Frobenius algebras & two dimensional TQFT's

Fagprojekt ved Matematisk Institut, Københavns Universitet

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Abstract

In this project equivalence between the category, $\mathbf{cFA}_{\Bbbk}$, of commutative frobenius algebras and category, $\mathbf{2dTQFT}_{\Bbbk}$, of symmetric monoidal functors from the category, $\mathbf{2Cob}$, of two dimensional cobordisms to the category, $\mathbf{Vect}_{\Bbbk}$, of vector spaces over a field will be shown. This is done through a graphical description of commutative frobenius algebras that allows for immediate comparing with the structure and description that we give of the cobordisms.

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I dette projekt vil ækvivalens mellem kategorien, $\mathbf{cFA}_{\Bbbk}$, af kommutative frobenius algebraer og kategorien, $\mathbf{2dTQFT}_{\Bbbk}$, af symmetrisk monoidale funktorer fra kategorien, $\mathbf{2Cob}$, af to dimensionelle cobordismer til kategorien, $\mathbf{Vect}_{\Bbbk}$, af vektorrum over et legeme blive vist. Dette er gjort via en grafisk beskrivelse af kommutative frobenius algebraer, som lægger op til en umiddelbar sammenligning med strukturen og beskrivelsen, vi giver af cobordismerne.

1 Introduction

The goal of this project is to prove the following main theorem.

Theorem 1.1 (Main theorem). $\mathbf{2dTQFT}_{\Bbbk}$ *and* $\mathbf{cFA}_{\Bbbk}$ *are equivalent as categories.*

The approach of proving this will be "top-down". The reason for this is that the main reference that is used, [Kock], is a 200 page elaborate book on the topic. So in order to cut away enough material but still present a detailed version of the proof, the presentation need to be one-eyed with respect to only present material sufficient for giving the proof. A top down approach helps doing that while also assisting the reader in keeping track of where we are and what we need next.

In this section we will translate the notions involved in stating the theorem to highlight the first layer of "dependencies" that we are looking for.

The definition of the two categories, $\mathbf{2dTQFT}_{\mathbb{k}}$ and $\mathbf{cFA}_{\mathbb{k}}$, involved in the statement of the main theorem is something we obviously want to know. We will now give the definitions that will be used, but we will through out the rest of this presentation elaborate on them and show that they makes sense. In fact, apart from proving the claimed equivalence, describing these categories is almost the entire content of this presentation. From this elaboration the equivalence will become a lot easier to deal with, as seen in the last section of this presentation.

Definition 1.2 ($\mathbf{cFA}_{\mathbb{k}}$)**.** The category of commutative frobenius algebras, called $\mathbf{cFA}_{\mathbb{k}}$, has commutative frobenius algebras over a field $\mathbb{k}$ as objects and frobenius algebra homomorphisms between them as morphisms.

Definition 1.3 ($\mathbf{2dTQFT}_{\mathbb{k}}$)**.** The category of 2 dimensional topological quantum field theories, called $\mathbf{2dTQFT}_{\mathbb{k}}$, has symmetric monoidal functors from $\mathbf{2Cob}$ to $\mathbf{Vect}_{\mathbb{k}}$ as objects and monoidal natural transformations between those as morphisms.

Definition 1.4 ($\mathbf{2Cob}$)**.** The category of 2 dimensional cobordisms, called $\mathbf{2Cob}$, has natural numbers representing disjoint union of circles as object's and cobordisms between them as morphisms.

Definition 1.5 ($\mathbf{Vect}_{\mathbb{k}}$)**.** The category of vector spaces over a field $\mathbb{k}$, called $\mathbf{Vect}_{\mathbb{k}}$, has vector spaces over $\mathbb{k}$ as objects and $\mathbb{k}$-linear maps between them as morphisms.

The rest of the presentation is divided into three sections. In the first we will describe the category $\mathbf{cFA}_{\mathbb{k}}$. We will show a series of equivalent formulations of what it is to be a frobenius algebra. We start from a standard formulation and move towards a formulation from which we can do a graphical representation of a frobenius algebra. Given this we show the central formulation that we apply in the proof of the main theorem. In the second section we describe the category $\mathbf{2dTQFT}_{\mathbb{k}}$ and according to the definition above[1], it especially involves a study of structures of the category $\mathbf{2Cob}$. Finally in the last section we give the proof of the main theorem.

Before we get into action I would like to thank: Professor Nathalie Wahl for kind and insightful supervision, the topology group/SYM centre for providing a motivating ambiance, fellow student Marc Stephan for helpful conversations and my family for coping with me in general.

2 Commutative frobenius algebra

In this section we will describe the category of commutative frobenius algebra. Especially we will show tree equivalences of frobenius algebras, hence

[1]

on object level of the category. This embraces the object level of the main theorem.

First we will recall some algebraic preliminaries.

Definition 2.1 ($\Bbbk$-vector space)**.** A $\Bbbk$-vector space is an abelian group, A, equipped with a map $A \times \Bbbk \to A$ such that it's actually just a $\Bbbk$-module over A. They are the objects in $\mathbf{Vect}_{\Bbbk}$.

Definition 2.2 ($\Bbbk$-algebra)**.** A $\Bbbk$-algebra is a $\Bbbk$-vector space, A, equipped with maps $\mu \colon A \otimes A \to A$ and $\eta \colon \Bbbk \to A$ such that the diagrams

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu \otimes id_A"'] \arrow[dr, "id_A \otimes \mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk \times A \arrow[r, "\eta \otimes id_A"] & A \otimes A \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "id_A \times \eta"'] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes.

Now we will describe frobenius algebras and study those, then later we will add some extra requirement on it in order to define what we mean by commutative frobenius algebra, which is the objects of $\mathbf{cFA}_{\Bbbk}$.

Definition 2.3 (Null-space)**.** For a linear functional $f \colon A \to \Bbbk$ over $\Bbbk$-algebra A we call $\{x \in A \mid f(x) = 0\}$ the *null-space* and refer to it as $Null(f)$.

Then the following definition of a frobenius algebra makes sense.

Definition 2.4 (Frobenius algebra by ε)**.** A frobenius algebra, (A, ε), is a finite $\Bbbk$-algebra equipped with a linear functional $\varepsilon \colon A \to \Bbbk$, such that there is no non-zero ideal of A contained in $Null(\varepsilon)$. We call ε the frobenius form.

We want to arrive at an equivalent definition of a frobenius algebra, namely the following, where we do not specify a frobenius form but rather a *frobenius paring.* We want this equivalence not only to give depth to the object level of the main theorem, but also to give rise to a graphical representation of the structure that set a frobenius algebra aside from just being an (finite) $\Bbbk$-algebra.

Definition 2.5 (Frobenius algebra by β)**.** A frobenius algebra, (A, β), is a finite $\Bbbk$-algebra equipped with an associative non-degenerate paring $\beta\colon A \otimes A \to \Bbbk$. β is called the frobenius paring.

So what we set out for now is to show this equivalence. In order to do that, and before we present the necessary algebraic preliminaries, we give in the following lemma another equivalent formulation on the requirement on ε in definition (2.4).

Lemma 2.6. *Let $f\colon A \to \Bbbk$ be a linear functional. Then no non-trivial left ideal of A is in $Null(f)$ iff $f(Ay) = 0 \Rightarrow y = 0$.*

Proof. We show both directions by contra position. " $\Rightarrow$ ": Assume $\exists y \neq 0$ ($y \in A$) st. $f(Ay) = 0$. Then Ay is a non-zero principle ideal of A, hence especially and ideal, where $f(Ay) = 0$, so $Ay \subseteq Null(f)$. " $\Leftarrow$ ": Assume I non-zero left ideal of A st. $I \subseteq Null(f)$. That means $ax \in I$, hence $f(ax) = 0$, for all $x \in I$ and $a \in A$. Since I is non-zero, $\exists y \neq 0$ in I s.t. $f(ay) = 0$ for all $a \in A$, hence $f(Ay) = 0$. □

In order to show the equivalence between definition (2.4) and (2.5) we first define the needed notions, namely pairing, non-degeneracy and associativity of a pairing. And then show the equivalence.

Definition 2.7 ((Co-)Pairing)**.** A pairing between two vector spaces, V and W, is a linear map $\beta\colon V \otimes W \to \Bbbk$. On elements we write it as the map that $v \otimes w \mapsto \langle v, w\rangle$. Similarly a *co-pairing* is a linear map $\gamma\colon \Bbbk \to V \otimes W$.

Definition 2.8 (Non-degenerate pairing)**.** A pairing, β, as above, is called non-degenerate in W if there exists a corresponding co-pairing, γ, as above, such that

$$W \xrightarrow{\gamma \otimes id_W} W \otimes V \otimes W \xrightarrow{id_W \otimes \beta} W$$

is the identity on W. And similarly it is called non-degenerate in V if $(\beta \otimes id_V) \circ (id_V \otimes \gamma) = id_V$. It is called, just, non-degenerate if it is non-degenerate in both variables.

Definition 2.9 (Associative pairing)**.** A pairing, $\beta\colon M \otimes N \to \Bbbk$ is called associative if $\langle ma, n\rangle = \langle m, an\rangle$ for $m \in M$, $n \in N$ and $a \in A$ all $\Bbbk$-algebras.

We will use two small lemmas, which relate pairings non-degeneracy and linear functionals, to show the equivalence.

Lemma 2.10. *Let V and W be vector spaces and let $\beta\colon V \otimes W \to \Bbbk$ by $v \otimes w \mapsto \langle v, w\rangle$ be a pairing, then the following is equivalent*

(i) *β is non-degenerate,*

(ii) *$\langle v, w\rangle = 0 \forall v \in V \Rightarrow w = 0$ and W is finite,*

(iii) $\langle v, w\rangle = 0 \forall w \in W \Rightarrow v = 0$ *and* V *is finite.*

Proof. We will just show $(i) \Leftrightarrow (ii)$ as $(i) \Leftrightarrow (iii)$ is analogue.

"$(i) \Rightarrow (ii)$". Assume β is non-degenerate, then by the given co-pairing, γ, we can look at the general element in $W \otimes V$ that is the image of $1_{\Bbbk}$ by γ, that is $\sum_{i=1}^{n} w_i \otimes v_i$ for some $w_i \in W$ and $v_i \in V$. The non-degeneracy property then say that when we map w through the composition we will get the following:

$$w \overset{\gamma \otimes id_W}{\longmapsto} \sum_{i=1}^{n} w_i \otimes v_i \otimes w \overset{id_W \otimes \beta}{\longmapsto} \sum_{i=1}^{n} w_i \langle v_i, w\rangle = w.$$

As this is true for any $w \in W$ imply that W is spanned by the w_i's hence of finite dimension. Now when we assume $\langle v, w\rangle = 0\ \forall v \in V$ then in particular $\langle v_i, w\rangle = 0$, since not all $v_i = 0$ we conclude $w = 0$.

"$\neg(ii) \Rightarrow \neg(i)$". We assume $\exists w \neq 0$, $w \in W$ with W finite such that $\langle v, w\rangle = 0\ \forall v \in V$. We assume β is non-degenerate, then as above when we map the general element of W through the composition given by the non-degeneracy requirement we get that $\sum_{i=1}^{n} w_i \langle v_i, w\rangle = w$. Independently of choice of γ, we get by our assumption that there exists a non-zero $w \in W$ such that $w = 0$. This contradiction then imply that β is not non-degenerate (in W). □

Lemma 2.11. *There is a one-to-one correspondence between linear functionals and associative pairings.*

Proof. Given a linear functional, $\varepsilon\colon A \to \Bbbk$, we can define a pairing $A \otimes A \to \Bbbk$ by $x \otimes y \mapsto \varepsilon(xy)$. Given a $z \in A$, then as $xz \otimes y \mapsto \varepsilon((xz)y)$ and $x \otimes zy \mapsto \varepsilon(x(zy)$ and $\varepsilon((xz)y) = \varepsilon(x(zy)$ we see it is associative.

On the other hand, given a associative pairing, $\beta\colon A \otimes A \to \Bbbk$ by $x \otimes y \mapsto \langle x, y\rangle$, we can define a linear functional $A \to \Bbbk$ by $a \mapsto \langle a, 1_A\rangle = \langle 1_A, a\rangle$.

So given such ε we get by this construct a new linear functional by $a \mapsto \varepsilon(1_A a) = \varepsilon(a 1_A)$, that is we get exactly ε back. Similarly, given such associative pairing β we get by the constructions a associative pairing by $x \otimes y \mapsto \langle xy, 1_A\rangle = \langle 1_A, xy\rangle$, that is we get β back. □

Theorem 2.12. *Definition (2.4) and (2.5) are equivalent.*

Proof. By lemma (2.11) it follows that the definitions are equivalent if it follows $Null(\varepsilon)$ contains no non-zero left-ideal of A iff β is non-degenerate. We denote the pairing as $\langle \cdot, \cdot\rangle$. From lemma (2.10) we know that β is non-degenerate iff $\langle A, y\rangle = 0 \Rightarrow y = 0$ which again is the same as saying $\varepsilon(Ay) = 0 \Rightarrow y = 0$ (be the previous lemma). Finally we apply lemma (2.6) to see that this is the same as saying $Null(\varepsilon)$ does not contain any non-zero left ideal of A. □

2.1 Graphical representation

Now we are ready to construct a graphical representation of these algebraic data and structures of the frobenius algebras. The aim is to use these to prove more facts about them, that we need to show the main theorem. To be more precise; we want to show that a frobenius algebra has a co-algebra structure where the co-unit is the frobenius form. We want to do that via calculations on the object we get from our graphical representation. Further we will show that this is actually one half of yet another equivalent definition of a frobenius algebra, namely that if we are given a vector space with multiplication and co-multiplication and corresponding units that satisfy that the multiplication and co-multiplication commutes, called the frobenius relation, then it is a frobenius algebra. We will show this equivalence.

But first let us recall what a co-algebra is.

Definition 2.13. A co-algebra over a field $\mathbb{k}$ is a $\mathbb{k}$-vector space, A, equipped with linear maps $\delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to \mathbb{k}$ such that the diagrams

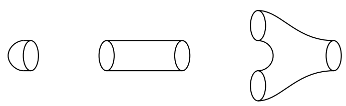
$$\begin{tikzcd} & A \otimes A \otimes A & \\ A \otimes A \arrow[ur, "\delta \otimes id_A"] & & A \otimes A \arrow[ul, "id_A \otimes \delta"'] \\ & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

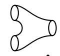
$$\begin{tikzcd} \mathbb{k} \times A & A \otimes A \arrow[l, "\varepsilon \otimes id_A"'] \arrow[r, "id_A \times \varepsilon"] & A \otimes \mathbb{k} \\ & A \arrow[ul] \arrow[u, "\delta"] \arrow[ur] & \end{tikzcd}$$

commutes.

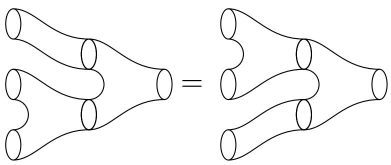
So we want to construct such δ and show that our frobenius form, ε, satisfy the co-unit condition.

The starting point is the morphisms of the $\mathbb{k}$-algebra, say, A: unit η, identity id_A and multiplication μ. We will represent graphically as pictured here:

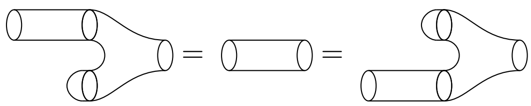


It is in order to explain how they make sense. For each of them we have the domain of the map they represent on the left and the co-domain on the right. We will often refer to them as input and output. Input and outputs that come from A is pictured with a circle. The map  is a map $A \otimes A \to A$ and, in the picture that represent it, the upper input is the first factor in $A \otimes A$ and the lower input is the second factor. When we map form the ground field, $\Bbbk$, the input is not represented with a circle, but with a cap. In other words, if we let A^n be A tensored with it self n times, then we can express maps $A^n \to A^m$ for $n, m \geq 0$ where we also pay attention to the order of the factors in the tensor power. This explains the machinery for constructing the graphical representation, and that it is well-defined.

Then the requirements on these morphisms to be a $\Bbbk$-algebra is represented as: First the associativity:



Then the unit requirement:



We still have the frobenius form, ε and the frobenius pairing, β, to work with. We will represent these as:

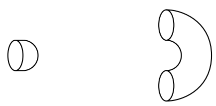
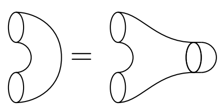
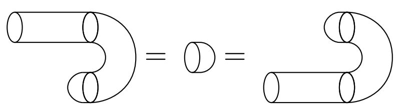


Figure 2.1: Frobenius form, Pairing

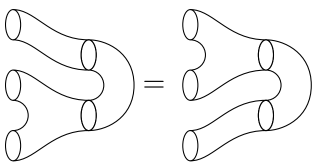
Remark 2.14. We note that these two representation is compatible with the statement in lemma (2.11). That is:





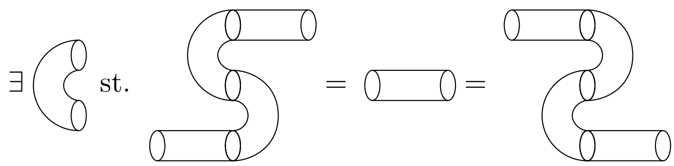
We can not show either of the equivalent conditions from lemma (2.6) of a frobenius algebra expressed via the frobenius form hold. I.e. the ideal we can not present, also implications. But we can express the one involving the associative pairing, since it is essential involving commutative diagrams, which we can express without graphical representation.

The associativity of the frobenius pairing is expressed as:



and follows from the associativity of the multiplication, μ.

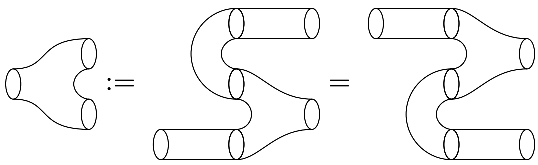
Then we need it to be non-degenerate. We recall from definition 2.8 that this means that for a pairing, β, there exists a co-pairing, γ, such that $(id_W \otimes \beta) \circ (\gamma \otimes id_W) = id_W$ and $(\beta \otimes id_V) \circ (id_V \otimes \gamma) = id_V$. Which is translated to the following:



The relation we will refer to as the snake relation.

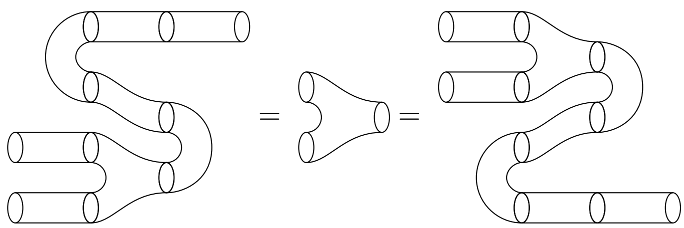
Now we will define the co-multiplication.

Definition 2.15. The co-multiplication, δ, is defined as the following:

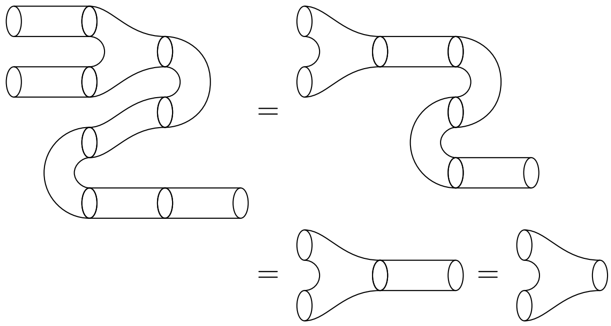


As we see it is a composition of co-pairing and multiplication, and and we use the two equalities since we need this to be co-associative, since that the requirement on the co-multiplication. We first show that the right most equality holds then show that it satisfy the co-associativity. To show the right most equation we first need the following lemma.

Lemma 2.16. *The following equations holds*

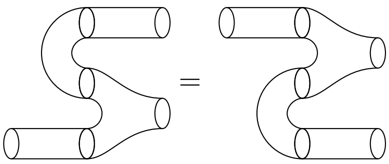


Proof. The proof is simply to use the snake relation by adding and removing identities as needed. I will do one side, the other is analogous just using the other part of the snake relation. So:

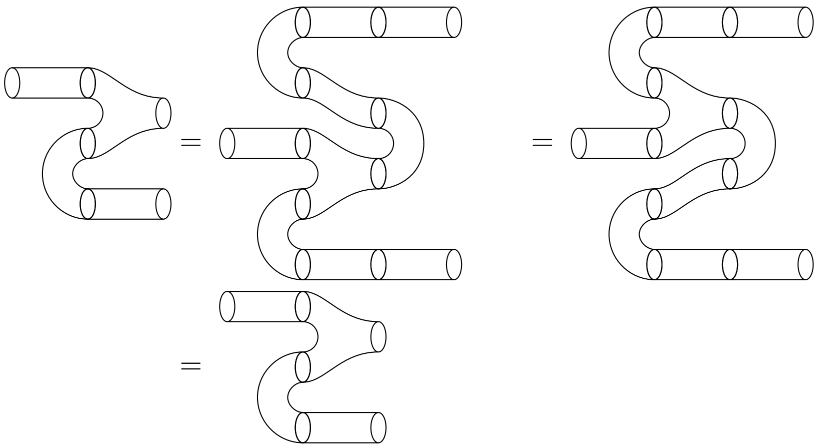


□

Lemma 2.17. *The co-multiplication is well defined. I.e. the following hold*



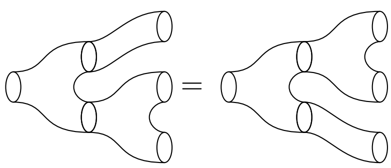
Proof. The proof follows by the previous lemma, that is used in the first and last equality below, and the associativity of the pairing, that is used in the middle equality.



□

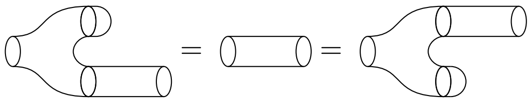
We now want to show that this co-multiplication satisfy the co-associativity regiment as formulated in definition 2.13. Namely:

Lemma 2.18. *It holds that*

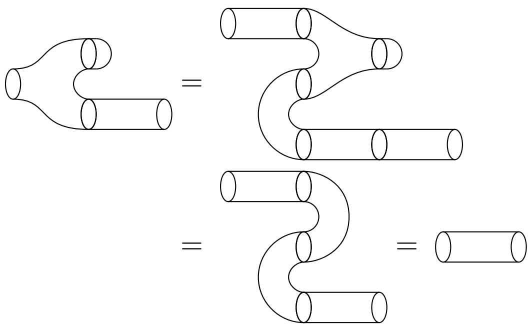


Proof. We just use the definition of our co-multiplication to write out what it means in terms of co-paring and multiplication, then we apply the relation in the definition twice and finally translate back via the ostensive part of the definition to get the wanted. □

Lemma 2.19 (ε is co-unit for δ)**.** *The frobenius form, ε, is in fact co-unit for the co-multiplication, δ; meaning:*

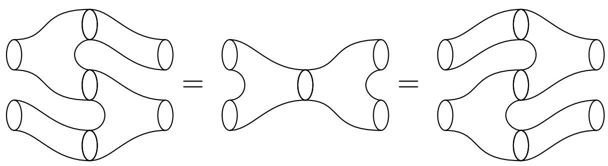


Proof. First we note that this relation exactly express the co-unit condition from definition (2.13). Then we show it holds:

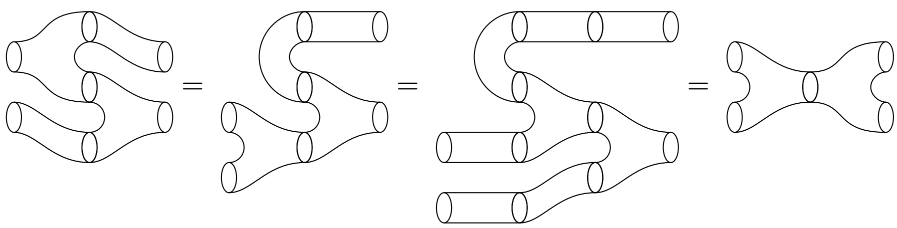


That is, first we use definition (2.15) of co-multiplication, then we use the first relation of remark (2.14) and finally the snake relation. The other equality in the lemma is analogous. □

Lemma 2.20. *The co-multiplication, δ, satisfy the frobenius relation:*



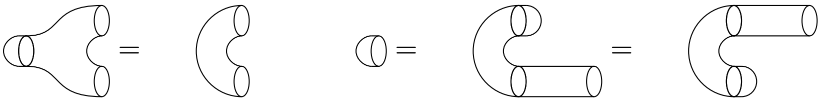
Proof. This is again two equalities that are analogous; we show the first. We use the definition of co-multiplication to get an expression in term of multiplication, then we apply associativity of multiplication and finally translate back with the definition of co-multiplication:



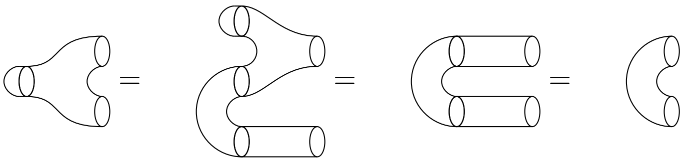
□

Now we have constructed what we set out for, but we need a few more lemmas before we get truly happy. First:

Lemma 2.21. *The dual of remark 2.14 holds; that is:*

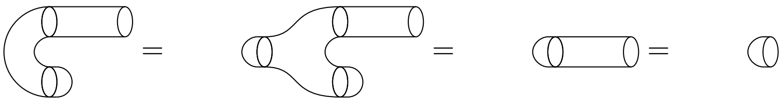


Proof. First



Which follows from first using the definition of co-multiplication and then the unit requirement for the multiplication.

Then



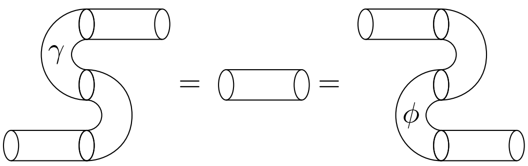
Which follows from first the above relation and then the co-unit requirement as displayed in lemma (2.19). ☐

And secondly, long overdue:

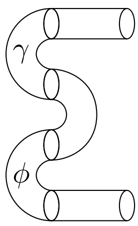
Lemma 2.22. *The co-pairing, γ, is unique.*

Proof. What this means is that when we talk about a pairing, β, to be non-degenerate, then we mean that it is so in both variables - as mentioned. For each variable it is non-degenerate, it induces a co-pairing. What we want now, is to show, that this is actually the same co-pairing. And this is why we sloppily allowed to use the same name in both cases previously.

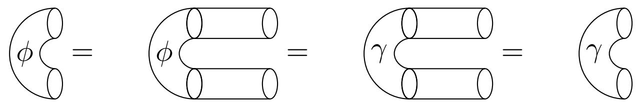
Now assume that the two co-pairings are not the same and name them γ and ϕ, then the relation in the non-degeneracy condition looks as:



Then observe the following composition:



When we apply both sides of the snake relation (by adding need identity) to the above we get:



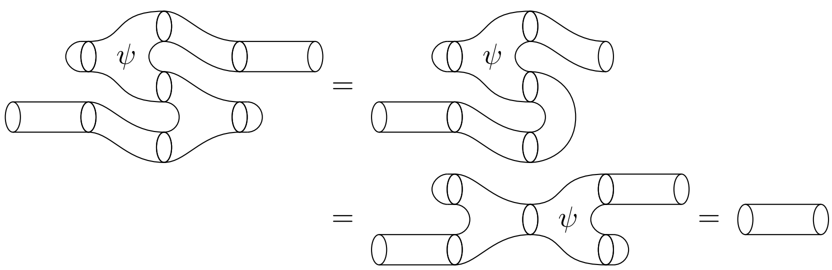
□

We are now ready to harvest the fruits of our labour with this graphical representing of structures in frobenius algebras. We will show the equivalence, mentioned in the beginning of section 2.1. The statement and proof is divided into the following two theorems.

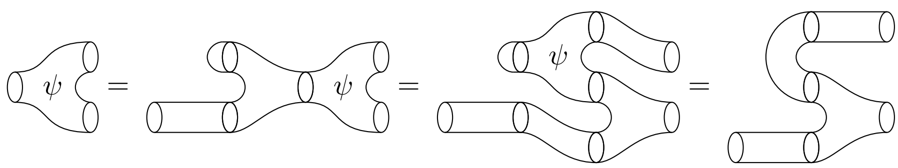
Theorem 2.23. *Given a frobenius algebra, (A, ε), then there exist a unique co-associative co-multiplication, δ, to which the frobenius form, ε, is the co-unit such that also the frobenius relation is satisfied.*

Proof. By definition 2.15, lemma 2.17, 2.19, 2.18 we only need to ensure that the constructed co-multiplication, δ, is unique. So assume there is another co-associative co-multiplication, ψ, that has ε as co-unit and also satisfy the frobenius relation. As in multiple other cases we only show one side of things as the other is analogue.

Now we compose the ψ'ifyed frobenius relation with co-unit and unit and get:



The first equality follows from applying lemma (2.21) and remark (2.14), the second is the frobenius relation and the last is the unit and co-unit condition. Then since the co-pairing, γ, is unique (lemma (2.22)) and by the snake relation we get that $\psi \circ \eta = \gamma$. Now we can rewrite ψ in the following way, using the above to get the last equality:



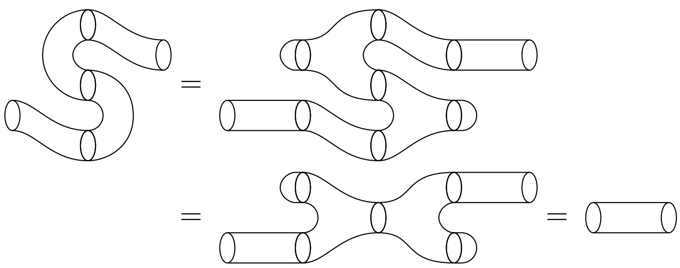
In other words; ψ is compatible with our definition of co-multiplication. In fact $\psi = \delta$. □

The other way show the interest in the frobenius relation; how it characterises the frobenius algebra.

Theorem 2.24. *Given a vectors space, A, equipped with a multiplication, μ, with unit, η, co-multiplication, δ, with co-unit, ε, such that it satisfies the frobenius relation, then it forms a frobenius algebra, (A, ε), where ε is the frobenius form.*

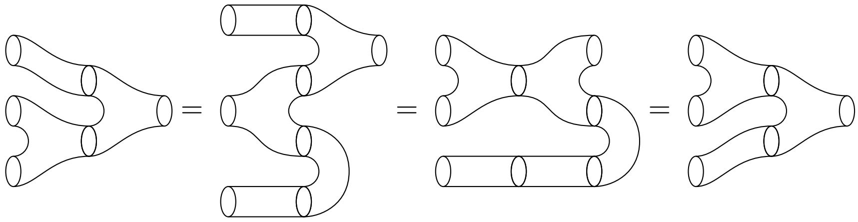
Proof. First, from η and μ we construct a paring, β, by $\beta := \mu \circ \varepsilon$. We then note that the unit condition on the multiplication μ then implies that relation in remark (2.14) are satisfied. This means that the co-unit ε is a candidate to be the frobenius form if we can show that β is a frobenius pairing.

In particular we want to show that β is non-degenerate. This is equivalent to show that the snake relation holds when we define the co-pairing to be $\gamma = \delta \circ \eta$:

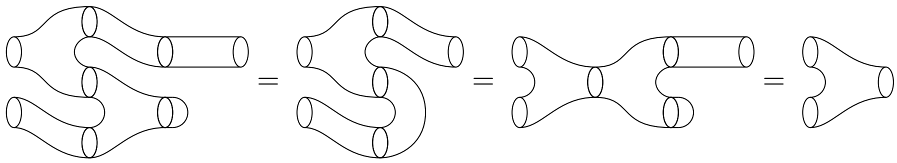


Where we only apply the frobenius relation and the unit and co-unit conditions. Then we note that non-degeneracy by lemma (2.10) implies that A is finite, so we have that requirement of a frobenius algebra settled too.

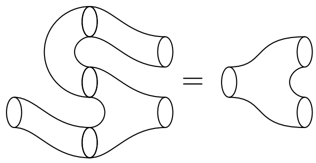
Now we also want β to be associative for it to be a frobenius form. We also want the multiplication be be associative and the co-multiplication be co-associative. We will show that the multiplication is associative. From that, as noted previously, associativity of the pairing follows directly (by construction of the pairing). Co-associativity of the co-multiplication is shown in an analogue manner hence we will omit it here. All in all we will then have showed that (A, ε) is a frobenius algebra. So; associativity of μ:



The middle equality follows from the frobenius relation directly, while the to others follow from relations we establish by composing the frobenius relation with unit and co-unit:



And similarly:

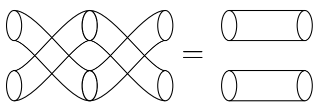


□

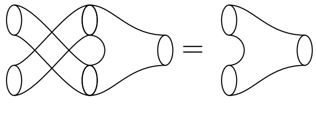
2.2 Commutativity

Up until now we have looked at frobenius algebras while not discussing the commutativity aspect present in $\mathbf{cFA}_{\mathbb{k}}$. We will do that now. This involves especially the introduction of a *twist map* which is the last thing we need settled in this category for showing the main theorem.

Definition 2.25. For two vector spaces, V, V', the twist map, $\sigma_{V,V'} : V \otimes V' \to V' \otimes V$, is the maps that interchanges factors such that $\sigma_{V',V}\sigma_{V,V'} = id_{V\otimes V'}$. We represent this graphically as:

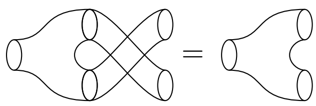


Definition 2.26. A frobenius algebra is commutative if the under laying algebra is commutative. In graphical terms this means:



And similarly we have for a co-algebra:

Definition 2.27. A co-algebra is co-commutative if



Finally we state this relation between the two:

Proposition 2.28. *The multiplication of a frobenius algebra is commutative iff the co-algebra is co-commutative.*

Proof. We leave out the proof as we do not really need this. □

2.3 Categorical perspective

The remaining part is to show $\mathbf{cFA}_{\Bbbk}$ is a category.

First we recall that the algebra A of a frobenius algebra (A, ε) is also a co-algebra which is fact from theorem (2.23). Then we are ready to define a *frobenius algebra homomorphism*:

Definition 2.29. Given two frobenius algebras then a homomorphisms between them is an algebra homomorphism that is also a co-algebra homomorphism. By diagrams this is the same as saying: $f\colon (A, \mu, \eta, \delta, \varepsilon) \to (A', \mu', \eta', \delta', \varepsilon')$ is a frobenius algebra homomorphism if the following diagrams

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "f \otimes f"'] & A \arrow[r, "\delta"] \arrow[d, "f"] & A \otimes A \arrow[d, "f \otimes f"] \\
A' \otimes A' \arrow[r, "\mu'"'] & A' \arrow[r, "\delta'"'] & A' \otimes A'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dd, "f"] \arrow[dr, "\varepsilon"] & \\
\Bbbk \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & \Bbbk \\
 & A' \arrow[ur, "\varepsilon'"'] &
\end{tikzcd}$$

commutes. And such that the (co-)multiplication is (co-)commutative. Note the left part of the two diagrams correspond to f being a algebra homomorphism, and the right to f being a co-algebra homomorphism.

Theorem 2.30. *The category of commutative frobenius algebras,* $\mathbf{cFA}_{\Bbbk}$*, as given in definition 1.2 is valid.*

Proof. We have a set of objects being frobenius algebras, and a set of morphisms between. It follows directly from the looking at diagrams in definition 2.29 that the properties of being a category is satisfied. □

3 Two dimensional TQFT

In this section I will go into detail about the category $\mathbf{2dTQFT}_{\Bbbk}$ especially we will discuss **2Cob**.

3.1 Two dimensional cobordisms

Proposition 3.1. **2Cob** *is in fact a category.*

Proof. We have stated in definition 1.4 that **2Cob** is the category where the objects are natural numbers representing disjoint union of circles and the morphisms are two dimensional cobordisms between then. From now on we will just call them cobordisms. And the objects will be denoted with bold face, so the disjoint union of four circles will be denoted **4**. The sets of objects and morphisms form a category since: First, the set of objects is settled, and for any two objects a cobordisms between them is a well defined morphism. Also for any three objects we can compose the morphisms in an associative manner. We have an identity morphism (the tube) and the morphisms completely determined the domain and co-domain (ie. the pair-of-pants goes from **2** to **1**). □

It is relevant to note that this definition is not the standard definition, but rather a slightly more abstract or generalised version of it. The complete translation of this is beyond the scope of this presentation. It is interesting in its own right, but here, while focusing on showing the main theorem, we need to omit that differential topological aspect by making this generalisation. Details about it is presented in the first part of [Kock]. In fact the definition we work with here is a special kind of subcategory, called the skeleton, of the standard category of cobordisms. We will now make it more precise what that means.

Definition 3.2. A *skeleton*, $\mathcal{S}$, of a category $\mathcal{C}$ is a sub-category of $\mathcal{C}$ that is equivalent to $\mathcal{C}$. The objects of $\mathcal{S}$ is a representative from each of the isomorphism classes of the objects of $\mathcal{C}$. The skeleton is the least sub-category such that they are equivalent.

So to give a little more depth to our definition we note that for the category of cobordisms, the objects are isomorphic if they are diffeomorphic. This also lead us to define what we mean by equivalent cobordisms.

Definition 3.3. Two cobordisms $M, M' \colon \mathbf{n} \to \mathbf{m}$ are equivalent if there is a diffeomorphism between them that respect the order of the boundaries.

In the next paragraph we exploit the following way of looking at a category. The skeleton idea mentioned above will, as we mentioned then, allow us to do so.

Definition 3.4. For a (small) category $\mathcal{C}$ we talk about *generating the category* from a set of generators and a set of relations. The generators are morphisms in the category and the relations are relations of morphisms such that any morphism in $\mathcal{C}$ is a composition of generators.

This notion of generating a category we will extend further giving the category is *monoidal*. For that we need the following definition.

Definition 3.5 (Monoidal category)**.** A category $\mathcal{C}$ equipped with two functors $\mu\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\eta\colon 1 \to \mathcal{C}$ such that the diagrams

$$\begin{tikzcd} & \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[dl, "\mu\times id_{\mathcal{C}}"'] \arrow[dr, "id_{\mathcal{C}}\times\mu"] & \\ \mathcal{C}\times\mathcal{C} \arrow[dr, "\mu"'] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\mu"] \\ & \mathcal{C} & \end{tikzcd}$$

$$\begin{tikzcd} 1\times\mathcal{C} \arrow[r, "\eta\times id_{\mathcal{C}}"] \arrow[dr] & \mathcal{C}\times\mathcal{C} \arrow[d, "\mu"] & \mathcal{C}\times 1 \arrow[l, "id_{\mathcal{C}}\times\eta"'] \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

commutes is called a (strict) monoidal category.

Remark 3.6. We will freely interchange between the notion $(\mathcal{C}, \mu, \eta)$ of a monoidal category and one noted as $(\mathcal{C}, \Box, I$. The dictionary for doing so is the following. On objects X, Y and f, g morphisms of $\mathcal{C}$ we define μ abstractly by $(X, Y) \xrightarrow{X} \Box Y$ and $(f, g) \xrightarrow{f} \Box g$. And I is the object in *calC* that η maps the object from the one-point category $\mathbf{1}$ to.

Then we can show that **2Cob** can be viewed as such:

Proposition 3.7. *(*$\mathbf{2Cob}, \sqcup, \emptyset$*) is a monoidal category.*

Proof. So we need to show that $\sqcup\colon \mathbf{2Cob} \times \mathbf{2Cob} \to \mathbf{2Cob}$ as a functor is well defined and that the condition expressed in definition 3.5, above, holds. So first the well-defindness part: For any two objects **n**,**m** in **2Cob** we see $\mathbf{n} \sqcup \mathbf{m}$ is an object in **2Cob**. Also for any two morphisms in **2Cob**, that is two cobordisms, $M\colon \mathbf{n} \to \mathbf{m}$ and $M'\colon \mathbf{n'} \to \mathbf{m'}$, then we get a cobordism $M \sqcup M'\colon \mathbf{n} \sqcup \mathbf{n'} \to \mathbf{m} \sqcup \mathbf{m'}$. Also it respects compositions as for cobordism $\mathbf{n} \xrightarrow{N} \mathbf{n'} \xrightarrow{N'} \mathbf{n''}$ and $\mathbf{m} \xrightarrow{M} \mathbf{m'} \xrightarrow{M'} \mathbf{m''}$ we get $(M' \circ M) \sqcup (N' \circ N) = (N' \sqcup M') \circ (N \sqcup M)$. And it deals with identity $I_{\mathbf{n}}\colon \mathbf{n} \to \mathbf{n}$, $I_{\mathbf{m}}\colon \mathbf{m} \to \mathbf{m}$ as it should: $I_{\mathbf{n}} \sqcup I_{\mathbf{m}} = I_{\mathbf{n} \sqcup \mathbf{m}}$. Secondly, we look at the associativity condition; $(\mathbf{n} \sqcup \mathbf{m}) \sqcup \mathbf{k} = \mathbf{n} \sqcup (\mathbf{m} \sqcup \mathbf{k})$. This equality holds in this version of **2Cob** but we have to note that in general category of cobordisms, ie. not the version we work with, it is only an isomorphism. Clearly also the conditions on the neutral object functor, $\emptyset$, is satisfied. $\Box$

With that in mind we can refine the generating principle even further:

Definition 3.8. A generating set of a monoidal category, $\mathcal{C}$, is a set of morphisms of $\mathcal{C}$ such that any morphism of $\mathcal{C}$ can be constructed by exploding the monoidal operator and composition of morphisms.

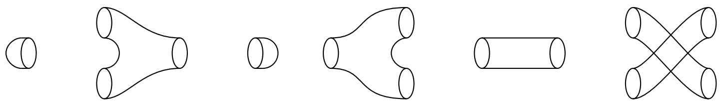
Lemma 3.9. *The twist is not a disjoint union of two identities.*

Proof. The point is that even though the twist is diffeomorphic to the disjoint union of to tubes, this diffeomorphism does not respect the order of the boundaries. And hence it is not equivalent as cobordisms. □

3.1.1 Generators

Our **2Cob** category is small and hence it makes sense to talk about a generating set and a set of relations. In this paragraph we will show that a specific set of cobordisms are a generating set:

Theorem 3.10. *The following generators describe* **2Cob**

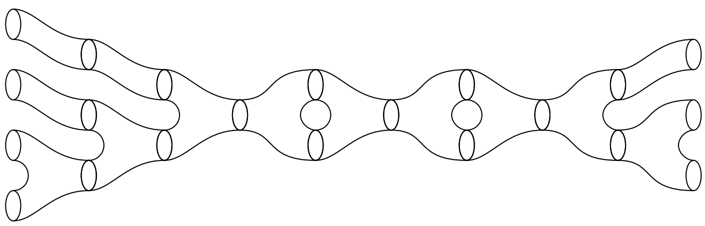


Proof. The proof follows directly from lemma 3.11, remark 3.12 and lemma 3.13 below. □

Lemma 3.11 (Connected cobordisms)**.** *Any connected cobordism can be constructed from a composition of disjoint unions of the connected generators: The tube, cap, co-cap, pants and co-pants.*

The proof is constructive in the sense that we describe an algorithm to decompose any connected cobordisms to what is called the *normal form.* As this is done in a unique way the normal form works as measure of equivalence of connected cobordisms.

Proof. The basic observation origin from classification of topological surfaces, namely: Any connected cobordisms are unique up to diffeomorphism provided it has the same genus, the same number of input boundaries and the same number of output boundaries. So it is an element of this class we bring to normal form. Consider the following example. Let $M\colon \mathbf{4} \to \mathbf{3}$ be a cobordism of genus 2. Then the normal form of M looks as:



With this picture in mind we now present the general algorithm. So let $M\colon \mathbf{n} \to \mathbf{m}$ of genus g be a connected cobordism. We think of the normal form as having three parts, one that takes care of the input, one for the topological part with the genus and one for the output. Lets start with the middle, topological, part. We can construct a cobordism with genus 1 by composing a pair of pants with a pair of co-pants. Then we can compose that composition by copies of itself g time to get one with genus g. If $g = 0$ we just do nothing, take the tube. The cobordism we then get is $G\colon \mathbf{1} \to \mathbf{1}$. Then the input part; we want to construct a cobordism $IN\colon \mathbf{n} \to \mathbf{1}$ with genus 0. Now if we compose $n-1$ pairs of pants in serial such that the output of the first pair goes into the lower input of the second, and the output of the second goes to the input of the third and so forth and the add tubes to make it well defined, then we get the wanted. And since we always choose to connect the output of one to the lower input of the next, this is unique. If $n = 0$, IN is just a cap. Similarly for the output side, we construct a cobordism $OUT\colon \mathbf{1} \to \mathbf{m}$ of genus 0. We do the same thing just with co-pants instead of pants and we connect the lower output to the input of the next. If $m = 0$, OUT is just a co-cap. And the composition $OUT \circ G \circ IN$ is in the same class as M. $\square$

Remark 3.12. By the monoidal structure we can construct non-connected cobordisms by disjoint union of connected cobordisms. This will cover all non-connected cobordisms generated by the set of generators listed in lemma 3.11, but it will not cover all non-connected cobordisms. This is where the last generator, the twist, enters the picture as expressed in the next lemma.

Lemma 3.13 (Non-connected cobordisms)**.** *Any morphism in* **2Cob** *that can not be expressed as a disjoint union of connected cobordisms can be expressed composing a disjoint union of connected cobordisms with the twist map.*

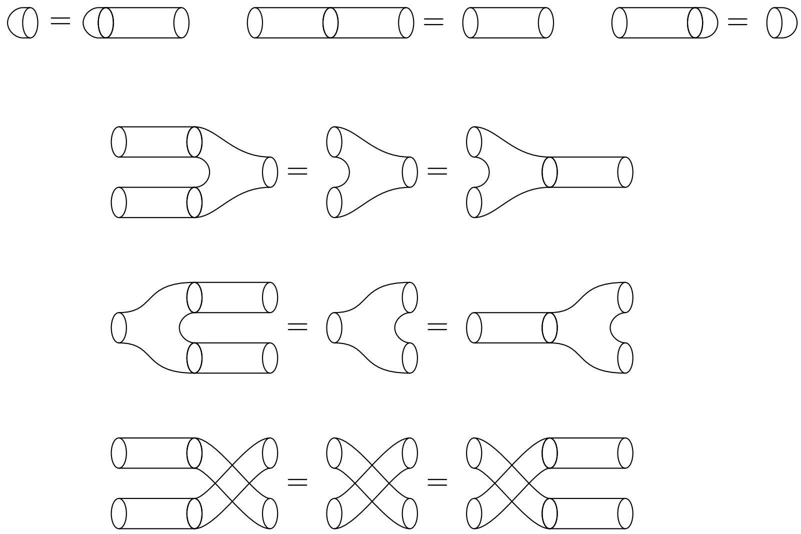
Proof. The proof is constructive and not as general as possible but easy to make general. So assume we are given a cobordism $M\colon \mathbf{n} \to \mathbf{m}$ that is non-connected but not a disjoint union of connected cobordisms. Further, and for making the method easy to display without loss of generality, assume there is two connected components, M_0 and M_1. Then the input boundary of M_0 is a disjoint union of p circles and the input boundary of M_1 is a disjoint union of q circles such that $p+q = n$. Similarly for the output boundary for M_0 and M_1 consists of a disjoint union of k and l circles respectively such that $k + l = m$.

The point is then, that since M is non-connected but not a disjoint union of M_0 and M_1, then not both the output and the input boundary of M_0 and M_1 are ordered such that we first have the boundaries for, say, M_0 and then those for M_1. Let us, for even further simplicity and still without loss of generality, assume the output boundaries are nicely ordered such that the output boundary of M consist of a disjoint union of the output boundary of M_0 and that of M_1. Then we can construct a diffeomorphism $\sigma\colon \mathbf{n} \to \mathbf{n}$ that reorders the input boundaries such that they become a disjoint union of the input boundaries of M_0 and M_1. This is essentially just a permutation which we can construct from transpositions, which is what the twist cobordisms actually do. In other words σ will be a composition of cobordisms that consist of a disjoint union of a twist and $n-2$ tubes. So now $M \circ \sigma$ is diffeomorphic to M and the input boundaries of $M \circ \sigma$ are ordered such that it is the disjoint union of the input boundary of the two connected components of $M \circ \sigma$. Hence we have transformed M diffeomorphicly into the desired form, namely one where it is a disjoint union of its connected components, by only using the twist cobordism. $\square$

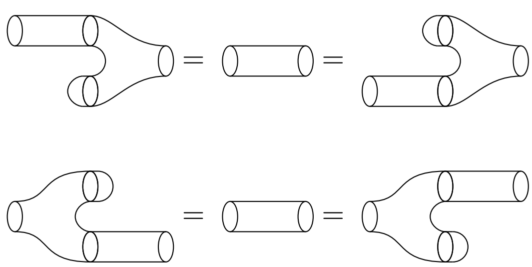
3.1.2 Relations

In this paragraph we present a series of lemmas about relations that hold in **2Cob**. These are needed to prove the main theorem and to describe **2Cob**.

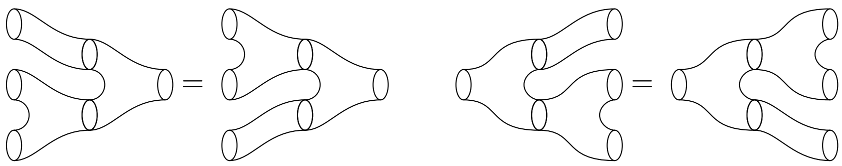
Lemma 3.14 (Identity relations)**.**



Lemma 3.15 ((co-)Unit realations)**.**

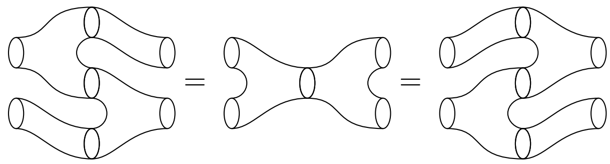


Lemma 3.16 ((co-)associativity relations)**.**



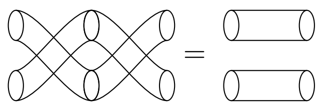
Lemma 3.17 ((co-)commutativity relations)**.**

Lemma 3.18 (Frobenius relation)**.**

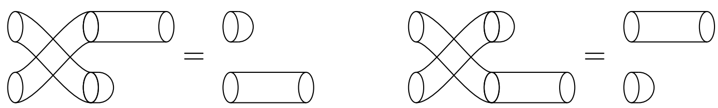


Lemma 3.19 (Twist relations)**.** *A list of relations involving the twist.*

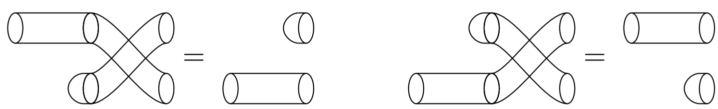
(i) The twist is inverse to itself:



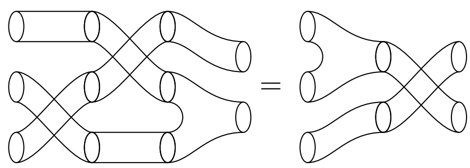
(ii) Twist and co-cap relations:

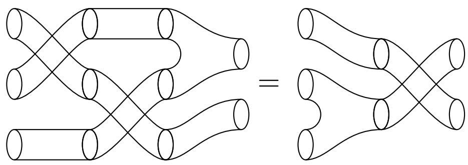


(iii) Twist and cap relation:

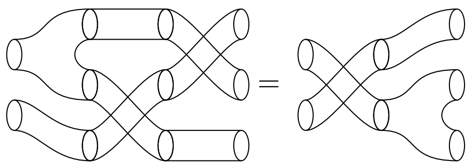


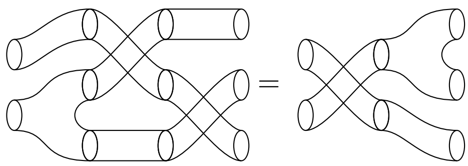
(iv) Twist and pants relations:



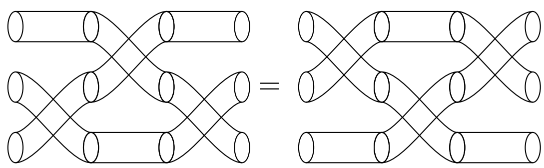


(v) Twist and co-pants relations:





(vi) Twist and twist relation:



Proof. The proof of all these relations is really just to show, that the one side (of the equality sign) is equivalent as cobordisms to the other side. Since equivalence as cobordisms means (see definition 3.3) that they are diffeomorphic respecting the boundaries. Clearly this is the case here. $\square$

Proposition 3.20 (Symmetry of **2Cob**)**.** *The twist cobordism, τ, makes the monoidal category* $(\mathbf{2Cob}, \sqcup, \emptyset)$ *into a symmetric monoidal category.*

Before we give the proof we need the following definition.

Definition 3.21 (Symmetric monoidal category)**.** A monoidal category $(\mathcal{C}, \Box, I)$ equipped also with a *twist map*, τ that for $X, Y \in \mathrm{Ob}(\mathcal{C})$ maps $\tau_{X;y} \colon X \Box Y \to Y \Box X$ such that

- for each pair $(f, g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$, say, $f\colon X \to X'$ and $g\colon Y \to Y'$ then

$$\begin{tikzcd} X \Box Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "f \Box g"'] & Y \Box X \arrow[d, "g \Box f"] \\ X' \Box Y' \arrow[r, "\tau_{X',Y'}"'] & Y' \Box X' \end{tikzcd}$$

commutes.

- for each $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ the following diagrams

$$\begin{tikzcd} X \Box Y \Box Z \arrow[rr, "\tau_{X,Y \Box Z}"] \arrow[dr, "\tau_{X,Y} \Box Id_Z"'] & & Y \Box Z \Box X \\ & Y \Box X \Box Z \arrow[ur, "Id_Y \Box \tau_{X,Y}"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \Box Y \Box Z \arrow[rr, "\tau_{X \Box Y,Z}"] \arrow[dr, "Id_X \Box \tau_{Y,Z}"'] & & Z \Box X \Box Y \\ & X \Box Z \Box Y \arrow[ur, "\tau_{X,Z} \Box Id_Y"'] & \end{tikzcd}$$

commutes.

- for each $X, Y \in \mathrm{Ob}(\mathcal{C})$ then $\tau_{Y,X} \circ \tau_{X,Y} = Id_{X \Box Y}$

is called a (strict) symmetric monoidal category. We refer to all of the data defining the symmetric monoidal category as $(\mathcal{C}, \Box, I, \tau)$.

Proof. Fresh in memory from the above definition we have the condition on the twist map. The relations (ii) to (vi) ensure that the naturality of the twist is satisfied, meaning the the twist commute with taking disjoint union. It obviously suffices to show that it holds for each of the generators[2] in disjoint union with the identity cobordisms. We will just explain this in one example to get a feel of it, the rest is done in the same manner. We take the first of cases with the pants, (iv): We let $M\colon \mathbf{1} \to \mathbf{1}$ and $M'\colon \mathbf{2} \to \mathbf{1}$ be

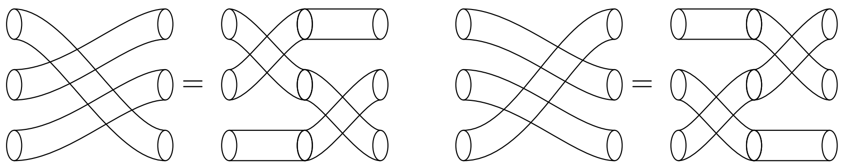
[2]Note that the tube is not a proper generator and this case would anyway be trivial, so it is omitted here.

given, then we are to show that

$$\begin{tikzcd}
\mathbf{1} \sqcup \mathbf{2} \arrow[r, "\tau_{\mathbf{1},\mathbf{2}}"] \arrow[d, "M \sqcup M'"'] & \mathbf{2} \sqcup \mathbf{1} \arrow[d, "M' \sqcup M"] \\
\mathbf{1} \sqcup \mathbf{1} \arrow[r, "\tau_{\mathbf{1},\mathbf{1}}"'] & \mathbf{1} \sqcup \mathbf{1}
\end{tikzcd}$$

commutes. The composition $\tau_{\mathbf{1},\mathbf{1}} \circ M \sqcup M'$ correspond to the right hand side directly. The other composition, $M' \sqcup M \circ \tau_{\mathbf{1},\mathbf{2}}$, correspond to the left hand side, since the twist of $\mathbf{1}$ and $\mathbf{2}$, that is like a permutation, factors though two twist as seen below.

Further we see that the relation (i) satisfies the last condition. The last condition that the twist should satisfy can be represented as follows:

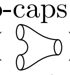
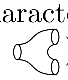


The unfamiliar picture, or notion, on the left hand side is to symbolise the twist of a circle, $\mathbf{1}$, with the disjoint union of two other circles, $\mathbf{2}$. $\square$

Remark 3.22. We note that an alternative route would be to first show that the twist made $(\mathbf{2Cob}, \sqcup, \emptyset)$ into a symmetric monoidal category, then the relations would follow directly. The proof is not hard, we just choose this other route.

Lemma 3.23. *The relations from lemma 3.14 to 3.19 spans* **2Cob**.

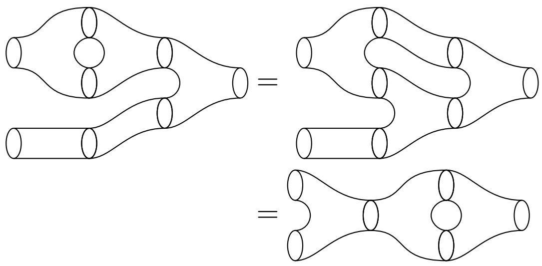
Proof. Our aim is to bring any cobordism to normal form. We will develop the complexity and hope to have a relation that helps up when we run into problems. If successful we have shown that the relations indeed span (though they the set of them might not be minimal). First we assume that we do not come across a twist when we bring it to normal form, then we explain what to do even though. And both cases for connected cobordisms, which is the only kind we have defined a normal form of. Then we treat the non-connected case by defining a normal for on such cobordism – that form will be only unique up to permutation though.

Assume $M : \mathbf{n} \to \mathbf{m}$ is a connected cobordism of genus g. Then the Euler characteristic of M is $\chi(M) = 2 - 2g - n - m$. We further assume M consists of a pants, b co-pants, p caps and q co-caps. The Euler characteristic of each of these are: $\chi($  $) = 1$, $\chi($  $) = 1$, $\chi($  $) = -1$ and $\chi($  $) = -1$, hence $\chi(M) = q + p - a - b$. Also we see that the equation $a + q + m = b + p + n$ holds. So we can express a and b in terms of m, n, p and q: $a = n - 1 + g + p$

and $b = m - 1 + g + q$. We observe that is is expected for the normal form of M.

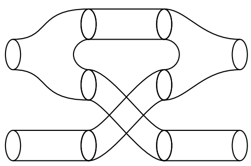
Assuming there is no twist in our decomposition, we will bring it to normal form in the following way. First we want to "move" $n - 1$ copies of pants to the far left. We will think about what the input of a pair of pants can meet. It can meet the tube, which is just the identity, so by identity relations we can just remove that tube. Then we can meet a cap, which by the unit relation gives us an identity, that we can just remove. In that case that pair of pants is eliminated, so this will happen p times, but we have $m - 1 + g$ copies left. The co-cap we can not meet due to connectivity assumption of the cobordisms. Should we meet a pair of pants we just continue moving that pair leftwards instead. Left is to meet a pair of co-pants, which can occur in the following ways:

The first case we don't have a dedicated relation for, but we can construct one:

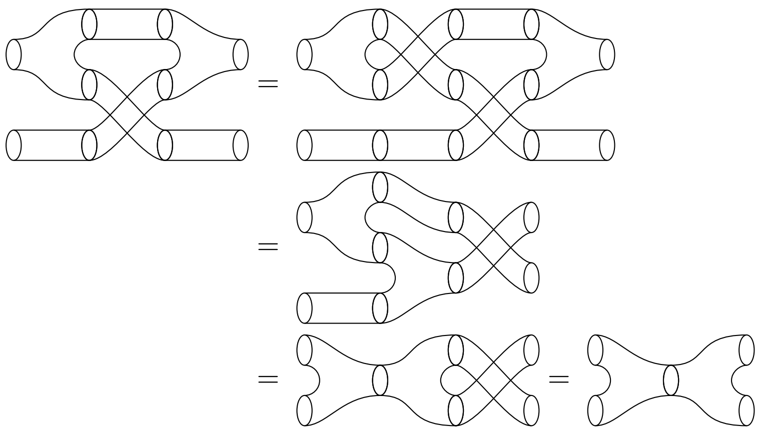


where the first equality stems from the associativity of pants and the last is applying the frobenius relation. The last case is treated by the frobenius relation. So far so good. Now we can do similarly the right, moving $m - 1$ copies of the co-pants to the right. What we are left with is g copies of pants and co-pants in the middle. Since both the left and right now are on normal form, the middle part has one input and output. Clearly this is of genus g and we can bring it to normal form by moving pants to the left applying the relation above.

Now we will examine what will happen if we meet a twist, while keeping the connectedness assumption of the cobordism. We will just meet one twist, then induction over twists will show it holds in general. So assume we meet a twist, then by the identity relations we can add tubes such there is only tubes above and below the twist. Now due to the connectedness assumption there is four possible situations that can occur around this twist: The left side can connect the input boundaries, and similarly the right side can connect the output boundaries. Also the top input and outputs can be connected and similarly for the bottom. We will just treat the left and top situation as the other are similar. So if the left side cobordism is connected we can bring it to normal form, and then we can permute the output side of it such that we get a pair of co-pants with both outputs connected to our twist. Then by the co-commutativity relation the twist will be eliminated. Now assume it is the top input and output that is connected. With the left side on normal form and a pair of co-pants permuted up such the lower outputs connected to the top input of the twist. Doing similar for the cobordism on the right side of the twist, we get the following situation:



Then by co-commutativity, twist relation (iv), the frobenius relation and again the co-commutativity relation we get the following sequence of equalities that removes the twist:



Finally we are ready to deal with the situation where the given cobordism is non-connected. As normal form in this case we will take the disjoint union of the normal forms of each component up to permutation of the components. The strategy of the proof is analogous to that of lemma 3.13, so we will not repeat it here, but note that in the process we apply the relations of the twist (i) and (vi) to construct the permutations. $\square$

3.2 Back to 2dTQFT$_{\Bbbk}$'s

First we note that $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$ is a symmetric monoidal category. We are not going to elaborate any further on that. The twist is induced by the standard interchanging of factors of a tensor product.

Now we will make precise and meaningful what $\mathbf{2dTQFT}_{\Bbbk}$ is. We recall from definition 1.3 the the set of objects are symmetric monoidal functors from **2Cob** to $\mathbf{Vect}_{\Bbbk}$. In the previous we have shown that these two categories are symmetric monoidal, so as such it makes sense to have functors between them with some extra structure. We want the symmetric monoidal functors to preserve the symmetric monoidal structure. We do it in steps:

Definition 3.24 (Monoidal functor)**.** A functor, F, between monoidal categories $(\mathcal{C}, \Box, I)$ and $(\mathcal{C}', \Box', I')$ is *monoidal* if it preserves the monoidal structure, that is, such that for $n \geq 0$ the diagram

$$\begin{tikzcd} \mathcal{C}^n \arrow[r, "F^n"] \arrow[d, "\mu^{(n)}"'] & \mathcal{C}'^n \arrow[d, "\mu'^{(n)}"] \\ \mathcal{C} \arrow[r, "\tau_F"'] & \mathcal{C}' \end{tikzcd}$$

commutes for any $n \geq 0$, where $\mu^{(0)} \colon \mathcal{C}^0 \to \mathcal{C}$ is $\eta \colon \mathbf{1} \to \mathcal{C}$.

And then the symmetric aspect:

Definition 3.25 (Symmetric monoidal functor)**.** A functor, F, between symmetric monoidal categories $(\mathcal{C}, \Box, I, \tau)$ and $(\mathcal{C}', \Box', I', \tau)$ is *symmetric monoidal* if it is monoidal and preserves the symmetric structure, that is, such that for $X, Y \in \mathrm{Ob}(\mathcal{C})$ then $F(\tau_{X,Y}) = \tau'_{F(X),F(Y)}$.

So this makes sense to the object level of $\mathbf{2dTQFT}_{\Bbbk}$. For the morphisms level we want to define what it means to be a monoidal natural transformation. The motivation is that it should preserve the monoidal structure. We recall that

Definition 3.26. A natural transformation u between functors $F, G \colon \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $u_X \colon F(X) \to G(X)$ in $\mathcal{D}$ over objects, X, in $\mathcal{C}$ such that for any morphism in $\mathcal{C}$, $f \colon X \to Y$, the following

$$\begin{tikzcd} F(X) \arrow[r, "u_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "u_Y"'] & G(Y) \end{tikzcd}$$

commutes.

But since we require further that F and G are monoidal between $(\mathcal{C}, \Box, I)$ and $(\mathcal{D}, \Box', I')$, then we have that $F(X \Box Y) = (F(X) \Box' F(Y))$, $F(I) = I'$ and similar $G(X \Box Y) = (G(X) \Box' G(Y))$, $G(I) = I'$. So for u to be called monoidal the following seems natural:

Definition 3.27. A natural transformation $u\colon F \Rightarrow G$ is called monoidal if for any objects X, Y in $\mathcal{C}$ morphism $u_{X\Box Y}\colon F(X\Box Y) \to G(X\Box Y)$ in $\mathcal{D}$ satisfy $u_{X\Box Y} = u_X\Box' u_Y$ and $u_I = id_{I'}$. Equivalent:

$$\begin{tikzcd}
F(X\Box Y) \arrow[r, "u_{X\Box Y}"] \arrow[d, equal, "F(f)"'] & G(X\Box Y) \arrow[d, equal, "G(f)"] \\
F(X)\Box' F(Y) \arrow[r, "u_X\Box' u_Y"'] & G(X)\Box' G(Y)
\end{tikzcd}$$

commutes.

All in all we have established what we need to make sense of the definition of $\mathbf{2dTQFT}_{\Bbbk}$ and also from this it is clear, by constructing grids of diagrams, that if actually forms a category.

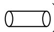
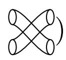
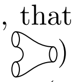
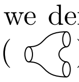
4 Main theorem

In this section we will prove the main theorem 1.1. We will first go through the general structure of such proof of equivalence, and then exhibit a proof for this case. This proof is going to, as promised, rely heavily on the descriptions we have given of the involved categories. Hence the motivation of describing them has not been solely to elaborate on the object level, and not only to show some needed results of the categories that we of cause needed for the formulation of the theorem to make sense either, but also it allows the proof to be quiet brief.

First the general – or categorical – setting. So let $\mathcal{D}, \mathcal{C}$ and $\mathcal{V}$ be categories. Then, if we want to show that the functors, $\Phi\colon \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \rightleftarrows \mathcal{D}\colon \Psi$, exhibit an equivalence, then we must show that $\Phi \circ \Psi \cong 1_{\mathcal{D}}$ and $\Psi \circ \Phi \cong 1_{\mathrm{Fun}\,(\mathcal{C},\mathcal{V})}$. To at all do that, we must define Φ and Ψ on both objects and morphisms of the involved categories. But to extend further to get closer to our setting, assume that $(\mathcal{C}, \Box, I)$ and $(\mathcal{V}, \Box', I')$ are symmetric monoidal categories such that it makes sense to consider the functor category between them as being symmetric monoidal, call it $\mathrm{SymMonFun}(-,-)$. And let us also assume $\mathcal{C}$ is small. This implies that any object of $\mathcal{C}$ can be constructed a generating object, call it $\mathbf{1}$, by composing it self with the monoidal operator, $\Box$. This means that when we want to define a functor from $\mathcal{C}$, which are objects in the functor category, then on object level, there we need only define it on this generating object. On a similar note, if we assume further that we have a generating set of morphism in $\mathcal{C}$, then when we want to define our functor from $\mathcal{C}$ on morphism level, we only need to define what it does for each of the generating morphism.

While this makes things easier on object level of defining Φ (and Ψ), it also does on morphism level. The morphisms in $\mathrm{SymMonFun}(\mathcal{C}, \mathcal{V})$ are monoidal natural transformations between symmetric monoidal functors; call a given one $u\colon F \Rightarrow G$. Then it consists of a collection of morphism $u_X\colon F(X) \to G(X)$ over objects, X, of $\mathcal{C}$. But now any object in $\mathcal{C}$ is of the form $\mathbf{1}^n := \mathbf{1}\Box\cdots\Box\mathbf{1}$ n-times, and hence the collection of morphisms in $\mathcal{V}$ that constitutes the natural transformation is $F(\mathbf{1}^n) \to G(\mathbf{1}^n)$ over $n \in \mathbb{N}$. By the monoidality of F, then $F(\mathbf{1}^n) = F(\mathbf{1})\Box'\cdots\Box' F(\mathbf{1})$ n-times. Since u is also a monoidal natural transformation it is clear from the diagram in definition 3.27 that we only need to specify what it does for the morphism $u_{\mathbf{1}}\colon F(\mathbf{1}) \to G(\mathbf{1})$. For example $u_{\mathbf{2}} = u_{\mathbf{1}}\Box' u_{\mathbf{1}}$ by said definition. From the definition of natural transformation to be fully satisfied, this $u_{\mathbf{1}}$ needs to make the square in definition 3.26 commute for each morphism $f\colon X \to Y$ in $\mathcal{C}$. So this is of the shape $f\colon \mathbf{1}^n \to \mathbf{1}^m$ where possible $n = m$. But since we a looking at a setting where we have a generating set of morphisms in $\mathcal{C}$, say, $\{f_1 \dots f_k\}$, then it suffices to require (or look at, depending on which of Ψ and Φ we talk about) commutativity for each of these generators, since commutativity of an arbitrary morphism will then be build up of commuting squares of those that is induced by the generators.

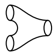
We are now ready to add content.

Proof of main theorem: We want to show that $\Phi\colon \mathbf{2dTQFT}_{\Bbbk} \to \mathbf{cFA}_{\Bbbk}\colon \Psi$ exhibit an equivalence of categories. We note that $\mathbf{2dTQFT}_{\Bbbk}$ is a symmetric monoidal functor category, so the setting from above holds. We want to define what Φ and Ψ do on both objects and morphism, and start with what it does on objects. An object in $\mathbf{2dTQFT}_{\Bbbk}$ is a symmetric monoidal functor, F, from $\mathbf{2Cob}$ to $\mathbf{Vect}_{\Bbbk}$. So to define such object, then according to the discussion above, we have to consider a generating object in $\mathbf{2Cob}$. From previous section we know, that $\mathbf{1}$ is that object. We let then $F(\mathbf{1} := A$ be a $\Bbbk$-vector space, ie. an object in $\mathbf{Vect}_{\Bbbk}$. And due to F being monoidal also $F(\mathbf{1}^n) := A^n$. By functoriality of F we get that $F($ $) := id_A$. Since F is symmetric monoidal it preserves the twist, by definition 3.25, so $F($ $) := \sigma_A\colon A^2 \to A^2$ where σ is the usual twist of factors on tensor product. This already took care of defining F on some of the generators of the generating set, that we have, from theorem 3.10. Here is how we defined on the rest. $F($ $) := \mu_A\colon A^2 \to A$, $F($ $) := \eta_A\colon \Bbbk \to A$, $F($ $) := \delta_A\colon A \to A^2$ and $F($ $) := \varepsilon_A\colon A \to \Bbbk$.

The next thing we observe is, that the relations of morphisms we have in $\mathbf{2Cob}$ translate into relations of morphisms in $\mathbf{Vect}_{\Bbbk}$ by the symmetric monoidality of F. Hence the relations listed in lemma 3.14 to 3.19 that holds in $\mathbf{2Cob}$ will be send to corresponding relations in $\mathbf{Vect}_{\Bbbk}$. Now from the relation in lemma 3.15 we see that η_A is unit to multiplication μ_A, and ε_A is co-unit to co-multiplication δ_A, and form lemma 3.18 the frobenius relations are satisfied. Hence by theorem 2.24 it follows that $(A, \mu_A, \eta_A, \delta_A, \varepsilon_A)$ is a frobenius algebra. This is why we spend the time to show that final equivalence of definitions of a frobenius algebras. By lemma 3.17 we see, it is also commutative. Hence we define Φ on objects as: $\Phi(F) := (A, \mu_A, \eta_A, \delta_A, \varepsilon_A)$.

Ψ we define on object completely similar but in reverse. So we are to send an object of $\mathbf{cFA}_{\Bbbk}$ to $\mathbf{2dTQFT}_{\Bbbk}$, meaning we have to construct a symmetric monoidal functor $\mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$ from a given commutative frobenius algebra. So let $\Psi((A, \mu_A, \eta_A, \delta_A, \varepsilon_A)) := F$. Clearly they satisfy equivalence on object level; they are constructed to be inverse of each other. We just have to check if F is well-defined, ie. that both sides of a relations in $\mathbf{2Cob}$ is send by F to morphisms in $\mathbf{Vect}_{\Bbbk}$ that are equivalent. This follows directly from our graphical representation of morphisms in $\mathbf{cFA}_{\Bbbk}$ and the relations between them, that we have shown in the frobenius algebra section.

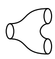
Now to the morphism level. So assume u is a monoidal natural transformation between symmetric monoidal functors F and G where $F(\mathbf{1}) = A$ and $G(\mathbf{1}) = B$. Then our list of generators of $\mathbf{2Cob}$ from theorem 3.10 provide us, due to naturality, with the following commutative diagrams, we exclude, however, the diagram involving the identity morphism as it does not provide any substantial information:

For $f =$ :

$$\begin{tikzcd} A \otimes A \arrow[r, equal] \arrow[d, "\mu_A"'] & F(\mathbf{2}) \arrow[r, "u_{\mathbf{2}} = u_{\mathbf{1}} \otimes u_{\mathbf{1}}"] \arrow[d, "F(f)"'] & G(\mathbf{2}) \arrow[r, equal] \arrow[d, "G(f)"] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, equal] & F(\mathbf{1}) \arrow[r, "u_{\mathbf{1}}"'] & G(\mathbf{1}) \arrow[r, equal] & B \end{tikzcd}$$

For $f =$ :

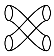
$$\begin{tikzcd} \Bbbk \arrow[r, equal] \arrow[d, "\eta_A"'] & F(\mathbf{0}) \arrow[r, "u_{\mathbf{0}} = id_{\Bbbk}"] \arrow[d, "F(f)"'] & G(\mathbf{0}) \arrow[r, equal] \arrow[d, "G(f)"] & \Bbbk \arrow[d, "\eta_B"] \\ A \arrow[r, equal] & F(\mathbf{1}) \arrow[r, "u_{\mathbf{1}}"'] & G(\mathbf{1}) \arrow[r, equal] & B \end{tikzcd}$$

For $f =$ :

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\delta_A"'] & F(\mathbf{1}) \arrow[r, "u_{\mathbf{1}}"] \arrow[d, "F(f)"'] & G(\mathbf{1}) \arrow[r, equal] \arrow[d, "G(f)"] & B \arrow[d, "\delta_B"] \\ A \otimes A \arrow[r, equal] & F(\mathbf{2}) \arrow[r, "u_{\mathbf{1}} \otimes u_{\mathbf{1}}"'] & G(\mathbf{2}) \arrow[r, equal] & B \otimes B \end{tikzcd}$$

For f =  :

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\varepsilon_A"'] & F(\mathbf{1}) \arrow[r, "u_{\mathbf{1}}"] \arrow[d, "F(f)"'] & G(\mathbf{1}) \arrow[r, equal] \arrow[d, "G(f)"] & B \arrow[d, "\varepsilon_B"] \\
\Bbbk \arrow[r, equal] & F(\mathbf{0}) \arrow[r, "id_{\Bbbk}"'] & G(\mathbf{0}) \arrow[r, equal] & \Bbbk
\end{tikzcd}$$

For f = :

$$\begin{tikzcd}
A \otimes A \arrow[r, equal] \arrow[d, "\sigma_A"'] & F(\mathbf{2}) \arrow[r, "u_{\mathbf{1}} \otimes u_{\mathbf{1}}"] \arrow[d, "F(f)"'] & G(\mathbf{2}) \arrow[r, equal] \arrow[d, "G(f)"] & B \otimes B \arrow[d, "\sigma_B"] \\
A \otimes A \arrow[r, equal] & F(\mathbf{2}) \arrow[r, "u_{\mathbf{1}} \otimes u_{\mathbf{1}}"'] & G(\mathbf{2}) \arrow[r, equal] & B \otimes B
\end{tikzcd}$$

We see, when we compare the first four diagrams above with those in definition 2.29, that they are exactly the same, meaning that u_1 corresponds to be a frobenius algebra homomorphism, and the last diagram involving the twist ensure that the multiplication is commutative. We see this for example when we fit the twist diagram into the multiplication diagram. Hence u_1 is a morphism in $\mathbf{cFA}_{\Bbbk}$. And similar in reverse direction: Given a frobenius algebra homomorphism (that preserves commutativity), then we get a monoidal natural transformation back, since the natural transformation is determined completely by this one morphism which in this case is a frobenius algebra homomorphism. So in other words: We define Φ and Ψ on morphisms as $\Phi(u) := u_1$ and $\Psi(u_1) := u$. Clearly either way we compose, they are inverse to each other. $\square$

References

[Kock] J. Kock: *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts (59), Cambridge University Press (2004).