3. It is closed under filtered colimits.
4. There exists a compact subset $W_0 \subset W$ such that every morphism from W, can be obtained as a filtered colimit of morphisms from W_0.

4.2 A Model Category Theorem

We are now able to state a very useful result, which will enable us to construct model structures, which enjoy many of the above properties. We will use this theorem to show that the Quillen-Kan model structure on simplicial sets is combinatorial, left proper. We will need the following lemma which is proved in [Lur09].

Lemma 4.11
Let $\mathscr{C}$ be a presentable category and let W and C be classes of morphisms in $\mathscr{C}$ such that

1. *The collection C is a weakly saturated class of $\mathscr{C}$, and there exists subset $C_0 \subset C$ which generates C as a weakly saturated class.*
2. *The intersection $C \cap W$ is weakly saturated.*
3. *The subcategory $W \subset \mathscr{C}^{[1]}$ is an accessible subcategory of $\mathscr{C}^{[1]}$.*
4. *W has the two-out-of-three property.*

5. If f has the right lifting property with respect to C, then $f \in W$.

Then $\mathscr{C}$ admits a combinatorial model structure, which may be described as follows

1. The cofibrations are C.

2. The weak equivalences are W.

3. If f has the right lifting property with respect to $C \cap W$, it is a fibration.

The proof of this lemma relies on a string of technical lemmas, which all has the same flavor as this one. However, the technical level of the proofs goes up the further down the rabbit hole one goes. Hence we will not enter the rabbit hole.

We now have the tools to state and prove the theorem which produces many of the model structures used in higher category theory.

Theorem 4.12
Let $\mathscr{C}$ be a presentable category. Suppose W is a class of morphisms, which we call weak equivalences. Let C_0 be a set of morphisms of $\mathscr{C}$, which we will call generating cofibrations. Suppose further

1. The class W is perfect.

2. For any diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X'' & \longrightarrow & Y'', \end{array}$$

where both squares are pushout, $f \in C_0$, and $g \in W$, then $g' \in W$.

3. If a morphism g of $\mathscr{C}$ has the right lifting property with respect to every morphism in C_0, then $g \in W$.

Then there exists a left proper combinatorial model structure on $\mathscr{C}$ which may be described as follows:

1. If a morphism f belongs to the weakly saturated class of morphisms generated by C_0, then $f \in Cof$.

2. The weak equivalences are W.

3. If a morphism f has the right lifting property with respect to every map in $Cof \cap W$ then $f \in Fib$.

Proof. Consider the collection of all morphisms f in $\mathscr{C}$ such that for two pushout squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X'' & \longrightarrow & Y'', \end{array}$$

where $g \in W$, then the map $g' \in W$, and denote this class by P. Note that $C_0 \subset P$ by assumption. The goal is to show that the weak saturation of C_0 has this property, and to show that, we show that P is weakly saturated.

That P is closed under pushouts is straight forward; assume f' arises as the pushout of f, and consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X''' & \longrightarrow & Y''' \end{array}$$

Now each square is a pushout square by assumption, hence the two upper squares constitute a single pushout square, with f on top, hence because $f \in P$, and $g \in W$, so is g'.

Next we show closure under transfinite composition. Let α be a ordinal, and let following be a α-indexed diagram in $\mathscr{C}$, where $X_\alpha = \operatorname{colim}_{\beta<\alpha} X_\beta$,

$$C \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_\alpha \qquad (\tilde{f} : C \to X_\alpha)$$

Where all morphisms are assumed to be in P, and $\tilde{f}$ is the transfinite composition induced from the directed system $\{X_\beta\}_{\beta<\alpha}$. Define Y as the pushout of the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & X_\alpha \\ \downarrow & & \\ X & & \end{array}$$

and note that Y is isomorphic to the colimit of the following diagram

$$\operatorname{colim}\Big(C \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots\Big) \longrightarrow X$$

It is clear that X is terminal within this diagram, hence the colimit of this diagram is isomorphic to X via the canonical map, so $Y \xrightarrow{\sim} X$. Thus we obtain the following diagram of pushout squares,

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & X_\alpha \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \longrightarrow & W, \end{array}$$

and we want to assert $g' \in W$. Now because $X \to Y$ is an isomorphism, the pushout of this map, the map $Z \to W$, is also an isomorphism. Hence both maps are in W, which by the two-out-of-three property implies that g' is in W.

To see that P is closed under retracts, let $f' : X' \to Y'$ be a retract of $f : X \to Y \in P$ so that

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X' \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f'} \\ Y' & \longrightarrow & Y & \longrightarrow & Y', \end{array}$$

commutes and the row compositions are the respective identities, and let the following be a pushout:

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

We claim that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a pushout, where the map $X \to Y$ is f, and the maps $X \to Z$ and $Y \to W$ are obtained from the retract assumption. To show this, we just check the relevant universal property. Given maps $Y \to Q$ and $Z \to Q$ such that the precomposition with the respective maps from X are equal, we can consider the diagram

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Y' \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & Z & \longrightarrow & Q & & \end{array}$$

Where the top row is a factorization of f'. Since the outer square is a pushout, we get a unique map $Q \to W$ compatible with the outer with the diagram, as we wanted. Since we have shown that P is weakly saturated, the weak equivalences are stable under pushouts by cofibrations.

We have yet to show that $\mathscr{C}$ is in fact a model category. This fact will be mediated by the lemma lemma 4.11, so we check that the conditions are satisfied. (1) is satisfied because C is generated by C_0 as a weakly saturated class, per. construction. (3) is true because W is a perfect class. (4) is per assumptions on W, and (5) is per assumption (3). It remains to check that (2) is satisfied, i.e., that $C \cap W$ is a weakly saturated class.

$C \cap W$ is closed under retracts because both W and C are closed under retracts. W is closed under retracts because W is a perfect class. C is closed under retracts because it is weakly saturated per. construction.

Next $C \cap W$ is closed under transfinite composition because C is weakly saturated, and W is closed under transfinite composition. The last fact follows from W being closed under composition and filtered colimits in the following way: Consider an ordinal α, and let the following diagram be a directed system in under category $\mathscr{C}_{c/}$,

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & ... & \longrightarrow & X_\alpha \\ \uparrow & \nearrow & & & & \nearrow & \\ c & & & & & & \end{array}$$

Where each map is in W. Now let $X_{<\alpha}$ be the colimit of the directed system on $\{c \to X_i\}_{i<\alpha}$. We wish to show that the induced map $c \to X_{<\alpha}$ is in W. Now note that the diagram can be rewritten

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & ... & \longrightarrow & X_\alpha \\ \uparrow & & \uparrow & & & & \uparrow \\ c & = & c & = & ... & = & c \end{array}$$

Where each map is in W. Now note that this can be considered as a α-indexed directed system in $\mathscr{C}^{[1]}$, where the colimit is the induced map $c \to X_{<\alpha}$, hence the induced map is W, because it was a obtained through a filtered colimit in W. It only remains to show that $C \cap W$ is closed under pushouts. Consider a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f''} \\ Y & \longrightarrow & Y'' \end{array}$$

in which $f \in C \cap W$, we show that $f'' \in C \cap W$. Because C is weakly saturated, it is enough to show $f'' \in W$. Apply the small object argurment to the map $X \to X''$, to factor it as

$$X \xrightarrow{g} X' \xrightarrow{h} X''$$

Where g is a cofibration and h has the right lifting property with respect to C_0. Now consider the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & X' & \xrightarrow{h} & X'' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f''} \\ Y & \longrightarrow & Y' & \xrightarrow{h'} & Y'' \end{array}$$

Where Y' is the pushout of f and g. Now note that Y'' recieves a map from Y which arise from the original pushout diagram which when composed with f, is equal to $f'' \circ h \circ g$, hence the h' exists by the universal property of the pushout. Now the two squares constitute a single pushout square, and so does the left square, hence the right square is also a pushout square. Since W is stable under the formation of pushouts by cofibrations, we have that $f' \in W$, $h \in W$ because of (3), and $h' \in W$ because it is a pushout of h by the cofibration f'. Hence by the two-out-of-three $h \circ f'' \in W$, and a second application of the two-out-of-three property shows that $f'' \in W$. Which ends the proof. □

This theorem has two purposes; one is to produce model structures given sets of morphisms that we might wish to use as equivalences and cofibrations, the other is to show that familiar model structures are combinatorial and left or right proper. An example of the second application is the following:

Theorem 4.13
The Quillen-Kan model structure on sSet *is a combinatorial, left proper and right proper.*

Proof. One can show Cof is generated by the collection of all inclusions $\partial\Delta^n \subseteq \Delta^n$. Given this fact, and the fact that sSet is presentable, (Lemma lemma 4.5), the conditions for theorem theorem 4.12 are satisfied. Now it only remains to verify that Fib for the resulting model structure are precisely the Kan-fibrations, which is *very* hard to show, and that sSet is right proper, both are proven by Quillen in [Qui67]. □

There is an alternative model structure on the category of simplicial sets, called the *Joyal model structure*, which we will describe now:

Theorem 4.14
There exists a left proper combinatorial model structure on sSet *with the following properties:*

1. *A map $p : S \to S'$ of simplicial sets is in Cof, if it is a monomorphism.*
2. *A map $p : S \to S'$ is a weak equivalence (which we will call a categorial equivalence) if the induced simplicial functor $\mathfrak{C}[S] \to \mathfrak{C}[S']$ is an equivalence of simplicial categories.*

Moreover the adjoint functors $\mathfrak{C}$ (simplicial category functor, see definition 3.4) and N (the homotopy coherent nerve, see definition 3.6) functor, determine a Quillen equivalence between sSet *and* Cat_Δ*, where* Cat_Δ *are simplicially enriched categories.*

Remark 4.15
We haven't formally defined the category Cat_Δ, but it will defined in the following section. Without the above statement the following discussion will seem very unmotivated.

Remark 4.16
In [Lur09] the proof of this fact relies on the Straightening and Unstraightning, but the fact that there exists a model structure can be proved using combinatorial methods by Joyal in [Joy07] and the fact that there is a Quillen equivalence can be proven using dendroidal sets, and is done by D.C. Cisinski and I. Moerdijk [DC11]. We will not delve deeper into the inner workings og any of these methods, but we will define the model structure on Cat_Δ. Before we begin this lenghty endeavor, we will collect some pleasant facts about the Joyal model structure, which we won't prove either.

Proposition 4.17
Every simplicial set is cofibrant in sSet*, and the fibrant objects are the* ∞*-categories, when* sSet *is endowed with the Joyal model structure.*

5 Simplicially enriched categories and their model structure.

5.1 Monoidal Model Categories

We will need the notion of monioidal categories, because we wish to make an enrichment which fits together with the monoidal structure. This section follows the appendices of [Lur09].

Definition 5.1
A monoidal category $\mathscr{C}$ is a category equipped with a "product"-functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a unit object 1 and a three natural isomorphisms such that the following conditions are satisfied,

$$\eta_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

and the unitality of 1 is expressed via isomorphisms

$$\alpha_A : A \otimes 1 \to A$$
$$\beta_A : 1 \otimes A \to A.$$

These isomorphisms are subject to the following axioms

1. Both $\eta_{A,B,C}$ depend functorially on the triple (A,B,C); η is a natural isomorphism between the functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathscr{C}$:

$$(A,B,C) \mapsto (A \otimes B) \otimes C \xrightarrow{\eta} (A,B,C) \mapsto A \otimes (B \otimes C).$$

   Likewise for α_A and β_A.

2. Given a quadruple (A,B,C,D) of objects in $\mathscr{C}$, the so called MacLane pentagon must commute,

$$\begin{array}{ccccc} & & (A\otimes B)\otimes C)\otimes D & & \\ & \swarrow^{\eta_{A,B,C}\otimes \mathrm{id}_D} & & \searrow^{\eta_{A\otimes B,C,D}} & \\ (A\otimes(B\otimes C))\otimes D & & & & (A\otimes B)\otimes(C\otimes D) \\ \downarrow{\scriptstyle \eta_{A,B\otimes C,D}} & & & & \downarrow{\scriptstyle \eta_{A,B,C\otimes D}} \\ A\otimes((B\otimes C)\otimes D) & & \xrightarrow{\mathrm{id}_A\otimes\eta_{B,C,D}} & & A\otimes(B\otimes(C\otimes D)). \end{array}$$

3. For any pair (A,B) of objects in $\mathscr{C}$, the triangle

$$\begin{array}{ccc} (A\otimes 1)\otimes B & \xrightarrow{\eta_{A,1,B}} & A\otimes(1\otimes B) \\ & {\scriptstyle \alpha_A\otimes\mathrm{id}_B}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_A\otimes\beta_B} & \\ & A\otimes B & \end{array}$$

Remark 5.2
So a monoidal category is the data $(\mathscr{C}, \otimes, 1, \eta, \alpha, \beta)$, subject to three axioms above. We will call the sextuple a monoidal structure on $\mathscr{C}$. We will abuse notation and say that $(\mathscr{C}, \otimes)$ is a monoidal category and $\otimes$ is a monoidal structure on $\mathscr{C}$. Furthermore we will neglect writing the subscript on η, α and β, in light of the MacLane's coherence theorem, since will be identities up to equivalence.

One can't have a discussion of monoidal categories without mentioning the coherence theorem due to MacLane and Kelly.

Theorem 5.3 (MacLane's coherence theorem)
In any monoidal category, any diagram written only using $\eta_{A,B,C}$, α_A and β_A is commutative. I.e. any monoidal category is equivalent to a strict monoidal category in which $\eta_{A,B,C}$, α_A and β_A are identity maps.

Proof. For a simple discussion see [Rie14] and a detailed discussion and proof see [Lan91] §XI.1. □

Example 5.4
It is clear that the category of simplicial sets sSet is monoidal with the product being Cartesian product.

5.2 Enrichment of a Monoidal Model Category and the Projective model structure

Our main motivation for introducing monoidal categories and model structures is because we want to define model structures on $\mathscr{C}$-enriched categories for $\mathscr{C}$ a monoidal category.

Definition 5.5
Let $(\mathscr{C}, \otimes)$ be a monoidal category. A $\mathscr{C}$-enriched category $\mathscr{D}$ consist of the following data.

1. A collection of objects.
2. For every pair of objects $X, Y \in \mathscr{D}$, a mapping object $\mathrm{Hom}_{\mathscr{D}}(X, Y)$ of $\mathscr{C}$.
3. For every triple $X, Y, Z \in \mathscr{D}$, a composition map

$$\mathrm{Hom}_{\mathscr{D}}(Y, Z) \otimes \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, Z),$$

which needs to be associative in the following sense; for any $W, X, Y, Z \in \mathscr{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(Y, Z) \otimes \mathrm{Hom}_{\mathscr{D}}(X, Y) \otimes \mathrm{Hom}_{\mathscr{D}}(W, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(X, Z) \otimes \mathrm{Hom}_{\mathscr{D}}(W, X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{D}}(Y, Z) \otimes \mathrm{Hom}_{\mathscr{D}}(W, Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(W, X) \end{array}$$

must commute.

4. For every object $X \in \mathscr{D}$, a unit map $1 \to \mathrm{Hom}_{\mathscr{D}}(X, X)$ making the following diagrams

$$\begin{array}{ccc} 1 \otimes \mathrm{Hom}_{\mathscr{D}}(Y, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(X, X) \otimes \mathrm{Hom}_{\mathscr{D}}(Y, X) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_{\mathscr{D}}(Y, X), & \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(Y, X) \otimes 1 & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(X, Y) \otimes \mathrm{Hom}_{\mathscr{D}}(X, X) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_{\mathscr{D}}(X, Y), & \end{array}$$

commute.

Example 5.6
Any monoidal category $(\mathscr{C}, \otimes)$ which is closed (the Hom-tensor adjunction exists) is enriched over itself, via the adjunction isomorphism $\mathrm{Hom}_{\mathscr{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathscr{C}}(A, \mathrm{Hom}_{\mathscr{C}}(B, C))$ with unit $\eta : A \to \mathrm{Hom}_{\mathscr{C}}(B, A \otimes B)$ and counit $\epsilon : \mathrm{Hom}_{\mathscr{C}}(B, C) \otimes B \to C$. We define composition as the adjunction applied to the following composition

$$\mathrm{Hom}_{\mathscr{C}}(B, C) \otimes \mathrm{Hom}_{\mathscr{C}}(A, B) \otimes A \xrightarrow{(\mathrm{id} \otimes \epsilon) \circ \eta} \mathrm{Hom}_{\mathscr{C}}(B, C) \otimes B \xrightarrow{\epsilon} c.$$

Hence we obtain $\mathrm{Hom}_{\mathscr{C}}(B, C) \otimes \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$. α in $\mathscr{C}$ correspond to a morphism $1 \to \mathrm{Hom}_{\mathscr{C}}(A, A)$ which we set as the unit map.

Example 5.7
The category of abelian groups is enriched over itself, this can be seen directly or via the previous example.

Example 5.8
Quillen [Qui67] showed that the category of small categories Cat can be enriched over sSet. We will denote the category of simplicially enriched categories Cat_{Δ}.

Definition 5.9
Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be model categories. We will say that a functor $F : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ is a left Quillen bifunctor if the following conditions are satisfied:

1. Let $i : C \to C'$ and $j : D \to D'$ be cofibrations in $\mathscr{C}$ and $\mathscr{D}$ respectively. Then the induced map
$$i \,\square\, j : F(C', D) \coprod_{F(A,B)} F(C, D') \to F(C', D')$$
is a cofibration in $\mathscr{E}$. Moreover, if either i or j is a trivial cofibration, then $i \wedge j$ is to.
2. The functor F preserve small colimits separately in each variable.

Definition 5.10
A monoidal model category $\mathscr{C}$ is a monoidal category equipped with a model structure, which satisfies the following conditions:

1. The tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a left Quillen bifunctor.
2. The unit object is cofibrant.
3. The monoidal structure on $\mathscr{C}$ is closed.

Example 5.11
The category of simplicial sets sSet is a monoidal model category with respect to Cartesian product and the Quillen-Kan model structure.

Definition 5.12
Let $(\mathscr{C}, \otimes)$ be a closed monoidal category and let $\mathscr{D}$ be a $\mathscr{C}$-enriched category. Let the functor $\mathscr{D} \to \mathscr{C}$ be defined as $Y \mapsto \mathrm{Hom}_{\mathscr{D}}(X, Y)^C$ where $X \mapsto X^C$, is the right adjoint to $A \mapsto A \otimes C$. If this functor is corepresentable, i.e. there exists an object $Z \in \mathscr{D}$ and an isomorphism of functors
$$\eta : \mathrm{Hom}_{\mathscr{D}}(X, -)^C \cong \mathrm{Hom}_{\mathscr{D}}(Z, -).$$
We denote the isomorphism by $X \otimes C$. If $X \otimes C$ exists for every $C \in \mathscr{C}$ and $X \in \mathscr{D}$, then we say that $\mathscr{D}$ is tensored over $\mathscr{C}$. Dually, let a functor $\mathscr{D} \to \mathscr{C}$ be defined $Y \mapsto {}^C\mathrm{Hom}_{\mathscr{D}}(Y, X)$, where $X \mapsto X^C$, is the right adjoint to $A \mapsto C \otimes A$. If this functor is representable, i.e. there exists an object $Z \in \mathscr{D}$ and an isomorphism of functors
$$\nu : {}^C\mathrm{Hom}_{\mathscr{D}}(-, X) \cong \mathrm{Hom}_{\mathscr{D}}(-, Z).$$
If this isomorphism exists for every $X \in \mathscr{D}$ and $C \in \mathscr{C}$, we say that $\mathscr{D}$ is cotensored over $\mathscr{C}$.

Definition 5.13
Let $\mathscr{C}$ be a monoidal model category. A $\mathscr{C}$-enriched model category is an $\mathscr{C}$-enriched category $\mathscr{D}$ equipped with a model structure such that

1. $\mathscr{D}$ is tensored and cotensored over $\mathscr{C}$.
2. The tensor product $\otimes : \mathscr{D} \times \mathscr{C} \to \mathscr{D}$, as defined above, is a left Quillen bifunctor.

Definition 5.14
Let $\mathscr{C}$ be an monoidal model category. A functor $F : \mathscr{D} \to \mathscr{E}$ in $\mathsf{Cat}_{\mathscr{C}}$ is a weak equivalence if the induced functor between homotopy categories $h\mathscr{D} \to h\mathscr{C}$ is an equivalence of $h\mathscr{C}$-enriched categories. I.e.

1. For every pair $X, Y \in \mathscr{D}$, the induced map
$$\mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{E}}(F(X), F(Y))$$
is a weak equivalence in $\mathscr{C}$.
2. Every object $Y \in \mathscr{E}$ is equivalent to $F(X)$ in the homotopy category $h\mathscr{E}$ for some $X \in \mathscr{D}$.

We now introduce a bit of notation for $\mathscr{C}$-enriched categories. Let $A \in \mathscr{C}$, then we let $[1]_A$ denote the $\mathscr{C}$-enriched category having two objects X and Y, with mapping object

$$\mathrm{Hom}_{[1]_A}(Z,W) = \begin{cases} \mathrm{id}_S & \text{if } Z = W = X, \\ \mathrm{id}_S & \text{if } Z = W = Y, \\ A & \text{if } Z = X,\ W = Y, \\ \varnothing & \text{if } Z = Y,\ W = X. \end{cases}$$

Where $\varnothing$ is the initial object of $\mathscr{C}$, and $\mathrm{id}_{\mathscr{C}}$ is the unit object with respect to the monoidal structure on $\mathscr{C}$. Let $[0]_{\mathscr{C}}$ denote the $\mathscr{C}$-enriched category having only a single object X and mapping object $\mathrm{id}_{\mathscr{C}}$. Let C_0 denotes the collection of all morphisms of $\mathscr{C}$ of the following type

1. The inclusion $\varnothing \to [0]_{\mathscr{C}}$.
2. The induced maps $[1]_S \to [1]_{S'}$, where $S \to S'$ range over a set of generators for the weakly saturated class of cofibrations in $\mathscr{C}$.

We wish to apply the following theorem to assert the existence of a model structure on Cat_Δ compatible with the sSet-enrichment, i.e. to assert that Cat_Δ is a sSet-enriched model category.

Theorem 5.15
Let $\mathscr{C}$ be a combinatorial monoidal model category. Assume that every object of $\mathscr{C}$ is cofibrant and that the collection of weak equivalences in $\mathscr{C}$ is stable under filtered colimits. Then there exists a left proper combinatorial model structure on $\mathsf{Cat}_{\mathscr{C}}$ (The category of categories enriched over $\mathscr{C}$) characterized by the following conditions:

- *The class of cofibrations in $\mathsf{Cat}_{\mathscr{C}}$ is the smallest weakly saturated class of morphisms containing the set og morphisms C_0, where C_0 is defined as above.*
- *The weak equivalences in $\mathsf{Cat}_{\mathscr{C}}$ are defined as in definition 5.14.*

Proof. We will only give a rough sketch of proof. We wish to apply theorem 4.12, so one checks the conditions. Condition (1) and (3) are fair straightforward, using many of the techniques used in the proof of theorem 4.12. Condition (2) is very technical. □

Definition 5.16
The above model structure when $\mathscr{C} = \mathsf{sSet}$, will be called the Bergner model structure for simplicially enriched categories, which we will denoted Cat_Δ.

Definition 5.17
Let $\mathscr{D}$ be a small category and let $\mathscr{C}$ be a model category. We will say that a natural transformation $\alpha : F \to G$ in $\mathsf{Fun}(\mathscr{D},\mathscr{C})$ is

- an projective fibration if the induced map $F(C) \to G(C)$ is a fibration in $\mathscr{C}$ for each $C \in \mathscr{D}$.
- A weak equivalence if the induced map $F(C) \to G(C)$ is a weak equivalence in $\mathscr{C}$ for each $C \in \mathscr{D}$.
- an projective cofibration if it has the left lifting property with respect to every morphism β in $\mathsf{Fun}(\mathscr{D},\mathscr{C})$ which is simultaneously a weak equivalence and a projective fibration.

Proposition 5.18
Let $\mathscr{C}$ be a combinatorial model category let $\mathscr{D}$ be a small category. Then there exists a combinatorial model structure on $\mathsf{Fun}(\mathscr{D},\mathscr{C})$, called the projective model structure determined by the projective cofibrations, weak equivalence, and projective fibrations.

Proof. The proof goes as usual for proving that a certain triple of classes of maps constitutes a model structure, up to minor details. □

5.3 The Covariant Model Structure

Next we describe the last model structure which will appear in this text, namely the covariant model structure.

Definition 5.19
If S and S' are simplicial sets, then the simplicial set $S \star S'$, called the join of S and S', is defined as follows. Let

$$(S \star S')_n = S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j.$$

The structure maps are induced by S and S'.

Definition 5.20
Let K be a simplicial set. The left cone $K^{\triangleleft}$ is defined to be the join $\Delta^0 \star K$.

Definition 5.21
Let $S \in sSet$, and form the overcategory $\mathsf{sSet}_{/S}$, and consider $f : X \to Y$ in $\mathsf{sSet}_{/S}$. We say that f is a

1. covariant cofibration if it is a monomorphism of simplicial sets.
2. covariant equivalence if the induced map

$$X^{\triangleleft} \coprod_X S \to Y^{\triangleleft} \coprod_Y S$$

is a categorical equivalence.
3. covariant fibration if it has the right lifting property with respect to every map which is both a covariant cofibration and a covariant equivalence.

Proposition 5.22
Let $S \in \mathsf{sSet}$. The covariant cofibrations, covariant equivalences, and covariant fibrations determine a left proper combinatorial model structure on $\mathsf{sSet}_{/S}$.

Proof. We show the conditions for theorem 4.12 are satisfied. Now given the following diagram

$$\begin{array}{ccccc} X^{\triangleleft} & \longleftarrow & X & \longrightarrow & S \\ \downarrow & & \downarrow & & \| \\ Y^{\triangleleft} & \longleftarrow & Y & \longrightarrow & S \end{array}$$

Where the left most map exists because join is functorial, we obtain an induced map

$$X^{\triangleleft} \coprod_X S \to Y^{\triangleleft} \coprod_Y S$$

If this is an equivalence in the Joyal model structure it is an covariant equivalence. We show this class is perfect. Now since the class of categorical equivalences are perfect, (1)-(2) are obvious and (3)-(4) follows from commutativity of filtered colimits.

Now consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \longrightarrow & W, \end{array}$$

Where f is a covariant cofibration, and g is a covariant equivalence, we must show that g' is a covariant equivalence. Applying the functor $\mathfrak{C}((-)^{\triangleleft} \coprod_{(-)} S)$ to the diagram induces the following diagram,

$$\begin{array}{ccc} \mathfrak{C}(X^{\triangleleft}\coprod_X S) & \longrightarrow & \mathfrak{C}(Y^{\triangleleft}\coprod_Y S) \\ \downarrow & & \downarrow \\ \mathfrak{C}(X'^{\triangleleft}\coprod_{X'} S) & \longrightarrow & \mathfrak{C}(Y'^{\triangleleft}\coprod_{Y'} S) \\ \downarrow & & \downarrow{\scriptstyle g_*} \\ \mathfrak{C}(Z^{\triangleleft}\coprod_Z S) & \longrightarrow & \mathfrak{C}(W^{\triangleleft}\coprod_W S), \end{array}$$

because $\mathfrak{C}$ preserves cofibrations, which it does because it of a Quillen equivalence, and it is the left adjoint hence it preserve colimits. $(-)^{\triangleleft}\coprod_{(-)} S$ preserves colimits, as described in **??**, and it can be shown that it sends covariant cofibrations to cofibration of the Joyal model structure. Per. assumption the induced map of g, g_* is a Bergner equivalence, and now because the Bergner equivalences satisfy (2), the induced map of g', g'_* is a Bergner equivalence. Because left Quillen functors, such as $\mathfrak{C}$, preserve weak equivalence between cofibrant object, and every object is cofibrant in the Joyal model structure, $g'^{\triangleleft}\coprod_f S$ is a categorial equivalence, and hence g' is a covariant cofibration.

Given a map $p : X \to Y$ in sSet, which has the right lifting property with respect to every covariant cofibration, we must show that p is a covariant equivalence.

In that case p is a Kan fibration, because C_0 contained horn inclusions and furthermore it is a trivial Kan fibration, because it has because every cofibration has the left lifting property with respect to it. Hence we can produce a section s, by considering the diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

We will show that p and s are mutual inverses in the homotopy category of enriched simplicial sets with the Bergner model structure. We already know that one composition is the identity, since s was a section of p, so it will suffice to show that the map $\mathfrak{C}(X^{\triangleleft}\coprod_X S) \to \mathfrak{C}(X^{\triangleleft}\coprod_X S)$ induced by the other composition is the identity in the homotopy category. We note again that p is a trivial Kan fibration, here with emphasis on the "trivial" part, so that we can choose a homotopy of p and the identity, $h : \Delta^1 \times X \to X$. One can show that the inclusion $\{0\} \times X \hookrightarrow \Delta^1 \times$ is left anodyne. All that is left is to show that a left anodyne map is a covariant equivalence. Which is shown in [Lur09] in lemma 2.1.4.6. □

We will need the following invariance result.

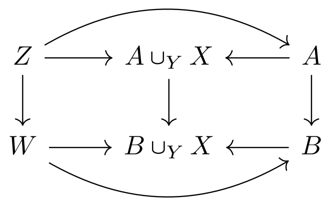
Lemma 5.23
For a categorical equivalence of simplicial sets $f : X \to Y$, the corresponding adjoint pair $f_! \dashv f^*$, where $f_! : \mathsf{sSet}_{/X} \to \mathsf{sSet}_{/Y}$ given by post composition and $f^* : \mathsf{sSet}_{/Y} \to \mathsf{sSet}_{/X}$ is given by pulling back along f, is a Quillen equivalence for the covariant model structure.

For the proof of this result, we will need the following lemma, which makes use of the notion of inner anodyne maps.

Definition 5.24
A map of simplicial sets $f : X \to Y$ is called an inner fibration if it has the right lifting property with respect to all horn inclusions $\Lambda^i_n \subset \Delta^n$, for $0 < i < n$. A morphism of simplicial set which has the left lifting property with respect to all inner fibrations is called inner anodyne. Analogously there are left and right fibrations which have the right lifting property with respect to all horn inclusions, for $0 \le i < n$ and $0 < i \le n$ respectively. Again there are also right and left anodyne maps.

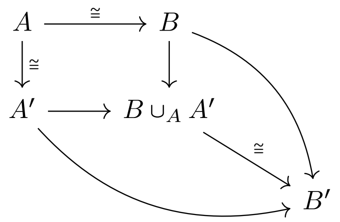
Lemma 5.25
Every inner anodyne map of simplicial sets is a categorical equivalence.

Proof. of lemma 5.23. Note first it is easy to see that $f_!$ and f^* form an adjunction, which in fact is a Quillen adjunction: it is obvious that cofibrations are preserved by $f_!$, we show that f^* preserve fibrations. Assume $f : A \to B$ is a fibration, i.e. has the right lifting property with respect to trivial cofibrations. Consider the diagram

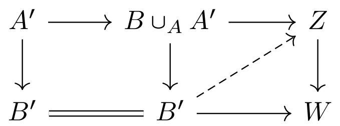


Where the map $Z \to W$ is a trivial cofibration, and the bended maps arise through the maps into the pushouts. There exists a lift $W \to A$, which produces the lift $W \to A \cup_Y X$, hence f^* preserves fibrations, hence the pair constitute a Quillen adjunction.

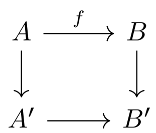
Now consider an inner anodyne map $A \to A'$ where A' is fibrant, and choose an inner anodyne map $B \cup_A A' \to B'$ where B' is fibrant, these fit together in the following pushout square.



Where the maps denote $\cong$ are categorical equivalences due to lemma 5.25. Now every map is an categorical equivalence if just one more map in the diagram is a categorical equivalence, via repeated use of the two-out-of-three-property. We show that $A' \to B'$ is inner anodyne. Consider the following diagram, where $Z \to W$ is a inner fibration



The map $B' \to Z$ exists because $B \cup_A A' \to B'$ was inner anodyne, and hence there exists a lift $B' \to Z$, which shows that $A' \to B'$ is inner anodyne, hence an categorical equivalence. So we've formed the following diagram, where every map is a categorical equivalence



One strategy, due to G. Heuts and I. Moerdijk [GH16a], is to show that the proposition holds for the two vertical maps, and for the lower horizontal one, hence for f. This strategy doesn't rely on Straightening and Unstraightening. [Lur09] also has a proof, but this relies on Straightening and Unstraightening.

□

Lemma 5.26
For an equivalence of simplicial categories $F : \mathscr{C} \to \mathscr{D}$, the corresponding adjoint pair

$$F_! : \mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet}) \rightleftarrows \mathsf{Fun}_\Delta(\mathscr{D}, \mathsf{sSet}) : F^*$$

is a Quillen equivalence for the projective model structures. sSet *is equipped with the Joyal model structure.*

Proof. To show that the pair is a Quillen equivalence, we will show that the derived unit and counit are projective weak equivalences. We first prove the theorem under the following assumption

- For every pair of objects $C, D \in \mathscr{D}$, the map

$$\mathrm{Hom}_{\mathscr{D}}(C, D) \to \mathrm{Hom}_{\mathscr{C}}(F(C), F(D))$$

  is a cofibration in sSet.

Note now that the functor F^* is essentially surjective on homotopy categories, as it is an equivalence of simplicial categories. This means that we can check whether a certain natural transformation η between two functors $K, K' : \mathscr{D} \to \mathsf{sSet}$ is a projective weak equivalence be checking whether the corresponding natural transformation $F^*\eta$ is a projective weak equivalence. To see this, first assume η is a projective weak equivalence. Because F^* is the right adjoint in a Quillen equivalence it preserves weak equivalences, hence $F^*\eta$ is a projective weak equivalence.

Now assume $F^*\eta$ is a projective weak equivalence. Now for all $x \in \mathscr{D}$ choose $y \in \mathscr{C}$ such that $x \simeq F(y)$. Since $F^*\eta$ is a projective weak equivalence it holds for all $y \in \mathscr{C}$ that $K(F(y)) \to K'(F(y))$ is a categorical equivalence, now pass to homotopy categories and note that we have the following isomorphisms $K(x) \cong K(F(y)) \cong K'(F(y)) \cong K'(x)$. Hence η is a projective weak equivalence.

□

6 Straightening and Unstraightening

6.1 Construction of the straightening functor

At this point we are ready to construct the straightening functors. The unstraightening functor will be constructed later via the adjoint functor theorem. We will also give some basic examples to clarify certain assertion concerning the construction. Per. construction these functors will form an adjoint pair, and the content of the straightening and unstraightening theorem is that they form a Quillen equivalence. We will prove this in the next section.

Construction 6.1
Fix a simplicial set S, a simplicial category $\mathscr{C}$, and a functor $\varphi : \mathfrak{C}[S] \to \mathscr{C}^{op}$. Given an object $X \in \mathsf{sSet}_{/S}$, let ∞ denote the cone point of $X^{\triangleright}$. Now we can view the ordinary pushout

$$\mathcal{M}_X = \mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} \mathscr{C}^{op}$$

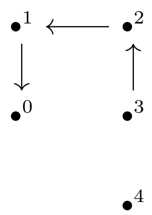
as a correspondence in the sense of [Lur09] from $\mathscr{C}^{op}$ to $\{\infty\}$, in the sense that it enables us, for each $c \in \mathscr{C}$, ask the question: "how do I get from c to ∞?". This path will necessarily go through the image of $\mathfrak{C}$.

Definition 6.2
Using the above construction we define the straigtening functor $\mathrm{St}_\varphi : \mathsf{sSet}_{/S} \to \mathsf{Fun}_{\mathsf{Cat}_\Delta}(\mathscr{C}, \mathsf{sSet})$, which on objects $X \in \mathsf{sSet}_{/S}$ is given as

$$\mathrm{St}_\varphi(X) := \mathrm{Hom}_{\mathcal{M}_X}(-, \infty) : \mathscr{C} \to \mathsf{sSet}.$$

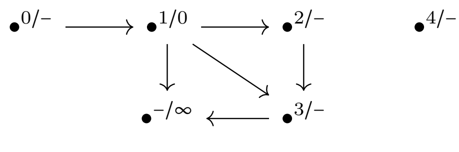
We will give a very simple example of this construction in the case where each piece of data comes from the very first examples of simplicial sets, categories and functors.

Example 6.3
Consider the one simplex $1 \to 3$ in Δ^3, and the representing map $(0,1) : \Delta^1 \to \Delta^3$. Consider Δ^1 as an element of the category $\mathsf{sSet}_{/\Delta^3}$. Note that $\mathfrak{C}(\Delta^3)$ has objects the 0-simplices of Δ^3, i.e., the set $\{0,1,2,3\}$. We take as the simplicial category, C, the a category with trivial mapping spaces (either empty or a point) and objects $\{0,1,2,3,4\}$, i.e the planar graph



together with the edges obtained by compostion.

We can consider the functor $\mathfrak{C}(\Delta^3) \to \mathscr{C}^{op}$ given by the inclusion on objects and the unique map to Δ^0 on the non-empty mapping spaces. This induces, by precomposition, the functor $\mathfrak{C}(\Delta^1) \to \mathscr{C}^{op}$ sending 0 to 1 and 1 to 3, with the only possible map on each mapping space. We can now form the pushout mentioned in the previous construction, denote it again by $\mathcal{M}$. Here we adopt some ad hoc notation; an element in $\mathcal{M}$ is denote i/j where i is the label in $\mathscr{C}$ and j is the label in $\Delta^1 \star \Delta^0$ (the adjoined terminal vertex is denoted ∞). In this notation, we can visualize $\mathcal{M}$ as follows:



We can now consider the functor $\mathscr{C} \to \mathsf{sSet}$ given by $c \mapsto \mathrm{Hom}_{\mathcal{M}}(c, \infty)$.

Armed with this framework, we are ready to explore some examples of straightening and give a sketch of proof which is due to [GH16b].

6.2 Proof of Straightening and Unstraightening

In this section we give a proof of straightening unstraightening in the sense of Lurie. We follow [GH16b], for the proof strategy. We will in the following section discuss in some detail several simple examples, to get a feeling for the structures involved.

Theorem 6.4
For a simplicial set X, a simplicial category $\mathscr{C}$ and a map $\mathfrak{C}(X) \to \mathscr{C}$ there is an adjoint pair of functors

$$St : \mathsf{sSet}_{/X} \to \mathsf{Fun}_{Cat_\Delta}(\mathscr{C}, \mathsf{sSet})$$
$$Un : \mathsf{Fun}_{Cat_\Delta}(\mathscr{C}, \mathsf{sSet}) \to \mathsf{sSet}_{/X}$$

which is a Quillen adjunction, where $\mathsf{sSet}_{/X}$ has the covariant model structure, and $\mathsf{Fun}_{Cat_\Delta}(\mathscr{C}, \mathsf{sSet})$ has the projective model structure. In the case where the map $\mathfrak{C}(X) \to \mathscr{C}$ is a categorical equivalence the pair is a Quillen equivalence.

If nothing else is stated, we will consider the identity map $\mathfrak{C}(X) \to \mathfrak{C}(X)$ instead of a general morphism of simplicial categories.

The overall strategy for the proof will be to construct left Quillen functors in both directions. As with lemma 5.26, this strategy is due to G. Heuts and I. Moerdijk. Armed with these functors, the proof simplifies quite a bit compared to the one originally given by Lurie in [Lur09]. We start by contructing the other Quillen pair we will need.

We wish to define a certain simplicial functor $h : \mathscr{C}^{op} \to \mathsf{sSet}_{/N(\mathscr{C})}$, for some simplicial category $\mathscr{C}$, from which we will construct a Quillen adjoint pair $(h_!, h^*)$, where $h_! : \mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet}) \to \mathsf{sSet}_{/N(\mathscr{C})}$ and $h^* : \mathsf{sSet}_{/N(\mathscr{C})} \to \mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet})$. That this pair is a Quillen adjunction is the content of the first proposition.

Definition 6.5
Let $\mathscr{C}$ be a simplicial category. Given a two simplicial functors $F, G : \mathscr{C} \to \mathsf{sSet}$ and a simplicial set M, we define the tensor product

$$(F \otimes M)(a) = F(a) \times M$$

and for simplicial natural transformations

$$\mathrm{Hom}(F, G)_n = \mathrm{Hom}_{\mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet})}(F \otimes \Delta^n, G).$$

Remark 6.6
$\mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet})$ with the projective model structure is simplicial with the simplicial structure given by the above.

Construction 6.7
The functor $h : \mathscr{C}^{op} \to \mathsf{sSet}_{/N(\mathscr{C})}$ is defined on objects $x \in \mathscr{C}$ as $x \mapsto h(x) : x_{/N(\mathscr{C})} \to N(\mathscr{C})$. The simplicial structure of $x_{/N(\mathscr{C})}$ is given as

$$(x_{/N(\mathscr{C})})_n = \{\varphi : \Delta^{n+1} \cong \Delta^0 \star \Delta^n \to N(\mathscr{C}) \mid \varphi|_{\Delta^0} = x\}$$

h is a simplicial functor: we define maps of simplicial sets $h_{x,y}$ where

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(y,x) \times x_{/N(\mathscr{C})} & \xrightarrow{h_{x,y}} & y_{/N(\mathscr{C})} \\ \downarrow & \swarrow & \\ N(\mathscr{C}), & & \end{array}$$

via considering the n-simplex ξ in $\mathrm{Hom}_{\mathscr{C}}(y,x) \times x_{/N(\mathscr{C})}$, i.e a map

$$\xi = (\xi_1, \xi_2) : \Delta^n \to \mathrm{Hom}_{\mathscr{C}}(y,x) \times x_{/N(\mathscr{C})}.$$

ξ_1 exists because $\mathscr{C}$ is simplicially enriched, and ξ_2 corresponds to a map $\varphi : \Delta^{n+1} \cong \Delta^0 \star \Delta^n \to N(\mathscr{C})$ such that $\varphi|_{\Delta^0} = x$. Using the $(\mathfrak{C}, N)$ adjunction ξ_2 corresponds to a simplicial functor $\tilde{\xi_2} : \mathfrak{C}[\Delta^{n+1}] \to$

$\mathscr{C}$. Now we define the n-simplex $h_{x,y}(\xi)$ of $y_{/N(\mathscr{C})}$ as the one which corresponds to the simplicial functor $\tilde{h}_{x,y}(\xi) : \mathfrak{C}[\Delta^{n+1}] \to \mathscr{C}$. $\tilde{h}_{x,y}(\xi)$ is on objects given by

$$\tilde{h}_{x,y}(\xi)(i) = \begin{cases} y & i = 0 \\ \tilde{\xi_2}(i) & 1 \leq i \leq n+1. \end{cases}$$

The mapping spaces between non-zero vertices is already determined since the functor is the identity on the on these vertices; so we can just use the mapping spaces we had before. Therefore for $i > 0$ we define

$$\tilde{h}_{x,y}(\xi)(i,j) : \mathrm{Hom}_{\mathfrak{C}\Delta^{n+1}}(i,j)] \to \mathrm{Hom}_{\mathscr{C}}(\tilde{h}_{x,y}(\xi)(i), \tilde{h}_{x,y}(\xi)(j))$$

to be the same as

$$\tilde{\xi_2}(i,j) : \mathrm{Hom}_{N(\mathscr{C})(\Delta^{n+1})}(i,j)] \to \mathrm{Hom}_{\mathscr{C}}(\tilde{\xi_2}(i), \tilde{\xi_2}(j)).$$

For $i = 0$ and $j > 0$ we need to do more, for now we won't because of its very technical nature, even though it is a very large part of what makes this strategy work, and it is very non-trivial to show that this is possible. The maps defined make $\tilde{h}_{x,y}(\xi)$ a simplicial functor. Hence h is a simplicial functor, we now construct $h_!$. Let $h_!$ be the simplicial left Kan extension of h along the Yoneda embedding:

$$\begin{array}{ccc} \mathscr{C}^{op} & \xrightarrow{h} & \mathsf{sSet}_{/N(\mathscr{C})} \\ \big\downarrow y & \nearrow_{h_!} & \\ \mathsf{Fun}_{\Delta}(\mathscr{C}, \mathsf{sSet}) & & \end{array}$$

For a simplical functor F we have $h_!(F) = h \otimes_{\mathscr{C}} F$, where we form the tensor product by regarding F as a left module (presheaf on $\mathscr{C}$) and h as a right module (an enriched functor to $\mathscr{C}$). This tensor product can be computed as the coequalizer of the following diagram in simplicial sets over $N(\mathscr{C})$,

$$\textstyle\coprod_{a,b \in \mathscr{C}} h(b) \otimes (\mathrm{Hom}_{\mathscr{C}}(a,b) \times F(a)) \rightrightarrows \coprod_{a \in \mathscr{C}} h(a) \otimes F(a).$$

Here the tensor product is the one from definition 6.5. The two arrows are the ones afforded by the left module structure of F and the right module structure of h. The associativity of $\otimes$ from definition 6.5 ensures the existence of a natural isomorphism $h_!(F \otimes M) \cong h_!(F) \otimes M$, hence $h_!$ is a simplicial functor. The general adjoint functor theorem ensures the existence of a right adjoint h^*.

Having defined this adjoint pair, we can state the first step in the proof:

Proposition 6.8
The functor constructed in construction 6.7 is a part of a Quillen adjunction, where $\mathsf{Fun}_{\Delta}(\mathscr{C}, \mathsf{sSet})$ *is given the projective model structure, and* $\mathsf{sSet}_{/N(A)}$ *is given the covariant model structure. We denote the right Quillen functor in this adjunction by* h^*.

To check that the adjunction is indeed a Quillen adjunction it would suffice to prove that it preserves cofibrations and trivial cofibrations, which can be checked directly on a generating set for each.

We now sketch a proof of the fact that the pair is in fact a Quillen adjunction. Crucially, we will see that the orientation of this adjunction is the opposite of what we saw in proposition 6.8.

Proposition 6.9
The functor pair of theorem 6.4 is a Quillen adjunction, when we consider the covariant and projective model structures.

The proof of this fact amounts to a slightly more careful version of the proof of proposition 6.8.

The final ingredient in the setup, is the Quillen equivalence mentioned in remark 3.7.

Definition 6.10
Through the general adjoint functor theorem St has a right adjoint $\mathrm{Un} : \mathsf{Fun}_{Cat_{\Delta}}(\mathscr{C}, \mathsf{sSet}) \to \mathsf{sSet}_{/X}$, which we call unstraigtening.

For the proof of theorem 6.4, we will need these two lemma

Lemma 6.11
For a fibrant simplicial category $\mathscr{C}$, the two functors

$$\begin{aligned} \mathbb{L}St \circ \mathbb{L}h_! &: h\mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet}) \to h\mathsf{Fun}_\Delta(\mathfrak{C}(N(\mathscr{C})), \mathsf{sSet}) \\ \mathbb{R}\epsilon^* &: h\mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet}) \to h\mathsf{Fun}_\Delta(\mathfrak{C}(N(\mathscr{C})), \mathsf{sSet}) \end{aligned}$$

are naturally isomorphic. Where the equivalence of simplicial categories $\epsilon : \mathfrak{C}(N(\mathscr{C})) \to \mathscr{C}$ is the counit of the Quillen equivalence $(\mathfrak{C}, N)$.

Note that, by lemma 5.23, we have that $\mathbb{R}\epsilon^*$ is an isomorphism. This lemma proves that $\mathbb{L}h_!$ has a left quasi-inverse, for concluding theorem 6.4 we need to show that it has a right inverse too. For this we need the following lemma

Lemma 6.12
Let $F : \mathscr{C} \to \mathscr{D}$ be a simplicial functor and consider the square

$$\begin{array}{ccc} \mathsf{Fun}_\Delta(\mathscr{C}, \mathsf{sSet}) & \xrightarrow{h_!^{\mathscr{C}}(-)} & \mathsf{sSet}_{/N(\mathscr{C})} \\ \Big\downarrow{\scriptstyle F_!} & & \Big\downarrow{\scriptstyle N(\mathscr{C})(F)} \\ \mathsf{Fun}_\Delta(\mathscr{D}, \mathsf{sSet}) & \xrightarrow{h_!^{\mathscr{D}}(-)} & \mathsf{sSet}_{/N(\mathscr{C})} \end{array}$$

Where the horizontal functors are $h_!$ applied to simplicial functors out of $\mathscr{C}$ and $\mathscr{D}$ respectively. There is a natural transformation $H : (N(F))_! \circ h_!^{\mathscr{C}} \to h_!^{\mathscr{D}} \circ F_!$, which is a covariant weak equivalence over $N(\mathscr{D})$ when evaluated on projectively cofibrant objects.

Remark 6.13
The square in lemma 6.12 does *not* commute up to natural isomorphism.

Lemma 6.14
Let X be a simplicial set, and consider the unit $\eta : X \to N(\mathfrak{C}(X))$. The functors

$$\begin{aligned} \mathbb{L}h_! \circ \mathbb{L}St &: h\mathsf{sSet}_{/X} \to h\mathsf{sSet}_{/N(\mathfrak{C}(X))} \\ \mathbb{L}\eta_! &: h\mathsf{sSet}_{/X} \to h\mathsf{sSet}_{/N(\mathfrak{C}(X)))} \end{aligned}$$

are naturally isomorphic.

Again we have that η is an equivalence, and so $\eta_!$ is part of a Quillen equivalence by lemma 5.26.

Lets construct the right quasi-inverse to $h_!$, pick $X \in \mathsf{sSet}$ and an equivalence $F : \mathfrak{C}(X) \to \mathscr{C}$. Write $\tilde{F} : X \to N(\mathscr{C})$ for the adjoint map, which is a categorical equivalence. Now

$$\begin{aligned} \mathbb{L}h_! \circ \mathbb{L}F_! \circ \mathbb{L}\mathrm{St}_! &\cong \mathbb{L}((N(F))_!) \circ \mathbb{L}h_! \circ \mathbb{L}\mathrm{St}_! \\ &\cong \mathbb{L}((N(F))_!) \circ \mathbb{L}\eta_! \\ &\cong \mathbb{L}\tilde{F}_! \end{aligned}$$

The first isomorphism follows from lemma 6.12, the second is lemma lemma 6.14, and the third is from the identification $\tilde{F} = N(F) \circ \eta$. Since $\tilde{F}$ is a categorical equivalence, lemma 5.23 gives that $\mathbb{L}\tilde{F}_!$ is an equivalence of categories, hence $\mathbb{L}h_!$ admits a right quasi-inverse. Hence it follows that $\mathbb{L}h_! \circ \mathbb{L}\mathrm{St}$ is an isomorphism.

Essentially the same calculation shows that $\mathbb{L}\mathrm{St}$ admits a left quasi-inverse, which is by the opposite argument is an equivalence of categories. This finishes the proof that in the special case of theorem 6.4 where the map $\mathfrak{C}(X) \to \mathscr{C}$ is indeed the identity map $\mathfrak{C}(X) \to \mathfrak{C}(X)$. To get the general result where we replace this identity map with a categorical equivalence is then just an application of lemma 5.26.

This finishes the sketch of the proof.

7 Some special cases

7.1 Straightening over Δ^0

We will now consider the following special case to indicate how these functors work. In particular we will show directly that we obtain a Quillen equivalence in this special case. This section is mainly based on [Lur09] and [JR16].

Corollary 7.1
There is a adjunction on the category simplicial sets afforded by straightening and unstraightening, by specializing theorem 6.4 with $S = \Delta^0$ and $\varphi = \mathrm{id}_{\mathfrak{C}[\Delta^0]}$. Furthermore for any simplicial set X,

$$St(X) = \mathrm{colim}_{\Delta^n \to X}(\mathfrak{C}[\mathrm{Hom}_{J^n}(x,\infty)]).$$

Where $J^n = \Delta^{n+1} \coprod_{\Delta^n} \Delta^0$.

Proof. Note that $\mathfrak{C}[\Delta^0]$ is the simplicial category with one object 0 and mapping space $\mathrm{Hom}_{\mathfrak{C}[\Delta^0]}(0,0) = \Delta^0$. Therefore there are isomorphisms

$$\mathsf{Fun}_{\mathsf{Cat}_\Delta}(\mathfrak{C}[\Delta^0], \mathsf{sSet}) \cong \mathsf{sSet},$$
$$\mathsf{sSet}_{/\Delta^0} \cong \mathsf{sSet}.$$

Which proves the first statement. Now straightening, being a left adjoint, preserves colimits, so it is determined by the cosimplicial object Q in sSet given by

$$Q^n = \mathrm{St}(\Delta^n).$$

Now per. definition Q, the fact that $\Delta^{n+1} \cong \Delta^n \star \Delta^0$, and $\mathfrak{C}$ preserving colimits, we have

$$Q^n = \mathrm{Hom}_{\mathcal{M}_{\Delta^n}}(x,\infty) \cong \mathrm{Hom}_{\mathfrak{C}[\Delta^{n+1}\coprod_{\Delta^n}\Delta^0]}(x,\infty) \coloneqq \mathrm{Hom}_{\mathfrak{C}[J^n]}(x,\infty).$$

Where x is the unique vertex of Δ^0. Now since every simplicial set can be described as a colimit over its simplices we have

$$\mathrm{St}(X) = \mathrm{colim}_{\Delta^n \to X}(\mathrm{Hom}_{\mathfrak{C}[J^n]}(x,\infty)).$$

□

We will now apply unstraigtening to a hom-set, and realize it as a more tangible object. Let $X \in \mathsf{sSet}$ then $\mathrm{Hom}^R_X(a,b)$ is the simplicial set given by the formula

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathrm{Hom}^R_X(a,b)),$$

which describes the set of all $z : \Delta^{n+1} \to X$ such that $z|_{\Delta^{\{n+1\}}} = b$ and $z|_{\Delta^{\{0,\dots,n\}}}$ is the constant simplex at the vertex a. An alternative description is

$$\mathrm{Hom}^R_X(a,b)_n = \{\sigma : J^n \to X | \sigma(x) = a, \sigma(\infty) = b\},$$

Where ∞ and x are as above. These are the same via the Yoneda lemma.

Lemma 7.2
Given a simplicial category $\mathscr{C}$, there is a natural isomorphism

$$\mathrm{Hom}^R_{N(\mathscr{C})}(a,b) \cong Un(\mathrm{Hom}_{\mathscr{C}}(a,b)).$$

Proof. Any simplex $\sigma \in \mathrm{Hom}_{N(\mathscr{C})}(a,b)_n$ correspond under the $(\mathfrak{C}, N)$ adjunction to a map $\tilde{\sigma} : \mathfrak{C}J^n \to \mathscr{C}$, again taking $\tilde{\sigma}(\infty) = b$ and $\tilde{\sigma}(x) = a$. This simplicial set is determined by the map $\mathrm{Hom}_{\mathfrak{C}[J^n]}(x,\infty) \to \mathrm{Hom}_{\mathscr{C}}(a,b)$. Hence we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}^R_{N(\mathscr{C})}(a,b)_n &\cong \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{Hom}_{\mathfrak{C}[J^n]}(x,\infty), \mathrm{Hom}_{\mathscr{C}}(a,b)) \\
&= \mathrm{Hom}_{\mathsf{sSet}}(Q^n, \mathrm{Hom}_{\mathscr{C}}(a,b)) \\
&= \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{St}(\Delta^n), \mathrm{Hom}_{\mathscr{C}}(a,b)) \\
&\cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathrm{Un}(\mathrm{Hom}_{\mathscr{C}}(a,b))) \\
&\cong \mathrm{Un}(\mathrm{Hom}_{\mathscr{C}}(a,b))_n.
\end{aligned}$$

□

We begin the proof that the adjunction of corollary 7.1 is in fact a Quillen equivalence.

Definition 7.3
For any $[n]$, denote $\mathcal{P}([n])$ for the poset associated to the powerset of $[n]$. Note that this is isomorphic as a poset to a cube

$$\begin{aligned} \mathcal{P}([n]) &\to [1]^{n+1} \\ S &\mapsto (e_0, ..., e_n) \end{aligned}$$

Where $e_i = 1$ if $i \in S$ and $e_i = 0$ if not. Define $\mathcal{P}_{[n]} \subset \mathcal{P}([n])$ to be the full subcategory of the power set on all nonempty subsets. The nerve preserves products, hence we obtain

$$K_{[n]} := N(\mathcal{P}_{[n]}) \subset N([1]^{n+1}) = N([1])^{n+1} = (\Delta^1)^{n+1}.$$

Which is the simplicial set $K_[n]$ is the barycentric subdivision of Δ^n.

Definition 7.4
For any $[n]$ taking the supremum gives a poset map

$$\begin{aligned} \mathcal{P}_{[n]} &\to [n] \\ S &\mapsto \sup(S). \end{aligned}$$

Lemma 7.5
Consider the specialization of corollary 7.1, then there is a natural transformation $\pi : St \Rightarrow \mathrm{id}$.

Proof. Consider the supremum map of definition 7.4, and apply nerves to obtain a map $K_{[n]} \to \Delta^n$. We wish to refine this map to a map of pushouts, $\pi^n : Q^n \to \Delta^n$. For each $i \in [n]$ one can consider a face of the cube,

$$(\Delta^1)^{\{0,...,i-1\}} \times \{1\} \times (\Delta^1)^{\{i+1,...,n\}} \subset K_{[n]}.$$

We obtain Q^n by collapsing $(\Delta^1)^{\{0,...,i-1\}} \times \{1\} \times (\Delta^1)^{\{i+1,...,n\}}$ onto $(\Delta^1)^{\{i+1,...,n\}}$, via the following pushout

$$\begin{array}{ccc} \coprod_{i\in[n]} (\Delta^1)^{\{0,...,i-1\}} \times \{1\} \times (\Delta^1)^{\{i+1,...,n\}} & \longrightarrow & K_{[n]} \\ \downarrow & & \downarrow \\ \coprod_{i\in[n]} (\Delta^1)^{\{i+1,...,n\}} & \longrightarrow & Q^n \end{array}$$

This construction is functorial, so we obtain a map $\pi^n : Q^n \to \Delta^n$. These maps induce a map of cosimplicial objects $\pi : Q^\bullet \to \Delta^\bullet$. Given any simplicial set X, π induces a map on colimits

$$\pi_X : \mathrm{colim}_(Q^n) \to \mathrm{colim}_{\Delta^n \to X}(\Delta^n)$$

Using the description developed in the proof of corollary 7.1 of $\mathrm{St}(X)$, we obtain the components of the desired natural transformation: $\pi_X : \mathrm{St}(X) \to X$. □

Lemma 7.6
The components of the natural transformation π from lemma 7.5, is a Kan equivalence.

Proof. Let $\mathcal{T}$ be the collection of all simplicial sets for which the result is true.

If $\mathcal{T}$ is closed under filtered colimits, and it contains the simplicial sets with finitely many nondegenerate simplices, we are done, because any simplicial set may be written as a filtered colimit of its finite subobjects.

We start by showing that $\mathcal{T}$ is closed under filtered colimits. Suppose we have a filtered diagram $F \in \mathsf{Fun}_\Delta(J, \mathsf{sSet})$, with $F_i \in \mathcal{T}$ for every $i \in J$. Note that π give a pointwise Kan equivalence $\mathrm{St} \circ F \to F$. Since J is filtered, the colimit functor $\mathrm{colim} : \mathsf{Fun}_\Delta(J, \mathsf{sSet}) \to \mathsf{sSet}$, takes pointwise Kan equivalences to Kan equivalences.

Now it only remains to prove that $\mathcal{T}$ contains the simplicial sets with finitely many nondegenerate simplices. We will proceed by induction on the dimension, and the number of nondegenerate simplices of that dimension. The base case $X = \varnothing$ is clear. Suppose X is obtained from Y by attaching n-cells,

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

Because the monomorphisms are the cofibrations of the Quillen-Kan model structure the left hand vertical map is a cofibration. Now straightening preserves colimits, hence it preserve the above pushout. Per. construction of straightening it preserves monomorphisms, hence cofibrations, so we obtain $\mathrm{St}(X)$ as the following homotopy pushout

$$\begin{array}{ccc} \mathrm{St}(\partial\Delta^n) & \longrightarrow & \mathrm{St}(Y) \\ \downarrow & & \downarrow \\ \mathrm{St}(\Delta^n) & \longrightarrow & \mathrm{St}(X) \end{array}$$

Now if Y, $\partial\Delta^n$ and Δ^n are contained in $\mathcal{T}$, we are done, because then we will have obtained a pointwise Kan equivalence between these two diagrams, which by the above will give a Kan equivalence on the colimits of the systems,

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \\ \Delta^n & & \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{St}(\partial\Delta^n) & \longrightarrow & \mathrm{St}(Y) \\ \downarrow & & \\ \mathrm{St}(\Delta^n). & & \end{array}$$

Note that $\partial\Delta^n$ and Y are in $\mathcal{T}$ by induction hypothesis. Note that $\Delta^n \in \mathcal{T}$ is the same as $\pi^n : Q^n \to \Delta^n$, is a Kan equivalence. The latter is true, because Q^n, was constructed as a pushout of $\Delta^n \to \Delta^0$ and $\Delta^n \to \Delta^n \star \Delta^0$, hence it has a terminal object, hence it is Kan equivalent to a point, and so is Δ^n. □

Theorem 7.7
The adjunction (St, Un) is a Quillen equivalence for the Quillen-Kan model structure on simplicial sets.

Proof. St preserve cofibrations. Kan equivalences are also preserved, to see this apply the 2-out-of-3 property to the naturality square

$$\begin{array}{ccc} \mathrm{St}(X) & \longrightarrow & \mathrm{St}(Y) \\ \downarrow \simeq & & \downarrow \simeq \\ X & \longrightarrow & Y \end{array}$$

Here we've applied lemma 7.6. Hence St and Un is a Quillen pair. Now we wish to show that the left derived of St, $\mathbb{L}\mathrm{St}$ is naturally isomorphic to the identity $\mathrm{id}_{h\mathsf{sSet}}$. Consider $X \in \mathsf{sSet}$, it is its own cofibrant replacement because every object is cofibrant. From lemma 7.6 we have that $\mathrm{St}(X) \to X$ is a Kan equivalence, hence it is inverted in the homotopy category, therefore $\mathbb{L}\mathrm{St}(X) \to X$ is an isomorphism, hence $\mathbb{L}\mathrm{St} \cong \mathrm{id}_{h\mathsf{sSet}}$, and therefore St and Un constitute a Quillen equivalence. □

7.2 Straightening induced by the identity on $\mathfrak{C}X$

We will now consider another special case, for which we will obtain a way to compare straightening and unstraightening in terms of $\mathrm{Hom}^R_X(x, y)$, for $X \in \mathsf{sSet}$. This comparison will turn out to be quite good in the case where X is an ∞-category. We will consider the case when $\varphi = \mathrm{id}_{\mathfrak{C}X}$, for which we obtain a adjoint pair

$$\mathrm{St}_K : \mathsf{sSet}_{/X} \to \mathsf{Fun}_{Cat_\Delta}(\mathfrak{C}X, \mathsf{sSet})$$
$$\mathrm{Un}_K : \mathsf{Fun}_{Cat_\Delta}(\mathfrak{C}X, \mathsf{sSet}) \to \mathsf{sSet}_{/X}$$

We will need some parts of the theory of anodyne maps, all of which is stated and proved in [Lur09].

Lemma 7.8 (Prop. 2.1.4.9)
Let $X \in \mathsf{sSet}$. Every right anodyne map is a trivial covariant cofibration on $\mathsf{sSet}_{/X}$.

The following lemma is due to Joyal.

Lemma 7.9 (Cor. 2.1.2.2)
For X an ∞-category, and $p : D \to X$ a diagram. Then the projection from the under category $X_{/p} \to X$ is a right fibration. In particular $X_{/p}$ is an ∞-category.

Definition 7.10
Let $p : S \to T$ be a map of simplcial sets. We say that p is cofinal if, for any right fibration $X \to T$ where we view S, X as objects of sSet_T, the induced map of simplicial sets

$$X^T \times_{T^T} \{\mathrm{id}_T\} \to X^S \times_{T^S} \{p\}$$

is a homotopy equivalence.

Lemma 7.11 (Cor. 4.1.1.3)
An inclusion $i : X \to X'$ of simplicial sets is cofinal if and only if it is right anodyne.

Definition 7.12
Let $X \to K$ be a map of simplicial sets. Given a vertex $k \in K$, we define the fiber over k as the pullback X_k

$$\begin{array}{ccc} X_k & \longrightarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow k \\ X & \longrightarrow & K \end{array}$$

Lemma 7.13 (Prop 2.2.3.15)
There is a canonical map of pushouts

$$St_k(X_k) \to (St_K(X))(k)$$

If the map $X \to K$ which constructs the fiber, is a right fibration, then this map is a Kan equivalence.

Lemma 7.14 (Prop 4.1.3.1)
Let $f : X \to Y$ be a map of simplicial sets with Y a ∞-category. Then f is cofinal if and only if for every object $y \in Y$, the simplicial set $X \times_Y Y_{Y/}$ is weakly contractible.

Corollary 7.15
The inclusion of the terminal object $\bullet \to \mathscr{C}$ is cofinal.

Proof. By lemma 7.14 the inclusion is cofinal precisely if for all $c \in \mathscr{C}$, the hom-set $Hom_{\mathscr{C}}(c, \bullet)$ is contractible. Which is true per. definition of being terminal. □

Definition 7.16
Let $X \in \mathsf{sSet}$. Via lemma 7.2 we obtain an isomorphism $\mathrm{Hom}^R_{N\mathfrak{C}X}(x,y) \cong \mathrm{Un}(\mathrm{Hom}_{\mathfrak{C}X}(x,y))$ for any $x, y \in X$. The unit of the $(\mathfrak{C}, N)$ adjunction gives a map

$$\mathrm{Hom}^R_X(x,y) \to \mathrm{Hom}^R_{N\mathfrak{C}X}(x,y) \to \mathrm{Un}(\mathrm{Hom}_{\mathfrak{C}X}(x,y))$$

which under the $(\mathrm{St}, \mathrm{Un})$ adjunction correspond to a map,

$$c : \mathrm{St}(\mathrm{Hom}^R_X(x,y)) \to \mathrm{Hom}_{\mathfrak{C}X}(x,y)$$

Theorem 7.17
If X is an ∞-category the map c from definition 7.16 is a Kan equivalence.

Proof. We will give a factorisation of c, and show that both c' and c'' are Kan equivalence, which will imply that c is a Kan equivalence, using the 2-out-of-3-property,

$$\begin{array}{ccc} \mathrm{St}(\mathrm{Hom}^R_X(x,y)) & \xrightarrow{c} & \mathrm{Hom}_{\mathfrak{C}X}(x,y) \\ \downarrow{\scriptstyle c'} & \nearrow{\scriptstyle c''} & \\ (\mathrm{St}_X(X_{/y}))(x) & & \end{array}$$

Consider the map $c' : \mathrm{St}(\mathrm{Hom}^R_X(x,y)) \to (\mathrm{St}_X(X_{/y}))(x)$ which is given as the following composition of maps

$$\mathrm{St}(\mathrm{Hom}^R_X(x,y)) \to \mathrm{St}_{\{x\}}((X_{/y})_x) \to (\mathrm{St}(X_{/y}))(x)$$

Where

$$(X_{/y})_n = \{\sigma : \Delta^{n+1} \to X \mid \sigma(n+1) = y\}.$$

The first map of this composition is an isomorphism, which follows from the definition, and the second map is the one which appear in lemma 7.13. Now via lemma 7.9 for any ∞-category X, the map $X_{/y} \to X$ is a right fibration, which together with lemma 7.13, gives that c' is a Kan equivalence.

Furthermore consider the map $c'' : (\mathrm{St}_X(X_{/y}))(x) \to \mathrm{Hom}_{\mathfrak{C}(X)}(x,y)$ which is induced by the canonical map

$$\mathfrak{C}((X_{/y})^{\triangleright} \coprod_{X_{/y}} X) \to \mathfrak{C}(X).$$

Note that c'' makes the diagram

$$\begin{array}{ccc} (\mathrm{St}_X(\{y\}))(x) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathfrak{C}X}(x,y) \\ {\scriptstyle (\mathrm{St}_X(i))(x)}\downarrow & \nearrow{\scriptstyle c''} & \\ (\mathrm{St}_X(X_{/y}))(x) & & \end{array}$$

commute. The isomorphism follows from the definition of $(\mathrm{St}_X(\{y\}))(x)$. Now $i : \{y\} \to X_{/y}$ is cofinal via corollary 7.15, hence it is right anodyne via lemma 7.11. Now via lemma 7.8 i is a trivial covariant cofibration on $\mathsf{sSet}_{/X}$, which therefore is sent to a pointwise Kan equivalence by St_X. Therefore by the 2-out-of-3-property c'' is a Kan equivalence. □

Corollary 7.18
If X is a ∞-category, then for any $x, y \in X$, we have the following Kan equivalence

$$\mathrm{Hom}^R_X(x,y) \to \mathrm{Hom}_{\mathfrak{C}(X)}(x,y).$$

Proof. We have Kan equivalences, from theorem 7.17, and lemma 7.6,

$$\mathrm{Hom}^R_X(x,y) \to \mathrm{St}(\mathrm{Hom}^R_X(x,y)) \to \mathrm{Hom}_{\mathfrak{C}X}(x,y).$$

□

Remark 7.19
Comparing lemma 7.2 and corollary 7.18, for X a ∞-category. Passing to the homotopy category, one can see the tight relationsship between the two adjoint pairs $(\mathfrak{C}, N)$ and $(\mathrm{St}, \mathrm{Un})$, in the two isomorphisms,

$$\mathrm{Hom}^R_{N(\mathfrak{C}(X))}(x,y) \to \mathrm{Un}(\mathrm{Hom}_{\mathfrak{C}(X)}(x,y)),$$
$$\mathrm{St}(\mathrm{Hom}^R_X(x,y)) \to \mathrm{Hom}_{\mathfrak{C}(X)}(x,y).$$

Instead of taking $\mathfrak{C}(X)$ of some ∞-category X, one could take $\mathscr{C}$ a simplicial category, for which we were certain that $N(\mathscr{C})$ would be an ∞-category, e.q the hom-sets are Kan-complexes, we could simplify the above, because then we would have an isomorphism from lemma 7.2,

$$\mathrm{St}(\mathrm{Hom}^R_{N(\mathscr{C})}(x,y)) \to \mathrm{St}(\mathrm{Un}(\mathrm{Hom}_{\mathscr{C}}(x,y)). \tag{2}$$

This isomorphism alludes to the following corollary.

Corollary 7.20
If $\mathscr{C}$ is a locally Kan simplicial category, then the counit of the $(\mathfrak{C}, N)$ induces Kan equivalences

$$\epsilon_{x,y} : \mathrm{Hom}_{\mathfrak{C}(N(\mathscr{C}))}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,y)$$

for any $x, y \in \mathscr{C}$.

Proof. If $\mathscr{C}$ is locally Kan, then the hom-sets are Kan complexes, hence when applying the homotopy coherent nerve we obtain an ∞-category this is shown in [Lur09]. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{St}(\mathrm{Hom}^R_{N(\mathscr{C})}(x,y)) & \xrightarrow{\cong} & \mathrm{St}(\mathrm{Un}(\mathrm{Hom}_{\mathscr{C}}(x,y))) \\ \downarrow{\scriptstyle c} & & \downarrow \\ \mathrm{Hom}_{\mathfrak{C}(N(\mathscr{C}))} & \xrightarrow{\epsilon} & \mathrm{Hom}_{\mathscr{C}}(x,y). \end{array}$$

The top map comes from (1), the left vertical map is a Kan equivalence from theorem 7.17, and the right vertical map is the counit of the $(\mathrm{St}, \mathrm{Un})$ adjunction, which is a Kan equivalence because every object of sSet is cofibrant. Hence, by the 2-out-of-3-property, the desired map is an Kan equivalence. □

References

[AG04] M. Raynaud A. Grothendieck. Revêtements étales et groupe fondamental (SGA 1). arXiv:math/0206203, 2004.

[DC11] I. Moerdijk D. Cisinski. “Dendroidal sets and simplicial operads.” In: (2011).

[GH16a] I. Moerdijk G. Heuts. “Left fibrations and homotopy colimits”. In: (2016).

[GH16b] I. Moerdijk G. Heuts. “Left fibrations and homotopy colimits II”. In: (2016).

[JA94] J. Rosicky J. Adamek. Locally presentable and accessible categories. Cambridge University Press, 1994.

[Joy07] A. Joyal. Quasi-categories vs simplicial categories. Unpublished, 2007.

[JR16] A. Dhillon J. Richardson. Mapping Spaces and Straightening-Unstraightening. 2016.

[Lan91] S. Mac Lane. Categories for the Working Mathematician. Springer Verlag, 1991.

[LH] I. Madsen L. Hesselholt. Untitled notes Bicategories and Pseudofunctors.

[Lur09] J. Lurie. Higher Topos Theory. Princeton University Press, 2009.

[Qui67] D. Quillen. Homotopical Algebra. 1967.

[Rie14] E. Riehl. Categories in Context. Cambridge University Press, 2014.