2-Dimensional Topological Quantum Field Theories

Project outside course scope

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Contents

1 Introduction

1.1 Introduction to TQFTs

The purpose of this paper is to achieve the knowledge and mathematical tools needed to define and classify 2-dimensional topological quantum field theory (2D TQFTs). The axiomatic formulation of an n-dimensional TQFT (proposed by Atiyah [Ati86]) is that of a rule which assigns finite-dimensional vector spaces to closed oriented $(n-1)$-manifolds Σ and linear maps between vector spaces to n-dimensional oriented cobordisms (up to boundary preserving diffeomorphism) with boundary Σ. However, using category theory, a clearer formulation of TQFTs can be given. We will speak of a 2-dimensional TQFT as a functor Z from the category of oriented cobordisms $\mathbf{Bord}_{12}^{\mathrm{OR}}$ to the category of finite vector spaces $\mathbf{Vect}_{\Bbbk}$. This functor will precisely provide the mentioned assignments.

It turns out that 2D TQFTs has a close connection to commutative Frobenius algebras. A Frobenius algebra can be characterized as an algebra equipped with the special stucture of a co-algebra. In fact our main goal is to establish a one-to-one correspondence between 2D TQFTs and commutative Frobenius algebras. The main source of this paper is the excellent book by Joachim Kock [Koc03]. John M. Lee's book on smooth manifolds [Lee02] has been used to gain an understanding of the differential geometry involved in defining a smooth structure and orientation of manifolds. A thesis by Arik Wilbert [Wil11] has furthermore been used as a helping hand getting an overview of the most important results in [Koc03] relevant to this paper.

Finally a big thank you to Thomas Anton Wasserman for his help and advice during his supervision of this paper.

1.2 Analogy to D_6

We will here present an analogy of how we will use relations between generators in $\mathbf{Bord}_{12}^{OR}$ to examine structures of linear maps in $\mathbf{Vect}_{\Bbbk}$.
Recall the dihedral group D_6, the group of symmetries of an equilateral triangle. The group is generated by a counterclockwise rotation of $120° = \frac{2\pi}{3}$ and reflection in a line through a vertex and the midpoint of the opposite edge. An equivalence relation applies to these two operations: rotating the triangle three times is equivalent to doing nothing to the triangle. This is also the case if we reflect the triangle two times in the same line. Also if we rotate, reflect and rotate once more, it's the same as if we only reflect once. These three relations are sufficient in the sense that, any other relation can be built from these three relations. If r denotes rotation and s denotes reflection, the *representation* of the group is then

$$\mathrm{D}_6 =< r, s \mid r^3 = e, s^2 = e, rs = sr^{-1} >,$$

where e is the identity element of the group.
Words i.e. elements in D_6 can be written as combinations of the operations r and s, and the relations give us a way of comparing words: any two words in D_6 are equal if both can be reduced to the same word by using the equivalence relations from the representation. For example, the two words $sssr$ and $ssr^{-1}s$ are equal since they both reduce down to sr by the use of $s^2 = e$ and $rs = sr^{-1}$.

Now let O(2) be the group of all real invertible 2×2 matrices, whose transpose equals their inverse, $\mathrm{O}(2) = \{M \in \mathrm{GL}_2(\mathbb{R}) \mid M^T = M^{-1}\}$.

We then define a map

$$\phi : \mathrm{D}_6 \to \mathrm{O}(2),$$

to take the generators of D_6 to the following rotation and reflection matrices:

$$r \mapsto \begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$
$$s \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We would like to see whether the relations between the generators in D_6 also hold for these matrices in O(2), i.e. whether ϕ is a homeomorphism.
Any word x in D_6 can be written as a combination of the generators by the use of $rs = sr^{-1}$, i.e. $x = r^a s^b$ for any $a, b \in \mathbb{Z}$. We define the value of ϕ on x by:

$$\phi(x) := \phi(r)^a \phi(s)^b = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^a \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^b.$$

This definition is well-defined, as we can see by showing that $\phi(r)$ and $\phi(s)$ satisfy the same relations as r and s:

$$\phi(r)^3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = I_2 \ , \ \phi(s)^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = I_2,$$

where I_2 is the 2×2 identity matrix, and

$$\phi(r)\phi(s)\phi(r) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \phi(s).$$

Thus we see that ϕ is a homomorphism, when the image of r of s are defined to be these specific matrices. Hence the image of ϕ obeys the same structure as r and s.

Now define the following set

$$R := \{(M, N) \in \mathrm{O}(2) \times \mathrm{O}(2) \mid M^3 = I_2, N^2 = I_2, MNM = N\}.$$

We would now like to see whether if we take two matrices M and N in $O(2)$ that obey the same relations as the generators in D_6, then we can construct a homeomorphism from D_6 to $O(2)$. We see that R is isomorphic to $\mathrm{Hom}(\mathrm{D}_6, \mathrm{O}(2))$, the set of all homeomorphism from D_6 to O(2): Take the following map f, that sends any tuple $(M, N) \in R$ to the homeomorphism taking the generators of D_6 to M and N in the following way:

$$f : R \to \mathrm{Hom}(\mathrm{D}_6, \mathrm{O}(2))$$
$$(M, N) \mapsto \begin{pmatrix} \psi : \mathrm{D}_6 \to \mathrm{O}(2) \\ r \mapsto M \\ s \mapsto N \end{pmatrix}.$$

We now show that the following map g is the inverse to f,

$$\begin{aligned} g : \mathrm{Hom}(\mathrm{D}_6,\mathrm{O}(2)) &\to R \\ (\phi : \mathrm{D}_6 \to \mathrm{O}(2)) &\mapsto (\phi(r), \phi(s)). \end{aligned}$$

Take $(M, N) \in R$ under the composition $g \circ f$:

$$(M, N) \overset{f}{\mapsto} \begin{pmatrix} \psi : \mathrm{D}_6 \to \mathrm{O}(2) \\ r \mapsto M \\ s \mapsto N \end{pmatrix} \overset{g}{\mapsto} (\psi(r), \psi(s)) = (M, N).$$

Now take $\phi \in \mathrm{Hom}(\mathrm{D}_6,\mathrm{O}(2))$ and note that $(\phi(r), \phi(s)) \in R$ since ϕ is a homeomorphism. Take ϕ under the composition $f \circ g$:

$$(\phi : \mathrm{D}_6 \to \mathrm{O}(2)) \overset{g}{\mapsto} (\phi(r), \phi(s)) \overset{f}{\mapsto} \begin{pmatrix} \psi : \mathrm{D}_6 \to \mathrm{O}(2) \\ r \mapsto \phi(r) \\ s \mapsto \phi(s) \end{pmatrix} = (\phi : \mathrm{D}_6 \to \mathrm{O}(2)).$$

Thus $R \cong \mathrm{Hom}(\mathrm{D}_6,\mathrm{O}(2))$. By describing R we would then be able to find the homeomorphisms $\psi : \mathrm{D}_6 \to \mathrm{O}(2)$ determined by

$$\{(\psi(r), \psi(s)) \in \mathrm{O}(2) \times \mathrm{O}(2) \mid \psi(r)^3 = I_2, \psi(s)^2 = I_2, \psi(r)\psi(s)\psi(r) = \psi(s)\}.$$

In the same spirit, given a relation preserving functor $Z : \mathbf{Bord}_{12}^{\mathrm{OR}} \to \mathbf{Vect}_{\Bbbk}$, we will from the relations between generators of $\mathbf{Bord}_{12}^{\mathrm{OR}}$ examine what the vector space $Z(\Sigma)$, Σ being a circle, carries in structure: by defining the image of the generators under Z to be certain linear maps between tensor products of a vector space A, the functor Z allows us to translate the relations between cobordisms in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ to relations between linear maps in $\mathbf{Vect}_{\Bbbk}$. It turns out that when the relations in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ are translated to the linear maps, the vector space A gets equipped with a certain structure, turning it into a commutative Frobenius algebra. To prove our main result, namely functors like Z having a one-to-one correspondance to Frobenius algebras, we need to acquire knowledge about categories, vector spaces, algebraic structures and topological spaces.

2 Basic category theory

In this chapter we will go over definitions of categories, maps between categories and symmetric monoidal structures.

Definition 2.1. A category $\mathbf{C}$ consists of

- a collection of *objects* denoted $\mathbf{C}_0$;
- for every pair of objects X, Y, a set of *morphisms* from X to Y denoted $\mathbf{C}(X, Y)$;
- for every triple of objects X, Y, Z, an *associative composition*, $\circ : \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$;

- for every object X, an *identity morphism* $\mathrm{id}_X \in \mathbf{C}(X, X)$, that maps an object X to itself. The identity morphism acts as neutral element for the composition.

An element in $\mathbf{C}(X, Y)$ is written $f : X \to Y$, and the composition $X \to Y \to Z$ of two morphisms $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ is written gf. The associativity of composition gives that for three morphisms $e : W \to X, f : X \to Y, g : Y \to Z$, we have $(fg)e = f(ge)$. That the identity morphism acts as a neutral element for the composition means that for every morphism $f : X \to Y$ and every morphism $e : W \to X$, we have $f\,\mathrm{id}_X = f$ and $\mathrm{id}_X\, e = e$.

Some simple examples of categories include **Set**, the category where objects are sets and morphisms are functions between the sets, and **Top** where objects are topological spaces and morphisms are continuous maps between these topological spaces. Another important example which this project will rely on is the category $\mathbf{Vect}_{\Bbbk}$, where the objects are vector spaces over the field $\Bbbk$ and the morphisms are $\Bbbk$-linear maps. We will return to this category later.

A study of maps between categories seems useful: a functor between two categories maps objects to objects and morphisms to morphisms. A more formal definition:

Definition 2.2. A functor F between two categories **C** and **D** consists of

- a map $F : \mathbf{C}_0 \to \mathbf{D}_0$ assigning to every object X in **C** an object $F(X)$ in **D**;
- for each pair of objects X, Y in **C** a map $F_{X,Y} : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ assigning to every morphism f in **C** a morphism $F_{X,Y}(f)$ in **D** such that
  - $F_{X,Z}(gf) = F_{Y,Z}(g)F_{X,Y}(f)$ for any morphisms $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, Z)$;
  - for each object X in **C**, we have $F_{X,X}(id_X) = id_{F(X)}$.

id denotes the identity map.

Definition 2.3. Let F, G be two functors between categories **C** and **D**. Then a natural transformation η is a collection of morphisms η_X where

- $\eta_X : F(X) \to G(X)$ is a morphism assigned to every object X in **C**;
- the following diagram commute for all morphisms $f : X \to Y$ in **C**:

$$\begin{array}{ccc} F(X) & \xrightarrow{F_{X,Y}(f)} & F(Y) \\ \big\downarrow{\scriptstyle \eta_X} & & \big\downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G_{X,Y}(f)} & G(Y). \end{array}$$

If η_X is an isomorphism in **D** for every X, then η is called a natural isomorphism.

We are now ready to define a specific structure on a category:

Definition 2.4. A monoidal structure on a category **C** consists of the following:

- A functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$;
- An object $\mathbb{1}$ in $\mathbf{C}$ called the unit;
- A natural isomorphism

$$\alpha : ((-) \otimes (-)) \otimes (-) \overset{\cong}{\Rightarrow} (-) \otimes ((-) \otimes (-))$$

  called the *associator* with components

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \overset{\cong}{\to} X \otimes (Y \otimes Z);$$

- Two natural isomorphisms

$$l : (\mathbb{1} \otimes (-)) \overset{\cong}{\Rightarrow} (-)$$
$$r : ((-) \otimes \mathbb{1}) \overset{\cong}{\Rightarrow} (-)$$

  called the *left and right unitors* respectively with components

$$l_X : \mathbb{1} \otimes X \overset{\cong}{\to} X$$

$$r_X : X \otimes \mathbb{1} \overset{\cong}{\to} X,$$

such that the following diagrams commute for all objects involved:

$$\begin{array}{ccc} & X \otimes Y & \\ {}^{r_X \otimes id_Y}\nearrow & & \nwarrow^{id_X \otimes l_Y} \\ (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \end{array} ,$$

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & {}^{\alpha_{W \otimes X,Y,Z}}\nearrow & & \searrow^{\alpha_{W,X,Y \otimes Z}} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle \alpha_{W,X,Y} \otimes id_Z} & & & & \uparrow{\scriptstyle id_W \otimes \alpha_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\alpha_{W,X \otimes Y,Z}} & & W \otimes ((X \otimes Y) \otimes Z) \end{array} .$$

The first is called the triangle identity and the second the pentagon identity.

When defining structures, we will denote them by n-tuples consisting of the objects and the structures associated with them. For example we denote a monoidal category by the triple $(\mathbf{C}, \otimes, \mathbb{1})$.

A monoidal category is *strict* if its associator, left unitor and right unitor are identity maps, which implies the equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$. In fact Mac Lane's Coherence Theorem [Lan98] is a result, which states that every monoidal category is in fact equivalent to a strict monoidal category. We will throughout this paper only encounter monoidal categories, that are strict.

Definition 2.5. A *symmetric monoidal category* is a monoidal category equipped with an additional natural isomorphism

$$B_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

called the braiding, such that

$$B_{Y,X} B_{X,Y} = \mathrm{id}_{X \otimes Y},$$

and the associator and braiding make the following diagram commute:

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{B_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\ \Big\downarrow{\scriptstyle B_{X,Y} \otimes id_Z} & & & & \Big\downarrow{\scriptstyle \alpha_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{id_Y \otimes B_{X,Z}} & Y \otimes (Z \otimes X) \end{array} .$$

This is called the hexagon identity. The braiding also has compatibility with the unit:

$$\begin{array}{ccc} X \otimes \mathbb{1} & \xrightarrow{B_{X,\mathbb{1}}} & \mathbb{1} \otimes X \\ & {\scriptstyle l_X} \searrow & \Big\downarrow{\scriptstyle r_X} \\ & & X \end{array} .$$

We will denote a symmetric monoidal category by the quadruple $(\mathbf{C}, \otimes, \mathbb{1}, B_{X,Y})$

2.1 $\mathbf{Vect}_{\Bbbk}$ as a symmetric monoidal category

$\mathbf{Vect}_{\Bbbk}$ can be equipped with a symmetric monoidal structure of great importance throughout this paper. In order to describe this structure it is assumed that the reader is familiar with the notion of a vector space. Now the functor of the monoidal structure will be the usual tensor product between vector spaces, so first a definition of the tensor product:

Definition 2.6. If V and W are two vector spaces with common ground field $\Bbbk$, then the tensor product $\otimes$ is defined as

$$V \otimes W := F(V \times W)/\sim,$$

where $F(V \times W)$ is the vector space generated by $V \times W$, and $\sim$ denotes the equivalence relation generated by the relation that for all $v, v' \in V, w, w' \in W$ and $c \in \Bbbk$:

$$\begin{aligned} (v + v', w) &\sim (v, w) + (v', w), \\ (v, w + w') &\sim (v, w) + (v, w'), \\ c(v, w) &\sim (cv, w), \\ c(v, w) &\sim (v, cw). \end{aligned}$$

We will denote the equivalence class of $(v, w) \in V \times W$ by $v \otimes w$. If $f : V \to W$ and $g : S \to T$ are two linear maps with $V, W, S, T \in \mathbf{Vect}_{\Bbbk}$, we denote the tensor product of the maps by:

$$\begin{aligned} f \otimes g : V \otimes S &\to W \otimes T \\ v \otimes s &\mapsto f(v) \otimes g(s). \end{aligned}$$

Note that the tensor product of two vector spaces is itself a vector space.

By looking at the maps $\Bbbk \otimes V \to V$ and $V \otimes \Bbbk \to V$ it should be clear that they are isomorphisms with inverse maps taking $v \mapsto 1 \otimes v$ and $v \mapsto v \otimes 1$, where 1 is the multiplicative identity of $\Bbbk$. We then have $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$.
Taking the ground field $\Bbbk$ (which is also a vector space, hence an object in $\mathbf{Vect}_\Bbbk$) as the unit object, the structure of the tensor product gives the unitors

$$\begin{aligned} l_V : \Bbbk \otimes V &\overset{\cong}{\to} V \\ c \otimes v &\mapsto cv \ , \end{aligned}$$

and

$$\begin{aligned} r_V : V \otimes \Bbbk &\overset{\cong}{\to} V \\ v \otimes c &\mapsto cv. \end{aligned}$$

The tensor product is also symmetric, so it has a braiding

$$\begin{aligned} \sigma : V \otimes W &\overset{\cong}{\to} W \otimes V \\ v \otimes w &\mapsto w \otimes v \ . \end{aligned}$$

The tensor product is also associative:

$$\begin{aligned} (V \otimes W) \otimes Z &\overset{\cong}{\to} V \otimes (W \otimes Z) \\ (v \otimes w) \otimes z &\mapsto v \otimes (w \otimes z), \end{aligned}$$

which is a result of the definition

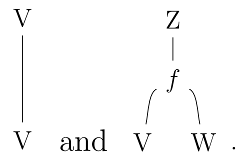
$$V \otimes W \otimes Z := F(V \times W \times Z)/\sim,$$

where $\sim$ denotes the equivalence relation described above but with three coordinates in the cartesian product instead of two. Because of this, we will omit the parentheses and just write $V \otimes W \otimes Z$.

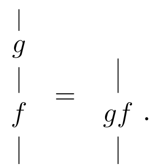
Thus $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk, \sigma)$ is a symmetric monoidal category.

2.2 Short introduction to string diagrams

At this point and from now on it would be beneficial to have axioms and proofs visualised in some way other than commuting diagrams, in order to get intuition of what is going on. Hence we introduce string diagrams, which is a graphical way of expressing morphisms in a monoidal category.
The overall idea is to think of objects in categories as strings and morphisms between the categories as nodes, where the source strings enter and target strings exit. We will in the following couple of sections work with $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk.\sigma)$, so the strings represent vector spaces and nodes represent linear maps. The diagrams will be read from bottom to top with respect to composition, and tensor products between vector spaces are represented by strings beside each other. We will start off with some simple examples. Let $V, W, Z \in \mathbf{Vect}_\Bbbk$. The identity map $id_V : V \to V$ is represented by a single string, and a morphism $f : V \otimes W \to Z$ is represented by two strings merging into a single string:



The identity map can be added or taken out of a string diagram, leaving the diagram unchanged, just as it would leave any composition unchanged.

There are equivalences between string diagrams, the most important one involving composition: 

We will from now on omit the labels when the maps and vector spaces are clear from context.

We now move on to new structures, that will hopefully be more intuitive with the help from string diagrams. It will indeed be helpful when we arrive at our main result, where we need to realise similarities between cobordisms and linear maps.

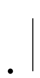
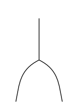
3 Algebras in $\mathbf{Vect}_{\mathbb{k}}$

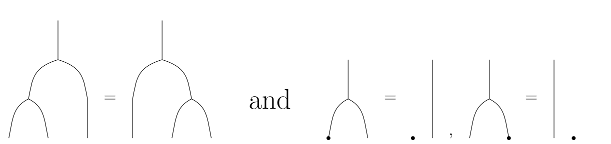
We will now move on to algebras and their dual structure, which leads us to Frobenius algebras. We start with defining the notion of an algebra in the category of vector spaces over a field $\mathbb{k}$:

Definition 3.1. An *algebra* is an object A in $\mathbf{Vect}_{\mathbb{k}}$ together with a multiplication map $\mu : A \otimes A \to A$ and a unit map $u : \mathbb{k} \to A$, such that associativity and unitality hold i.e. the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id_A \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes id_A} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbb{k} \otimes A & \xrightarrow{u \otimes id_A} & A \otimes A & \xleftarrow{id_A \otimes u} & A \otimes \mathbb{k} \\ & \searrow{\scriptstyle l_A} & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle r_A} & \\ & & A & & \end{array} .$$

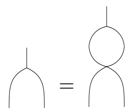
We will denote an algebra A by a triple (A, μ, u).

In terms of string diagrams, the unit object $\mathbb{k}$ is represented as a dot, so for example the unit map $u : \mathbb{k} \to V$ is represented as , and the right unitor $\mathbb{k} \otimes V \to V$ is represented as . The multiplication map $\mu : A \otimes A \to A$ is represented as . The diagrams from the definition above can be expressed as



Since $\mathbf{Vect}_{\mathbb{k}}$ possesses a braiding, we get the notion of commutativity of an algebra:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{B_{A,A}} & A\otimes A \\ & \searrow^{\mu} & \downarrow \mu \\ & & A \end{array} \quad ,$$

This is expressed with string diagrams as .

If we write the multiplication by juxtaposition

$$X\otimes Y \mapsto XY,$$

and let $\mathbb{1}_A$ be the image of 1 under u, then we can write the associativity and unitality axioms as such:

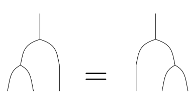
$$(XY)Z = X(YZ) \quad , \quad \mathbb{1}_A X = X = X\mathbb{1}_A.$$

Example: $\mathbb{C}$ as an $\mathbb{R}$-algebra. The complex numbers $\mathbb{C}$ as a vector space over the ring $\mathbb{R}$ has an algebraic structure with the following multiplication and unit map:

$$\begin{aligned} \mu : \mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C} &\to \mathbb{C} \\ (x+iy)\otimes(x'+iy') &\mapsto (xx'-yy'+i(xy'+x'y)), \end{aligned}$$

where the unit map is given by

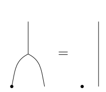
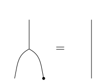
$$\begin{aligned} u : \mathbb{R} &\to \mathbb{C} \\ x &\mapsto x. \end{aligned}$$

We need to check that . We start with the composition on left hand side. For $(x+iy),(x'+iy'),(x''+iy'')\in\mathbb{C}$ we get:

$$\begin{aligned} (x+iy)\otimes(x'+iy')\otimes(x''+iy'') &\overset{\mu\otimes id}{\longmapsto} (xx'-yy'+i(xy'+x'y))\otimes(x''+iy'') \\ &\overset{\mu}{\mapsto} ((xx'-yy')x''-(xy'+x'y)y''+i((xx'-yy')y''+x''(xy'+x'y)) =: z, \end{aligned}$$

and on the right hand side we get:

$$\begin{aligned} (x+iy)\otimes(x'+iy')\otimes(x''+iy'') &\overset{id\otimes\mu}{\longmapsto} (x+iy)\otimes(x'x''-y'y''+i(x'y''+x''y')) \\ &\overset{\mu}{\mapsto} (x(x'x''-y'y'')-y(x'y''+x''y')+i(x(x'y''+x''y')+(x'x''-y'y'')y)) = z. \end{aligned}$$

So the associativity axiom holds. Next we check  and . For $r\in\mathbb{R}$ and $(x+iy)\in\mathbb{C}$ we look at the two mappings,

$$r\otimes(x+iy) \overset{u\otimes id}{\longmapsto} r\otimes(x+iy) \overset{\mu}{\mapsto} rx+iry = r(x+iy)$$

and

$$(x+iy)\otimes r \xmapsto{id\otimes u} (x+iy)\otimes r \overset{\mu}{\mapsto} xr + iry = r(x+iy).$$

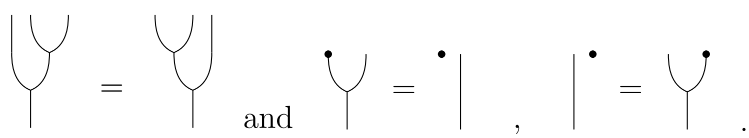
In each case end up with scalar multiplication, so the unitality axiom holds.

A structure that is dual to the associative unital algebra, is that of a coalgebra, which also has a form of multiplication and unit maps but with the arrows reversed:

Definition 3.2. A *co-algebra* is an object $A \in \mathbf{Vect}_{\mathbb{k}}$ together with co-multiplication $\delta : A \to A \otimes A$ and co-unit $e : A \to \mathbb{k}$, such that co-associativity and co-unitality holds i.e. the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\delta} & A\otimes A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta\otimes id_A} \\ A\otimes A & \xrightarrow{id_A\otimes\delta} & A\otimes A\otimes A \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbb{k}\otimes A & \xleftarrow{e\otimes id_A} & A\otimes A & \xrightarrow{id_A\otimes e} & A\otimes\mathbb{k} \\ & {\scriptstyle l_A^{-1}}\nwarrow & \uparrow{\scriptstyle\delta} & \nearrow{\scriptstyle r_A^{-1}} & \\ & & A & & \end{array} .$$

We will denote a co-algebra A by a triple (A, δ, e). The diagrams are represented with string diagrams as such:



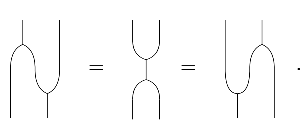
3.1 Frobenius Algebras

A Frobenius algebra is a vector space that is both an algebra and co-algebra with a relation involving the two.

Definition 3.3. A *Frobenius algebra* is a quintuple (A, μ, δ, u, e) such that

- (A, μ, u) is an algebra with multiplication $\mu : A \otimes A \to A$ and unit $u : \mathbb{k} \to A$;
- (A, δ, e) is a co-algebra with co-multiplication $\delta : A \to A \otimes A$ and co-unit $e : A \to \mathbb{k}$;
- $(\mu \otimes id_A)(id_A \otimes \delta) = \delta\mu = (id_A \otimes \mu)(\delta \otimes id_A)$.

The last condition is called the *Frobenius axiom* and is expressed through string diagrams as such:

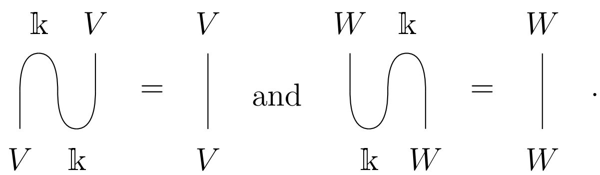


There exist multiple equivalent definitions of Frobenius algebras, and while we will primarily be working with the one above, other definitions could be more handy in other cases. For example the following alternative definition below avoids explicit construction of coevaluation and evaluation. However we will not be proving the equivalence nor work with this definition henceforth.

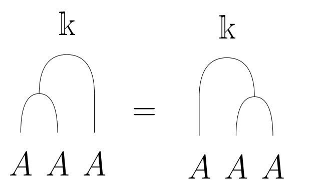
Definition 3.4. Let $V, W \in \mathbf{Vect}_{\Bbbk}$. A *pairing* of V and W is a linear map

$$\begin{aligned} \beta : V \otimes W &\to \Bbbk \\ v \otimes w &\mapsto < v \mid w >, \end{aligned}$$

where $< v \mid w >$ is notation the pairing acting on an element. A pairing is *nondegenerate* if there exists a linear map $\gamma : \Bbbk \to W \otimes V$ such that



Definition 3.5 (Alternative definition of Frobenius algebra)**.** A Frobenius algebra is an algebra (A, μ, u) of finite dimension, equipped with an nondegenerate pairing $\beta : A \otimes A \to \Bbbk$, which is associative with μ:



The pairing is called the *Frobenius pairing*.

Example: ($Mat_n(\mathbb{R})$ as a Frobenius algebra with nondegenerate pairing) **.** In the case of $Mat_n(\mathbb{R})$, it is more convenient to use the alternative definition. $Mat_n(\mathbb{R})$ is the ring of $n \times n$ matrices over $\mathbb{R}$. It is especially a finite vector space with standard unit basis E_{ij} of matrices with 1 in entry ij and 0 in any other entry. We have $\dim E_{ij} = n^2$. $Mat_n(\mathbb{R})$ is then an algebra $(Mat_n(\mathbb{R}), \mu, u)$ with the usual matrix multiplication as μ and unit

$$\begin{aligned} u : \mathbb{R} &\to M_n(\mathbb{R}) \\ r &\mapsto rI_n, \end{aligned}$$

where I_n denotes the $n \times n$ identity matrix. The associativity of μ comes from the associativity of matrix multiplication. It would be tricky to find the coevaluation, so instead we induce a pairing from a well known map, namely the trace of a matrix. The pairing is the map taking two matrices from $Mat_n(\mathbb{R})$ to the trace of their matrix multiplication:

$$\begin{aligned} \beta : Mat_n(\mathbb{R}) \times Mat_n(\mathbb{R}) &\to \mathbb{R} \\ M \otimes N &\mapsto \mathrm{Tr}(MN), \end{aligned}$$

where

$$\begin{aligned} \mathrm{Tr} : Mat_n(\mathbb{R}) &\to \mathbb{R} \\ (a_{ij}) &\mapsto \sum_{i=1}^{n} a_{ii}. \end{aligned}$$

Note that $\mathrm{Tr}(E_{ij}E_{kl})$ is nonzero only when $kl = ji$: $\mathrm{Tr}E_{ij}E_{kl} = 1$. Define the linear map,

$$\gamma : \mathbb{R} \to M_n(\mathbb{R}) \otimes M_n(\mathbb{R})$$
$$1 \mapsto \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}.$$

A matrix $M \in Mat_n(\mathbb{R})$ can be written as a linear combination of the E_{ij}: $M = \sum_{i,j=1}^{n} c_{ij}E_{ij}$. We send M through the composition $(\beta \otimes id_{M_n(\mathbb{R})})(id_{M_n(\mathbb{R})} \otimes \gamma)$:

$$\sum_{k,l} c_{kl}E_{kl} \mapsto \sum_{i,j,k,l} E_{kl} \otimes E_{ij} \otimes c_{kl}E_{ji} \mapsto \sum_{i,j,k,l} \mathrm{Tr}(E_{kl}E_{ij}) \otimes c_{kl}E_{ji} = \sum_{i,j} 1 \otimes c_{ji}E_{ji} = \sum_{i,j} c_{ji}E_{ji}.$$

The other composition from Definition 3.4 is checked in the same way. Hence β is nondegenerate, and $Mat_n(\mathbb{R})$ is a Frobenius algebra.

4 Duality

Definition 4.1. Let $(\mathbf{C}, \otimes, \mathbb{1})$ be a monoidal category, and let A be an object in $\mathbf{C}$. A right dual for A is an object $A^* \in \mathbf{C}$ together with:

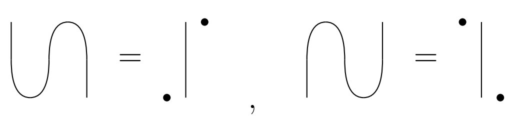
- a morphism $ev_A : A^* \otimes A \to \mathbb{1}$ called the evaluation map represented by $\cap$;
- a morphism $coev_A : \mathbb{1} \to A \otimes A^*$ called the coevaluation map represented by $\cup$;

such that the following diagrams commute:

$$\begin{array}{ccc} \mathbb{1} \otimes A & \xrightarrow{r_A^{-1} l_A} & A \otimes \mathbb{1} \\ & \searrow^{coev_A \otimes id_A} & \uparrow{\scriptstyle id_A \otimes ev_A} \\ & & A \otimes A^* \otimes A \end{array} \quad , \quad \begin{array}{ccc} A^* \otimes \mathbb{1} & \xrightarrow{l_{A^*}^{-1} r_{A^*}} & \mathbb{1} \otimes A^* \\ {\scriptstyle id_{A^*} \otimes coev_A}\downarrow & \nearrow_{ev_A \otimes id_{A^*}} & \\ A^* \otimes A \otimes A^* & & \end{array} \quad .$$

A is called the left dual of A^*.

In string diagrams the two diagrams above are represented by



giving them the name *snake relations*.

When right duals exist, all of them are canonically isomorphic to each other and so are left duals: Let $(B, ev, coev)$ be a right dual of A and suppose $(B', ev', coev')$ is another right dual of A. Then

$$(ev \otimes id_{B'})(id_B \otimes coev') : B \to B'$$

is the map taking B to B' as follows:

$$B = B \otimes \mathbb{1} \xrightarrow{id_B \otimes coev'} B \otimes A \otimes B' \xrightarrow{ev \otimes id_{B'}} \mathbb{1} \otimes B' = B'.$$

Likewise the following map takes B' to B:

$$(ev' \otimes id_B)(id_{B'} \otimes coev) : B' \to B.$$

It is obvious that they are each others inverse, hence B and B' are isomorphic to each other.

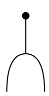
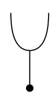
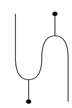
Note that when **C** is a symmetric monoidal category, every right dual is also a left dual and the other way around, since $A^* \otimes A \cong A \otimes A^*$. In this case we will simply call them duals.

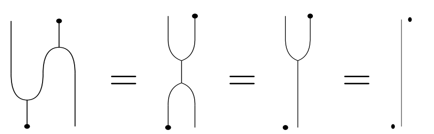
Theorem 4.2. *A Frobenius algebra F in a monoidal category is dual to itself.*

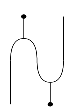
Proof. Let (F, μ, δ, u, e) be a Frobenius algebra in a monoidal category, and let $\mathbb{1}$ be the monoidal unit. If F is a dual to itself, there must exist two morphisms $ev : F \otimes F \to \mathbb{1}$ and $coev : \mathbb{1} \to F \otimes F$ such that the diagrams from Definition 4.1 commute. Let these maps be defined as:

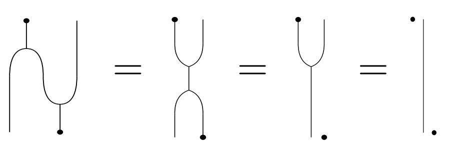
$$ev : F \otimes F \xrightarrow{\mu} F \xrightarrow{e} \mathbb{1},$$

$$coev : \mathbb{1} \xrightarrow{u} F \xrightarrow{\delta} F \otimes F,$$

i.e we define  :=  and  := . With these we compose . We then use the Frobenius axiom, and afterwards the unit and counit axioms:



In the same way we compose , and use the Frobenius axiom together with the unitality and co-unitality axiom:



Thus we have that the evaluation and coevalution satisfies the snake relations. □

Remark: Theorem 4.2 implies that Definition 3.3 leads to Definition 3.5.

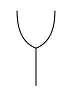
Example: $\mathbb{C}$ as a dual to itself. We have previously shown that $\mathbb{C}$ is an algebra, and we actually also know that $\mathbb{C}$ is in fact a Frobenius algebra. Now we are curious to find the co-multiplication and counit maps, an obvious counit being

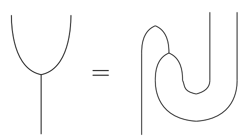
$$\begin{aligned} e : \mathbb{C} &\to \mathbb{R} \\ z &\mapsto \Re z. \end{aligned}$$

We will now put the previous theorem to use and use the fact that $\mathbb{C}$ is a Frobenius algebra in order to construct the corresponding co-multiplication.
The theorem and proof gives us that $\mathbb{C}$ is its own dual with the following evaluation and coevaluation maps for $(x + iy), (x' + iy') \in \mathbb{C}$ and $r \in \mathbb{R}$:

$$ev : (x + iy) \otimes (x' + iy') \overset{\mu}{\mapsto} (xx' - yy' + i(xy' + x'y)) \overset{e}{\mapsto} xx' - yy',$$

$$coev : r \overset{u}{\mapsto} r \overset{\delta}{\mapsto} \delta(r).$$

Now the trick is to use  and  from the duality and construct . We define the co-multiplication in the following way:

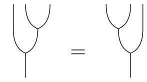
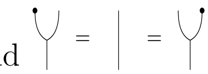


The right hand side is a composition of the evaluation, coevaluation and multiplication maps, with the same number of input and output strings as the co-multiplication. It's not very giving to explicitly go through every mapping of the composition, but since it is our first complicated string diagram, here's an overview of the composition from the bottom and up:

$$\begin{array}{c} \mathbb{C} \otimes \mathbb{C} \\ {\scriptstyle r \otimes id}\uparrow \\ \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{C} \\ {\scriptstyle ev \otimes id \otimes id}\uparrow \\ \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \\ {\scriptstyle id \otimes \mu \otimes id \otimes id}\uparrow \\ \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \\ {\scriptstyle id \otimes id \otimes coev \otimes id}\uparrow \\ \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{C} \\ {\scriptstyle id \otimes r^{-1} \otimes id}\uparrow \\ \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \\ {\scriptstyle id \otimes coev}\uparrow \\ \mathbb{C} \otimes \mathbb{R} \\ {\scriptstyle r^{-1}}\uparrow \\ \mathbb{C} \end{array} .$$

With this we end up getting any $z \in \mathbb{C}$ mapped to $(\Re z + i\Im z) \otimes 1 + (\Im z - i\Re z) \otimes i$, so we now define the co-multiplication to be the map

$$\delta : z \mapsto (\Re z + i\Im z) \otimes 1 + (\Im z - i\Re z) \otimes i.$$

It is now fairly straightforward to check  and  , concluding our search for a co-multiplication.

4.1 Dual vector spaces

We now move on to describe a specific dual-admitting category, namely $\mathbf{Vect}_{\Bbbk}$. In $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$ a dual object V^* to a vector space V is the linear dual.
A *linear functional* ψ is a $\Bbbk$-linear map from a vector space V to it's ground field $\Bbbk$, and the space of linear functionals of a vector space V is denoted $\mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. We will from now on take $\Bbbk = \mathbb{C}$.

Definition 4.3 (Dual vector spaces)**.** Let V be a vector space over $\mathbb{C}$. The space of linear functionals on V is called the *dual* of V and is denoted

$$V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

If $\{e_1, ..., e_n\}$ is a basis in V, then one can construct a dual basis $\{e^1, ..., e^n\}$ of linear functionals defined by $e^i(c^1 e_1 + ... + c^n e_n) = c^i,\ i = 1, ..., n$ for any coefficients $c^i \in \mathbb{C}$. Letting in turn each of the c^is be one and the others zero, we get the following

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

An element $x \in V$ can be written as a linear combination for fitting $c^i \in \mathbb{C}$, $x = \sum_{i=1}^n c^i e_i$. We can then write any element $x \in V$ as $\sum_{i=1}^n e^i(x) e_i$ since

$$e^i(x) = e^i\big(\sum_{k=1}^n c^k e_k\big) = \sum_{k=1}^n c^k e^i(e_k) = \sum_{k=1}^n c^k \delta^i_k = c^i,$$

hence

$$\sum_{i=1}^n e^i(x) e_i = \sum_{i=1}^n c^i e_i = x.$$

Knowing this, we can define and prove the evaluation and co-evaluation maps for the dual:

Theorem 4.4. *The following two morphisms are evaluation and coevaluation maps that exhibit V^* as the dual for a vector space, $V \in (\mathbf{Vect}_{\mathbb{C}}, \otimes, \mathbb{C})$:*

$$\begin{aligned} ev : V^* \otimes V &\to \mathbb{C} \\ \psi \otimes v &\mapsto \psi(v), \end{aligned}$$

$$\begin{aligned} coev : \mathbb{C} &\to V \otimes V^* \\ c &\mapsto \sum_{i=1}^n c e_i \otimes e^i. \end{aligned}$$

Proof. We need to check if the two maps satisfy the snake relations, and we will in the following write $\sum_i$ instead of $\sum_{i=1}^n$ to ease notation. We will start with the left snake relation from Definition 4.1.
Let $\{e_1, ..., e_n\}$ be a basis for V and $\{e^1, ..., e^n\}$ a basis for the dual space V^*. Let $v = \sum_k c^k e_k \in V$, for $c^k \in \mathbb{C}$. Then for any $c \in \mathbb{C}$ we see that the map

$$c \otimes v \xmapsto{coev \otimes id_V} \sum_i ce_i \otimes e^i \otimes v \xmapsto{id_V \otimes ev} \sum_i ce_i \otimes e^i(v)$$

is equivalent to

$$c \otimes v \xmapsto{l_v} cv \xmapsto{r_v^{-1}} v \otimes c$$

by using linearity of the functional and the scalar multiplication of the tensor product:

$$\sum_i ce_i \otimes e^i(v) = \sum_i ce_i \otimes \sum_k c^k e^i(e_k) = \sum_i ce_i \otimes \sum_k c^k \delta_k^i = \sum_i ce_i \otimes c^i = \sum_i c^i e_i \otimes c = v \otimes c.$$

Likewise to check the right snake relation, let $\psi = \sum_k c^k e^k \in V^*$ and $c \in \mathbb{C}$. Then for $\psi \otimes c \in V^* \otimes \mathbb{C}$ we get

$$\psi \otimes c \xmapsto{id_\psi \otimes coev} \psi \otimes \sum_i ce_i \otimes e^i \xmapsto{ev \otimes id_{e^i}} \psi\big(\sum_i ce_i\big) \otimes e^i$$

is equivalent to the mapping

$$\psi \otimes c \xmapsto{l_\psi^{-1} r_\psi} c \otimes \psi$$

by the following:

$$\psi\big(\sum_i ce_i\big) \otimes e^i = c \sum_i \psi(e_i) \otimes e^i = c \sum_i \sum_k c^k \delta_i^k \otimes e^i = c \sum_i c^i \otimes e^i = c \otimes \sum_i c^i e^i = c \otimes \psi.$$

Thus the two maps satisfy the snake relations. ☐

We note that the sum in the co-evaluation only makes sense when the basis is finite and it turns out that V can only admit a dual if it is finite dimensional. Assume V has basis $\{e_i\}, i \in I$, and admits a dual V^* with basis $\{e^j\}, j \in I$. Then $\{e_i \otimes e^j\},\ i, j \in I$, is a basis for $V \otimes V^*$. The co-evaluation is determined by it's value of 1, $coev(1) \in V \otimes V^*$, and the most general form of $coev(1)$ is a linear combination of basis elements. Hence we get:

$$coev : 1 \mapsto \sum_{i \in I, j \in J} c_{ij} e_i \otimes e^j.$$

This linear combination is necessarily finite, which means there exist finitely many $c_{ij} \neq 0$, which in turn means that there is finitely many indices $i \in I$ such that $c_{ij} \neq 0$. Now take an index, say $k \in I$, that satisfies $c_{kj} = 0$ for all $j \in J$. Then consider the composition

$$V \xrightarrow{l_V^{-1}} \Bbbk \otimes V \xrightarrow{coev_V \otimes id_V} V \otimes V^* \otimes V \xrightarrow{id_V \otimes ev_A} V \otimes \Bbbk \xrightarrow{r_V} V,$$

which is supposed to be the identity. If we send e_k through the above map we get:

$$e_k \mapsto 1 \otimes e_k \mapsto \sum_{i,j} c_{ij} e_i \otimes e^j \otimes e_k \mapsto \sum_{i,j} c_{ij} e_i \otimes ev(e^j \otimes e_k) \mapsto \sum_{i,j} c_{ij} ev(e^j \otimes e_k) e_i.$$

Then the coefficient of e_i when $i = k$ in the last sum above is $c_{kj} ev(e^j \otimes e_k) = 0$ and not 1, which it should be in order for the map to be the identity. Hence we achieve contradiction for an infinite basis.

5 Manifolds

We will now work our way up to the definition of the category of n-dimensional oriented cobordisms.

A manifold is a topological space, that locally resembles Euclidian space - in this case the real coordinate space of n dimensions, $\mathbb{R}^n$. A manifold of dimension one, written 1-manifold, is a topological space such that when you zoom in close enough, it looks like $\mathbb{R}^1$. So a circle, that you would usually see as an object embedded in $\mathbb{R}^2$, actually looks like a line when you look at a very small part of it, hence a circle is a 1-manifold. A sphere as we would normally imagine it, as the surface of a three dimensional ball, is a 2-manifold, since when you take a small enough part of it, it looks like a surface in $\mathbb{R}^2$. Another obvious example of an n-manifold is $\mathbb{R}^n$.

Definition 5.1. A *topological n-manifold* is a paracompact, second countable, Hausdorff space M, where every point has a neighbourhood homeomorphic to an open subset in $\mathbb{R}^n$. That is, for any point $p \in M$ there exists an open subset U of M containing p and a homeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^n$.

The above mentioned homeomorphic map is called a *chart* or a *coordinate map* and is denoted by the pair, (U, ϕ). A collection of charts $\mathcal{A}$, whose domain covers M is called an *atlas*, i.e. $\mathcal{A} = \{(U_a, \phi_a) \mid \cup_{a \in A} U_a = M\}$ for some index set A.

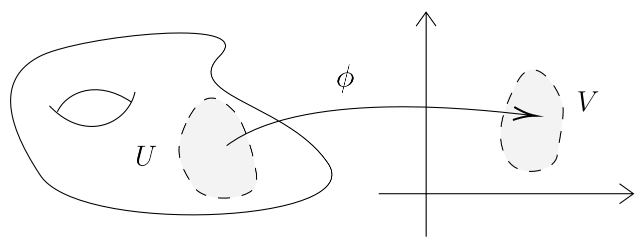


Figure 1: Chart from a 2-manifold to $\mathbb{R}^2$.

In order to have a sense of smoothness of manifolds, we need an additional structure, namely transition maps. Suppose $\phi : U \to V$ and $\psi : \tilde{U} \to \tilde{V}$ are two charts of a point x, such that $U \cap \tilde{U} \neq \emptyset$. We then have a *transition map* between subsets of $\mathbb{R}^n$:

$$\psi \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \psi(V \cap \tilde{V}),$$

See figure 2. Since ϕ and ψ are both homeomorphic, $\psi \circ \phi^{-1}$ is also homeomorphic. We now say, that an atlas $\mathcal{A}$ is *smooth* if for any two charts (U, ϕ) and $(\tilde{U}, \psi)$, either $U \cap \tilde{U} = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is smooth. Since the transition map is between subsets of $\mathbb{R}^n$, the smoothness we require should be understood as the smoothness of maps we are used to: $\psi \circ \phi^{-1}$ is smooth if every component function of $\psi \circ \phi^{-1}$ has continuous partial derivatives of all orders.
A smooth atlas is *maximal*, if it is not contained in any strictly larger smooth atlas. A *smooth structure* on a topological manifold is a smooth maximal atlas, and if a topological manifold is equipped with a smooth structure, we call the manifold *smooth*.

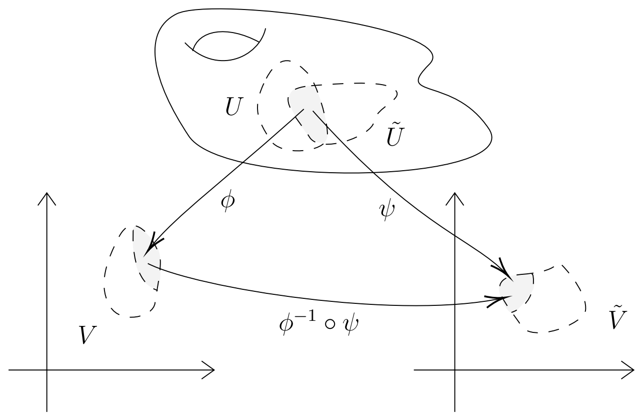


Figure 2: Transition map.

Manifolds will throughout this paper be smooth and compact, but not necessarily connected - from now on an n-manifold means a smooth n-manifold.

Charts define a notion of derivatives on an n-manifold M, and we will later use how differentiability of a curve on M is defined through charts:

Definition 5.2. Let M be an n-dimensional manifold. A curve on M

$$\gamma : I \subset \mathbb{R} \to M$$

is *differentiable*, if for any chart (U, ϕ) of x the composition $\phi \circ \gamma : I \to \phi(U)$ is a differentiable curve in $\mathbb{R}^n$.

Note that the derivative of a curve $\phi \circ \gamma$ lies in the affine subspace of $\mathbb{R}^n$ placed tangentally on $\phi(U)$ at $\phi(x)$.

We will now go in depth with a simple example to get more comfortable with the technical definition of a manifold.

Example 5.3 (The unit circle)**.** A typical example of a 1-manifold is a circle. Every point of the circle has a neighbourhood homeomorphic to an open interval on the real line, and the entire circle can be covered by an atlas of four charts:
Consider the unit circle, and divide it by the x-axis into an upper arc and a lower arc, both open subsets of the circle, leaving behind the points $(1, 0)$ and $(-1, 0)$ of the circle in doing so. The points on the upper arc is uniquely determined by its x-coordinate, so we can take $\phi_{upper} : (x, y) \mapsto x$ to be a chart mapping the top arc to the open interval $(-1, 1) \subset \mathbb{R}^1$. The bottom arc can in the same way be mapped to the same interval by $\phi_{lower} = \phi_{top}$. But we haven't covered the entire manifold yet, we are missing the two points. Splitting the unit circle down the y-axis into left and right arcs also being open sets of the circle, we include the points $(1, 0)$ and $(-1, 0)$. We can now take the chart $\phi_{right} = \phi_{left} : (x, y) \mapsto y$ so the left and right arc both get mapped to $(-1, 1) \subset \mathbb{R}^1$. This way all points of the circle are covered by the four charts, making the four of them an atlas. To include transition maps into this example, notice that there are four overlaps in the charts. One of them is the piece of the circle from $(1, 0)$ to $(0, 1)$ (going anti-clockwise), where ϕ_{top} and ϕ_{right} overlaps in the

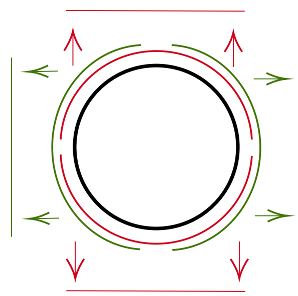


Figure 3: The unit circle and the four charts.

interval $(0,1) \in \mathbb{R}^1$. We can then construct a transition map for any element $a \in (0,1)$:

$$a \mapsto \phi_{upper}\phi_{right}^{-1}(a) = \phi_{upper}(\sqrt{1-a^2}, a) = \sqrt{1-a^2}.$$

The atlas we have constructed is not unique. You could make other partitions of the circle and charts of an atlas do not need to be projections. You can for example cover the entire circle with the union of two arcs: the arc going from $(0,0)$ to $(-1,0)$ and the arc from $(-1,0)$ to $(0,1)$. In this case you can consider the unit circle to be embedded in $\mathbb{C}$ and chart in this case can be $Arg : z = x + iy \mapsto Arg(z)$.

5.1 Manifolds with boundary

In the way we have defined manifolds, every point x has a neighbourhood homeomorphic to $\mathbb{R}^n$. But there also exist *manifolds with boundary*, for example the closed disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, where the point $(1, 1)$ does not have a neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$.
We then define an n-manifold with boundary to be a topological space, where the neighbourhood is homeomorphic to an open subset in the *half-space* H^n instead of $\mathbb{R}^n$. The half-space is defined as

$$H^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}.$$

An example is $H^2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, -\infty < y < \infty\}$. From the definition it is clear, that the H^n inherits the subspace topology from $\mathbb{R}^n$. The interior is defined as $\text{Int}H^n = \{x \in \mathbb{R}^n \mid x_1 > 0\} \subset \mathbb{R}^n$, and the boundary is defined as $\partial H^n = \{x \in \mathbb{R}^n \mid x_1 = 0\}$. The boundary ∂H^n is isomorphic to $\mathbb{R}^{n-1}$ - for example $\partial H^2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0\} = \{(0, y) \in \mathbb{R}^2\} \cong \mathbb{R}$.

We say that a point $x \in M$ is a *boundary point*, if it for some chart corresponds to a point on ∂H^n. The boundary points do not have any neighbourhood homeomorphic to an open set in $\mathbb{R}^n$, but the set of all the boundary points on M is covered by open sets isomorphic to $\mathbb{R}^{n-1} \cong \partial H^n$. In this way the boundary points become an $(n-1)$-manifold.

Example 5.4. Taking the unit disc as a manifold, the interior of the disc is the 2-manifold $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and the boundary of the disc is the unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ - a 1-manifold.

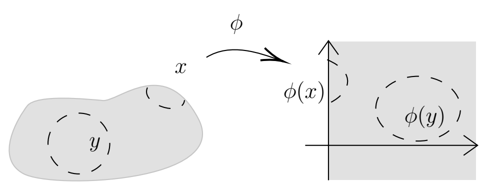


Figure 4: A boundary point x and interior point y in open subsets of a manifold being mapped to points $\phi(x) \in \partial H^2$ and $\phi(y) \in \text{Int}(H^2)$ in open subsets of the half-space H^2.

We note that the boundary of a manifold can also be empty, and in this way any manifold can be considered a manifold with boundary. We will however be explicit whether we consider a manifold to have boundary or not.

Example 5.5. Let Σ be an $(n-1)$-manifold without boundary and let I be an $(n-1)$-manifold with boundary. Then the product matrix $\Sigma \times I$ has boundary $\Sigma \times \partial I$. In general we call these product manifolds cylinders if $I = [0, 1]$. For example let $\Sigma = S^1$ and $I = [0, 1]$. Then the product manifold $S^1 \times I$ is a hollow cylinder with boundary $S^1 \times \{0\}$ and $S^1 \times \{1\}$.

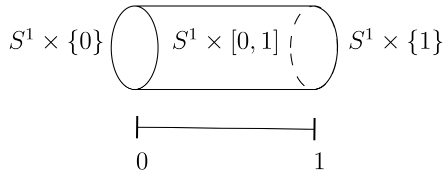


Figure 5: The product manifold $S^1 \times [0,1]$.

5.2 Orientation

In a coordinate system like $\mathbb{R}^2$ we are used to the notion of a positive and negative orientation. When moving anti-clockwise, we are going in the positive direction, and when moving clockwise, we are going in the negative direction. This is because $\left[\binom{1}{0},\binom{0}{1}\right]$ is the ordered basis that determines the standard orientation of $\mathbb{R}^2$: The positive orientation is defined to be the direction of rotation from $\binom{1}{0}$ to $\binom{0}{1}$ through an angle less than π. Likewise when going "to the right" on the real line, we go in a positive direction, since $\left[\binom{1}{0}\right]$ is the basis determining the standard orientation of $\mathbb{R}$. Also in $\mathbb{R}^3$ we are used to a positive and negative orientation, also known as the right-hand and left-hand orientation, but when we get up in higher dimensions, it gets harder to have the same notion of orientation. Instead we find a way to compare bases, determining whether two bases have the same orientation. We will from now on use square brackets to denote an ordered basis.

5.2.1 Orientation of vector spaces

Assume we have a vector space V of dimension $n \geq 1$, and let $B = [b_1, ..., b_n]$ and $\tilde{B} = [\tilde{b}_1, ..., \tilde{b}_n]$ be two ordered bases for V. Then a result in linear algebra says that there exist a unique linear transformation matrix $T_{B,\tilde{B}}$ from B to $\tilde{B}$. We say that the two bases represent the same orientation, if $\det(T_{B,\tilde{B}}) > 0$. This gives an equivalence relation $\sim$ on the set of all ordered bases of V.

Definition 5.6. An *orientation* of V is a choice of an equivalence class of $\sim$, and is referred to as the positive orientation.

Definition 5.7. Let B be an ordered basis of the chosen equivalence class. Then for any other ordered basis $\tilde{B}$, we say $\tilde{B}$ is a *positive basis* if $\det(T_{B,\tilde{B}}) > 0$, and $\tilde{B}$ is a *negative basis* if $\det(T_{B,\tilde{B}}) < 0$.

5.2.2 Orientation of manifolds

It would be nice to be able to use this theory of orientation of vector spaces on manifolds. For this we introduce tangent spaces of manifolds with the help from charts as a form of translation between the rather abstract n-manifold and the more intuitive $\mathbb{R}^n$, which we are used to working in.

We can for any point x on an n-manifold M talk about the tangent space of M at x, denoted T_xM. The most intuitive way to define these tangent spaces is through the use of curves. Let $\gamma : (-1,1) \subset \mathbb{R} \to M$ be a curve on M through x i.e. $\gamma(0) = x$. We define the derivative $\gamma'(0)$ in terms of charts as in Definition 5.2, which allows us to give the following definition of tangent spaces of M. $C^1(M)$ will denote the set of all differentiable real valued curves on M in the sense of Definition 5.2.

Definition 5.8. The tangent space of a point x of an n-manifold M is defined as

$$T_xM = \{\gamma : (-1,1) \to M,\ \gamma \in C^1(M),\ \gamma(0) = x\}/\sim,$$

where $\sim$ denotes the following equivalence relation: Two curves $\gamma_1, \gamma_2 : (-1,1) \to M$, $\gamma_1, \gamma_2 \in C^1(M)$, are equivalent if for all charts (U, ϕ) of x: $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$.

An equivalence class in the above relation is called a *tangent vector to M at x*, so the tangent space at x is the set of all tangent vectors to M at x. What we do is look at the derivatives of $\{\phi \circ \gamma\}$ equal at the point $\phi(x)$. We then track this collection of derivatives back to the corresponding curves $\{\gamma\}$ on M, which we then define as the corresponding *tangent vector on M*.

In this way we transfer vector space operations from $\mathbb{R}^n$ to the tangent space of M. For any $x \in M$ we now make the identification that $T_xM \cong T_{\psi(x)}\psi(U)$ for any chart (U, ψ) of x.

Remark. We quickly note two things about tangent spaces in $\mathbb{R}^n$. The tangent space $T_x\mathbb{R}^n$ for any $x \in \mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$: Attaching a copy of $\mathbb{R}^n$ tangentially to x is the same as shifting the origin of $\mathbb{R}^n$ from 0 to x. Also, if x lies in an open subset $V \subset \mathbb{R}^n$, then $T_xV = T_x\mathbb{R}^n \cong \mathbb{R}^n$. So $T_xV \cong \mathbb{R}^n$.

This remark allows us to think of T_xM as a copy of $\mathbb{R}^n$ attached tangentially to M at x. Now, if we want orientation of a manifold, we can choose orientations for all tangent spaces of the manifold (note that when we choose orientation for a tangent space T_xM, we actually choose an orientation for the tangent space $T_{\psi(x)}\psi(U)$ for any (U, ψ) of x). But we do need to make sure that these orientations are related to each other in some smooth way, to avoid the orientations of tangent spaces of points close to each other switching randomly. We do this by making sure that the differentials of the transition maps are orientation preserving.

Definition 5.9. Let $\psi \circ \phi^{-1}$ be a transition map. The differentials of $\psi \circ \phi^{-1}$ is represented in the matrix of derivatives, $D_x(\psi \circ \phi^{-1})$, which is a linear map between two tangent spaces:

$$D_x(\psi \circ \phi^{-1}) : T_{\psi(x)}\mathbb{R}^n \to T_{\phi(x)}\mathbb{R}^n = T_{(\psi \circ \phi^{-1})(x)}\mathbb{R}^n.$$

See Figure 6. $D_x(\psi \circ \phi^{-1})$ is orientation preserving, if $\det(D_x(\psi \circ \phi^{-1})) > 0$.

We are now ready for the final definition of what this section is about:

Definition 5.10. An *orientation* of a manifold is a choice of orientation of each of its tangent spaces, such that the differentials of the transition maps preserve orientation. A manifold is called *oriented* if there has been chosen an orientation of it.

Example 5.11. An orientation of a point, which is a 0-manifold, is given by assigning the sign $+$ or $-$ to it. Here the orientation is not as much of a visual orientation as in higher dimensions, but is more of a technicality. The tangent space at a point is the trivial vector space $\{0\}$, which has the empty set as a unique basis. We have to choose a sign for this basis.

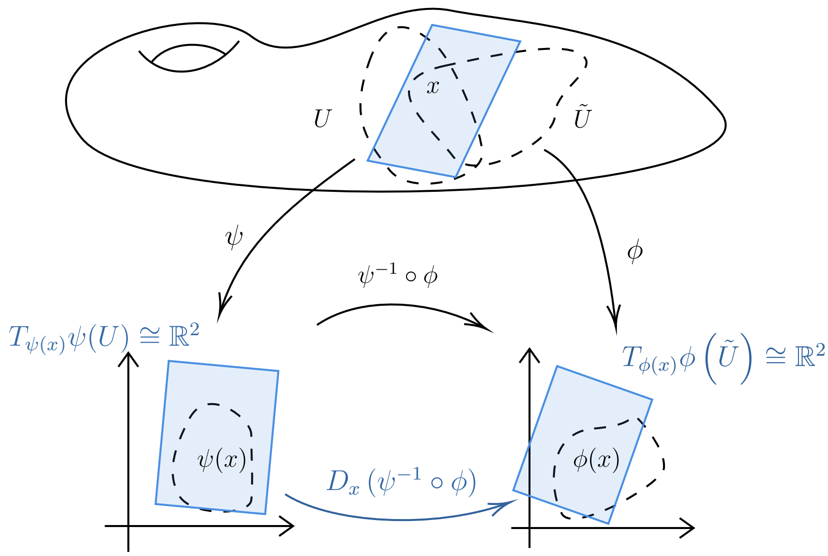


Figure 6: Differentials of a transition map. The tangent space of the manifold is drawn in the way we think of it: as a tangentially attached real vector space.

Example 5.12. Let X and Y be two oriented manifolds, X without boundary. Let (x, y) be a point on the product manifold $X \times Y$. If $[v_1, ..., v_n]$ is a positive basis for T_xX and $[u_1, ..., u_n]$ is a positive basis for T_yY, then we choose $[v_1, ..., v_n, u_1, .., u_n]$ as a positive basis for $T_{(x,y)}X \times Y$. In this way the product manifold obtains orientation.

Example 5.13. Let Σ be an oriented manifold without boundary. Let the unit interval $I = [0, 1]$ have standard orientation induced from $\mathbb{R}$ with positive basis $[e_1]$. If at a point $x \in \Sigma$ $T_x\Sigma$ has positive basis $[v_1, ..., v_{n-1}]$, then the product orientation of $\Sigma \times I$ has positive basis $[v_1, ..., v_{n-1}, e_1]$.

5.3 In- and out-boundaries

Definition 5.14. Let Σ be an $(n-1)$-manifold embedded in an n-manifold M. Assume they are both oriented. For an $x \in \Sigma$ let $[v_1, ..., v_{n-1}]$ be a positive basis for $T_x\Sigma$. A *positive normal* is a vector $w \in T_xM$ such that $[v_1, ..., v_{n-1}, w]$ is a positive basis for T_xM.

If Σ is the boundary of M, M could be a product manifold $M = \Sigma \times I$, where I is a manifold with boundary. Then w in the above definition corresponds to a vector on the real tangent space of H^n at $\phi(x)$ for some chart (U, ϕ). Then this vector will either point in towards H^n or out from H^n, which can be interpreted as w pointing in to M or out from M.

Definition 5.15. If a positive normal points in towards M, we call Σ an *in-boundary*, and if it points out from M, we call Σ an *out-boundary*.

If some positive normal points inward, then any other positive normal at $z \in \Sigma$ points inward as well. This is true for positive normals that point outward as well.

Example 5.16. Consider the unit interval $I = [0,1]$ with orientation induced from the standard orientation of $\mathbb{R}$. Assign to $\{0\}$ and $\{1\}$ the orientation $+$. Then $\{0\}$ is an in-boundary of I and $\{1\}$ is an out-boundary of I. Consider now the product manifold $\Sigma \times I = [0,1]$, where Σ is a compact, oriented 1-manifold without boundary, and let $\Sigma \times I = [0,1]$ be equipped with the product orientation from Example 5.12. Then $\Sigma \times \{0\}$ is an in-boundary and $\Sigma \times \{1\}$ is an out-boundary.

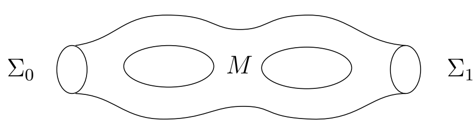
6 Cobordisms

By definition a *closed* manifold is a compact manifold without boundary. This definition can seem quite counterintuitive, since the unit interval $I = [0,1]$ consequently is not a closed 1-manifold. But a circle which has no boundary is indeed a closed 1-manifold.

Definition 6.1. A *cobordism* between two closed $(n-1)$-manifolds Σ_0 and Σ_1 is an n-manifold M with boundary diffeomorphic to the disjoint union $\Sigma_0 \sqcup \Sigma_1$.

More loosely, a cobordism connects two manifolds, which in turn becomes the boundary of the cobordism.

Example 6.2. A very simple example of a cobordism is the closed interval, for example $[0,1]$, where $\partial[0,1] = \{0\} \sqcup \{1\}$. We have also already looked at cylinders, specifically $S^1 \times [0,1]$, which is a cobordism between $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Below are two other examples of 2-dimensional cobordisms:



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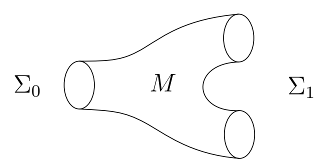

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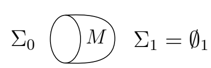
Notice that the right cobordism has boundary $\Sigma_0 \sqcup \Sigma_1$, where Σ_1 is disconnected: it consists of two disjoint manifolds. A cobordism need not be connected either. A cobordism between a circle and the empty 1-manifold could be



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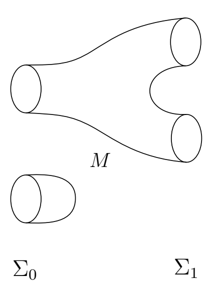
Then below is a disconnected cobordism from two circles to two circles:



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We think of cobordisms as a form of evolution in time, for example the right cobordism in the first example above is a circle splitting in to two circles. The cobordism between a circle and the empty 1-manifold is called the death of a circle since the circle can be seen as collapsing into a single point. This way of thinking of cobordisms gives rise to a sense of direction or an arrow describing time. Orientation of cobordisms is therefore natural to define.

6.1 Oriented cobordisms

If we take a look at Definition 6.1 and now let M, Σ_0 and Σ_1 be oriented manifolds and Σ_0 and Σ_1 be the in- and out-boundary of M respectively, then M becomes an oriented cobordism. This is a nice intuitive definition, but it is not enough in our case. We want a cobordism from a given manifold Σ to itself, and this is not possible in the sense of this intuitive definition, since a manifold cannot be both the in- and out-boundary of a manifold M.

Definition 6.3. Let Σ_0 and Σ_1 be two closed, oriented $(n-1)$-manifolds. Then an *oriented cobordism* from Σ_0 to Σ_1 is an oriented n-manifold M together with two smooth, diffeomorphic orientation preserving embeddings:

$$\Sigma_0 \to M,$$
$$\Sigma_1 \to M,$$

such that Σ_0 is mapped onto the in-boundary of M and Σ_1 is mapped onto the out-boundary of M.

We will draw oriented coborisms with the in-boundary on the left and the out-boundary on the right, and denote a cobordism M from Σ_0 to Σ_1 as $M : \Sigma_0 \Rightarrow \Sigma_1$.

We can now, provided with two diffeomorphisms, embed two copies of a given manifold Σ in an oriented manifold M, ending up with a cobordism from Σ to Σ.

Provided we find the right diffeomorphisms, we can view a single manifold as a cobordism between various distinct objects.

Example 6.4. Let $I = [0, 1]$ have orientation induced from the standard orientation of $\mathbb{R}$. We've established through Example 5.16, that with orientation $+$ assigned to boundaries $\{0\}$ and $\{1\}$, $\{0\}$ is an in-boundary and $\{1\}$ is an out-boundary. Thus I defines an oriented cobordism from 0 to 1. We can generalize this example. Take two arbitrary one-point manifolds p_0 and p_1 with positive orientation $+$. Then two orientation preserving diffeomorphisms taking p_0 and p_1 to $\{0\}$ and $\{1\}$ respectively:

$$p_0 \to I \leftarrow p_1,$$

allow us to use I as a cobordism between p_0 and p_1.
We can also replace the unit interval with any oriented injective, continuous path M in a topological space, by taking an orientation preserving diffeomorphism from I to M. By composing this diffeomorphism with the one before, we get a new cobordism, see figure 7.

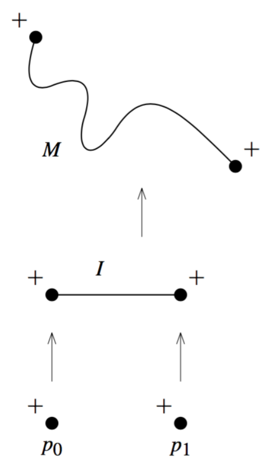


Figure 7: Construction of new cobordism from the unit interval[Koc03].

Example 6.5. For any two diffeomorphic manifolds Σ_0 and Σ_1 there exists a cobordism from Σ_0 to Σ_1:
Take Σ_0 and cross it with the unit interval I with standard orientation and standard orientation of the boundary points. As we have established in Example 5.16, the boundary of $\Sigma_0 \times I$ then consists of $\Sigma_0 \times \{0\}$ as an in-boundary and $\Sigma_0 \times \{1\}$ as an out-boundary. A cobordism from Σ_0 to Σ_0 is then constructed by taking the maps

$$\Sigma_0 \xrightarrow{\sim} \Sigma_0 \times \{0\} \subset \Sigma_0 \times I,$$
$$\Sigma_0 \xrightarrow{\sim} \Sigma_0 \times \{1\} \subset \Sigma_0 \times I,$$

and taking the diffeomorphism between Σ_0 and Σ_1 we embed Σ_1 in the out-boundary of $\Sigma_0 \times I$:

$$\Sigma_1 \xrightarrow{\sim} \Sigma_0 \xrightarrow{\sim} \Sigma_0 \times \{1\} \subset \Sigma_0 \times I.$$

Thus via the cylinder construction, we now have a cobordism $\Sigma_0 \times I : \Sigma_0 \Rightarrow \Sigma_1$. Any orientation-preserving diffeomorphism $\Sigma_0 \times I \xrightarrow{\sim} M$ with M being another n-dimensional manifold with boundary, will also define a cobordism $M : \Sigma_0 \Rightarrow \Sigma_1$.

7 Category of cobordisms

We have thought about oriented cobordisms as manifolds going from one manifold to another, so a natural way of defining the category of n-cobordisms would be with oriented $(n-1)$-manifolds as objects and oriented n-cobordisms as morphisms between the objects. We then need an associative composition and an identity morphism.

An obvious choice of composition would be "gluing". Take for example closed, oriented 1-manifolds Σ_0, Σ_1 and Σ_2 and 2-cobordisms $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$. Then the composition of M_0 and M_1 would be gluing the two cobordisms together along Σ_1, such that Σ_1 is a manifold inside of the composed corbodism M_0M_1. The composed corbodism M_0M_1 is denoted the disjoint union along Σ_1 $M_0 \sqcup_{\Sigma_1} M_1 : \Sigma_0 \Rightarrow \Sigma_2$. This potential composition is obviously associative, as you can imagine gluing to be associative. The problem is whether the gluing can be equipped with a smooth structure.

It can be shown that $M_0 \sqcup_{\Sigma_1} M_1$ in fact can be equipped with a smooth structure unique up to diffeomorphism (for in depth explanation see [Koc03], chapter 1.3.). We introduce corbordism classes:

Definition 7.1. Let M and M' be two cobordisms from Σ_0 to Σ_1:

$$\begin{array}{ccccc} & \nearrow & M & \nwarrow & \\ \Sigma_0 & & & & \Sigma_1 \\ & \searrow & M' & \swarrow & \end{array} \quad .$$

M and M' are equivalent if there exists an orientation preserving diffeomorphism $\psi : M \xrightarrow{\sim} M'$, such that the following diagram commutes:

$$\begin{array}{ccccc} & \nearrow & M & \nwarrow & \\ \Sigma_0 & & \simeq \big\downarrow \psi & & \Sigma_1 \\ & \searrow & M' & \swarrow & \end{array} \quad .$$

An equivalence class of this relation is called a *cobordism class*.

Note that if M and M' are equivalent, ψ induces the identity on the boundaries Σ_0 and Σ_1. The idea is now to compose cobordism classes instead of cobordisms. This notion of composition turns out to be well-defined and also associative since gluing cobordisms is associative.

We will from now on only work with the category of 2-cobordisms, but all is true for the category of n-cobordisms.

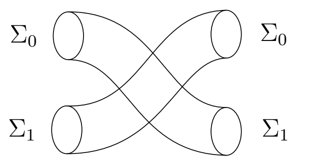
Definition 7.2. $\mathbf{Bord}_{12}^{\mathrm{OR}}$ denotes the category consisting of closed oriented 1-manifolds as objects and oriented 2-cobordism classes in the sense of Definition 7.1 as morphisms.

We will denote the objects of $\mathbf{Bord}_{12}^{\mathrm{OR}}$ as $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$, where $\mathbf{0}$ denotes the empty 1-manifold, $\mathbf{1}$ denotes a circle Σ, and $\boldsymbol{n}$ denotes the disjoint union of n copies of Σ.

7.1 The monoidal structure of $\mathbf{Bord}_{12}^{\mathbf{OR}}$

We can equip $\mathbf{Bord}_{12}^{\mathrm{OR}}$ with a monoidal structure: The disjoint union acts as the functor in the following way. Given two objects, i.e. closed oriented 1-manifolds, their tensor product is defined to be the disjoint union, which is again a closed oriented 1-manifold. Given two cobordism classes their tensor product is given by the class of the oriented cobordism obtained from taking the disjoint union of a representing cobordism from each class. The unit object is given by the empty 1-manifold $\emptyset$.

Given two oriented closed 1-manifolds Σ_0 and Σ_1 there exist a *twist*-cobordism, $\Sigma_0 \sqcup \Sigma_1 \Rightarrow \Sigma_1 \sqcup \Sigma_0$:



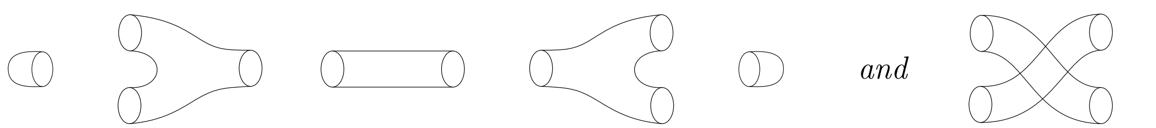
This gives the monoidal category $(\mathbf{Bord}_{12}^{\mathrm{OR}}, \sqcup, \emptyset)$ a symmetric structure.

7.2 Generators of $\mathbf{Bord}_{12}^{\mathbf{OR}}$

In group theory we are used to the notion of generators and relations between the generators. In category theory there is a corresponding notion.

Generators of a symmetric monoidal category is a set of morphisms, such that any other morphism can be obtained by composing arrows from this set. Relations are equalities between two ways of writing a given morphism obtained from generators.

Proposition 7.3. The symmetric monoidal category $\mathbf{Bord}_{12}^{\mathrm{OR}}$ is generated under composition and disjoint union by the following six cobordisms:

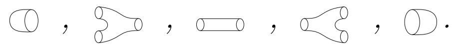


In the visual sense, by composition we mean gluing cobordisms in series, and by disjoint union we mean stacking cobordisms on top of each other.

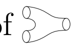
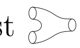
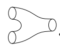
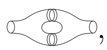
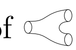
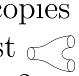
We will not provide direct proof of the proposition above, but we will go over important results underlying the proof. The proof relies directly on the classification of surfaces, for which we will quickly remind ourselves of the following: The *genus* of a compact, connected, oriented surface is intuitively the number of holes. For a surface with boundary (the reader can for this imagine the cylinder) the genus is defined as the genus of the closed surface obtained by sewing discs to the boundaries. For example sewing discs to the in- and out-boundary of the cylinder gives a closed sausage, which has no holes. Thus a cylinder has genus 0.

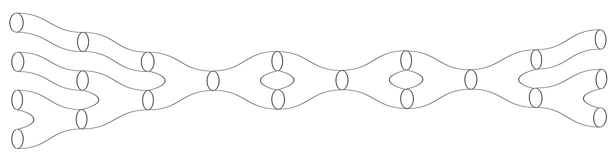
Theorem 7.4 (Topological classification of surfaces, (Kock [Koc03], 1.4.15))**.** *Two connected, compact, oriented surfaces with oriented boundaries are diffeomorphic relative to the boundary if and only if they have same genus, same number of in-boundaries and same number of out-boundaries.*

This theorem gives us that the number of in-boundaries, out-boundaries and genus determines the topological type of 2-cobordisms.

Lemma 7.5. *Every connected 2-cobordism can be obtained by composition and disjoint union of the generators* .

This lemma is a consequence of the *normal form* of a connected surface, which is both a way of decomposing a connected surface into a number of basic cobordisms and a way to construct any connected cobordism from the generators listed in Lemma 7.5.

Normal form. Assume we want to construct a connected cobordism with m in-boundaries, n out-boundaries and genus g. The normal form consists of three parts, the first part which is a cobordism with m in-boundaries and 1 out-boundary, the middle part which is a cobordism with 1 in-boundary and 1 out-boundary, and the last part which is a cobordisms with 1 in-boundary and n out-boundaries. The first part is a composition of $m-1$ copies of  and an appropriate number of cylinders, such that the output of the first  should be connected with the lower input of the following . The middle part is then a serial connection of g times , which has genus 1 each. The last part is similar to the first part, it is a composition of $n-1$ copies of  and an appropriate number of cylinders, such that the lower output of the first  is connected to the input of the following. Here is an example with $m = 4$, $n = 3$, $g = 2$:

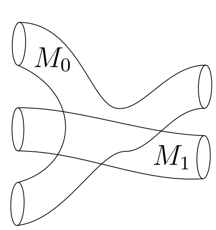
 .

We need that also 2-cobordisms which are *not* connected can be obtained from the generators in Proposition 7.3. There is a distinction between a cobordism (seen as a manifold) being a disjoint union of manifolds and then a cobordism being a disjoint union of cobordisms. For example the twist cobordism from the symmetric structure in $\mathbf{Bord}_{12}^{OR}$: as a manifold it is the disjoint union of two cylinders, but it is not the disjoint union of two identity cobordisms.

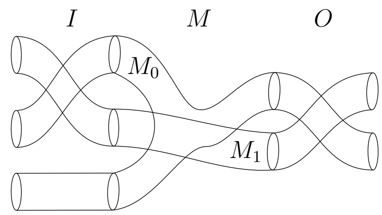
For the next result, we introduce *permutation cobordisms* which are disjoint unions of compositions of twist cobordisms and cylinders.

Lemma 7.6. *Every* 2*-cobordism factors as a permutation cobordism, followed by a disjoint union of connected corbordisms, followed by a permutation of cobordisms.*

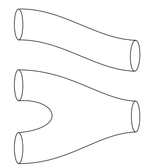
We will quickly go through this lemma with an example. Take a cobordism M which has two connected components M_0 and M_1.

M 

We can permute the boundaries by composing with two permutation cobordisms I and O to the in- and out-boundary respectively:

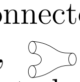
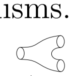
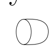


We then end up with a cobordism which is a disjoint union of its connected components:

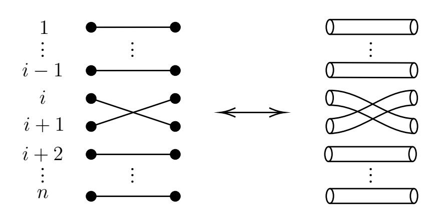
IMO  .

The permutations of the in-boundary $\boldsymbol{m}$ is a diffeomorphisms $\boldsymbol{m} \xrightarrow{\sim} \boldsymbol{m}$, which orders the components of the in-boundary. The diffeomorphisms induces the cobordism $I : \boldsymbol{m} \Rightarrow \boldsymbol{m}$. Likewise the permutation of the out-boundary n is a diffeomorphisms $\boldsymbol{n} \xrightarrow{\sim} \boldsymbol{n}$, which orders the components of the out-boundary and induces the cobordism $O : \boldsymbol{n} \Rightarrow \boldsymbol{n}$, such that the cobordism $IMO : \boldsymbol{m} \Rightarrow \boldsymbol{n}$ is a disjoint union of cobordisms.

7.3 Sketch of proof of Proposition 7.3

We want to show that every 2-cobordism can be obtained by composing and taking the disjoint union of the generators in 7.3. By Lemma 7.6 any 2-cobordism factors into permutation cobordisms and a disjoint union of connected cobordisms. By Lemma 7.5 the latter connected cobordisms are generated by  ,  , ,  , .

We still need the permutation cobordisms to be written in terms of the listed generators. For this we recall the symmetric group S_n on a finite set X of n symbols $\{x_1, ..., x_n\}$, where the elements are bijections from X to itself, i.e. the elements are permutations of the n symbols. The symmetric group is generated by transpositions $\tau_i : (x_i, x_{i+1}),\ i = 1, ..., n$, that interchange two adjacent letters. Denote the set of permutation cobordisms with n in- and out-boundaries by PermCob(n). A permutation cobordism is a composition of the twist cobordism and the cylinder determined by its permutation of the circles. Hence there exists an isomorphism between PermCob(n) and S_n. Under this isomorphism the transposition corresponds to the twist cobordism:



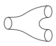
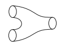
.

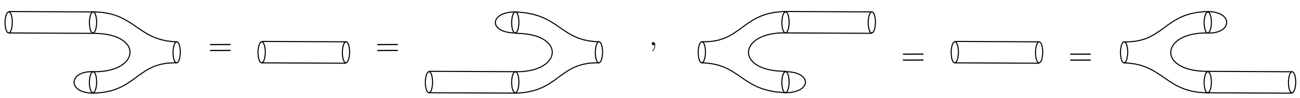
Hence the permutation cobordisms can be obtained by composition and disjoint union of the twist cobordism and cylinders. The classification theorem gives that the factorization of a 2-cobordism is diffeomorphic to the disjoint union of cobordisms from the factorization, since it has the same genus and number of in- and out-boundaries.

7.4 Relations of $\mathbf{Bord}_{12}^{\mathbf{OR}}$

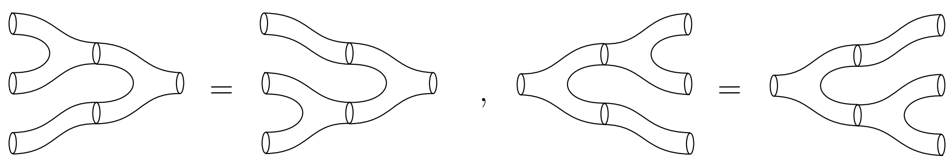
We first list a set of relations between the generators, and afterwards use the classification theorem to prove that they hold. Lastly we will sketch a proof of sufficiency of the relations.

The most obvious relations are the identity relations, using that the cylinder is an identity, which we will not picture.

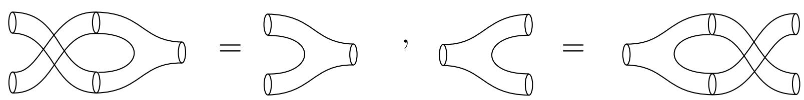
We have the following relations from sewing a disc in one of the holes of the pair-of-pants  and  and composing with a cylinder:



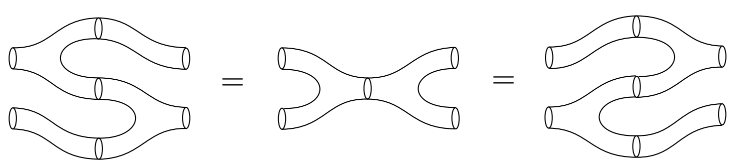
We have the relations, 'associativity' and 'co-associativity':



We have the relations, 'commutativity' and 'co-commuativity':

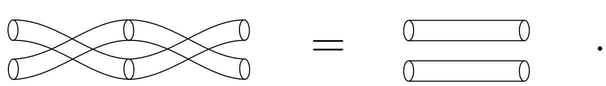


And lastly the 'Frobenius relation':



To prove that the above relations hold, note that they are all of same topological type. Then the Classification Theorem 7.4 gives that such two cobordisms are diffeomorphic, belonging to the same cobordism class.

We have before noted that the twist cobordism turns $\mathbf{Bord}_{12}^{\mathrm{OR}}$ into a symmetric monoidal category together with $\sqcup$ as composition and the empty manifold as unit object. This fact gives rise to a set of relations, the basic relation being the fact that the twist cobordism is its own inverse:

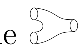
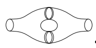
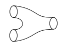
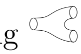


A bunch of other relations involving the twist cobordism express the hexagon identity (See Definition 2.5). The hexagon identity states that for any two cobordisms, it does not make any difference whether we apply the twist cobordism before their disjoint union or after. Since we will not need these relations, we will just note that they exist and refer to [Koc03] 1.4.35, where the relations are described.

7.5 Sufficiency of the relations

We have come up with a bunch of relations above, and the question is now whether there are other relations that we have not found yet. We would like to end up with a set of relations that are sufficient, in the sense that given any other relation it can be built from our set of relations. Having a set of relations sufficient to describe all relations in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ allows us to to handle any kind of relation that we encounter, having full control of what generators and relations that span $\mathbf{Bord}_{12}^{\mathrm{OR}}$.

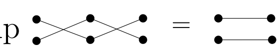
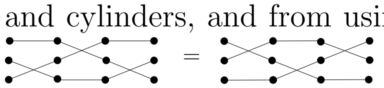
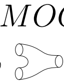
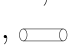
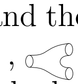
Having sufficiency of the relations corresponds to being able to transform any given decomposition of a cobordism to normal form. We can encounter two kinds of cobordisms, connected and nonconnected surfaces.

The case of connected cobordisms is fairly straightforward. Assuming we have an arbitrary decomposition of a connected surface, we can use the listed relations to move the  pieces way to the left. Doing this the pair-of-pants can encounter  and vanish due to the relations of sewing in discs onto the pair-of-pants. Some of the pair-of-pants will get stuck together and form handles , but  can still pass through this handle to the left with the use of associativity and the Frobenius relations. We can do the same thing moving  to the right.

It turns out, that if twist maps are a part of the decomposition, they can be eliminated using the listed relations and some additional relations stemming from relations of the symmetric group.

Moving pieces and eliminating twist maps, we end up with a surface on normal form.

The case of nonconnected surfaces are a bit trickier. We need to define a normal form for nonconnected surfaces, since the one mentioned above is for connected surfaces. It could be something factorised in three parts: a permutation cobordism, a disjoint union of connected surfaces in normal form, and a permutation cobordism again. We already know from Lemma 7.6 that there exists permutation cobordisms I and O such that IMO is a disjoint union of connected cobordisms. From the discussion of the normal form of a connected surface, we then know that each of the connected components of IMO can be brought on normal form.

The four permutation cobordisms I, I^{-1} [1], O and O^{-1} can be built of the twist cobordisms and cylinders, and from using the relations of the symmetric group  and  (see Kock [Koc03], 1.4.2 and 1.4.40), it can be shown that $I^{-1}I =$ id and $OO^{-1} =$ id. We can then write $M = I^{-1}IMOO^{-1}$, and the components of the middle part IMO are on normal form generated by  ,  ,  ,  and . Then M has been brought in 'normal form', where every of the three part can be built from the six generators. For more details consult [[Koc03], p.73-77].

8 Classification of 2-dimensional TQFTs

A *strict monoidal functor* between two (strict) monoidal categories is one that preserves all monoidal structure i.e. the associator and the left and right unitor. A *symmetric* monoidal

[1] An *inverse* M^{-1} to an oriented cobordism M is an oriented cobordism obtained by changing the orientation of both M and the in- and out-boundaries.

functor is one that takes the braiding (Definition 2.5) of one monoidal category to the braiding of the other. The symmetric monoidal functors we use are always considered strict.

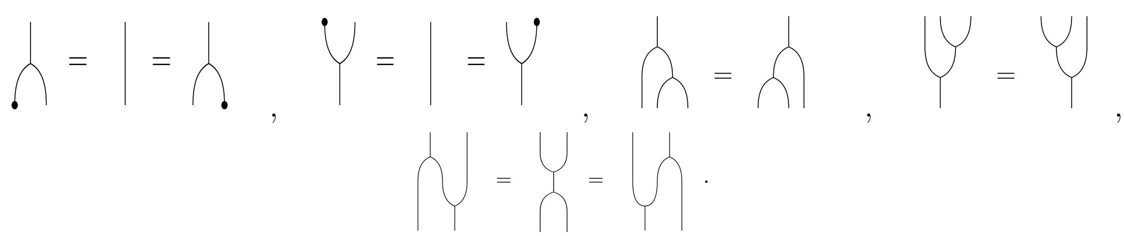
Definition 8.1. An *n-dimensional topological quantum field theory* is a symmetric monoidal functor $Z : (\mathbf{Bord}^{\mathrm{OR}}_{(n-1)n}, \bigsqcup, \emptyset, T) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, where T denotes the twist cobordism from Section 7.1 and σ denotes the braiding from Section 2.1.

What we get from the definition is that for any two cobordisms $M : \Sigma \Rightarrow \Sigma'$ and $M' : \Sigma' \Rightarrow \Sigma''$ in $\mathbf{Bord}^{\mathrm{OR}}_{12}$, we get

$$Z(\Sigma \sqcup \Sigma') = Z(\Sigma) \otimes Z(\Sigma') \quad , \quad Z(M \sqcup M') = Z(\Sigma) \otimes Z(\Sigma')$$

$$Z(M \sqcup_{\Sigma} M') = Z(M') \circ Z(M) \;\text{ and }\; Z(\emptyset) = \Bbbk.$$

We will soon show our main result which establishes a bijectivity between 2-dimensional TQFTs and commutative Frobenius algebras. Recall first that if (F, μ, u, η, e) is a commutative Frobenius algebra, it must satisfy the following relations: unitality, co-unitality, associativity, co-associativity axioms and the Frobenius axiom - represented below accordingly:



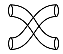
All the maps of F are maps between tensor powers of F i.e. $F^n := \underbrace{F \otimes \cdots \otimes F}_{n \text{ times}}$ is n copies of F. It is a convention that $F^0 = \Bbbk$. Taking for example the unit map and rotating it 90 degrees, we can associate it with the graphical representation  instead. We can do this with all the maps of F such that are represented in the same graphical way as cobordisms. One can then easily convince themselves that the relations of $\mathbf{Bord}^{\mathrm{OR}}_{12}$ described in Section 7.4 correspond precisely to the axioms for F being a commutative Frobenius algebra.

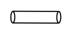
We have now come to our main theorem.

Theorem 8.2. *There is a one-to-one correspondence between 2-dimensional TQFTs and commutative Frobenius algebras.*

Proof. Given a 2-dimensional TQFT Z, one can define A to be the finite vector space that is the image of $\mathbf{1}$, i.e. $A := Z(\mathbf{1})$. The idea is now to show that A is a Frobenius algebra. The monoidality of Z implies that the image of $\boldsymbol{n}$ is A^n, and the symmetry of Z implies that the image of the twist cobordism is the braiding σ of the tensor product. Hence the following definition of the images of the objects of $\mathbf{Bord}^{\mathrm{OR}}_{12}$, twist cobordism and identity cobordisms follows automatically from Z:

$$\mathbf{Bord}_{12}^{\mathrm{OR}} \to \mathbf{Vect}_{\Bbbk}$$
$$\mathbf{1} \mapsto A$$
$$\boldsymbol{n} \mapsto A^n$$

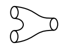
 $\mapsto [id_A : A \to A]$

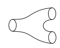
 $\mapsto [\sigma : A^2 \to A^2]$.

Then define the images of the generators as following linear maps:

$$\mathbf{Bord}_{12}^{\mathrm{OR}} \to \mathbf{Vect}_{\Bbbk}$$

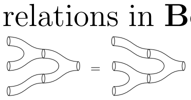
 $\mapsto [u : \Bbbk \to A]$

 $\mapsto [\mu : A^2 \to A]$

 $\mapsto [\mu : A \to A^2]$

 $\mapsto [e : A \to \Bbbk]$.

Since Z preserves relations, the relations that hold among the cobordisms in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ now hold among the linear maps defined above. As we noted previously, these relations translate precisely into the axioms for a commutative Frobenius algebra. Thus A is a commutative Frobenius algebra.

Conversely let A be a commutative Frobenius algebra, (A, μ, u, η, e). We can then construct a 2-dimensional TQFT Z, by defining the images of the generators under Z in accordance with the above mappings of $\mathbf{Bord}_{12}^{\mathrm{OR}} \to \mathbf{Vect}_{\Bbbk}$. We here need to check whether the relations in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ are respected by Z. We must for example check that for the cobordism  it makes no difference whether we set image to $\mu(\mu \otimes id_A)$ or $(id_A \otimes \mu)\mu$. But since the relations in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ corresponds to the axioms for a Frobenius algebra this is automatically achieved. Then any relation in $\mathbf{Bord}_{12}^{\mathrm{OR}}$ is respected by Z, since this set of relations is sufficient and we can relate every possible decomposition, turning Z into a symmetrical monoidal functor. Hence Z is well-defined.

It is clear that the two constructions are each others inverse: Given a 2-dimensional TQFT Z, we can construct a commutative Frobenius algebra $A : Z(\mathbf{1})$. If we then construct a 2-dimensional TQFT such that $\mathbf{1} \mapsto A$, this TQFT is equal to Z. This concludes the proof of our main theorem. □

References

[Ati86] Michael Atiyah. *Topological quantum field theories*. Inst. Hautes Etudes Sci. Publ. Math. 68, 1989, 175–186.

[Koc03] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Cambridge University Press, 2003.

[Lan98] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, 2 edition, 1998.

[Lee02] John M. Lee. *Introduction to smooth manifolds*. Graduate texts in mathematics 218. Springer, 2002.

[Wil11] Arik Wilbert. *Algebraic aspects of topological quantum field theories*. University of Bonn, 2011.