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Algebraic K-Theory of Permutative Categories

Project outside course scope
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Contents

Introduction

The main goal with this project is to develop a model for algebraic K-theory for permutative categories using the infinite loop space machinery developed by G. Segal in [1]. Another major goal is to relate this notion of K-theory to the usual K-theory developed by D. Quillen in [2].

The infinite loop space machinery relies on a generalization of monoids called Γ-spaces and their associated spectra. We will therefore develop the theory for both Γ-spaces and spectra, although both are "bare-bone" introductions.

Any model for K-theory should intrinsically be a homotopical notion, therefore we also develop the homotopy theory for Γ-space, i.e. we give model structures on the category of Γ-spaces.

We begin by recalling some classical definitions and ideas from algebraic K-theory. For a detailed account see fx. J. Rosenberg [3].

Definition ($K_0(R)$) *Let R be a ring, and denote by iR the set of isomorphism classes of finitely generated projective R-modules. iR is an abelian monoid under direct sum. Then the 0'th K-theory group of R is defined to be the Grothendieck group of iR, i.e.*

$$K_0(R) = \mathbb{Z}[iR]/\sim \quad \textit{where } [P] + [Q] \sim [P \oplus Q].$$

Where $\mathbb{Z}[iR]$ is the free abelian group generated by the elements of iR.

Considering this construction, and the ad hoc definitions of K_1 and Milnor's K_2, the idea that spectra should play an integral role seems rather ex nihilo. This magnificent insight is due to Quillen.

Definition (The K-theory spectrum of R) *Given a commutative ring R, there is a spectrum $K(R)$ which arises from the infinite loop space $B\,\mathrm{GL}(R)^+$, i.e. $B\,\mathrm{GL}(R)^+$ is weakly equivalent to 0'th space of a Ω-spectrum. Here the "+" is Quillens plus-construction. The higher algebraic K-theory groups of R are the homotopy groups of the infinite loop space $B\,\mathrm{GL}(R)^+$, i.e. $\pi_n(B\,\mathrm{GL}(R)^+)$ for $n > 0$.*

The K-theory spectrum which we develop will contain the K-theory of a commutative ring through realizing iR as a permutative category denoted $\mathbb{P}_R$. The machinery which we develop will show that $B\,\mathrm{GL}(R)^+$ is an infinite loop space.

Chapter 1

Γ-spaces and their associated spectra

In this chapter we introduce spectra and Γ-spaces, and we will show how to obtain one from the other. Furthermore we motivate spectra by explaining their relation to the more familiar concept of cohomology theories. This chapter is based on J.F. Adams' [4], E. Browns [5], A.K. Bousfield & E.M. Friedlanders [6], and G. Segal [1].

1.1 Spectra and cohomology theories

We will see that cohomology theories are represented by spectra through the classical result due to E. Brown often called Browns representability theorem. Whenever we write space we will mean a topological space which is compactly generated, Hausdorff, and pointed. Furthermore continuous maps will also be pointed. Let S^1 be the simplicial circle $\Delta^1/\partial\Delta^1$, and let S^n be the n-fold smash product of S^1.

Definition 1.1.1. A *spectrum*, A, is a sequence of pointed simplicial sets $\{A_n\}_{n\in\mathbb{N}}$ and morphisms $\sigma_n : S^1 \wedge A_n \to A_{n+1}$, which we will call structure maps. A morphism of spectra $f : A \to B$ is a sequence of maps $(f_i)_{n\in\mathbb{N}}$ such that the following diagram commutes,

$$\begin{array}{ccc} S^1 \wedge A_n & \xrightarrow{S^1\wedge f_n} & S^1 \wedge B_n \\ \big\downarrow{\scriptstyle \sigma_n^A} & & \big\downarrow{\scriptstyle \sigma_n^B} \\ A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1}. \end{array}$$

The collection of spectra and morphisms of spectra assemble into a category, which we will denote Sp.

Note that there are many different models for spectra, e.g. taking spaces instead of simplicial sets. We shall occasionally consider this model, mainly for certain motivating examples, which are more easily accessed using this model. We will write $\mathsf{Sp(Top)}$ for the category of spectra in topological spaces.

Definition 1.1.2. Given a $A \in \mathsf{Sp}$ with structure maps $\sigma_n : \Sigma A_n \to A_{n+1}$, then for $k \in \mathbb{Z}$ the *n'th homotopy group of* A, is $\pi_n(A) = \operatorname{colim}_k \pi_{n+k}(A_k)$. The colimit is formed along the maps

$$\pi_{n+k}(A_k) \xrightarrow{\Sigma(-)} \pi_{n+k+1}(\Sigma(A_k)) \xrightarrow{\pi_{n+k+1}(\sigma_k)} \pi_{n+k+1}(A_{k+1})$$

We will say that a map of spectra is an *equivalence* if it induces isomorphisms on homotopy groups.

As already mentioned cohomology theories are represented by spectra in the following sense.

Proposition 1.1.3. *Given a spectrum A, write $A^n(-) : \mathsf{sSet}^{op} \to \mathsf{Ab}$ for the functor $X \mapsto [X, A_n]$. Here sSet^{op} is the opposite category of sSet. This functor satisfies the Eilenberg-Steenrod axioms, hence it defines a cohomology theory.*

Proof. We verify the axioms. We start with the suspension isomorphism, which is a consequence of the suspension loop space adjunction $\Sigma \dashv \Omega$.

$$A^n(X) = [X, A_n] \simeq [X, \Omega A_{n+1}] \simeq [\Sigma X, A_{n+1}] = A^{n+1}(\Sigma X).$$

Next we verify the wedge isomorphism, which is a consequence of the mapping objects turning coproducts into products

$$A^n(\bigvee_{i\in I} X_i) = [\bigvee_{i\in I} X_i, A_n] \simeq \prod_{i\in I}[X_i, A_n] = \prod_{i\in I} A^n(X_i).$$

Homotopy invariance is clear. Let

$$Y \xrightarrow{i} X \xrightarrow{p} C,$$

be a cofiber sequence. We expand it to a homotopy pushout square, and obtain the following diagram of homotopy pushout squares,

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d] & X \arrow[r] \arrow[d, "p"] & \bullet \arrow[d] & & \\
\bullet \arrow[r] & C \arrow[r, "\eta"] \arrow[d] & \Sigma Y \arrow[r] \arrow[d, "-\Sigma i"] & \bullet \arrow[d] & \\
 & \bullet \arrow[r] & \Sigma X \arrow[r, "-\Sigma p"] \arrow[d] & \Sigma C \arrow[r] \arrow[d, "-\Sigma\eta"] & \dots \\
 & & \vdots & \vdots & \ddots
\end{tikzcd}$$

Which gives rise to a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & A^n(X) \arrow[r] & A^n(Y) \arrow[r] \arrow[d, "\cong"] & A^{n+1}(C) \arrow[r, "p^*"] & A^{n+1}X \arrow[r, "-i^*"] & A^{n+1}(Y) \arrow[r] & \dots \\
 & & A^{n+1}(\Sigma Y) \arrow[ur, "\eta^*"'] & & & &
\end{tikzcd}$$

□

The converse statement, that cohomology theories give rise to spectra, is also true and is called the Brown representability theorem. The following is a slight reformulation of the original result proved in [5].

Theorem 1.1.4. *Let $\mathcal{H}o(\mathsf{Top}^c_\bullet)$ denote the homotopy category of connected pointed topological spaces. A functor $F : \mathcal{H}o(\mathsf{Top}^c_\bullet)^{op} \to \mathsf{Set}_*$ is representable precisely if*

- *F takes coproducts to products,*
- *F takes weak pushouts to weak pullbacks.*

Since we chose simplicial sets as our model, the classical result does not apply, we therefore use a refinment due to J. Jardine [7].
Note that sSet_* equipped with Quillen-Kan model structure satisfies the conditions of Brown representability in the sense of [7]. Let $X \in \mathsf{sSet}_*$, and $H^n(X, R) \coloneqq H^n(Z(X), R)$, where $Z(X)$ is chain complex associated to the simplicial abelian group $\mathbb{Z}[X]$ which n'th level is the free abelian group generated by the n-simplicies of X. The theorem applies, and it secures the existence of a simplicial set, E_n, and a natural equivalences $[X, E_n] \cong [X, \Omega E_{n+1}]$, where Ω is adjoint to Σ, which are induced from weak equivalences $E_n \to \Omega E_{n+1}$, hence $\{E_n\}_{n\in\mathbb{N}}$ assemble into a spectrum.

We will now give some examples of spectra.

Example 1.1.5 (The suspension spectrum). Given a pointed simplicial set X, the suspension spectrum $\Sigma^\infty X$ is the sequence of pointed simplicial sets $\{\Sigma^n X\}_{n\in\mathbb{N}}$ with natural isomorphisms $\sigma : \Sigma(\Sigma^n X) \to \Sigma^{n+1}X$ as the structure maps. Thus for each pointed simplicial set X we have a spectrum $\Sigma^\infty X$. This assignment is functorial: Let $\phi : X \to Y$ be a morphism of pointed simplicial set, then we define $(\Sigma^\infty\phi)_n := \Sigma^n\phi = \phi \wedge \mathrm{id}_{S^n} : X \wedge S^n \to Y \wedge S^n$, which assemble to a morphism of spectra $\Sigma^\infty\phi : \Sigma^\infty X \to \Sigma^\infty Y$. Hence we have a functor $\Sigma^\infty : \mathsf{sSet}_* \to \mathsf{Sp}$.

There is a functor $\Omega^\infty : \mathsf{Sp} \to \mathsf{sSet}_*$. Let $X \in \mathsf{Sp}$, then $\Omega^\infty(X)$ is given by $\mathrm{colim}_n\, \Omega^n X_n$. It can be shown that Σ^∞ and Ω^∞ form an adjunction $(\Sigma^\infty, \Omega^\infty)$.

Example 1.1.6 (Sphere spectrum). The suspension spectrum of the 0-sphere S^0, is denoted $\mathbb{S}$. The natural isomorphisms $S^1 \wedge S^n \to S^{n+1}$ are the structure maps.

Example 1.1.7 (Eilenberg-Mac Lane spectrum). For this particular example consider Top as the model for spectra. Let $A \in \mathsf{Ab}$. The Eilenberg-Mac Lane spectrum is defined as the sequence $\{K(A,n)\}_{n\in\mathbb{N}}$ of Eilenberg-Mac Lane spaces, hence the name. The structure maps are given as the adjoint maps to the homotopy equivalence $\sigma : K(A, n-1) \to \Omega K(A,n)$. It is denoted HA. An element in $(HA)_n$ can be viewed as a formal A-linear combination of points in S^n, in the sense of Dold-Thom [8]. Due to this viewpoint it is easy to see that the assignment of an Eilenberg-Mac Lane spectrum to an abelian group is functorial: given $\phi : A \to B$, we obtain $(H\phi)_n : (HA)_n \to (HB)_n$, given by $(H\phi)_n(\sum a_i x_i) = \sum \phi(a_i)x_i$. This defines a morphism of spectra $H\phi : HA \to HB$. Hence we obtain a functor $H : \mathsf{Ab} \to \mathsf{Sp}(\mathsf{Top})$.

Definition 1.1.8. A Ω-*spectrum* E is a spectrum, such that the adjoint maps of the structure maps $\tilde{\sigma_n} : E_n \to \Omega E_{n+1}$ is a weak equivalence.

Example 1.1.9. The Eilenberg-Mac Lane spectrum is an example of an Ω-spectrum, because homotopy equivalences are the weak equivalences in the classic model structure on topological space.

1.2 Γ-spaces

The usual K_0-group of a ring R, is constructed from an abelian monoid, which is group completed by the Grothendieck construction. The analog of the monoid in our setting is that of a Γ-space. In this section we will define Γ-spaces, see how they are generalizations of monoids, and consider a long list of examples. All of the examples will play a role later in the text.

Definition 1.2.1 (Segals Category). Let Fin_* denote the category of finite pointed sets. *Segals category* is defined as

$$\Gamma^{op} := i\mathsf{Fin}_*$$

Where $i\mathsf{Fin}_*$ is isormorphism classes of finite pointed sets.

We denote by $n_+ := \{\bullet, 1, 2, ..., n\}$ the representative for the isomorphism class of sets with n non-basepoint elements, where $0_+ := \{\bullet\}$. A map $m_+ \to n_+$ in Γ^{op} can be specified by giving preimages of non-basepoint elements. Hence it is convenient to employ the following notation, for $0 < i \leq n$ let $p_i : n_+ \to 1_+$ be the map given by $p^{-1}(\{1\}) = \{i\}$.

Definition 1.2.2 (Γ-object). Let $\mathscr{C}$ be a pointed category, with initial/terminal object $\bullet$. A Γ-*object* is a $\mathscr{C}$- valued presheaf on Γ. A Γ-space is in explicit terms a functor $A : \Gamma^{op} \to \mathsf{sSet}_*$. These evidently assemble into a category, which we will denote it by $\Gamma(\mathsf{sSet}_*)$.

The following example shows that Γ-spaces are generalizations of abelian monoids.

Example 1.2.3. Every abelian monoid gives a Γ-space. Consider M an abelian monoid. It defines a Γ-space M with

$$M(m_+) = \prod_{i \in m_+ \setminus \{\bullet\}} M$$

for $m_+ \in \Gamma^{op}$ and $n \geq 0$.

Example 1.2.4. Γ-spaces are special cases of simplicial sets. Consider the following functor from $\Delta \to \Gamma$ given by $[m] \mapsto m_+$ and $f : [m] \to [n]$ to $\theta : m_+ \to n_+$ where $\theta(i) = \{j \in n_+ | f(i-1) < j \leq f(i)\}$. Hence the (co)limits in $\Gamma(\mathsf{sSet}_*)$ are calculated levelwise, just as in sSet_*.

Example 1.2.5. [The Γ-space $\mathbb{S}$] There is an inclusion of Γ^{op} into sSet_* by sending $n_+ \mapsto \Delta^n$, we call this inclusion the sphere spectrum Γ-space, and denote it $\mathbb{S}$.

Example 1.2.6 (Eilenberg-Mac Lane objects)**.** Consider $A \in \mathsf{Ab}$ and think of it as a pointed set $(0 = \bullet)$. We can associate to A a Γ-set HA (Γ-object in Set) in the following way,

$$n_+ \mapsto HA(n_+) = A \otimes \mathbb{Z}[n_+ \setminus \{\bullet\}] \cong A \otimes (\mathbb{Z} \oplus ... \oplus \mathbb{Z}) \cong A^{\oplus n}.$$

Let $\phi : n_+ \to m_+$ be a morphism in Γ^{op}, then we obtain a homomorphism $HA(\phi) : HA(n_+) \to HA(m_+)$, given by

$$(a_1, ..., a_n) \mapsto \Big(\Big(\sum_{j \in \phi^{-1}(1)} a_j \Big), ..., \Big(\sum_{j \in \phi^{-1}(m)} a_j \Big) \Big)$$

where $m_0 = \bullet$. HA will be referred to as Eilenberg-Mac Lane objects associated to A, because we shall later relate spectra and Γ-objects, and under this relation these objects give rise to the Eilenberg-Mac Lane spectra.

We will now give some examples of how to form new Γ-spaces from old. These will turn out to play an integral role when we define the strict model structure on $\Gamma(\mathsf{sSet}_*)$.

Example 1.2.7. [n-truncation] Let $i_n : \Gamma_n \to \Gamma$ denote the inclusion of the full subcategory of Γ with no more than n non-basepoint elements. We define the n-truncation functor $T_n : \Gamma(\mathsf{sSet}_*) \to \Gamma_n(\mathsf{sSet}_*)$, where $\Gamma_n(\mathsf{sSet}_*)$ is defined analogously to $\Gamma(\mathsf{sSet}_*)$. T_n is defined by sending $A : \Gamma \to \mathsf{sSet}_*$ to $A \circ i_n : \Gamma_n \to \mathsf{sSet}_*$.

Example 1.2.8. [Skeleton of Γ-space] Consider the functor

$$\mathrm{sk}_n : \Gamma_n(\mathsf{sSet}_*) \to \Gamma(\mathsf{sSet}_*),$$

which is called the n-skeleton functor, and it is given for $A \in \Gamma_n(\mathsf{sSet}_*)$ by

$$(\mathrm{sk}_n\, A)(m_+) = \mathrm{colim}_{k_+ \to m_+, k \geq n}\, A(k_+).$$

This functor is the left adjoint of T_n.

Example 1.2.9. [Coskeleton of Γ-space] Consider the functor

$$\mathrm{csk}_n : \Gamma_n(\mathsf{sSet}_*) \to \Gamma(\mathsf{sSet}_*),$$

which is called the n-coskeleton functor, and it is given $A \in \Gamma_n(\mathsf{sSet}_*)$ by

$$(\mathrm{csk}_n\, A)(m_+) = \lim_{m_+ \to j_+, j \geq n} A(j_+).$$

This functor is the right adjoint of T_n.

Example 1.2.10. [Γ^{n_+}] For $n_+ \in \Gamma$, we define a Γ-space $\Gamma^{n_+} \in \Gamma(\mathsf{sSet}_*)$, which is given by

$$\Gamma^{n_+}(m_+) = \mathrm{Hom}_{\Gamma^{op}}(n_+, m_+).$$

Note that $\Gamma^{1_+} \cong \mathbb{S}$.

We want Γ-spaces to be a homotopical version of abelian monoids, hence we need a neutral element. The first guess is to set $A(0_+)$ as the unit; for this to fit into a homotopy theoretical setting, $A(0_+)$ should be weakly equivalent to $\bullet$, which in general is not true. So to obtain a useful notion we will have to require a few conditions on our Γ-spaces.

Definition 1.2.11. [Special Γ-space] A Γ-space A is *special* if for each $n \in \mathbb{N}$ the maps

$$\varphi_n := \prod_{i=1}^{n} p_i : A(n_+) \to \prod_{i=1}^{n} A(1_+),$$

are weak equivalences. The maps φ_n are often called Segal maps. Furthermore we require $A(\bullet)$ to be weakly equivalent to $\bullet$. The category of special Γ-spaces is the full subcategory generated by the special Γ-spaces, we denote it $s\Gamma(\mathsf{sSet}_*)$.

Intuitively one should think of $A(1_+)$ as the "underlying space" of a Γ-space A, and $A(n_+)$ as "a model for the n-fold product of A".

There is the following alternative definition of special.

Proposition 1.2.12. *A Γ-space A is special if and only if*

$$(p_n^*, p_m^*) : A(n_+ \vee m_+) \simeq A(n_+) \times A(m_+).$$

Where $p^n : n_+ \vee m_+ \to n_+$ is the map which sends m_+ to the basepoint, and is the identity on n_+.

Proof. An easy induction argument, using $n_+ \vee 1_+ \cong (n+1)_+$, gives the desired result. □

In [1] the definition of Γ-spaces is our notion of special Γ-space. We choose to make this distinction because $\Gamma(\mathsf{sSet}_*)$, as we shall see has the structure of a closed simplicial model category, while $s\Gamma(\mathsf{sSet}_*)$ is not even (co)complete, see [6].

The following proposition solidifies our intuition that special Γ-spaces should be a homotopical version of abelian monoids.

Proposition 1.2.13. *If A is a special Γ-space, then $\pi_0(A(1_+))$ is an abelian monoid.*

Proof. Because π_0 preserve finite products, we can define the multiplication as

$$\pi_0(A(1_+)) \times \pi_0(A(1_+)) \simeq \pi_0(A(2_+)) \xrightarrow{\mu_*} \pi_0(A(1_+)),$$

where $\mu : 2_+ \to 1_+$ is defined by $\mu^{-1}(\{1\}) = \{1, 2\}$. That $\pi_0(A(1_+))$ is abelian follows by the following diagram

$$\begin{tikzcd} 2_+ \arrow[d, "s"] \arrow[dr, "\mu"] & \\ 2_+ \arrow[r, "\mu"] & 1_+ \end{tikzcd}$$

Where s defines as $1 \mapsto 2$ and $2 \mapsto 1$, which makes the following diagram commutative

$$\begin{tikzcd} \pi_0(A(1_+)) \times \pi_0(A(1_+)) \arrow[r, "\cong"] \arrow[d, "S"] & \pi_0(A(2_+)) \arrow[d, "s_*"] \\ \pi_0(A(1_+)) \times \pi_0(A(1_+)) \arrow[r, "\cong"] & \pi_0(A(2_+)) \end{tikzcd}$$

Where S swaps the factors in the product. Hence $\pi_0(A(1_+))$ is a abelian monoid. $\square$

Definition 1.2.14. A special Γ-space A is called *very special* (or *group-like*) if the above multiplication gives $\pi_0(A(1_+))$ an abelian group structure.

It turns out that there is an equivalent definition. Let $m_n : n_+ \to 1_+$ be given by $m_n^{-1}(\{1\}) = \{1, ..., n\}$ and let $p_n : n_+ \to (n-1)_+$ be given by $p_n^{-1}(\{i\}) = \{i\}$ for $1 \leq i \leq n-1$ and $p_n^{-1}(\{n\}) = \bullet$.

Proposition 1.2.15. *A Γ-space A is very special if and only if for each $n \in \mathbb{N}$ the diagram*

$$\begin{tikzcd} A(n_+) \arrow[r, "m_n"] \arrow[d, "p_n"] & A(1_+) \arrow[d] \\ A((n-1)_+) \arrow[r] & \bullet \end{tikzcd}$$

is a homotopy pullback.

Proof. We are going to proceed by induction on n. The induction start will by far be the most challenging part. Consider the diagram when $n = 2$, then we need to show that A is very special if and only if the map

$$(p_1, m_2) : A(2_+) \to A(1_+) \times A(1_+),$$

is a weak equivalence. At this point we will use the following fact: If P is a homotopy associative H-space, then P is an H-group, if and only if the shear map $\sigma : P \times P \to P \times P$, which is given by $\sigma(x, y) = (x, xy)$ is a weak equivalence, see [9] 2.4.28 for a proof. Note that the special condition on A ensures that this fact applies to $A(1_+)$. Consider the following diagram in which (p_1, p_2) is a weak equivalence because we assumed that A was special,

$$\begin{tikzcd} A(2_+) \arrow[r, "(p_1,p_2)"] \arrow[dr, "(p_1,m_2)"'] & A(1_+) \times A(1_+) \arrow[d, "\sigma"] \\ & A(1_+) \times A(1_+) \end{tikzcd}$$

From which we see that (p_1, m_2) is a weak equivalence if and only σ is. We have a homotopy inverses if and only if σ is a weak equivalence, i.e. $\pi_0(A(1_+))$ is an abelian group. Hence $\pi_0(A(1_+))$ is an abelian group if and only if σ is a weak equivalence if and only if (p_1, m_2) is which is true if and only if the diagram is a homotopy pullback for $n = 2$. $n > 2$ follows easily by induction. $\square$

1.3 The spectrum associated to a Γ-space

In this section we explain how one can associate a spectrum to a Γ-space A. We do this by first prolonging A to a functor $\mathsf{sSet}_* \to \mathsf{sSet}_*$ and then to a functor $\mathsf{Sp} \to \mathsf{Sp}$.

As mentioned it turns out that the category of spectra and the category of Γ-spaces are Quillen equivalent. To make sense of this we must equip both categories with model structures, and construct an adjunction. The above prolongation will be one of the functors in this adjunction. Before undergoing the mayor task of defining the model structures, and thereby formulating their homotopy theories, we construct the adjunction. We will not show that it constitutes a Quillen equivalence.

Lemma 1.3.1. *Let A be a Γ-space. Then it induces a functor $A : \mathsf{sSet}_* \to \mathsf{sSet}_*$. Given a Γ-space A, we will abuse notation and denote the induced endofunctor on sSet_* as A.*

Proof. Lets first extend A from Γ to Fin_*. For each pointed set S of cardinality $n+1$ choose a pointed isomorphism $\alpha_S : S \to n_+$, and then set $A(S) = A(n_+)$, and if $f : S \to T$ is a function of pointed finite sets let $A(f) = A(\alpha_t f \alpha_S^{-1})$. Hence $A : \mathsf{Fin}_* \to \mathsf{sSet}_*$.

Now given a pointed set X view it as a poset over its finite subsets, and form the filtered colimit indexed over the subsets, and set

$$A(X) \coloneqq \mathrm{colim}_{Y \subseteq X} A(Y).$$

Note that for this to be well-defined we require a axiom of choice, since each colimit has to be chosen, and not just remain a representative of the isomorphism class of choices. This extends A to a functor $A : \mathsf{Set}_* \to \mathsf{sSet}_*$ in the following way.

For each $X \in \mathsf{sSet}_*$, using A we can produce a pointed bisimplicial set from it, given by $([n],[m]) \mapsto A(X_n)_m$. We define

$$A(X) = \{[n] \mapsto A(X_n)_n\}.$$

Which extends A to a functor $A : \mathsf{sSet}_* \to \mathsf{sSet}_*$. ☐

Remark 1.3.2. Let K be a finite based simplicial set, we can view it as a functor $K : \Delta^{op} \to \Gamma$. Let X be a Γ-space. Using an prolongation as in 1.3.1, one can show that $X(|K|)$ defined as the coend

$$X(|K|) = \int^{n_+ \in \Gamma^{op}} F(n_+, |K|) \wedge X(n_+)$$

where $F(-,-)$ is based mapping space, is homeomorphic to the geometric realization of the simplicial space $[k] \mapsto X(K_k)$. We will use this fact in the proof for many of our results, e.g 3.2.1.

Example 1.3.3. Consider $C \in \Gamma_{n-1}(\mathsf{sSet}_*)$, as defined in 1.2.7. Now there is a canonical map between the $n-1$-skeleton and $n-1$-coskeleton, because of them being adjoint pairs,

$$(\mathrm{sk}_{n-1} C)(n_+) \to (\mathrm{csk}_{n-1} C)(n_+).$$

Now as seen in 1.2.4 and the above it is natural to view Γ-spaces as sSet_*, so lets consider the factorization coming from a model structure on sSet_*,

$$(\mathrm{sk}_{n-1} C)(n_+) \to K \to (\mathrm{csk}_{n-1} C)(n_+). \tag{1.1}$$

We can prolong C to an object C_n in $\Gamma_n(\mathsf{sSet}_*)$ by setting $C_n(n_+) = K$. The factorization 1.1 agrees with the one obtained via the same procedure applied to C_n. This fact will let us construct factorizations inductively on the n-truncations of a Γ-space.

Another formulation of the prolongation defined in 1.3.1 is in terms of left Kan extension, which extends a Γ-space A to an endofunctor on sSet_* in a single swoop. Consider the inclusion of $i : \Gamma^{op} \to \mathsf{sSet}_*$, and a given Γ-space A, and form the left Kan extension of A along i

$$\begin{tikzcd} \Gamma^{op} \arrow[r, "A"] \arrow[d, "i"] & \mathsf{sSet}_* \\ \mathsf{sSet}_* \arrow[ur, dashed, "\mathrm{Lan}_i A"'] & \end{tikzcd}$$

Where i is the inclusion constructed via the inclusion described in 1.2.5. Yet another way is that of the following coend $A : \mathsf{sSet}_* \to \mathsf{sSet}_*$,

$$A(X) = \int^{k_+ \in \Gamma^{op}} \prod^{k} X \wedge A(k_+).$$

This construction is not so enlightning, but it is very useful.

Lemma 1.3.4. *Given a* Γ*-space* A*, its induced endofunctor on* sSet_* *induces an endofunctor on* Sp*.*

Proof. Given $X, Y \in \mathsf{sSet}_*$. The endofunctor A is obtain via lemma 1.3.1. Consider the map

$$\begin{aligned} X &\to F(A(Y), A(X \wedge Y)) \\ x &\mapsto A(Y \to X \wedge Y), \end{aligned}$$

where $Y \to X \wedge Y$ is defined as $y \mapsto x \wedge y$, which is a map $A(Y) \to A(X \wedge Y)$. Consider the adjoint of this map,

$$X \wedge A(Y) \to A(X \wedge Y)$$

Now for a spectrum P, with n'th level P_n we define $A(P) \in \mathsf{Sp}$ as $(A(P))_n = A(P_n)$ equipped with the structural maps constructed via the above simplicial map

$$S^1 \wedge A(P_n) \to A(S^1 \wedge P_n) \to A(P_{n+1}).$$

This extends the endofunctor A on sSet_* to an endofunctor on Sp. □

Theorem 1.3.5. *A* Γ*-space* A *determines a spectrum, denoted* $A(\mathbb{S})$*. Hence we have a functor* $(-)(\mathbb{S}) : \Gamma(\mathsf{sSet}_*) \to \mathsf{Sp}$*.*

Proof. Lemma 1.3.4 A induces an endofunctor on Sp, apply this to $\mathbb{S}$, to obtain the desired spectrum. □

Other than being part of a Quillen adjunction this functor has many interesting consequences, for one via Brown representability, we have that Γ-spaces give cohomology theories. We can also defined the associated (reduced) homology theory: let $K \in \mathsf{sSet}_*$,

$$\tilde{H}_*(K, A) := \pi_*(A(\mathbb{S})) \wedge K.$$

At this point we have constructed one of the functors for our adjoint pair, the other one is significantly easier to define.

Definition 1.3.6. Given $X, Y \in \mathsf{Sp}$, and $n_+ \in \Gamma^{op}$ consider

$$\Phi(X, Y)(n_+) = \mathrm{Hom}_{\mathsf{Sp}}(\prod^n X, Y)$$

$\Phi(X, Y)$ is a Γ-space.

The following is an elaboration of the proof given in [6].

Lemma 1.3.7. *Let* A *be a* Γ*-space. There is an isomorphism in* Sp*, given as*

$$A(X) \cong \coprod_{n \geq 0} \Big(\prod^n X \Big) \wedge A(n_+) / \sim$$

Where $\sim$ *is given by relating* $\phi_*(x)$ *and* $\phi^*(x)$ *where* $\phi : m_+ \to n_+$ *in* Γ^{op} *and* $x \in (\prod^n X) \wedge A(m_+)$*, via*

$$(\textstyle\prod^m X) \wedge A(m_+) \xleftarrow[\phi^*]{} (\textstyle\prod^n X) \wedge A(m_+) \xrightarrow{\phi_*} (\textstyle\prod^n X) \wedge A(n_+).$$

Proof. Consider a Γ-space A, via the coend construction discussed earlier we may prolong A to a functor $\mathsf{sSet}_* \to \mathsf{sSet}_*$, furthermore per. definition of the coend we have an isomorphism of simplicial sets for each $S \in \mathsf{sSet}_*$,

$$A(S) \cong \int^{k_+ \in \Gamma^{op}} \prod^k S \wedge A(k_+) \cong \coprod_{n \geq 0} \Big(\prod^k S \Big) \wedge A(k_+) / \sim$$

Where $\sim$ is a relation analogous to that given in the lemma, just for simplicial sets. Now prolonging A to a functor $\mathsf{Sp} \to \mathsf{Sp}$, gives the desired isomorphism. □

Theorem 1.3.8. *For $X, Y \in \mathsf{Sp}$ and $A \in \Gamma(\mathsf{sSet}_*)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Sp}}(A(X), Y) \cong \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(A, \Phi(X, Y))$$

Proof. Consider A, X, Y as in the theorem. Let $n_+ \in \Gamma^{op}$, and let $\eta_{n_+} \in \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(A(n_+), \Phi(X,Y)(n_+))$ be the n_+-component of a natural tranformation $\eta \in \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(A, \Phi(X,Y))$. Note that if we can show η corresponds to a map $A(X) \to Y$ in Sp, we are done. Let $\phi : n_+ \to m_+$, then we have naturality squares for η, because maps of $\Gamma(\mathsf{sSet}_*)$ are natural transformations. Furthermore we have the corresponding naturality square, under the $\wedge$-Hom adjunction:

$$\begin{array}{ccc} A(n_+) & \xrightarrow{\eta_{n_+}} & \mathrm{Hom}(\prod^n X, Y) \\ \downarrow & & \downarrow \\ A(m_+) & \xrightarrow{\eta_{m_+}} & \mathrm{Hom}(\prod^m X, Y), \end{array} \qquad \begin{array}{ccc} A(n_+) \wedge \prod^n X & \xrightarrow{\overline{\eta_{n_+}}} & Y \\ \downarrow & & \downarrow \mathrm{id} \\ A(m_+) \wedge \prod^m X & \xrightarrow{\overline{\eta_{m_+}}} & Y. \end{array}$$

Note that the square to the right, incodes that the maps $\overline{\eta} : A(-) \wedge \prod^{(-)} X \to Y$ is $\sim_W$-invariant, hence via the universal property of the quotient we obtain a unique map

$$\coprod_{n \geq 0} \Big(\prod^n X \Big) \wedge A(n_+) / \sim_W \to Y$$

which under the isomorphism given in 1.3.7, gives a map $A(X) \to Y$, hence we obtain the desired natural isomorphism. $\square$

Plugging $\mathbb{S}$ into the above theorem, gives the following corollary which is of great interest to us, since it is the key-result in relating Γ-spaces and spectra.

Corollary 1.3.9. *There is an adjunction*

$$(-)(\mathbb{S}) : \Gamma(\mathsf{sSet}_*) \to \mathsf{Sp},$$
$$\Phi(\mathbb{S}, -) : \mathsf{Sp} \to \Gamma(\mathsf{sSet}_*).$$

Chapter 2

Model structures on the category of Γ-spaces

We will now develop a model structure for the category of Γ-spaces. The model structure we will construct is called stable model structure. We will not prove every detail during this endeavor, but will provide references or sketch of proofs when we do not. Our model structure is going to be a Bousfield localization of $\Gamma(\mathsf{sSet}_*)$ equipped with another model structure called the strict model structure, we consider this one first.

2.1 The strict model structure

Definition 2.1.1. A map of $\Gamma(\mathsf{sSet}_*)$ $f : A \to B$ is called a

- *Strict weak equivalence* if $f(n_+) : A(n_+) \to B(n_+)$ is weak equivalence in sSet_* for $n \geq 1$.
- *Strict cofibration* if the induced map

$$(\mathrm{sk}_{n-1} B)(n_+) \coprod_{(\mathrm{sk}_{n-1} A)(n_+)} A(n_+) \to B(n_+)$$

  is injective and Σ_n acts freely on the simplices of $B(n_+)$ not in the image. sk was defined in 1.2.8.

- *Strict fibration* if the induced map

$$A(n_+) \to (\mathrm{csk}_{n-1} A)(n_+) \prod_{(csk_{n-1} B)(n_+)} B(n_+),$$

  is a fibration in sSet_*. csk was defined in 1.2.9.

These classes of maps constitute a model structure. We are not going to give all details in the proof, see [6] for the remaining details .

Theorem 2.1.2. *The category of Γ-spaces becomes a proper closed simplicial model category, when equipped with the strict weak equivalences, strict cofibrations and strict fibrations. This is called the strict model structure.*

Proof. Because (co)limits and weak equivalences are defined levelwise, and thus defined in sSet_*, which is bicomplete, and has the 2-out-of-3-property for weak equivalences, it is elementary to show that this is also true for $\Gamma(\mathsf{sSet}_*)$.
The fact that if f is a retract of g and g is a strict weak equivalence, strict fibration, or strict cofibration, then so is f, follows from it being true in a certain subcategory of sSet_*, denoted $\Sigma_n(\mathsf{sSet}_*)$, see [6] proposition 3.3, equipped with a closed model structure very akin to the one developed by Quillen [10] in II.4. Consider the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a strict cofibration and p a strict fibration, then we wish to show that the filler map $B \to X$ exists if either i or p is a strict weak equivalence. The two cases are very similiar, so we will omit one half. Let i be a strict trivial cofibration. For each $n \geq 0$ we apply the n-truncation, as defined in 1.2.7, to the diagram above

$$\begin{tikzcd} T_n(A) \arrow[r] \arrow[d, "T_n(i)"] & T_n(X) \arrow[d, "T_n(p)"] \\ T_n(B) \arrow[r] \arrow[ur, dashed] & T_n(Y) \end{tikzcd}$$

We can construct each of these truncated fillers $u_n : T_n(B) \to T_n(X)$, and then inductively construct the filler $u : B \to X$. This a delicate (and technical) procedure and is, in many ways, the main difficulty in the proof.

We only need to show that every map $f : A \to B$ in $\Gamma(\mathsf{sSet}_*)$ can be factored $f = p \circ i$ where i is a strict cofibration and p is a strict fibration, and one of i or p is trivial. Assume i is trivial.

Suppose inductively that we have a factorization for the $n-1$-truncation,

$$T_{n-1}(A) \to T_{n-1}(C) \to T_{n-1}(B) \in \Gamma(\mathsf{sSet}_*),$$

for some $n \geq 1$. Now using the before mentioned closed model category structure on $\Sigma_n(\mathsf{sSet}_*)$ one may obtain a factorization of canonical map between the skeleton and the coskeleton, in $\Sigma_n(\mathsf{sSet}_*)$ given as

$$(\mathrm{sk}_{n-1} C)(n_+) \coprod_{(\mathrm{sk}_{n-1} A(n_+))} \to K \to (\mathrm{csk}_{n-1} C)(n_+) \prod_{\mathrm{csk}_{n-1} B)(n_+)} B(n_+).$$

The desired factorization $A \to C \to B$ is obtained using an inductive procedure and 1.3.3.

We omit the proof that the closed model structure, is also simplicial, as it is a elementary consequence of the closed model category properties of sSet_*. $\square$

2.2 The stable model structure

We now have the model structure which we are going to localize, lets define the weak equivalence and fibrations of our stable model structure.

Definition 2.2.1. A map of $\Gamma(\mathsf{sSet}_*)$ $f : A \to B$, is called a

- *Stable weak equivalence* if $f_* : \pi_*(A(\mathbb{S})) \to \pi_*(B(\mathbb{S}))$ is an isomorphism.
- *Stable fibration* if it has the right lifting property for the strict trivial cofibrations.

Example 2.2.2. We give here an elementary example of a stable equivalence. Consider the Γ-space constructed from an abelian monoid in example 1.2.3. Let $\tilde{M}$ denote the universal abelian group generated by M, we note that the Γ-space map $M \to \tilde{M}$ is a stable equivalence, because we have isomorphisms $\pi_*(M(S^n)) \to \pi_*(\tilde{M}(S^n))$ for $n \geq 1$, this fact is shown in [11] Cor. 5.7.

Our main goal is to prove the following theorem, which is theorem 5.2 of [6].

Theorem 2.2.3. *The category of Γ-spaces has the structure of a closed simplicial model category, when equipped with the stable weak equivalences, the strict cofibrations and the stable fibrations. This model structure is called the stable model structure.*

Before we do this lets first remind ourselves of the notion of a Bousfield localization.

Theorem 2.2.4. *Let $\mathscr{C}$ be a proper model category with a functor $T : \mathscr{C} \to \mathscr{C}$ and a natural transformation $\eta : 1 \to T$ such that the following axioms hold*

(1) If $f : X \to Y$ is a weak equivalence, then so is $Tf : TX \to TY$.

(2) For each $X \in \mathscr{C}$, the maps $\eta_{TX}, T_{\eta_X} : TX \to TTX$ are weak equivalences.

(3) Consider a pullback square in $\mathscr{C}$

$$\begin{array}{ccc} V & \xrightarrow{k} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{h} & Y \end{array}$$

If f is a fibration between fibrant objects such that $\eta_X : X \to TX$, $\eta_Y : Y \to TY$, and $Th : TW \to TY$ are weak equivalences, then $Tk : TV \to TX$ is a weak equivalence.

Then the following notions define a proper model structure on $\mathscr{C}$: a morphism is a T-cofibration if and only if it is a cofibration, a T-equivalence if and only if $Tf : TX \to TY$ is a weak equivalence, and T-fibration if and only if f is a fibration and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tf} \\ Y & \xrightarrow{\eta_X} & TY \end{array}$$

is homotopy cartesian.

Before we move on to the proof of 2.2.3, we need to construct the endofunctor T. We will need an intermediate construction which in fact is a localization functor, but in the framework of symmetric spectra. Note that $A(\mathbb{S})$ is naturally a symmetric spectrum via the action of Σ_n on S^n. This construction uses the stable model structure defined in [6]. Let $Q : \mathsf{Sp}^\Sigma \to \mathsf{Sp}^\Sigma$, where Sp^Σ is the category of symmetric spectra, and $\eta : 1 \to Q$ be such that for each spectrum X, $\eta_X : X \to QX$ is a stable weak equivalence of spectra and QX is Ω-spectrum, i.e. a fibrant object in the stable model structure for spectra. A specific model for Q is given as

$$(QX)^n = \lim_{i \to \infty} \operatorname{Sing} \Omega^i |X^{n+1}|$$

Where Sing is the singular chains functor $\mathsf{Top} \to \mathsf{sSet}_*$. For details see [12] Lemma 2.1.3. An alternative construction which is very well described can be found in [13] p. 79.

Finally lets define $T : \Gamma(\mathsf{sSet}_*) \to \Gamma(\mathsf{sSet}_*)$ as $T(A) = \Phi(\mathbb{S}, Q(A(\mathbb{S})))$, where Φ and A are defined in 1.3.5 and 1.3.6. Let $\eta : 1 \to T$ be the canonical transformation.

Now that we have the localization functor T one would think that we are ready to prove 2.2.3, but unfortunately the following theorem is true.

Lemma 2.2.5. *Axiom* (3) *of 2.2.4 does not hold for* $\Gamma(\mathsf{sSet}_*)$ *equipped with the strict model structure.*

Proof. Consider the abelian monoid given by

$$M = \{n \in \mathbb{Z} | n \geq 0\} \cup \{\mathbb{O}\}$$

with the usual addition for the non-negative integers and with $\mathbb{O} + \mathbb{O} = 0$, $\mathbb{O} + 0 = \mathbb{O}$, and $\mathbb{O} + n = n$ for $n \geq 1$. Note that the universal abelian group generated by M, $\tilde{M}$ is $\mathbb{Z}$. Now consider $D = \{0, \mathbb{O}\} \subset M$. Consider the pullback square in $\Gamma(\mathsf{sSet}_*)$ where we consider each abelian monoid as a Γ-space via 1.2.3,

$$\begin{array}{ccc} D & \longrightarrow & 0 \\ \downarrow{\scriptstyle\subseteq} & & \downarrow \\ M & \longrightarrow & \mathbb{Z} \end{array}$$

The inclusion $D \subseteq M$ is a stable fibration. From 2.2.2 $M \to \mathbb{Z}$ is a stable equivalence, but $D \to 0$ is not, hence this shows that axiom (3) fails. □

Because of this very unfortunate fact we need to do some more work to employ 2.2.4. The following lemma will remedy this problem.

Lemma 2.2.6. *For a pullback square in* $\Gamma(\mathsf{sSet}_*)$,

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{k} & Y \end{array}$$

suppose j is a strict fibration with X and Y very special and that

$$\pi_0(X(1_+)) \cong \pi_0(X(\mathbb{S})) \xrightarrow{j_*} \pi_0(Y(\mathbb{S})) \cong \pi_0(Y(1_+))$$

is surjective. If k is a stable equivalence, then so is h.

We will not prove this lemma, the details can be seen in [6] B.3 and 4.3.

Sketch of proof of 2.2.3. We will prove this fact by applying 2.2.4 to $\Gamma(\mathsf{sSet}_*)$ with the strict model structure described in 2.1.2. Again we will implicitly use the stable model structure on spectra defined in [6], since our definition of T relied on that of Q. It can be shown that for each Γ-space A, $\eta_A : A \to T(A)$ is a stable weak equivalence and $T(A)$ is stricly fibrant and very special. Note that the T-(co)fibrations and T-equivalences of 2.2.4, are the same as the stable fibrations, strict cofibration and stable weak equivalences defined above. Hence we only need to show that the axioms of 2.2.4 are satisfied. The first two axioms hold because Q satisfies them. But as seen in lemma 2.2.5 axiom (3) fails. When we are in this situation it turns out that all the closed model category axioms hold for $\Gamma(\mathsf{sSet}_*)$ with the three classes described above, except the factoring axiom. To be precise we still need to show that a morphism can be factored as a trivial strict cofibration and a stable fibration. This axiom is usually proven using axiom (3), which is not available to use, but we can use 2.2.6 instead. We will not give the details that show we are justified in substituting axiom (3) with 2.2.6, this is described in [6] Lemma 5.4. □

We end this section with a characterisation of the stably fibrant objects of $\Gamma(\mathsf{sSet}_*)$, and give a large class of Γ-spaces which are cofibrant. Furthermore we give a construction which can be thought of as a cofibrant replacement.

Lemma 2.2.7. *Let X be a Γ-space. Then X is stably fibrant if and only if it is very special and the map $X(n_+) \to \bullet$ is a fibration of simplicial sets for all $n_+ \in \Gamma$.*

Remark 2.2.8. In some places in the literature X is called pointwise fibrant when the map $X(n_+) \to \bullet$ is a fibration of simplicial sets for all $n_+ \in \Gamma$. Likewise $X \to Y$ is called a pointwise fibration if $X(n_+) \to Y(n_+)$ is a fibration of simplicial sets, for each $n_+ \in \Gamma$, e.g. [14]. Analogously for cofibration and weak equivalences. The pointwise weak equivalences, and (co)fibrations, constitute a closed simplicial model category for the detail see [15].

Proof of 2.2.7. Assume first that X is stably fibrant. We start by showing that X is pointwise fibrant. $X \to \bullet$ has the lifting property with respect to all maps that are strict trivial cofibrations, hence in particular pointwise cofibrations, i.e X is pointwise fibrant. We show that X is special.

Consider the map $\Gamma^{n_+} \vee \Gamma^{m_+} \to \Gamma^{n_+ \vee m_+} \cong \Gamma^{n_+} \times \Gamma^{m_+}$, it can be shown that it is a cofibration and a homotopy equivalence, hence a stable equivalence. From this it follows that

$$\mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\Gamma^{n_+ \vee m_+}, X) \to \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\Gamma^{n_+} \vee \Gamma^{m_+}, X) \tag{2.1}$$

is a stable equivalence and a stable fibration, hence a pointwise equivalence and a pointwise fibration. We have the following elementary isomorphism

$$X(n_+ \vee m_+) \cong \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\Gamma^{n_+ \vee m_+}, X),$$
$$X(n_+) \times X(m_+) \cong \mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\Gamma^{n_+} \vee \Gamma^{m_+}, X).$$

Hence because (2.1) is an equivalence, X is special. As mentioned in example 1.2.10, $\Gamma^{1_+} \cong \mathbb{S}$. From this fact and similarly to the above, the map

$$\mathbb{S} \vee \mathbb{S} \xrightarrow{i \vee \Delta} \mathbb{S} \times \mathbb{S}$$

is a stable equivalence. Consider the induced map

$$\pi_0(\mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\mathbb{S} \vee \mathbb{S}, X)) \xrightarrow{(i \vee \Delta)_*} \pi_0(\mathrm{Hom}_{\Gamma(\mathsf{sSet}_*)}(\mathbb{S} \times \mathbb{S}, X))$$

Using the isomorphisms above, and $\Gamma^{1_+} \cong \mathbb{S}$ we obtain

$$\pi_0(X(1_+)) \times \pi_0(X(1_+)) \xrightarrow{(a,b) \mapsto (a,ab)} \pi_0(X(1_+)) \times \pi_0(X(1_+))$$

This map has an inverse because X is fibrant: it has the lifting property with respect to strict trivial cofibrations, which allow us to construct an inverse, analogously to how the inverse was constructed in the proof of 1.2.15. Hence X is very special.

Now assume conversely that X is pointwise fibrant and very special. Let $X \to Y \to \bullet$ be a factorization into a map that is a strict trivial cofibration $i : X \to Y$ followed by a stable fibration $Y \to \bullet$. Since X is very special and Y is stably equivalent it is also very special, hence i must be a pointwise equivalence. Because of this i has a section, which comes from the pointwise model structure described in 2.2.8, i.e. X is a retract of a stably fibrant object, and hence stably fibrant itself, because (in the pointwise model structure) there exists a lift in the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "i"] & X \arrow[d] \\ Y \arrow[r] & \bullet. \end{tikzcd}$$

$\square$

Example 2.2.9. Γ^{n_+} is cofibrant, which can be seen by directly plugging $\bullet \to \Gamma^{n_+}$ into the definition of strict cofibration.

The above example is essentially what makes the "cofibrant replacement" work. We will need the following lemma for the proof. See [16] Chapter IV, Proposition 1.9.

Lemma 2.2.10. *Suppose that $X \to Y$ is a map of bisimplicial sets such that $X_n \to Y_n$ are weak equivalences of simplicial sets. Then the induced map of associated diagonal simplicial sets $d(X) \to d(Y)$ is a weak equivalence.*

Theorem 2.2.11. *There exists a endofunctor $\mathbb{B} : \Gamma(\mathsf{sSet}_*) \to \Gamma(\mathsf{sSet}_*)$ for which $\mathbb{B}X$ is pointwise cofibrant for every X, and has a simplicial structure. Furthermore there exists a natural tranformation $\eta : \mathbb{B} \to 1$ such that $\eta_X : \mathbb{B}X(M) \to X(M)$ is a weak equivalence of simplicial sets.*

Proof. Consider a Γ-space X. Lets construct a simplicial Γ-space, given by

$$\mathbb{B}X(n_+)_p = \bigvee_{(n_+^0,\dots,n_+^p)\in\prod^p \Gamma^{op}} X(n_+^0) \wedge \mathrm{Hom}_{\Gamma^{op}}(n_+^0, n_+^1) \wedge \dots \wedge \mathrm{Hom}_{\Gamma^{op}}(n_+^{p-1}, n_+^p) \wedge \mathrm{Hom}_{\Gamma^{op}}(n_+^p, n_+).$$

With face and degeneracy maps

$$d_i(f \wedge \alpha_1 \wedge \dots \wedge \alpha_p \wedge \beta) = \begin{cases} (X(\alpha_1)(f) \wedge \alpha_2 \wedge \dots \wedge \alpha_p \wedge \beta) & i = 0, \\ (f \wedge \alpha_1 \wedge \dots \wedge \alpha_{i+1} \circ \alpha i \wedge \dots \wedge \beta & i \leq i \leq p-1 \\ (f \wedge \alpha_1 \wedge \dots \wedge \alpha_{p-1} \wedge (\beta \circ \alpha_p) & i = p \end{cases}$$
$$s_j(f \wedge \alpha_1 \wedge \dots \wedge \alpha_p \wedge \beta) = (f \wedge \dots \wedge \alpha_j \wedge \mathrm{id} \wedge \alpha_{j+1} \wedge \dots \wedge \beta)$$

$\mathbb{B}X$ is an example of the simplicial bar construction, which will play an integral role in the proofs of the next section. Note that because Γ^{n_+} is cofibrant, $\mathbb{B}M$ is cofibrant for every $n_+ \in \Gamma$, i.e. pointwise cofibrant.

Consider the natural tranformation

$$\begin{aligned} \eta_X : \mathbb{B}X &\longrightarrow X \\ (f \wedge \alpha_1 \wedge \dots \wedge \beta) &\longmapsto X(\beta \circ \alpha_p \circ \dots \circ \alpha_1)(f) \end{aligned}$$

It is elementary to check that for each $n_+ \in \Gamma^{op}$ we obtain a simplicial homotopy inverse to $\eta_X(n_+)$ by sending $f \in X(n_+)$ to $(f \wedge \mathrm{id}_{n_+} \wedge \dots \wedge \mathrm{id}_{n_+})$. $\mathbb{B}X$ commutes with filtered colimits, because it is a bar construction (i.e. a colimit), and so does X, hence η_X is an equivalence on all pointed sets and so by 2.2.10, η_X is an equivalence for all pointed simplicial sets because $\mathbb{B}X$ and X are applied degreewise. Thus, for all pointed simplicial sets $S \in \mathsf{sSet}_*$ the map $\eta_X(S) : \mathbb{B}X(S) \to X(S)$ is a weak equivalence. □

Chapter 3

Main theorems on Γ-spaces

This section is dedicated to proving two of the main theorems of [1]. The first theorem, which is theorem 3.1.6, gives a criterion on the Γ-space which insures that its associated spectrum is an Ω-spectrum. This will be proved using an inductive procedure using the shift functor defined in 3.1.4.

As already mentioned to obtain the K_0-group of a ring R, one takes an abelian monoid, and group complete it via the Grothendieck construction. The second theorem, theorem 3.2.2, is the homotopical analog of group completion. The theorem gives a criterion on the Γ-space which secures that its associated spectrum is almost an Ω-spectrum. Furthermore it insures that one of the structure maps in the associated spectrum is a group completion map. Lets begin with the first theorem.

3.1 $A(\mathbb{S})$ is an Ω-spectrum

The proofs are based on the proofs given in [13]. We begin by collecting useful results, some of which prove and some of which we give references to. For later reference we state the following technical lemma.

Lemma 3.1.1. *Let $f : X \to Y$ be a morphism of simplicial spaces, such that for every face map $d_i[n-1] \to [n]$ the square*

$$\begin{tikzcd} X_n \arrow[r, "d_i^*"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\ Y_n \arrow[r, "d_i^*"] & Y_{n-1} \end{tikzcd}$$

is a homotopy pullback. Then for every $m \geq 0$, the square

$$\begin{tikzcd} X_m \times \nabla^m \arrow[r] \arrow[d, "f_m \times \nabla^m"] & \|X\| \arrow[d] \\ Y_m \times \nabla^m \arrow[r] & \|Y\| \end{tikzcd}$$

is a homotopy pullback. Here $\|-\|$ is the fat realization, for details see the appendices of [1].

We are going to define two functors which both are going to be integral parts of the proof of the first main theorem. Both are sorts of shift functors, very akin to the shift functor for spectra.

Definition 3.1.2. [Translate of simplicial space] Let $T : \Delta \to \Delta$ be the functor, which shifts each object up, $T([n]) = [n+1]$ and likewise for a morphism α, we have $T(\alpha)(0) = 0$ and $T(\alpha)(i) = \alpha(i-1) + 1$ for $i \geq 1$. Let X be a simplicial space, then the *translate of* X is TX.

Lemma 3.1.3. *Let X_0 be the 0th level of a simplicial space X. Then $|TX|$ has the same homotopy type as X_0.*

Proof. We are simply going to write down an explicit homotopy. Note that there are simplicial maps $X_0 \to TX$, and $TX \to X_0$ respectively, given as $s_0^{n+1} : X_0 \to X_{n+1}$ and $d_1^{n+1} : X_{n+1} \to X_0$. Note that the composite $X_0 \to TX \to X_0$ is the identity. On the other hand, the collection $h_i : T_n X \to T_{n+1} X$ defined by $h_i = s_0^{i+1} \circ d_1^i$ for $0 \le i \le n$, defines a simplicial homotopy in the sense of [17]. Explicitly $d_1 \circ h_0 = \mathrm{id}$ and $d_{n+2} \circ h_n = s_0^{n+1} \circ d_1^{n+1}$ and

$$d_{i+1} \circ h_j = \begin{cases} h_{j-1} \circ d_{i+1} & i < j \\ d_{i+1} \circ h_{i-1} & i = j \\ h_j \circ d_i & i > j \end{cases} \quad \text{and} \quad s_{i+1} \circ h_j = \begin{cases} h_{j+1} \circ s_{i+1} & i \ge j \\ h_j \circ s_i & i > j \end{cases}.$$

which means that the two maps are homotopic, the result follows from corollary 11.10 in [18]. □

Definition 3.1.4. [Shift of a Γ-space] Given a Γ-space A, then the *shifted Γ-space* $\mathrm{sh}(A)$ is defined as $\mathrm{sh}(A) = A(S^1 \wedge -)$.

Lemma 3.1.5. *If A is special, then* $\mathrm{sh}(A)$ *is special. Furthermore if* $\mathrm{sh}(A)$ *is connected,* $\mathrm{sh}(A)$ *is very special.*

Proof. Assume A is special. Consider n_+ and m_+ finite pointed sets, then we have

$$\begin{aligned} \mathrm{sh}(A)(n_+ \vee m_+) &= A(S^1 \wedge (n_+ \vee m_+)) \\ &\cong A((S^1 \wedge n_+) \vee (S^1 \wedge m_+)) \\ &\simeq A(S^1 \wedge n_+) \times A(S^1 \wedge m_+) = \mathrm{sh}(A)(n_+) \times \mathrm{sh}(A)(m_+). \end{aligned}$$

Thus $\mathrm{sh}(A)$ is special. We omit the remainder of the proof. □

Theorem 3.1.6. *Let A be a cofibrant very special Γ-space. Then the spectrum $A(\mathbb{S})$ is an Ω-spectrum.*

Proof. Consider the structure map $\sigma_0 : S^1 \wedge A(S^0) \to A(S^1)$ of the spectrum $X(\mathbb{S})$, and apply the suspension loop space adjunction to obtain $\tilde{\sigma}_0 : A(S^0) \to \Omega A(S^1)$. We start by showing that $\tilde{\sigma}_0$ is a weak equivalence. Consider the simplicial space $BA : \Delta^{op} \to \mathsf{Top}$ defined as the composite

$$\Delta^{op} \xrightarrow{S^1} \Gamma^{op} \xrightarrow{A} \mathsf{Top}.$$

Note by 1.3.2 the space $A(S^1)$ is homeomorphic to $|BA|$.

We define a new simplicial space $EA : \Delta^{op} \to \mathsf{Top}$ as $EA := BA \circ T$, where T is the translate defined in 3.1.2. There is a morphism of simplicial spaces $d_0 : EA \to BA$, which arises in the following way. The morphism in Δ, $d_0 : [n] \to [n+1] = T([n])$ lets us form the following diagram

$$\begin{tikzcd} {[n]} \arrow[r, "\mathrm{id}"] \arrow[d, "d_0"] & {[n]} \arrow[d, "d_0"] \\ T([n]) \arrow[r, "\mathrm{id}"] & T([n]). \end{tikzcd}$$

which shows that d_0 is a component of a natural transformation, which we will also denote by $d_0 : \mathrm{id}_\Delta \to T$. Hence precomposing with d_0, gives morphisms $d_0^* : (EA)_n = (BA)_{n+1} \to (BA)_n$ which assemble into a morphism of simplicial spaces $d_0 : EA \to BA$. For every morphism $v : [0] \to [n] \in \Delta$, we can form the following square

$$\begin{tikzcd} (EA)_n \arrow[r, "v^*"] \arrow[d, "d_0^*"] & (EA)_0 \arrow[d, "d_0^*"] \\ (BA)_n \arrow[r, "v^*"] & (BA)_0. \end{tikzcd}$$

Now note that $(BA)_0 = A(S^1_0) = \bullet$, because A was a special Γ-space. A homotopy pullback over a point, is weakly equivalent to a product of the two off-diagonal objects. Hence the square is a homotopy pullback if the following map is a weak equivalence,

$$(d_0^*, T(v)^*) : A(S^1_{n+1}) \to A(S^1_1) \times A(S^1_n). \tag{3.1}$$

Here we have used that $(EA)_n = A(S^1_{n+1})$ and $(BA)_n \times (EA)_0 = A(S^1_1) \times A(S^1_n)$. There are two cases, either $v(0) = 0$ or not. First if $v(0) = 0$, then one can show that $d_0^* : S^1_{n+1} \to S^1_n$ and $T(v)^* : S^1_{n+1} \to S^1_1$ are complementary projections, hence the map is a weak equivalence, because A was special. Now if $v(0) \geq 1$, then one may also show that $T(v)^* : S^1_{n+1} \to S^1_1$ is a iterated fold map, which makes the map from (3.1) a weak equivalence, because A was very special.

Let $d_i : [n-1] \to [n]$ be any face map in Δ. Consider the following diagram

$$\begin{tikzcd}
(EA)_n \arrow[r, "d_i^*"] \arrow[d, "d_0^*"] & (EA)_{m+1} \arrow[r, "v^*"] \arrow[d, "d_0^*"] & (EA)_0 \arrow[d, "d_0^*"] \\
(BA)_n \arrow[r, "d_i^*"] & (BA)_m \arrow[r, "v^*"] & (BA)_0.
\end{tikzcd}$$

By the above the right square, and the outer rectangle are homotopy pullbacks, which implies that the left square is too, for a proof of this fact see [16] chapter IV. Now apply 3.1.1 to the morphism $d_0 : EA \to BA$ obtained earlier. This implies that the following square is a homotopy pullback

$$\begin{tikzcd}
(EA)_0 \arrow[r] \arrow[d] & \|EA\| \arrow[d, "\|d_0\|"] \\
(BA)_0 \arrow[r] & \|BA\|.
\end{tikzcd}$$

Here $\|-\|$ is the fat realization. Since A is a cofibrant Γ-space the simplicial spaces EA and BA the inclusions of degenerate simplices $\bigcup_i s_i(EA_{n-1}) \to (EA)_n$ and $\bigcup_i s_i(BA_{n-1}) \to (BA)_n$ are closed cofibrations in the classical model structure on topological spaces, i.e EA and BA are proper. Because of this the natural maps $\|EA\| \to |EA|$ and $\|BA\| \to |BA|$ are weak equivalences, for details see the appendices [1]. Which implies that the following diagram is a homotopy pullback

$$\begin{tikzcd}
(EA)_0 \arrow[r] \arrow[d] & |EA| \arrow[d] \\
(BA)_0 \arrow[r] & |BA|.
\end{tikzcd}$$

Note that $(EA)_0 = A(S^1_1) = A(S^0)$, $|BA| = A(S^1)$ and, as already noted, $(BA)_0 = \bullet$. Furthermore because $(BA)_0$ is contractible, 3.1.2 implies that $|EA| = |T(BA)|$ is also contractible, therefore we obtain a homotopy pullback diagram

$$\begin{tikzcd}
A(S^0) \arrow[r] \arrow[d] & \bullet \arrow[d, "t"] \\
\bullet \arrow[r] & A(S^1).
\end{tikzcd}$$

Hence we obtain a weak equivalence from $A(S^0) \to \mathrm{hofib}(t)$, and thus a weak equivalence $A(S^0) \to \Omega A(S^1)$, which in fact is the adjoint structure map $\tilde{\sigma}$.

At this point we still need to handle the higher structure maps of $A(\mathbb{S})$. Consider the n-fold shifted Γ-space from 3.1.4, which can be seen as a functor on finite sets through a prolongation of the Γ-space $\mathrm{sh}^n(A)$, from a functor $\Gamma \to \mathsf{sSet}_*$ to a functor $\mathsf{Fin}_* \to \mathsf{sSet}_*$, see the proof of 1.3.1. By the first part the adjoint structure map $|\mathrm{sh}^n(A)(S^0)| \to \Omega|\mathrm{sh}^n(A)(S^1)|$ is a weak equivalence, but this map is isomorphic to the adjoint structure map $\tilde{\sigma}_n : A(S^n) \to \Omega A(S^{n+1})$, hence $A(\mathbb{S})$ is a Ω-spectrum. $\square$

3.2 $A(\mathbb{S})$ is a positive Ω-spectrum

The proof of the second main theorem, is a consequence of the following lemma.

Lemma 3.2.1. *Given a A a cofibrant special Γ-space, there exists a Γ-space $\tilde{A}$ and a morphism $f : A \to \tilde{A}$ which satisfies the following properties*

1. *The map $f(1_+) : A(1_+) \to \tilde{A}(1_+)$ induces an isomorphism*
$$H_*(A(1_+), \mathbb{Z})[(\pi_0(A(S^0))^{-1}] \to H_*(\tilde{A}(1_+), \mathbb{Z}).$$
2. *The map $A(S^1) \to \tilde{A}(S^1)$ is a weak equivalence.*
3. *The Γ-space $\tilde{A}$ is very special.*

We will postpone the lengthy proof of the lemma, until after we see its application, namely the second main theorem.

Theorem 3.2.2. *Let A be a cofibrant special Γ-space. The adjoint structure map $\tilde{\sigma} : A(S^0) \to \Omega A(S^1)$ is a group completion map, hence it induces an isomorphism,*

$$H_*(A(S^0), \mathbb{Z})[(\pi_0(A(S^0))^{-1}] \to H_*(\Omega A(S^1), \mathbb{Z}).$$

Furthermore $A(\mathbb{S})$ is a positive Ω-spectrum, i.e. an Ω-spectrum from the 1st level and up.

Proof. We will for notational ease denoted $(\pi_0(A(S^0))$ simply as π. Let $\tilde{A}$ be as in 3.2.1. Consider the commutative square

$$\begin{array}{ccc} A(1^+) & \xrightarrow{f(1_+)} & \tilde{A}(1_+) \\ \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma_0} \\ \Omega A(S^1) & \xrightarrow{\Omega f(S^1)} & \Omega\tilde{A}(S^1). \end{array}$$

Note that $\tilde{A}(1_+) = \tilde{A}(S^0)$, hence the right vertical map is a weak equivalence via 3.1.6 because $\tilde{A}$ is very special according to 3.2.1(3). The lower horizontal map is a weak equivalence because of 3.2.1(2), and the fact that Ω preserve weak equivalences. Hence we get a commutative diagram of homology rings,

$$\begin{array}{ccc} H_*(A(1_+), \mathbb{Z})[\pi^{-1}] & \xrightarrow{f(1_+)_*} & H_*(\tilde{A}(1_+), \mathbb{Z}) \\ \downarrow{\scriptstyle (\sigma_0)_*} & & \downarrow{\scriptstyle (\sigma_0)_*} \\ H_*(\Omega A(S^1), \mathbb{Z}) & \xrightarrow{(\Omega f(S^1))_*} & H_*(\Omega(\tilde{A}(S^1), \mathbb{Z}), \end{array}$$

Property 3.2.1(1) implies that the top horizontal map is an isomorphism, and because the right vertical map and the lower horizontal map was induced from weak equivalences, they are also isomorphisms. Hence the left vertical map $(\sigma_0)_*$ is an isomorphism, which when noting that $A(1_+) = A(S^0)$ gives the desired isomorphism.

Via 3.2.1(2) $A(S^1) \to \tilde{A}(S^1)$ is a weak equivalence. Ω preserve weak equivalences so $\Omega A(S^1) \to \Omega\tilde{A}(S^1)$ is a weak equivalence. Furthermore note that $\tilde{A}(\mathbb{S})$ is an Ω-spectrum by 3.2.1(3) and 3.1.6. These two facts implies that A is a positive Ω-spectrum. □

Remark 3.2.3. Now if M is an H-space the multiplication induces a ring structure in its homology, called the Pontrjagin ring of M, and lets view $\pi_0(M)$ as a multiplicative subset of the Pontrjagin ring $H_\bullet(M, \mathbb{Z})$.

Lemma 3.2.4. *Let X be a homotopy commutative H-space. Then $H_*(X, \mathbb{F})[\pi_0(X)^{-1}]$ is a Hopf algebra.*

Proof. The proof that $H_*(X,\mathbb{F})$ is an bialgebra will be given in the proof of lemma 3.2.1. We will only define the antipode on $H_*(X,\mathbb{F})$ here. Suppose first that X is path connected. Let $\Delta : X \to X \times X$ be the diagonal map. Let $\phi : H_*(X,\mathbb{F}) \to H_*(X,\mathbb{F}) \otimes H_*(X,\mathbb{F})$, be the comultiplication. Consider $1 \in H_0(X,\mathbb{F})$ and let it be represented by $\Delta^0 \to X$, then we have $\psi(x) = 1 \otimes 1$. To analyze the effect of ψ on the higher homology groups we consider the following diagram, and one where we collapse the first factor instead of the second.

$$\begin{tikzcd}
H_*(X,\mathbb{F}) \arrow[r, "\psi"] \arrow[ddr, bend right, "\mathrm{id}"'] & H_*(X \times X,\mathbb{F}) \arrow[d] & H_*(X,\mathbb{F}) \otimes H_*(X,\mathbb{F}) \arrow[l, "\cong"'] \arrow[d] \\
& H_*(X \times \{\bullet\},\mathbb{F}) \arrow[d, "\cong"] & H_*(X,\mathbb{F}) \otimes H_*(\{\bullet\},\mathbb{F}) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
& H_*(X,\mathbb{F}) & H_*(X,\mathbb{F}) \otimes \mathbb{Z} \arrow[l, "\cong"']
\end{tikzcd}$$

Where we've used the Künneth isomorphism, and the rest of the maps are the obvious ones. From the above diagram it is evident that for $x \in H_*(X,\mathbb{F})$ for $\deg(x) > 0$

$$\psi(x) = x \otimes 1 + \sum x' \otimes x'' + 1 \otimes x.$$

Where we sum over tensors $x' \otimes x''$ for which x' and x'' satisfy $\deg(x') > 0$ and $\deg(x'') > 0$. Now in the case where X is not path connected, we consider $g : \Delta^0 \to X$ represent a path component $X_g \subseteq X$. Assume that $x \in \operatorname{im}(H_*(X_g,\mathbb{F}) \to H_*(X,\mathbb{F}))$ and $\deg(x) > 0$. Then we have that $\psi(g) = g \otimes g$, and

$$\psi(x) = x \otimes g + \sum x' \otimes x'' + g \otimes x$$

which follows from the connected case and naturality of ψ with respect to inclusions. Now since X was an H-space it has a Pontrjagin product $\bullet : H_*(X,\mathbb{F}) \otimes H_*(X,\mathbb{F}) \to H_*(X,\mathbb{F})$. Because X is homotopy commutative we may assume the Pontrjagin ring $H_*(X,\mathbb{F})$ is graded commutative. We will now define the map $c : H_*(X,\mathbb{F}) \to H_*(X,\mathbb{F})[\pi_0(X)^{-1}]$ which is going to lift to the antipode. Let $g : \Delta^0 \to X$ represent a path component X_g, then g is invertible in $H_*(X)[\pi(X)^{-1}]$ and we define $c(g) = g^{-1}$, and extend this map to all of $H_0(X)$ by $\mathbb{F}$-linearity. For $x \in H_*(X_g,\mathbb{F}) \subseteq H_*(X,\mathbb{F})$ with $\deg(x) > 0$, define inductively

$$c(x) = -x \bullet g^{-2} - \sum x' \bullet c(x'') \bullet g^{-1}$$

and extend this to all of $H_*(X,\mathbb{F})$ by $\mathbb{F}$-linearity. It is elementary to check that c is a ring homomorphism with respect to the Pontrjagin product and that $c(\pi_0(X)) \subseteq \pi_0(X)$. Hence by the universal property of localization c induces an $\mathbb{F}$-algebra homomorphism

$$c : H_*(X,\mathbb{F})[\pi_0(X)^{-1}] \to H_*(X,\mathbb{F})[\pi_0(X)^{-1}].$$

Now let $\epsilon : H_*(X,\mathbb{F})[\pi_0(X)^{-1}] \to \mathbb{F}$ be the unit induced by $X \to \{e\}$ where e is the unit with respect to the Pontrjagin product, and let $\eta : \mathbb{F} \to H_*(X,\mathbb{F})[\pi_0(X)^{-1}]$ be the counit induced by $\{e\} \to X$. Note that for all $g : \Delta^0 \to X$ we have that $\epsilon(g) = 1$, and for all $x \in H_*(X,\mathbb{F})$ with $\deg(x) > 0$ we have that $\epsilon(x) = 0$.

We wish to show that $(\bullet(\mathrm{id} \otimes x)\psi) = \eta\epsilon$, namely that c constitutes an antipode. Consider $g : \Delta^0 \to X$, then

$$(\bullet(\mathrm{id} \otimes x)\psi)(g) = g \bullet c(g) = g \bullet g^{-1} = e = \eta(1) = (\eta\epsilon)(g).$$

Let $x \in H_*(X)$ and assume $\deg(x) > 0$, then we have

$$\begin{aligned}
(\bullet(\mathrm{id} \otimes x)\psi)(x) &= x \bullet x^{-1} + \sum x' \bullet c(x'') + g \cdot c(x) \\
&= x \bullet x^{-1} + \sum x' \bullet c(x'') + g \cdot (-x \bullet g^{-2} - \sum x' \bullet c(x'') \bullet g^{-1}) \\
&= 0.
\end{aligned}$$

which proves the claim. □

Proof of 3.2.1. Consider a finite pointed set X, and form the following homotopy pullback

$$\begin{tikzcd} P_k(X) \arrow[r] \arrow[d] & A(\Delta[1]_k \wedge X) \arrow[d] \\ A(\Delta[1]_k \wedge X) \arrow[r] & A(S^1_k \wedge X) \end{tikzcd}$$

Where lower left copy of $\Delta[1]$ is pointed at 0, and the upper right copy of $\Delta[1]$ is pointed at 1. As k varies, the structure maps of $\Delta[1]$ and S^1, and the functoriality of homotopy pullback makes $P_k(X)$ into a simplicial space $[k] \mapsto P_k(X)$. Let $\tilde{A}(X) = |[k] \mapsto P_k(X)|$.

Proof of 3.2.1(1). Let $\mathbb{F}$ denote any field, and denote the homology of $A(1_+)$ with $\mathbb{F}$-coefficients $H_*(A(1_+), \mathbb{F})$ as H. Consider diagonal map $\Delta : \tilde{A}(1_+) \to \tilde{A}(1_+) \times \tilde{A}(1_+)$, and consider its induced map

$$\begin{tikzcd} H_*(\tilde{A}(1_+), \mathbb{F}) \arrow[r, "\Delta_*"] \arrow[dr, dashed, "\psi"] & H_*(\tilde{A}(1_+) \times \tilde{A}(1_+), \mathbb{F}) \arrow[d, "\cong"] \\ & H_*(\tilde{A}(1_+), \mathbb{F}) \otimes H_*(\tilde{A}(1_+), \mathbb{F}). \end{tikzcd}$$

Where the vertical isomorphism is the Künneth isomorphism which applies because Tor-term vanishes in $\mathbb{F}$-coefficients, hence we obtain a comultiplication on $H_*(\tilde{A}(1_+), \mathbb{F})$, denote it ψ. Now as already noted in the proof of 1.2.15, $\tilde{A}(1_+)$ is an H-space, hence via 3.2.3, we obtain multiplication on $H_*(\tilde{A}(1_+), \mathbb{F})$. The multiplication m is given as

$$\begin{tikzcd} H_*(\tilde{X}(1_+) \times \tilde{A}(1_+), \mathbb{F}) \arrow[r, "H_*((p_1)_*, (p_2)_*)^{-1}"] \arrow[d, "\cong"] & H_*(\tilde{A}(2_+), \mathbb{F}) \arrow[r, "H_*((p_1)_*, \mu_*)"] & H_*(\tilde{A}(1_+) \times \tilde{A}(1_+), \mathbb{F}) \arrow[d, "(\mathrm{pr}_2)_*"] \\ H_*(\tilde{A}(1_+), \mathbb{F}) \otimes H_*(\tilde{A}(1_+), \mathbb{F}) \arrow[rr, dashed, "m"] & & H_*(\tilde{A}(1_+), \mathbb{F}) \end{tikzcd}$$

Where both isomorphisms are the Künneth isomorphism. The multiplication and comultiplication grants $H_*(\tilde{A}(1_+), \mathbb{F})$ the structure of a $\mathbb{F}$-bialgebra, i.e it is (co)associative, (co)commutative and has (co)unit. The above shows the missing part of 3.2.4.

Since $\tilde{A}(1_+)$ is defined as the geometric realization of a simplicial space

$$|[k] \mapsto P_k(1_+)| = \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) / \sim$$

Where $\sim$ is the usual relation $(x, d_i(y)) \sim (s_i(x), y)$. Consider its n'th truncation, defined as $i_n := \coprod_{n \geq i \geq 0} X_i \times \Delta^i$. There is a canonical map $i_n \to |P_\bullet(1_+)|$, consider the n'th truncations image under this map and denote it $F_n|P_\bullet(1_+)|$. These give a filtation

$$\ldots \subseteq F_n|P_\bullet(1_+)| \subseteq F_{n+1}|P_\bullet(1_+)| \subseteq \ldots$$

This filtration gives rise to a spectral sequence of $\mathbb{F}$-algebras converging to the homology of $\tilde{A}(1_+)$ with coefficients in $\mathbb{F}$, as described in [19] p. 109.

$$E^1_{p,q} = H_p(P_q(1_+), \mathbb{F}) \Rightarrow H_{p+q}(\tilde{A}(1_+), \mathbb{F}). \tag{3.2}$$

According to [18] Theorem 11.14 the E^2-page is

$$E^2_{p,q} = H_p(H_q(P_\bullet(1_+), \mathbb{F})),$$

where for each fixed $q \geq 0$ $H_q(P_\bullet(1_+), \mathbb{F})$ is regarded as a chain complex with differential $\partial_n = \sum^n (-1)^i (d_i)_*)$ where d_i are the face maps in the simplicial space $P_\bullet(1_+)$. Now consider the Γ-space A, for which we have weak equivalences $A(\Delta[1]_k) \to A(1_+)^{k+1}$ which follows from the isomorphism $\Delta[1]_k \cong \bigvee_{i=1}^{k+1} S^0$, and 1.2.12. Therefore we obtain the following commutative diagram

$$\begin{tikzcd}
A(\Delta[1]_k) \arrow[r] \arrow[d, "\sim"] & A(S^1_k) \arrow[d, "\sim"] & A(\Delta[1]_k) \arrow[l] \arrow[d, "\sim"] \\
A(1_+)^{k+1} \arrow[r] & A(1_+)^k & A(1_+)^{k+1} \arrow[l]
\end{tikzcd}$$

Hence we obtain the following diagram of homotopy pullbacks

$$\begin{tikzcd}
& P_k(1_+) \arrow[rr] \arrow[dl, "\sim"'] \arrow[dd] & & A(\Delta[1]_k) \arrow[dl, "\sim"'] \arrow[dd] \\
P \arrow[rr, crossing over] \arrow[dd] & & A(1_+)^{k+1} & \\
& A(\Delta[1]_k) \arrow[rr] \arrow[dl, "\sim"] & & A(S^1_k) \arrow[dl, "\sim"] \\
A(1_+)^{k+1} \arrow[rr] & & A(1_+)^k \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Note that that the homotopy pullback P is weakly equivalent to $A(1_+)^{k+2}$. Therefore we obtain isomorphisms

$$H_*(P_k(1_+),\mathbb{F}) \cong H_*(A(1_+)^{k+2},\mathbb{F}) \cong H_*(A(1_+),\mathbb{F}) \otimes \ldots \otimes H_*(A(1_+),\mathbb{F}), \tag{3.3}$$

where the second isomorphism is the Künneth isomorphism. This is a description of the chain complex appearing on the E^2-page of the spectral sequence in 3.2. Under this isomorphism the simplicial structure maps of $H_*(P_k(1_+),\mathbb{F})$ become the face and degeneracy maps of the simplicial bar construction for the bialgebra H, $\mathbb{B}(H\otimes H, H, \mathbb{F})$, namely the simplicial $\mathbb{F}$-module $[k] \mapsto (H \otimes H) \otimes (\bigotimes_{i=1}^k H) \otimes \mathbb{F}$. Explicitly they become the maps

$$d_i((h\otimes h')\otimes(h_1\otimes\ldots\otimes h_k)\otimes v) = \begin{cases} m((h,h'),\psi(h_1))\otimes h_2\otimes\ldots\otimes h_k\otimes v & \text{if } i=0 \\ (h\otimes h')\otimes(h_1\otimes\ldots\otimes m(h_i,h_{i+1})\otimes\ldots\otimes h_k)\otimes v & \text{if } 0<i<k \\ (h\otimes h')\otimes(h_1\otimes\ldots\otimes\epsilon(h_k)\cdot v) & \text{if } i=k. \end{cases}$$

$$s_i((h\otimes h')\otimes(h_1\ldots\otimes h_k)\otimes v) = (h\otimes h')\otimes(h_1\otimes\ldots\otimes h_{i-1}\otimes 1\otimes h_i\otimes\ldots\otimes h_k)\otimes v$$

Here $m : H\otimes H \to H$ is the multiplication induced on H through $A(1_+)$'s H-space structure analogous to how the multiplication on the homology of $\tilde{A}(1_+)$ was defined earlier.

Therefore, the chain complex needed on the E^2-page of the spectral sequence in 3.2 is

$$\ldots \xrightarrow{\partial_n} (H\otimes H)\otimes H\otimes\ldots\otimes H\otimes\mathbb{F} \xrightarrow{\partial_n} \ldots \xrightarrow{d_0-d_1} (H\otimes H)\otimes H\otimes\mathbb{F} \longrightarrow 0$$

Let us denote this chain complex $\mathbb{B}_\bullet(H\otimes H,H,\mathbb{F})$. $\mathbb{B}_\bullet(H\otimes H,H,\mathbb{F})$ is the chain complex whose homology defines the relative Tor-group, see [20] p. 288. I.e there is an isomorphism

$$\mathrm{Tor}^H_{p,*}(H\otimes H,\mathbb{F}) \cong H_*(\mathbb{B}_\bullet(H\otimes H,H,\mathbb{F})). \tag{3.4}$$

Hence from the definition of the spectral sequence we have

$$E^2_{p,*} = \mathrm{Tor}^H_{p,*}(H\otimes H,\mathbb{F}).$$

The multiplicative subset π acts trivially on $\mathbb{F}$, because the action is the one of $H_*(A(1_+),\mathbb{F})$ on $H_*(\{\bullet\},\mathbb{F})$ via the augmentation map ϵ. Hence $\mathbb{F}$ is π-local. Now localization is an exact functor, so it commutes with homology, therefore it commutes with Tor, so we obtain a natural map for every $p\geq 1$ induced by the localization morphism of H-modules $H\otimes H \to (H\otimes H)[\pi^{-1}]$, where we view $H\otimes H$ as a H-module via the diagonal action using the coproduct ψ.

$$\mathrm{Tor}^H_{p,*}(H\otimes H,\mathbb{F}) \to \mathrm{Tor}^{H[\pi^{-1}]}_{p,*}((H\otimes H)[\pi^{-1}],\mathbb{F})$$

These localization maps are isomorphisms because the left hand side was local. Now via 3.2.4 we have that $H[\pi^{-1}]$ is an Hopf algebra with antipode c as defined in the proof of 3.2.4. We will use this fact to realize that $(H\otimes H)[\pi^{-1}]$ is flat, ensuring that $\mathrm{Tor}_{p,*}^{H[\pi^{-1}]}((H\otimes H)[\pi^{-1}],\mathbb{F})=0$, such that the spectral sequence 3.2 collapses at the E^2-page.

Consider the following composition of maps

$$H\otimes H \xrightarrow{1\otimes\psi} H\otimes H\otimes H \xrightarrow{1\otimes c\otimes 1} H\otimes H[\pi^{-1}]\otimes H \xrightarrow{m\otimes 1} H[\pi^{-1}]\otimes H[\pi^{-1}],$$

We denote the composition Φ, which via the universal property of localization induces a map

$$\Phi:(H\otimes H)[\pi^{-1}]\to H[\pi^{-1}]\otimes H[\pi^{-1}].$$

Next consider the composition

$$H\otimes H \xrightarrow{1\otimes\psi} H\otimes H\otimes H \xrightarrow{m\otimes 1} (H\otimes H)[\pi^{-1}]$$

Which also induces a map

$$\Psi: H[\pi^{-1}]\otimes H[\pi^{-1}]\to (H\otimes H)[\pi^{-1}].$$

It can be shown that both Φ and Ψ are H-linear with the diagonal action on $(H\otimes H)[\pi^{-1}]$ and the action on the hand fact of $H[\pi^{-1}]\otimes H[\pi^{-1}]$, and that they are inverse to one another. Φ is an isomorphism of $H[\pi^{-1}]$-modules, and because the right hand side is free, we deduce that $(H\otimes H)[\pi^{-1}]$ is free, which implies that it is flat. So as mentioned earlier we have

$$\mathrm{Tor}_{p,*}^{H}(H\otimes H,\mathbb{F})=0$$

for $p\geq 1$, hence it is only the 0'th column of the E^2-page of the spectral sequence 3.2 that is non-zero. Since the differentials on page $k>1$ has positive horizontal degree $E^2=E^\infty$. Therefore the $\bigoplus_{p+q=n}E^\infty_{p,q}$ is isomorphic to the associated graded homology, but we actually get more, we obtain a map from $\oplus_p E^\infty_{p,n-p}\to H_n$, because the succesive quotient which defines $E^\infty_{p,n-p}$ is trivial because there is only a single non-zero column. Hence we obtain an isomorphism of algebras,

$$(H\otimes H)\otimes_H\mathbb{F}\cong \mathrm{Tor}_{0,*}^{H}(H\otimes H,\mathbb{F})\cong H_*(\tilde{A}(1_+),\mathbb{F}).$$

Furthermore consider the following string of isomorphisms

$$(H\otimes H)\otimes_H\mathbb{F}\xrightarrow{(1)}(H\otimes H)[\pi^{-1}]\otimes_{H[\pi^{-1}]}\mathbb{F}\xrightarrow{(2)}H[\pi^{-1}]\otimes(H[\pi^{-1}]\otimes_{H[\pi^{-1}]}\mathbb{F})\xrightarrow{(3)}H[\pi^{-1}].$$

Where (1) is the $p=0$ of 3.4, (2) is Φ and reordering of the localizations, and (3) is via first considering $\mathbb{F}$ as a trivial $H[\pi^{-1}]$-module, and consider $H[\pi^{-1}]$ as a $\mathbb{F}$-module, which we can because it was a $\mathbb{F}$-bialgebra. Combining these two isomorphisms shows property 1.

Proof of 3.2.1(3). Consider finite pointed sets B and C, and form the following commutative diagram using that $\wedge$ distributes over $\vee$,

$$\begin{tikzcd}
A(\Delta[1]_k\wedge(B\vee C)) \arrow[r] \arrow[d, "\sim"] & A(S^1_k\wedge(B\vee C)) \arrow[d, "\sim"] & A(\Delta[1]_k\wedge(B\vee C)) \arrow[l] \arrow[d, "\sim"] \\
A(\Delta[1]_k\wedge B)\times A(\Delta[1]_k\wedge C) \arrow[r] & A(S^1_k\wedge B)\times A(S^1_k\wedge C) & A(\Delta[1]_k\wedge B)\times A(\Delta[1]_k\wedge C) \arrow[l]
\end{tikzcd}$$

The vertical maps are weak equivalences because A is special. Hence we obtain the following diagram of homotopy pullbacks

$$
\begin{tikzcd}
 & P_k(A \vee B) \arrow[rr] \arrow[dl, "\sim"'] \arrow[dd] & & A(\Delta[1]_k \wedge (B \vee C)) \arrow[dl, "\sim"'] \arrow[dd] \\
P \arrow[rr] \arrow[dd] & & A(\Delta[1]_k \wedge B) \times A(\Delta[1]_k \wedge C) \arrow[dd] & \\
 & A(\Delta[1]_k \wedge (B \vee C)) \arrow[rr] \arrow[dl, "\sim"'] & & A(S^1_k \wedge (B \vee C)) \arrow[dl, "\sim"] \\
A(\Delta[1]_k \wedge B) \times A(\Delta[1]_k \wedge C) \arrow[rr] & & A(S^1_k \wedge B) \times A(S^1_k \wedge C) &
\end{tikzcd}
$$

Note that the homotopy pullback P is weakly equivalent to $P_k(B) \times P_k(C)$, hence we have a weak equivalence $P_k(B \vee C) \to P_k(B) \times P_k(C)$. Geometric realization commutes with products, so we obtain a weak equivalence

$$\tilde{A}(B \vee C) \to \tilde{A}(B) \times \tilde{A}(C),$$

hence $\tilde{A}$ is special. From the proof of 3.2.1(1), it is clear that the homology algebra structure $\tilde{H} := H_*(\tilde{A}(1_+), \mathbb{F})$ is defined such that the following diagram commutes

$$
\begin{tikzcd}
\tilde{H} \otimes \tilde{H} \arrow[rr, "\Phi"] \arrow[d, "\cong"] & & \tilde{H} \otimes \tilde{H} \arrow[d, "\cong"] \\
H_*(\tilde{A}(1_+) \times \tilde{A}(1_+), \mathbb{F}) \arrow[r, "\cong", "H_*(p^1_*, p^2_*)^{-1}"'] & H_*(\tilde{A}(2_+), \mathbb{F}) \arrow[r, "H_*(p^1_*, (m_2)_*)"] & H_*(\tilde{A}(1_+) \times \tilde{A}(1_+), \mathbb{F})
\end{tikzcd}
$$

Here the vertical maps are the Künneth isomorphism. Via property 3.2.1(1), $\tilde{H}$ is isomorphic to $H_*(A(1_+), \mathbb{F})[\pi^{-1}]$, in particular $\tilde{H}$ is an Hopf algebra. Denote the antipode by $c : \tilde{H} \to \tilde{H}$. Define Φ and its inverse Ψ as in 3.2.1(1). Because the map $(p^1_*, p^2_*) : A(2_+) \to A(1_+) \times A(1_+)$ is a weak equivalence, we deduce from 1.2.15 that the induced map of $(p^1_*, (m_2)_*) : A(2_+) \to A(1_+) \times A(1_+)$ is an isomorphism on homology. Now this map is an isomorphism of homology groups and the fundamental group, hence via [21] Corollary 1, page 79 it is a weak equivalence. Which shows that the Γ-space $\tilde{A}$ is very special.

Proof of 3.2.1(2). $A(S^1)$ is isomorphic to the geometric realization of the simplicial space $[k] \mapsto A(S^1_k)$, hence as described before the simplicial skeletal filtration comes with a spectral sequence

$$E^1_{p,q} = H_p(A(S^1_q), \mathbb{F}) \Rightarrow H_{p+q}(A(S^1), \mathbb{F}),$$

Because A is special $A(S^1_k)$ is weakly equivalent to $A(1_+)^k$, hence analogous to 3.3, we have

$$H_*(A(S^1_k), \mathbb{F}) \cong H_*(A(1_+)^k, \mathbb{F}) \cong H_*(A(1_+), \mathbb{F}) \otimes \ldots \otimes H_*(A(1_+), \mathbb{F}).$$

Again via an analogous procedure as in the proof of 3.2.1(1), the simplicial structure maps become the maps in the simplicial bar construction $\mathbb{B}(\mathbb{F}, H, \mathbb{F})$. From which we have

$$E^2_{p,*} = \mathrm{Tor}^H_{p,*}(\mathbb{F}, \mathbb{F}).$$

$\tilde{A}$ was special via property 3.2.1(3), so the above argument applies to $\tilde{A}$, so we have

$$H_*(\tilde{A}(S^1_k), \mathbb{F}) \cong H_*(\tilde{A}(1_+)^k, \mathbb{F}) \cong H_*(\tilde{A}(1_+), \mathbb{F}) \otimes \ldots \otimes H_*(\tilde{A}(1_+), \mathbb{F}).$$

Applying the identification $H_*(\tilde{A}(1_+),\mathbb{F}) \cong H_*(A(1_+),\mathbb{F})[\pi^{-1}]$ given in the proof of 3.2.1(1), the structure maps become the maps in $\mathbb{B}(\mathbb{F},H[\pi^{-1}],\mathbb{F})$, so we obtain

$$\tilde{E}^2_{p,*} = \mathrm{Tor}^{H[\pi^{-1}]}_{p,*}(\mathbb{F},\mathbb{F}).$$

The map $f : A \to \tilde{A}$ induces a map between these two spectral sequence,

$$\begin{tikzcd}
E^2_{p,*} = \mathrm{Tor}^{H}_{p,*}(\mathbb{F},\mathbb{F}) \arrow[r, Rightarrow] \arrow[d] & H_*(A(S^1),\mathbb{F}) \arrow[d, "f(S^1)_*"] \\
\tilde{E}^2_{p,*} = \mathrm{Tor}^{H[\pi^{-1}]}_{p,*}(\mathbb{F},\mathbb{F}) \arrow[r, Rightarrow] & H_*(\tilde{A}(S^1),\mathbb{F})
\end{tikzcd}$$

Because π act invertibly on $\mathbb{F}$, $\mathbb{F}$ as a H-module is already π-local, hence the induced map on Tor-groups is an isomorphism. The spectral sequences are concentrated in the first quadrant, hence we can conclude that the induced map $f(S^1)_*$ is an isomorphism for all fields $\mathbb{F}$. This implies that $f(S^1)_*$ is a isomorphism of integral homology. Now $A(S^1)$ and $\tilde{A}(S^1)$ are loop spaces, hence they are H-spaces. They are furthermore path connected, which implies they are simple spaces. The well known fact that $H_*(-,\mathbb{Z})$-equivalence between simple spaces are weak equivalences, implies that $f(S^1)_*$ is an weak equivalence. □

Chapter 4

The K-theory spectrum of a permutative category

In this chapter we finally define the K-theory spectrum associated to a permutative category. We will also deal with the main example, where we consider the permutative category of finitely generated projective R-modules, $\mathbb{P}_R$.

4.1 Γ-categories and the K-theory spectrum

Lets first define permutative categories and see how they fit into the Γ-object formalism.

Definition 4.1.1. A *permutative category* $\mathscr{C}$ is a symmetric monoidal category $(\mathscr{C}, \oplus, 0, \alpha, l, r, b)$ where the associator a, the left and right unitors l and r, and the braiding b isomorphisms are identities.

Construction 4.1.2. We will now construct a category $\overline{\mathscr{C}}(n_+)$ from a permutative category $\mathscr{C}$ and $n_+ \in \Gamma$.

- An object X of $\overline{\mathscr{C}}(n_+)$ is a collection $X = \{X_S, \rho_{S,T}\}$ consisting of
  - an object X_S of $\mathscr{C}$ for all $S \subseteq n_+ \setminus \{\bullet\}$.
  - an isomorphism $\rho_{S,T} : X_S \oplus X_T \to X_{S \cup T}$ for every pair of disjoint subsets S and T of n_+.
- A morphism $f : X \to X'$ in $\overline{\mathscr{C}}(n_+)$ consists of morphisms $f_S : X_S \to X'_S$ for all S as above, such that $f_\varnothing = \mathrm{id}_0$ and such that the following square commutes for every pair of disjoint subsets S and T,

$$\begin{tikzcd} X_S \oplus X_T \arrow[r, "\rho_{S,T}"] \arrow[d, "f_S \oplus f_T"] & X_{S \cup T} \arrow[d, "f_{S \cup T}"] \\ X'_S \oplus X'_T \arrow[r, "\rho'_{S,T}"] & X'_{S \cup T}. \end{tikzcd}$$

This data is subject to the following conditions

- $X_\varnothing = 0$ and $\rho_{S,\varnothing} = \mathrm{id}_{X_S} : X_S \oplus X_\varnothing \to X_S$ for all S.
- For all mutually disjoint subsets S, T, U of n_+ the following squares commute

$$\begin{tikzcd} X_S \oplus X_T \arrow[r, "\rho_{S,T}"] \arrow[d, "b_{X_S, X_t}"] & X_{S \cup T} \arrow[d, "\mathrm{id}"] \\ X_T \oplus X_S \arrow[r, "\rho_{T,S}"] & X_{T \cup S} \end{tikzcd} \qquad \begin{tikzcd} X_S \oplus X_T \oplus X_U \arrow[r, "\rho_{S,T} \oplus \mathrm{id}_{X_U}"] \arrow[d, "\mathrm{id}_{X_S} \oplus \rho_{T,U}"] & X_{S \cup T} \oplus X_U \arrow[d, "\rho_{S \cup T, U}"] \\ X_S \oplus X_{T \cup U} \arrow[r, "\rho_{S, T \cup U}"] & X_{S \cup T \cup U}. \end{tikzcd}$$

Lemma 4.1.3. *The $\overline{\mathscr{C}}$ construction is a covariant functor $\overline{\mathscr{C}} : \Gamma^{op} \to \mathsf{Cat}$, hence give rise to Γ-category.*

Proof. Let $\alpha : m_+ \to n_+$ be a morphism in Γ^{op}, we define $\alpha_* : \overline{\mathscr{C}}(m_+) \to \overline{\mathscr{C}}(n_+)$ in the following way. Let $X \in \overline{\mathscr{C}}(m_+)$, then

$$\alpha_*(X) = \{(\alpha_*(X))_S, (\alpha_*(\rho))_{S,T}\} = \{X_{\alpha^{-1}(S)}, \rho_{\alpha^{-1}(S),\alpha^{-1}(T)}\}.$$

Note that α is based, hence $\alpha^{-1}(S)$ does not contain the basepoint. Consider $f : X \to X'$, then we define $\alpha_*(f) : \alpha_*(X) \to \alpha_*(X')$ at each $S \subseteq n_+ \setminus \{\bullet\}$ as $(\alpha_*(f))_S = f_{\alpha^{-1}(S)}$. It is straightforward to check that this gives rise to a functor $\alpha_* : \overline{\mathscr{C}}(m_+) \to \overline{\mathscr{C}}(n_+)$. Given another map $\beta : k_+ \to m_+$ in Γ^{op}, we force the last functor condition to hold, by setting $(\alpha\beta)_*(S) = \beta_*(\alpha_*(S))$ for all $S \in n_+ \setminus \{\bullet\}$, hence $(\alpha\beta)_* = \alpha_* \circ \beta_* : \overline{\mathscr{C}}(k_+) \to \overline{\mathscr{C}}(n_+)$. □

The following theorem shows that given a permutative category $\mathscr{C}$, then the Γ-category $\overline{\mathscr{C}}$ is a special Γ-space.

Proposition 4.1.4. *Consider a small permutative category $\mathscr{C}$, then $\overline{\mathscr{C}}$ satisfies the following properties*

(1) $\overline{\mathscr{C}}(0_+)$ *is terminal in* Cat.

(2) For every pair n_+, m_+ in Γ, the functor

$$(p^*_{n_+}, p^*_{m_+}) : \overline{\mathscr{C}}(n_+ \vee m_+) \to \overline{\mathscr{C}}(n_+) \times \overline{\mathscr{C}}(m_+),$$

induced from the morphisms $p_{n_+} : n_+ \vee m_+ \to n_+$ which sends m_+ to the basepoint, and is the identity on n_+, is a equivalence of categories.

(3) $\overline{\mathscr{C}}(1_+) \cong \mathscr{C}$

Proof. **(1)** Per. definition of $\overline{\mathscr{C}}$ the objects of $\overline{\mathscr{C}}(0_+)$ are collections $\{X_S, \rho_{S,T}\}$ indexed by $S \subset 0_+$ not containing the basepoint. There is only one such S, namely $S = \varnothing$, which by the axioms of $\overline{\mathscr{C}}$ gives $X_\varnothing = 0$ which is additive neutral element of the permutative category $\mathscr{C}$. Hence $\overline{\mathscr{C}}(0_+) = \bullet$, i.e. the one point category, which is terminal in Cat.
(2) Because $(n_+ \vee m_+) \cong (n+m)_+$ showing the above equivalence is equivalent to showing that

$$P_* = \prod_{i=1}^{n} (p_i)_* : \overline{\mathscr{C}}(n_+) \to \overline{\mathscr{C}}(1_+) \times \ldots \times \overline{\mathscr{C}}(1_+)$$

where $p_i : n_+ \to 1_+$ is the map which sends all element but the i'th element to the basepoint. Let $X \in \overline{\mathscr{C}}(n_+)$ then P_* is defined as

$$P_*(X) = P_*(\{X_S, \rho_{S,T}\}) = (X_{\{1\}}, \ldots, X_{\{n\}})$$

We define the inverse, $Q : \overline{\mathscr{C}}(1_+) \times \ldots \times \overline{\mathscr{C}}(1_+) \to \overline{\mathscr{C}}(n_+)$ as

$$Q((X_{\{1\}}, \ldots, X_{\{n\}})) = \left\{ \bigcup_{S \subset n_+ \setminus \{\bullet\}} \left\{ \bigoplus_{i \in S} X_{\{i\}} \right\}, \bigcup_{i,j \in n_+} \rho_{\{i\},\{j\}} : X_{\{i\}} \oplus X_{\{j\}} \to X_{\{i,j\}} \right\}$$

Now note that we may construct $\rho_{S,T}$ for any disjoint $S, T \subset n_+$, by iteratively applying $\rho_{\{i\},\{j\}}$ according to S and T. We will now describe this a bit more precise, first note that

$$\begin{tikzcd} X_S \oplus X_T \arrow[rrrr, dashed, bend right, "\rho_{S,T}"'] & \ldots . \arrow[l, "\cong"'] & \bigoplus_{p \in S} X_{\{p\}} \oplus \bigoplus_{q \in T} X_{\{q\}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & \ldots \arrow[r, "\cong"] & X_{S \cup T} \end{tikzcd}$$

This means that

$$\left\{ \bigcup_{S \subset n_+ \setminus \{\bullet\}} \left\{ \bigoplus_{i \in S} X_{\{i\}} \right\}, \bigcup_{i,j \in n_+} \rho_{\{i\},\{j\}} : X_{\{i\}} \oplus X_{\{j\}} \to X_{\{i,j\}} \right\} \cong \{X_S, \rho_{S,T}\}$$

Knowing this it is easy to see that P_* and Q are inverses. This amounts to an equivalence of categories and *not* an isomorphism, because there is a choice associated to the construction of $\rho_{S,T}$, namely the order of application of $\rho_{\{i\},\{j\}}$.

(3) There is only one subset of $1_+ \setminus \{\bullet\}$ besides the emptyset, hence an object $X \in \overline{\mathscr{C}}(1_+)$, is the "collection" $\{X_{\{1\}}\}$ where the data of the isomorphisms is superfluous, hence the objects are in one-to-one correspondence with the objects of $\mathscr{C}$. □

At this point we wish to obtain a spectrum associated to a given permutative category $\mathscr{C}$. If we apply the nerve functor N to our Γ-category $\overline{\mathscr{C}}$ we obtain a Γ-space. We can then evaluate on spheres according to 1.3.5, to obtain a spectrum.

Definition 4.1.5. The *K-theory spectrum* of a permutative category $\mathscr{C}$ is the spectrum,

$$K(\mathscr{C}) = |N_\bullet(\overline{\mathscr{C}})|(\mathbb{S}).$$

We define the i'th K-group of the permutative category $\mathscr{C}$ as

$$\pi_i(K(\mathscr{C})) := \mathrm{colim}_n\, \pi_{i+n}(K(\mathscr{C})_n)$$

Lemma 4.1.6. *For every permutative category $\mathscr{C}$ the K-theory spectrum $K(\mathscr{C})$ is a Ω-spectrum from the 1st level and up.*

Proof. Note that the nerve functor is covariant $\mathsf{Cat} \to \mathsf{sSet}_*$ and it is continuous. In particular it preserves products and terminal objects. Furthermore it sends equivalences of categories to homotopy equivalences of simplicial sets. Geometric realization is also a covariant continuous functor $\mathsf{sSet}_* \to \mathsf{Top}_*$, which sends homotopy equivalences of simplicial sets to homotopy equivalences, hence 4.1.4 gives us that $|N_\bullet\overline{\mathscr{C}}|$ is a special Γ-space. Apply 3.2.2 to obtain the desired result. □

4.2 K-theory of rings

Consider a associative unital ring R. Consider the category of finitely generated projective R-modules $\mathbb{P}_R$, whose objects are isomorphism classes of projective R-modules and morphisms are R-linear isomorphisms. The usual direct sum of modules $\oplus : \mathbb{P}_R \times \mathbb{P}_R \to \mathbb{P}_R$ gives $(\mathbb{P}_R, \oplus)$ the structure of a permutative category.

Via the above the K-theory spectrum of $\mathbb{P}_R$, $K(\mathbb{P}_R)$ is a Ω-spectrum from the 1st level and above.

We will in this section try to identify the infinite loop space of the K-theory spectrum $K(\mathbb{P}_R)$. For the sake of notational simplicity we set $P = |N_\bullet(\overline{\mathbb{P}_R})|$. Observe that we constructed P from a permutative category, this together with $N_\bullet$ and $|-|$ being continuous functors, hence preserves finite products, implies that $P(1_+)$ is topological monoid. Furthermore we have $\Omega^\infty(K(\mathbb{P}_R)) \simeq \Omega(P(S^1))$ per. definition.

Lemma 4.2.1. *We have the following equivalence of spaces $P(S^1) \cong B(P(1_+))$.*

Proof. From the proof of 4.1.6 we see that P is special, hence the n'th simplicial level of $P(S^1)$ is

$$P(S^1_n) \simeq P(1_+) \times \ldots \times P(1_+).$$

This implies that we have the following equivalence in each simplicial level of $P(S^1_\bullet) \cong \mathbb{B}_\bullet(P(1_+))$, where $\mathbb{B}_\bullet$ is the simplicial bar construction which we've already utilized a couple of times in the previous proofs. The structure maps of $\mathbb{B}_\bullet(P(1_+))$ are given as described in the proof of 2.2.11. If we apply geometric realization to this equivalence, we obtain the desired equivalence of spaces $P(S^1) \cong B(P(1_+))$. This is a consequence of [22] theorem A.4. $\square$

This lemma lets us get a better handle on $P(S^1)$, which will turn out to give us a better handle on its loop space $\Omega(P(S^1))$.

Lemma 4.2.2. *There is an isomorphism of spaces $P(1_+) \cong \bigsqcup_{[P]} B(\mathrm{Aut}(P))$.*

Proof. Note that P can be seen as the following composition of functors

$$\Gamma \xrightarrow{\overline{\mathbb{P}_R}} \mathsf{Cat} \xrightarrow{N_\bullet} \mathsf{sSet}_* \xrightarrow{|-|} \mathsf{Top}_*$$

If we consider $\overline{\mathbb{P}_R}(1_+)$, then via property (3) of 4.1.4, we obtain $\overline{\mathbb{P}_R}(1_+) \cong \mathbb{P}_R$, and then we are applying the classifying space functor $B = |N_\bullet(-)|$. Lets analyze the nerve of $\mathbb{P}_R$. Given two isomorphisms classes of projective R-modules $[P]$ and $[Q]$, there are no morphisms between them, because the morphisms in $\mathbb{P}_R$ are the R-linear isomorphisms, hence there are only compositions of automorphisms from each isomorphism class. Hence we have an isomorphism of categories $\mathbb{P}_R \cong \bigsqcup_{[P]} \mathrm{Aut}(P)$ given by sending $[P] \mapsto \mathrm{Aut}(P)$, and sending $\varphi : [P] \to [P]$ to the corresponding element in $\mathrm{Aut}(P)$. The classifying space functor commutes with disjoint unions, hence we obtain the desired isomorphism. $\square$

Before proceeding, we need a small intermezzo on cofinality of permutative categories, to better understand $P(1_+)_\infty$. This discussion is based on [23] p. 115-116.

Let $\mathscr{D}$ be a full subcategory of a symmetric monoidal category $\mathscr{C}$. If $\mathscr{D}$ contain 0 and it is closed under finite products it is also symmetric monoidal.

Definition 4.2.3. We say that $\mathscr{D}$ is *cofinal* in $\mathscr{C}$ if for every object $C \in \mathscr{C}$ there exists a $C' \in \mathscr{C}$ such that $C \oplus C'$ is isomorphic to an element of $\mathscr{D}$. If one considers isomorphism classes of objects in $\mathscr{C}$ and $\mathscr{D}$, this becomes the usual cofinality notion for abelian monoids.

Lemma 4.2.4. *Let $\mathbb{F}_R$ be the subcategory of $\mathbb{P}_R$ consisting of finitely generated free R-modules. Denote by $F = |N_\bullet(\mathbb{F}_R)|$. Then the monoid $\pi_0(F(1_+))$ is cofinal in $\pi_0(P(1_+))$.*

Proof. Note that $\mathbb{F}_R$ is cofinal in $\mathbb{P}_R$, because every projective R-module is a summand of a free R-module. The above result easily follows. $\square$

Now lets pick up where we left. Combining the lemma 4.2.1 and 4.2.2 we obtain

$$\Omega(P(S^1)) \cong \Omega(B(\bigsqcup_{[P]} B(\mathrm{Aut}(P)))). \tag{4.1}$$

The symmetry functors in $\mathbb{P}_R$ give $P(1_+)$ the structure of a homotopy commutative topological monoid. We now invoke the following version of the group completion theorem which due to Oscar Randal-Williams [24], which now applies to $P(1_+)$. The language and notation is that of G.Segal and D.McDuff's [25].

Theorem 4.2.5. *Let M be a homotopy commutative topological monoid, and denote by $[x]$ the path component of an element $x \in M$. Let $m_1, m_2, ... \in M$, be a sequence of elements such that for every $m \in M$ and $n \in \mathbb{N}$, there exists $k \geq 0$ such that $[m]$ is a right fact of $[m_{n+1} \cdot m_{n+2} \cdot ... \cdot m_{n+k}]$ in the discrete monoid $\pi_0(M)$. Form*

$$M_\infty = \mathrm{hocolim}(M \xrightarrow{\cdot m_1} M \xrightarrow{\cdot m_2} ...).$$

Then the McDuff-Segal comparison map

$$M_\infty \xrightarrow{s} \mathrm{hofib}_*(\pi) \xleftarrow[t]{} \Omega(B(M))$$

induces an isomorphism on homology with all systems of local coefficients on $\Omega(B(M))$.

Let consider $P(1_+)_\infty$, recall that the directed homotopy colimit is weakly equivalent to a mapping telescope. Because $\pi_0(F(1_+)) \to \pi_0(P(1_+))$ is cofinal, we may choose the sequence $m_1, m_2, \ldots$ to be the constant sequence $[R], [R], \ldots$, where $[R] \in \pi_0(P(1_+))$ is the path component of the free module of rank 1. Hence we may consider the following mapping telescope

$$P(1_+)_\infty \simeq \mathrm{Tel}\big(P(1_+) \xrightarrow{\oplus[R]} P(1_+) \xrightarrow{\oplus[R]} \ldots\big).$$

Theorem 4.2.5 now implies that

$$P(1_+)_\infty \to \Omega(B(P(1_+))$$

is a homology equivalence with all local coefficients on $\Omega(B(P(1_+)))$, i.e. it is an acyclic map.

As explained in [24] a consequence of this is the fact that the fundamental group of $P(1_+)_\infty$, for every choice of basepoint, has a perfect commutator subgroup. Now apply the plus-construction with respect to this perfect subgroup to each path-component of $P(1_+)_\infty$ separately. The result is an acyclic map $P(1_+)^+_\infty \to \Omega(B(P(1_+))$ which in addition induces an isomorphism of fundamental groups hence it is a weak homotopy equivalence. $\Omega(B(P(1_+))$ is an grouplike homotopy commutative H-space, therefore all its path components are homotopy equivalent.

This implies that we have the following equivalence

$$P(1_+)^+_\infty \simeq \pi_0(\Omega BP(1_+)) \times \Omega_0(BP(1_+)), \tag{4.2}$$

where $\Omega_0(BP(1_+))$ is the path component of the identity. Recall that $\pi_0(\Omega(B(M)) \cong \mathcal{G}(\pi_0(M))$ for any homotopy commutative topological monoid M, where $\mathcal{G}$ is the Grothendieck group construction. Applying this to our situation yields

$$\pi_0(\Omega(B(P(1_+))) = \mathcal{G}(\pi_0(P(1_+))) \cong \mathcal{G}(iR) := K_0(R).$$

Via this, 4.2.5, and (4.2), we have that

$$P(1_+)^+_\infty \simeq K_0(R) \times \Omega_0(BP(1_+)), \tag{4.3}$$

The path component of the identity $\Omega_0(BP(1_+))$ is given by the infinite mapping telescope

$$\mathrm{Tel}\big(B\,\mathrm{Aut}(0) \xrightarrow{\oplus[R]} B\,\mathrm{Aut}(R) \xrightarrow{\oplus[R]} \ldots\big)$$

Note that the automorphisms of R^n are $\mathrm{GL}_n(R)$, hence the telescope is $B\,\mathrm{GL}(R)$. Hence

$$P(1_+)^+_\infty \simeq K_0(R) \times B\,\mathrm{GL}(R)^+. \tag{4.4}$$

Combining by (4.1), (4.4), and (4.3)

$$K_0(R) \times B\,\mathrm{GL}(R)^+ \simeq P(1_+)^+_\infty \simeq \Omega(B(\bigsqcup_{[P]} B(\mathrm{Aut}(P)))) \simeq \Omega^\infty(K(\mathbb{P}_R))$$

This in particular shows that $B\,\mathrm{GL}(R)^+$ is homotopy equivalent to an infinite loop space.

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