Algebraic K-theory of a Finite Field

Daniela Egas Santander

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Abstract

The main goal of the present project is to give a sketch of the calculation of the algebraic K-theory of a finite field. Not all the details are developed, but several references are given where detailed proofs may be found. However, the main source of the project is D.J. Benson's book on "Representations and Cohomolgy: Cohomolgy of groups and modules" [4]. Section 1 is an introduction to principal G-bundles and uses them in order to define classifying spaces. Special attention is given to the fiber bundles coming from the Stiefel and Grassmann manifolds. These are later used to define the space BU as the union of complex Grassmann manifolds of with the weak topology. Section 2 goes further and uses BU to give a general definition of topological K-theory for paracompact spaces. Certain fundamental properties of topological K-theory are presented in this section as Bott periodicity and Adam's operations. Section 3 gives a brief introduction to homotopy fixed points and how to give to this space and additive structure. Furthermore, it is shown that one can think of the Adam's operations ψ^q as self-maps of BU, and the space $F\psi^q$ is defined as the homotopy fixed points of such self-maps. Moreover, it is shown that $F\psi^q$ is the fiber of a give fibration $BU \to BU$ and thus its homotopy groups are easily calculated from the long exact sequence in homotopy. At the end of the section, it is shown that one can construct a map $BGL(\mathbb{F}_q) \to F\psi^q$ well defined up to homotopy, that induces a homology isomorphism. Section 4 defines Quillen's plus construction and his definition of the algebraic K-groups for an arbitrary ring. Finally, Section 5 blends the results of all the previous sections in order to calculate the algebraic K-groups of a finite field by means of the map $BGL(\mathbb{F}_q) \to F\psi^q$ constructed previously.

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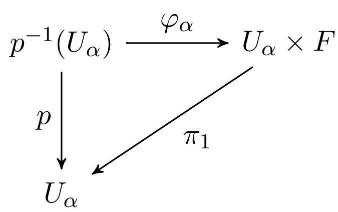
1 Principal G-bundles and the Classifying Space

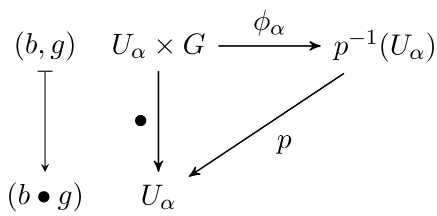
1.1 Principal G-bundles

Definition 1.1 (Fiber bundle)**.** A *fiber bundle* structure on a space E with fiber F consists on a projection map $p : E \to B$ and an open cover $\{U_\alpha\}$ of B together with homeomorphisms $\varphi_\alpha : p^{-1}(U) \to U \times F$ such that the following diagram commutes



where π_1 is the projection map onto the first factor.

Definition 1.2 (Principal G-bundle)**.** Let G be a topological group. A (locally trivial) *principal G-bundle* is a fiber bundle $p : E \to B$ with fiber homeomorphic to G and a right G action on the total space ($\bullet : E \times G \to E$) such that there is an open cover $\{U_\alpha\}$ of B, together with homeomorphisms $\phi_\alpha : U \times F \to p^{-1}(U)$ such that the following diagram commutes



Note that the action on the total space induces an action on the base space.

Definition 1.3 (Structure group)**.** Let $\xi = (p : E \to B)$ be a principal G bundle such that the action of G over F defines a continuous map $G \times F \to F$. Define the space $E \times_G F = (E \times F)/\sim$, where $\sim$ is the equivalence relation generated by $(xg, y) \sim (x, gy)$. Define $\xi[F]$ to be the bundle $\xi[F] = (p_F : E \times_G F \to B)$ where the projection map is given by $p_F(x, y) = x$. This construction makes $\xi[F]$ into a fiber bundle with fiber F. Thus $\xi[F]$ is called the fiber bundle with *structure group G and fiber F.*

Definition 1.4 (Pullback bundle)**.** Let $\xi = (p : E \to B)$ be a fiber bundle. Given a map $f : B' \to B$ we define the pullback bundle $f^*(\xi)$ to be the bundle with total space $E' = \{(x, y) \in B' \times E | f(x) = p(y)\}$ and projection map $p' : E' \to B'$ given by $p'(x, y) = x$.

The pullback bundle $f^*(\xi)$ is a fiber bundle with the same fiber as ξ. Moreover, if ξ is a principal bundle then the same is true for $f^*(\xi)$.

Remark 1.5. Let $p : E \to B$ and $p' : E' \to B'$ be n and m dimensional vector bundles respectively. Then the pullback

$$\begin{array}{ccc} E'' & \longrightarrow & E' \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ E & \underset{p'}{\longrightarrow} & B \end{array}$$

gives a $(n+m)$-dimensional vector bundle $E'' \to B$ which is called the Whitney sum of the bundles and denoted as $E \oplus E'$.

Proposition 1.6. *Let $\xi = (p : E \to B)$ be a principal G-bundle. Let $f, g : B' \to B$ be maps of base spaces. If B' is paracompact and f and g are homotopic, then $f^*(\xi) \cong g^*(\xi)$ as principal G-bundles over B'.*

Proof. Let $\xi' = p : E' \to B' \times I$ be a be a bundle over $B' \times I$. Define a map $\rho : B' \times I \to B' \times I$ as $\rho(b, t) = (b, 1)$. We will construct a map θ such that the following diagram commutes.

$$\begin{array}{ccc} E' & \overset{\theta}{\longrightarrow} & E' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B' \times I & \underset{\rho}{\longrightarrow} & B' \times I \end{array} \tag{1.1}$$

Note that this commutative diagram induces a bundle isomorphism $\xi' \cong \rho^*(\xi')$ since the maps θ and p in the diagram, factor through the total space of the pullback bundle $\rho^*(\xi')$.

Since I is a compact space and B' is paracompact, then we may choose an open cover $\{U_\alpha\}$ of B such that the restriction of ξ' to each subspace $\{U_\alpha\}$ is a product bundle i.e. we can choose homeomorphisms $\varphi_\alpha : p^{-1}(U_\alpha \times I) \to U_\alpha \times I \times G$ such that $p = \pi \circ \varphi_\alpha$, where π denotes the projection onto $U_\alpha \times I$. Let $\phi_\alpha : U_\alpha \times I \times G \to p^{-1}(U_\alpha \times I)$ be the inverse of φ_α, then we have the following commutative diagram.

$$\begin{array}{ccc} p^{-1}(U_\alpha \times I) & \overset{\phi_\alpha}{\longleftarrow} & U_\alpha \times I \times G \\ {\scriptstyle p}\big\downarrow & \swarrow{\scriptstyle \pi} & \\ U_\alpha \times I & & \end{array}$$

Moreover, again by paracompactness of B' we can assume that the cover U_α is locally finite. Then we can find a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. Since $f_\alpha(x)$

is different from zero for finitely many α we can define a new family of functions $\{f'_\alpha\}$ by setting $f'_\alpha(x) = \max_\alpha \{f_\alpha(x)\}$. Note that $\max_\alpha \{f'_\alpha(x)\} = 1$. Then define

$$\theta_\alpha(\phi_\alpha(x,t,g)) = \phi_\alpha(x, \max\{f'_\alpha(x), t\}, g)$$

$$\rho_\alpha(x,t) = (x, \max\{f'_\alpha(x), t\})$$

Notice that θ_α and ρ_α fit into a commutative diagram like the one given in diagram 1.1. Outside of U_α the diagram commutes trivially and inside U_α we have the following by diagram 1.1

$$\begin{array}{ccc} \phi_\alpha(x,t,g) & \xrightarrow{\theta_\alpha} & \phi_\alpha(x, \max\{f'_\alpha(x), t\}, g) \\ \downarrow p & & \downarrow p \\ p(\phi_\alpha(x,t,g)) = (x,t) & \xrightarrow[\rho_\alpha]{} & (x, \max\{f'_\alpha(x), t\}) = p(\phi_\alpha(x, \max\{f'_\alpha(x), t\}, g)) \end{array}$$

We can order the index family $\{\alpha\}$ and define θ and ρ via infinite compositions

$$\theta = \theta_{\alpha_1} \circ \theta_{\alpha_2} \circ \theta_{\alpha_3} \dots$$

$$\rho = \rho_{\alpha_1} \circ \rho_{\alpha_2} \circ \rho_{\alpha_3} \dots$$

Note that since $\{U_\alpha\}$ is locally finite then these compositions make sense, since for every point we can find a neighborhood where only finitely many maps are not the identity map. Note that this defines ρ as desired and it makes diagram 1.1 commute.

Now to proof the proposition, it's enough to consider a homotopy $h : B' \times I \to B$ between f and g and let $\xi' = h^*(\xi)$, then we get that $\xi' \cong \rho^*(h^*(\xi)) \cong g^*(\xi)$. By considering the inverse homotopy $\overline{h}(x,t) = h(x, 1-t)$ we get that $\xi' \cong f^*(\xi)$ proving the statement. □

Corollary 1.7. *Let B be a contractible paracompact space. Then every principal G-bundle over B is isomorphic to the product bundle $B \times G$*

Proof. Let $\xi = p : E \to B$ be a principal G-bundle over B. Since B is contractible, then have that the map $f : B \to B$ given by $f(b) = b_0$ is homotopic to the identity map. Then by the proposition the bundle ξ is isomorphic to the trivial bundle □

1.2 Classifying Spaces

Definition 1.8. Let $[B; BG]$ denote the homotopy classes of maps $B \to BG$ and $Princ_G(B)$ denote all principal G-bundles over B. A *universal G-bundle* $\xi_G : EG \to BG$ is a principal G-bundle such that for all paracompact spaces B the following map is a bijection.

$$[B; BG] \longrightarrow Princ_G(B)$$
$$f \longmapsto f^*(\xi_G)$$

Proposition 1.9. *Let B and B' be paracompact spaces. Let $\xi_G : E \to B$ and $\xi'_G : E' \to B'$ be universal G-bundles. Then there is a homotopy equivalence $f : B \to B'$ such that $\xi_G = f^*(\xi'_G)$*

Proof. We have bijections $[B', B] \leftrightarrow Princ_G(B')$ and $[B, B'] \leftrightarrow Princ_G(B)$. Let $f : B' \to B$ and $f' : B \to B'$ be the maps corresponding to ξ'_G and ξ_G respectively, i.e. $f^*(\xi_G) = \xi'_G$ and $f'^*(\xi'_G) = \xi_G$. Then we have that $(f \circ f')^*(\xi'_G) = f^*(\xi_G) = \xi'_G = id^*_{B'}(\xi'_G)$ and $(f' \circ f)^*(\xi_G) = f'^*(\xi'_G) = \xi_G = id^*_B(\xi_G)$. Thus $f \circ f' \simeq id_B$ and $f' \circ f \simeq id_{B'}$. □

Definition 1.10. Let X and Y be topological spaces. The *join* $X{*}Y$ is defined to the the quotient space $X \times I \times Y/\sim$ where $\sim$ is the equivalence relation generated by

$$(x, 0, y) \sim (x', 0, y) \qquad \forall x, x' \in X, y \in Y$$
$$(x, 1, y) \sim (x, 1, y') \qquad \forall x \in X, y, y' \in Y$$

Definition 1.11 (Milnor's construction of EG)**.** Let G be a topological space. Define the space EG to be the infinite join $G * G * G * G * \ldots$.

Note the elements of EG can be regarded to be of the form $(t_1g_1, t_2g_2, t_3g_3, \ldots)$ for $t_i \in I$, $t_i \neq 0$ for finitely many i and $\sum t_i = 1$, where we identify the pairs $(t_1g_1, t_2g_2, t_3g_3, \ldots) \sim (t_1g'_1, t_2g'_2, t_3g'_3, \ldots)$, if when $t_i \neq 0$ then $g_i = g'_i$.

Remark 1.12. If the topological group G is a CW-complex then EG is also a CW-complex. Moreover there is a free action of G on EG given by $(t_1g_1, t_2g_2, t_3g_3, \ldots)g = (t_1g_1g, t_2g_2g, t_3g_3g, \ldots)$

Definition 1.13 (Milnor's construction of BG)**.** Let G be as above. The classifiying space of G is the quotient space $BG = EG/G$.

Remark 1.14. Let p_G be the quotient map from EG to BG. Then the construction above makes $\xi_G = p_G : EG \to BG$ a locally trivial principal G-bundle.

Remark 1.15. Let EG and BG be given as in definitions 1.11 and 1.13 the principal G-bundle $\xi_G = p_G : EG \to BG$ is a universal G-bundle

Proposition 1.16. *The space EG as constructed in definition 1.11 is weakly contractible.*

Proof. Let $f : S^n \to EG$ be a map of spaces. Since S^n is a compact space, then the image of f lies on a compact subspace of $EG = G * G * G \ldots$ so it is contained in a some finite join $G_n = G * G * G \ldots * G$ n-times. However the cone $Gn * 1_G$ is naturally contained in G_{n+1} and then it is contractible. So the map f can be factored as $f : S^n \to G_n \to Gn * 1_G \hookrightarrow G_{n+1} \hookrightarrow EG$ where the first inclusion map is nullhomotopic, thus f is nullhomotopic. □

Remark 1.17. Note that if G has the homotopy type of a CW-complex then the same holds for EG and so by Whitehead's theorem this later space is contractible.

Theorem 1.18. *Let G have the homotopy type of a CW-complex. Let $\xi = E(\xi) \to B(\xi)$ be a principal G-bundle where $B(\xi)$ has the homotopy type of a CW-complex. Then ξ is a universal G-bundle if and only if $E(\xi)$ is contractible.*

Proof. Let $\xi = E(\xi) \to B(\xi)$ be a universal bundle, then by proposition 1.9 there is a homotopy equivalence $BG \overset{\simeq}{\to} B(\xi)$ and then EG and $E(\xi)$ are homotopy equivalent. Thus, by proposition 1.16 $E(\xi)$ is contractible.

Now, let $\xi = E(\xi) \to B(\xi)$ be a bundle with $E(\xi)$ contractible. Then we have the following commutative diagram.

$$\begin{array}{ccccc} E(\xi) & \longleftarrow & E(\xi) \times EG & \longrightarrow & EG \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_G} \\ B(\xi) & \longleftarrow & (E(\xi) \times EG)/G & \longrightarrow & BG \end{array}$$

Notice that the horizontal maps are fibrations with fiber EG and $E(\xi)$ which are contractible. Thus in the long exact sequence in homotopy these maps are weak homotopy equivalences. Since by theorem 1.15 the bundle $p_G : EG \to BG$ is universal, then ξ is also a universal bundle.

□

1.3 Stiefel and Grassman Manifolds

Definition 1.19 (Stiefel and Grassman Manifolds)**.** The space of n-frames in $\mathbb{R}^k$ i.e. n-tuples of orthonormal vectors in $\mathbb{R}^k$ with the subspace topology with respect to n copies of the unit sphere is the real Stiefel manifold $V_n(\mathbb{R}^k)$. The real Grassman manifold $G_n(\mathbb{R}^k)$ is defined as the space of n-dimensional vector subspaces of $\mathbb{R}^k$. The complex Stiefel manifold $V_n(\mathbb{C}^k)$ and the complex Grassman manifold $G_n(\mathbb{C}^k)$ are defined in a similar way using the hermitian inner product in $\mathbb{C}^k$.

Now, note that there are natural projections $p : V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ and $p' : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ by sending each n frame to the subspace it spans in $\mathbb{R}^k$ and $\mathbb{C}^k$. Thus we can topologize $G_n(\mathbb{R}^k)$ and $G_n(\mathbb{C}^k)$ as quotient spaces of $V_n(\mathbb{R}^k)$ and $V_n(\mathbb{C}^k)$ respectively. Now the fiber of p are n-tuples of orthonormal vectors in a fixed n-plane in $\mathbb{R}^k$. Thus the fiber is homemorphic to $V_n(\mathbb{R}^n)$. An n-tuple of orthonormal vectors in $\mathbb{R}^n$ is equivalent to an orthogonal matrix and thus the fiber of p is the orthogonal group $O(n)$. In the same way, the fiber of p' is the unitary group $U(n)$. Note that we can extend this definitions to the case where $k = \infty$ by setting $V_n(\mathbb{R}^\infty) = \cup_k V_n(\mathbb{R}^k)$, $G_n(\mathbb{R}^\infty) = \cup_k G_n(\mathbb{R}^k)$, $V_n(\mathbb{C}^\infty) = \cup_k V_n(\mathbb{C}^k)$ and $G_n(\mathbb{C}^\infty) = \cup_k G_n(\mathbb{C}^k)$. Moreover, the projection map p is actually a principal $O(n)$-bundle and the projection map p' is a principal $U(n)$-bundle.

Lemma 1.20. *The $O(n)$-bundle $V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ and the $U(n)$-bundle $V_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ are universal. Thus, we denote $EO(n) = V_n(\mathbb{R}^\infty)$, $BO(n) = G_n(\mathbb{R}^\infty)$, $EU(n) = V_n(\mathbb{C}^\infty)$ and $BU(n) = G_n(\mathbb{C}^\infty)$.*

Proof. Given that $V_n(\mathbb{R}^\infty)$, $G_n(\mathbb{R}^\infty)$, $V_n(\mathbb{C}^\infty)$ and $G_n(\mathbb{C}^\infty)$ can all be given a CW-complex structure, then it's enough to show that $V_n(\mathbb{R}^\infty)$ and $V_n(\mathbb{C}^\infty)$ are contractible spaces. The argument is exactly the same in both the real and the complex spaces. So we will analize the real case. The idea is to write a contraction of $V_n(\mathbb{R}^\infty)$. To do this first define a homotopy $h_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $h_t(x_1, x_2, x_3, ...) = (1-t)(x_1, x_2, x_3, ...) + t(0, x_1, x_2, x_3, ...)$. Note that h_t is linear with trivial kernel. Thus, we can apply h_t to an n-frame and obtain and n-tuple of linearly independent vectors to which we can apply the Gram-Schmidt process in order to make it orthonormal and thus get a deformation retraction of $V_n(\mathbb{R}^\infty)$ onto the space of n-frames with first coordinate zero. We can iterate this process n times in order to get a deformation retraction of $V_n(\mathbb{R}^\infty)$ into the space of n-frames with first n coordinates equal to zero. Now, we will compose this deformation with the deformation that takes one of this n-frames, say $(v_1, v_2, ..., v_n)$ to $(1-t)(v_1, v_2, ..., v_n) + t(e_1, e_2, ..., e_n)$ where e_i is the i-th standard vector basis in $\mathbb{R}^\infty$. This last deformation sends linearly independent vectors to linearly independent vectors, so via de Gram-Schmidt process we can write this deformation via n-frames. Then, the composition of this two deformations gives a contraction of $V_n(\mathbb{R}^\infty)$. □

Definition 1.21. We will denote $\widehat{V_n}(\mathbb{R}^\infty)$ and $\widehat{V_n}(\mathbb{C}^\infty)$ for the space of n-tuples of all linearly independent vectors in $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$ respectively.

Now, in a similar way than before we have natural projections $p : \widehat{V_n}(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ and $p' : \widehat{V_n}(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ which are then principal $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ bundles respectively. Also, $\widehat{V_n}(\mathbb{R}^\infty)$ and $\widehat{V_n}(\mathbb{C}^\infty)$ are contractible spaces, by the same arguments as for $V_n(\mathbb{R}^\infty)$ and $V_n(\mathbb{C}^\infty)$. So, these bundles are also universal. Thus we have that $BGL_n(\mathbb{R}) = G_n(\mathbb{R}^\infty) = BO(n)$ and $BGL_n(\mathbb{C}) = G_n(\mathbb{C}^\infty) = BU(n)$.

Lemma 1.22. *There is a one to one correspondance between principal $GL_n(\mathbb{R})$-bundles over B and rank n real vector bundles over B. There is also one to one correspondance between principal $GL_n(\mathbb{C})$-bundles over B and rank n complex vector bundles over B.*

Proof. Let $p : E \to B$ be a principal $GL_n(\mathbb{R})$-bundle. Then we can form the bundle with structure group $GL_n(\mathbb{R})$ and fiber $\mathbb{R}^n$, $\xi[\mathbb{R}^n]$. Then the homeomorphisms $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ are linear on each fiber and thus the fiber bundle $\xi[\mathbb{R}^n] \to B$ is a real rank n vector bundle. Now, on the opposite way let $p' : E' \to B$ be a rank n real vector bundle i.e. a vector bundle with fiber $\mathbb{R}^n$. We can form the vector bundle over B with fibers $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong M_n(\mathbb{R})$ in a canonical way. Then, the subbundle whose fibers are isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ is the principal $GL_n(\mathbb{R})$-bundle $p : E \to B$. The complex case follows in the same way. □

Theorem 1.23. *Let B be a paracompact space. There is a one to one correpondance between rank n real vector bundles over B and homotopy classes of maps $f : B \to G^n(\mathbb{R}^\infty)$.*

Proof. From lemma 1.22 we know that there is a one to one correspondance between principal $GL_n(\mathbb{R})$-bundles over B and rank n real vector bundles over B given by associating a principal $GL_n(\mathbb{R})$-bundle ξ with the bundle $\xi[\mathbb{R}^n]$ with structure group

$GL_n(\mathbb{R})$ and fiber $\mathbb{R}^n$. Let $\xi = \widehat{V_n}(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$. Then, $\xi[\mathbb{R}^n] = \widehat{G_n}(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ where

$$\widehat{G_n}(\mathbb{R}^\infty) = (\widehat{V_n}(\mathbb{R}^\infty) \times \mathbb{R}^n)/GL_n(\mathbb{R}) = (\widehat{V_n}(\mathbb{R}^\infty) \times \mathbb{R}^n)/O(n)$$

Thus to the map $f : B \to G^n(\mathbb{R}^\infty)$ we can associate the rank n-vector bundle correspoding to the one obtained by pulling back the cannonical bundle $\widehat{G^n}(\mathbb{R}^\infty) \to G^n(\mathbb{R}^\infty)$. □

Theorem 1.24. *Let B be a paracompact space. There is a one to one correpondance between rank n complex vector bundles over B and homotopy classes of maps $f : B \to G^n(\mathbb{C}^\infty)$.*

Proof. The same proof as in the real case holds, where the vector bundle corresponding to f is the pullback bundle obtained from the cannonical bundle $\xi[\mathbb{C}^n] = \widehat{G^n}(\mathbb{C}^\infty) \to G^n(\mathbb{C}^\infty)$. In this case, the canonical bundle is bundle of structure group $GL_n(\mathbb{C})$ and fiber $\mathbb{C}^n$ obtained from the bundle $\xi = \widehat{V_n}(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$. Thus we have that:

$$\widehat{G_n}(\mathbb{C}^\infty) = (\widehat{V_n}(\mathbb{C}^\infty) \times \mathbb{C}^n)/GL_n(\mathbb{C}) = (\widehat{V_n}(\mathbb{C}^\infty) \times \mathbb{C}^n)/O(n)$$

□

1.4 Properties of the Classifying Space

Theorem 1.25. *The Milnor's construction of the classifying space BG has the following properties:*

i. There is a homotopy equivalence $\Omega BG \xrightarrow{\simeq} G$.

ii. $\pi_i(BG) \cong \pi_{i-1}(G)$.

iii. If G is discrete then BG is an Eilenberg-MacLane space i.e. $BG = K(G, 1)$ and EG is it's universal cover.

Proof. To prove i consider the path-loop fibration $\Omega BG \to PBG \to BG$ and the pullback

$$\begin{array}{ccc} X & \longrightarrow & EG \\ \downarrow & & \downarrow p_G \\ PBG & \longrightarrow & BG \end{array}$$

Then we have the following diagram of fibrations

$$
\begin{array}{ccccc}
 & & G & \xrightarrow{id_G} & G \\
 & & \downarrow & & \downarrow \\
\Omega BG & \longrightarrow & X & \longrightarrow & EG \\
{\scriptstyle id_{\Omega BG}}\downarrow & & \downarrow & & \downarrow {\scriptstyle p_G} \\
\Omega BG & \longrightarrow & PBG & \longrightarrow & BG
\end{array}
$$

Thus $\Omega BG \simeq \Omega BG \times EG \simeq X \simeq G \times PBG \simeq G$

To prove *ii* consider the long exact sequence for the fibration $G \to EG \to BG$

$$\ldots \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \pi_{n-1}(EG) \to \ldots$$

Since EG is contractible we have that $\pi_i(BG) \cong \pi_{i-1}(G)$.

To prove *iii* if G is discrete we have a map $K(G,1) \to BG$ which is an isomorphism on the fundamental group. Let $\tilde{K}$ be the universal cover of $K(G,1)$, then this is a contractible space as well as EG. Thus we have a lift $\tilde{K} \to EG$ of the map $K(G,1) \to BG$. This induces a commutative square in higher homotopy groups, showing that $\pi_n(BG) = 0 \quad \forall n \geq 2$. Thus BG and $K(G,1)$ are weakly equivalent and by Whiteheads theorem we have the desired result. □

Thus we can define the cohomology of an arbitrary topological group G by $H^n(G,R) = H^n(BG,R)$ and the previous theorem shows that this is in fact a generalization of the definition of group cohomology given through Eilenberg-MacLane spaces for discrete groups.

2 Topological K theory

2.1 Basic Constructions

Definition 2.1. Let B be a compact space. We write $K(B)$ for the additive group with generators $[E]$ correponding to the complex vector bundles $E \to B$ and with relators $[E] + [E'] = [E'']$ whenever $E'' \to B$ is the Whitney sum of $E \to B$ and $E' \to B$

We have canonical inclusion $G_n(\mathbb{C}^m) \hookrightarrow G_{n+1}(\mathbb{C}^{m+1})$ giving rise to inclusion maps $BU(n) \hookrightarrow BU(n+1)$. Thus we set $BU = \bigcup_{n\geq 1} BU(n)$ with the weak topology coming from these inclusion maps.

Proposition 2.2. *If B is a compact space, then $K(B) \cong [B; BU \times \mathbb{Z}]$*

Proof. Note that both $K(B)$ and $[B; BU \times \mathbb{Z}]$ are additive in each connected component. So we can assume with out loss of generality that B is connected. In this scenario, a map $B \to BU \times \mathbb{Z}$ is represented by a map $B \to BU$ and an integer. Now, recall that there is a one to one correspondance between rank n complex vector bundles and maps $B \to BU(n)$ where $BU(n) = G_n(\mathbb{C}^\infty)$. Thus there is clearly a map $K(B) \to [B; BU \times \mathbb{Z}]$ that takes a generator $[E]$ that corresponds to a complex vector bundle $E \to B$ it's corresponding map $B \to BU(n) \hookrightarrow BU$ and to the constant map $B \to \mathbb{Z}$ at $n \in \mathbb{Z}$. Now, to obtain the inverse map $[B; BU \times \mathbb{Z}] \to K(B)$ take an element of $[B; BU \times \mathbb{Z}]$ given by a map $B \to BU$ and an integer m. Since B is compact, then the image of $B \to BU$ lies on $BU(n)$ for some n and thus corresponds to a complex vector bundle $E \to B$ of dimension n. Then, we can add or subtract a trivial bundle in order to obtain a bundle of dimension m and thus a well defined element of $K(B)$. □

Remark 2.3. Consider a map of compact spaces $f : B' \to B$. Then we can take the pullback of vector bundles and obtain a map $f^* : K(B) \to K(B')$.

Definition 2.4 (Reduced K-theory)**.** Let B be a compact space with base point x. The maps $x \hookrightarrow B \twoheadrightarrow x$ gives rise to maps $\mathbb{Z} = K(x) \to K(B) \to K(x) = \mathbb{Z}$ whose composition equals the identity map. Define $\tilde{K}(B)$ to be the kernel fo the map $K(B) \to \mathbb{Z}$. Note that $K(B) \cong \tilde{K}(B) \oplus \mathbb{Z}$

2.2 Fundamental Properties of $K(B)$

Lemma 2.5. *Let B be a compact space, and let B' be a closed subspace of B. Then the maps $B' \hookrightarrow B \twoheadrightarrow B/B'$ induce the following exact sequence*

$$\tilde{K}(B/B') \to \tilde{K}(B) \to \tilde{K}(B')$$

Proof. It is clear that by construction the composition map is zero. Now, every element of $\tilde{K}(B)$, $\tilde{K}(B')$ and $\tilde{K}(B/B')$ are represented by actual bundles. Consider a bundle over B, whose restriction as a bundle over B' is represented by the zero element in $\tilde{K}(B')$. Then, this restricted bundle is equivalent to a trivial bundle by adding a suitable trivial bundle. From this trivialization (which is not unique) we can contruct a vector bundle over B/B' such that when taking the pullback over the map $B \to B/B'$ we obtain a bundle over B which is represented by the original element chosen in $K(B)$. □

This definition of $K(B)$ and the last stated property allows us to understand $\tilde{K}(B)$ as the zeroth part of a generalized cohomology theory and thus define higher K-groups via the following definition.

Definition 2.6 (Higher K-groups)**.** Let B be a compact space and $*$ be an independent point. We define for $n \geq 0$

$$\tilde{K}^{-n}(B) = \tilde{K}(S^n B), \qquad K^{-n}(B) = \tilde{K}^{-n}(B \dot{\cup} \{*\})$$

Now, that we have defined the topological K-groups as part of a reduced cohomology theory we need a way to calculate $K^n(B)$ for n sufficiently large, as the definition given in 2.6 allows to calculate these groups in the negative direction. We are able to do so by means of the following theorem, which is deep result in topological K-theory. The proof of this theorem goes beyond the scope of this project, for a detailed proof refer for example to [12] or [7].

Theorem 2.7 (Bott Periodicity)**.** *There are natural isomorphisms*

$$K^{-n-2}(B) \cong K^{-n}(B)$$

Moreover, via the adjunction between S and Ω together with proposition 2.2 the Bott Periodicity theorem can be restated as:

$$\Omega^2 BU \simeq BU \times \mathbb{Z}$$

One can see that:

$$K^0(\{*\}) \cong [*, BU \times \mathbb{Z}] \cong \mathbb{Z} \oplus \mathbb{Z} \cong \tilde{K}(\{*\}) \oplus \mathbb{Z}$$

Thus we have that $\tilde{K}^0(\{*\}) \cong \mathbb{Z}$. Moreover, in the same way, we have that:

$$\tilde{K}(S^q) \cong [S^q; BU] = \pi_q(BU)$$

Theorem 2.7 shows that these homotopy groups are periodic (of period two). Thus in order to calculate all the coefficient groups $\tilde{K}(S^q)$ it's enough to calculate $\pi_1(BU)$. The homotopy equivalence $G \simeq \Omega BG$ gives:

$$\pi_q(BG) = [S^q; BG] \cong [S^{q-1}; \Omega BG] \cong \pi_{q-1}(G)$$

Therefore, it is enough to calculate $\pi_0(U, 1)$. We have an epimorphism $\pi_q(U(n-1), 1) \to \pi_q(U(n), 1)$. Moreover, $U(1) = S^1$ and $\pi_0(U(1), 1) = 0$. Thus we have that:

$$\tilde{K}^0(\{*\}) = \mathbb{Z}$$
$$\tilde{K}^{-1}(\{*\}) = \tilde{K}^0(S^1) = 0$$

Thus the Bott periodicity theorem gives us a way to define functors K^n from the category of compact spaces to the category of groups $\forall n \geq 0$; and understanding

$\tilde{K}^0$ as the zeroth part of a reduced cohomology theory, we can extend this definition in the negative direction.

The next goal is to calculate the K-theory of BG for G a compact Lie group, and Atiya's completion theorem will provide an answer to this question. Firstly, we have defined $K(B)$ for B a compact space only. However, BG is a CW-complex and thus paracompact and we can extended the previous definition to all paracompact spaces following result 2.2.

Definition 2.8. Let B be a paracompact space. We define

$$K(B) = [B; BU \times \mathbb{Z}]$$

However, if B is a CW-complex and B^n is it's n-th squeleton it is not true that $K(B) = \varprojlim K(B^n)$. Nevertheless, we can apply Milnor's sequence [4], [12] to definition 2.8 and obtain

$$0 \to \varprojlim{}^1 K^{-1}(B^n) \to K(B) \to \varprojlim K(B^n)$$

Moreover, if B is the classifying space of a finite group or a compact Lie group, then the term $\varprojlim^1 K^{-1}(BG^n)$ vanishes [4] and thus we have that $K(B) \cong \varprojlim K(B^n)$. Thus, we can relate each element of $K(BG)$ with an element of the $K(BG^n)$ for some n. Thus, each vector bundle over B gives rise to an element of $K(B)$. Now, let $\mathcal{R}(\mathbb{C}G)$ be the Grothendieck ring of finite dimensional complex representations [3], then we have a natural map

$$\mathcal{R}(\mathbb{C}G) \to K(BG)$$

This map is not an isomorphism, however it induces an isomorphism when taking the completion of this ring as it is stated in the following important theorem. The proof of this theorem is out of the scope of this project see for example [1] or [2].

Theorem 2.9 (Atiyah completion theorem)**.** *Let $\mathcal{R}(\mathbb{C}G)$ be the Grothendieck ring of finite dimensional complex representations. Let I_G be the kernel of the augmentation map that sends each representation to it's dimension $\mathcal{R}(\mathbb{C}G) \to \mathbb{Z}$. Let $\mathcal{R}(\mathbb{C}G)^\wedge$ denote the completion of $\mathcal{R}(\mathbb{C}G)$ with respect to this ideal. That is*

$$\mathcal{R}(\mathbb{C}G)^\wedge = \varprojlim \mathcal{R}(\mathbb{C}G)/I_G^n$$

Then, $K^1(BG) = 0$ and the natural map $\mathcal{R}(\mathbb{C}G) \to K(BG)$ induces an isomorphim $\mathcal{R}(\mathbb{C}G)^\wedge \cong K(BG)$

Via the Whitney sum and tensor product of vector bundles we can give an algebra structure to $K(B)$ and $\tilde{K}(B)$. Moreover, we can also define a special kind of ring homomorphisms the Adam's operations which we will define now.

Proposition 2.10 (Adam's operations)**.** *Let B be a compact Hausdorff space. Then $\forall n \geq 0$ $n \in \mathbb{Z}$ there are ring homomorphims $\psi^k : K(B) \to K(B)$ with the following properties:*

1. *Let f be a map $f : B \to B'$, then $\psi^k f^* = f^* \psi^k$.*

2. *If* L *is a line bundle, then* $\psi^k(L) = L^k$*.*

3. $\psi^k \circ \psi^l = \psi^{kl}$

Proof. A sketch of the proof will be presented now. The basic idea is to construct these ring homomorphisms based on the exterior powers of E, $\lambda^i(E)$. Recall that from the properties of vector spaces and exterior powers [11] that we know that:

- $\lambda^k(E \oplus E') \approx \oplus_i(\lambda^i(E) \otimes \lambda^{k-i}(E'))$
- $\lambda^0(E) = 1$ where 1 represents the trivial line bundle
- $\lambda^1(E) = E$
- $\lambda^k(E) = 0$ for k greater than the maximum dimension of the fibers of E

If $E = L_1 \oplus L_2 ... \oplus L_n$ where L_i is a line bundles $\forall\ 1 \geq i \geq n$, one can construct a polynomial with integral coefficients s_k such that $s_k(\lambda^1(E), \lambda^2(E)...., \lambda^n(E)) = L_1^k + L_2^k + ... + L_n^k$ for $k \geq 0$ [8]. Then, we define for an arbitrary element $[E]$ of $K(B)$ $\psi^k(E) = s_k(\lambda^1(E), \lambda^2(E)...., \lambda^n(E))$. Now, in order to show that this definition satisfies the required properties, we shall make use of the following principle.

Lemma 2.11 (Splitting Principle)**.** *Let* B *be a compact Hausdorff space and* $E \to B$ *be a given vector bundle. There is a compact Hausdorff space* B' *and a map* $p : B' \to B$ *such that the induced map* $p^* : K(B) \to K(B')$ *is injective and the pullback* $p^*(E)$ *splits as the sum of line bundles.*

For a proof of the splitting principle refer for example to [8] . From the properties of exterior algebras we know that $f^*(\lambda^i(E)) = \lambda^i(f^*(E))$ and thus property 1 follows clearly. Property 2 follows from the defintion of ψ^k and the construction of s_k. Now, to observe that property 3 holds one must first notice that this definition of the Adam's operation is additive, in the sense that $\psi^k(E \oplus E') = \psi^k(E) + \psi^k(E')$. This holds since by the splitting principle one can pullback and split E and then pullback again in order to split E'. Now, since this operations are additive, it is enought to show that property 3 holds for an arbitraty line bundle L; but in this case we have that $\psi^k(\psi^l(L)) = \psi^k(L^l) = L^{kl} = \psi^{kl}(L)$ □

3 The spaces $F\psi^q$ and $BGL(\mathbb{F}_q)$

We have defined the Adam's operations on $K(B)$ in the previous section, for B compact. In a similar fashion we can define Adam's operations on $\tilde{K}(B)$ as homomorphisms $\psi^q : \tilde{K}(B) \to \tilde{K}(B)$. Now, we can interpret ψ^q as natural transformations on $\tilde{K}(B)$ and since $\tilde{K}(B)$ is representable as $[B; BU]$ we wish to apply Yoneda's lemma in order to view the Adam's operations ψ^q as maps $BU \to BU$. However, BU is not a compact space and thus we can not directly apply our definition of Adam's operations to $\tilde{K}(BU)$. Nevertheless, BU can be given a CW-complex structure with cells only in even dimension [8]. Then we have that $BU = \bigcup X^m$ where $X^m = X^{m+1}$ $\forall m \in 2\mathbb{Z}, m \geq 0$. Now we can apply an inductive argument in order to show that $\tilde{K}^{-1}(X^m) = 0\ \forall m \in \mathbb{Z}, m \geq 0$. First, via the Bott periodicity theorem we have shown that:

$$\tilde{K}^{-1}(X^0) = \tilde{K}^{-1}(\{*\}) = 0$$
$$\tilde{K}^{-1}(X^1) = \tilde{K}^{-1}(X^0) = 0$$

Let $\tilde{K}^{-1}(X^{2m}) = 0$, then we have that $\tilde{K}^{-1}(X^{2m+1}) = \tilde{K}^{-1}(X^{2m}) = 0$. Now consider the inclusion $X^{2m} \hookrightarrow X^{2m+2}$. Since BU has only cells in even dimension, then we have that $X^{2m+2}/X^{2m} \simeq \vee_\alpha S^{2m+2}$ where α runs over the $(2m+2)$-cells of BU. Then by lemma 2.5 and since $\tilde{K}^n$ defines a reduced cohomology theory we have the following exact sequence:

$$\tilde{K}^{-1}(\vee_\alpha S^{2m+2}) \to \tilde{K}^{-1}(X^{2m+2}) \to \tilde{K}^{-1}(X^{2m})$$

Moreover, by the wedge axiom we have that

$$\tilde{K}^{-1}(\vee_\alpha S^{2m+2}) \cong \prod_\alpha \tilde{K}^{-1}(S^{2m+2}) \cong \prod_\alpha \tilde{K}^{-1}(\{*\}) = 0$$

Then, we have that $\tilde{K}^{-1}(X^{2m+2}) = 0$. Thus, the functor $\tilde{K}^{-1}$ vanishes in the squeleta of BU and thus we can apply the Milnor exact sequence to this scenario

$$0 \to \varprojlim{}^1 \tilde{K}^{-1}(X^m) \to \tilde{K}(BU) \to \varprojlim \tilde{K}(X^m) \to 0$$

and obtain that $\tilde{K}(BU) \cong \varprojlim \tilde{K}(X^m)$. Thus we can define the Adam's operations on $\tilde{K}(BU)$ via this isomorphism and then interpret them as maps $BU \to BU$.

3.1 Homotopy fixed points

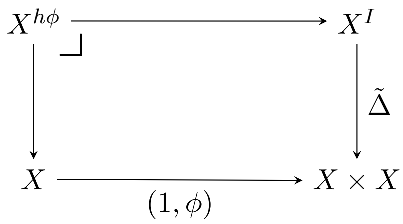
Recall the definition of fixed points of a self-map

Definition 3.1. Let $\Delta : X \to X \times X$ denote the diagonal map. Let $\phi : X \to X$ be a self-map of a space. The fixed points of ϕ denoted as X^ϕ are obtained by taking the following pullback.

$$\begin{array}{ccc} X^\phi & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \Delta \\ X & \xrightarrow[(1,\phi)]{} & X \times X \end{array}$$

In a similar fashion, we will define the homotopy fixed points of a map by replacing the diagonal map by a homotopy equivalent fibration. Let $\tilde{\Delta} : X^I \to X \times X$ denote the map that takes a path to it's end points. Note that this map is a fibration. Moreover, recall that the inclusion $X \hookrightarrow X^I$ which takes a point in X to the constant path in X^I is a homotopy equivalence; and note that the composition of this inclusion with $\tilde{\Delta}$ i.e. $X \hookrightarrow X^I \overset{\tilde{\Delta}}{\to} X \times X$ gives the diagonal map Δ.

Definition 3.2. Let $\tilde{\Delta}$ be defined as above. Let $\phi : X \to X$ be a self-map of a space. The homotopy fixed points of ϕ denoted as $X^{h\phi}$ are obtained by taking the following pullback.



Note that the vertical maps in this diagram are fibrations with fiber the loop space of X i.e. ΩX.

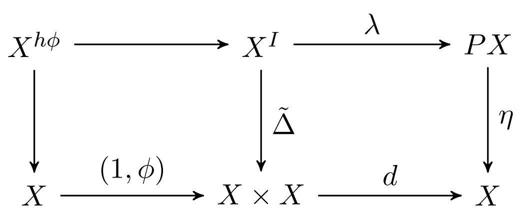
We want to give more structure to the space of homotopy fixed points.

Definition 3.3. Let $(X, *)$ be a space with base point. We say $(X, *)$ has an additive structure if there is map $+ : X \times X \to X$, the addition map, that is associative and commutative up to homotopy and a map $\epsilon : X \to X$ which behaves as an additive inverse up to homotopy.

Note that the composition of these two maps $d : X \times X \overset{(1,\epsilon)}{\to} X \times X \overset{+}{\to} X$ gives a subtration map d on $(X, *)$.

Lemma 3.4. *If X has an additive struture, then $X^{h\phi}$ is the homotopy fiber of the map $1 - \phi$.*

Proof. Let PX be the path space of X, which is contractible. Let $\lambda : X^I \to PX$ be the map given by $\lambda([t \mapsto \omega(t)]) = [t \mapsto d(\omega(t), \omega(0))]$; this is the map that changes the starting point of a path to the origin $\{*\}$. Let $\eta : PX \to X$ be the map that sends each path to it's end point i.e. $\eta([t \mapsto \omega(t)]) = \omega(1)$. Then we can extend the previous pullback diagram to the following diagram:



Thus we can see that the space of homotopy fixed points $X^{h\phi}$ is the homotopy fiber of the map $1 - \phi = d \circ (1, \phi)$.

□

Lemma 3.5. *If X has and additive struture, then so does $X^{h\phi}$.*

Proof. Let $P_{1-\phi} = \{(x, \omega)|x \in X, \omega : I \to X, \omega(1) = d(x, \phi(x))\}$. Then $P_{1-\phi}$ is homotopy equivalent to X and the map $P_{1-\phi} \to X$ that sends $(x, \omega) \mapsto \omega(1) = (1-\phi)(x)$ is a fibration with fiber $X^{h\phi}$. Then, the addition map on X gives addition maps as in the following diagram which thus define an addition map on $X^{h\phi}$.

$$\begin{array}{ccc}
X^{h\phi} \times X^{h\phi} & \overset{+}{\dashrightarrow} & X^{h\phi} \\
\downarrow & & \downarrow \\
P_{1-\phi} \times P_{1-\phi} & \longrightarrow & P_{1-\phi} \\
\downarrow & & \downarrow \\
X \times X & \overset{+}{\longrightarrow} & X
\end{array}$$

In the same way, the inverse map on X, gives maps as in the following diagram and defines an inverse map in $X^{h\phi}$.

$$\begin{array}{ccc}
X^{h\phi} & \overset{\epsilon}{\dashrightarrow} & X^{h\phi} \\
\downarrow & & \downarrow \\
P_{1-\phi} & \longrightarrow & P_{1-\phi} \\
\downarrow & & \downarrow \\
X & \overset{\epsilon}{\longrightarrow} & X
\end{array}$$

□

Proposition 3.6. *Let X be a space with additive structure and $\phi : X \to X$ be a self-map. Let Y be a space such that every map $Y \to \Omega X$ is homotopic to the constant map. Let $[Y; X]^\phi$ denote the fixed points of $[Y; X]$ under composition with ϕ. Then $[Y; X^{h\phi}] \cong [Y; X]^\phi$*

Proof. Note first that a map $\tilde{f} : Y \to X^{h\phi}$ is equivalent to a map $f : Y \to X$ and a homotopy between f and $\phi \circ f$. Thus, we can construct a surjective map $\Phi : [Y; X^{h\phi}] \to [Y; X]^\phi$ by setting $\Phi(\xi) = p \circ \xi$ where p is the fibration $p : X^{h\phi} \to X$. Now, by using the additive structures in X and $X^{h\phi}$ it can be seen that this map is in fact a surjective homomorphism. Let $\xi \in Ker\Phi$, then by the homotopy lifting property of the fibration $p : X^{h\phi} \to X$, the map ξ is homotopic to a map

$\tilde{\xi} : Y \to X^{h\phi}$ with image in the fiber of p which is the loop space ΩX.

$$\begin{array}{ccc} Y & \overset{\tilde{\xi}}{\dashrightarrow} & \Omega X \\ \downarrow & & \downarrow \\ Y \times 0 & \longrightarrow & X^{h\phi} \\ \downarrow & & \downarrow \\ Y \times I & \longrightarrow & X \end{array}$$

Since all maps $Y \to \Omega X$ are homotopic to the constant map, then so is ξ and thus the kernel is trivial showing that $[Y; X^{h\phi}] \cong [Y; X]^{\phi}$. □

3.2 The space $F\psi^q$

In the previous section we have shown that we can interpret the Adam's operations ψ^q on $\tilde{K}(BU)$ as self-maps $BU \to BU$. Thus we can consider the space of homotopy fixed points of these maps, Quillen denoted this space $F\psi^q$

$$\begin{array}{ccc} F\psi^q & \longrightarrow & BU^I \\ \downarrow & \lrcorner & \downarrow \tilde{\Delta} \\ BU & \xrightarrow{(1,\psi^q)} & BU \times BU \end{array} \tag{3.1}$$

Proposition 3.7. *The homotopy groups of $F\psi^q$ are given by*

$$\pi_{2j-1}(F\psi^q) = \mathbb{Z}/(q^{2j} - 1) \qquad \pi_{2j}(F\psi^q) = 0$$

Proof. As seen in the previous section, $F\psi^q$ is the homotopy fiber of the map $1-\psi^q : BU \to BU$. Thus we have a long exact sequence in homotopy

$$... \to \pi_j(BU) \overset{(1-\psi^q)^*}{\longrightarrow} \pi_j(BU) \to \pi_{j-1}(F\psi^q) \to \pi_{j-1}(BU) \overset{(1-\psi^q)^*}{\longrightarrow} ...$$

Now recall that:

$$\pi_j(BU) = [S^j; BU] = \tilde{K}(S^j) = \begin{cases} \mathbb{Z} & \text{if j is even} \\ 0 & \text{if j is odd} \end{cases}$$

Moreover, by our definition of ψ^q one can see that this map acts as multiplication by q on $\tilde{K}(S^2)$ and then by Bott periodicity it acts as multiplication by q^j on $\tilde{K}(S^j)$ which gives the desired result. □

Lemma 3.8. *$F\psi^q$ is a simple space*

Proof. We can analize the long exact sequence above as a sequence of modules over $\pi_1(F\psi^q)$ and since $\pi_1(BU) = \tilde{K}(S^1) = 0$ the action of $\pi_1(F\psi^q)$ is trivial on $\pi_n(F\psi^q)$. □

3.3 Relation between $BGL(\mathbb{F}_q)$ and $F\psi^q$

The aim of this section is to find a well defined map between these two spaces $BGL(\mathbb{F}_q) \to F\psi^q$ which we will later show is a homology isomorphism. This map is fundamental in the calculation of the algebraic K-groups of the finite field $\mathbb{F}_q$.

Lemma 3.9. *There is a well defined map* $\mathcal{R}(\mathbb{F}_q G) \to [BG; F\psi^q]$

Proof. Recall that there is a well defined map $\mathcal{R}(\mathbb{C}G) \to K^0(BG) = [BG; BU \times \mathbb{Z}]$. We can compose this map with the projection map to reduced K-theory $K^0(BG) \to \tilde{K}^0(BG) = [BG; BU]$ to obtain a map $\mathcal{R}(\mathbb{C}G) \to \tilde{K}^0(BG) = [BG; BU]$. We have defined Adam's operations on $K^0(BG)$ and $\tilde{K}^0(BG)$ and in a similar fashion they can be defined in $\mathcal{R}(\mathbb{C}G)$ [3]. In this sense, this composition map commutes with the Adam's operations i.e. we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\mathbb{C}G) & \longrightarrow & \tilde{K}^0(BG) \\ \downarrow \psi^q & & \downarrow \psi^q \\ \mathcal{R}(\mathbb{C}G) & \longrightarrow & \tilde{K}^0(BG) \end{array}$$

Thus, we we a well defined map $\mathcal{R}(\mathbb{C}G)^{\psi^q} \to [BG; BU]^{\psi^q}$. Now by Atiyah's completion theorem (2.9) together with the fact that $\Omega BG \cong G$ we have that

$$0 = \tilde{K}^1(BG) = [BG; U] \cong [BG; \Omega BU]$$

Thus we can apply proposition 3.6 and hence $[BG; BU]^{\psi^q} \cong [BG; BU^{\psi^q}]$ thus we have a well defined map $[BG; BU]^{\psi^q} \to [BG; BU^{\psi^q}]$. Moreover, the Brauer lift ([3] gives an isomorphism $\mathcal{R}(\mathbb{F}_q G) \cong \mathcal{R}(\mathbb{C}G)^{\psi^q}$. Then the composition

$$\mathcal{R}(\mathbb{F}_q G) \overset{\cong}{\to} \mathcal{R}(\mathbb{C}G)^{\psi^q} \to [BG; BU]^{\psi^q} \to [BG; BU^{\psi^q}] = [BG; F\psi^q]$$

gives the desired map. Note then, that give a finitely generated $\mathbb{F}_q G$-module, we can obtain a map $BG \to F\psi^q$ well defined up to homotopy. □

Now, we can embedd $GL_n(\mathbb{F}_q) \hookrightarrow GL_{n+1}(\mathbb{F}_q)$ by sending a matrix A to the augmented matrix which agrees with the identity matrix in the last row and colum. Then set $GL(\mathbb{F}_q) = \bigcup_{n \geq 1} GL_n(\mathbb{F}_q)$ with the given inclusions. There is a natural $(n+1)$-dimensional module struture for $GL_{n+1}(\mathbb{F}_q)$ which restricts to the natural n-dimensional module structure of $GL_n(\mathbb{F}_q)$ and a trivial module. Thus, we can construct a map $BGL_n(\mathbb{F}_q) \to BGL_{n+1}(\mathbb{F}_q)$ and we can compose this map with the map obtained above via the Brauer lift $BGL_{n+1}(\mathbb{F}_q) \to F\psi^q$. Thus with these definitions we get a map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ well defined up to homotopy.

Theorem 3.10 (Quillen)**.** *Let p be a primer number and let $q = p^d$ for some $d \geq 1$. Let l be a prime number that does not divide q and let r be the multiplicative order of q modulo l. Let $n = mr + e$ such that $0 \leq e < r$. Let C be a cyclic group of order $q^r - 1$. Then:*

i The ring $H^*(F\psi^q; \mathbb{F}_l)$ is generated by classes c_{jr} and e_{jr} with degrees $deg(c_{jr}) = 2jr$ and $deg(e_{jr}) = 2jr - 1$ under the relations:

$$e_{jr}^2 = \begin{cases} 0 & \text{for the typical case} \\ \sum_{a=0}^{jr} c_a c_{2jr-1-a} & \text{for the exceptional case} \end{cases}$$

where the typical case is give by l odd or $l = 2$, $q \equiv 1$ mod 4 and the exceptional case is give by $l = 2$ and $q \equiv 3$ mod 4.

ii There are restriction maps

$$H^*(F\psi^q; \mathbb{F}_l) \to H^*(GL_n(\mathbb{F}_q); \mathbb{F}_l) \to (\otimes^m H^*(C; \mathbb{F}_l)^{\mathbb{Z}/r})^{\Sigma_m}$$

The first map is surjective. The second map is injective and for $l \neq 2$ it is an isomorphism.

iii The ring $H^*(GL_n(\mathbb{F}_q))$ is generated by the classes c_{jr} and e_{jr} for $1 \leq j \leq m$ subject to the same relations as in i.

Proof. A proof of this theorem will not be provided, refer to [9] and [4]. However, the idea of the proof is the following. The space $F\psi^q$ has been defined as a given pullback as shown in diagram 3.1. Thus we can apply the Eilenberg-Moore spectral sequence to this scenario. Via the Serre spectral sequence, one can calculate the cohomology of BU, BU^I and $BU \times BU$ and use this result to build the E_2 page of the the Eilenberg-Moore spectral sequence of the pullback diagram 3.1. This is a second quadrant spectral sequence, and the differentials go from the second to the first cuadrant, thus we have that $E_2 = E_\infty$ from which one can obtain the cohomology groups of $F\psi^q$ i.e. $H^n(F\psi^q; \mathbb{F}_l)$. Thus, it is left to find the ring structure of $H^*(F\psi^q; \mathbb{F}_l)$. In order to do this, consider a cyclic group C as given above; then $C \cong \mathbb{F}_{q^r}^\times$. Thus C^m has a faithful representation of dimension mr over $\mathbb{F}_q$ giving rise to an embedding $C^m \hookrightarrow GL_n(\mathbb{F}_q)$. Now, this together with the map coming from the Brauer lift as shown above gives us a sequence of maps

$$B(C^m) \to BGL_n(\mathbb{F}_q) \to F\psi^q$$

This sequence induces the desired restriction in cohomology. □

Theorem 3.11. *The map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ as defined above induces isomophisms $H_n(BGL(\mathbb{F}_q)) \to H_n(F\psi^q)$ $\forall n \geq 0, n \in \mathbb{Z}$*

Proof. Quillen's calculation given in theorem 3.10 gives that the map θ is a mod l cohomology isomorphism for all primes $l \neq p$. Now, Quillen also shows [9] that the mod p cohomology vanishes in positive degrees of at most d(p-1). On the other hand, from proposition 3.7 it follows that there is no p-torsion in $\pi_*(F\psi^q)$ and then by the p-local version of the Hurewicz theorem it follows that $H_i(F\psi^q; \mathbb{F}_p) =$

0 for $i > 0$. In the same way, the Hurewicz theorem in characteristic zero gives that $H_i(F\psi^q;\mathbb{Q}) = 0$ for $i > 0$. Therefore, the map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ induces cohomology isomorphims $H^n(F\psi^q;\mathbb{F}_p) \to H^n(BGL(\mathbb{F}_q);\mathbb{F}_p)$ for all primes p and all $n \geq 0$ and a cohomology isomorphism $H^n(F\psi^q;\mathbb{Q}) \to H^n(BGL(\mathbb{F}_q);\mathbb{Q})$ for all $n \geq 0$. Then, by the relative version of the universal coefficient theorem, the map θ induces cohomology isomorphisms $H^n(F\psi^q;\mathbb{Z}) \to H^n(BGL(\mathbb{F}_q);\mathbb{Z})$ and thus it induces homology isomorphisms $H_n(BGL(\mathbb{F}_q);\mathbb{Z}) \to H_n(F\psi^q;\mathbb{Z})$ for all $n \geq 0$ as desired. □

4 Quillen's Algebraic K-theory

In this section we will define algebraic K-theory for an arbitrary ring and order to do so we will first introduce Quillen's plus construction.

4.1 The plus constuction

Given a CW-complex X, the aim of Quillen's plus construction is to build a new space X^+ by attaching cells to X in order to kill a perfect subgroup of $\pi_1(X)$ while preserving homology.

Theorem 4.1 (Quillen)**.** *Let X be a connected CW-complex. Let H be a perfect normal subgroup of $\pi_1(X)$ (i.e. $H \trianglelefteq \pi_1(X)$ and $[H;H] = H$). Then there is an inclusion $\iota : X \hookrightarrow X^+$ with the following properties:*

i. X^+ is a CW-complex construted from X by attaching 2-cells and 3-cells only.

ii. $\iota_* : \pi_1(X) \to \pi_1(X^+)$ is surjective with kernel $Ker(\iota) = H$.

iii. Let $p^+ : \tilde{X^+} \to X^+$ and $p : \tilde{X} \to X$ be a covering spaces, such that $\iota_* \circ p_* \pi_1(\tilde{X}) = p^+_* \pi_1(\tilde{X^+})$. Then the lift $\tilde{\iota} : \tilde{X} \to \tilde{X^+}$ is a homology isomorphism.

Moreover, let $f : (X, xo) \to (Z, zo)$ be a map of connected spaces with $Ker(f_* : \pi_1(X) \to \pi_1(Z)) = H$. Then:

iv. There is a map $f' : (X^+, xo) \to (Z, zo)$ such that $f' \circ i \simeq f$

v. If f_* is surjective and has the properties given in iii for i_*, then the map f' of iv is a homotopy equivalence. This implies in particular that X^+ is unique up to homotopy.

The following lemma will be useful for the proof of the theorem

Lemma 4.2. *Let $f : X \to Y$ be a map between connected CW-complexes that induces an isomorphim on fundamental groups. If the lift to universal covers $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a homology isomorphism, then f is a homotopy equivalence.*

Proof. The universal covers $\tilde{X}$ and $\tilde{Y}$ are simply connected, thus a lift $\tilde{f}$ exist. Moreover, since $\tilde{f}$ is a homology isomorphim then by Hurewicz theorem it is also a homotopy equivalence and thus it induces isomorphisms in all homotopy groups. Now, the projection maps induce isomorphisms in all higher homotopy groups $\pi_n \quad \forall n \geq 2$.

$$
\begin{array}{ccc}
\pi_n(\tilde{X}) & \xrightarrow[\cong]{\tilde{f}_*} & \pi_n(\tilde{Y}) \\
{\scriptstyle p_{X*}} \downarrow \cong & & \cong \downarrow {\scriptstyle p_{Y*}} \\
\pi_n(X) & \xrightarrow[f_*]{} & \pi_n(Y)
\end{array}
$$

Therefore f induces isomorphisms in all homotopy groups and by Whitehead's theorem it is a homotopy equivalence. $\square$

Proof of theorem 4.1. The space X^+ will be constructed from the space X by first attaching 2-cells in order to kill the required subgroup of the fundamental group, and then attaching 3-cells to this later space in order to reconstruct the homology of X. The details are as follows.

Choose generators of H of the form $[\tilde{\alpha_j}; \tilde{\beta_j}]$ where $\tilde{\alpha_j}, \tilde{\beta_j} \in H \trianglelefteq \pi_1(X)$. Then, $\tilde{\alpha_j}, \tilde{\beta_j}$ define homotopy classes of maps $\alpha_j, \beta_j : (S^1, s_0) \to (X, xo)$. Define via composition of loops the map $\lambda_j : \alpha_j \centerdot \beta_j \centerdot \overline{\alpha_j} \centerdot \overline{\beta_j}$ where $\overline{\alpha_j}$ and $\overline{\beta_j}$ denote the inverse loops of α_j and β_j respectively. Note that λ_j is well defined up to homotopy. Let Y be obtained from X by attaching one 2-cell for each generator of H via the attaching map λ_j, that is

$$Y = X \bigcup_{\substack{\lambda_j \\ j \in J}} e_j^2$$

Moreover, note that by cellular approximation, the maps λ_j can be choosen to have image lying in the 1-skeleton of X making the space Y a CW-complex. Then, the inclusion $X \hookrightarrow Y$ induces a map in fundamental groups that sends each element of H to a trivial loop in Y by Van Kampen's theorem. Thus this later map is just the quotient map $\pi_1(X) \to \pi_1/H$ (i.e. $\pi_1(X) \to \pi_1(Y)$ is a surjective map with kernel H).

Now, the attaching maps λ_j define characteristic maps $\Phi_j : e_j^2 \to Y$ by considering the composition of maps $\Phi_j : e_j^2 \hookrightarrow X \coprod e_j^2 \xrightarrow{q_{1,j}} X \cup_{\lambda_j} e_j^2 \hookrightarrow Y$ where $q_{1,j}$ is the quotient map defined by λ_j. Let B_j be the quotient space of e_j^2 formed by identifying the boundary parts correspoding to α_j and $\overline{\alpha_j}$ in opposite directions. Then the characteristic map Φ_j factors through $X \coprod B_j$ as shown in diagram 4.1 where $q_{2,j}$ is the quotient map that identifies an interval of the open cylinder B_j with X via the loop α_j and the boundary circles of B_j to X via the loops β_j and $\overline{\beta_j}$.

$$\begin{array}{ccccccc} e_j^2 & \hookrightarrow & X \coprod e_j^2 & \xrightarrow{q_{1,j}} & X \cup_(\lambda_j) e_j^2 & \hookrightarrow & Y \\ & & \downarrow & \nearrow_{q_{2,j}} & & & \\ & & X \coprod B_j & & & & \end{array} \qquad (4.1)$$

Then q_{2j} defines a map $\tilde{\mu_j} : B_j \to Y$. Moreover since $\tilde{\beta_j} \in H$ then the loops β_j and $\overline{\beta_j}$ are nullhomotopic in Y, thus the map $\tilde{\mu_j}$ can be extended to a map $\mu_j : (S^2, so) \to (Y, yo)$ by filling the boundary circles of B_j (i.e. capping the open ends of B_j). Let X^+ be obtained from Y by attaching one 3-cell for each generator of H via the attaching map μ_j, that is

$$X^+ = Y \bigcup_{\substack{\mu_j \\ j \in J}} e_j^3$$

Again by cellular approximation X^+ can be construted as a CW-complex.

Now we will show that X^+ has the desired properties. Property i is fulfilled by construction. Moreover, by Van-Kampen's theorem, attaching cells of dimension 3 and higher to a CW-complex does not affect the fundamental group. Therefore the inclusion $\iota : X \hookrightarrow X^+$ enjoys property ii.

Now, to show property iii, let $\tilde{X}$ be a covering space of X of covering group G. Let $\tilde{Y}$ and $\tilde{X^+}$ be the correponding covering spaces of Y and X^+ respectively i.e. both with covering group $G/H \geq \pi_1(X)/H$. Recall that the number of sheets in $\tilde{X}$ is $[G : \pi_1(X)]$. Thus the covering translations are in one to one correspondance with the number of cosets of G in $\pi_1(X)$. Let φ_g be covering translation corresponding to a coset of G with representative g. Then, by the constructions above we can build $\tilde{Y}$ and $\tilde{X^+}$ from $\tilde{X}$ as follows.

$$\tilde{Y} = \tilde{X} \bigcup_{\substack{\varphi_g(\lambda_j) \\ j \in J \\ g \in \pi_1(X)/G}} e^2_{j,g}$$

$$\tilde{X^+} = \tilde{Y} \bigcup_{\substack{\varphi_g(\mu_j) \\ j \in J \\ g \in \pi_1(X)/G}} e^3_{j,g}$$

Since $\tilde{X^+}$ is build from $\tilde{X}$ by attaching 2-cells and 3-cells only, then the relative homology groups of the pair $(\tilde{X^+}, \tilde{X})$ are concentrated in levels 2 and 3. More explicitly, the cellular chain complex of the pair is given by

$$\ldots \to C_3(\tilde{X^+}, \tilde{X}) \xrightarrow{\partial_3} C_2(\tilde{X^+}, \tilde{X}) \to 0 \to 0$$

Let e^3_i for $i \in I$ be the 3-cells of X. By definitions of the n-chains of a pair and cellular homology [6] we have that

$$\begin{aligned} C_3(\tilde{X^+}, \tilde{X}) &= C_3(\tilde{X^+})/C_3(\tilde{X}) \\ &= H_3((\tilde{X^+})^3, (\tilde{X^+})^2)/H_3((\tilde{X})^3, (\tilde{X})^2) \\ &= \mathbb{Z}[e^3_{j,g}] \bigoplus \mathbb{Z}[e^3_i]/\mathbb{Z}[e^3_i] \\ &\cong \mathbb{Z}[e^3_{j,g}] \end{aligned}$$

Then, $C_3(\tilde{X^+}, \tilde{X})$ is the free abelian group (or free $\mathbb{Z}$-module) with generators $e^3_{j,g} \quad j \in J, \quad g \in \pi_1(X)/G$. In the same way $C_2(\tilde{X^+}, \tilde{X})$ is the the free abelian group with generators $e^2_{j,g} \quad j \in J, \quad g \in \pi_1(X)/G$. Thus it is clear that both groups are isomorphic. Finally, by the way the attaching maps of e^3_j where constructed, one can see that the boundary map takes generators to generators i.e. $\partial_3(e^3_{j,g}) = e^2_{j,g}$. Thus, ∂_3 is an ismorphism. Then $H_n(\tilde{(X^+)}, \tilde{(X)}) = 0 \quad \forall n \geq 0$; thus the inclusion map $\tilde{\imath} : \tilde{(X)} \hookrightarrow \tilde{(X^+)} \quad \forall$ is a homology isomorphism $\forall n \geq 0$.

Now, to show property iv let $f : (X, x_0) \to (Z, z_0)$ such that $Ker(f_* : \pi_1(X) \to \pi_1(Z)) = H$. Let (Z^+, z_0) be the following pushout

$$\begin{array}{ccc} (X, x_0) & \overset{\iota}{\hookrightarrow} & (X^+, x_0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^+} \\ (Z, z_0) & \underset{\iota'}{\hookrightarrow} & (Z^+, z_0) \end{array}$$

Note that

$$\begin{aligned} Z^+ &= X^+ \coprod Z/\iota(x) \sim f(x) \\ &= X \bigcup_{\substack{\lambda_j \\ j\in J}} e_j^2 \bigcup_{\substack{\mu_j \\ j\in J}} e_j^3 \coprod Z/\iota(x) \sim f(x) \\ &= Z \bigcup_{f\circ\lambda_j, j\in J} e_j^2 \bigcup_{\substack{f\circ\mu_j \\ j\in J}} e_j^3 \end{aligned}$$

Thus the space Z^+ follows the construction of X^+ via the attaching maps $f \circ \lambda_j$ and $f \circ \mu_j$ which are nullhomoptopic since $\lambda_j, \mu_j \in H$ and $Ker(f_*) = H$. Then the map $\iota' : Z \to Z^+$ enjoys properties i and ii with respect to the trivial subgroup of $\pi_1(Z)$. Thus, $\iota'_* : \pi_1(Z) \to \pi_1(Z^+)$ is an isomorphism. Moreover, let $\tilde{Z}$ and $\tilde{Z^+}$ be the universal covers of Z and Z^+ respectively. Since the covers are simply connected, inclusion map ι lifts to a map of universal covers $\tilde{\iota'}$.

$$\begin{array}{ccc} (\tilde{Z}, \tilde{z_0}) & \overset{\tilde{\iota'}}{\hookrightarrow} & (\tilde{Z^+}, \tilde{z_0}) \\ \downarrow & & \downarrow \\ (Z, z_0) & \underset{\iota'}{\hookrightarrow} & (Z^+, z_0) \end{array}$$

By property ii the map ι' is a homology isomorphism and and thus by lemma 4.2 the inclusion ι' is a homotopy equivalence. Now let $\gamma : (Z^+, z_0) \to (Z, z_0)$ be the homotopy inverse of ι'. Let $f' = \gamma \circ f^+$, then $f' \circ \iota = \gamma \circ f^+ \iota = \gamma \circ \iota' \circ f \simeq f$

Moreover, by construction $f'_* : \pi_1(X+, xo) \to \pi_1(Z, z_0)$ has trivial kernel. Hence, if f has the properties ii and iii then f' is an isomorphism on fundamental groups and it's lift to universal covers $\tilde{f'} : \tilde{X+} \to \tilde{(Z)}$ is a homology isomorophism, then again by lemma 4.2 f' is a homotopy equivalence, which proves property v. □

4.2 Algebraic K-groups

Let R be a ring, and $GL_n(R) \hookrightarrow GL_{n+1}(R)$ be the inclusion using the upper left corner. Let $GL(R)$ denote the direct limit $\lim_n GLn(R)$ as a discrete group i.e. regard $GL(R)$ as the union of $GL_n(R)$ with the given inclusions. Let $E(R)$ denote the normal subgroup generated by elementary matrices (i.e. matrices that differ from the identity matrix in a single non-diagonal entry).

Lemma 4.3. *The commutator subgroup of $GL(R)$ is $E(R)$, that is $E(R) = [GL(R); GL(R)]$*

Proof. First recall that the commutator of $GL(R)$ is a normal subgroup and that every elementary matrix in $GL(R)$ can be expressed as the commutator of two matrices. Since $E(R)$ is the smallest normal subgroup generated by elementary matrices then $E(R) \subseteq [GL(R); GL(R)]$. Now, for the other inclusion it's enough to show that every commutator in $GL_n(R)$ can be writen as a product of elementary matrices in $GL_{2n}(R)$. Now let $A, B \in GL_n(R)$, we have that

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

Thus $E(R) \supseteq [GL(R); GL(R)]$ □

Definition 4.4. The algebraic K *groups* of a given ring R as defined as

$$K_i(R) = \pi_i(BGL(R)^+)$$

where the plus construction is taken with respect to $E(R)$ which is a perfect normal subgroup of $\pi_1(BGL(R)) = GL(R)$.

Remark 4.5. It can be shown that this construction defines a sequence of functors K_n from the category of rings to the category of groups [4], [10].

In the next chapter we will sketch the calculation of the algebraic K-groups of a finite field. In order to do so the following lemma will prove very useful.

Lemma 4.6. *$BGL^+(R)$ is a simple space.*

Proof. It's enough to show that $BGL^+(R)$ is an H-space and for this is enough to note that multiplication by the identity element is nullhomotopic. Multiplication on $BGL^(R)$ comes from multiplication in $GL^(R)$. In this scenario, when multipliying an matrix A with the identity matrix we spread the original matrix in between colums and lines that coincide with that identity matrix. Now, note that such an operation can be written in terms of conjugation with elementary matrices and thus by definition of $BGL^+(R)$ this map is nullhomotopic. □

5 K-theory of a finite field

The main goal of the present project is to sketch a calculatio of the algebraic K-theory of a finite field. In this section, all the results of the previous sections will be blended together in order to obtain the desired result.

5.1 Algebraic K groups of a finite field

We will calculate the algebraic K-groups of a finite field by means of the map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ constructed theorem 3.11.

Theorem 5.1. *There is a homotopy equivalence $BGL(\mathbb{F}_q)^+ \to F\psi^q$ and thus the algebraic K-groups of a finite field are given by*

$$K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}/(q^{2j} - 1) \qquad K_{2j-1}(\mathbb{F}_q) = 0$$

Proof. Theorem 3.11 give a map $\theta : BGL(\mathbb{F}_q) \to F\psi^q$ that is a homology isomorphism. These spaces are connected, thus the Hurewicz map from the fundamental group to the first homology group is the abelianization homomorphism[**?**]. Since the Hurewicz map is natural we have the following commuting diagram.

$$\begin{array}{ccc} \pi_1(BGL(\mathbb{F}_q)) & \xrightarrow{\theta_*} & \pi_1(F\psi^q) \\ \downarrow h_1 & & \cong \downarrow h_2 \\ H_1(BGL(\mathbb{F}_q)) & \xrightarrow[\theta_*]{\cong} & H_1(F\psi^q) \end{array}$$

Since $\pi_1(BGL(\mathbb{F}_q)) = GL(\mathbb{F}_q)$ and $\pi_1(F\psi^q) = \mathbb{Z}/(q-1)$, then h_2 is an isomorphism. Thus, h_1 is surjective and by lemma 4.3 $Ker(h_1) = [GL(\mathbb{F}_q); GL(\mathbb{F}_q)] = E(\mathbb{F}_q)$. So $\theta_* : \pi_1(BGL(\mathbb{F}_q)) \to \pi_1(F\psi^q)$ is surjective with $Ker(\theta_*) = Ker(h_1) = E(\mathbb{F}_q)$, thus by theorem 4.1 *iv* there is a map $\theta' : BGL(\mathbb{F}_q)^+ \to F\psi^q$ such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} BGL(\mathbb{F}_q)^+ & & \\ \uparrow \iota & \searrow^{\theta'} & \\ BGL(\mathbb{F}_q) & \xrightarrow[\theta]{} & F\psi^q \end{array}$$

Since ι and θ are homology isomorphisms, then θ' is also an homology isomorphism. Furthermore, by lemmas 4.6 and 3.8, $BGL(\mathbb{F}_q)^+$ and $F\psi^q$ are simple spaces. So, by Whitehead's theorem, θ' is a homotopy equivalence. Thus, $K_i(\mathbb{F}_q) = \pi_1(BGL(\mathbb{F}_q)^+) = \pi_1(F\psi^q)$ and proposition 3.7 gives the desired result. ∎

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