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Symmetric Spectra and ∞-Categorical Spectra

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Introduction

In algebraic topology one of the core problems is to understand topological spaces up to homotopy equivalence. The perhaps most fruitful way to approach this problem is by constructing algebraic invariants from the topological spaces, which have the property that two homotopy equivalent spaces have isomorphic algebraic invariants. Some of the most celebrated algebraic invariants are the reduced cohomology theories. It has long been known that these give rise to sequences of spaces (or simplicial sets), which are called *spectra*. These sequences have for a couple of decades had an ever increasing importance in homotopy theory. Certain reduced cohomology theories, e.g. the reduced singular cohomology $\tilde{H}^*(-, R)$ with coefficients in a commutative ring R, are naturally graded commutative rings. This leads to the associated spectrum having certain algebraic properties: they are in some sense "homotopy coherent rings". This led to the realization that a robust generalization of classical algebra to a homotopical setting might be obtained. Historically one of the major goals was to construct a category of spectra which enjoyed the same properties as the category of abelian groups, namely that it should be a closed symmetric monoidal category. We show in first part of this text that this goal can be obtained. Furthermore we shall further develop this generalization of classical algebra, which is called *higher algebra*, for which we shall have the analogies given in the table below.

Another important requirement for the category of spectra, is that it should have an intrinsic notion of homotopy theory itself, i.e. it should have a model structure. But this model structure should be compatible with the tensor product associated to the symmetric monoidal structure. We also show that this is possible in part 1. The tensor product will descend to the homotopy category of this model strucutre and it will give rise to a symmetric monoidal structure here. This will allow us to do algebra in the homotopy category of spectra.

The theory developed in part 1 is what is called a 1-categorical approach to a robust generalization of classical algebra. A more modern approach is to develop the generalization in the formalism of $(\infty, 1)$-categories. This approach tackles the overwhelming task of handling "higher homotopies", i.e. homotopies between homotopies. This generalization is often called *∞-categorical higher algebra*. We shall in part 2 of this text introduce the theory of ∞-categories, which is one model for $(\infty, 1)$-categories. At a very brisk pace we will introduce the necessary theorems and notions from ∞-category theory needed to consider the ∞-category of spectra, which is the starting point of ∞-categorical higher algebra. One of the main goals for the text is to compare these two generalizations. This is done as the very last theorem, theorem 7.3.9. Due to time constraints we shall not introduce the notion of an E_∞-ring, but we include it in the table of analogies for completeness.

Classical algebra	Higher algebra	∞-categorical higher algebra
Sets	Simplicial sets	Simplicial sets
Category	Model category	∞-category
Abelian group	Symmetric spectrum	Spectrum
Commutative ring	Commutative ring spectrum	E_∞-ring
$\mathbb{Z}$	The symmetric sphere spectrum $\mathbb{S}$	The sphere spectrum $\mathbb{S}^\infty$.

Part I

The Category of Symmetric Spectra

1 Symmetric Spectra

This part is mainly based on [1] and [2]. Lets begin with a naive definition of spectra, which historically was also the first. We will in the following section see that this is not the correct notion for a number of reasons. Let S^1 denote the simplicial circle $\Delta^1/\partial\Delta^1$ and write S^n for the n-fold smash product of S^1.

1.1 Spectra

Definition 1.1.1. A *spectrum* E is a sequence of based simplicial sets $\{E_n\}_{n\in\mathbb{N}}$, with structure maps $\sigma_n : E_n \wedge S^1 \to E_{n+1}$. Given two spectra E and D, a *morphism of spectra* is a collection of morphisms $\{f_n : E_n \to D_n\}_{n\in\mathbb{N}}$ such that the following diagram commutes

$$\begin{array}{ccc} E_n \wedge S^1 & \xrightarrow{f_n \wedge S^1} & D_n \wedge S^1 \\ \downarrow{\scriptstyle \sigma_n^E} & & \downarrow{\scriptstyle \sigma_n^D} \\ E_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1}. \end{array}$$

This gives a category of spectra denoted $\mathsf{Sp}^{\mathbb{N}}$.

Remark 1.1.2. Instead of simplicial sets, one might take pointed compactly generated weak Hausdorff topological spaces as a model, to obtain the original definition due to E.L. Lima [3].

There is a tight relationship between cohomology theories and spectra.

Proposition 1.1.3. *Given a spectrum A, write $A^n(-) : \mathsf{sSet}^{op} \to \mathsf{Ab}$ for the functor $X \mapsto [X, A_n]$. Here sSet_*^{op} is the opposite category of the category of pointed simplicial sets. This functor satisfies the Eilenberg-Steenrod axioms, hence it defines a cohomology theory.*

Proof. The proof is straightforward checking of the Eilenberg-Steenrod axioms. □

Via the above proposition we are implicitly giving examples of cohomology theories in the following examples.

Example 1.1.4. The trivial spectrum is given as the constant sequence consisting of the initial object in sSet_*, $\bullet$, in each level, where the structure maps are isomorphisms.

Example 1.1.5 (The suspension spectrum)**.** Given a pointed simplicial set X, the suspension spectrum $\Sigma^\infty X$ is the sequence of pointed simplicial sets $\{X \wedge S^n\}_{n\in\mathbb{N}}$ with natural isomorphisms $\sigma : X \wedge S^n \wedge S^1 \to X \wedge S^{n+1}$ as the structure maps. Thus for each pointed simplicial set X we have a spectrum $\Sigma^\infty X$. This assignment is functorial: let $\phi : X \to Y$ be a morphism of simplicial sets, then we define $(\Sigma^\infty \phi)_n = \phi \wedge \mathrm{id}_{S^n} : X \wedge S^n \to Y \wedge S^n$, which assemble to a morphism of spectra $\Sigma^\infty \phi : \Sigma^\infty X \to \Sigma^\infty Y$. Hence we have a functor $\Sigma^\infty : \mathsf{sSet}_* \to \mathsf{Sp}^{\mathbb{N}}$.

Example 1.1.6 (Loops infinty)**.** Given a spectrum X, there is a right adjoint to Σ^∞ which is given on objects as $X \mapsto X_0$, where X_0 is the 0'th level simplicial set of the spectrum, and on morphisms $f : X \to Y$ as $\Omega^\infty := f_0 : X_0 \to Y_0$. We denote this functor Ω^∞. This functor assigns a spectrum E to its associated infinite loop space $\Omega^\infty E$.

Example 1.1.7 (Sphere spectrum)**.** The suspension spectrum of the simplicial 0-sphere S^0, is denoted $\mathbb{S}$. The natural isomorphisms $S^1 \wedge S^n \to S^{n+1}$ are the structure maps. This spectrum is called the *sphere spectrum*.

Example 1.1.8 (Eilenburg-Mac Lane spectrum)**.** The Eilenburg-Mac Lane spectrum, denoted $H\mathbb{Z}$, is the sequence of Eilenburg-Mac Lane spaces $\{K(n,\mathbb{Z})\}_{n\in\mathbb{N}}$, and the structure maps are the adjoint maps of the weak equivalences $K(n,\mathbb{Z}) \to \Omega K(n+1,\mathbb{Z})$. One could replace $\mathbb{Z}$ with any abelian group.

The above examples of spectra gives two classes of spectra and thus of cohomology theories. But a priori it was not clear that it would be this easy to define concrete examples of spectra. The following theorem shows that we in fact should expect to be able to construct a plethora of spectra. This theorem is called Brown representability which was proved by E. H. Brown in [4], we give a more modern version due to R. Jardine [5], which applies spectra of simplicial sets.

Theorem 1.1.9. *Suppose $\mathscr{C}$ is a closed model category which is cocomplete, pointed, and compactly generated. Suppose that the functor $G : \mathscr{C}^{op} \to \mathsf{Set}_*$ takes weak equivalences to bijections, $G(\bullet)$ is a singleton, it takes coproducts to products for cofibrant objects, and given a pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow \\ B & \longrightarrow & B \cup_A X \end{array}$$

where i is a cofibration and all objects are cofibrant, then the induced function $G(B \cup_A X) \to G(B) \times_{G(A)} G(X)$, is surjective. Then there is an object $Y \in \mathscr{C}$ and a natural bijection

$$G(X) \cong [X, Y].$$

for all objects X of $\mathscr{C}$.

Remark 1.1.10. Note sSet_* equipped with Quillen-Kan model structure satisfies the conditions. A generalized cohomoloy theory on sSet_* satisfies the conditions of the functor G, hence the theorem applies. Therefore there exists spaces E_n for $n \in \mathbb{N}$, and natural equivalences $[X, E_n] \cong [X, \Omega E_{n+1}]$, where Ω is adjoint to Σ, which are induced from weak equivalences $E_n \to \Omega E_{n+1}$, hence $\{E_n\}_{n\in\mathbb{N}}$ assemble into a spectrum.

Remark 1.1.11. The assumptions in this version of Brown representability, will turn up again in part 2 when we define the notion of spectra in ∞-categories, because they quantify what a spectrum, in some sense, should be.

From this point of view, spectra are interesting simply because they correspond to cohomology theories, which are interesting in themselves. Lets collect a couple of well known cohomology theories and their corresponding spectra.

Example 1.1.12. Ordinary cohomology is represented by the Eilenberg-Maclane spectrum with $A = \mathbb{Z}$.

Example 1.1.13. Using topological spaces as a model, complex K-theory correspond to the BU-spectrum, where the even terms are $BU := \operatorname{colim}_n BU(n)$, and the odd terms are $U := \operatorname{colim}_n U(n)$.

Note that the natural equivalences $[X, E_n] \cong [X, \Omega E_{n+1}]$ was induced by weak equivalences $E_n \to \Omega E_{n+1}$, this is a very special property.

Definition 1.1.14. A Ω*-spectrum* E is a spectrum where each level E_n is a Kan complex, and the adjoint maps of the structure maps $\tilde{\sigma_n} : E_n \to \Omega E_{n+1}$ is a weak equivalence.

The main problem with the category of spectra Sp is that it does not have all the right formal properties one would like from a category of spectra. As a start one wants to equip it with a model structure such that its homotopy category has a monoidal product which comes from a monoidal product on the model category. This is not possible because it turns out that the sphere spectrum, in some some sense yet to be defined, is not a commutative object. This is an essential property because the sphere spectrum $\mathbb{S}$ should serve as the unit for the monoidal product. One can remedy this fault by constructing the smash product only on the homotopy category, see [6], but it is rather awkward. We will take a different route, which will ensure that the sphere spectrum is a commuative object.

1.2 Symmetric Spectra

In this section we will give the following refinement of spectra which will give rise to the right notion of a category of spectra, i.e. it has the desired formal properties one would like. A symmetric spectrum is a spectrum which is endowed with an action by the symmetric group in each level. Let Σ_n be the symmetric group on the set $\overline{n} := \{1, ..., n\}$. We embed $\Sigma_n \times \Sigma_m$ as a subgroup of Σ_{n+m} with Σ_n acting on the first n elements of $\overline{n+m}$ and Σ_m on the last m elements.

Definition 1.2.1. A *symmetric spectrum* E is a sequence of pointed simplicial sets $\{E_n\}_{n\in\mathbb{N}}$, structure maps $\sigma : E_n \wedge S^1 \to E_{n+1}$ for each $n \in \mathbb{N}$, and a basepoint preserving left action of Σ_n on E_n, such that

$$E_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{id}} E_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1}\wedge \mathrm{id}} \dots \xrightarrow{\sigma_{n+m-1}} E_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant for all $n, m \in \mathbb{N}$. A *morphism of symmetric spectra* $f : E \to D$ is a collection of Σ_n-equivariant morphisms $\{f_n : E_n \to D_n\}_{n\in\mathbb{N}}$ such that the following diagram commutes for all $n \in \mathbb{N}$.

$$\begin{array}{ccc} E_n \wedge S^1 & \xrightarrow{f_n \wedge \mathrm{id}} & D_n \wedge S^1 \\ \downarrow{\sigma_n^E} & & \downarrow{\sigma_n^D} \\ E_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1}. \end{array}$$

This gives a category of symmetric spectra denoted Sp^Σ.

We will now revisit the examples from the previous section and modify them, to obtain analogous symmetric spectra.

Example 1.2.2 (The symmetric suspension spectrum)**.** The symmetric suspension spectrum $\Sigma^\infty X$ is the suspension spectrum in $\mathsf{Sp}^\mathbb{N}$ equipped with an action of Σ_n at each level: Σ_n acts via the diagonal action on $S^n \wedge X$ with left permutation on S^n and the trivial action on X. Thus for each pointed simplicial set X we have a symmetric spectrum $\Sigma^\infty X$. This assignment is again functorial. Hence we have a functor $\Sigma^\infty : \mathsf{sSet}_* \to \mathsf{Sp}^\Sigma$. Ω^∞ is defined as above, and we still obtain a functor $\Omega^\infty : \mathsf{Sp}^\Sigma \to \mathsf{sSet}_*$, which is right adjoint to Σ^∞.

Example 1.2.3 (Sphere spectrum)**.** The action of Σ_n on the usual sphere spectrum $\mathbb{S}$ is action by left permutation on S^n. From now on $\mathbb{S}$ will refer to the symmetric variant.

Suspension spectra are not the only way to go from a pointed simplicial set to a symmetric spectrum.

Example 1.2.4. [Free symmetric spectrum] We shall later need this construction for pointed topological spaces, so we will give the construction in this case. The one for simplicial sets is analogous. Let K be a pointed topological space (or simplical set) and let $m \geq 0$, we then define the *free symmetric spectrum* as the symmetric spectrum $F_m K$, which is trivial below level m, and is given by

$$(F_m K)_{m+n} = \Sigma_{m+n}^\bullet \wedge_{1\times\Sigma^n} K \wedge S^n = ((\Sigma_{m+n}^\bullet \wedge K)/\sim) \wedge S^n.$$

Where $\Sigma_n^\bullet$ is seen as a discrete topological space, with a basepoint $\bullet$ attached, and $\sim$ is the actions of the subgroup of Σ_{n+m} consisting of permutation which fix the first m elements. The structure map $\sigma_{m+n} : (F_m K)_{m+n} \wedge S^1 \to (F_m K)_{m+n+1}$ is given by $i \wedge \mathrm{id}_K \wedge f$, where $i : \Sigma_{n+m} \to \Sigma_{n+m+1}$ is the inclusion, and f is the isomorphism $S^n \wedge S^1 \cong S^{n+1}$.

Ω^∞ send a symmetric spectrum to its 0'th level, hence send a symmetric spectrum to a simplicial set. There are other ways to do this.

Example 1.2.5. [Evaluation of symmetric spectra] Consider the evaluation functor $\mathrm{Ev}_n : \mathsf{Sp}^\Sigma \to \mathsf{sSet}_*$ for all $n \geq 0$, defined as $\mathrm{Ev}_n(X) = X_n$ and $\mathrm{Ev}_n(g) = g_n$. It turns out that F_m is left adjoint to Ev_m, we will not prove this fact, see [1] for details.

In the previous section we did not need to pick a model for the Eilenberg-Mac Lane spaces $K(A, n)$ while defining the Eilenberg-Mac Lane spectrum. Because we want to describe the symmetric group action, we need to pick a model.

Example 1.2.6 (Eilenburg-Mac Lane spectrum)**.** The Eilenburg-Mac Lane spectrum, denoted $H\mathbb{Z}$, is the sequence of simplicial abelian groups $\{\mathbb{Z} \otimes S^n\}_{n\in\mathbb{N}}$, where the simplicial structure is given as

$$(Z \otimes S^n)_k = \mathbb{Z}[\{\sigma \in (S^n)_k | \bullet \notin \sigma\}].$$

Where the basepoint $\bullet$ is identified with 0. The symmetric group acts by permuting the generators, which is Σ_n-equivariant. Here $\mathbb{Z}$ could be replaced by any abelian group A. An element in $(HA)_n$ can be viewed as a formal A-linear combination of points in S^n, in the sense of Dold-Thom [7]. From this point of view it is easy to see that the assignment of an Eilenberg-Mac Lane spectrum to an abelian group is functorial: given $\phi : A \to B$, we obtain $(H\phi)_n : (HA)_n \to (HB)_n$, given by $(H\phi)_n(\sum a_i x_i) = \sum \phi(a_i)x_i$. The defines a morphism of symmetric spectra $H\phi : HA \to HB$. Hence we obtain a functor $H : \mathsf{Ab} \to \mathsf{Sp}^\Sigma$.

Example 1.2.7. [Mapping cone] The mapping cone $C(f)$ of a morphism of symmetric spectra $f : X \to Y$ is defined by

$$C(f)_n = C(f_n) = ([0,1] \wedge X_n) \cup_f Y_n,$$

i.e levelwise as the reduced mapping cone of $f_n : X_n \to Y_n$. The symmetric group Σ_n acts on $C(f)_n$ through the given action on X_n and Y_n and the trivial action on the interval.

Example 1.2.8. [Simplicial structure on Sp^Σ] Recall that Δ_+ is the full subcategory of the simplex category Δ containing only the face maps. For $X, Y \in \mathsf{Sp}^\Sigma$, we define $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(X, Y)$ as the pointed simplicial set whose n-simplices are the spectrum morphisms from $X \wedge \Delta^-_+$ to Y. For $f : [m] \to [n]$ in Δ we define $f^* : \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\Delta^n_+ \wedge X, Y) \to \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\Delta^m_+ \wedge X, Y)$ to be precomposition with $f_* \wedge \mathrm{id}_X : \Delta^m_+ \wedge X \to \Delta^n_+ \wedge X$. From this it is evident that the hom-sets of Sp^Σ are simplicial sets.

Example 1.2.9 (Hom-Tensor)**.** We will define two functors which will play an important role when we show that Sp^Σ is a so called sSet_*-category. Let $K \in \mathsf{sSet}_*$, then $K \wedge - : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$, and $(-)^K : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$. The structure of $K \wedge X$ is given by $(K \wedge X)_n := K \wedge X_n$, and the structure maps are $\sigma_n : K \wedge X_n \wedge S^1 \to K \wedge X_{n+1}$ are given by $(K \wedge X_n) \wedge S^1 \cong K \wedge (X_n \wedge S^1) \to K \wedge X_n + 1$. Σ_n acts via left permutation X_n and trivially on K, which constitutes an action on $K \wedge X_n$ for all n.

The structure of X^K is given as $(X^K)_n := \mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_n)$. The structure maps σ_n : $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_n) \wedge S^1 \to \mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_{n+1})$ is the composite

$$\mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_n) \wedge S^1 \xrightarrow{\overline{\phi}} \mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_n \wedge S^1) \xrightarrow{(\sigma_n)_*} \mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_{n+1}).$$

$\overline{\phi}$ is the adjoint (the $\wedge$-Hom-adjunction in sSet_*) of $\phi : \mathrm{Map}_{\mathsf{Sp}^\Sigma}(K, X_n) \wedge S^1 \wedge K \to X_n \wedge S^1$. Σ_n acts trivially in K and via left permuation on X_n, which is an action on $(X^K)_n$. One can easily show that $K \wedge X$ and X^K are symmetric spectra, with the above actions of Σ_n. On morphisms we have $(K \wedge f)_n = \mathrm{id}_K \wedge f_n$, and $(f^K)_n = f_n \circ -$. $(-) \wedge K$ is left adjoint to $(-)^K$.

Definition 1.2.10. A Ω-*spectrum* E is a symmetric spectrum, and the adjoint maps of the structure maps $\tilde{\sigma_n} : E_n \to \Omega E_{n+1}$ are a weak equivalence.

Remark 1.2.11. We will later see that Ω-spectra are the fibrant objects in the stable model structure on Sp^{Σ}. It is customary to require that E_n is a Kan complex for every n. We follow the conventions of [1].

Example 1.2.12. Both BU and HA are Ω-spectra. Not all symmetric spectra are Ω-spectra, the symmetric suspension spectrum for a general simplicial set is far from being an Ω-spectrum.

Example 1.2.13. [Shift] The *Shift* of a symmetric spectrum X is given by

$$(\operatorname{sh} X)_n = X_{n+1}$$

with Σ_n action via the restriction of the action of Σ_{n+1} on X_{n+1} along the injection $1 + (-) : \Sigma_n \to \Sigma_{n+1}$ which is given as $(1+\gamma)(1) = 1$ and $(1+\gamma)(i) = \gamma(i-1)+1$ for $2 \le i \le n+1$. The structure maps of $\operatorname{sh} X$ are the reindexed structure maps for X. There is a natural morphism $\lambda_X : S^1 \wedge X \to \operatorname{sh} X$ whose n'th component is given as the composite

$$S^1 \wedge X_n \xrightarrow{\cong} X_n \wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow{\chi_{n,1}} X_{n+1}.$$

Where $\chi_{n,1}$ is an endomorphism which shuffles the action of Σ_{n+1} on X_{n+1}. $\chi(n,1)$ is defined as

$$\chi_{m,n}(i) = \begin{cases} i+m & 1 \le i \le n, \\ i-n & n+1 \le i \le n+m. \end{cases}$$

Example 1.2.14. [Internal Hom Spectrum] Symmetric spectra have internal mapping objects. Let X and Y be symmetric spectra. We define a symmetric spectrum $\operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X,Y)$ as

$$\operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X,Y)_n = \operatorname{Map}_{\mathsf{Sp}^{\Sigma}}(X, \operatorname{sh}^n Y)$$

with Σ_n acting through the action on $\operatorname{sh}^n Y$. The structure map $\sigma_n : \operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X,Y) \wedge S^1 \to \operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X,Y)_{n+1}$ is the composite

$$\operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X, \operatorname{sh}^n Y) \wedge S^1 \xrightarrow{a} \operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X, \operatorname{sh}^n Y \wedge S^1) \xrightarrow{\phi} \operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X, \operatorname{sh}^{n+1} Y).$$

The map a is defined by $f \wedge t \mapsto (x \mapsto f(x) \wedge t)$ for $f : X \to \operatorname{sh}^n Y$ and $t \in S^1$, and $\phi := \operatorname{Hom}_{\mathsf{Sp}^{\Sigma}}(X, \lambda_{\operatorname{sh}^n Y})$. We will not verify that this indeed gives a symmetric spectrum.

Remark 1.2.15. The shift of a spectrum X enables us to do arguments in an inductive fashion. There is no direct examples in this text, but an example where the shift of a symmetric spectrum is used in an essential manner can be found in [1] theorem 7.17, in the proof that a cofibrant very special Γ-space, induces a symmetric spectrum which is an Ω-spectrum.

As mentioned before if one try to develop a homotopy theory for Sp one will run into a lot of trouble, so we have substituted Sp with Sp^{Σ}, to obtain the right homotopy theory. We will in the rest of this section describe some homotopy theoretic notions for symmetric spectra.

Definition 1.2.16. Two morphisms of symmetric spectra $f_0, f_1 : X \to Y$ are called *homotopic* if there is a morphism

$$H : \Delta^1 \wedge X \to Y$$

called a *homotopy*, such that $f_0 = H \circ i_0$, and $f_1 = H \circ i_1$. $\Delta^1 \wedge X$ is the symmetric spectrum given by $(\Delta^1 \wedge X)_n := \Delta^1 \wedge X_n$, with the trivial action on Δ^1. The morphisms $i_j : X \to \Delta \wedge X$ for $j = 0, 1$ are induced by the fact maps $\Delta^0 \to \Delta^1$ and the identification $X \cong \Delta^0 \wedge X$.

Remark 1.2.17. As is evident from the definition a homotopy between spectrum morphisms is really the same as levelwise based homotopies between $(f_0)_n : X_n \to Y_n$ and $(f_1)_n : X_n \to Y_n$, which are compatible with the Σ_n-action and structure maps.

Definition 1.2.18. Given a symmetric spectrum X, *the (naive) homotopy group of* X is defined as

$$\tilde{\pi}_k(X) = \operatorname{colim}_n \pi_{k+n}(X_n)$$

Where the colimit is taken over the composite

$$\pi_{k+n}X_n \xrightarrow{-\wedge S^1} \pi_{k+n+1}(X_n \wedge S^1) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1}X_{n+1}.$$

For large enough n, the set $\pi_{k+n}X_n$ has a natural abelian group structure, hence the above maps are homomorphisms, and therefore the colimit $\tilde{\pi}_k X$ also has abelian group structure.

Definition 1.2.19. A morphism of symmetric spectra $f : X \to Y$ which induces an isomorphism $\tilde{\pi}_*(f) : \tilde{\pi}_*(X) \to \tilde{\pi}_*(Y)$ is called a $\tilde{\pi}_*$-isomorphism.

We shall later need the following example of an $\tilde{\pi}_*$-isomorphism.

Example 1.2.20. For every morphism $f : X \to Y$ of spectra the morphism $h : \Sigma(F(f)) \to C(f)$ is a $\tilde{\pi}_*$-isomorphism.

We omit the elementary proof. It can be found in [1] proposition 2.17.

2 Symmetric Monoidal Structure on Sp^Σ

One of the important formal properties one would like to hold for Sp^Σ is that it has a symmetric monoidal structure. This means we have to construct a tensor-product. We shall use both tensor product and smash product interchangably throughout the text. One way to construct the smash-product on Sp^Σ is via Day convolution which is a tensor product in the sense of a symmetric monoidal category on the category of simplicially enriched functors and natural isomorphisms $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)$. In the following section we will construct the Day convolution, and many other objects. But first lets recall a few formal properties, which we shall largely take for granted.

2.1 Preliminaries

Recall that a *symmetric monoidal category* is a tuple $(\mathscr{C}, \otimes, 1, \alpha, r, l, b)$, satisfying some commutativity axioms, see [8]. We shall agree that when we are giving a symmetric monoidal category, we will omit the data of α, r, l, b, and only give the unit and tensor product. sSet_* is symmetric monoidal $(\mathsf{sSet}_*, \wedge, S^0)$, which is easily verified. If $\mathscr{C}$ is a (symmetric) monoidal category then a *$\mathscr{C}$-category $\mathscr{D}$* is loosely speaking a category where the mapping objects belong to $\mathscr{C}$ satisfying some associativity and unit axioms, see [9] for the complete definintion.

Proposition 2.1.1. *Sp^Σ is a sSet_*-category*

Proof. In example 1.2.8 we saw that the mapping-sets are objects in sSet_*, hence per. definition so are the hom-sets. Given $X, Y, Z \in \mathsf{Sp}^\Sigma$ We need to define a composition law $C : \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(Y, Z) \wedge \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y) \to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Z)$. From the definition of $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(-,-)$ it is clear that it will suffice to define C on simplices. Let $\delta : \Delta^n_+ \to \Delta^n_+ \wedge \Delta^n_+$ be the smash diagonal on Δ, and define

$$\begin{aligned} C_n : \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\Delta^n_+ \wedge Y, Z) \wedge \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\Delta^n_+ \wedge X, Y) &\to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\Delta^n_+ \wedge X, Z) \\ \phi \wedge \psi &\mapsto \phi \circ (\mathrm{id}_{\Delta^n_+}) \wedge \psi) \circ (\delta \wedge \mathrm{id}_X), \end{aligned}$$

It is straight forward to check the commutativity of the associativity-, and unity-diagrams. □

A *$\mathscr{C}$-symmetric monoidal category* is a $\mathscr{C}$-category which satisfies the axioms of a symmetric monoidal category. Such a category is called closed if $- \otimes X : \mathscr{D} \to \mathscr{D}$ has a right adjoint for all $X \in \mathscr{D}$. It turns out that $(\mathsf{sSet}_*, \wedge, S^0)$ is a closed sSet_*-symmetric monoidal category, again we omit the details.

If $\mathscr{D}$ and $\mathscr{E}$ are $\mathscr{C}$-categories, there is category of $\mathscr{C}$-functors and $\mathscr{C}$-natural transformations $\mathsf{Fun}_{\mathscr{C}}(\mathscr{D}, \mathscr{E})$ which itself is a $\mathscr{C}$-category. The special case $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)$, will be of particularly high interest, because it is here we define the Day convolution. In other words we can give the set of sSet_*-natural transformations between two sSet_*-functors the structure of a simplicial set. We do this using the special (co)limits in sSet_* called (simplicial) ends.

Definition 2.1.2. [Ends and Coends] Let $\mathscr{C}$ be a sSet_*-category and let $F : \mathscr{C}^{op} \times \mathscr{C} \to \mathsf{sSet}_*$ be a sSet_*-functor. Let $x \in \mathscr{C}$ and consider the following sSet_*-functor $\mathscr{C} \to \mathsf{sSet}_*$ and its adjoint under the ($\wedge$-Hom)-adjunction in sSet_*,

$$\begin{aligned} F(x,-)_{x,y} &: \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathsf{sSet}_*}(F(x,x), F(x,y)) \\ \overline{F(x,-)_{x,y}} &: \mathrm{Hom}_{\mathscr{C}}(x, y) \wedge F(x,x) \to F(x,y) \end{aligned}$$

Now because the monoidal structure on sSet_* was symmetric, we can apply the braiding to $\overline{F(x,-)_{x,y}}$ and the use the adjunction again to obtain the following morphism in sSet_*

$$f_{x,y} : F(x,x) \to \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(x,y), F(x,y))$$

Similiarly we have $F(-,y)_{x,y} : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathsf{sSet}_*}(F(y,y), F(x,y))$, which corresponds to

$$g_{x,y} : F(y,y) \to \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(x,y), F(x,y))$$

An *end* of F denoted $\int_{x\in\mathscr{C}} F(x,x)$ is an equalizer of the following diagram in sSet_*,

$$\prod_{x\in\mathscr{C}} F(x,x) \underset{\prod_{y,z\in\mathscr{C}} g_{y,z}}{\overset{\prod_{y,z\in\mathscr{C}} f_{y,z}}{\rightrightarrows}} \prod_{y,z\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(y,z), F(y,z)).$$

Dually using $F(x,-)_{y,x}$ we obtain $\overline{f_{y,x}}$ and $\overline{g_{y,x}}$. Then a *coend* of F denoted $\int^{x\in\mathscr{C}} F(x,x)$ is a coequalizer of the following diagram in sSet_*,

$$\prod_{y,z\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(z,y), F(y,z)) \underset{\prod_{y,z\in\mathscr{C}} \overline{g_{z,y}}}{\overset{\prod_{y,z\in\mathscr{C}} \overline{f_{z,y}}}{\rightrightarrows}} \prod_{x\in\mathscr{C}} F(x,x).$$

Because sSet_* is bicomplete these always exist and are unique up to canonical isomorphisms in sSet_*.

Lastly we realize the simplicial left Kan extension of a functor F along a precomposition functor as a coend.

Theorem 2.1.3. *[Simplicial Left Kan extensions] Let $\mathscr{C}$ and $\mathscr{D}$ be sSet_*-categories, and let $p : \mathscr{C} \to \mathscr{D}$ be a sSet_*-functor. Let*

$$\begin{aligned} - \circ p : \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{D}, \mathsf{sSet}_*) &\to \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*), \\ (F : \mathscr{D} \to \mathsf{sSet}_*) &\mapsto (F \circ p : \mathscr{C} \to \mathsf{sSet}_*), \\ (\eta : F \to G) &\mapsto (\eta \circ p) : F \circ p \to G \circ p \quad \textit{Where } (\eta \circ p)_x := \eta_{px}. \end{aligned}$$

This functor has a left adjoint

$$\mathrm{Lan}_p : \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*) \to \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{D}, \mathsf{sSet}_*),$$

called the left simplicial Kan extension along p. Consider an object in $F \in \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)$, then the left simplicial Kan extension of F along p is for $a \in \mathscr{D}$ given by

$$(\mathrm{Lan}_p F)(a) = \int^{x\in\mathscr{C}} \mathrm{Hom}_{\mathscr{D}}(px, a) \wedge Fc.$$

This relies on Fubini for coends and ends, and the simplicial (co)-Yoneda lemma, for both of these see [10]. The results and notions of this section will play an integral part of the rest of the construction of the smash product on Sp^{Σ}.

2.2 Day Convolution

Let $\mathscr{C}$ be a sSet_*-symmetric monoidal category. In this section we will show that the Day convolution product, yet to be defined, will endow $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)$ with a closed sSet_*-symmetric monoidal category structure. Given two sSet_*-functors $F, G : \mathscr{D} \to \mathscr{C}$, of sSet_*-categories, they induce a simplicial functor $(F,G) : \mathscr{D} \times \mathscr{D} \to \mathscr{C} \times \mathscr{C}$.

Definition 2.2.1. [External tensor product] Let $(\mathscr{C}, \otimes, 1)$ be a sSet_*-symmetric monoidal category. Consider sSet_*-functors $F, G : \mathscr{C} \to \mathsf{sSet}_*$. The *external tensor product* is the sSet_*-functor

$$\boxtimes : \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*) \times \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*) \to \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C} \times \mathscr{C}, \mathsf{sSet}_*)$$

defined as $F \boxtimes G := \wedge \circ (F, G)$ for two sSet_*-functors $F, G : \mathscr{C} \to \mathsf{sSet}_*$.

Definition 2.2.2. [Day convolution] Let $(\mathscr{C}, \otimes, 1)$ be a sSet_*-symmetric monoidal category. If F and G are sSet_*-functor from $\mathscr{C}$, then the *Day convolution of F and G* is the left Kan extension of the external tensor product $F \boxtimes G$, along $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$.

$$(F \otimes_D G) = \mathrm{Lan}_{\otimes}(F \boxtimes G)$$

Denote the Day convolution of F and G as $F \otimes_D G$.

Remark 2.2.3. Note that per. definition and 2.1.3, the Day convolution of $F, G : \mathscr{C} \to \mathsf{sSet}_*$, is given as

$$(F \otimes_D G)(c) \cong \int^{a,b \in \mathscr{C} \times \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a \otimes b, c) \wedge F(a) \wedge G(b)$$

Theorem 2.2.4. *Let $(\mathscr{C}, \otimes, 1)$ be a sSet_*-symmetric monoidal category. Then $(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*), \otimes_D, \mathrm{Hom}_{\mathscr{C}}(1, -))$, is a closed sSet_*-symmetric monoidal category and the internal hom denoted $[-,-]_{\mathsf{Fun}}$ is given as*

$$[F, G]_{\mathsf{Fun}}(c) \cong \int_{a,b \in \mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(c \otimes a, b), \mathrm{Hom}_{\mathsf{sSet}_*}(F(a), G(b))$$

for any $F, G \in \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)$ and any $c \in \mathscr{C}$.

Proof. Consider $\mathrm{Hom}_{\mathscr{C}}(1, -)$, and $F : \mathscr{C} \to \mathsf{sSet}_*$, we wish to show that $\mathrm{Hom}_{\mathscr{C}}(1, -)$ is the unit with respect to $\otimes_D$ then

$$\begin{aligned} F \otimes_D \mathrm{Hom}_{\mathscr{C}}(1, -) &\cong \int^{a,b \in \mathscr{C} \times \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a \otimes b, -) \wedge F(a) \wedge \mathrm{Hom}_{\mathscr{C}}(1, b) \\ &\cong \int^{a \in \mathscr{C}} \int^{b \in C} \mathrm{Hom}_{\mathscr{C}}(a \otimes b, -) \wedge F(a) \wedge \mathrm{Hom}_{\mathscr{C}}(1, b) \end{aligned}$$

Where we've used Fubini for coends, and then via the simplicial Yoneda lemma, we have

$$\int^{b \in \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a \otimes b, -) \wedge \mathrm{Hom}_{\mathscr{C}}(1, -) \cong \mathrm{Hom}_{\mathscr{C}}(a, -)$$

applying the simplicial Yoneda lemma again we obtain

$$\int^{a \in \mathscr{C}} \int^{b \in C} \mathrm{Hom}_{\mathscr{C}}(a \otimes b, -) \wedge F(a) \wedge \mathrm{Hom}_{\mathscr{C}}(1, b) \cong \int^{a \in \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a, -) \wedge F(a) \cong F$$

Hence $\mathrm{Hom}_{\mathscr{C}}(1, -)$ is the unit with respect to $\otimes_D$. Let us define sSet_*-natural transformation l, r left and right unitors. We start with $r_F : F \otimes_D \mathrm{Hom}_{\mathscr{C}}(1, -) \to F$. Consider the right unitor for $\mathscr{C}$, $r_{\mathscr{C}}$, this is a sSet_*-natural transformation. Then define the component r_F as the composite

$$\begin{array}{ccccc} (F \otimes_D \mathrm{Hom}_{\mathscr{C}}(1, -))(c) & \xrightarrow{\cong} \int^{a \in \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a \otimes 1, c) \wedge F(a) & \xrightarrow{(r_{\mathscr{C}})^*} & \int^{a \in \mathscr{C}} \mathrm{Hom}_C(a, c) \wedge F(a) \\ & & & \downarrow \cong \\ & & & F(c). \end{array}$$

There is a similiar definition of $l_F : \mathrm{Hom}_{\mathscr{C}}(1, -) \otimes_D F \to F$. Now $\otimes_D$ is symmetric via a routine calculation which ultimately boils down to $\otimes$ and $\wedge$ are symmetric on $\mathscr{C}$ and sSet_* respectively. The braiding $b_{F,G}$ is defined using the same technique as the one for r_F and l_F.

$\otimes_D$ is associative, consider $H : \mathscr{C} \to \mathsf{sSet}_*$ a sSet_*-functor, then using the definition and Fubini for coends, we obtain

$$(F \otimes_D (G \otimes_D H))(x) \cong \int^{a,c,d \in \mathscr{C}} \int^{b \in \mathscr{C}} \mathrm{Hom}_C(a \otimes b, x) \wedge \mathrm{Hom}_{\mathscr{C}}(c \otimes d, b). \wedge F(a) \wedge (G(c) \wedge H(d)).$$

Applying the simplicial Yoneda lemma yet again, we have

$$\int^{b\in\mathscr{C}} \mathrm{Hom}_C(a\otimes b, x)\wedge \mathrm{Hom}_{\mathscr{C}}(c\otimes d, b) \cong \mathrm{Hom}_{\mathscr{C}}(a\otimes c\otimes d, x)$$

hence associativity follows from associativity of $\wedge$ and $\otimes$ on sSet_* and $\mathscr{C}$. From this it is also clear how to define the associator α, namely using the same technique as the one used for l, r, and b, where we use the associator on sSet_* and $\mathscr{C}$. It remains to see that $[-,-]_{\mathsf{Fun}}$ is given by the end construction given in the theorem.

$$\begin{aligned}
&\mathsf{Fun}_{\mathsf{sSet}_*}(F,[G,H]_{\mathsf{Fun}})\\
&\qquad\cong \int_{a\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(F(a), \int_{b,c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_C(a\otimes b, c), \mathrm{Hom}_{\mathsf{sSet}_*}(G(b), H(c)))\\
&\qquad\cong \int_{a\in\mathscr{C}}\int_{b,c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(F(a), \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(a\otimes b, c), \mathrm{Hom}_{\mathsf{sSet}_*}(G(b), H(c)))\\
&\qquad\cong \int_{a\in\mathscr{C}}\int_{b,c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(F(a), \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(a\otimes b, c)\wedge G(b), H(c)))\\
&\qquad\cong \int_{a\in\mathscr{C}}\int_{b,c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(a\otimes b, c)\wedge F(a)\wedge G(b), H(c)))\\
&\qquad\cong \int_{c\in\mathscr{C}}\int_{a,b\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\mathrm{Hom}_{\mathscr{C}}(a\otimes b, c)\wedge F(a)\wedge G(b), H(c)))\\
&\qquad\cong \int_{c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}(\int^{a,b\in\mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(a\otimes b, c)\wedge F(a)\wedge G(b), H(c)))\\
&\qquad\cong \int_{c\in\mathscr{C}} \mathrm{Hom}_{\mathsf{sSet}_*}((F\otimes_D G)(c), H(c))\\
&\qquad\cong \mathsf{Fun}_{\mathsf{sSet}_*}(\mathscr{C}, \mathsf{sSet}_*)(F\otimes_D G, H)
\end{aligned}$$

□

2.3 Smash Product on Sp^Σ

Via 2.2.4 we have gotten a source of symmetric monoidal categories. We will in this subsection realize the category of symmetric spectra Sp^Σ as a category of right modules over a commutative monoid in a functor category (see, 2.3.1, 2.3.3) of the form from 2.2.4. We begin by giving the definition of monoid objects, and right modules.

Definition 2.3.1. [Monoid in $\mathscr{C}$] A *monoid* in a monoidal category $\mathscr{C}$ is an object $M\in\mathscr{C}$ equipped with a multiplication morphism $\mu: M\otimes M\to M$ in $\mathscr{C}$, and a unit morphism $u: 1\to M$ such that the multiplication is associative and the unit has left and right cancellation in the following sense,

$$\begin{array}{ccccc}
M\otimes(M\otimes M) & \xrightarrow{\alpha} & (M\otimes M)\otimes M & \xrightarrow{\mu\otimes\mathrm{id}_M} & M\otimes M\\
\downarrow{\scriptstyle \mathrm{id}_M\otimes\mu} & & & & \downarrow{\scriptstyle\mu}\\
M\otimes M & & \xrightarrow{\qquad\mu\qquad} & & M
\end{array}
\qquad
\begin{array}{ccccc}
1\otimes M & \xrightarrow{u\otimes\mathrm{id}_M} & M\otimes M & \xleftarrow{\mathrm{id}_M\otimes u} & M\otimes 1\\
& \searrow^{l} & \downarrow{\scriptstyle\mu} & \swarrow_{r} & \\
& & M & &
\end{array}$$

Definition 2.3.2. [Morphism of monoids in $\mathscr{C}$] Let (M,μ,u) and (N,μ',u') be monoids in $\mathscr{C}$ (not necessarily symmetric). A *morphism of monoids* is a morphism in $\mathscr{C}$ $f: M\to N$, such that the following diagrams commute

$$\begin{array}{ccc}
M\otimes M & \xrightarrow{f\otimes f} & N\otimes N\\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu'}\\
M & \xrightarrow{f} & S
\end{array}
\qquad
\begin{array}{ccc}
& 1 & \\
\swarrow^{u} & & \searrow^{u'}\\
R & \xrightarrow{\quad f\quad} & S
\end{array}$$

The notion of commutative only works in the setting of a *symmetric* monoidal category $(\mathscr{C},\otimes,1,\alpha,l,r,b)$.

Definition 2.3.3. [Commutative monoid in $\mathscr{C}$] Let $\mathscr{C}$ be a symmetric monoidal category. A commutative monoid in $\mathscr{C}$ is a monoid (M, μ, u) in $\mathscr{C}$ such that the unit and the braiding makes the following diagram commute

$$\begin{array}{ccc} M \otimes M & \xrightarrow{b_{M,M}} & M \otimes M \\ & \searrow^{\mu} & \downarrow{\scriptstyle \mu} \\ & & M \end{array}$$

Definition 2.3.4. [Right M-modules and M-module homomorphisms] Let (M, μ, u) be a monoid in $\mathscr{C}$. A *right module* over M is an object $\mathcal{M}$ of $\mathscr{C}$ equipped with an action of M, $m : \mathcal{M} \otimes M \to \mathcal{M}$ in $\mathscr{C}$ such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{M} \otimes M \otimes M & \xrightarrow{m \otimes \mathrm{id}} & \mathcal{M} \otimes M \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mu} & & \downarrow{\scriptstyle m} \\ \mathcal{M} \otimes M & \xrightarrow{m} & \mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathcal{M} \otimes 1 & \xrightarrow{\mathrm{id} \otimes \mu} & \mathcal{M} \otimes M \\ & \searrow^{r} \quad \swarrow_{m} & \\ & \mathcal{M} & \end{array}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be right M-modules. An M-module homomorphism is a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathscr{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} \otimes M & \xrightarrow{m} & \mathcal{M} \\ \downarrow{\scriptstyle f \otimes \mathrm{id}} & & \downarrow{\scriptstyle f} \\ \mathcal{N} \otimes M & \xrightarrow{m} & \mathcal{N} \end{array}$$

We denote the category of right M-modules in a monoidal category $\mathscr{C}$ as $\mathrm{Mod}_M(\mathscr{C})$.

We shall also need a notion of tensor product for right M-modules.

Definition 2.3.5. [Tensor product of right M-modules] Let $\mathscr{C}$ be a symmetric monoidal category and let M be a commutative monoid in $\mathscr{C}$. Let $\mathcal{M}$ and $\mathcal{N}$ be right M-modules with M-actions $m_{\mathcal{M}}$ and $m_{\mathcal{N}}$. The *tensor product* $\mathcal{M} \otimes_M \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$, is a coequalizer of the following diagram in $\mathscr{C}$,

$$\mathcal{M} \otimes M \otimes N \underset{\mathrm{id} \otimes (m_{\mathcal{N}} \circ b_{M,\mathcal{N}})}{\overset{m_{\mathcal{M}} \otimes \mathrm{id}}{\rightrightarrows}} \mathcal{M} \otimes \mathcal{N}$$

We want to build a category, which is going to replace $\mathscr{C}$ in 2.2.4, which encodes the data of a symmetric spectrum. The remainder of this section is based mainly on [2] with some inspiration from [10].

Definition 2.3.6. [The category $\mathcal{S}$] The category $\mathcal{S}$ has as objects the finite sets $[m] = \{1, ..., m\}$ for all $m \geq 0$ where $[0]$ is empty. The morphisms are

$$\mathrm{Hom}_{\mathcal{S}}([m],[n]) = \begin{cases} (\Sigma_m)_+ & \text{if } m = n \\ \bullet & \text{otherwise.} \end{cases}$$

Proposition 2.3.7. *$\mathcal{S}$ is sSet_*-symmetric monoidal category.*

Proof. The simplicial structure of $\mathrm{Hom}_{\mathcal{S}}([m],[n])$ is given as a constant pointed simplicial set. The monoidal product is the disjoint union of sets, and the unit it the empty set. The braiding $b_{[m],[n]} : [m] \coprod [n] \to [n] \coprod [m]$, is the permutation in Σ_{m+n} which switches the order of the first m elements, and the remaining n elements in $[m] \coprod [n]$. □

Corollary 2.3.8. *, $(Fun_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*), \otimes_D, \mathrm{Hom}_{\mathcal{S}}([0], -))$ is a closed sSet_*-symmetric monoidal category.*

Proof. This follows directly from 2.2.4. □

We shall need the following technical result.

Lemma 2.3.9. *If X,Y and Z are objects in $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$, then there is a natural bijection*

$$\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)(X \otimes_D Y, Z) \cong \prod_{p,q\geq 0} \mathrm{Hom}^{\Sigma_p\times\Sigma_q}_{\mathsf{sSet}_*}(X_p \wedge Y_q, Z_{p+q})$$

Where $\mathrm{Hom}^{\Sigma_p\times\Sigma_q}_{\mathsf{sSet}_*}(-,-)$ denote the set of pointed $\Sigma_p \times \Sigma_q$-equivariant maps.

Lemma 2.3.10. *Let $S : \mathcal{S} \to \mathsf{sSet}_*$ be the sSet_*-functor given by $[n] \mapsto S^n$, and a permutation σ of $[n]$ to the map $S^n \to S^n$ which permutes the smash factors via σ. Then S is a commutative monoid in $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$.*

Proof. The collection of $\Sigma_p\times\Sigma_q$-equivariant isomorphisms $S^p \wedge S^q \to S^{p+q}$ constitutes a morphism $S \otimes_D S \to S$ via 2.3.9, which also provides the multiplication morphism. □

Corollary 2.3.11. *Let S be as in 2.3.10. The category $Mod_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*))$ is a closed symmetric monoidal category with $\otimes_S$ as the closed symmetric monoidal product.*

This corollary will follow from the following proposition, which is proven in detail in [2].

Proposition 2.3.12. *[Symmetric monoidal structures on $Mod_M(\mathscr{C})$] Let $\mathscr{C}$ be a (co)complete symmetric monoidal category and let M be a commutative monoid in $\mathscr{C}$ such that $-\otimes M : \mathscr{C} \to \mathscr{C}$ preserve coequalizers. Then the tensor product $\otimes_M$ is a symmetric monoidal product on $Mod_M(\mathscr{C})$ with M as the unit. If in addition $\mathscr{C}$ is closed, then there exists an internal hom-module denoted $\mathrm{Hom}_M(\mathcal{M}, -) : Mod_M(\mathscr{C}) \to Mod_M(\mathscr{C})$ which is right adjoint to $-\otimes_M \mathcal{M} : Mod_M(\mathscr{C}) \to Mod_M(\mathscr{C})$ for all M-modules $\mathcal{M}$.*

We wish to define an isomorphism of categories from $\mathrm{Mod}_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$ to Sp^Σ, lets define the functors which will constitute an isomorphism. For this note that given a right S-module X, the morphism $X \otimes_D S \to X$ determines a collection of $\Sigma_p \times \Sigma_q$-equivariant maps $m_{p,q} : X_p \wedge S^q \to X_{p+q}$. Hence to there is a underlying object of $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*$ of a symmetric spectrum $\{X_n\}_{n\in\mathbb{N}}$ with $m_{n,1}$ as the structure maps. Given S-modules X and Y, a morphism of S-modules (see 2.3.2) $f : X \to Y$, is a sequence of pointed Σ_n-equivariant maps $f_n : X_n \to Y_n$, such that

$$\begin{array}{ccc} X_p \wedge S^q & \xrightarrow{m_{p,q}} & X_{p+q} \\ \downarrow{\scriptstyle f_p\wedge \mathrm{id}} & & \downarrow{\scriptstyle f_{p+q}} \\ Y_p \wedge S^q & \xrightarrow{m_{p,q}} & Y_{p+q}. \end{array}$$

commute for all $p, q \geq 0$.

Definition 2.3.13. For an S-module X, define

$$\Phi : \mathrm{Mod}_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)) \to \mathsf{Sp}^\Sigma$$

by letting $\Phi(X)$ be the symmetric spectrum which has X as its underlying object of $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*$, as described above. This is indeed a symmetric spectrum because the composite $\sigma^p = m_{p,q}$ is $\Sigma_p \times \Sigma_q$-equivariant. Let Y be a S-module, and let $f : X \to Y$ be a S-module homomorphism, then by the above discussion, each $f_n : X_n \to Y_n$ is a pointed Σ_n-equivariant map compatible with the structure maps σ_n, hence it constitutes a morphism of symmetric spectra $\Phi(X) \to \Phi(Y)$. This defines the functor Φ.

Definition 2.3.14. Let X be a symmetric spectrum. We define a functor

$$\Psi : \mathsf{Sp}^\Sigma \to \mathrm{Mod}_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$$

Given X the collection of $\Sigma_p \times \Sigma_q$-equivariant maps from $X_p \wedge S^n \to X_{p+q}$, gives a pairing $X \otimes_D S \to S$, via 2.3.9. Given Y a symmetric spectrum, a map of symmetric spectra $f : X \to Y$, gives a the left hand diagram, which gives rise the right hand diagram through the pairing

$$\begin{array}{ccc} X_p \wedge S^q & \xrightarrow{m_{p,q}} & X_{p+q} \\ \downarrow{\scriptstyle f_p \wedge \mathrm{id}} & & \downarrow{\scriptstyle f_{p+q}} \\ Y_p \wedge S^q & \xrightarrow{m_{p,q}} & Y_{p+q}. \end{array} \qquad \begin{array}{ccc} X \otimes_D S & \xrightarrow{m} & X \\ \downarrow{\scriptstyle f \otimes \mathrm{id}} & & \downarrow{\scriptstyle f} \\ Y \otimes_D S & \xrightarrow{m} & Y \end{array}$$

The diagram on the right evidently gives rise to a morphism of right S-modules. This defines the functor Ψ.

Theorem 2.3.15. *There is an isomorphism* $Mod_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)) \to \mathsf{Sp}^\Sigma$.

Proof. Φ and Ψ from 2.3.13 and 2.3.14 respectively, are clearly inverse to each other. □

Definition 2.3.16. [Smash product of symmetric spectra] If X and Y are symmetric spectra, then the *smash product of X and Y* is given as

$$X \otimes Y = \Phi(\Psi(X) \otimes_S \Psi(Y)).$$

We finally have the main theorem of the section.

Theorem 2.3.17. *[Symmetric spectra is closed symmetric monoidal]* $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$ *is a closed symmetric monoidal category. The internal hom-spectrum of the symmetric spectra X and Y is* $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y)$.

Proof. Note first that $\Phi(S)$ is the sphere spectrum $\mathbb{S}$. 2.3.8 gives that $(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*), \otimes_D, \mathrm{Hom}_{\mathcal{S}}([0], -))$ is a closed symmetric monoidal category. Therefore by 2.3.15 $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$ is isomorphic to $\mathrm{Mod}_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$, which has the structure of a closed symmetric monoidal category by 2.3.12. □

Remark 2.3.18. Under this isomorphism $[-,-]_{\mathsf{Fun}}$ is send to $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y)$ as defined in 1.2.14. Hence we obtain the $(\mathrm{Hom}_{\mathsf{Sp}^\Sigma}, \otimes)$-adjunction, hence Sp^Σ is closed.

Remark 2.3.19. Note that theorem 2.2.4 ensures that $(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*), \otimes_D, \mathrm{Hom}_{\mathcal{S}}([0], -))$ is enriched in sSet along with it being a closed symmetric monoidal category. Hence we have developed stronger theory than what is needed for introducing the smash product on Sp^Σ.

3 Model Structure on Sp^Σ.

We will assume familiarity with the theory of model structures, and Quillens homotopical algebra, and in particular the Quillen-Kan model structure on sSet_*. It is possible to give many different model structures on Sp^Σ, a plethora of examples is given in [2] and [1], we will turn our attention to the stable model structure, because the associated homotopy category is going to be our model for the stable homotopy category.

We will need many results concerning the weak equivalences of the stable model structure, called the stable equivalence.

3.1 Stable Equivalences

We collect the following obvious, but crucial result about Sp^Σ.

Theorem 3.1.1. Sp^Σ *is bicomplete.*

Proof. sSet_* is bicomplete, hence $\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*)$ is bicomplete, from which we know that $\mathrm{Mod}_S(\mathsf{Fun}_{\mathsf{sSet}_*}(\mathcal{S}, \mathsf{sSet}_*))$ is bicomplete, then theorem 2.3.15 gives the desired. □

The many different model structures on Sp^Σ, often rely on each other. As we shall see, the stable model structure will rely on the projective model structure. We begin by describing the (co)fibrations and weak equivalences of the projective model structrure.

Definition 3.1.2. [Level structure maps] Let $f : X \to Y$ be a map of symmetric spectra.

- The map f is *level equivalence* if each map $f_n : X_n \to Y_n$ is a weak equivalence in the Quillen-Kan model structure on sSet_*. Denote the class of morphisms W. We will say a spectrum is *level contractible* if it is level equivalent to the trivial spectrum.
- The map f is *(trivial) level cofibration* if each map $f_n : X_n \to Y_n$ is a (trivial) cofibration in the Quillen-Kan model structure on sSet_*.
- The map f is *(trivial) level fibration* if each map $f_n : X_n \to Y_n$ is a (trivial) fibration in the Quillen-Kan model structure on sSet_*.

Furthermore $f : X \to Y$ is a

- *injective fibration* if it has the right lifting property with respect to all level trivial cofibrations.
- *projective cofibration* if it has the left lifting property with respect to all level trivial fibrations.

Remark 3.1.3. Note that f_n is a trivial level fibration if it has the right lifting property with respect to the set of maps $(\Lambda^k_r)_+ \to \Delta^r_+$, and a level fibration if it has the right lifting property with respect to the set of maps $\partial\Delta^r_+ \to \Delta^r_+$ for $r \geq 0$.

Remark 3.1.4. The level structure maps does not constitute a model structure, because it doesn't satisfy the lifting axiom. See p. 32 [2].

One can make atleast two model structures using these classes of maps. If one take the level equivalences, level cofibrations, and injective fibrations one obtains the *injective level model structure*. It can be shown it is not monoidal, see [1], hence all our hard work constructing the monoidal structure on Sp^Σ would have been for nothing. If one instead take the level equivalences, the projective cofibrations and the level fibrations gives a model structure on Sp^Σ which is called the *projective model structure*. We will take for granted that this model structure is monoidal and proper, see [1]. We record here the following result for later referencing.

Theorem 3.1.5. Sp^Σ *equipped with the class of level equivalence, the class of projective cofibrations, and the class of level fibrations, has the structure of a model category.*

We will use the projective model structures as an intermediate model structure for constructing the stable model structure. We shall later in the chapter give a symmetric monoidal structure to the homotopy category associated to the stable model structure. Consider the localization at the level equivalences W of the category of symmetric spectra $\mathsf{Sp}^\Sigma[\mathsf{W}^{-1}]$, we will call the resulting category the level homotopy category. We have a localization map $p : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma[\mathsf{W}^{-1}]$. We are now at this point ready to make the definition of our weak equivalences.

Definition 3.1.6. A map of symmetric spectra $f : A \to B$ is a *stable equivalence* if for all Ω-spectra Z the induced map

$$[p(f), Z] : [B, Z] \to [A, Z],$$

is a bijection. Where $[-,-]$ is the internal hom-space in $\mathsf{Sp}^\Sigma[\mathsf{W}^{-1}]$. We say that a symmetric spectrum A is *stably contractible*, if the unique morphism to the trivial spectrum is a stable equivalence.

We shall need the following two theorems, we won't prove either. The following is proposition 4.17 of [1].

Theorem 3.1.7. *The following are equivalent*

(1) $f : A \to B$ *is a stable equivalence.*

(2) For every level fibrant Ω-spectrum X the induced map $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(f), X) : \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(B), X) \to \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(A), X)$ *is a weak equivalence of simplicial sets. Where $\mathcal{C}$ is cofibrant replacement functor for the projective level model structure.*

(3) The mapping cone $C(f)$ is stably contractible.

(4) For every level fibrant Ω-spectrum X the induced map $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(f), X) : \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(B), X) \to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(A), X)$ *is a level equivalence of symmetric spectra.*

Theorem 3.1.8. *We have the following implications*

Homotopy equivalence $\Rightarrow$ *Level equivalence* $\Rightarrow$ $\tilde{\pi}_*$*-isomorphism* $\Rightarrow$ *Stable equivalence.*

The first implication follows from 1.2.17. The second implication is proposition 4.6 of [1], and the third is proposition 4.23 [1]. There are conditions which ensure the opposite implication, these are implications of proposition 4.13 [1]. We omit the proofs.

Lemma 3.1.9. *If $f : X \to Y$ is a map of level fibrant spectra, f is a level equivalence if and only if f is a homotopy equivalence.*

It turns out that the two last implications are biimplications if we restrict ourselves to Ω-spectra.

Theorem 3.1.10. *Let $X, Y \in \mathsf{Sp}^\Sigma$ are Ω-spectra, then $f : X \to Y$ is a stable equivalence if and only if it is a level equivalence.*

We shall need a handful of regularity results for stable equivalence. In summary these result gives that stable equivalences are closed under wedges, filtered colimits, and that they satisfy a sort of 2-out-of-3-property on mapping cones and homotopy fibers.

Lemma 3.1.11. *Let $(f_i : X_i \to Y_i)_{i \in I}$ be stable equivalences, then $\bigvee_{i\in I} f_i : \bigvee_{i \in I} X_i \to \bigvee_{i\in I} Y_i$ is a stable equivalence.*

Proof. We prove this fact for both models of spectra, because we will need the topological version, and its prove relies on the simplicial version. We argue via the definition for symmetric spectra. For every famility $\{X_i\}_{i\in I}$ of symmetric spectra of simplicial sets and some test Ω-spectrum Z, the natural map

$$[\bigvee_{i\in I} X_i, Z] \to \bigvee_{i\in I}[X_i, Z]$$

is bijective by the universal property of the wedge.

Consider a family $\{X_i\}$ of symmetric spectra of topological spaces, then the canonical map

$$\bigvee_{i\in I} \mathrm{Sing}(X_i) \to \mathrm{Sing}(\bigvee_{i\in I} X_i), \tag{3.1}$$

is a $\tilde{\pi}_*$-isomorphism. This is because naive homotopy groups takes wedges to sums, and taking $\mathrm{Sing}(-)$ preserve naive homotopy groups. We won't show either of these facts, both can be found in [1] proposition 2.19. We have now reduced this case to the above case after applying 3.1.8, to see that 3.1 is a stable equivalence. □

Lemma 3.1.12. *Let I be a filtered category and let $A, B : I \to \mathsf{Sp}^\Sigma$ be functors which takes all morphisms in I to monomorphisms of symmetric spectra. If $\tau : A \to B$ is a natural transformation such that $\tau(i) : A(i) \to B(i)$ is a stable equivalence for every object $i \in I$, then the induced morphism*

$$\mathrm{colim}_I\, \tau : \mathrm{colim}_I\, A \to \mathrm{colim}_I\, B$$

is a stable equivalence.

Proof. For every test Ω-spectrum Z the simplicial set $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathrm{colim}_I\, A, Z)$ is isomorphic to $\lim(\mathrm{Map}_{\mathsf{Sp}^\Sigma}(A, Z))$, and similarly for the functor B. Since the image of A and B consists of injective morphisms, all morphisms in the limit systems $\lim(\mathrm{Map}_{\mathsf{Sp}^\Sigma}(A, X))$ and $\lim(\mathrm{Map}_{\mathsf{Sp}^\Sigma}(B, X))$ are Kan fibrations by the above lemma. Recall that filtered limits of weak equivalences along Kan fibrations are again weak equivalences, so the map $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathrm{colim}_I\, B, Z) \to \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathrm{colim}_I\, A, Z)$ is a weak equivalence, which via lemma 4.4(i) of [1] means that $\mathrm{colim}_I\, A \to \mathrm{colim}_I\, B$ is a stable equivalence. □

Lemma 3.1.13. *Consider the following commutative square of symmetric spectra*

$$\begin{array}{ccc} V & \xrightarrow{i} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{j} & Y \end{array}$$

Let $h : C(f) \cup g : C(i) \to C(j)$ be the map induced by f and g on mapping cones. Then if two of the three morphisms f, g and h are stable equivalences, so is the third.

Proof. Consider first the following special case. Let W and Y be trivial spectra. I.e. we show first that given any morphism $i : V \to X$, then if two of the spectra V, X and $C(i)$ are stably contractible, then so is the third. If $C(i)$ is stably contractible, then i is a stable equivalence via 3.1.7, hence V is stably contractible if and only if X is stably contractible. If V and X are stably contractible, then i is a stable equivalence, so $C(i)$ is stably contractible again via 3.1.7.

Let us return to the general case. We utilize the following result concerning mapping cones

$$(h = C(f) \cup g : C(i) \to C(j)) \cong (k = C(i) \cup j : C(f) \to C(g))$$

namely that the map induced by f and g on mapping cones is isomorphic to the map induced by i and j on mapping cones. Hence via 3.1.7 the general case follows from the special case, by applying the special case to the morphism $k : C(f) \to C(g)$. □

Definition 3.1.14. Let $f : X \to Y$ be a morphism of symmetric spectra. Then we define the homotopy fiber $F(f)$ as the spectrum defined as levelwise homotopy fibers $F(f)_n = F(f_n)$ where $f_n : X_n \to Y_n$. Σ_n acts on $F(f)_n$ through the given action on X_n and Y_n.

Lemma 3.1.15. *Consider the following commutative square of symmetric spectra*

$$\begin{array}{ccc} V & \xrightarrow{i} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{j} & Y \end{array}$$

and suppose that all four spectra are levelwise Kan. Let $e : F(i) \to F(j)$ be the map induced by f and g on homotopy fibers. Then if two of the three morphisms e, f and g are stable equivalences, so is the third.

Proof. $\Sigma(F(i))$ is naturally $\tilde{\pi}_*$-isomorphic to the mapping cone $C(i)$ from 1.2.20. Similiarly for j. Moreover, $e : F(i) \to F(j)$ is a stable equivalence if and only its suspension $\Sigma(e) : \Sigma(F(i)) \to \Sigma(F(j))$ is a stable equivalence. So e is a stable equivalence if and only if the morphism $h : C(i) \to C(j)$ is a stable equivalence. Now this follows from 3.1.13. □

3.2 Stable Model Structure

We will employ Bousfield localization to the projective model structure, to obtain the stable model structure. Let us remind ourselves what Bousfield localization is.

Theorem 3.2.1. *Let $\mathscr{C}$ be a proper model category with a functor $Q : \mathscr{C} \to \mathscr{C}$ and a natural transformation $\eta : 1 \to Q$ such that the following axioms hold*

(1) If $f : X \to Y$ is a weak equivalence, then so is $Qf : QX \to QY$.

(2) For each $X \in \mathscr{C}$, the maps $\eta_{QX}, Q_{\eta_X} : QX \to QQX$ are weak equivalences.

(3) Consider a pullback square in $\mathscr{C}$

$$\begin{array}{ccc} V & \xrightarrow{k} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{h} & Y \end{array}$$

If f is a fibration between fibrant objects such that $\eta_X : X \to QX$, $\eta_Y : Y \to QY$, and $Qh : QW \to QY$ are weak equivalences, then $Qk : QV \to QX$ is a weak equivalence.

Then the following classes of maps define a proper model structure on $\mathscr{C}$: a morphism is a Q-cofibration if and only if it is a cofibration, a Q-equivalence if and only if $Qf : QX \to QY$ is a weak equivalence, and Q-fibration if and only if f is a fibration and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & QX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Qf} \\ Y & \xrightarrow{\eta_X} & QY \end{array}$$

is homotopy cartesian.

Let construct the Q-functor, from the theorem. We will need the following lemma in the construction.

Lemma 3.2.2. *Let X be a symmetric spectrum. The adjoint of λ_X defined in 1.2.13, $\tilde{\lambda}_X : X \to \Omega(\operatorname{sh} X)$ is a level equivalence if and only if X is an Ω-spectrum.*

Proof. The n'th component of the morphism $\tilde{\lambda}_X$ is the composite

$$X_n \xrightarrow{\tilde{\sigma}_n} \Omega(X_{n+1}) \xrightarrow{\Omega(\chi_{n,1})} \Omega(X_{n+1}) = (\Omega(\operatorname{sh} X))_n.$$

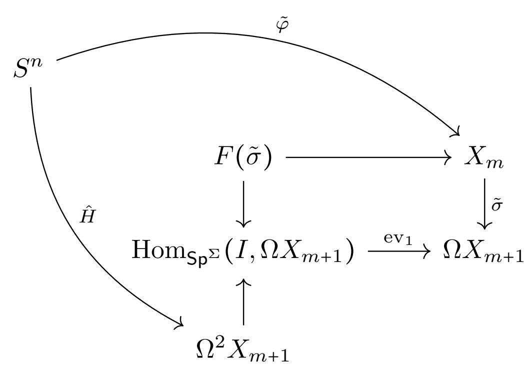
Since $\Omega(\chi_{n,1})$ is an isomorphism, $(\tilde{\lambda_X})_n$ is a weak equivalence of simplicial sets if and only if $\tilde{\sigma}$ is, hence $\tilde{\lambda_X}$ is a level equivalence if and only if X is an Ω-spectrum. □

Lemma 3.2.3. *There is an endofunctor $Q : \mathsf{Sp}^{\Sigma} \to \mathsf{Sp}^{\Sigma}$ with values in Ω-spectra, together with a natural stable equivalence $\eta_A : A \to QA$.*

Proof. **Construction of Q** The construction is most easily done for spectra with topological spaces as a model, and then modified to spectra with simplicial sets as model.
Consider the inclusion of topological spaces $S^{n+1} \to (F_m S^n)_{m+1} = \Sigma^{\bullet}_{m+1} \wedge S^n \wedge S^1$, where $(F_m S^n)$ is the free symmetric spectrum of the n-sphere defined in 1.2.4 and $\Sigma^{\bullet}_{m+1}$ is the symmetric group on $m+1$ elements, with a basepoint attached viewed as a discrete group. This inclusion is adjoint to $\lambda^n_m : F_{m+1} S^{n+1} \to F_m S^n$, via the evaluation, free symmetric spectrum adjunction. Consider the mapping cone $C(\lambda^n_m)$ of the morphism λ^n_m. Then as usual for mapping cones there is a corresponence between morphisms $f : C(\lambda^n_m) \to X$ for X a symmetric spectrum, and pairs (φ, H), consisting of a morphism $\varphi : F_m S^n \to X$ and a null-homotopy $H : [0,1] \wedge F_{m+1} S^{n+1} \to X$ of the composite $\varphi \circ \lambda^n_m$. Applying the Ω-Σ adjunction, this pair correspond bijectively to a based map $\tilde{\varphi} : S^n \to X_m$ and a null-homotopy $\tilde{H} : [0,1] \wedge S^n \to \Omega X_{m+1}$, of the composite $\tilde{\sigma} \circ \tilde{\varphi}$, where $\tilde{\sigma}$ is the mth adjoint structure map of X. Apply the Ω-Σ adjunction to $[0,1]$, to obtain

$$\hat{H} : S^n \to \Omega^2 X_{m+1},$$
$$\tilde{\varphi} : S^n \to X_m.$$

Consider the homotopy fiber of $\tilde{\sigma}$, $F(\tilde{\sigma})$ and its defining diagram, which we have pasted together with the defining diagram for $\Omega^2 X_{m+1}$, and the above maps,



From which we see that the data of $(\hat{H}, \tilde{\varphi})$ corresponds to a map $\tilde{f} : S^n \to F(\tilde{\sigma})$. The same reasoning applies to homotopies between morphisms going out of the mapping cone $C(\lambda^n_m)$. Hence we conclude that

$$[C(\lambda^n_m), X] \to \pi_n(F(\tilde{\sigma})), \quad [f] \mapsto [\tilde{f}] \tag{3.2}$$

is a bijection from the set of homotopy classes of morphisms from $C(\lambda^n_m)$ to X, to the nth homotopy group of $F(\tilde{\sigma})$. Consider a symmetric spectrum X, from which we define an object GX, through the following cone

$$GX = C\Bigg(ev : \bigvee_{m,n \geq 0} \Big(\bigvee_{f : C(\lambda^n_m) \to X} C(\lambda^n_m) \to X \Big)\Bigg)$$

Let $i_X : X \to GX$ denote the inclusion into the cone. Then iterating this process and forming the colimit gives

$$G^\infty X \coloneqq \operatorname{colim}\left(X \xrightarrow{i_X} GX \xrightarrow{i_{GX}} G^2X \xrightarrow{i_{G^2X}} \dots \right).$$

From which we finally obtain $QX = \Omega(\mathrm{sh}(G^\infty X)$. Then Q comes with a natural transformation $X \to QX$ which is defined as the composite

$$X \longrightarrow G^\infty X \xrightarrow{\tilde{\lambda}_{G^\infty X}} \Omega(\mathrm{sh}(G^\infty X)) = QX$$

To obtain a Q-functor for symmetric spectra with simplicial sets as a model, we set $QX = \mathrm{Sing}(Q(|X|))$ for X a simplicial set, where $\mathrm{Sing}(-)$ and $|-|$ are taken levelwise. The natural transformation is then obtained as the composite

$$\eta_X : X \xrightarrow{\text{unit}} \mathrm{Sing}(|X|) \xrightarrow{\mathrm{Sing}(\eta_{|X|})} \mathrm{Sing}(Q(|X|)),$$

where the second η is the natural transformation associated to the topological variant of Q.

QX is an Ω-spectrum Assume first that we have shown that the topological variant of QX is an Ω-spectrum, then we can easily derive the simplicial variant. The singular complex of the Ω-spectrum of spaces $Q|X|$ is an Ω-spectrum of simplicial sets. Recall first that Sing sends weak homotopy equivalences to weak equivalences in the Quillen-Kan model structure, hence the induced structure maps are weak equivalences. Every level equivalence is a stable equivalence from 3.1.8 and $\mathrm{Sing}(-)$ preserve stable equivalence per. definition. Hence we simply need to show that QX is an Ω-spectrum of spaces.

Consider $f : C(\lambda^n_m) \to G^\infty X$ for any $m, n \geq 0$, such a morphism factors through $G^k X$ for some finite number k, i.e as a map $\tilde{f} : C(\lambda^n_m) \to G^k X$. This map is one of the maps which was used to construct $G^{k+1}X$, hence the composite $i_{G^k X} \circ \tilde{f} : C(\lambda^n_m) \to G^{k+1}X$ is null homotopic, which means that so is the original map f. Since every morphism from $C(\lambda^n_m) \to G^\infty X$ is null-homotopic the bijection 3.2 applied to $G^\infty X$, gives that nth homotopy group of the homotopy fiber of the adjoint structure map $\tilde{\sigma} : (G^\infty X)_m \to \Omega(G^\infty X)_{m+1}$ vanishes i.e $\pi_n(F(\tilde{\sigma})) = 0$. In essence this shows that $\tilde{\sigma}$ gives an injection on π_0, and that all homotopy groups π_i for $i \geq 1$ are isomorphic. This implies that $\Omega\tilde{\sigma}_m : \Omega((G^\infty X)_m) \to \Omega^2((G^\infty X)_{m+1})$ is a weak equivalence. Note that $\Omega\tilde{\sigma}_m$ is the $(m-1)$th adjoint structure map of the spectrum $\Omega(\mathrm{sh}(G^\infty X)) = QX$, which shows that QX is an Ω-spectrum.

η_X is a stable equivalence Consider the morphism $\lambda^n_m : F_{m+1}S^{n+1} \to F_m S^n$, it can be shown that this map is a stable equivalence. This implies that its mapping cone is stably contractible via 3.1.7, hence every wedge of $C(\lambda^n_m)$ for varying m and n is stably contractible via 3.1.11. Which implies that we get a weak equivalence $X \to GX$ for every $X \in \mathsf{Sp}^\Sigma$. Each $i_{G^k X}$ for $k \in \mathbb{N}$ has the homotopy lifting property and it is a stable equivalences, so the canonical morphism to the colimit $X \to G^\infty X$ is a stable equivalence by 3.1.12. The morphism $\tilde{\lambda}_{G^\infty X}$ is a level equivalence via 3.2.2, thus a stable equivalence via 3.1.8. So η_X is a stable equivalence. □

Lets employ Bousfield localization with the above Q-functor.

Theorem 3.2.4. Sp^Σ *equipped with the class of stable equivalence, the class of stable cofibrations, and the class of level fibrations, has the structure of a model category.*

Proof. In lemma 3.2.3 we constructed an endofunctor $Q : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ which values was Ω-spectra, and a natural tranformation $\eta_X : X \to QX$. Note that $f : X \to Y$ is a stable equivalence if and only if $Qf : QX \to QY$ is a level equivalence. Hence 3.1.10 gives that f is a stable equivalence if and only if Qf is a stable equivalence. We check the axioms for 3.2.1. Consider the following commutative square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & QX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Qf} \\ Y & \xrightarrow{\eta_Y} & QY \end{array}$$

Via the discussion above, if f is a level equivalence, then Qf is a stable equivalence, and via 3.1.10 a level equivalence, because QX and QY are Ω-spectra by 3.2.3. Hence axiom (1) of 3.2.1 holds.

η_{QX} is a stable equivalence between Ω-spectra from 3.2.3, hence a level equivalence. Q takes all stable equivalences to level equivalence, in particular η_X, hence $Q\eta_X : QX \to QQX$ is a level equivalence. Hence axiom (2) of 3.2.1 holds.

Consider the following pullback square

$$\begin{array}{ccc} V & \xrightarrow{i} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{j} & Y \end{array}$$

of symmetric spectra in which X and Y are Ω-spectra, f is levelwise a Kan fibration and j is a stable equivalence. Now we showed in 3.1.15 that then i is also a stable equivalence. Hence axiom (3) holds. At this point 3.2.1 now provides a model structure with weak equivalences the stable equivalences, the cofibration the projective cofibrations. The fibrations are those morphisms $f : X \to Y$ which are level fibrations and such that the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & QX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Qf} \\ Y & \xrightarrow{\eta_Y} & QY \end{array}$$

is homotopy cartesian in the projective model structure. □

Definition 3.2.5. The *stable homotopy category* $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ of symmetric spectra is the homotopy category of Sp^Σ in the stable model structure. We denote the localization functor $\gamma : \mathsf{Sp}^\Sigma \to \mathcal{H}o(\mathsf{Sp}^\Sigma)$.

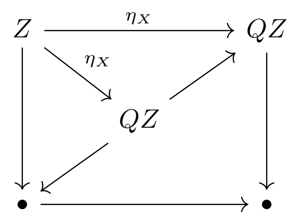
We have finally managed to complete the first mayor goal of this chapter. The rest of this section is devoted to get a handle on the fibrant and cofibrant objects, and to justify the "stable" in the name for the model structure and define the stable homotopy category $\mathcal{H}o(\mathsf{Sp}^\Sigma)$. Once these two things are done, we move on to the next mayor goal of the chapter, namely to describe the symmetric monoidal structure of the stable homotopy category $\mathcal{H}o(\mathsf{Sp}^\Sigma)$.

Corollary 3.2.6. *The fibrant objects of the stable model structure on the category of symmetric spectra are the Ω-spectra which are projective level fibrant.*

Proof. The proof will be a series of biimplication. If Z is fibrant then the unique map $Z \to \bullet$ is a stable fibration. From 3.2.4 we have a description of the stable fibrations, from which we gather that $Z \to \bullet$ is a projective level fibration and

$$\begin{array}{ccc} Z & \xrightarrow{\eta_X} & QZ \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{\mathrm{id}} & \bullet \end{array}$$

is a homotopy fiber square in the projective level model structure. Because of this we get



Now $\eta_X : Z \to QZ$ is a level equivalence, and because QZ is a projective level fibrant Ω-spectrum, so is Z. □

Lemma 3.2.7. *The functor $\Sigma^\infty : \mathsf{sSet}_* \to \mathsf{Sp}^\Sigma$ takes cofibrations of the Quillen-Kan model structure on sSet_* to stable cofibrations.*

Proof. Let $i : X \to Y$ be a cofibration between two pointed simplicial sets and let $f : Z \to W$ be a stable trivial fibration in Sp^Σ. We wish to show that $\Sigma^\infty(i)$ has the lifting property with respect to f. Because of the Σ^∞-Ω^∞-adjunction, supplying lifts in the diagram

$$\begin{array}{ccc} \Sigma^\infty X & \longrightarrow & Z \\ \Big\downarrow{\scriptstyle \Sigma^\infty(i)} & & \Big\downarrow{\scriptstyle f} \\ \Sigma^\infty Y & \longrightarrow & W \end{array}$$

is equivalent to providing a lift in the following diagram

$$\begin{array}{ccc} X & \longrightarrow & Z_0 \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle f_0} \\ Y & \longrightarrow & W_0 \end{array}$$

We assert that the stable trivial fibration f is a level trivial fibration.

f has a factorization in the projective model structure $p \circ i$ such that i is a projective cofibration and p is a level trivial fibration. Since p is a level equivalence it is a stable equivalence via 3.1.8, hence by the 2-out-of-3-property for stable equivalences i is a stable equivalence. Therefore, i is a stable trivial cofibration, and hence it has the left lifting property with respect to f. By the Retract argument see [Quillen Homotopical algebra], g is a retract of p, and so f is a trivial fibration.

Because f is a level trivial fibration, $f_0 : X_0 \to Y_0$ is a trivial Kan fibration. Hence there exists a lift in the second diagram, which gives a lift in the first diagram. □

Remark 3.2.8. In the proof above we showed that the stable trivial fibration f is a level trivial fibration. The other direction also hold: Suppose f is a level trivial fibration. Hence by definition, every stable cofibration has the left lifting property with respect to f, and in particular every stable trivial cofibration has the left lifting property with respect to f. So f is a stable fibration which is a level equivalence and hence a stable equivalence. So f is a stable trivial fibration.

Corollary 3.2.9. *The sphere spectrum $\mathbb{S}$ is stably cofibrant.*

Proof. The unique pointed morphism $\bullet \to S^0$ in sSet_* is a cofibration in the Quillen-Kan model structure, because every object is cofibrant. Hence $\bullet \to \Sigma^\infty(S^0) = \mathbb{S}$ is a stable cofibration. □

The rest of this section will be dedicated to justify the use of stable in the definition of the stable homotopy category of Sp^Σ, namely lets show that suspension descends to an equivalence in stable homotopy category. Our main goal is to show the following theorem, which in fact will show more than we've alluded to.

Theorem 3.2.10. *$\Sigma : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ and $\Omega : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ constitute a Quillen equivalence of the category of symmetric spectra.*

We will need a handful of results, for the proof.

Lemma 3.2.11. *Let $f : A \to B$ be a morphism of symmetric spectra. If f is a stable equivalence, then $\Sigma(f) : \Sigma A \to \Sigma B$ is a stable equivalence.*

Proof. We are going to make extensive use of 3.1.7. Let X be an level fibrant Ω-spectrum. Consider the simplicial set $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(C(f), X)$ where $C(-)$ is the mapping cone. Consider the 0'th level of the morphism of symmetric spectra, $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(f, X) : \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(B, X)_0 \to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)_0$, this is a map of Kan complexes. It can be shown that there is a isomorphism of simplicial sets

$$F(\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(f, X)_0) \cong \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(C(f), X).$$

So condition (2) of 3.1.7 holds, hence we can conclude that $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(C(f), X)$ is contractible, and thus $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(f, X) : \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(B, X) \to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)$ induces a bijection on fundamental groups. Because the simplicial set $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)$ is Kan, we have the following natural bijection $\pi_1(\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)) \cong [S^1, \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)]$ from the fundamental group to the set of homotopy classes of morphisms from the circle. Now because sSet_* is closed we have the following natural bijection $[S^1, \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(A, X)] \cong [S^1 \wedge A, X] \coloneqq [\Sigma A, X]$, which shows that the map $[\Sigma f, X] : [\Sigma B, X] \to [\Sigma A, X]$ is bijective. Hence the suspension of f is a stable equivalence. □

Remark 3.2.12. One can similiarly show that $\Omega : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ preserve stable equivalences for symmetric spectra of spaces. We will omit this.

Lemma 3.2.13. *$\Sigma(\Omega(X)) \to X$ and $X \to \Omega(\Sigma(X))$ are $\tilde{\pi}_*$-isomorphisms.*

Proof. We show that $\tilde{\pi}_n(\Omega(X)) \cong \tilde{\pi}_{n+1}(X)$ and $\tilde{\pi}_n(\Sigma(X)) \cong \tilde{\pi}_{n-1}(X)$. Note that

$$\begin{aligned} \tilde{\pi}_n(\Omega(X)) &= \mathrm{colim}_k([S^{n+k}, \Omega(X_k)]) \\ &\cong \mathrm{colim}_k([S^{n+k+1}, X_k]) \\ &\cong \mathrm{colim}_k([S^{n+1+k}, X_k]) \\ &\cong \tilde{\pi}_{n+1}(X). \end{aligned}$$

The other isomorphisms is only slightly harder to obtain, but is also formal, hence we omit it. □

At this point the proof 3.2.10 is easy.

Proof of 3.2.10. We will complete the proof for symmetric spectra of spaces. The proof for symmetric spectra of simplicial sets is more technical because $\Omega : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ does not preserve stable equivalences in this setting. Now because both Σ and Ω preserve stable equivalence via **??** and 3.2.12, the unit and counit of their adjunction is also the derived unit and counit. From 3.2.13 $\Sigma(\Omega(X)) \to X$ and $X \to \Omega(\Sigma(X))$ are $\tilde{\pi}_*$-isomorphisms, hence they are stable equivalences, therefore they constitute a Quillen equivalence, which in particular show the desired result. □

Corollary 3.2.14. *The right derived of Ω and the left derived of Σ are equivalences of the stable homotopy category $\mathcal{H}o(\mathsf{Sp}^\Sigma)$.*

This corollary shows that Sp^Σ and $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ has some things in common, in the next section we shall see that they have quite a few things in common, and that this in fact is a general phenomenon.

3.3 Monoidal Structure of the Stable Model Structure on Sp^Σ

In this section we will see that the smash product of symmetric spectra defined in the previous chapter is compatible with the stable model structure defined in the previous section. We will see that the stable model structure on symmetric spectra Sp^Σ is monoidal, which in turn will endow the stable homotopy category $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ with the structure of a symmetric monoidal category.

Definition 3.3.1. Let $(\mathscr{C}, \otimes, 1)$ be a symmetric monoidal category. If $f : X \to Y$ and $g : A \to B$ are two morphism in $\mathscr{C}$, then the *pushout product* $f \square g$ of f and g is the morphism

$$f \square g : X \otimes B \coprod_{X \otimes A} Y \otimes A \to Y \otimes B,$$

out of the pushout induced from the following commutative diagram

$$\begin{array}{ccc} X \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & Y \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes g} & & \downarrow{\scriptstyle \mathrm{id}_Y \otimes g} \\ X \otimes B & \xrightarrow{f \otimes \mathrm{id}_B} & Y \otimes B. \end{array}$$

Remark 3.3.2. Recall that it is proved in Quillens homotopical algebra (II.3) that if f and g are monomorphisms then $f \square g$ is to, and it is a weak equivalence if either f or g is a weak equivalence.

Definition 3.3.3. Let $(\mathscr{C}, \otimes, 1)$ be a symmetric monoidal category. A model structure on $\mathscr{C}$ is called *monoidal* if $f : X \to Y$ and $g : X' \to Y'$ are cofibration in $\mathscr{C}$, then the pushout product $f \square g$ of f and g is a cofibration in the model structure on $\mathscr{C}$, and $f \square g$ is a trivial cofibration if f or g is a trivial cofibration in model structure on $\mathscr{C}$.

Our main goal is to prove the following theorem.

Theorem 3.3.4. *The stable model structure on the category of symmetric spectra* $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$ *is monoidal.*

Before we begin the proof of this theorem we will need the following notions.

Definition 3.3.5. Let I be a class of maps a category $\mathscr{C}$. A map is called I-injective if it has the right lifting property with respect to every map in I. A map is an I-cofibration if it has the left lifting property with respect to every I-injective map. The class of I-cofibrations is denoted $\mathrm{cof}(I)$

Lemma 3.3.6. *Let I_∂ be the set of maps $\partial\Delta^m_+ \to \Delta^m_+$ for $m \geq 0$. Let $FI_\partial = \bigcup_{n\geq 0} F_n(I_\partial)$ where F_n is free symmetric spectrum functor defined in 1.2.4. The level trivial fibrations are the FI_∂-injective maps.*

Proof. A map g is a level trivial fibration if and only if $\mathrm{Ev}_n(g) = g_n$ is a trivial Kan fibration for all $n \geq 0$. But $\mathrm{Ev}_n(g)$ is a trivial Kan fibration if and only if it has the right lifting property with respect to the class I_∂. Now apply the (F_n, Ev_n)-adjunction, to obtain that g is a level trivial fibration if and only if g has the right lifting property with respect to the class FI_∂. □

Corollary 3.3.7. *The stable cofibrations are the FI_∂-cofibrations.*

Lemma 3.3.8. *If f is a cofibration in* sSet_* *then $F_n(f)$ is a stable cofibration for $n \geq 0$.*

Proof. Suppose g is a map of symmetric spectra which is a level trivial fibration, and f is a map of simplicial sets which is a cofibration. Then f has the left lifting property with respect to the trivial Kan fibration $\mathrm{Ev}_n(g)$. By adjunction, $F_n(f)$ has the left lifting property with respect to g, then $F_n(f)$ is a stable cofibration. □

Lemma 3.3.9. *Let I,J and K be classes of maps in Sp^Σ. Let $f \square g \in K$ for all $f \in I$ and $g \in J$. Then $f' \square g' \in \mathrm{cof}(K)$ for all $f' \in \mathrm{cof}(I)$ and $g' \in \mathrm{cof}(J)$.*

The real content of theorem 3.3.10 is in fact contained the following theorem.

Theorem 3.3.10. *Let f and g be maps of symmetric spectra.*

(1) If f and g are stable cofibrations then $f \square g$ is a stable cofibration.

(2) If f and g are stable cofibrations, and either f or g is a level equivalence, then $f \square g$ is a level equivalence.

(3) If f and g are stable cofibrations, and either f or g is a stable equivalence, then $f \square g$ is a stable equivalence.

Proof of 3.3.4. This is an immediate consequence of theorem 3.3.10. □

Proof of 3.3.10. **(1)** In lemma 3.3.9 let $I = J = K = FI_\partial$, then $\mathrm{cof}(K)$ is the class of stable cofibration via 3.3.7. Because the free symmetric spectrum functor is a left adjoint it preserve colimits, hence we have a natural isomorphism

$$F_p(f) \square F_q(g) \cong F_{p+q}(f \square g), \tag{3.3}$$

for $f, g \in \mathsf{sSet}_*$. By 3.3.8 we have that $f \square g \in \mathrm{cof}(K)$ for all $f \in I$ and $g \in J$. Now we apply lemma 3.3.9 to obtain the desired result.

(2) As in part 1, set $I = J = K = FI_\partial$, to obtain the isomorphism from 3.3. By 3.3.2 and 3.3.8 and the following isomorphism $F_q(f) \cong \Omega^q(\Sigma^\infty(f))$, which we take for granted, we get that $f \square g$ is a monomorphism and that it induces a level equivalence under $F_*(-)$ if and only if f and g are monomorphisms and either f or g is a weak equivalence, for all $f \in I$ and $g \in J$. Again 3.3.9 gives the desired result.

(3) We begin by noting that a level cofibration $i : X \to Y$ is a stable equivalence if and only if its cofiber $\mathrm{Co}(i) \cong Y/X$ is stably contractible. By part 1 the map $f \square g$ is a stable cofibration. Now since colimits commute the cofiber of $f \square g$ is the smash product $\mathrm{Co}(f) \otimes \mathrm{Co}(g)$ of the cofiber of f and the cofiber of g. Because f is a stable cofibration, then $\mathrm{Co}(f)$ is stably cofibrant. Let E be a level fibrant Ω-spectrum. We will show that the internal hom spectrum $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathcal{C}(\mathrm{Co}(f) \otimes \mathrm{Co}(g)), E)$ is a level contractible spectrum, which amounts to show that $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathrm{Co}(f) \otimes \mathrm{Co}(g), E)$ is level contractible, because they are level equivalent. Given this we may conclude via 3.1.7 that $\mathrm{Co}(f) \otimes \mathrm{Co}(g)$ is stably contractible, which then implies $f \square g$ is a stable equivalence.

Suppose f is a stable equivalence, then it can be shown that $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathrm{Co}(f), E)$ is a level contractible spectrum. Therefore $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathrm{Co}(f) \otimes \mathrm{Co}(g), E) \cong \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathrm{Co}(g), \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(\mathrm{Co}(f), E))$ is level contractible, hence $f \square g$ is a stable equivalence.

□

3.4 Symmetric Monoidal Structure of $\mathcal{H}o(\mathsf{Sp}^\Sigma)$

We will in this section give the stable homotopy category a symmetric monoidal structure. This will largely be a consequence of the fact that the stable model structure was monoidal. We will realize the monoidal product on $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ as a derived version of the usual smash product. As already mentioned we will assume knowledge of many facts about model categories and their properties. The main result of this section, discussed above, will be a corollary of a very general theorem concerning closed symmetric monoidal model categories, and because of this this section will mostly be independent of the previous section.

Let $\mathscr{C}$ and $\mathscr{D}$ be model categories. Recall that the left derived functor $\mathbb{L}F$ of $F : \mathscr{C} \to \mathscr{D}$ is the left Kan extension along the localization functor associated to the homotopy category of $\mathscr{C}$, $\lambda : \mathscr{C} \to \mathcal{H}o(\mathscr{C})$. Explicitly $\mathbb{L}F : \mathcal{H}o(\mathscr{C}) \to \mathscr{D}$ is a functor together with a natural transformation $\epsilon : \mathbb{L}F \circ \lambda \to F$ such that if $G : \mathcal{H}o(\mathscr{C}) \to \mathscr{D}$ is another functor together with a natural transformation $\eta : G \circ \lambda \to F$, then there exists a unique natural transformation $\Theta : G \to \mathbb{L}F$ such that

$$G \circ \lambda \xrightarrow{\Theta_\lambda} \mathbb{L}F \circ \lambda \xrightarrow{\epsilon} F,$$

where Θ_λ is the natural transformation with components $(\Theta_\lambda)_X = \Theta_{\lambda(X)}$.

The following three lemmas will be key ingredients in the proof of the theorem mentioned above. These are standard theorems found in fx. [11] or [12].

Lemma 3.4.1. *Let $\mathscr{C}$ be a model category, and $F : \mathscr{C} \to \mathscr{D}$ a functor. If F takes weak equivalences between cofibrant objects in $\mathscr{C}$ to isomorphisms in $\mathscr{D}$, then the left derived functor $\mathbb{L}F$ exists.*

Proof. Consider the functor $\phi : \mathscr{C} \to \mathscr{D}$ which on $X \in \mathscr{C}$ is defined as $\phi(X) = F(QX)$ where Q is cofibrant replacement in $\mathscr{C}$. On morphisms $f : X \to Y$ it is given as $\phi(f) = F(Qf)$. If f is a weak equivalence, then Qf is a weak equivalence between cofibrant objects hence $\phi(f)$ is an isomorphism in $\mathscr{D}$ per. assumption on F. Now the universal property of the localization, there exists a unique functor $\mathbb{L}F : \mathcal{H}o(\mathscr{C}) \to \mathscr{D}$ for which it holds that $\mathbb{L}F \circ \lambda = \phi$. Let $\epsilon : \mathbb{L}F \circ \lambda \to F$ be a natural transformation which components are $\epsilon_X = F(p_X)$, where p_X is the trivial fibration $QX \to X$. We show that the pair $(\mathbb{L}F, \epsilon)$ is the left derived functor of F.

Let $G : \mathcal{H}o(\mathscr{C}) \to \mathscr{D}$ be a functor with a natural transformation $\eta : G \circ \lambda \to F$. For each $X \in \mathscr{C}$ consider the diagram

$$\begin{array}{ccc} (G \circ \lambda)(QX) & \xrightarrow{\eta_{QX}} & (\mathbb{L}F \circ \lambda)(X) \\ \downarrow{\scriptstyle (G\circ\lambda)(p_X)} & & \downarrow{\scriptstyle \epsilon_X} \\ (G \circ \lambda)(X) & \xrightarrow{\eta_X} & F(X) \end{array}$$

$p_X : QX \to X$ is a trivial fibration, hence in particular a weak equivalence, hence $\lambda(p_X)$ is an isomorphism in $\mathcal{H}o(\mathscr{C})$ which means that $(G \circ \lambda)(p_X)$ is an isomorphism in $\mathscr{D}$. Now defined a natural transformation $\Theta : G \circ \lambda \to \mathbb{L}F \circ \lambda$ by $\Theta_X = \eta_{QX} \circ ((G \circ \lambda)(p_X))^{-1}$. Now note that Θ_X is the unique morphism making the diagram above commute which proves the desired result. □

Lemma 3.4.2. $(\mathscr{C}, \otimes, 1)$ *closed symmetric monoidal category. If X is cofibrant, then $X \otimes - : \mathscr{C} \to \mathscr{C}$ preserve (trivial) cofibrations.*

Proof. Let X be cofibrant in $\mathscr{C}$ and let $f : Y \to Z$ be a cofibration. For every object $C \in \mathscr{C}$ the endofunctor $C \otimes -$ preserve colimits because it is a left adjoint, hence $C \otimes \bullet$ is weakly equivalent to the initial object $\bullet$ in $\mathscr{C}$ for every $C \in \mathscr{C}$. Note that $Y \otimes X$ is a pushout of the diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ Y \otimes X & & \end{array}$$

From this it is clear that pushout product of $f : Y \to Z$ and the cofibration $\bullet \to X$ is $f \otimes \mathrm{id}_X$. Since the model structure on $\mathscr{C}$ is assumed to be symmetric monoidal the pushout product $\mathrm{id}_X \otimes f$ is a cofibration as wanted. If f is also a weak equivalence, then so is $\mathrm{id}_X \otimes f$. □

The following theorem is due to K. Brown.

Lemma 3.4.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. If F takes trivial cofibrations between cofibrant object in $\mathscr{C}$ to weak equivalences in $\mathscr{D}$, then F takes all weak equivalence between cofibrant objects in $\mathscr{C}$ to weak equivalences in $\mathscr{D}$.*

Proof. Let $f : X \to Y$ be a weak equivalence between cofibrant objects in $\mathscr{C}$. We show that $F(f)$ is a weak equivalence in $\mathscr{D}$. Because $\mathscr{C}$ is a model category it satisfies the factorization axiom, thus there exists cofibrant $Z \in \mathscr{C}$, a cofibration $q : X \coprod Y \to Z$, and a trivial fibration $p : Z \to Y$ such that $f \coprod \mathrm{id}_Y = p \circ q$. $\iota_0 : X \to X \coprod Y$ and $\iota_1 : Y \to X \coprod Y$ are cofibration, hence $q \circ \iota_0 : X \to Z$ and $q \circ \iota_1 : Y \to Z$ are cofibrations. We have $p \circ q \circ i_0 = (f \coprod \mathrm{id}_Y) \circ \iota_0 = f$, hence $q \circ \iota_0$ is a weak equivalence by the 2-out-of-3-property. Similiarly $p \circ q \circ \iota_1 = \mathrm{id}_Y$, hence $q \circ \iota_1$ is a weak equivalence again by the 2-out-of-3-property. Per. assumption on F, $F(q \circ \iota_0)$ and $F(q \circ \iota_1)$ are weak equivalences. Note that $F(p) \circ F(q \circ \iota_1) = F(p \circ q \circ \iota_1) = F(\mathrm{id}_Y)$, hence the 2-out-of-3-property gives that $F(p)$ is a weak equivalence. From this we conclude that $F(f) = F(p \circ q \circ \iota_0) = F(p) \circ F(q \circ \iota_0)$ is a weak equivalence as wanted. □

Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ and $\gamma : \mathscr{C} \to \mathscr{E}$ be functors. Recall that the right Kan extension $\mathrm{Ran}_\gamma F : \mathscr{E} \to \mathscr{D}$ of F along γ is right adjoint to the functor $- \circ \gamma : \mathsf{Fun}(\mathscr{E}, \mathscr{D}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{D})$. If $G : \mathscr{C} \to \mathscr{D}$ is another functor and $\eta : F \to G$ is a natural isomorphisms, then $\mathrm{Ran}_\gamma \eta : \mathrm{Ran}_\gamma F \to \mathrm{Ran}_\gamma G$ is also a natural isomorphism.

Theorem 3.4.4. *$\mathscr{C}$ be a closed symmetric monoidal model category and assume the tensor unit is cofibrant. The total left derived functor $\otimes^L$ of $\otimes$ exists and gives $\mathcal{H}o(\mathscr{C})$ the structure of a symmetric monoidal category.*

Proof. Lets first show existence of $\otimes^L$. By 3.4.1 and 3.4.3 it suffices to show that $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ takes two trivial cofibrations between cofibrant objects in $\mathscr{C}$ to a weak equivalence in $\mathscr{C}$. Let $f : A \to A'$ and $g : B \to B'$ be trivial cofibrations between cofibrant objects in $\mathscr{C}$. Lemma 3.4.2 both $A \otimes -$ and $- \otimes B'$ preserve trivial cofibrations, hence $A \otimes g : A \otimes B \to A \otimes B'$ and $f \otimes B' : A \otimes B' \to A' \otimes B'$ are trivial cofibrations. Note that $f \otimes g = (f \otimes B') \circ (A \otimes g)$, hence $f \otimes g$ is a weak equivalence, this shows the existence of $\otimes^L$. It remains to show that $\otimes^L$ is associative, unital, and symmetric up to coherence isomorphism. To do this we will employ the fact about right Kan extensions, hence about left derived functors, mentioned above. Note that the universal property of the localization gives an equivalence $\mathcal{H}o(\mathscr{C} \times \mathscr{C} \times \mathscr{C}) \cong \mathcal{H}o(\mathscr{C}) \times \mathcal{H}o(\mathscr{C}) \times \mathcal{H}o(\mathscr{C})$. The above fact shows that the associator natural isomorphism $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ in $\mathscr{C}$ induces a natural isomorphism $\alpha^L : (- \otimes^L -) \otimes^L - \to - \otimes^L (- \otimes^L -)$ in $\mathcal{H}o(\mathscr{C})$. The unitor and symmetry natural isomorphisms induce natural isomorphisms using the same technique. We omit the straight forward proof that these make the defining diagrams for a symmetric monoidal category commute. □

Corollary 3.4.5. *The total left derived functor $\otimes^L : \mathcal{H}o(\mathsf{Sp}^\Sigma) \times \mathcal{H}o(\mathsf{Sp}^\Sigma) \to \mathcal{H}o(Sp^\Sigma)$ of $\otimes : \mathsf{Sp}^\Sigma \times \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$ exists and gives the stable homotopy category of symmetric spectra the structure of a symmetric monoidal category $(\mathcal{H}o(\mathsf{Sp}^\Sigma), \otimes^L, \gamma(\mathbb{S}))$.*

We end this section with the following theorem, which together with the above corollary shows that $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ is an extremely structured category. The above states that $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ has an internal notion of algebra, and the following shows that $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ has an internal notion of homotopical algebra.

Theorem 3.4.6. *$\mathcal{H}o(\mathsf{Sp}^\Sigma)$ is a triangulated category.*

This is Theorem 2.9 in [1].

4 Algebra Objects in Sp^Σ and $\mathcal{H}o(\mathsf{Sp}^\Sigma)$

One of the reasons for introducing the smash product on Sp^Σ and deriving it to obtain the smash product on $\mathcal{H}o(\mathsf{Sp}^\Sigma)$ is to give a robust generalization of classical algebra to the setting of symmetric spectra. This generalization is often called *higher algebra*. Symmetric spectra is to higher algebra what abelian groups are to classical algebra, since they both assemble to a category having the right formal properties (symmetric monoidal model category and abelian category respectively.). We shall develop other analogies between higher algebra and classical algebra.

4.1 Ring Spectra

In this section we define the notion of a ring spectrum, and then we give some examples. Ring spectra are to higher algebra what rings are to classical algebra.

Definition 4.1.1. A *(symmetric) ring spectrum* R is a monoid in the symmetric monoidal category $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$. A *commutative (symmetric) ring spectrum* is a commutative monoid in $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$. A *ring spectrum momorphism* $f : R \to S$ between ring spectra R and S, is a monoid morphism of monoids in $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$.

Usually we will omit "symmetric" and simply say ring spectrum. The ring spectra and ring spectra morphisms assemble in to a category, namely the category of ring spectra which we denote $\mathcal{R}\mathrm{ing}(\mathsf{Sp}^\Sigma)$. We also have a category of commutative ring spectra denoted $\mathcal{CR}\mathrm{ing}(\mathsf{Sp}^\Sigma)$. We begin by presenting some examples, where we omit the details of check commutativity of the diagrams described in definition 2.3.1 and 2.3.3.

Example 4.1.2. Since $\mathbb{S}$ is the unit for the monoidal product $\otimes$ on Sp^Σ, we have a natural isomorphism $\lambda : \mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$ which provides the multiplication map. The identity on $\mathbb{S}$ is the unit. Now because $\otimes$ is symmetric, these two maps gives the sphere spectrum the structure of a commutative ring spectrum. This fails if we did not consider symmetric spectra, but only spectra. Just like in classical algebra $\mathbb{S}$ is the initial object of the category $\mathcal{CR}\mathrm{ing}(\mathsf{Sp}^\Sigma)$: Let R be a commutative ring spectrum, then the unit map $\iota : \mathbb{S} \to R$ is unique, because if there was another $\iota' : \mathbb{S} \to R$, then $\iota = \iota'$ by unitality.

Example 4.1.3. Let A be a commutative ring, then the Eilenberg-Mac Lane spectrum HA turns out to be a commutative ring spectrum. In each level let $S^n \to HA_n$ be defined by sending $x \in S^n$ to $1 \otimes x$ in $HA_n = \{A \otimes S^n\}$, these constitute a map $\mathbb{S} \to HA$, which is the unit map. We define $\Sigma_p \times \Sigma_q$-equivariant morphisms of simplicial sets for all $m, n \geq 0$, $\mu_{n,m} : HA_n \wedge HA_m \to HA_{n+m}$ defined as

$$\mu_{n,m}\Big(\sum_i r_i x_i \wedge \sum_j r'_j y_j\Big) = \sum_{i,j} (r_i r'_j)(x_i \wedge y_j).$$

These maps give HA the structure of a commutative ring spectrum. In fact HRA constitutes an endofunctor on $\mathcal{CR}\mathrm{ing}(\mathsf{Sp}^\Sigma)$. To see this, let S be another ring, and let $f : R \to S$ be a ring homomorphism. Then we define $Hf : HA \to HS$, in each level by

$$(Hf)_n(\sum_i a_i x_i) = \sum_i f(a_i) x_i.$$

Example 4.1.4. Let X be a symmetric spectrum. We define the endomorphism ring spectrum of X by $\mathrm{End}(X) = \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, X)$. Consider the identity $\mathrm{id} : \mathrm{End}(X) \to \mathrm{End}(X)$, let $e : \mathrm{End}(X) \otimes \mathrm{End}(X) \to X$ denote the adjoint map. We define the multiplication $\mu : \mathrm{End}(X) \otimes \mathrm{End}(X) \to \mathrm{End}(X)$, as the adjoint of the map

$$\mathrm{End}(X) \otimes \mathrm{End}(X) \otimes X \xrightarrow{\mathrm{id} \otimes c} \mathrm{End}(X) \otimes X \xrightarrow{e} X.$$

Let the unit be the adjoint of the left unitor $\mathbb{S} \otimes X \to X$.

Example 4.1.5. Consider M a simplicial monoid, with multiplication $\eta : M \times M \to M$, we define the spherical monoid ring $\mathbb{S}M$ by

$$(\mathbb{S}M)_n = M^+ \wedge S^n.$$

Here M^+ denotes the underlying simplicial set with a disjoint basepoint added. The symmetric group action is given via permuting the sphere coordinates and the trivial action on M^+. The spherical monoid ring is a symmetric ring spectrum. The unit is $\mathbb{S} \to \mathbb{S}M$ is defined in each level as $1 \wedge - : S^n \to M^+ \wedge S^n$ given as $x \mapsto 1 \wedge x$. The multiplication map $\mu : \mathbb{S}M \otimes \mathbb{S}M \to \mathbb{S}M$ is in each level defined as

$$\mu_{n,m} : (M^+ \wedge S^n) \wedge (M^+ \wedge S^m) \cong (M \times M)^+ \wedge (S^n \wedge S^m) \xrightarrow{\eta \wedge f} M^+ \wedge S^{n+m}.$$

Where f is the usual isomorphism.

Example 4.1.6. The above construction works more generally if we start with a commutative ring spectrum R. Again let M be a simplicial monoid. We define the a symmetric ring spectrum called the monoid ring spectrum RM by $(RM)_n = M^+ \wedge R_n$. We define the unit map as the composite

$$\mathbb{S} \to R \cong \{1\}^+ \wedge R \to M^+ \wedge R,$$

where the first map exist because $\mathbb{S}$ is initial in $\mathcal{CR}\mathrm{ing}(\mathsf{Sp}^\Sigma)$ and the last map is induced by the unit of M. The multiplication in each level is given completely analogously to the multiplication on $\mathbb{S}M$.

Definition 4.1.7. Let R be a ring spectrum. A *Left R-module* is a left R-module in $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$. An *R-module homomorphism* $\varphi : X \to Y$ between the R-modules X, Y, is a R-module homomorphism between X and Y in $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$. *Right R-modules* are defined analogously.

The notion of a R-module is not suprising the higher algebra analog of R-modules from classical algebra.

Example 4.1.8. Consider $X, Y \in \mathsf{Sp}^\Sigma$, then $\mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y)$ is a left module over $\mathrm{End}(Y)$. The map $m : \mathrm{End}(Y) \otimes \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y) \to \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y)$ is the adjoint of the following composite

$$\mathrm{End}(Y) \otimes \mathrm{Hom}_{\mathsf{Sp}^\Sigma}(X, Y) \otimes X \xrightarrow{e} \mathrm{End}(Y) \otimes Y \xrightarrow{e} Y$$

Where e was defined in 4.1.4.

4.2 Homotopy Ring Spectra

In the classical theory of algebraic topology there is an homotopical analog of algebra objects, namely that of a H-objects, which are algebra objects in $\mathcal{H}o(\mathsf{Top})$. This idea is easily applicable to our situation. It will turnout with only minimal formal input that there is an abundance of such objects.

Definition 4.2.1. A *homotopy ring spectrum R* is a monoid in the symmetric monoidal category $(\mathcal{H}o(\mathsf{Sp}^\Sigma), \otimes^L, \gamma(\mathbb{S}))$. homotopy commutative ring spectra, homotopy ring spectrum homomorphisms, left (right) homotopy modules, are defined analogously.

We shall see that a ring spectrum gives rise to homotopy ring spectrum, which turns out to be a formal consequence of γ being a lax monoidal functor. A definition of what it means for a functor to be lax monoidal can be seen at [13].

Lemma 4.2.2. *The localization functor* $\gamma : \mathsf{Sp}^{\Sigma} \to \mathcal{H}o(\mathsf{Sp}^{\Sigma})$ *is lax monoidal.*

We omit the proof.

Theorem 4.2.3. *The localization functor* $\gamma : \mathsf{Sp}^{\Sigma} \to \mathcal{H}o(\mathsf{Sp}^{\Sigma})$ *takes symmetric ring spectra to homotopy ring spectra.*

Proof. Consider R a ring spectrum, with $m : R \otimes R \to R$ being the multiplication, and $i : \mathbb{S} \to R$ being the unit, then R becomes a homotopy ring spectrum with respect to the multiplication map

$$\gamma(R) \otimes^L \gamma(R) \xrightarrow{\mu_{R,R}} \gamma(R \otimes R) \xrightarrow{\gamma(m)} \gamma(R),$$

here μ is the natural transformation which exists because γ is lax monoidal. The unit map is $\gamma(i) : \gamma(\mathbb{S}) \to \gamma(R)$. $\square$

This gives us a source of homotopy ring spectra. Further strengthening the analogy between homotopy ring spectra and H-monoids, is the fact that one can't in general rigidify a homotopy spectrum to obtain a ring spectrum, just like one in general can't rigidify a H-monoid to obtain a topological monoid. For the details of this obstruction see remark 4.14 of [1].

Part II

The ∞-Category of Spectra

5 Introduction to ∞-Categories

We saw in the previous part of the text that it was possible to endow the category of symmetric spectra with a smash product such that it got the structure of a closed symmetric monoidal category. Furthermore we gave it a model structure, namely the stable model structure, which we showed gave Sp^Σ the structure of a stable closed symmetric monoidal model category.

In this chapter we will consider spectra from a different view, namely that of $(\infty,1)$-categories. In this viewpoint what we developed in the previous part will be a presentation, or a model for the 1-categorical part of the $(\infty,1)$-category of spectra. Our ultimate goal is to introduce this new viewpoint, and not to give a comprehensible account of $(\infty,1)$-categories or the $(\infty,1)$-category of spectra. Hence we will omit proofs, to a larger degree than what we did in the previous part.

5.1 Definitions and Ideas

The observant reader will at this point have noted the use of ∞-category and $(\infty,1)$-category. ∞-categories are models for $(\infty,1)$-categories, the same way S^1 is a model for the Eilenburg-MacLane space $K(\mathbb{Z},1)$, or symmetric spectra is a model for a category spectra. The Eilenburg-MacLane space is an idea, it is a set of desirable properties for which there are different instantiations. The same holds true for $(\infty,1)$-categories, so lets first describe some of the desirable properties, again we will do this by way of analogy.

In normal categories, e.g. topological spaces Top, we have objects and morphisms connecting them, one might call the objects 0-morphisms and the morphisms 1-morphisms. These are often called 1-categories. So we have 1-morphisms connecting 0-morphisms. In the category of categories Cat, besides categories (0-morphisms) and functors (1-morphisms) we also have natural transformations (2-morphisms) which connect functors. Cat is an example of a 2-category, collectively k-categories for $k > 1$ are called higher categories. The idea of a $(\infty,1)$-category is an expansion of this idea, namely that we should have a higher category $\mathscr{C}$ for which the k-morphisms of $\mathscr{C}$ should be invertible and they should connect $(k-1)$-morphisms for $k > 1$. Following [14] we will in this text limit ourselves to the following model for $(\infty,1)$-categories.

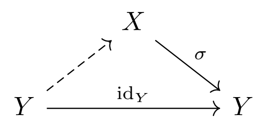
Definition 5.1.1. An *∞-category* is a simplicial set K for which $f_0 : \Lambda^n_i \to K$ admits an extension $f : \Delta^n \to K$.

One might summerize this definition as "∞-categories are simplicial sets which has not-necessarily unique inner horn-fillers". Lets fix some conventions. Let $\mathscr{C}$ be a ∞-category. The objects of $\mathscr{C}$ are the 0-simplices, and the morphisms are the 1-simplices in $\mathscr{C}$. Given a morphism f in $\mathscr{C}$, $d_1(f)$ is the domain of f, and $d_0(f)$ is the codomain; if $d_1(f) = X$ and $d_0(f) = Y$, then we write $f : X \to Y$. The identity id_X is the image of X under the degeneracy map $s_0 : \mathscr{C}_0 \to \mathscr{C}_1$. Given $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, then they assemble to an inner horn $h : \Lambda^2_1 \to \mathscr{C}$, for which $d_2(h) = f$ and $d_0(h) = g$. This admits and extension to a 2-simplex $H : \Delta^2 \to \mathscr{C}$, and we think of $d_1(H)$ as the composition of f and g in $\mathscr{C}$, and we denote it $g \circ f$.

We will now try to motivate this model for $(\infty,1)$-categories, and explain what place in the theory it should hold. In some sense we want to obtain a formalism which captures the classical homotopy theory and the ordinary 1-categorical perspective. These are two useful extremes to consider. Let us try to assimilate the these two extremes to the context of simplicial sets, to better see that ∞-categories lie somewhere between the two. Lets begin with the classical homotopy theory.

Definition 5.1.2. A *Kan complex* is a simplicial set which has horn fillers.

Recall that a *groupoid* is loosely speaking a category for which every morphism is invertible, analogously to the above a $(\infty,1)$-groupoid is a $(\infty,1)$-category for which every k-morphism is invertible for $k > 0$. Kan complexes are good models for $(\infty,1)$-groupoids: Consider a Kan complex K, and two 1-simplices $\sigma : \Delta^1 \to K$ and $\sigma' : \Delta^1 \to K$, for which $d_1(\sigma) = X$, $d_0(\sigma) = Y$, $d_i(\sigma') = Y$ for X and Y 0-simplices. Using the above notation convention, one might visualize this as



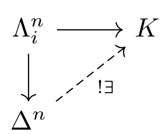
where the Kan condition ensures the existence of a left inverse to σ, analogously one obtains a right inverse. Note that neither are unique! This idea can be utlized in any simplicial level, hence giving inverses to any k-morphism. Therefore one might take Kan complexes as the definition of ∞-groupoids. One obtains the classical homotopy theory through the fundamental ∞-groupoid of a topological space, defined as $\mathrm{Sing}(X)$ (This is originally due to A. Joyal [15], it is proved in Proposition 1.2.5.1 [14]). The take away from this discussion is that it is not reasonable to require the notion of an ∞-category to have outer horn fillers.

Given a category $\mathscr{C}$ one obtains a simplicial set $N(\mathscr{C})$ by applying the nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$. Up to isomorphism one might recover $\mathscr{C}$ from $N(\mathscr{C})$. The following theorem characterizes the simplicial sets which arise as nerves of categories. The theorem is proposition 1.1.2.2 of [14].

Theorem 5.1.3. *Let K be a simplicial set. Then the following are equivalent:*

(1) There exists a category $\mathscr{C}$ and an isomorphism $K \cong N(\mathscr{C})$.

(2) For each $0 < i < n$ and each diagram



is commutative.

Here the uniqueness is a crucial part. The take away from the theorem is that to capture this extreme without killing the other extreme, we can not require our extensions to be unique. The result is that the composite of two morphisms f and g in a ∞-category is not uniquely defined as in ordinary category theory, but it is defined "up to contractible space of choices" which we shall ellaborate in the following section. The above discussion alludes to the following two examples, which also shows that our definition in fact captures the two extremes.

Example 5.1.4. Any Kan complex is an ∞-category. In particular $\mathrm{Sing}(X)$ for $X \in \mathsf{Top}$.

Example 5.1.5. The nerve of any category is an ∞-category. By identifying $\mathscr{C}$ with its nerve $N(\mathscr{C})$ we may view ordinary category theory as a special case of ∞-category theory.

Because ∞-categories are certain simplicial sets it is rather easy to define functors between them.

Definition 5.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ ∞-categories. A ∞-*functor* (or simply *functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a morphism $\mathscr{C} \to \mathscr{D}$ of simplicial sets. If K is a simplicial set, then we let $\mathsf{Fun}(K, \mathscr{C})$ denote the simplicial set $\mathrm{Hom}_{\mathsf{sSet}}(K, \mathscr{C})$, following [14].

If both $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories then $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ turn out to be an ∞-category. We shall prove this in a following section.

Definition 5.1.7. Let $\mathscr{C}$ and $\mathscr{D}$ be two ∞-categories. A *natural equivalence* btween two functors from $\mathscr{C}$ to $\mathscr{D}$ is an equivalence in $\mathsf{Fun}(\mathscr{C},\mathscr{D})$. A functor $F:\mathscr{C}\to\mathscr{D}$ is an *equivalence of ∞-categories* if there exists another functors $G:\mathscr{D}\to\mathscr{C}$ together with natural equivalence between $F\circ G$ and $\mathrm{id}_{\mathscr{D}}$, and $G\circ F$ and $\mathrm{id}_{\mathscr{C}}$.

5.2 Uniqueness of Composition up to Contractible Space of Choices

As mentioned in previous section the composite of two morphisms in an ∞-category is not uniquely defined, but it is rather defined up to a contractible space of choices. The following theorem (Corollary 2.3.2.2 [14]), due to A.Joyal, solidifies this intuition.

Theorem 5.2.1. *A simplicial set $\mathscr{C}$ is an ∞-category if and only if the restriction map*

$$\mathsf{Fun}(\Delta^2,\mathscr{C})\to\mathsf{Fun}(\Lambda^2_1,\mathscr{C})$$

is a trivial Kan fibration.

We think of the space of compositions as the fiber of the map, which is contractible because the map is a trivial Kan fibration. This section is dedicated to proving this theorem.

Definition 5.2.2. A model category $\mathscr{C}$ is *cofibrantly generated* if there are sets of morphisms I,J in $\mathscr{C}$ such that

- $\mathrm{cof}(I)$ is precisely the collection of cofibrations of $\mathscr{C}$;
- $\mathrm{cof}(J)$ is precisely the collection of trivial cofibrations in $\mathscr{C}$, and
- I and J permit the small object argument.

Remark 5.2.3. Recall that sSet_*, Top and Sp^Σ equipped with the Quillen-Kan, classic, and stable model structure respectively are cofibrantly generated. We have implicitly used this fact when we asserted that the stable cofibrations are the FI_∂-cofibrations in 3.3.7, because we used that $\mathrm{cof}(FI_\partial)$ was the morphisms with the right lifting property with respect to the FI_∂-injective morphisms **??**. For now the important part is that sSet_* is cofibrantly generated, with generating collections I and J which respectively consists of maps $\partial\Delta_n\to\Delta_n$ and $\Lambda^n_i\to\Delta^n$.

Definition 5.2.4. Let $\mathscr{C}$ be a cocomplete cofibrantly generated model category. A class of maps I in $\mathscr{C}$ is called *weakly saturated* if it is closed under the operations of forming, pushouts, transfinite composition and retract. The smallest weakly saturated class containing I, is said to be *generated by I*.

We need the following classes of maps, which shall be important in the proofs to come.

Definition 5.2.5. A morphism $f:K\to S$ of simplicial sets is

- a left fibration if f has the right lifting property with respect to all left horn inclusions, i.e. $\Lambda^n_i\subseteq\Delta^n$, $0\le i<n$.
- a right fibration if f has the right lifting property with respect to all right horn inclusions, i.e. $\Lambda^n_i\subseteq\Delta^n$, $0<i\le n$.
- a inner fibration if f has the right lifting property with respect to all inner horn inclusions, i.e. $\Lambda^n_i\subseteq\Delta^n$, $0<i<n$.
- a left anodyne morphism if f has the left lifting property with respect to all left fibrations.

- a right anodyne morphism if f has the left lifting property with respect to all right fibrations.
- a inner anodyne morphism if f has the left lifting property with respect to all inner fibrations.

The following theorem is due to A.Joyal, it is going to be a key part of the proof of theorem 5.2.1.

Lemma 5.2.6. *The following collections all generate the same weakly saturated class of morphisms of simplicial sets*

1. *The collection A of all inner horn inclusions.*
2. *The collection B of all inclusions*

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2. \tag{5.1}$$

3. *The collection C of all inclusions*

$$(S' \times \Lambda^2_1) \coprod_{S \times \Lambda^2_1} (S \times \Delta^2) \subseteq S' \times \Delta^2,$$

where $S \subseteq S' \in \mathsf{sSet}$.

Remark 5.2.7. One might reformulate the above classes in terms of the pushout product in the symmetric monoidal category $(\mathsf{sSet}, \times, \Delta^0)$. A morphism f in C is then $i \,\square\, u$ for i a cofibration and $u : \Lambda^2_1 \to \Delta$. This point of view is not going to enter the picture before later.

Proof. Lets first show that every morphism of C belongs to the weakly saturated class of morphisms generated by B. Consider a pair of simplicial sets $S \subseteq S'$. Consider the pushout diagram definning the n-skeleton of S

$$\begin{array}{ccc} \coprod_{x\in S_n} \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1} S \\ \downarrow & & \downarrow \\ \coprod_{x\in S_n} \Delta^n & \longrightarrow & \operatorname{sk}_n S \end{array}$$

Here the coproduct is only taken over all non-degenerate $x \in S_n$. Which inductively shows that replacing $\operatorname{sk}_n S$ and $\operatorname{sk}_n S'$ with S and S' in (5.1), is contained in the weakly saturated class generated by B, because of the closure under pushout. Now closure under transfinite induction gives that every morphism of C belongs to the weakly saturated class of morphisms generated by B.

Next we show that every morphism of A is a retract of a morphism belonging to C. Explicitly we show that for $0 < i < n$, the inclusion $\Lambda^n_i \subseteq \Delta^n$ is a retract of the inclusion

$$(\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) \subseteq \Delta^n \times \Delta^2.$$

Consider the embedding $\Delta^n \to \Delta^n \times \Delta^2$ given via the map of partially ordered sets $s : [n] \to [n] \times [2]$, and the retraction $\Delta^n \times \Delta^2 \to \Delta^n$ given via the map of partially ordered sets $r : [n] \times [2] \to [n]$, which are defined as

$$s(j) = \begin{cases} (j,0) & j < i \\ (j,1) & j = i \\ (j,2) & j > i \end{cases} \qquad r(j,k) = \begin{cases} j & j < i,\ k = 0 \\ j & j > i,\ k = 2 \\ i & \text{otheriwse} \end{cases}$$

respectively. Now consider the following retract diagram

$$\begin{array}{ccccc}
\Lambda^n_i & \longrightarrow & (\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) & \longrightarrow & \Lambda^n_i \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\quad s \quad} & \Delta^n \times \Delta^2 & \xrightarrow{\quad r \quad} & \Delta^n
\end{array}$$

which shows every morphism of A is a retract of a morphism belonging to C.

Next we show that every morphism of B is inner anodyne, i.e. it lies in the weakly saturated class of morphisms generated by A. Choose $m \geq 0$. For each $0 \leq i \leq j < m$, we let σ_{ij} and τ_{ij} denote the $(m+1)$-simplex and the $(m+2)$-simplex of the m-prism $\Delta^m \times \Delta^2$, which corresponds to the maps $f_{ij} : [m+1] \to [m] \times [2]$ and $g_{ij} : [m+2] \to [m] \times [2]$, defined as

$$f_{ij}(k) = \begin{cases} (k, 0) & 0 \leq k \leq i \\ (k-1, 1) & i+1 \leq k \leq j+1 \\ (k-1, 2) & j+2 \leq k \leq m+1. \end{cases} \qquad g_{ij}(k) = \begin{cases} (k, 0) & 0 \leq k \leq i \\ (k-1, 1) & i+1 \leq k \leq j+1 \\ (k-2, 2) & j+2 \leq k \leq m+2. \end{cases}$$

respectively. Let $X(0) = (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2)$. For $0 \leq j < m$, we define inductively

$$X(j+1) = X(j) \cup \sigma_{0j} \cup \ldots \cup \sigma_{jj}.$$

This gives rise to a chain of inclusions

$$X(j) \subseteq X(j) \cup \sigma_{0j} \subseteq \ldots \subseteq X(j+1),$$

where each level is a pushout of a morphism in A, hence they are inner anodyne, from which we gather that $X(j) \subseteq X(j+1)$ is inner anodyne. Set $Y(0) = X(m)$, so that the inclusion $X(0) \subset Y(0)$ is inner anodyne. We set inductively $Y(j+1) = Y(j) \cup \tau_{0j} \cup \ldots \cup \tau_{jj}$ for $0 \leq j \leq m$. As before we have a chain of inclusions

$$Y(j) \subseteq Y(j) \cup \tau_{0j} \subseteq \ldots \subseteq Y(j+1),$$

again each level is a pushout of a morphism in A, again $Y(j) \subseteq Y(j+1)$ is inner anodyne. Because the composite of inner anodyne maps is inner anodyne, we conclude that $X(0) \subseteq Y(m+2)$ is inner anodyne. We conclude the proof by noting that $Y(m+2) = \Delta^m \times \Delta^2$. □

Given this lemma the proof of theorem 5.2.1 is reduced to a series of biimplications.

Proof of 5.2.1. Assume $\mathscr{C}$ is a an ∞-category, then $f_0 : \Lambda^n_i \to \mathscr{C}$ extends to $f : \Delta^n \to \mathscr{C}$ for $0 < i < n$. This happens if and only if $g : \mathscr{C} \to \bullet$ has the right lifting property with respect to all inner horn inclusions, which by 5.2.6 happens if and only if $g : \mathscr{C} \to \bullet$ has the right lifting property with respect to the class B. Explicitly the dotted arrow exists in the following commutative diagram

$$\begin{array}{ccc}
(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) & \longrightarrow & \mathscr{C} \\
\downarrow{\scriptstyle i} & \overset{t}{\nearrow} & \downarrow \\
\Delta^m \times \Delta^2 & & \bullet
\end{array}$$

Where i is the inclusion. Consider the pushout diagram defining $(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2)$, and augment it with the above diagram,

$$\begin{array}{ccccc}
\partial\Delta^m \times \Lambda^2_1 & \longrightarrow & \partial\Delta^m \times \Delta^2 & & \\
\downarrow & & \downarrow{\scriptstyle i_1} & & \\
\Delta^m \times \Lambda^2_1 & \longrightarrow & (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) & \longrightarrow & \mathscr{C} \\
& & \downarrow{\scriptstyle \subseteq} & \nearrow & \downarrow \\
& & \Delta^m \times \Delta^2 & & \bullet
\end{array}$$

There is a natural restriction $\partial\Delta^m \times \Delta^2 \to \Delta^m \times \Lambda^2_1$, this restriction together with the maps from $\partial\Delta^m \times \Delta^2$ and $\Delta^m \times \Lambda^2_1$ into $\mathscr{C}$ which factor over $\Delta^m \times \Delta^2$, gives an extension in the following diagram

$$\begin{array}{ccc} \partial\Delta^m \times \Delta^2 & \longrightarrow & \mathscr{C} \\ \downarrow & \nearrow & \\ \Delta^m \times \Lambda^2_1 & & \end{array}$$

which via adjunction gives the desired result. □

This theorem gives us that the class of ∞-categories $\mathscr{C}$ is characterized by the requirement that one can compose morphisms in $\mathscr{C}$, and that the composition is well-defined up to a contractible space of choices. We now collect a very fundamental result.

Theorem 5.2.8. *Let K be an arbitrary simplicial set, then for every ∞-category $\mathscr{C}$, the simplicial set $\mathsf{Fun}(K, \mathscr{C})$ is an ∞-category.*

We will obtain it via a result concerning inner anodyne maps, which will be a corollary of lemma 5.2.6. Again this is due to A.Joyal [15].

Corollary 5.2.9. *Let $i : X \to X'$ be an inner anodyne map of simplicial sets and let $j : Y \to Y'$ be a cofibration. Then the induced map*

$$(X \times Y') \coprod_{X \times Y} (X' \times Y) \to X' \times Y'$$

is inner anodyne.

Proof. The induced map is the pushout product $i \square j$. Observe that this pushout product is associative: $(i \square j) \square k \cong i \square (j \square k)$ and commutative: $i \square j \cong j \square i$. Recall from remark 3.3.2 that if i and j are monomorphisms then so is $i \square j$. Lemma 5.2.6 characterizes the inner anodyne maps as the weakly saturated class generated by C. Let $\mathcal{C}$ be the class generated by C. We wish to show that $\mathcal{C} \square j \subseteq \mathcal{C}$, hence it suffices to check $C \square j \subset \mathcal{C}$. Let $i \in C$. Then as remarked just after lemma 5.2.6, $i = i' \square u$ for i' a cofibration, and $u : \Lambda^2_1 \to \Delta^2$. Using the above observations we get

$$i \square j = (i' \square u) \square j \cong i' \square (u \square j) \cong i' \square (j \square u) \cong (i' \square j) \square u$$

which is in C because $i' \square j$ is a monomorphism. □

Proof of 5.2.8. To show that $\mathsf{Fun}(K, \mathscr{C})$ is an ∞-category it suffices to show that it has the extension property with respect to all inner anodyne inclusions $A \subseteq B$. Via adjunction this is equivalent to $\mathscr{C}$ having the right lifting property with respect to $A \times K \subseteq B \times K$, where we've used that id_K is a cofibration. Corollary 5.2.9 gives that $A \times K \subseteq B \times K$ is inner anodyne, and $\mathscr{C}$ is an ∞-category. Hence $\mathsf{Fun}(K, \mathscr{C})$ is an ∞-category. □

5.3 Simplicial Categories and Their Underlying ∞-Categories

Recall that the 2-category of categories Cat can be enriched over simplicial sets, this was shown by D.Quillen in [16].

Definition 5.3.1. A *simplicial category* is a simplicially enriched category. We will denote the category of simplicial categories as Cat_Δ following [14].

It turns out that simplicial categories has underlying ∞-categories which we obtain by a modification of the usual nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$, to a functor $N_\Delta : \mathsf{Cat}_\Delta \to \mathsf{sSet}$ called the simplicial nerve. The usual nerve is build from $[n]$, and the main idea is to replace $[n]$ with a simplicial category $\mathfrak{C}[\Delta^n]$ containing more combinatorial data. We start by introducing $\mathfrak{C}[\Delta^n]$.

Definition 5.3.2. Let J be a finite nonempty linearly ordered set. We define the simplicial category $\mathfrak{C}[\Delta^J]$.

- Its objects are the elements of J.
- For $i, j \in J$, then

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) \begin{cases} N(P_{i,j}) & \text{if } i \leq j, \\ \varnothing & \text{if } j < i. \end{cases}$$

  where is the poset $P_{i,j} = \{I \subset J : i, j \in J \text{ and } k \in I \text{ such that } i \leq k \leq j\}$ and N is the usual nerve.
- If $i_0 \leq i_1 \leq \cdots \leq i_n$ then the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_n) \times \cdots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1}, i_n) \to \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_n)$$

  is induced by the map of partially ordered sets

$$\begin{aligned} P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} &\to P_{i_0,i_n}, \\ (I_1, \ldots, I_n) &\mapsto I_1 \cup \ldots \cup I_n. \end{aligned}$$

Furthermore let $f : J \to J'$ be a monotone map between linearly ordered sets. The simplicial functor $\mathfrak{C}[f] : \mathfrak{C}[\Delta^J] \to \mathfrak{C}[\Delta^{J'}]$ is defined as

- For each $i \in \mathfrak{C}[\Delta^J]$, $\mathfrak{C}[f](i) = f(i) \in \mathfrak{C}[\Delta^{J'}]$.
- If $i \leq j$ in J, then the map $\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) \to \mathrm{Map}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j))$ induced by f is the nerve of the map

$$\begin{aligned} P_{i,k} &\to P_{f(i),f(j)}, \\ I &\mapsto f(I). \end{aligned}$$

Remark 5.3.3. This construction defines a functor from $\mathfrak{C} : \Delta \to \mathsf{Cat}_\Delta$, given by $\Delta^n \mapsto \mathfrak{C}[\Delta^n]$, and because Cat_Δ admits all colimits, the functor $\mathfrak{C}$, extends uniquely to a functor $\mathsf{sSet} \to \mathsf{Cat}_\Delta$. Note that the simplicial structure on sets of morphisms is evident from the application of the usual nerve.

We shall need this lemma later. We omit the elementary proof.

Lemma 5.3.4. *We may regard $\mathfrak{C}[\Lambda^n_i]$ as a simplicial subcategory of $\mathfrak{C}[\Delta^n]$ for $0 < i < n$. For $0 \leq j \leq k \leq n$, the simplicial set $Map_{\mathfrak{C}[\Lambda^n_i]}(j,k)$ coincides with $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j,k)$ unless $j = 0$ and $k = n$.*

Lemma 5.3.5. *There are isomorphisms $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(0,n) = N(P_{0,n}) \cong \prod^{n-1} \Delta^1$ for each $n > 0$.*

Proof. Note that $P_{0,n}$ is the poset category of $[n]$. We indicate the result in the case $n = 3$, which is the first non-trivial example. One might visualize $P_{0,3}$ as

$$\begin{array}{ccc} \{0,3\} & \longrightarrow & \{0,1,3\} \\ \downarrow & \searrow & \downarrow \\ \{0,2,3\} & \longrightarrow & \{0,1,2,3\} \end{array}$$

Here all the arrows are inclusions. With two 2-simplexes coming from the two chains of inclusions in the poset

$$\begin{aligned} \{0,3\} \subset \{0,1,3\} \subset \{0,1,2,3\}, \\ \{0,3\} \subset \{0,2,3\} \subset \{0,1,2,3\} \end{aligned}$$

There are no non-degenerate k-simplicies for $k > 2$. From which we see $\mathrm{Map}_{\mathfrak{C}[\Delta^3]}(0,3) = N(P_{0,3}) \cong \Delta^1 \times \Delta^1$. $\square$

Now we are ready for the simplicial nerve construction.

Definition 5.3.6. Let $\mathscr{C} \in \mathsf{Cat}_\Delta$. We define the simplicial nerve $N_\Delta(\mathscr{C})$ as the simplicial set described by the formula

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, N_\Delta(\mathscr{C})) = \mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathscr{C}). \tag{5.2}$$

N defines a functor

$$N_\Delta : \mathsf{Cat}_\Delta \to \mathsf{sSet}.$$

It turns out that $N_\Delta(\mathscr{C})$ of a simplicial category is not in general an ∞-category, but if one further assumes that $\mathscr{C}$ is enriched in the full subcategory Kan of sSet it holds, or more precisely formulated:

Theorem 5.3.7. *Let $\mathscr{C}$ be a simplicial category for which the simplicial set $\mathrm{Map}_{\mathscr{C}}(X,Y)$ is a Kan complex for every pair of objects $X, Y \in \mathscr{C}$. Then the simplicial nerve $N_\Delta(\mathscr{C})$ is an ∞-category.*

Proof. We must show $N_\Delta(\mathscr{C})$ has the extension property with respect to all inner horn inclusions. Per. construction N_Δ is adjoint to $\mathfrak{C}$, hence it is equivalent to show that $\mathscr{C}$ has the extension property with respect to the simplicial functor $\mathfrak{C}[\Lambda^n_i] \to \mathfrak{C}[\Delta^n]$ for $0 < i < n$. Hence the following two extension problems are equivalent

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{F} & N(\mathscr{C}) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \qquad \begin{array}{ccc} \mathrm{Map}_{\mathfrak{C}[\Lambda^n_i]}(j,k) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(F(j),F(k)) \\ \downarrow & \nearrow & \\ \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j,k) & & \end{array}$$

For $0 \le j \le k \le n$. Lemma 5.3.4 and the fact that $\mathrm{Map}_{\mathscr{C}}(F(j),F(k))$ is a Kan complex gives us that it suffices to find a solution to the lifting problem on the right in the case $j = 0$, and $k = n$, i.e. to check if $\mathrm{Map}_{\mathfrak{C}[\Lambda^n_i]}(0,n) \to \mathbb{N}_\Delta(P_{0,n})$ is anodyne. Via lemma 5.3.5, this is equivalent to showing that the map from the inclusion of the $(n-1)$-cube without the interior and a $(n-2)$-cube into the $(n-1)$-cube is anodyne. More precisely that

$$(\prod^{n-1} \Delta^1) \smallsetminus ((\partial(\prod^{n-1}(\Delta^1)))^c \cup \prod^{n-2} \Delta^1) \to \prod^{n-1} \Delta^1$$

is anodyne. It can shown that this is in fact true. □

Example 5.3.8. Let Kan be the full subcategory of sSet spanned by the Kan complexes. Recall that if $X, Y \in \mathsf{Kan}$, then $\mathrm{Map}_{\mathsf{sSet}}(X,Y) \in \mathsf{Kan}$, hence proposition 5.3.7 implies that $N_\Delta(\mathsf{Kan})$ is an ∞-category. We define the *∞-category of spaces* to be $N_\Delta(\mathsf{Kan})$ and we denote it by S.

Example 5.3.9. Recall that Sp^Σ was enriched in sSet, hence it is a simplicial category. Furthermore Sp^Σ satisfies that if $i : A \to B$ is a stable cofibration and $p : X \to Y$ is a stable fibration, then the induced map of simplicial sets

$$\mathrm{Map}_{\mathsf{Sp}^\Sigma}(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}_{\mathsf{Sp}^\Sigma}(A,X) \times_{\mathrm{Map}_{\mathsf{Sp}^\Sigma}(A,Y)} \mathrm{Map}_{\mathsf{Sp}^\Sigma}(B,Y)$$

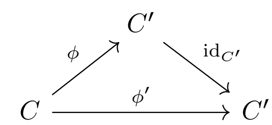
is a Kan fibration and a trivial Kan fibration if either i or p is a stable equivalence. Consider X a stably cofibrant spectrum, and Y a stably fibrant object. Then we have stable cofibration $i : \bullet \to X$ and stable fibration $p : Y \to \bullet$, then this condition ensures that $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(X,Y) \to \bullet$ is a Kan fibration, and hence $\mathrm{Map}_{\mathsf{Sp}^\Sigma}(X,Y)$ is a Kan complex.

Let Sp^Σ_{fc} denote the full subcategory of Sp^Σ generated by the stably fibrant-cofibrant symmetric spectra. We've just shown that Sp^Σ_{fc} is a Kan-enriched simplicial category, hence theorem 5.3.7 applies, and we obtain a ∞-category $N_\Delta(\mathsf{Sp}^\Sigma_{fc})$, we define this as *the underlying ∞-category of* Sp^Σ and denote it $\mathsf{Sp}^\Sigma_\infty$.

5.4 The Homotopy Category of an ∞-Category

In this section we give a description of the homotopy category of a ∞-category. There are two equivalent descriptions, we will present the more explicit of the two, and refer the reader to [14] sections 1.1.4 and parts of 1.2.3 for the other. We will start by defining the notion of homotopic maps.

Definition 5.4.1. Let $\mathscr{C}$ be an ∞-category. Let $\phi : C \to C'$ and $\phi' : C \to C'$ be a pair of morphisms in $\mathscr{C}$. We say that ϕ and ϕ' are *homotopic* if there is a two simplices $\sigma : \Delta^2 \to \mathscr{C}$:

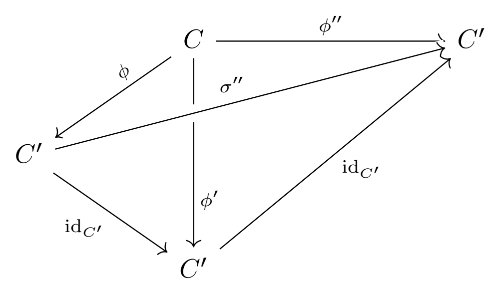


We call σ a *homotopy* between ϕ and ϕ'.

This homotopy relation is in fact a equivalence relation.

Proposition 5.4.2. *Let $\mathscr{C}$ be an ∞-category and let $\mathscr{C}$ and $\mathscr{C}'$ be objects in $\mathscr{C}$. Then the homotopy relation is an equivalence relation.*

Proof. Let $\phi : \Delta^1 \to \mathscr{C}$ be an edge. Then $s_1(\phi)$ is a homotopy from ϕ to itself. So the homotopy relation is reflexive. Consider ϕ, ϕ' and $\phi'' : C \to C'$ morphisms in $\mathscr{C}$. Let σ be a homotopy between ϕ and ϕ' and let σ' be a homotopy between ϕ and ϕ''. Let σ'' be the identity from the codomain of ϕ and ϕ''. We may visualize this as



Note that this is the horn $\Lambda^3_1 \to \mathscr{C}$, which has an extension $\tau : \Delta^3 \to \mathscr{C}$. Now $d_1(\tau)$ is a homotopy form ϕ' to ϕ'', which shows that homotopy is transitive. Now set $\phi = \phi''$, to deduce that the relation is symmetric. □

Definition 5.4.3. Let $\mathscr{C}$ be an ∞-category. We define the homotopy category of $\mathscr{C}$ as $\mathcal{H}o(\mathscr{C})$. The objects of $\mathcal{H}o(\mathscr{C})$ are the objects of $\mathscr{C}$. Given X and Y in $\mathcal{H}o(\mathscr{C})$ we define $\mathrm{Hom}_{\mathcal{H}o(\mathscr{C})}(X, Y)$ as the set of homotopy classes of morphisms $\phi : X \to Y$. Let Z be an object in $\mathcal{H}o(\mathscr{C})$. ϕ and ψ determines a map $\Lambda^2_1 \to \mathscr{C}$, which extends to $\sigma : \Delta^2 \to \mathscr{C}$, because $\mathscr{C}$ was an ∞-category. We define $[\psi] \circ [\phi] = [d_1(\sigma)]$.

Lemma 5.4.4. *Let $\mathscr{C}$ be an ∞-category. The composition law on $\mathcal{H}o(\mathscr{C})$ is well-defined.*

Proof. We must show that the composition does not depend on the choice of either σ, ϕ or ψ up to homotopy. We begin with the former. Suppose we are given $\sigma, \sigma' : \Delta^2 \to \mathscr{C}$ such that $d_0(\sigma) = d_0(\sigma') = \psi$ and $d_2(\sigma) = d_2(\sigma') = \phi$. Consider the degenerate 2-simplex $s_1(\psi)$. σ, σ' and $s_1(\psi)$ determines a map $\Lambda^3_1 \to \mathscr{C}$ which extends to a 3-simplex $\tau : \Delta^3 \to \mathscr{C}$. Now $d_1(\tau)$ is a homotopy between $d_1(\sigma)$ and $d_1(\sigma')$.

We show that $[\psi] \circ [\phi]$ depends on ψ and ϕ only up to homotopy. It can be shown that that this statement is symmetric with respect to ψ and ϕ. Hence it suffices to show that the composition does not change when we change ϕ to a homotopic morphism ϕ'. Let $\sigma : \Delta^2 \to \mathscr{C}$

be such that $d_0(\sigma) = \psi$ and $d_2(\sigma) = \phi$, and let σ' be a homotopy between ϕ and ϕ'. Consider the degenerate 2-simplex $s_0(\psi)$. σ, σ' and $s_0(\psi)$ determines a horn $\Lambda^3_1 \to \mathscr{C}$ which extends 3-simplex $\tau : \Delta^3 \to \mathscr{C}$. Set $\sigma'' = d_1(\tau)$. Then $[\psi] \circ [\phi'] = [d_1(\sigma')]$, but $d_1(\sigma) = d_1(\sigma')$ so therefore $[\psi] \circ [\phi] = [\psi] \circ [\phi']$. □

We now show that the homotopy category of $\mathscr{C}$, $\mathcal{H}o(\mathscr{C})$ in fact is a category.

Proposition 5.4.5. *If $\mathscr{C}$ is an ∞-category, then $\mathcal{H}o(\mathscr{C})$ is a category.*

Proof. Let $C \in \mathscr{C}$. We need to show that $[\mathrm{id}_C]$ is the identity with respect to the composition law in $\mathcal{H}o(\mathscr{C})$, and that the composition is associative. We omit the latter. For every morphism $\phi : C' \to C$ in $\mathscr{C}$, the degenerate 2-simplex $s_1(\phi)$ satisfies the equation $[\mathrm{id}_C] \circ [\phi] = [\phi]$. This shows that $[\mathrm{id}_C]$ is a left identity. Dually it is also a right identity. □

Definition 5.4.6. A morphism $f : X \to Y$ in $\mathscr{C}$ is an *equivalence* if $[f] : X \to Y$ is an isomorphism in $\mathcal{H}o(\mathscr{C})$.

5.5 Limits and Colimits

We shall need the notion of limits and colimits in ∞-categories, as we shall realize the ∞-category of spectra as a certain limit in the ∞-category of ∞-categories. The definition of (co)limits of ∞-categories, is in many ways analogous to (co)limits in the 1-categorical setting. We begin by constructing the analog of cones.

Definition 5.5.1. Consider $X, Y \in \mathsf{sSet}$, then we defined the join $X \star Y$ of X and Y as the simplicial set given in each level as

$$(X \star Y)_n = X_n \cup Y_n \cup \bigcup_{i+j=n-1} X_i \times Y_j.$$

We denote the inclusion of X and Y into $X \star Y$ as $i_X : X \to X \star Y$ and $i_Y : Y \to X \star Y$ respectively. Note that $\star$ determines a functor $\mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$.

Lemma 5.5.2. *If X and Y are ∞-categories, then $X \star Y$ is an ∞-category.*

Proof. Consider the map $p : \Lambda^n_i \to X \star Y$ for $0 < i < n$. If p factors as either $i_X \circ p$ or $i_Y \circ p$, we immediately obtain an extension of p to Δ^n using the assumption that X and Y are ∞-categories. Suppose otherwise, i.e. p imbeds $\{0, ..., j\}$ into X and $\{j+1, ..., n\}$ into Y. Hence we may restrict p as

$$p_X : \Delta^j \to X$$
$$p_Y : \Delta^{n-(j+1)} \to Y$$

These two restrictions determines a map $p_X \star p_Y : \Delta^j \star \Delta^{n-(j+1)} \to X \star Y$, and because $\Delta^j \star \Delta^{n-(j+1)} \cong \Delta^{n-(j+1)+j+1} = \Delta^n$, this determines the desired extension. □

Definition 5.5.3. Let X be a simplicial set. The *left cone* $X^{\triangleleft}$ is defined as the join $\Delta^0 \star X$. Dually the *right cone* $X^{\triangleright}$ is the join $X \star \Delta^0$. The distinguished vertex in Δ^0 is in both cases refered to as the *cone point*.

The following proposition will allow us to determine the overcategory of a ∞-category via a universal property. We shall need the following notation: for simplicial sets X, Y, Z and a morphism $p : X \to Y$, we denote the set of morphisms $f : Z \star X \to Y$ such that $f|_X = p$ by $\mathrm{Hom}_p(Z \star X, Y)$.

Proposition 5.5.4. *Let X and Y be simplicial sets, and let $p : X \to Y$ be a map. There exists a simplicial set $Y_{/p}$ with the following universal property*

$$\mathrm{Hom}_{\mathsf{sSet}}(Z, Y_{/p}) = \mathrm{Hom}_p(Z \star X, Y).$$

Proof. Recall that $\mathrm{Hom}_{\mathsf{sSet}}(Z, Y_{/p})$ is cocontinuous in Z. The same can be shown to hold for $\mathrm{Hom}_p(Z \star X, Y)$. Hence it will suffice to show the property for $Z = \Delta^n$. Lets define $(Y_{/p})_n$ as $\mathrm{Hom}_p(\Delta^n \star X, Y)$, hence by the Yoneda embedding the universal property holds for $Z = \Delta^n$. □

If one replaces $Z \star X$ with $X \star Z$ we obtain analogously another simplicial set, which we denote $Y_{p/}$.

Definition 5.5.5. Consider $p : X \to Y$, for Y an ∞-category, we refer to $Y_{/p}$ as an *overcategory* of Y, and to $Y_{p/}$ as an *undercategory* of Y.

Definition 5.5.6. Let $\mathscr{C}$ be an ∞-category. An object $X \in \mathscr{C}$ is called *terminal* if the projection $\mathscr{C}_{/X} := \mathscr{C}/p \to \mathscr{C}$, where $p : X \to \bullet$, is a trivial Kan fibration. Dually an object $Y \in \mathscr{C}$ is called *initial* if the projection $\mathscr{C}_{Y/} \to \mathscr{C}$ is a trivial Kan fibration.

For simplicial sets this is the notion of *strongly (co)final*, which for general simplicial sets is stronger than the usual notion of terminal/initial, but for ∞-categories they coincide, this is Corollary 1.2.12.5 [14], which justifies our choice of definition.

Definition 5.5.7. Let $\mathscr{C}$ be an ∞-category and let $p : K \to \mathscr{C}$ be a map of simplicial sets. A *colimit* for p is an initial object of $\mathscr{C}_{p/}$, and a *limit* for p is a terminal object of $\mathscr{C}_{p/}$. An ∞-category $\mathscr{C}$ is called *(co)complete* if it admits all (co)limits. $\mathscr{C}$ is called *finitely (co)complete* if (co)limits of $p : K \to \mathscr{C}$ exists for all K with finitely many non-degenerate simplicies.

6 The ∞-Category of ∞-Categories

To make sense of spectra in the setting of ∞-categories, we will need to understand (co)limits taken in the ∞-category of ∞-categories. This chapter is dedicated to defining this ∞-category and to show that it is bicomplete. Again we follow [14]. We shall realize the ∞-category of ∞-categories as the simplicial nerve of a simplicial category $\mathsf{Cat}_\infty^\Delta$.

Definition 6.0.1. The simplicial category $\mathsf{Cat}_\infty^\Delta$ has as its objects the ∞-categories. Given two ∞-categories $\mathscr{C}$ and $\mathscr{D}$, we define $\mathrm{Map}_{\mathsf{Cat}_\infty^\Delta}(\mathscr{C}, \mathscr{D})$ to be the largest Kan complex contained in the ∞-category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$.

We define Cat_∞ as $N_\Delta(\mathsf{Cat}_\infty^\Delta)$, and refer to Cat_∞ as the *∞-category of ∞-categories.*

Proposition 6.0.2. Cat_∞ *is an ∞-category.*

Proof. This is an immediate consequence of 5.3.7, because the mapping spaces in $\mathsf{Cat}_\infty^\Delta$ are Kan complexes. □

Note that by construction Cat_∞ has 2-morphisms which are given as homotopies between functors. This makes Cat_∞ a model for the $(\infty, 2)$-category of $(\infty, 1)$-categories. The ∞-category inside of it should have a presentation given by a simplicial model category. We shall find another fitting model in the section "Marked simplicial sets". That we may use this other model, will be one of the crucial points in the proof of bicompleteness.

The proof of bicompleteness of Cat_∞ will rely on a rather large and involved chunk of theory regarding Cat_∞ and its simplicial category presentation. To this end we shall need a few model structures. All of them are essentielly consequences of the same theorem, which we show in the next section.

6.1 Main Model Theorem

This section is devoted to prove the following theorem, which will produce the four model structure which we will need in this chapter. The theorem is theorem A.2.6.13 in [14]. We will prove this theorem in detail, and refrain from giving the details of how to obtain the model structures from it. In the interest of not over encumbering ourselves with definitions, we shall refer the reader to appendix A.2 of [14] for undefined notions in this section.

Theorem 6.1.1. *Let $\mathscr{C}$ be a presentable category. Suppose W is a class of morphisms, which we call weak equivalences. Let C_0 be a set of morphisms of $\mathscr{C}$, which we will call generating cofibrations. Suppose further*

1. *The class W is perfect.*
2. *For any diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X'' & \longrightarrow & Y'', \end{array}$$

where both squares are pushout, $f \in C_0$, and $g \in W$, then $g' \in W$.

3. *If a morphism g of $\mathscr{C}$ has the right lifting property with respect to every morphism in C_0, then $g \in W$.*

Then there exists a left proper combinatorial model structure on $\mathscr{C}$ which may be described as follows:

1. *If a morphism f belongs to the weakly saturated class of morphisms generated by C_0, then $f \in Cof$.*

2. *The weak equivalences are W.*

3. *If a morphism f has the right lifting property with respect to every map in $Cof \cap W$ then $f \in Fib$.*

The following proof was worked out with a fellow student (Magnus Kristensen), and is heavily inspired by the proof given in [14].

Proof. Consider the collection of all morphisms f in $\mathscr{C}$ such that for two pushout squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X'' & \longrightarrow & Y'', \end{array}$$

where $g \in W$, then the map $g' \in W$, and denote this class by P. Note that $C_0 \subset P$ by assumption. The goal is to show that the weak saturation of C_0 has this property, and to show that, we show that P is weakly saturated.

That P is closed under pushouts is straight forward; assume f' arises as the pushout of f, and consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X''' & \longrightarrow & Y''' \end{array}$$

Now each square is a pushout square by assumption, hence the two upper squares constitute a single pushout square, with f on top, hence because $f \in P$, and $g \in W$, so is g'.

Next we show closure under transfinite composition. Let α be a ordinal, and let following be a α-indexed diagram in $\mathscr{C}$, where $X_\alpha = \operatorname{colim}_{\beta<\alpha} X_\beta$,

$$C \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_\alpha, \qquad \tilde{f} : C \to X_\alpha$$

Where all morphisms are assumed to be in P, and $\tilde{f}$ is the transfinite composition induced from the directed system $\{X_\beta\}_{\beta<\alpha}$. Define Y as the pushout of the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & X_\alpha \\ \downarrow & & \\ X & & \end{array}$$

and note that Y is isomorphic to the colimit of the following diagram

$$\operatorname{colim}\Big(C \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots\Big) \longrightarrow X$$

It is clear that X is terminal within this diagram, hence the colimit of this diagram is isomorphic to X via the canonical map, so $Y \xrightarrow{\sim} X$. Thus we obtain the following diagram of pushout squares,

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & X_\alpha \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \longrightarrow & W, \end{array}$$

and we want to assert $g' \in W$. Now because $X \to Y$ is an isomorphism, the pushout of this map, the map $Z \to W$, is also an isomorphism. Hence both maps are in W, which by the two-out-of-three property implies that g' is in W.

To see that P is closed under retracts, let $f' : X' \to Y'$ be a retract of $f : X \to Y \in P$ so that

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X' \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f'} \\ Y' & \longrightarrow & Y & \longrightarrow & Y', \end{array}$$

commutes and the row compositions are the respective identities, and let the following be a pushout:

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

We claim that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a pushout, where the map $X \to Y$ is f, and the maps $X \to Z$ and $Y \to W$ are obtained from the retract assumption. To show this, we just check the relevant universal property. Given maps $Y \to Q$ and $Z \to Q$ such that the precomposition with the respective maps from X are equal, we can consider the diagram

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Y' \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & Z & \longrightarrow & Q & & \end{array}$$

Where the top row is a factorization of f'. Since the outer square is a pushout, we get a unique map $Q \to W$ compatible with the outer with the diagram, as we wanted. Since we have shown that P is weakly saturated, the weak equivalences are stable under pushouts by cofibrations.

We have yet to show that $\mathscr{C}$ is in fact a model category. This fact will be mediated by the lemma A.2.6.8 [14], which says that it is enough to check that

1. The collection C is a weakly saturated class of morphisms, and there exists a subset $C_0 \subseteq C$ which generates C.

2. The intersection $C \cap W$ is a weakly saturated class of morphisms.

3. W is an accessible subcategory of the morphisms of $\mathscr{C}$.

4. W has the 2-out-of-3-property.

5. If f has the right lifting property with respect to each element of C, then $f \in W$.

(1) is satisfied because C is generated by C_0 as a weakly saturated class, per. construction. (3) is true because W is a perfect class. (4) is per assumptions on W, and (5) is per. assumption (3). It remains to check that (2) is satisfied, i.e., that $C \cap W$ is a weakly saturated class.

$C \cap W$ is closed under retracts because both W and C are closed under retracts. W is closed under retracts because W is a perfect class. C is closed under retracts because it is weakly saturated per. construction.

Next $C \cap W$ is closed under transfinite composition because C is weakly saturated, and W is closed under transfinite composition. The last fact follows from W being closed under composition and filtered colimits in the following way: Consider an ordinal α, and let the following diagram be a directed system in under category $\mathscr{C}_{c/}$,

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow & X_\alpha \\ \uparrow & \nearrow & & & & \nearrow & \\ c & & & & & & \end{array}$$

Where each map is in W. Now let $X_{<\alpha}$ be the colimit of the directed system on $\{c \to X_i\}_{i<\alpha}$. We wish to show that the induced map $c \to X_{<\alpha}$ is in W. Now note that the diagram can be rewritten

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow & X_\alpha \\ \uparrow & & \uparrow & & & & \uparrow \\ c & = & c & = & \dots & = & c \end{array}$$

Where each map is in W. Now note that this can be considered as a α-indexed directed system in $\mathscr{C}^{[1]}$, where the colimit is the induced map $c \to X_{<\alpha}$, hence the induced map is W, because it was a obtained through a filtered colimit in W. It only remains to show that $C \cap W$ is closed under pushouts. Consider a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y & \longrightarrow & Y'' \end{array}$$

in which $f \in C \cap W$, we show that $f'' \in C \cap W$. Because C is weakly saturated, it is enough to show $f'' \in W$. Apply the small object argurment to the map $X \to X''$, to factor it as

$$X \xrightarrow{g} X' \xrightarrow{h} X''$$

Where g is a cofibration and h has the right lifting property with respect to C_0. Now consider the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & X' & \xrightarrow{h} & X'' \\ \downarrow f & & \downarrow f' & & \downarrow f'' \\ Y & \longrightarrow & Y' & \xrightarrow{h'} & Y'' \end{array}$$

Where Y' is the pushout of f and g. Now note that Y'' recieves a map from Y which arise from the original pushout diagram which when composed with f, is equal to $f'' \circ h \circ g$, hence the h' exists by the universal property of the pushout. Now the two squares constitute a single pushout square, and so does the left square, hence the right square is also a pushout square. Since W is stable under the formation of pushouts by cofibrations, we have that $f' \in W$, $h \in W$ because of (3), and $h' \in W$ because it is a pushout of h by the cofibration f'. Hence by the two-out-of-three $h \circ f'' \in W$, and a second application of the two-out-of-three property shows that $f'' \in W$. Which ends the proof. □

In the following sections we will introduce four different model structure, all of which are constructed via this theorem.

6.2 Model Structures

In this section we will introduce three different model structures, on three different categories in the following chapters. We shall endow the ∞-category $\mathsf{Fun}(K, \mathscr{C})$ with a model structure. We will endow sSet with another model structure, for which the fibrant objects are the ∞-categories.Lastly we give a model structure on the category of simplicial categories. Because they arise through 6.1.1 they will all be left proper combinatorial.

Projective Model Structure

At this point we have not given sufficient criteria to ensure that $\mathsf{Fun}(K, \mathscr{C})$ is bicomplete. Analogous to ordinary category theory, the following theorem holds, which is Corollary 5.1.2.3 [14].

Proposition 6.2.1. *Let $S \in \mathsf{sSet}$ and let $\mathscr{C}$ be an ∞-category which admits all (co)limits. Let $\mathscr{D}$ be an ∞-category, then the ∞-category $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ has all (co)limits.*

Definition 6.2.2. Let $\mathscr{D}$ be a category and let $\mathscr{C}$ be a model category. We will say that a natural transformation $\alpha : F \to G$ in $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ is

- an projective fibration if the induced map $F(C) \to G(C)$ is a fibration in $\mathscr{C}$ for each $C \in \mathscr{D}$.
- A weak equivalence if the induced map $F(C) \to G(C)$ is a weak equivalence in $\mathscr{C}$ for each $C \in \mathscr{D}$.
- an projective cofibration if it has the left lifting property with respect to every morphism β in $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ which is simultaneously a weak equivalence and a projective fibration.

The following theorem is ultimately a consequence of 6.1.1.

Theorem 6.2.3. *Let $\mathscr{C}$ be a combinatorial model category let $\mathscr{D}$ be a small category. Then there exists a combinatorial model structure on $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$, called the projective model structure determined by the projective cofibrations, weak equivalence, and projective fibrations.*

Joyal Model Structure

The Joyal model structure was first defined by Joyal, and proven to be left proper combinatorial, using only combinatorial methods, but one can also show the following theorem using 6.1.1. It is theorem 2.2.5.1 [14].

Theorem 6.2.4. *There exists a left proper combinatorial model structure on the category of simplicial sets, called the Joyal model structure, with the following properties*

- *A map $p : S \to S'$ is a cofibration if and only if it is a monomorphism.*
- *A map $p : S \to S'$ is a categorical equivalence if and only if the induced simplicial functor $\mathfrak{C}[S] \to \mathfrak{C}[S']$ is an equivalence of simplicial categories.*

Moreover, the adjoint functors $(\mathfrak{C}, N_\Delta)$ determine a Quillen equivalence between sSet *with the Joyal model structure, and* Cat_Δ*.*

One of the most important properties for the Joyal model structure is the following, which is propositio 2.4.6.1 [14].

Proposition 6.2.5. *Let $\mathscr{C}$ be a simplicial set. Then $\mathscr{C}$ is Joyal fibrant if and only if $\mathscr{C}$ is an ∞-category.*

The Bergner Model Structure

We shall also need a model structure on the category of simplicial categories. We begin by describing a model structure on $\mathscr{C}$-enriched categories.

Definition 6.2.6. Let $\mathscr{C}$ be an monoidal model category. A functor $F : \mathscr{D} \to \mathscr{E}$ in $\mathsf{Cat}_{\mathscr{C}}$ is a weak equivalence if the induced functor between homotopy categories $\mathcal{H}o(\mathscr{D}) \to \mathcal{H}o(\mathscr{C})$ is an equivalence of $\mathcal{H}o(\mathscr{C})$-enriched categories. I.e.

1. For every pair $X, Y \in \mathscr{D}$, the induced map
$$\mathrm{Hom}_{\mathscr{D}}(X,Y) \to \mathrm{Hom}_{\mathscr{E}}(F(X), F(Y))$$
is a weak equivalence in $\mathscr{C}$.
2. Every object $Y \in \mathscr{E}$ is equivalent to $F(X)$ in the homotopy category $\mathcal{H}o(\mathscr{E})$ for some $X \in \mathscr{D}$.

We now introduce a bit of notation for $\mathscr{C}$-enriched categories. Let $A \in \mathscr{C}$, then we let $[1]_A$ denote the $\mathscr{C}$-enriched category having two objects X and Y, with mapping object

$$\mathrm{Hom}_{[1]_A}(Z,W) = \begin{cases} \mathrm{id}_S & \text{if } Z = W = X, \\ \mathrm{id}_S & \text{if } Z = W = Y, \\ A & \text{if } Z = X,\ W = Y, \\ \varnothing & \text{if } Z = Y,\ W = X. \end{cases}$$

Where $\varnothing$ is the initial object of $\mathscr{C}$, and $\mathrm{id}_{\mathscr{C}}$ is the unit object with respect to the monoidal structure on $\mathscr{C}$. Let $[0]_{\mathscr{C}}$ denote the $\mathscr{C}$-enriched category having only a single object X and mapping object $\mathrm{id}_{\mathscr{C}}$. Let C_0 denotes the collection of all morphisms of $\mathscr{C}$ of the following type

1. The inclusion $\varnothing \to [0]_{\mathscr{C}}$.
2. The induced maps $[1]_S \to [1]_{S'}$, where $S \to S'$ range over a set of generators for the weakly saturated class of cofibrations in $\mathscr{C}$.

We wish to apply the following theorem to assert the existence of a model structure on Cat_Δ compatible with the sSet-enrichment, i.e. to assert that Cat_Δ is a sSet-enriched model category.

Theorem 6.2.7. *Let $\mathscr{C}$ be a combinatorial monoidal model category. Assume that every object of $\mathscr{C}$ is cofibrant and that the collection of weak equivalences in $\mathscr{C}$ is stable under filtered colimits. Then there exists a left proper combinatorial model structure on* $\mathsf{Cat}_{\mathscr{C}}$ *(The category of categories enriched over $\mathscr{C}$) characterized by the following conditions:*

- *The class of cofibrations in* $\mathsf{Cat}_{\mathscr{C}}$ *is the smallest weakly saturated class of morphisms containing the set og morphisms* C_0*, where* C_0 *is defined as above.*
- *The weak equivalences in* $\mathsf{Cat}_{\mathscr{C}}$ *are defined as in 6.2.6.*

Again the theorem is a consequence of 6.1.1.

Definition 6.2.8. The above model structure when $\mathscr{C} = \mathsf{sSet}$, will be called the *Bergner model structure* for simplicially enriched categories.

6.3 Marked Simplicial sets

We wish to find another simplicial category which presents Cat_∞ under the coherent nerve, is bicomplete, and which cofibrant-fibrant objects are the ∞-categories. This category will be the category of marked simplicial sets.

Definition 6.3.1. A *marked simplicial set* is a pair $(X, \mathscr{E})$ where X is a simplicial set, and $\mathscr{E}$ is a set of edges of X which contains every degenerate edge. We will say an edge of X is *marked* if it belongs to $\mathscr{E}$. A morphism $f : (X, \mathscr{E}) \to (X', \mathscr{E}')$ of marked simplicial sets is a map $f : X \to X'$ having the property that $f(\mathscr{E}) \subseteq \mathscr{E}'$. The category of marked simplicial sets will be denoted by sSet^+.

Given $S \in \mathsf{sSet}$, we shall denote $S^\#$ by the marked simplicial set (S, S_1), i.e where every edge is marked, and we shall denote $S^\flat$ by the marked simplicial set $(S, s_0(S_0))$, i.e. where it is only the degenerate edges which are marked.

We will show that sSet^+ is sSet-enriched, with the Quillen-Kan model structure. We begin by defining the mapping objects. The following is as consequence of sSet being cartesian-closed.

Lemma 6.3.2. *The category* sSet^+ *is cartesian-closed.*

Definition 6.3.3. Let $X, Y \in \mathsf{sSet}^+$. Consider the internal mapping object Y^X, which exists by the above lemma. We let $\mathrm{Map}^\flat(X, Y)$ denote the underlying simplicial set of Y^X, and $\mathrm{Map}^\#(X, Y) \subseteq \mathrm{Map}^\flat(X, Y)$ the simplicial subset consisting of all simplices $\sigma \in \mathrm{Map}^\flat(X, Y)$ such that every edge of σ is a marked edge of Y^X.

At this point it is not clear wether we wish to take $\mathrm{Map}^\flat(X, Y)$ or $\mathrm{Map}^\#(X, Y)$ as the mapping objects, this will become apparent soon. One may endow the category of marked simplicial sets with a model structure. We begin by describing the weak equivalences, for this we need the following auxillary type of morphism.

Definition 6.3.4. A morphism $p : X \to Y$ in sSet^+ is a *Cartesian fibration* if it is an inner fibration, and for every $f : x \to y$ of Y, and every $\overline{y} \in X$ such that $p(\overline{y}) = y$, there is a Cartesian morphism $\overline{f} : \overline{x} \to \overline{y}$ in X such that $p(\overline{f}) = f$.

This lifting property seems rather ex nihilo, but Cartesian fibrations play a vital role in the $(\infty, 1)$-categorical analog of the Grothendieck construction which is an equivalence of ∞-categories called straightening and unstraightning, see [14], just like Cartesian morphisms play an integral role in the ordinary Grothendieck construction, see [17] or [18].

Definition 6.3.5. Let $p : X \to S$ be a Cartesian fibration of simplicial sets. We let $X^\sqcup$ denote the marked simplicial set $(X, \mathscr{E})$, where $\mathscr{E}$ is the set of Cartesian morphisms in X.

Note that we here choose to diverge from [14] in notation, as it to the untrained eye it can be hard to distinquish the notation in [14] for the maximally marked simplicial set, which we denote $X^\#$ and the notation for marked simplicial set associated to the domain of a Cartesian fibration, which we denote $X^\sqcup$. We choose $\sqcup$ to remind ourselves that the marked edges are the Cartesian edges, which intuitively lets us fill in the top part of square-diagrams. The following lemma is rather important, and is in many ways the first essential property of Cartesian fibrations.

Lemma 6.3.6. *Let $\mathscr{C}$ be a simplicial set. If $\mathscr{C} \to \Delta^0$ is a Cartesian fibration, then $\mathscr{C}$ is an ∞-category and $\mathscr{C} \simeq \mathscr{C}^{\sqcup}$.*

Proof. $\mathscr{C} \to \Delta^0$ is an inner fibration, hence its fibers are ∞-categories, i.e. $\mathscr{C}$ is an ∞-category. Per. definition the Cartesian edges, become the equivalences of $\mathscr{C}$. □

The following lemma alludes to the fact that sSet^+ should be another presentation of $\mathsf{Cat}_\infty^\Delta$, and if one compares the definition of $\mathsf{Cat}_\infty^\Delta$ it also tells us which mapping space to take.

Lemma 6.3.7. *If $X \in \mathsf{sSet}^+$ and $p : Y \to \Delta^0$ is a Cartesian fibration, then $\mathrm{Map}^\flat(X, Y^\sqcup)$ is an ∞-category and $\mathrm{Map}^\#(X, Y^\sqcup)$ is the largest Kan complex contained in $\mathrm{Map}^\flat(X, Y^\sqcup)$.*

Definition 6.3.8. Let $p : X \to Y$ be a morphism in sSet^+. $p : X \to Y$ is a *Cartesian equivalence* if for every Cartesian fibration $Z \to \Delta^0$, the induced map

$$\mathrm{Map}^\#(Y, Z^\sqcup) \to \mathrm{Map}^\#(X, Z^\sqcup),$$

is a homotopy equivalence of Kan complexes. Equivalently if the induced map

$$\mathrm{Map}^\flat(Y, Z^\sqcup) \to \mathrm{Map}^\flat(X, Z^\sqcup)$$

is an equivalence of ∞-categories.

That these two are equivalent requires an argument, which is proposition 3.1.3.3 in [14]. The following is again a consequence of 6.1.1, it is theorem 3.1.3.7 of [14].

Theorem 6.3.9. *There exists a left proper combinatorial model structure on sSet^+ which may be described as follows.*

- *The cofibrations are those morphisms $p : X \to Y$ in sSet^+ which are cofibrations, when regarded as morphisms of simplicial sets.*
- *The weak equivalences in sSet^+ are the Cartesian equivalences.*
- *The fibrations are those maps which have the right lifting property with respect to every map which is a cofibration, and a Cartesian equivalence.*

The following is corollary 3.1.4.4 [14] with $S = \Delta^0$, which is a consequence of theorem 6.3.9.

Corollary 6.3.10. *Setting the mapping objects in sSet^+ to be $\mathrm{Map}^\#(X, Y)$, then sSet^+ is sSet-enriched with the Quillen-Kan model structure.*

Instead if one picks the $\mathrm{Map}^\flat(X, Y)$ as the mapping objects, one obtains an enrichment in sSet with the Joyal model structure. The following proposition characterizes the fibrant objects of sSet^+.

Proposition 6.3.11. *An object $X \in \mathsf{sSet}^+$ is fibrant if and only if $X \simeq Y^\sqcup$, where $Y \to \Delta^0$ is a Cartesian fibration.*

Using 6.3.6 we see that the fibrant objects of sSet^+ are precisely the ∞-categories in which the marked edges are the equivalences, and because all objects in sSet^+ are cofibrant we obtain the following identification.

Theorem 6.3.12. *Consider sSet^+ enriched in sSet with the Quillen-Kan model structure. Then we obtain the following simplicial equivalence $\mathsf{Cat}_\infty^\Delta \simeq \mathsf{sSet}^+_{fc}$.*

6.4 Cat_∞ is Bicomplete

In this section we show that Cat_∞ is bicomplete. This is a very deep result which relies on all of the above theory. Beyond the above we will still need a few auxillary results, which we state now.

Definition 6.4.1. Let $p : S \to T$ be a map of simplicial sets. We shall say that p is *cofinal* if, for any right fibration $X \to T$, the induced map of simplicial sets

$$\mathrm{Map}_T(T, X) \to \mathrm{Map}_T(S, X)$$

is a homotopy equivalence. Here $\mathrm{Map}_T(T, X)$ is defined as the fiber of the map $X^S \to T^S$ which is induced from the object of T^S corresponding to p.

Remark 6.4.2. A property of cofinal morphisms is that composing with them is (co)continuous, this is an easy corollary of 4.1.1.8 [14].

This fact, together with the following theorem, will allow us to restrict ourselves to diagrams of shape $N(\mathscr{J})$ where N is usual nerve and $\mathscr{J}$ is a category, instead of diagrams with shape of a general simplicial set. It is theorem 4.2.3.14 [14].

Proposition 6.4.3. *For every simplicial set K, there exists a category $\mathscr{J}$ and a cofinal map $f : N(\mathscr{J}) \to K$.*

This theorem will let us pass between the nerve of a mapping object, and a mapping object evaluated at a nerve, which will prove extremely useful. In particular it will allow us to compare diagrams in the ∞-categorical and simplicial setting.

Proposition 6.4.4. *Let S be a simplicial set, $\mathscr{D}$ a excellent model category, and $u : \mathfrak{C}[S] \to \mathscr{D}$ an equivalence. Suppose that $\mathscr{C}$ is a combinatorial simplicial model category. Then the induced map*

$$N_\Delta(\mathsf{Fun}(\mathscr{D}, \mathscr{C}_{fc})) \to \mathsf{Fun}(S, N_\Delta(\mathscr{C}_{fc})),$$

is a categorical equivalence of simplicial sets.

Here we have equipped $\mathsf{Fun}(\mathscr{D}, \mathscr{C}_{fc})$ with the projective model structure. This is a specialization of proposition 4.2.4.4 [14], where we have used that a combinatorial simplicial model category is a $\mathscr{D}$-chunk of itself if the model structure on $\mathscr{D}$ is excellent, this is example A.3.4.4 of [14]. All cases which are interesting to us, are covered by this version, because the Quillen-Kan model structure on sSet is excellent.

The following theorem gives a criterion which lets us pass between homotopy colimits in Bergner fibrant simplicial categories, and colimits in $N_\Delta(\mathscr{C})$. It is theorem 4.2.4.1 [14].

Proposition 6.4.5. *Let $\mathscr{C}$ and $\mathscr{J}$ be Bergner fibrant simplicial categories and $F : \mathscr{J} \to \mathscr{C}$ a simplicial functor. Suppose we are given an object $C \in \mathscr{C}$ and a compatible family of maps $\{\eta_I : F(I) \to C\}_{I \in \mathscr{J}}$. The following are equivalent.*

- *The maps η_I witnesses C as a homotopy colimit of the diagram F.*
- *Let $f : N_\Delta(\mathscr{J}) \to N_\Delta(\mathscr{C})$ be the simplicial nerve of F, and $\overline{f} : N_\Delta(\mathscr{J})^\triangleright \to N_\Delta(\mathscr{C})$ the extension of f determined by $\{\eta_I\}$. Then $\overline{f}$ is a colimit diagram in $N_\Delta(\mathscr{C})$.*

The following theorem will be the centerpiece in the proof of bicompleteness.

Corollary 6.4.6. *Let $\mathscr{C}$ be a combinatorial simplicial model category. The underlying ∞-category $N_\Delta(\mathscr{C}_{fc})$ admits all limits and colimits.*

Proof. The argument for limits is dual to that for colimits, so we give the argument for colimits. Let $p : K \to N_\Delta(\mathscr{C}_{fc})$ be a diagram in $N_\Delta(\mathscr{C}_{fc})$. By proposition 6.4.3, there exists a category $\mathscr{J}$ and a cofinal map $q : N_\Delta(\mathscr{J}) \to K$. q is cofinal, which by remark 6.4.2 implies p has a colimit in K if and only if $p \circ q$ has a colimit in $N(\mathscr{J})$. Hence it suffices to show that we have colimits for diagrams of the form $N(\mathscr{J}) \to N_\Delta(\mathscr{C}_{fc})$.
By 6.4.4 we may suppose p is the nerve of a projectively fibrant diagram $p' : \mathscr{J} \to \mathscr{C}_{fc}$, where we have equipped $\mathsf{Fun}(\mathscr{J}, \mathscr{C}_{fc})$ with the projective model structure. Here we use that p' in particular is a diagram in $\mathscr{C}$. Now let $\overline{p}' : \mathscr{J} \star \{x\} \to \mathsf{Fun}(\mathscr{J}, \mathscr{C})$ be a colimit of p', such that $\overline{p}'$ is a homotopy colimit of $\mathscr{C}$. Now choose a trivial projective fibration $\overline{p}'' \to \overline{p}'$ in $\mathsf{Fun}(\mathscr{J}, \mathscr{C})$, where $\overline{p}''$ is projectively cofibrant. Applying 6.4.5 we see that $N_\Delta(\overline{p}'')$ determines a colimit diagram $\overline{f} : N(\mathscr{J})^{\triangleright} \to N_\Delta(\mathscr{C}_{fc})$. We now observe that $f = \overline{f}|_{N(\mathscr{J})}$ is equivalent to p, so that p also admits a colimit in $N_\Delta(\mathscr{C}_{fc})$. □

Theorem 6.4.7. Cat_∞ *is bicomplete.*

Proof. We have the following equivalences

$$\mathsf{Cat}_\infty = N_\Delta(\mathsf{Cat}_\infty^\Delta) \simeq N_\Delta(\mathsf{sSet}^+_{fc}).$$

By corollary 6.4.4 $N_\Delta(\mathsf{sSet}^+_{fc})$ is bicomplete, hence Cat_∞ is bicomplete. □

6.5 Adjoint functors

In this section we shall describe the notion of adjoint functors of ∞-categories, and describe the adjoint functor theorem. The adjoint functor theorem is going to be the theorem which we invoke most often throughout the rest of this text.

Definition 6.5.1. We will say that a map $p : X \to S$ of simplicial sets is a coCartesian fibration if the opposite map $p^{op} : X^{op} \to S^{op}$ is a Cartesian fibration.

Remark 6.5.2. coCartesian fibrations are just as important to the theory as coCartesian fibration, and give rise to model structure, the *coCartesian model structure.*

Definition 6.5.3. Let $p : \mathcal{M} \to \Delta^1$ be a Cartesian fibration and suppose we are given equivalences of ∞-categories $h_0 : \mathscr{C} \to p^{-1}(\{0\})$ and $h_1 : \mathscr{D} \to p^{-1}(\{1\})$. We will say that a functor $g : \mathscr{D} \to \mathscr{C}$ is *associated* to $p : \mathcal{M} \to \Delta^1$ if there is a commutative diagram

$$\begin{array}{ccc} \mathscr{D} \times \Delta^1 & \xrightarrow{\quad s \quad} & \mathcal{M} \\ & \searrow \quad \swarrow & \\ & \Delta^1 & \end{array}$$

such that $s|_{\mathscr{D}\times\{1\}} = h_1$, $s|_{\mathscr{D}\times\{0\}} = h_0 \circ g$, and $s|_{\{x\}\times\Delta^1}$ is a Cartesian morphism of $\mathcal{M}$ for every object $x \in \mathscr{D}$.

The following proposition is a consequence of 5.2.1.3 and 5.2.1.4 of [14].

Proposition 6.5.4. *There is a bijective correspondence between equivalence classes of functors $\mathscr{D} \to \mathscr{C}$ and equivalence classes of Cartesian fibrations $p : \mathcal{M} \to \Delta^1$ equipped with equivalences $\mathscr{C} \to p^{-1}(\{0\})$ and $\mathscr{D} \to p^{-1}(\{1\})$.*

Hence a functor $g : \mathscr{D} \to \mathscr{C}$ which is associated to $p : \mathcal{M} \to \Delta^1$ is uniquely determined up to equivalence. We have a dual correspondence if $p : \mathcal{M} \to \Delta^1$ is a coCartesian fibration, which shows that a functor $f : \mathscr{C} \to \mathscr{D}$ which is associated to $p : \mathcal{M} \to \Delta^1$ is uniquely determined up to equivalence.

Definition 6.5.5. Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories. An *adjunction* between $\mathscr{C}$ and $\mathscr{D}$, is a map $p : \mathcal{M} \to \Delta^1$ which is both a Cartesian fibration and a coCartesian fibration, together with equivalences $\mathscr{C} \to \mathcal{M}_{\{0\}} := p^{-1}(\{0\})$ and $\mathscr{D} \to \mathcal{M}_{\{1\}}$. $p : \mathcal{M} \to \Delta^1$ be an adjunction between $\mathscr{C}$ and $\mathscr{D}$ and let $f : \mathscr{C} \to \mathscr{D}$ and $g : \mathscr{D} \to \mathscr{C}$ be functors associated to $p : \mathcal{M} \to \Delta^1$. In this case, we will say that f is *left adjoint* to g and g is *right adjoint* to f.

Another consequence of 5.2.1.3 and 5.2.1.4 of [14] is that $f : \mathscr{C} \to \mathscr{D}$ has a right adjoint $g : \mathscr{D} \to \mathscr{C}$, then g is uniquely determined up to homotopy. Analogously to the classical category theory, we have an adjoint functor theorem, which is 5.5.2.9 [14], with the small caveat that it only holds for functors between *presentable* ∞-categories, and *accessible functors*. We will not give the definitions given in [14], because they use notions and notation which we have not introduced.

Definition 6.5.6. An ∞-category is *presentable* if it cocomplete, and its objects are presented under colimits by a set of objects.

This definition is equivalent to the one given in [14] via the main theorem of [19].

Definition 6.5.7. An ∞-category $\mathscr{C}$ is *accessible* if it has all filtered colimits, and there is some subcategory $\mathscr{C}' \subseteq \mathscr{C}$ spanned by compact objects, which generate $\mathscr{C}$ through filtered colimits. A functor out of an accessible ∞-category $F : \mathscr{C} \to \mathscr{D}$, which preserve filtered colimits, is called *accessible*.

This definition is equivalent to the one given in [14] via 5.4.2.2, and section 5.4.3 of [14]. Finally we can state the adjoint functor theorem.

Theorem 6.5.8. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable ∞-categories.*

(1) The functor F has a right adjoint if and only if it preserves colimits

(2) The functor F has a left adjoint if and only if it is accessible and preserve limits.

Again we omit the proof. Another important theorem in the classical category theory is the Yoneda lemma, which we describe in the following section.

6.6 The Yoneda lemma

In classical category theory the category of sets Set plays a vital role, which is largely due to the Yoneda lemma, which allows for objects of a category $\mathscr{C}$ to be thought of generalized sets, i.e. $C \mapsto \mathrm{Hom}_{\mathscr{C}}(\bullet, \mathscr{C})$ is fully faithful. Hence we may often pass questions about an abstract category $\mathscr{C}$ to the more well-behaved category of sets. The ∞-categorical analog to the category of sets, is the ∞-category of spaces S.

Definition 6.6.1. Let S be a simplicial set. We let $\mathcal{P}(S)$ denote the simplicial set $\mathsf{Fun}(S^{op}, \mathsf{S})$. We will refer to $\mathcal{P}(S)$ as the *∞-category of presheaves on S.*

Proposition 6.6.2. *Let S be a simplicial set. Then $\mathcal{P}(S)$ is bicomplete.*

Proof. Let $\phi : \mathfrak{C}[S]^{op} \to \mathscr{C}$ be an equivalence of simplicial categories, then we may identify $\mathcal{P}(S)$ with the underlying ∞-category of the simplicial model category $\mathsf{sSet}^{\mathfrak{C}[S]^{op}}$. This is a consequence of theorem 5.1.1.1 [14]. Note that $\mathsf{sSet}^{\mathfrak{C}[S]^{op}}$ is bicomplete, hence we invoke 6.4.6 to see that $\mathcal{P}(S)$ is bicomplete. ☐

This propositon lets us obtain a very simple proof for the bicompleteness of S.

Corollary 6.6.3. *The ∞-category of spaces S is bicomplete.*

Proof. This is obtained by setting $S = \bullet$ in 6.6.2, to obtain (analogously to classic category theory) $\mathcal{P}(\bullet) \simeq \mathsf{S}$. ☐

Definition 6.6.4. Let K be a simplicial set, and set $\mathscr{C} = \mathfrak{C}[K]$. Per. construction $\mathscr{C}$ is a simplicial category, so $(X,Y) \mapsto \mathrm{Sing}\,|\mathrm{Hom}_{\mathscr{C}}(X,Y)|$ determines a simplicial functor $\mathscr{C}^{op} \times \mathscr{C} \to \mathsf{Kan}$. There exists a natural map $\mathfrak{C}[K^{op} \times K] \to \mathscr{C}^{op} \times \mathscr{C}$, composing these two maps we obtain a simplicial functor

$$\mathfrak{C}[K^{op} \times K] \to \mathsf{Kan}$$

Using the adjunction $(\mathfrak{C}, N_\Delta)$, which holds per. construction, we get a map of simplicial sets $K^{op} \times K \to N_\Delta(\mathsf{Kan})$, which by the adjunction $(\mathsf{Fun}, \times)$ in sSet, can be identified with

$$j : K \to \mathsf{Fun}(K^{op}, \mathsf{S}) := \mathcal{P}(K).$$

We shall refer to j as the *Yoneda embedding.*

Proposition 6.6.5. *Let K be a simplicial set. Then the Yoneda embedding $j : K \to \mathcal{P}(K)$ is fully faithful.*

Proof. Let $\mathscr{C}' = \mathrm{Sing}\,|\mathfrak{C}[K^{op}]|$. We endow $\mathsf{sSet}^{\mathscr{C}'}$ with the projective model structure, described earlier in the chapter. Using 6.4.4 we may factor j, as

$$K \xrightarrow{j'} N_\Delta(\mathsf{sSet}^{\mathscr{C}'}_{fc}) \xrightarrow{j''} \mathsf{Fun}(K^{op}, \mathsf{S}).$$

j'' is the map from 6.4.4, hence it is a categorical equivalence, therefore it suffices to prove that j' is fully faithful. We show that the adjoint map under the $(\mathfrak{C}, N_\Delta)$-adjunction, $J : \mathfrak{C}[K] \to \mathsf{sSet}^{\mathscr{C}'}$ is a fully faithful functor betwen simplicial categories. We might factor J as

$$\mathfrak{C}[K] \xrightarrow{f} (\mathscr{C}')^{op} \xrightarrow{y} \mathsf{sSet}^{\mathscr{C}'},$$

where f is an equivalence, and y is the usual simplicial Yoneda embedding, which is fully faithful by the classical theory. ☐

We collect the following regularity result for the Yoneda embedding. It is proposition 5.1.3.2 [14].

Proposition 6.6.6. *Let $\mathscr{C}$ be a ∞-category and $j : \mathscr{C} \to \mathcal{P}(\mathscr{C})$ the Yoneda embedding. Then j preserves all small limits which exists in $\mathscr{C}$.*

7 Stable ∞-Categories and the ∞-Category of Spectra

In this chapter we will define the ∞-category of spectra, and give a series of general formal properties, which this ∞-category will posses. We shall see that its homotopy category will agree with the stable homotopy category of symmetric spectra defined in the previous part of the text. We will aim to formulate a universal property of the stable ∞-category of spectra, and compare it to the underlying ∞-category of Sp^Σ defined in the previous chapter. We shall in this chapter follow selected parts of chapter 1 of [20].

7.1 Stable ∞-Categories

The ∞-category presented by a category of chain complexes with values in an abelian category $\mathrm{Ch}(\mathscr{A})$, which homotopy category is the derived category $D(\mathscr{A})$ of the underlying abelian category is an example of great importance, and in many ways it motivates the definition of a *stable* ∞-category, which is what this section explores. We will begin to quantify some of the notions a *stable* ∞-category should posses. The idea is to endow a category with properties such that, when we pass to its homotopy category we will obtain the properties that $D(\mathscr{A})$ posses, namely the triangulated structure.

Definition 7.1.1. Let $\mathscr{C}$ be an ∞-category. A *zero object* of $\mathscr{C}$ is an object which is both initial and terminal. We say $\mathscr{C}$ is *pointed* if it contains a zero object.

Note that if 0 is a zero object in $\mathscr{C}$, then the natural map

$$\mathrm{Map}_{\mathscr{C}}(X,0) \times \mathrm{Map}_{\mathscr{C}}(0,Y) \to \mathrm{Map}_{\mathscr{C}}(X,Y)$$

has contractible domain, which implies that we obtain a well defined morphism $X \to Y$ in the homotopy category $\mathcal{H}o(\mathscr{C})$.

Definition 7.1.2. Let $\mathscr{C}$ be a pointed ∞-category. A *triangle* in $\mathscr{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathscr{C}$, which we illustrate as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of $\mathscr{C}$. We will say that a triangle in $\mathscr{C}$ is a *fiber sequence* if it is a pullback square, and a *cofiber sequence* if it is a pushout square.

A triangle can alternatively be described as two morphisms $f : X \to Y$ and $g : Y \to Z$ which form a horn Λ^2_1, a morphism $h : X \to Z$ which together with f and g form a 2-simplex which identifies h with the composition $g \circ f$, and lastly a 2-simplex formed from h, and $0 \to Z$ and $X \to 0$.

Definition 7.1.3. Let $\mathscr{C}$ be a pointed ∞-category containing a morphism $f : X \to Y$. A *fiber* of g is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y. \end{array}$$

Dually, a *cofiber* of f is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

We shall usually refer to W and Z as the fiber and cofiber of g, and denote them $\mathrm{fib}(g)$ and $\mathrm{cof}(g)$.

Remark 7.1.4. A cofiber of a morphism $f : X \to Y$ is uniquely determined up to equivalence. In the context of stable ∞-categories, it will be apparent that it is enough that only cofibers are unique.

Definition 7.1.5. An ∞-category $\mathscr{C}$ is *stable* if it satisfies the following conditions:

(1) $\mathscr{C}$ is pointed.

(2) Every morphism in $\mathscr{C}$ admits a fiber and a cofiber.

(3) A triangle in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Example 7.1.6. Sp^{Σ} and even the topological variant, where we have spectra of topological spaces, can be organized into a stable ∞-category Sp. The homotopy category $\mathcal{H}o(\mathsf{Sp})$ can be identified with the stable homotopy category $\mathcal{H}o(\mathsf{Sp}^{\Sigma})$ which we defined in the first part of this text. This justifies the terminology *stable*. We shall utilize a more conceptual definition of spectra, from which all the desired properties shall be formal consequences of the theory we shall develop. We shall see that Sp in a certain sense is the universal stable ∞-category.

Recall that $\mathcal{H}o(\mathsf{Sp}^{\Sigma})$ is triangulated (Theorem 3.4.6). If we take the above comment seriously, we should suspect that the homotopy category of a stable ∞-category should be triangulated. This is in fact true, and it is theorem 1.1.2.15 [20].

Theorem 7.1.7. *Let $\mathscr{C}$ be a stable ∞-category, then $\mathcal{H}o(\mathscr{C})$ has the structure of a triangulated category.*

We will not prove this fact, but we shall define the translation functors, which will determine the class of distinguished triangles which will endow $\pi(\mathscr{C})$ with a triangulated structure. The translation functors will be of vital importance to us. Their construction will utilize a central result concerning left/right Kan extensions of ∞-categories. The result is proposition 4.3.2.15 [14], which we state in the form we shall need

Proposition 7.1.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories, and consider a full subcategory $\mathscr{C}_0$ of $\mathscr{C}$. Let $K \subset \mathsf{Fun}(\mathscr{C}, \mathscr{D})$ be the full subcategory spanned by the functors f which are left Kan extensions along $f|_{\mathscr{C}_0}$. Furthermore let $K \subset \mathsf{Fun}(\mathscr{C}_0, \mathscr{D})$ be the full subcategory spanned by the functors for which the left Kan extension exists. Then the forgetful functor $K \to K'$ is a trivial fibration.*

Construction of Ω and Σ For a stable ∞-category $\mathscr{D}$, we will construct the two translation functors $\Omega : \mathscr{D} \to \mathscr{D}$ and $\Sigma : \mathscr{D} \to \mathscr{D}$ which for $X \in \mathscr{D}$, will send $X \mapsto X[-1]$ and $X \mapsto X[1]$, and hence form the distinquished triangles. The constructions are dual, hence we will only give the one for $\Sigma : \mathscr{D} \to \mathscr{D}$.

Let $\mathscr{C}$ and $\mathscr{C}_0$ be the following categories

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \qquad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

respectively. Let $K \subset \mathsf{Fun}(\mathscr{C},\mathscr{D})$ be the full subcategory spanned by the functors which are left Kan extensions along the inclusion $\mathscr{C}_0 \to \mathscr{C}$, and let $K' \subseteq \mathsf{Fun}(\mathscr{C}_0,\mathscr{D})$ be the full subcategory spanned by functors for which the left Kan extension exists. Proposition 7.1.8 implies that $K \to K'$ is a trivial fibration, hence we obtain a section $s : K' \to K$. Now if we post-compose with the functor $\mathrm{ev} : K \to \mathscr{D}$ which evaluates at the terminal vertex, we obtain a functor $\mathsf{Fun}(\mathscr{C}_0,\mathscr{D}) \to \mathscr{D}$.

Consider the functor $X : \bullet \to \mathscr{D}$, which picks out an element of $\mathscr{D}$, say X for the sake of reference. Furthermore consider the inclusion $\bullet \to \mathscr{C}_0$, which includes into the initial vertex. Taking the right Kan extension along this inclusion, we obtain a functor which assigns X to the following diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0, & & \end{array}$$

where 0 is a zero object of $\mathscr{D}$. Let $K'' \subseteq \mathsf{Fun}(\mathscr{C}_0,\mathscr{D})$ be the full subcategory spanned by the functors which are right Kan extensions along the inclusion $\bullet \to \mathscr{C}_0$. Because $\mathsf{Fun}(\bullet,\mathscr{D}) \simeq \mathscr{D}$, the dual of proposition 7.1.8 implies that $K'' \to \mathscr{D}$ is a trivial fibration. Again we obtain a section $\mathscr{D} \to K''$. Hence we obtain an endofunctor on $\mathscr{D}$

$$\mathscr{D} \simeq \mathsf{Fun}(\bullet,\mathscr{D}) \rightleftarrows K'' \rightleftarrows K \xrightarrow{\mathrm{ev}} \mathscr{D}$$

which is given by sending X to the colimit of the diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0. & & \end{array}$$

We define this functor to be $\Sigma : \mathscr{D} \to \mathscr{D}$, and write $X \mapsto \Sigma X$ or $X \mapsto X[1]$. The functor obtained from the dual process we will denote by $\Omega : \mathscr{D} \to \mathscr{D}$, and write $X \mapsto \Omega X$ or $X \mapsto X[-1]$. Note by the stability of $\mathscr{D}$ the full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by those diagrams which are pushouts, is the same as the full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by those diagrams which are pullbacks. From this it is elementary to show that Σ and Ω are mutually inverse equivalences $\mathscr{D} \to \mathscr{D}$.

Using exactly the same technique we may also show that in a pointed ∞-category $\mathscr{C}$ we can associate to a morphism $f : X \to Y$ its cofiber in a functorial way. We denote this assignment by $\mathrm{cof} : \mathsf{Fun}(\Delta^1,\mathscr{C}) \to \mathscr{C}$. It is well-defined up to a contractible space of choices. This is a proof of the fact stated in 7.1.4. A corollary of this construction is the following.

Corollary 7.1.9. *The functor* $\mathrm{cof} : \mathsf{Fun}(\Delta^1,\mathscr{C}) \to \mathscr{C}$ *preserve all colimits which exists in* $\mathsf{Fun}(\Delta^1,\mathscr{C})$

Proof. By the construction of cof, it can be identified with a left adjoint to the left Kan extension functor $\mathscr{C} \simeq \mathsf{Fun}(\bullet,\mathscr{C}) \to \mathsf{Fun}(\Delta^1,\mathscr{C})$. Theorem 6.5.8 give the desired result. □

Another interesting consequence of the existence of the functors $\Omega, \Sigma : \mathscr{C} \to \mathscr{C}$ on a stable ∞-category $\mathscr{C}$ is that its homotopy categor $\mathcal{H}o(\mathscr{C})$, is naturally enriched over abelian groups, which is an essentiel part of the proof of theorem 7.1.7. When $\mathscr{C}$ is stable Σ is an equivalence, hence we may choose Z such that $Z[2] \cong X$ for any X. This enrichment is implied from the abelian group structure on $\mathrm{Hom}_{\mathcal{H}o(\mathscr{C})}(X[2],Y) \simeq \pi_2(\mathrm{Map}_{\mathscr{C}}(X,Y))$. In particular we have a group structure on $\mathrm{Hom}_{\mathcal{H}o(\mathscr{C})}(X[1],Y) \simeq \pi_1(\mathrm{Map}_{\mathscr{C}}(X,Y))$. Given a map $f : X \to Y$ we write $-f : X \to Y$ for the inverse of f with respect to this group structure.

We end this section with a couple of results concerning stability, which we shall need in the following section. This theorem is 1.1.3.4 [20].

Theorem 7.1.10. *Let $\mathscr{C}$ be a pointed ∞-category. Then $\mathscr{C}$ is stable if and only if*

(1) $\mathscr{C}$ admits finite limits and colimits.

(2) A square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

in $\mathscr{C}$ is a pushout if and only if it is a pullback.

Proof. Assume the conditions hold. Condition (1) implies pullbacks and pushouts exists, which implies fibers and cofibers exists. Condition (2) implies that a triangle in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence. This proves that $\mathscr{C}$ is stable.

Assume $\mathscr{C}$ is stable. We shall prove that $\mathscr{C}$ admits finite limits and colimits, and omit the proof of condition (2), due to its technical nature, and the fact that condition (1) is the most important in the next section. We prove $\mathscr{C}$ admits finite colimits, the dual argument will show that $\mathscr{C}$ admits finite limits. Analogously to classical category theory, the existence of coequalizers and pairwaise coproducts will suffice to show that $\mathscr{C}$ admits finite colimits. Note this uses that $\mathscr{C}$ is pointed. We begin by constructing the pairwise coproducts.

Let $X, Y \in \mathscr{C}$, and consider $u : \Omega X \to 0$ and $v : 0 \to Y$. Note that $X \simeq \mathrm{cof}(u : \Omega X \to 0)$ and $Y \simeq \mathrm{cof}(v : 0 \to Y)$. Proposition 5.1.2.2 [14]gives that u and v admit a coproduct in $\mathsf{Fun}(\Delta^1, \mathscr{C})$. Corollary 7.1.9 gives that cof preserve colimits, from which we conclude that X and Y admit a coproduct. Observe, that a coequalizer for a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

can be identified with $\mathrm{cof}(f - g)$. Hence condition (1) holds. □

We shall need the following result in the following section. The following is proposition 1.4.2.11 [20].

Proposition 7.1.11. *Let $\mathscr{C}$ be a pointed ∞-category which admits finite limits and colimits, then if $\Omega : \mathscr{C} \to \mathscr{C}$ is an equivalence of ∞-categories, then $\mathscr{C}$ is stable.*

Now that we have defined the notion of a stable ∞-category, and we have seen a few first properties, we shall define the ∞-category spectra in a slightly different way.

7.2 The Stable ∞-Category of Spectra

As was discussed in chapter 1 section 1 in classic stable homotopy theory we may associate a spectrum to a generalized cohomology theory and vice versa using Brown representability. The key assumptions on the functor in Brown representability are that it should send pushouts to pullbacks, and preserve basepoint. These assumptions are imposed to mimic the prototypical cohomology theory namely, singular cohomology $H^\bullet(X, \mathbb{Z})$, which is defined as the cohomology of the simplicial abelian group $\mathbb{Z}\,\mathrm{Sing}(X)_\bullet$, which has these property. We shall employ the same intuition, and define spectra as functors of ∞-categories having exactly these properties.

Definition 7.2.1. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories.

(1) If $\mathscr{C}$ admits pushouts, then we will say that F is *excisive* if F sends pushout squares in $\mathscr{C}$ to pullback squares to $\mathscr{D}$.

(2) If $\mathscr{C}$ admits a zero object $\bullet$, we will say that F is *reduced* if $F(\bullet)$ is a final object of $\mathscr{D}$.

If $\mathscr{C}$ admits pushouts and a zero object, then we let $\mathsf{Exc}_\bullet(\mathscr{C}, \mathscr{D})$ denote the full subcategory of $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ spanned by the excisive and reduced functors.

Note that in the following that if one considers a stable ∞-category, all of the definitions will be valid, in light of 7.1.10.

Remark 7.2.2. Let $\mathscr{D}$ be a presentable ∞-category. Then it turns out that $\mathsf{Exc}_*(\mathscr{C}, \mathscr{D})$ is also a presentable ∞-category. This is a very deep result, which relies 5.5.4.18 and 5.5.4.19 [14]. Ultimately it also relies on the fact that $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ is bicomplete.

In Brown representability the functors are evaluated in pointed categories of homotopical interest, such as sSet_* or Top_*, it is much the same in the ∞-categorical setting. Our excisive and reduced functors shall be evaluated in the ∞-category of finite pointed spaces.

Definition 7.2.3. Let $\mathsf{S}_\bullet$ denote the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{S})$ spanned by those morphisms $f : X \to Y$ in S, where X is a final object of S. Let $\mathsf{S}_*^{\mathrm{fin}}$ denote the smallest full subcategory of S_* which contains the final object $\bullet$ and is stable under finite colimits.

Via the constructions in the previous section, we may construct a suspension functor $\Sigma : \mathsf{S}_*^{\mathrm{fin}} \to \mathsf{S}_*^{\mathrm{fin}}$. For each $n \le 0$, we let $S^n \in \mathsf{S}_*^{\mathrm{fin}}$ denote the n'th suspension of $\bullet$.

Definition 7.2.4. Let $\mathscr{C}$ be an ∞-category which admits finite limits. A *spectrum* object of $\mathscr{C}$ is a reduced, and excisive functor $X : \mathsf{S}_*^{\mathrm{fin}} \to \mathscr{C}$. Let $\mathsf{Sp}(\mathscr{C}) = \mathsf{Exc}_*(\mathsf{S}_*^{\mathrm{fin}}, \mathscr{C})$ denote the full subcategory of $\mathsf{Fun}(\mathsf{S}_*^{\mathrm{fin}}, \mathscr{C})$ spanned by the spectrum objects of $\mathscr{C}$.

Example 7.2.5. Let $\mathscr{C}$ be an ∞-category which admits finite limits. Consider a spectrum object $X \in \mathsf{Sp}(\mathscr{C})$, i.e. a reduced excisive functor $X : \mathsf{S}_*^{\mathrm{fin}} \to \mathscr{C}$. For each $n \in \mathbb{N}$, write $S^n \in \mathsf{S}_*^{\mathrm{fin}}$ for the n-sphere $(\Sigma^n(\bullet))$, and write $X_n := X(S^n)$. Via the (Σ, Ω)-adjunction in $\mathsf{S}_*^{\mathrm{fin}}$, we have homotopy pushout squares

$$\begin{array}{ccc} S^n & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & S^{n+1} \end{array}$$

and since X is excisive it sends them homotopy pullbacks, with $\bullet$ being preserved, which gives equivalences $X_n \to \Omega X_{n+1}$. Hence the data of an reduced excisive functor gives rise to an Ω-spectrum.

In the first part of the text we showed that there are two compatible ways to think of spectra. The first was a generalized cohomology theories and Brown representability, which motivated our definition. The other point of view is for Ω-spectra, which always can arranged up to weak equivalence, is that the simplicial set X_0 contains the homotopical information of spectrum X in dimensions $k \le 0$, and X_1 contains the homotopical information of X in dimension $k \le 1$ and so on. Hence $\Omega^\infty X$ will contain all the stable information of X. This viewpoint is also a valid one in the ∞-categorical setting, we quickly mention this point of view, before continuing the discussion of the universal property of $\mathsf{Sp}(\mathscr{C})$.

Definition 7.2.6. Regard $\bullet = S^0$ as an object of $\mathsf{S}_*^{\mathrm{fin}}$. If $\mathscr{C}$ is an ∞-category which admits finite limits, we let $\Omega^\infty : \mathsf{Sp}(\mathscr{C}) \to \mathscr{C}$ denote the functor given by evaluation at S^0.

The following is proposition 1.4.2.24 [20].

Proposition 7.2.7. *Let $\mathscr{C}$ be a pointed ∞-category which admits finite limits. Then the functor $\Omega^\infty : \mathsf{Sp}(\mathscr{C}) \to \mathscr{C}$ can be lifted to an equivalence of $\mathsf{Sp}(\mathscr{C})$ with the homotopy limit of the tower of ∞-categories*

$$... \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C}.$$

Corollary 7.2.8. *Let $\mathscr{C}$ be a pointed ∞-category which admits finite limits. We can identify the ∞-category $\mathsf{Sp}(\mathscr{C})$ with the homotopy limit of the tower of ∞-categories*

$$... \xrightarrow{\Omega} \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_*.$$

Where $\mathscr{C}_*$ is the category of pointed objects of $\mathscr{C}$.

Proof. This follows from 7.2.7 and the fact that the forgetful functor $\mathscr{C}_* \to \mathscr{C}$ induces an equivalence of ∞-categories $\mathsf{Sp}(\mathscr{C}_*) \to \mathsf{Sp}(\mathscr{C})$. The latter statement follows from the isomorphism of simplicial sets $\mathsf{Sp}(\mathscr{C}_*) \cong \mathsf{Sp}(\mathscr{C})_*$, and the fact the forgetful functor $\mathsf{Sp}(\mathscr{C})_* \to \mathsf{Sp}(\mathscr{C})$ is an equivalence of ∞-categories (This is a consequence of the stability of $\mathsf{Sp}(\mathscr{C})$ which we prove later in this section). □

We shall need the following identification.

Corollary 7.2.9. *Let $\mathscr{C}$ be an ∞-category which admits finite limits, and K a simplicial set. Then we have a canonical isomorphism $\mathsf{Sp}(\mathsf{Fun}(K,\mathscr{C})) \simeq \mathsf{Fun}(K,\mathsf{Sp}(\mathscr{C}))$.*

This follows from the fact that (homotopy) limits in a functor category is computed pointwise. We omit the verification.

The special case of the ∞-category of spectra, is spectra of spaces, i.e when $\mathscr{C} = \mathsf{S}$. We shall denote $\mathsf{Sp} \coloneqq \mathsf{Exc}_*(\mathsf{S}_*^{\mathrm{fin}}, \mathsf{S})$. This special case recovers the classic stable homotopy theory. One of the main goals for this text is to make this precise. We shall do this by showing that Sp is equivalent to the underlying ∞-category of symmetric spectra, namely that we an equivalence $\mathsf{Sp} \simeq \mathsf{Sp}^\Sigma_\infty$, we show this as the last theorem. The rest of this chapter is devoted to formulating this universal property. We begin by showing that $\mathsf{Sp}(\mathscr{C})$ is stable if $\mathscr{C}$ admits finite limits. For this we will need the following result, which is a consequence of 6.6.6.

Proposition 7.2.10. *Let $\mathscr{C}$ be a stable ∞-category and $j : \mathscr{C} \to \mathcal{P}(\mathscr{C})$ the Yoneda embedding. Then j commutes with finite limits.*

Theorem 7.2.11. *Let $\mathscr{C}$ be an ∞-category which admits finite limits. Then the ∞-category $\mathsf{Sp}(\mathscr{C})$ is stable.*

Proof. We shall show the more general statement that $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$, where $\mathscr{C}$ admit finite limits, and $\mathscr{D}$ admits finite colimits, is stable. We will show this first in the case where $\mathscr{C}$ is a presentable ∞-category. Let $S : \mathsf{Fun}(\mathscr{D},\mathscr{C}) \to \mathsf{Fun}(\mathscr{C},\mathscr{D})$ be given by $F \mapsto F \circ \Sigma$, where Σ is the suspension on $\mathscr{D}$. Note here that we have not assumed $\mathscr{D}$ to be a stable ∞-category, which was the setting in which we constructed Σ, but it is in fact possible to carry out the exact same construction of Σ while only requiring $\mathscr{D}$ to be pointed and admitting cofibers. S carries $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ to itself, hence S is a homotopy inverse to the functor Ω on $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$. It can be shown (Lemma 1.4.2.10 [20]) that $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ is pointed. $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ admits finite limits and colimits by 7.2.2, which is where we use that $\mathscr{C}$ was assumed to be a presentable ∞-category. We've found a homotopy inverse to Ω on the stable ∞-category $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$, hence by 7.1.11 $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ is stable.

To handle the general case where $\mathscr{C}$ admits finite colimits, consider the ∞-category of presheaves on $\mathscr{C}$, $\mathcal{P}(\mathscr{C})$, and let $j : \mathscr{C} \to \mathcal{P}(\mathscr{C})$ be the Yoneda embedding. Now via 7.2.10 and 6.6.5 j induces a fully faithful embedding $\mathsf{Exc}_*(\mathscr{D},\mathscr{C}) \to \mathsf{Exc}_*(\mathscr{D},\mathcal{P}(\mathscr{C}))$. Then $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ is equivalent to a full subcategory of the ∞-category $\mathsf{Exc}_*(\mathscr{D},\mathcal{P}(\mathscr{C}))$, which closed under finite limits and suspensions. Now $\mathcal{P}(\mathscr{C})$ is presentable (see section 5.5 [14]), hence via 7.2.2, $\mathsf{Exc}_*(\mathscr{D},\mathcal{P}(\mathscr{C}))$ is presentable. Full subcategories of stable ∞-categories, which are closed under finite limits and suspensions, are stable, this is proposition 1.1.3.3 [20]. Hence $\mathsf{Exc}_*(\mathscr{D},\mathscr{C})$ is stable. Set $\mathscr{D} = \mathsf{S}_*^{fin}$, to obtain the desired result. □

7.3 The Universal Property For $\mathsf{Sp}(\mathscr{C})$

The following section follows section 1.4.2 and parts of 1.4.4 of [20]. The essential ingredient of the proof of the universal property for $\mathsf{Sp}(\mathscr{C})$ will be the existence of a left adjoint to Ω^∞, which we, analogously to the classic stable homotopy theory, will denote Σ^∞. Σ^∞ will turn out to induce a certain equivalence, which will be a consequence of Ω^∞ inducing an equivalence under certain conditions. We begin by quantifying the conditions which will ensure that Ω^∞ itself is an equivalence, to this end we shall need the following universal property of $\mathsf{S}^{\mathrm{fin}}$.

Lemma 7.3.1. *For every ∞-category $\mathscr{D}$ which admits finite colimits, evaluation at $\bullet$ induces an equivalence of ∞-categories $\mathsf{Fun}'(\mathsf{S}^{fin}, \mathscr{D}) \to \mathscr{D}$. Where $\mathsf{Fun}'(\mathsf{S}^{fin}, \mathscr{D})$ denotes the full subcategory of $\mathsf{Fun}(\mathsf{S}^{fin}, \mathscr{D})$ spanned by functors which which commute with colimits.*

I.e. $\mathsf{S}^{\mathrm{fin}}$ is freely generated by the space $\bullet \in \mathsf{S}^{\mathrm{fin}}$ under finite colimits. This lemma will be an integral ingredient in the proof of to the following proposition.

Proposition 7.3.2. *Let $\mathscr{C}$ be an ∞-category which admits finite limits. The following conditions are equivalent:*

(1) The ∞-category $\mathscr{C}$ is stable.

(2) The functor $\Omega^\infty : \mathsf{Sp}(\mathscr{C}) \to \mathscr{C}$ is an equivalence of ∞-categories.

Proof. The implication (2) implies (1) follows from 7.2.11. Assume (1) holds. Consider the forgetful functor $F : \mathsf{S}^{\mathrm{fin}}_* \to \mathsf{S}^{\mathrm{fin}}$. F admits a left adjoint $f : \mathsf{S}^{\mathrm{fin}} \to \mathsf{S}^{\mathrm{fin}}_*$, which is defined by $X \mapsto X \cup \{\bullet\}$, i.e. adding a disjoint basepoint. Let $\mathsf{Exc}'(\mathsf{S}^{\mathrm{fin}}, \mathscr{C})$ be the full subcategory of $\mathsf{Exc}(\mathsf{S}^{\mathrm{fin}}, \mathscr{C})$ spanned by those functor which carry $\bullet \in \mathsf{S}^{\mathrm{fin}}$ to the zero object of $\mathscr{C}$. From lemma 1.4.2.21 of [20], we get that evaluation at the object $\bullet \in \mathsf{S}^{\mathrm{fin}}$, induces an equivalence of the ∞-categories $\mathsf{Sp}(\mathscr{C}) \coloneqq \mathsf{Exc}_*(\mathsf{S}^{\mathrm{fin}}_*, \mathscr{C}) \to \mathsf{Exc}'(\mathsf{S}^{\mathrm{fin}}, \mathscr{C})$. One may identify the objects of $\mathsf{Exc}'(\mathsf{S}^{\mathrm{fin}}, \mathscr{C})$ with the functors $X : \mathsf{S}^{\mathrm{fin}} \to \mathscr{C}$ which commute with finite colimits. Hence by the above universal property of $\mathsf{S}^{\mathrm{fin}}$, we obtain an equivalence of ∞-categories $\mathsf{Exc}'(\mathsf{S}^{\mathrm{fin}}, \mathscr{C}) \simeq \mathscr{C}$. Hence evaluation at the object $\bullet \coloneqq S^0$ in $\mathsf{Sp}(\mathscr{C})$ to $\mathscr{C}$ is an equivalence of ∞-categories, hence Ω^∞ is an equivalence of ∞-categories. □

Now that we have quantified the properties to impose on $\mathscr{C}$ which ensure that $\Omega^\infty : \mathsf{Sp}(\mathscr{C}) \to \mathscr{C}$ is an equivalence of ∞-categories, we move on to proving that it induces an equivalence on excisive reduced functors.

Proposition 7.3.3. *Let $\mathscr{C}$ be a pointed ∞-category which admits finite colimits and $\mathscr{D}$ an ∞-category which admits finite limits. Then composition with the functor $\Omega^\infty : \mathsf{Sp}(\mathscr{D}) \to \mathscr{D}$ induces an equivalence of ∞-categories*

$$\Theta : \mathsf{Exc}_*(\mathscr{C}, \mathsf{Sp}(\mathscr{D})) \to \mathsf{Exc}_*(\mathscr{C}, \mathscr{D}).$$

Proof. Using 7.2.9 we obtain the following canonical isomorphism $\mathsf{Exc}_*(\mathscr{C}, \mathsf{Sp}(\mathscr{D})) \cong \mathsf{Sp}(\mathsf{Exc}_*(\mathscr{C}, \mathscr{D}))$. Θ corresponds to $\Omega^\infty : \mathsf{Sp}(\mathsf{Exc}_*(\mathscr{C}, \mathscr{D})) \to \mathsf{Exc}_*(\mathscr{C}, \mathscr{D})$. In the proof of 7.2.11 we in fact showed that $\mathsf{Exc}_*(\mathscr{C}, \mathscr{D})$ was stable, under the assumptions of this proposition, hence 7.3.2 implies that Θ is an equivalence of ∞-categories. □

We shall need the following proposition which is 1.4.4.4 in [20]. We omit the proof.

Proposition 7.3.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be presentable ∞-categories, and suppose $\mathscr{D}$ is stable. Then*

(1) $\mathsf{Sp}(\mathscr{C})$ is presentable.

(2) The functor $\Omega^\infty : \mathsf{Sp}(\mathscr{C}) \to \mathscr{C}$ admits a left adjoint $\Sigma^\infty : \mathscr{C} \to \mathsf{Sp}(\mathscr{C})$.

(3) A functor $G : \mathscr{D} \to \mathsf{Sp}(\mathscr{C})$ which commutes with finite limits and colimits admits a left adjoint if and only if $\Omega^\infty \circ G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint.

Definition 7.3.5. Let $\mathscr{C}$ and $\mathscr{D}$ be presentable ∞-categories. Let $\mathsf{Fun}^L(\mathscr{C},\mathscr{D})$ denote the full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by those functors which admit *right* adjoints, and let $\mathsf{Fun}^R(\mathscr{C},\mathscr{D})$ denote the full subcategory spanned by those functors which admit *left* adjoints.

The following corollary is the last statement we need, before we are able to give the universal property of the ∞-category of spectra Sp.

Corollary 7.3.6. *Let $\mathscr{C}$ and $\mathscr{D}$ be presentable ∞-categories, and suppose that $\mathscr{D}$ is stable. Then composition with $\Sigma : \mathscr{C} \to \mathsf{Sp}(\mathscr{C})$ induces an equivalence of ∞-categories,*

$$\mathsf{Fun}^L(\mathsf{Sp}(\mathscr{C}),\mathscr{D}) \to \mathsf{Fun}^L(\mathscr{C},\mathscr{D}).$$

Proof. (3) from 7.3.4 gives that there is a one-to-one correspondence between functors in $\mathsf{Fun}^R(\mathscr{D},\mathsf{Sp}(\mathscr{C}))$ and functors in $\mathsf{Fun}^R(\mathscr{D},\mathscr{C})$. Using 7.2.9 and the strategy of the proof of 7.3.3, we may promote this correspondence to an equivalence

$$\mathsf{Fun}^R(\mathscr{D},\mathsf{Sp}(\mathscr{C})) \to \mathsf{Fun}^R(\mathscr{D},\mathscr{C}).$$

This is equivalent to the statement in corollary, via (2) from 7.3.4. □

Definition 7.3.7. Let $\mathbb{S} \in \mathsf{Sp}$ denote the image under $\Sigma^\infty : \mathsf{S} \to \mathsf{Sp}$ of the terminal object $\bullet \in \mathsf{S}$. We refer to $\mathbb{S}^\infty$ as the *sphere spectrum*.

We are now finally able to prove the universal property of Sp.

Theorem 7.3.8. *Let $\mathscr{D}$ be a presentable stable ∞-category. Then evaluation on the sphere spectrum induces an equivalence of ∞-categories*

$$\Theta : \mathsf{Fun}^L(\mathsf{Sp},\mathscr{D}) \to \mathscr{D}$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} & & \mathsf{Fun}^L(\mathsf{S},\mathscr{D}) & & \\ & \overset{\Theta'}{\nearrow} & & \overset{\Theta''}{\searrow} & \\ \mathsf{Fun}^L(\mathsf{Sp},\mathscr{D}) & & \xrightarrow{\quad\Theta\quad} & & \mathscr{D} \end{array}$$

where Θ' is given by composition with Σ, which is an equivalence of ∞-categories by 7.3.6, and Θ'' is given by evaluation at the terminal object of S. Θ'' is an equivalence via, theorem 5.1.5.6 [14], where we set $S = \bullet$. Here we've also used theorem 6.5.8. □

Theorem 7.3.9. $\mathsf{Sp}^\Sigma_\infty \simeq \mathsf{Sp}$

Proof. We equipped Sp^Σ with a combinatorial simplicial model structure, namely the stable model structure, this implies that $\mathsf{Sp}^\Sigma_\infty$ is a presentable ∞-category, which follows from theorem A.3.7.6 [14]. $\mathsf{Sp}^\Sigma_\infty$ is stable because it arises as the coherent nerve of a stable model category. Sp is presentable from 7.3.4, and stable from 7.2.11. Let $\mathbb{S}^\Sigma \in \mathsf{Sp}^\Sigma_\infty$ be the image of the bifibrant replacement of the symmetric sphere spectrum $\mathbb{S} \in \mathsf{Sp}^\Sigma$ under N_Δ. The universal property of Sp gives a unique colimit preserving functor $F : \mathsf{Sp} \to \mathsf{Sp}^\Sigma_\infty$ such that $F(\mathbb{S}^\infty) = \mathbb{S}^\infty$. $F : \mathsf{Sp} \to \mathsf{Sp}^\Sigma_\infty$ admits a right adjoint, because colimit preserving is equivalent to admitting a right adjoint via 6.5.8. Denote the right adjoint $G : \mathsf{Sp}^\Sigma_\infty \to \mathsf{Sp}$. We argue that it suffices to show the three following three conditions.

1. G preserve geometric realizations of simplicial objects.

2. G is conservative.

3. The map $\mathbb{S}^\infty \to G(\mathbb{S}^\Sigma)$ is an equivalence.

Assume these conditions hold. Because we are considering presentable stable ∞-categories it is enough to show an equivalence on their homotopy categories. Let $X \in \mathsf{Sp}$, then we show that $X \to G(F(X))$ is an equivalence, for all $X \in \mathsf{Sp}$. Consider the class $\mathcal{C}$, of $X \in \mathsf{Sp}$ for which this is true. From condition (1) above $\mathcal{C}$ is closed under coproducts. Because Sp is stable ∞-categories, $\mathcal{H}o(\mathsf{Sp})$ is triangulated, hence we may consider a distinguished triangle $X \to Y \to Z$, where Z is the mapping cone of $X \to Y$, for $X, Y \in \mathcal{C}$. Consider the following diagram

$$\begin{array}{ccccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Y \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow & & \downarrow{\simeq} & & \downarrow{\simeq} \\ G(F(X)) & \longrightarrow & G(F(Y)) & \longrightarrow & G(F(Z)) & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Y \end{array}$$

Which shows that $Z \to G(F(Z))$ is an equivalence, and hence $\mathcal{C}$ is closed under mapping cones. Analogously one might show that $\mathcal{C}$ is closed under suspensions. Per. condition (3) we see that $\mathbb{S}^{\infty} \in \mathcal{C}$, because $G(\mathbb{S}^{\Sigma}) = G(F(\mathbb{S}^{\infty}))$, per. construction of F. Any class in $\mathcal{H}o(\mathsf{Sp})$ which is closed under coproducts, mapping cones, suspensions and which contains the sphere spectrum is all of $\mathcal{H}o(\mathsf{Sp})$, hence we've obtain an equivalence $\mathsf{Sp}^{\Sigma}_{\infty} \simeq \mathsf{Sp}$. Now we simply need to show the conditions.

We first give a explicit definition of G. Up to homotopy G is on objects $X \in \mathsf{Sp}^{\Sigma}_{\infty}$ given as

$$\mathrm{Map}_{\mathsf{Sp}^{\Sigma}_{\infty}}(\mathbb{S}^{\Sigma}, X \otimes -) : \mathsf{S}^{\mathrm{fin}}_{*} \to \mathsf{S}$$

Which follows from the fact that $\mathsf{Sp}^{\Sigma}_{\infty}$ is tensored over S. We will take this for granted. For a $A \in \mathsf{S}^{\mathrm{fin}}_{*}$ we define $X \otimes A$, as $\mathrm{hocolim}_{[n] \in \Delta^{op}} A_n \otimes X$. For convenience we will still refer to this functor as G. We shall need that $\mathbb{S}^{\Sigma}$ is an compact generator of $\mathsf{Sp}^{\Sigma}_{\infty}$.

We know that $\mathbb{S} \in \mathsf{Sp}^{\Sigma}$ is a compact generator, i.e. $\mathrm{Map}_{\mathsf{Sp}^{\Sigma}}(\mathbb{S}, -)$ preserve filtered colimits, and we may generate every object of Sp^{Σ} from $\mathbb{S}$ through colimits. Hence if N_{Δ} preserve colimits we may also generate every object of $\mathsf{Sp}^{\Sigma}_{\infty}$ from $\mathbb{S}^{\Sigma}$, and $\mathrm{Map}_{\mathsf{Sp}^{\Sigma}_{\infty}}(\mathbb{S}^{\Sigma}, -)$ preserve filtered colimits, hence $\mathbb{S}^{\Sigma}$ is a compact generator. That N_{Δ} preserve colimits, follows from lemma 1.3.4.26 [20] together with the fact that $N_{\Delta}(\mathsf{Sp}^{\Sigma}_{fc})$ and $N_{\Delta}(\mathsf{Sp}^{\Sigma}_{c})[\mathsf{W}^{-1}]$ exhibit the same ∞-category (lemma 1.3.4.20 [20]), where Sp^{Σ}_{c} is the full subcategory of Sp^{Σ} spanned by the stably cofibrant objects, and W is the class of stable equivalences.

We are now ready to begin showing the conditions. We begin with (1). We will show that G preserve colimits, which implies that it preserve geometric realization. G admits a left adjoint, hence by theorem 6.5.8 it is accessible, i.e. G preserves filtered colimits, and G preserve limits. In particular G preserve fiber sequences, in this case we say that G is *exact*. Preservation of filtered colimits, exactness of G, and the fact that $\mathbb{S}^{\Sigma}$ is compact implies that G preserve colimits. Which shows (1).

Let $\alpha : D \to D'$ be a morphism in $\mathsf{Sp}^{\Sigma}_{\infty}$ such that $G(\alpha)$ is an equivalence in Sp. Let D'' be the cofiber of α, because Sp is stable (theorem 7.2.11) $G(\alpha)$ being an equivalence is equivalent to $G(D'') \simeq 0$. Because G is exact, we are now reduced to showing that $D'' \simeq 0$. $G(D'') \simeq 0$, means that we have the following equivalences for every $A \in \mathsf{S}^{\mathrm{fin}}_{*}$,

$$\begin{aligned} 0 \simeq G(D'')(A) &\simeq \mathrm{Map}_{\mathsf{Sp}^{\Sigma}_{\infty}}(\mathbb{S}^{\Sigma}, D'' \otimes A) \\ &\simeq \mathrm{Map}_{\mathsf{Sp}^{\Sigma}}(\mathbb{S}, (D'' \otimes A)_f) \end{aligned}$$

Here $(D'' \otimes A)_f$ means the fibrant replacement of $D'' \otimes A$. The second equivalence follows from the fact that mapping objects in a presentable ∞-category, may be computed in its associated combinatorial model category, if the domain is cofibrantly replaced, and the codomain is fibrantly replaced. Note that $\mathbb{S}$ is stably cofibrant, hence we obtain the above equivalence. Consider

$A = S^n$ for every $n \in \mathbb{Z}$. Up to stable equivalence we may consider $(D'' \otimes S^n)$ instead of $(D'' \otimes A)_f$, we have the following equivalences of homotopy groups

$$\pi_m(\mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathbb{S}, (D'' \otimes S^n)) \simeq \pi_m(D'' \otimes S^n) \simeq \pi_{m-n}(D'')$$

Hence $\pi_k(D'')$ vanishes for every k. Because $\mathbb{S}$ is a compact generator in Sp^Σ, this shows that $D'' \simeq 0$, and hence α is an equivalence, which shows (2).

The last thing we need to show is that $\mathbb{S}^\infty \to G(\mathbb{S}^\Sigma)$ is an equivalence. From the universal property we have that $\mathbb{S}^\infty : \mathsf{S}_*^{\mathrm{fin}} \to \mathsf{S}_*$, is given as $A \mapsto \Omega^\infty\Sigma^\infty A$. If we can show that $\mathbb{S}^\infty(A)$ may be calculated as $\mathrm{hocolim}_n\, \Omega^n\Sigma^n A$, we are done, because

$$\begin{aligned} G(\mathbb{S}^\Sigma)(A) &= \mathrm{Map}_{\mathsf{Sp}^\Sigma_\infty}(\mathbb{S}^\Sigma, A \otimes \mathbb{S}^\Sigma) \\ &\simeq \mathrm{Map}_{\mathsf{Sp}^\Sigma}(\mathbb{S}, (A \otimes \mathbb{S})^f) \\ &\simeq (A \otimes \mathbb{S})^f \\ &\simeq \mathrm{hocolim}_n\, \Omega^n\Sigma^n A. \end{aligned}$$

Here the last equivalence is via the fibrant replacement for semistable symmetric spectra, which we describe now. Consider a symmetric spectrum X, and let $l : X \to \mathrm{sh}\, X$. The fibrant replacement is given as the homotopy colimit of the system

$$X \xrightarrow{\Omega \circ l} \Omega\, \mathrm{sh}\, X \xrightarrow{\Omega \circ l} \Omega^2\, \mathrm{sh}^2\, X \longrightarrow \ldots$$

Which is $\Omega^\infty\, \mathrm{sh}^\infty\, X$. Now this homotopy colimit is in general neither an Ω-spectrum nor is $X \to \Omega^\infty\, \mathrm{sh}^\infty\, X$ an equivalence. But if $X = \Sigma^\infty A$, it is both. Under this assumption $\mathrm{sh}^n\, X \simeq \Sigma^\infty A \otimes S^n$, hence we obtain the desired formula for the fibrant replacement via the above system.

We now sketch the argument that $\mathbb{S}^\infty(A)$ may be calculated as $\mathrm{hocolim}_n\, \Omega^n\Sigma^n A$. Recall that Sp could be identified as the homotopy limit of iterated loop functors $\Omega : \mathsf{S}_* \to \mathsf{S}_*$, 7.2.7. If one instead took the so called *lax limit* instead of the homotopy limit, one would obtain the ∞-category of *prespectra*, denoted $P\mathsf{Sp}$. One may view Sp as a localization of $P\mathsf{Sp}$. Analogously to 7.2.5 the data of a functor in $P\mathsf{Sp}$ gives rise to spectrum, where the structure maps are not equivalence, i.e. the data of a functor $\mathsf{S}_*^{\mathrm{fin}} \to \mathsf{S}_*$ which reduced, but not necessarily excisive. Furthermore we may realize Ω^∞ as the composite

$$\mathsf{Sp} \longrightarrow P\mathsf{Sp} \xrightarrow{S^0} \mathsf{S}_*$$

where S^0 is evaluation at S^0, and the first is a forgetful functor. Analogously we may also realize Σ^∞ as the composite

$$\mathsf{S}_* \xrightarrow{I} P\mathsf{Sp} \xrightarrow{\gamma} \mathsf{Sp}$$

where I is iterated suspensions, i.e. $A \mapsto \{\Sigma^k A\}_{k \in \mathbb{N}}$, and the second is the localization. Hence we are reduced to showing that the localization of a prespectrum $X = \{X_n\}_{n\in\mathbb{N}}$ with structure maps $\{X_n \to \Omega X_{n+1}\}_{n\in\mathbb{N}}$, is given as $\mathrm{hocolim}_k\, \Omega^k X$. This follows from the fact that there exists an endofunctor T on $P\mathsf{Sp}$ given as $X \mapsto \{\Omega X_{n+1}\}_{n\in\mathbb{N}}$, with a natural transformation $\mathrm{id} \to T$. Note from 7.2.5 T is an equivalence if X is a spectrum, this together with the naturality, shows the map $X \to T(X)$ is a local equivalence for the localization $\gamma : P\mathsf{Sp} \to \mathsf{Sp}$. Hence $X \to \mathrm{hocolim}_k\, T^k(X) = \mathrm{hocolim}_k\, \Omega^k X$ is a local equivalence, and it can be shown that the homotopy colimit is a spectrum, which shows it is the localization of X. From this we see that $\mathbb{S}^\infty(A) = \Omega^\infty\Sigma^\infty(A) = \mathrm{hocolim}_k\, \Omega^k\Sigma^k A$, which concludes the proof. □

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