DEPARTMENT OF MATHEMATICAL SCIENCES
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Lie Algebra Cohomology

Master's Project in Mathematics

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January 29th, 2016

Abstract

This project deals with Lie algebra cohomology. We define it in terms of the Chevalley-Eilenberg complex, but also consider the equivalent definition of the cohomology groups as certain Ext-groups. We examine the Lie derivative as a historical motivation for the definition of the Chevalley-Eilenberg complex, and look at the relationship between De Rham cohomology and Lie algebra cohomology; to this end we prove that we can calculate the De Rham cohomology of a smooth manifold by considering the complex of differential forms which are invariant under a smooth action of a compact connected Lie group on the manifold.

Resumé

Dette projekt omhandler Lie-algebra-kohomologi. Dette defineres ved hjælp af Chevalley-Eilenberg-komplekset, men vi ser også på den ækvivalente definition af kohomologigrupperne som Ext-grupper. Vi ser på Lie-derivatet som historisk motivation for definitionen af Chevalley-Eilenberg-komplekset og på forholdet mellem De Rham-kohomologi og Lie-algebra-kohomologi; undervejs viser vi, at De Rham-kohomologien af en glat mangfoldighed kan bestemmes ved at betragte komplekset af differentialformer, der er invariante under virkningen af en kompakt sammenhængende Lie-gruppe på mangfoldigheden.

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Preface

Introduction

Lie algebra cohomology was invented by Claude Chevalley and Samuel Eilenberg in the middle of the 20'th century in an attempt to compute the De Rham cohomology of compact connected Lie groups; it is dealt with in detail in their paper [3] and heavily influenced by the work of Élie Cartan. In this project, we define Lie algebra cohomology, consider the historical motivation for the theory, and look at some examples. In Chapter 1, we go over the definitions of Lie algebras, Lie algebra modules, and universal enveloping algebras. We go on to define Lie algebra cohomology in terms of a chain complex and finally show that one can equivalently define it in terms of derived functors. In Chapter 2, we define the Lie derivative with the aim of proving the invariant formula for the exterior derivative of differential forms; we see that we can view differential forms as alternating multilinear maps taking vector fields as variables, which motivates the attempt to compute the De Rham cohomology of a Lie group in terms of its Lie algebra (the left-invariant vector fields on the Lie group). In Chapter 3, we explore this relationship further: We show that if a compact connected Lie group acts on a manifold, then to compute the De Rham cohomology of the manifold it suffices to consider the differential forms which are invariant under the Lie group action. As a corollary, we will see that the De Rham cohomology of a compact connected Lie group is isomorphic to the cohomology of its Lie algebra. Finally, in Chapter 4, we consider some simple examples.

Prerequisites and Notation

This project assumes basic knowledge of the following differential geometry theory: Vector fields, differential forms, De Rham cohomology, Lie groups, and Lie algebras. We refer to [11, 8, 12] for definitions and the basic theory. To make sure we agree on definitions and notation and have the relevant basics fresh in the memory, we will here briefly recap the main things needed to understand this text:

$S_{k,m} \subseteq S_{k+m}$ will denote the set of permutations $\sigma \in S_{k+m}$ satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+m)$.

We will assume that all manifolds are smooth. Let M, N be manifolds. For a smooth map of manifolds, $f\colon M \to N$, we denote the derivative by $Df\colon TM \to TN$. We may alter between the notations $D_pf(v)$ and $D_{p,v}f$ for the derivative of f at $p \in M$ in the direction $v \in T_pM$. For $f \in C^\infty(M)$, we will sometimes write df instead of Df, where $df \in \Omega^1(M)$, is the exterior derivative of f, and $df_p = D_pf$.

In general, for a ring R and an R-module A, we denote the dual space as $A^* = Hom_R(A, R)$.

Given a vector space V over a field K, the exterior algebra of V, $\Lambda(V)$, is the quotient of the tensor algebra, $T(V)$, (see Definition 1.1.9 Chapter 1) by the ideal $\mathcal{I} = \langle v \otimes v \mid v \in V\rangle$. The exterior product is defined as $v \wedge w = [v \otimes w]_{\mathcal{I}}$. The k'th exterior power of V, denoted by $\Lambda^k(V)$, is the subspace of $\Lambda(V)$ spanned by elements of the form $v_1 \wedge \cdots \wedge v_k$, $v_i \in V$. The exterior product is then a bilinear map $\wedge\colon \Lambda^k(V) \times \Lambda^m(V) \to \Lambda^{k+m}(V)$. Given a map $f\colon V \to W$ of vector spaces, we can define a map of K-algebras $\Lambda(f)\colon \Lambda(V) \to \Lambda(W)$ by $\Lambda(f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$.

A k-linear map $f\colon V^k \to K$ is said to be alternating, if $f(v_1, \dots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. If we define the alternating algebra of V, $\mathrm{Alt}^k(V)$, as the set of alternating k-linear maps $V^k \to K$, and the exterior product, $\wedge\colon \mathrm{Alt}^k(V) \times \mathrm{Alt}^m(V) \to \mathrm{Alt}^{k+m}(V)$, by

$$\omega \wedge \eta(v_1, \dots, v_{k+m}) = \sum_{\sigma \in S_{k,m}} \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+m)}),$$

then we have a natural isomorphism of vector spaces

$$\varphi\colon \Lambda^k(V)^* \to \mathrm{Alt}^k(V), \quad \varphi(F)(v_1, \dots, v_k) = F(v_1 \wedge \cdots \wedge v_k).$$

If, in addition, V is finite dimensional, then $\Lambda^k(V)^*$ and $\Lambda^k(V^*)$ are naturally isomorphic via the map $\psi\colon \Lambda^k(V^*) \to \Lambda^k(V)^*$ given by

$$\psi(f_1 \wedge \cdots \wedge f_k)(v_1 \wedge \cdots \wedge v_k) = \sum_{\sigma \in S_k} \operatorname{sign} \sigma \, f_1(v_{\sigma(1)}) \cdots f_k(v_{\sigma(k)}).$$

Let M be a manifold of dimension n. We opt for the definition of differential k-forms on M as smooth sections

$$\omega\colon M \to \Lambda^k(M) = \bigcup_{p \in M} \Lambda^k(T_pM^*),$$

where $\Lambda^k(M)$ is equipped with the natural smooth structure such that the projection onto M is smooth — the charts are of the form

$$U \times \mathbb{R}^{\binom{n}{k}} \to \Lambda^k(M), \quad (x, v) \mapsto \Lambda^k((D_p\theta^{-1})^*) \circ \varphi(v) \in \Lambda^k(T_pM^*),$$

for a chart $\theta\colon U \to M$, and an isomorphism $\varphi\colon \mathbb{R}^{\binom{n}{k}} \to \Lambda^k((\mathbb{R}^n)^*)$. This definition is easily seen to be equivalent to the one given in [8], namely that a differential k-form is a family, $\{\omega_p\}_{p \in M}$, of alternating k-linear maps $T_pM^k \to \mathbb{R}$, such that the pullback $\theta^*\omega\colon U \to \Lambda^k(U) = U \times \Lambda^k((\mathbb{R}^n)^*)$ is smooth for all charts $\theta\colon U \to M$. The set of differential k-forms on M is denoted by $\Omega^k(M)$. Given a chart $\theta\colon \mathbb{R}^n \to U \subseteq M$, the maps $x_i := \mathrm{pr}_i \circ \theta^{-1}\colon U \to \mathbb{R}$ are local coordinates on V, where $\mathrm{pr}_i\colon \mathbb{R}^n \to \mathbb{R}$ is projection onto the i'th coordinate.
Consider the differentials $dx_i\colon U \times \mathbb{R}^n \to \mathbb{R}$, which we shall consider as differential 1-forms on U by evaluating in the first coordinate, $dx_i\colon U \to (\mathbb{R}^n)^* = \Lambda^1((\mathbb{R}^n)^*)$. For $p = \theta(x)$, $\{D_x\theta(e_i)\}_{i=1}^n$ form a basis of T_pM with dual basis $\{dx_i(p)\}_{i=1}^n$ of T_pM^*, where (e_i) is the standard basis of $\mathbb{R}^n$. Thus $\{dx_{\sigma(1)}(p) \wedge \cdots \wedge dx_{\sigma(k)}(p)\}_{\sigma \in S_{k,n-k}}$ form a basis of $\Omega^k(T_pM^*)$. Hence any map $\omega\colon M \to \Lambda^k(M)$ can be written locally on U as

$$\omega = \sum_{\sigma \in S_{k,n-k}} f_\sigma \, dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)},$$

for some functions $f_\sigma\colon U \to \mathbb{R}$, and ω is smooth on U if and only if all the f_σ are smooth. Moreover,

$$f_\sigma(p) = \omega_p(D_x\theta(e_{\sigma(1)}), \dots, D_x\theta(e_{\sigma(k)})), \quad \text{for } p = \theta(x).$$

We will sometimes write $dx_\sigma := dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}$.
A smooth vector field on M is a smooth section of the tangent bundle, $X\colon M \to TM$. As we are only interested in smooth vector fields, we let the smoothness assumption be implicit from now on in. We denote the set of vector fields on M by $\mathfrak{X}(M)$. One can identify a vector field with its action on $C^\infty(M)$: for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, $X(f)\colon M \to \mathbb{R}$ given by

$X(f)(p) = D_{p,X(p)}f$ is smooth. $\mathfrak{X}(M)$ is a Lie algebra when equipped with the commutator bracket with respect to the composition in $\mathrm{End}(C^\infty(M))$.
We may, for the sake of clarity, alter between the notations $f(p)$ and f_p for evaluation at p.
As usual, a hat denotes that an element is omitted, for example

$$(x_1, \dots, \hat{x}_i, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$
$$x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n = x_1 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_n.$$

$\mathbb{N}$ denotes the natural numbers including 0. We will write $\mathbb{N}_{>0}$ for $\mathbb{N} \setminus \{0\}$.
We denote De Rham cohomology by $H^*_{dR}(-)$.

Acknowledgements

I would like to thank Dustin for supervision, and Matias and Kevin for proofreading.

1 Lie Algebras, Universal Enveloping Algebras, and Lie Algebra Cohomology

In this chapter we briefly recall the definitions of Lie algebras and their universal enveloping algebras. We go on to define Lie algebra cohomology using the Chevalley-Eilenberg complex, and finally show that we can equivalently define it as the right derived functor of the invariants functor.

1.1 Basics

Let K be a field.

Definition 1.1.1. A *Lie algebra* over K is a K-vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the Lie bracket, satisfying

1. $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$ (anti-symmetry).
2. $[[X,Y],Z]=[X,[Y,Z]]-[Y,[X,Z]]$ for all $X,Y,Z\in\mathfrak{g}$ (Jacobi identity).

A *Lie algebra homomorphism* is a linear map between Lie algebras, $\varphi\colon\mathfrak{g}\to\mathfrak{h}$, which respects the Lie bracket, i.e. $\varphi([x,y]_{\mathfrak{g}})=[\varphi(x),\varphi(y)]_{\mathfrak{h}}$, for all $x,y\in\mathfrak{g}$.

Definition 1.1.2. Let R be a commutative ring. An associative R-algebra is an R-module, A, equipped with an associative multiplication operation with identity which respects the R-module structure, that is the multiplication map, $A\times A\to A$, is R-bilinear.

Remark 1.1.3. Given an associative algebra A over a field K, A may be turned into a Lie algebra by taking the commutator with respect to the associative product as the Lie bracket. For example, for any vector space V, the space of endomorphisms of V, $\mathrm{End}(V)$, becomes a Lie algebra with the commutator bracket.

Remark 1.1.4. If G is a Lie group, then the tangent space at $1\in G$, T_1G, is a Lie algebra with the Lie bracket defined as follows: Define $\mathrm{Ad}\colon G\to\mathrm{GL}(T_1G)$, $g\mapsto D_1\psi_g$, where $\psi_g\colon G\to G$ is conjugation by g; next, define $\mathrm{ad}:=D_1\mathrm{Ad}\colon T_1G\to\mathrm{End}(T_1G)$. Finally, set $[x,y]:=\mathrm{ad}(x)y$ for $x,y\in T_1G$. With this $\mathfrak{g}:=T_1G$ becomes a Lie algebra — the Lie algebra of G. One can also identify $\mathfrak{g}$ with the set of left-invariant vector fields on G, with Lie bracket the usual commutator.

Definition 1.1.5. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module is a K-vector space V together with a representation $\rho\colon\mathfrak{g}\to\mathrm{End}(V)$ of $\mathfrak{g}$, i.e. ρ is a Lie algebra homomorphism. In other words, a $\mathfrak{g}$-module is a K-vector space V and a linear action of $\mathfrak{g}$ on V, $-.-\colon\mathfrak{g}\times V\to V$, satisfying $[X,Y].v=X.(Y.v)-Y.(X.v)$ for all $X,Y\in\mathfrak{g}$, $v\in V$.

Example 1.1.6. Examples of $\mathfrak{g}$-modules are

1. K with the trivial action, $\mathfrak{g}\to\mathrm{End}(K)$ the zero map.
2. $\mathfrak{g}$ itself with the adjoint action; $\mathrm{ad}\colon\mathfrak{g}\to\mathrm{End}(\mathfrak{g})$, $\mathrm{ad}(x)(y)=[x,y]$.

3. If $\mathfrak{g}$ is the Lie algebra of a Lie group G then $C^\infty(G)$ is a $\mathfrak{g}$-module: We identify $\mathfrak{g}$ with the set of left-invariant vector fields on G, and these in turn with their action on $C^\infty(G)$.

Definition 1.1.7. Let $\mathfrak{g}$ be a Lie algebra over K. The universal enveloping algebra of $\mathfrak{g}$ is an associative K-algebra $U(\mathfrak{g})$ and a Lie algebra homomorphism $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$ satisfying the universal property pictured in the diagram below, where A is an associative K-algebra, φ is a Lie algebra homomorphism and ψ is a homomorphism of associative K-algebras:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\iota"'] & A \\ U(\mathfrak{g}) \arrow[ur, dashed, "\exists!\psi"'] & \end{tikzcd}$$

Proposition 1.1.8. Any Lie algebra has a universal enveloping algebra.

To prove this, we construct the universal enveloping algebra directly. This will be done in a few steps with some intermediary results.

Definition 1.1.9. Let V be any K-vector space. For $n \in \mathbb{N}_{>0}$, set $T^n(V) = V^{\otimes n}$, and set $T^0(V) := K$. The tensor algebra of V is $T(V) := \bigoplus_{n\in\mathbb{N}} T^n(V)$, with multiplication determined by the canonical isomorphisms $T^n(V) \otimes T^m(V) \to T^{n+m}(V)$. There is a canonical linear map $V \to T(V)$ mapping into the second term of the direct sum, $T^1(V) = V$. For any linear map, $f\colon V \to V'$, there is an induced map $T(f)\colon T(V) \to T(V')$ determined by $T(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$.

Proposition 1.1.10. The tensor algebra satisfies the following universal property, where A is an associative algebra, φ is linear and ψ is a homomorphism of associative K-algebras:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & A \\ T(V) \arrow[ur, dashed, "\exists!\psi"'] & \end{tikzcd}$$

Proof. Clearly, $T(V)$ is an associative algebra. Given $\varphi\colon V \to A$, define $\psi\colon T(V) \to A$ by $v_1 \otimes \cdots \otimes v_n \mapsto \varphi(v_1)\cdots\varphi(v_n)$. $\square$

Remark 1.1.11. T is a functor from the category of K-vector spaces to the category of associative K-algebras. It is left adjoint to the forgetful functor mapping an algebra to its underlying vector space.

Now, let $\mathfrak{g}$ be a Lie algebra over K, and consider the ideal

$$\mathcal{I} = \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle \lhd \mathcal{T}(\mathfrak{g}).$$

Proposition 1.1.12. $U(\mathfrak{g}) := T(\mathfrak{g})/\mathcal{I}$ is the universal enveloping algebra of $\mathfrak{g}$.

Proof. There is a canonical map $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$, namely the composition $\mathfrak{g} \to T(\mathfrak{g}) \to U(\mathfrak{g})$. This is a Lie algebra homomorphism by construction. Now, given an associative K-algebra A and a Lie algebra homomorphism $\mathfrak{g} \to A$, consider the diagram

$$\begin{tikzcd} & T(\mathfrak{g}) \arrow[d] \arrow[dd, bend left=60, "(1)"] \\ \mathfrak{g} \arrow[ur] \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & U(\mathfrak{g}) \arrow[d, "(2)"] \\ & A \end{tikzcd}$$

The map (1) exists and is unique by the universal property of the tensor algebra, and (2) exists and is unique by the universal property of the quotient, as φ is a Lie algebra homomorphism, so it is trivial on the generators of $\mathcal{I}$. □

Remark 1.1.13. The Birkhoff-Witt theorem (see [7]) implies that the map $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$ is injective, so $\mathfrak{g}$ can be seen as a subspace of $U(\mathfrak{g})$.

Remark 1.1.14. U is a functor from the category of Lie algebras over K to the category of associative K-algebras. It is left adjoint to the forgetful functor mapping an associative algebra to its underlying vector space equipped with the commutator bracket.

Example 1.1.15.

1. If $\mathfrak{g}$ is abelian, then the universal enveloping algebra of $\mathfrak{g}$ is the symmetric algebra, $\mathrm{Sym}(\mathfrak{g})$ — the free commutative algebra over $\mathfrak{g}$.

2. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then one can identify $\mathfrak{g}$ with the left-invariant first order differential operators on G (the vector fields on G), and the universal enveloping algebra can be identified with the set of left-invariant differential operators on G of all orders. This was how the universal enveloping algebra was originally introduced by Poincaré in 1899 (See [4] for details on this example).

Remark 1.1.16. The universal property of the universal enveloping algebra implies that a K-vector space Γ is a $\mathfrak{g}$-module, if and only if it is a $U(\mathfrak{g})$-module in the usual sense, when viewing $U(\mathfrak{g})$ as a ring. If $\rho\colon \mathfrak{g} \to \mathrm{End}(\Gamma)$ is a Lie algebra homomorphism, then the ring homomorphism $U(\mathfrak{g}) \to \mathrm{End}(\Gamma)$ is determined by $[x_1 \otimes \cdots \otimes x_n] = \rho(x_1) \circ \cdots \circ \rho(x_n)$.

1.2 Lie Algebra Cohomology

In this section, we define Lie algebra cohomology. To this end, we define the Chevalley-Eilenberg chain complex. Let K be a field, $\mathfrak{g}$ a Lie algebra over K, and Γ a $\mathfrak{g}$-module. Set

$$C^n(\mathfrak{g},\Gamma) := \mathrm{Hom}_K(\Lambda^n\mathfrak{g},\Gamma),\ n > 0, \qquad C^0(\mathfrak{g},\Gamma) := \Gamma.$$

That is, $C^n(\mathfrak{g},\Gamma)$ is the set of alternating n-linear maps $\mathfrak{g}^n \to \Gamma$. These are the n-cochains of the Chevalley-Eilenberg complex. We define the differential $d\colon C^n(\mathfrak{g},\Gamma) \to C^{n+1}(\mathfrak{g},\Gamma)$ as follows: Given $c \in C^n(\mathfrak{g},\Gamma)$, let $dc \in C^{n+1}(\mathfrak{g},\Gamma)$ be given by

$$\begin{aligned} dc(x_1,\ldots,x_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}x_i.c(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}) \\ &+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j}c([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1}), \end{aligned}$$

for all $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, where $x_i.c(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$ denotes the action of $x_i \in \mathfrak{g}$ on $c(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \in \Gamma$ according to the $\mathfrak{g}$-module structure of Γ. A long and tedious calculation shows that $d^2 = 0$ (see Proposition A.1.1).

Definition 1.2.1. $(C^*(\mathfrak{g}, \Gamma), d)$ is called the *Chevalley-Eilenberg complex* of $\mathfrak{g}$ with coefficients Γ, and the *cohomology* of $\mathfrak{g}$ with coefficients in Γ, $H^*(\mathfrak{g}, \Gamma)$, is defined as the homology of the Chevalley-Eilenberg complex.

We will give examples of Lie algebras and their cohomology in Chapter 4.

1.3 Derived Functor Approach

We will now look into a more categorical approach to Lie algebra cohomology. Let K be a field, $\mathfrak{g}$ a Lie algebra over K. We will show that the Lie algebra cohomology is the right derived functor of the invariants functor from the category of $\mathfrak{g}$-modules to itself:

$$I\colon \Gamma \mapsto \Gamma^{\mathfrak{g}} = \{\gamma \in \Gamma \mid x.\gamma = 0 \text{ for all } x \in \mathfrak{g}\}$$

Note that for any two $\mathfrak{g}$-modules, V and W, with representations $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$, $\pi\colon \mathfrak{g} \to \mathrm{End}(W)$, we have

$$\mathrm{Hom}_{U(\mathfrak{g})}(V, W) = \{T \in \mathrm{Hom}_K(V, W) \mid T \circ \rho(x) = \pi(x) \circ T \text{ for all } x \in \mathfrak{g}\}.$$

This follows directly from the $U(\mathfrak{g})$-module structure on V and W:

$$\begin{aligned} T([x_1 \otimes \cdots \otimes x_n].v) &= T(\rho(x_1) \circ \cdots \circ \rho(x_n)(v)), \\ [x_1 \otimes \cdots \otimes x_n].T(v) &= \pi(x_1) \circ \cdots \circ \pi(x_n)(T(v)), \end{aligned}$$

for any element of the form $[x_1 \otimes \cdots \otimes x_n] \in U(\mathfrak{g})$, $v \in V$. Hence, if Γ is a $\mathfrak{g}$-module, then, since $\mathrm{Hom}_K(K, \Gamma) \cong \Gamma$, $T \mapsto T(1)$, we have that $\Gamma^{\mathfrak{g}} \cong \mathrm{Hom}_{U(\mathfrak{g})}(K, \Gamma)$, where $\mathfrak{g}$ acts trivially on K. This isomorphism is natural, so the two functors, I and $\mathrm{Hom}_{U(\mathfrak{g})}(K, -)$, are isomorphic.

Theorem 1.3.1. The Lie algebra cohomology of $\mathfrak{g}$ with coefficients in Γ is the right derived functor of I and can also be described as an Ext-functor:

$$H^*(\mathfrak{g}, \Gamma) \cong \mathrm{Ext}^*_{U(\mathfrak{g})}(K, \Gamma) \cong R^*I(\Gamma).$$

Proof. By the above observations, $I(-) \cong \mathrm{Hom}_{U(\mathfrak{g})}(K, -)$, and thus $\mathrm{Ext}^*_{U(\mathfrak{g})}(K, \Gamma) \cong R^*I(\Gamma)$. Now, we calculate $\mathrm{Ext}_{U(\mathfrak{g})}(K, \Gamma)$ by finding a projective resolution of K. Consider the sequence

$$\cdots \longrightarrow \Lambda^3\mathfrak{g} \otimes U(\mathfrak{g}) \longrightarrow \Lambda^2\mathfrak{g} \otimes U(\mathfrak{g}) \longrightarrow \mathfrak{g} \otimes U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \longrightarrow K$$

where we tensor over K, and $\Lambda^n\mathfrak{g} \otimes U(\mathfrak{g}) \longrightarrow \Lambda^{n-1}\mathfrak{g} \otimes U(\mathfrak{g})$ is given by

$$\begin{aligned} (x_1 \wedge \cdots \wedge x_n) \otimes u \;\mapsto\; & \sum_{i=1}^{n} (-1)^i (x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n) \otimes x_i u \\ & + \sum_{1 \le i < j \le n} (-1)^{i+j+1} ([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n) \otimes u, \end{aligned}$$

here we identify $\Lambda^0(\mathfrak{g}) \otimes U(\mathfrak{g}) = K \otimes U(\mathfrak{g}) \cong U(\mathfrak{g})$ and $\Lambda^1\mathfrak{g} = \mathfrak{g}$, and $U(\mathfrak{g}) \to K$ is the augmentation map induced by the zero map $\mathfrak{g} \to K$.

This is a free and thus projective resolution of the trivial $U(\mathfrak{g})$-module K: Indeed, $\Lambda^n(\mathfrak{g})$ is a free K-module, being a vector space over K, and therefore $\Lambda^n(\mathfrak{g}) \otimes U(\mathfrak{g})$ is a free $U(\mathfrak{g})$-module. Now, for exactness of the sequence: This is a long and technical proof, so we have opted for a sketch of proof instead. For the fun of it, we prove exactness at the first two terms of the sequence: $\mathfrak{g} \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \to k \to 0$. The augmentation map is surjective, as it restricts to the identity on $K = \mathfrak{g}^{\otimes 0} \subseteq U(\mathfrak{g})$, and its kernel is the image of the subspace $\bigoplus_{n \in \mathbb{N}_{>0}} \mathfrak{g}^{\otimes n} \subseteq T(\mathfrak{g})$ in $U(\mathfrak{g})$; this is equal to $\iota(\mathfrak{g})U(\mathfrak{g}) \subseteq U(\mathfrak{g})$, which is exactly the image of the map $\mathfrak{g} \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$, $x \otimes u \mapsto xu$, given above. So the first part of the sequence is exact.
Now, to prove exactness of the complete sequence, first note that the composite of two succesive maps is zero by calculations similar to those used for proving that $d^2 = 0$ in the Chevalley-Eilenberg complex. Hence, it is a chain complex, and we can prove exactness by proving that its homology is trivial. To do this, one takes a filtration of the complex, $F_0 \subseteq F_1 \subseteq \cdots \subseteq C$, where C denotes the complex defined above — the Birkhoff-Witt theorem ([7]) is used to define this filtration. Then one defines complexes $W_k := F_k/F_{k-1}$ and shows that these have trivial homology by constructing a contracting chain homotopy. The short exact sequence of complexes $0 \to F_{k-1} \to F_k \to W_k \to 0$ induces a long exact sequence in homology, which yields isomorphisms $H^*(F_k) \cong H^*(F_{k-1})$. By definition of the filtration, F_0 is the complex $0 \to K \to K \to 0$, which has trivial homology, implying that all the F_k have trivial homology. Now, as homology respects direct limits in ${}_{U(\mathfrak{g})}\mathrm{Mod}$, it follows that $C = \bigcup F_k$ has trivial homology, as desired. We refer to [6] for the complete proof. This complex is called the Koszul complex.
Finally, we apply $\mathrm{Hom}_{U(\mathfrak{g})}(-, \Gamma)$ to the resolution and note that the map

$$\begin{aligned} \mathrm{Hom}_{U(\mathfrak{g})}(\Lambda^n \mathfrak{g} \otimes U(\mathfrak{g}), \Gamma) &\longrightarrow \mathrm{Hom}_K(\Lambda^n \mathfrak{g}, \Gamma) = C^n(\mathfrak{g}, \Gamma), \\ \text{given by} \quad T \mapsto \widetilde{T}, &\quad \text{with } \widetilde{T}(x_1 \wedge \cdots \wedge x_n) = T((x_1 \wedge \cdots \wedge x_n) \otimes 1) \end{aligned}$$

is an isomorphism of chain complexes, which implies the desired result, namely that

$$H^*(\mathfrak{g}, \Gamma) \cong \mathrm{Ext}^*_{U(\mathfrak{g})}(K, \Gamma).$$

$\square$

2 Historical Motivation I: The Lie Derivative

In this chapter we define the Lie derivative. It will not be used directly when dealing with Lie algebra cohomology, but it provides an important motivation for the definition of Lie algebra cohomology given in Chapter 1. More specifically, we will prove the invariant formula for the exterior derivative of differential forms, which is entirely dependent on the vector fields on the manifold in question (Proposition 2.2.14). Here we will see what originally motivated the definition of the Chevalley-Eilenberg complex.

2.1 Tensor Fields

Let M be a manifold of dimension n.

Definition 2.1.1. Set $T^{r,s}(M) := \bigcup_{p\in M}(T_pM)^{r,s}$, where $V^{r,s} = (V^{\otimes_K r}) \otimes_K (V^{*\otimes_K s})$ for any K-vector space V, and by convention $V^{\otimes_K 0} = K$. We give $T^{r,s}(M)$ the natural smooth structure such that the projection $T^{r,s}(M) \to M$ is smooth — this is a smooth vector bundle. A *tensor field* on M of type (r, s) is a smooth section $S\colon M \to T^{r,s}(M)$. We shall denote the set of tensor fields of type (r, s) on M by $\mathcal{T}^{r,s}(M)$, and the set of all tensor fields on M by $\mathcal{T}(M) = \bigoplus_{r,s\in\mathbb{N}} \mathcal{T}^{r,s}(M)$. This is a $C^\infty(M)$-module with the obvious operations; moreover, it has a multiplication operation, namely the tensor product: If $S, T \in \mathcal{T}(M)$ of type (r, s) respectively (r', s'), then $S\otimes T$, given by $p \mapsto S(p)\otimes T(p)$ is a tensor field of type $(r+r', s+s')$, where we use the canonical isomorphism $A \otimes B \cong B \otimes A$.

Proposition 2.1.2. The set of differential k-forms on M, $\Omega^k(M)$, is isomorphic as a $C^\infty(M)$-module to the set of alternating $C^\infty(M)$-multilinear maps $\mathfrak{X}(M)^k \to C^\infty(M)$, where the scalar multiplication in both cases is given by pointwise multiplication. In particular, $\Omega^1(M) \cong \mathfrak{X}(M)^*$.

Proof. Given $\omega \in \Omega^k(M)$, define $\widetilde{\omega}\colon \mathfrak{X}(M)^k \to C^\infty(M)$ by

$$\widetilde{\omega}(X_1, \ldots, X_k)(p) = \omega_p(X_1(p), \ldots, X_k(p)), \quad \text{for } X_i \in \mathfrak{X}(M),\ p \in M. \tag{2.1}$$

$\widetilde{\omega}$ is alternating and $C^\infty(M)$-multilinear, as ω_p is alternating and $\mathbb{R}$-multilinear for all $p \in M$. The map $\widetilde{\omega}(X_1, \ldots, X_k)\colon M \to \mathbb{R}$ is equal to the composite

$$M \xrightarrow{\omega\times X_1\times\cdots\times X_k} \Lambda^k(M) \times_M TM \times_M \cdots \times_M TM \xrightarrow{\text{ev}} \mathbb{R},$$

where the second map is the evaluation map. Locally the evaluation map is a $(k + 1)$-linear map of finite-dimensional vector spaces, $\mathrm{Alt}^k(\mathbb{R}^n) \times (\mathbb{R}^n)^k \to \mathbb{R}$, and as such it is smooth. Then, since $\omega\colon M \to \Lambda^k(M)$ is smooth, and all the $X_i\colon M \to TM$ are smooth, we conclude that $\widetilde{\omega}(X_1, \ldots, X_k) \in C^\infty(M)$.
Conversely, given $\eta\colon \mathfrak{X}(M)^k \to C^\infty(M)$ alternating and $C^\infty(M)$-multilinear, define

$$\hat{\eta}\colon M \to \Lambda^k(M) \quad \text{by} \quad \hat{\eta}_p(v_1, \ldots, v_k) = \eta(X_1, \ldots, X_k)(p),$$

for some choice of $X_i \in \mathfrak{X}(M)$ with $X_i(p) = v_i$. Such vector fields always exist, but we must show that the definition is independent of the choice of vector fields: $\eta(X_1, \ldots, X_k)(p)$ depends only on the values of the X_i at p. By multilinearity of η, it is enough to prove the case $k=1$, and it suffices to prove that if $X(p)=0$, then $\eta(X)(p)=0$. To see this, let $\theta\colon \mathbb{R}^n \to U$ be a chart around p, and define vector fields on U, $X_i\colon U \to TM$, as the composites $U \xrightarrow{\theta^{-1}} \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \{e_i\} \xrightarrow{D\theta} TM$, for $i=1,\dots,n$. As $\{D_x\theta(e_i)\}_{i=1}^n$ is a basis of $T_{\theta(x)}M$, we can uniquely write

$$X(q)=\sum_{i=1}^n a_i^q D_{\theta^{-1}(y)}\theta(e_i)=\sum_{i=1}^n a_i^q X_i(q), \quad \text{for all } q\in U, \text{ and some } a_i^q\in\mathbb{R}.$$

Defining $f_i\colon U\to\mathbb{R}$ by $f_i(q)=a_i^q$, we see that $f_i\in C^\infty(U)$, as X and the X_i are smooth. Assume now that $X(p)=0$; then $f_i(p)=0$ for all $i=1,\dots,n$. Hence,

$$\eta(X)(p)=\sum_{i=1}^n f_i(p)\eta(X_i)(p)=0.$$

We conclude that $\hat{\eta}$ is well-defined. Moreover, $\hat{\eta}_p$ is $\mathbb{R}$-linear and alternating for all $p\in M$. To see that it is smooth and thus a differential k-form, let $\theta\colon\mathbb{R}^n\to U$ be a chart on M with local coordinates $x_1,\dots,x_n$, and define vector fields, $X_i\colon U \xrightarrow{\theta^{-1}} \mathbb{R}^n \hookrightarrow \mathbb{R}^n\times\{e_i\} \xrightarrow{D\theta} TM$, as above. Then we can write $\hat{\eta}$ locally as

$$\hat{\eta}|_U=\sum_{\sigma\in S_{k,n-k}} f_\sigma dx_{\sigma(1)}\wedge\cdots\wedge dx_{\sigma(k)}, \qquad \text{for } f_\sigma=\eta\left(X_{\sigma(1)},\dots,X_{\sigma(k)}\right).$$

(Here we skip some technicalities: We can extend the X_i to vector fields on M by multiplying with a suitable bump function; to do this, we may have to shrink U a little.) Then $f_\sigma\in C^\infty(U)$, so $\hat{\eta}$ is smooth.
It is clear that the maps $\omega\mapsto\widetilde{\omega}$ and $\eta\mapsto\hat{\eta}$ are each other's inverses. □

Remark 2.1.3. In view of the above proposition, we will from now on in many cases simply interpret $\omega\in\Omega^k(M)$ as an alternating k-linear map of $C^\infty(M)$-modules $\mathfrak{X}(M)^k\to C^\infty(M)$.

We also need the following application of a version of the Serre-Swan Theorem:

LEMMA 2.1.4. $\Omega^1(M)$ and $\mathfrak{X}(M)$ are finitely generated projective $C^\infty(M)$-modules.

PROOF. We refer to Theorem 11.32 of [2]. □

COROLLARY 2.1.5. The set of smooth vector fields on M, $\mathfrak{X}(M)$, is isomorphic as a $C^\infty(M)$-module to the set of $C^\infty(M)$-linear maps $\Omega^1(M)\to C^\infty(M)$.

PROOF. By Proposition 2.1.2, $\Omega^1(M)\cong\mathfrak{X}(M)^*$. Being finitely generated projective, $\mathfrak{X}(M)$ is canonically isomorphic to its double dual (cf. Proposition A.2.1). Hence, $\mathfrak{X}(M)\cong\Omega^1(M)^*$, as desired. Tracing out the isomorphisms involved, we see that the isomorphism $\mathfrak{X}(M)\to\Omega^1(M)^*$ is explicitly given by $X\mapsto\widetilde{X}$, with $\widetilde{X}(\omega)=\widetilde{\omega}(X)=\omega_{(-)}(X(-))$, where $\widetilde{\omega}$ is defined in 2.1 in the proof of Proposition 2.1.2. □

PROPOSITION 2.1.6. The set of type (r,s) tensor fields on M, $\mathcal{T}^{r,s}(M)$, is isomorphic as a $C^\infty(M)$-module to the set of $C^\infty(M)$-multilinear maps $\Omega^1(M)^r\times\mathfrak{X}(M)^s\to C^\infty(M)$.

PROOF. In general, we have $\mathcal{T}^{r,s}(M) \otimes_{C^\infty(M)} \mathcal{T}^{r',s'}(M) = \mathcal{T}^{r+r',s+s'}(M)$, by definition of the product in $\mathcal{T}(M)$. In particular, $\mathcal{T}^{r,s}(M) = \mathcal{T}^{1,0}(M)^{\otimes r} \otimes \mathcal{T}^{0,1}(M)^{\otimes s} = \mathfrak{X}(M)^{\otimes r} \otimes \Omega^1(M)^{\otimes s}$, where we tensor over $C^\infty(M)$ (this will be implicit for the remainder of this proof). Hence, by Proposition 2.1.2 and corollary 2.1.5

$$\mathcal{T}^{r,s}(M) = \mathfrak{X}(M)^{\otimes r} \otimes \Omega^1(M)^{\otimes s} \cong \left(\Omega^1(M)^*\right)^{\otimes r} \otimes (\mathfrak{X}(M)^*)^{\otimes s}.$$

As $\mathfrak{X}(M)$ and $\Omega^1(M)$ are finitely generated projective, we have

$$\begin{aligned}\left(\Omega^1(M)^*\right)^{\otimes r} \otimes (\mathfrak{X}(M)^*)^{\otimes s} &\cong \left(\Omega^1(M)^{\otimes r} \otimes \mathfrak{X}(M)^{\otimes s}\right)^* \qquad \text{(cf. Proposition A.2.2)}\\ &\cong \{\Omega^1(M)^r \times \mathfrak{X}(M)^s \to C^\infty(M),\ C^\infty(M)\text{-multilinear}\},\end{aligned}$$

where the last isomorphism comes from the universal property of the tensor product. For $r = s = 0$, the first two tensor products are just $C^\infty(M)$ which is canonically isomorphic to $\{* \to C^\infty(M)\}$. □

Remark 2.1.7. We will henceforth readily switch between the identifications below and use the one most suitable to the given situation:

$$\begin{aligned}\mathcal{T}^{r,s}(M) = (\mathfrak{X}(M)^{\otimes r}) \otimes (\Omega^1(M)^{\otimes s}) &\cong (\Omega^1(M)^{*\otimes r}) \otimes (\mathfrak{X}(M)^{*\otimes s})\\ &\cong \{\Omega^1(M)^r \times \mathfrak{X}(M)^s \to C^\infty(M),\ C^\infty(M)\text{-multilinear}\}.\end{aligned}$$

The explicit isomorphism

$$(\mathfrak{X}(M)^{\otimes r}) \otimes (\Omega^1(M)^{\otimes s}) \to \{\Omega^1(M)^r \times \mathfrak{X}(M)^s \to C^\infty(M),\ C^\infty(M)\text{-multilinear}\}$$

is given by $X_1 \otimes \cdots \otimes X_r \otimes \omega_1 \otimes \cdots \otimes \omega_s \ \mapsto\ \Psi_{X_i,\omega_i}$, where

$$\Psi_{X_i,\omega_i}(\eta_1, \ldots, \eta_r, Y_1, \ldots, Y_s) = \eta_1(X_1)\cdots\eta_r(X_r)\omega_1(Y_1)\cdots\omega_s(Y_s).$$

Remark 2.1.8. It follows almost directly from the definition, that we can view $\Omega^k(M)$ as a submodule of $\mathcal{T}^{0,k}(M)$ — this can also be seen by Propositions 2.1.2 and 2.1.6. From this we deduce that $\Omega^*(M)$ is spanned as a $C^\infty(M)$-algebra by $\Omega^1(M)$: To see this, note first that the symmetric group on k symbols, S_k, acts on the set of type $(0,k)$ tensor fields, $\mathcal{T}^{0,k}(M)$, by $\sigma.S(X_1, \ldots, X_k) = S(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$, for $\sigma \in S_k$, $S \in \mathcal{T}^{0,k}(M)$; on $\Omega^k(M)$ this is simply acting by sign. Now, given $\omega \in \Omega^k(M) \subseteq \mathcal{T}^{0,k}(M)$, we can write ω as a linear combination of tensors $\omega_1 \otimes \cdots \otimes \omega_k$, $\omega_i \in \Omega^1(M)$. If $\omega = \sum_{i=1}^n f^i \omega_1^i \otimes \cdots \otimes \omega_k^i$, $\omega_j^i \in \Omega^1(M)$, $f^i \in C^\infty(M)$, then for any $\sigma \in S_k$.

$$\begin{aligned}\omega(X_1, \ldots, X_k) &= \operatorname{sign}(\sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\\ &= \sum_{i=1}^n \operatorname{sign}(\sigma)\, f^i \omega_1^i(X_{\sigma(1)}) \cdots \omega_k^i(X_{\sigma(k)})\\ &= \sum_{i=1}^n \operatorname{sign}(\sigma)\, f^i \omega_{\sigma(1)}^i \otimes \cdots \otimes \omega_{\sigma(k)}^i(X_1, \ldots, X_k).\end{aligned}$$

Hence,

$$k!\, \omega = \sum_{i=1}^n f^i \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \omega_{\sigma(1)}^i \otimes \cdots \otimes \omega_{\sigma(k)}^i = \sum_{i=1}^n f^i \omega_1^i \wedge \cdots \wedge \omega_k^i$$

Interpreting $\Omega^k(M)$ as a submodule of $\mathcal{T}^{0,k}(M)$ does not, however, relate the multiplication operations in the two modules, $\wedge$ respectively $\otimes$. We shall instead interpret $\Omega^k(M)$ as a quotient of $\mathcal{T}^{0,k}(M)$. We claim that $\Omega^k(M)$ is the largest quotient of $\mathcal{T}^{0,k}(M)$ on which S_k acts by sign, i.e. it satisfies the universal property pictured in the diagram below, where A is a $C^\infty(M)$-module on which S_k acts by sign and all maps are equivariant:

$$\begin{tikzcd}
\mathcal{T}^{0,k}(M) \arrow[r, two heads, "q"] \arrow[d, two heads, "p"'] & \Omega^k(M) \arrow[dl, dashed, two heads, "\exists!"] \\
A &
\end{tikzcd}$$

The map $q\colon \mathcal{T}^{0,k}(M) \to \Omega^k(M)$ is given by $\omega_1 \otimes \cdots \otimes \omega_k \mapsto \omega_1 \wedge \cdots \wedge \omega_k$, $\omega_i \in \Omega^1(M)$. This is surjective, as we observed above that $\Omega^k(M)$ is spanned by elements of the form $\omega_1 \wedge \cdots \wedge \omega_k$, $\omega_i \in \Omega^1(M)$. Note that q has a section, namely the map

$$s\colon \Omega^k(M) \to \mathcal{T}^{0,k}(M), \quad \text{given by } s(\omega_1 \wedge \cdots \wedge \omega_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(k)}.$$

Now, given p as in the diagram above, set $\varphi := ps$. Clearly, this is the unique map which makes the diagram commute, so all we need to show, is that it is surjective. By linearity, it suffices to show that φ hits $p(\omega_1 \otimes \cdots \otimes \omega_k)$ for any $\omega_1 \otimes \cdots \otimes \omega_k \in \mathcal{T}^{0,k}(M)$. Given $\omega_1 \otimes \cdots \otimes \omega_k \in \mathcal{T}^{0,k}(M)$, we see that

$$\begin{aligned}
\varphi(\omega_1 \wedge \cdots \wedge \omega_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, p(\omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \sigma.p(\omega_1 \otimes \cdots \otimes \omega_k) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sign}(\sigma))^2 p(\omega_1 \otimes \cdots \otimes \omega_k) = p(\omega_1 \otimes \cdots \otimes \omega_k).
\end{aligned}$$

As q respects the multiplication operations, we get the desired relationship between them. In view of this observation, we will often interpret a differential k-form as a vector field of type $(0, k)$, even though we are in fact looking at an equivalence class of vector fields.

2.2 The Lie Derivative

Let M be a manifold.

Definition 2.2.1. Let $X \in \mathfrak{X}(M)$. The *Lie derivative* with respect to X is the type-preserving map $L_X\colon \mathcal{T}(M) \to \mathcal{T}(M)$ satisfying the following axioms:

1. L_X is $\mathbb{R}$-linear.
2. $L_X(f) = X(f)$ for any $f \in C^\infty(M) = \mathcal{T}^{0,0}(M)$.
3. $L_X(S \otimes T) = (L_X S) \otimes T + S \otimes (L_X T)$ for any $S, T \in \mathcal{T}(M)$. (Leibniz' rule)
4. $L_X(\eta(Y)) = (L_X \eta)(Y) + \eta(L_X Y)$ for any $\eta \in \Omega^1(M)$, $Y \in \mathfrak{X}(M)$.
   (Leibniz' rule with respect to contractions)
5. L_X commutes with the exterior derivative on $C^\infty(M) = \Omega^0(M)$.

Before proving existence of such a map, we will show that given it exists, it must be unique.

Proposition 2.2.2. If the Lie derivative exists, it is unique.

Proof. Let $X \in \mathfrak{X}(M)$. Assume that $L_X : \mathcal{T}(M) \to \mathcal{T}(M)$ is type-preserving and satisfies axioms 1-5 of Definition 2.2.1. Being type-preserving and $\mathbb{R}$-linear, it suffices to show that it is unique on $\mathcal{T}^{r,s}(M)$ for any choice of r, s. Now, the fact that $\mathcal{T}^{r,s}(M) = (\mathfrak{X}(M)^{\otimes r}) \otimes (\Omega^1(M)^{\otimes s})$ and axiom 3 reduces the problem to showing that L_X is unique on $\mathfrak{X}(M)$ and $\Omega^1(M)$. If L_X is unique on $\mathfrak{X}(M)$, then axioms 2 and 4 imply uniqueness on $\Omega^1(M)$. So we just need to show that $L_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is unique. We shall identify $\mathfrak{X}(M)$ with its action on $C^\infty(M)$: $Z \in \mathfrak{X}(M)$ defines a map $C^\infty(M) \to C^\infty(M)$, $g \mapsto Z(g)$ given by $Z(g) : p \mapsto D_{p,Z(p)}\, g$. Note that for any $g \in C^\infty(M)$ and $Z \in \mathfrak{X}(M)$, we have $Z(g) = dg(Z)$, where $dg \in \Omega^1(M) \cong \mathfrak{X}(M)^*$. Using this and axioms 1, 4 and 5, we see that for $Y \in \mathfrak{X}(M)$ and $f \in \mathbb{C}^\infty(M)$,

$$\begin{aligned} XY(f) = X(Y(f)) &= L_X(Y(f)) = L_X(df(Y)) = (L_X df)(Y) + df(L_X Y) \\ &= (d(L_X f))(Y) + (L_X Y)(f) = d(X(f))(Y) + (L_X Y)(f) \\ &= Y(X(f)) + (L_X Y)(f) = YX(f) + (L_X Y)(f). \end{aligned}$$

Hence, $L_X Y = [X, Y]$. In particular, L_X is uniquely given on $\mathfrak{X}(M)$. □

We will need the following definition in the existence proof:

Definition 2.2.3. Given a vector field $X \in \mathfrak{X}(M)$ and a diffeomorphism $f : M \to N$, the *pushforward* of X along f is the vector field $f_* X := df \circ X \circ f^{-1} \in \mathfrak{X}(N)$.

Definition 2.2.4. Given a diffeomorphism $f : N \to M$, define a map $f^* : \mathcal{T}(M) \to \mathcal{T}(N)$, such that f^* is $\mathbb{R}$-linear and$f^*(S \otimes T) = f^*S \otimes f^*T$ for any $S, T \in \mathcal{T}(M)$, and such that f^* agrees with the usual pullback on $C^\infty(M)$ and $\Omega^1(M)$, and $f^*X = f_*^{-1} X$ for $X \in \mathfrak{X}(M)$. This uniquely determines f^*, and we define the *pullback* of a tensor field $S \in \mathcal{T}(M)$ along f as $f^*S \in \mathcal{T}(N)$.

Proposition 2.2.5. Given $X \in \mathfrak{X}(M)$, there exists a type-preserving map $L_X : \mathcal{T}(M) \to \mathcal{T}(M)$ satisfying the axioms of Definition 2.2.1.

Proof. Let $S \in \mathcal{T}^{r,s}(M)$. We will define $L_X S$ locally. Let $q \in M$, and let $U \subseteq M$ be a chart with $q \in U$. Consider a bump function $\lambda : M \to \mathbb{R}$ such that supp$\lambda \subseteq U$ is compact and there is an open $V \subseteq U$ such that $q \in V$ and $\lambda|_V = 1$; then λX is a compactly supported vector field on U, and therefore complete. Let $\Phi : U \times \mathbb{R} \to U$ denote the flow of λX, and define $\varphi_t := \Phi|_{U \times \{t\}} : U \to U$ for all $t \in \mathbb{R}$; $\{\varphi_t\}_{t \in \mathbb{R}}$ is a one-parameter group of diffeomorphisms on U, and $\varphi_0 = \mathrm{id}_U$. The map $(t, p) \mapsto (\varphi_t^* S)_p$ defines a smooth map $\mathbb{R} \times U \to T^{r,s}(M)$.
Define $L_X S \in \mathcal{T}^{r,s}(M)$ by

$$(L_X S)_p := \frac{d}{dt}(\varphi_t^* S)_p \Big|_{t=0}, \quad \text{for } p \in V.$$

Uniqueness of flows implies that $\Phi|_{V \times \mathbb{R}}$ coincides with the flow of X on V, which in turn implies that $L_X S$ is well-defined.
It is enough to show the that the axioms hold locally, so we fix a U, V and $\{\varphi_t\}_{t \in \mathbb{R}}$ as above.
Clearly, L_X is type-preserving and $\mathbb{R}$-linear, as the pullback is $\mathbb{R}$-linear. Note that

$$(L_X f)_p = \frac{d}{dt}(\varphi_t^* f)_p \Big|_{t=0} = \frac{d}{dt} f \circ \varphi_t(p) \Big|_{t=0} = D_{p,X(p)} f = X(f)(p),$$

for all $p \in V$. Hence, $L_X f = X(f)$, so axiom 2 holds.

Let $S, T \in \mathcal{T}(M)$ of type (r,s) respectively (r',s'). Then, using the chain rule and the fact that $-\otimes -\colon T^{r,s}(T_pM) \times T^{r',s'}(T_pM) \to T^{r+r',s+s'}(T_pM)$ is R-bilinear, we have that

$$\begin{aligned} L_X(S\otimes T)_p &= \frac{d}{dt}(\varphi_t^*(S\otimes T))_p\Big|_{t=0} = \frac{d}{dt}(\varphi_t^*S)_p \otimes (\varphi_t^*T)_p\Big|_{t=0} \\ &= \frac{d}{dt}(\varphi_t^*S)_p\Big|_{t=0} \otimes (\varphi_t^*T)_p\Big|_{t=0} + (\varphi_t^*S)_p\Big|_{t=0} \otimes \frac{d}{dt}(\varphi_t^*T)_p\Big|_{t=0} \\ &= (L_XS)_p \otimes T_p + S_p \otimes (L_XT)_p, \qquad \text{for all } p \in V. \end{aligned}$$

So axiom 3, Leibniz' rule, holds.
Let $c\colon \mathcal{T}^{1,1}(M) = \mathfrak{X}(M)\otimes \Omega^1(M) \to \mathcal{T}^{0,0}(M) = C^\infty(M)$ denote the contraction given by $c(Y\otimes\omega) = \omega(Y)$. We claim that c and L_X commute: Indeed, c commutes with the pullback (this is easily checked), and thus, as c is linear

$$L_Xc(Y\otimes\omega) = \frac{d}{dt}\varphi_t^*(c(Y\otimes\omega))\Big|_{t=0} = \frac{d}{dt}c(\varphi_t^*(Y\otimes\omega))\Big|_{t=0} = c\frac{d}{dt}\varphi_t^*(Y\otimes\omega)\Big|_{t=0} = cL_X(Y\otimes\omega).$$

Using that we have already proved axiom 3, it follows that

$$L_X(\omega(Y)) = cL_X(Y\otimes\omega) = c(Y\otimes L_X\omega + L_XY\otimes\omega) = (L_X\omega)(Y) + \omega(L_XY),$$

and thus axiom 4 holds.
To prove axiom 5, we need an intermediary result, namely that $L_XY = [X,Y]$ for any $Y \in \mathfrak{X}(M)$. Recall that $Y(f) = df(Y)$ for $f \in C^\infty(M)$, $Y \in \mathfrak{X}(M)$. Given $Y \in \mathfrak{X}(M)$, note that $d\varphi_t \circ (\varphi_{-t})_*Y = Y\circ\varphi_t$, as $\varphi_t^{-1} = \varphi_{-t}$, and thus for any $f \in C^\infty(M)$

$$((\varphi_{-t})_*Y)(f\circ\varphi_t) = df\circ d\varphi_t\circ(\varphi_{-t})_*Y = df\circ Y\circ\varphi_t = Y(f)\circ\varphi_t.$$

Given $p \in V$, note that the map $\epsilon\colon T_pM \times C^\infty(M) \to C^\infty(M)$, $(v,f) \mapsto D_{p,v}f = Y(f)_p$, $Y \in \mathfrak{X}(M)$ with $Y(p) = v$, is an $\mathbb{R}$-bilinear map of finite dimensional vector spaces. Differentiating both sides of the equation $(Y(f)\circ\varphi_t)_p = (((\varphi_{-t})_*Y)(f\circ\varphi_t))_p$, we get

$$\begin{aligned} X(Y(f))_p &= \frac{d}{dt}(Y(f)\circ\varphi_t)_p\Big|_{t=0} = \frac{d}{dt}(((\varphi_{-t})_*Y)(f\circ\varphi_t))_p\Big|_{t=0} \\ &= \frac{d}{dt}\epsilon((\varphi_{-t})_*Y(p), f\circ\varphi_t)\Big|_{t=0} = \epsilon\left(\frac{d}{dt}(\varphi_{-t})_*Y(p)\Big|_{t=0}, f\right) + \epsilon\left(Y(p), \frac{d}{dt}f\circ\varphi_t\Big|_{t=0}\right) \\ &= \epsilon((L_XY)_p, f) + \epsilon(Y(p), X(f)) = ((L_XY)(f))_p + Y(X(f))_p. \end{aligned}$$

This holds for all $p \in V$; thus $L_XY = [X,Y]$, as desired. Using this and axioms 1 and 4, we see that for $f \in C^\infty(M)$,

$$\begin{aligned} L_X(df)(Y) &= L_X(df(Y)) - df(L_XY) \\ &= X(Y(f)) - [X,Y](f) = Y(X(f)) = Y(L_Xf) = (dL_Xf)(Y). \end{aligned}$$

Thus, as desired, L_X commutes with d on $C^\infty(M)$, axiom 5. □

Definition 2.2.6. Given $X \in \mathfrak{X}(M)$, $L_X\colon \Omega^*(M) \to \Omega^*(M)$ is given by

$$L_X(\omega_1\wedge\cdots\wedge\omega_k) := q(L_X(\omega_1\otimes\cdots\otimes\omega_k)).$$

This is well-defined, as we have a section of q, and it is the unique type-preserving map satisfying:

1. L_X is $\mathbb{R}$-linear.

2. $L_X(f) = X(f)$ for any $f \in \Omega^0(M) = C^\infty(M)$.
3. $L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta)$ for any $\omega, \eta \in \Omega^*(M)$.
4. $L_X(\eta(Y)) = (L_X\eta)(Y) + \eta([X,Y])$ for any $\eta \in \Omega^1(M)$, $Y \in \mathfrak{X}(M)$.
5. L_X commutes with the exterior derivative on $\Omega^0(M) = C^\infty(M)$.

It is this Lie derivative, which we will need.

Remark 2.2.7. The definition given in the above existence proof could also be used to directly define the Lie derivative and is important in understanding the geometric interpretation: It measures the change of a tensor field along the flow of the given vector field X. Another interpretation is the following: In general, an action of a Lie algebra $\mathfrak{g}$ on a manifold M is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ such that the corresponding map $\mathfrak{g} \times M \to M$ is smooth. Such an action induces an action on $\Omega^k(M)$: Let $X \in \mathfrak{g}$, $\omega \in \Omega^*(M)$, and let V_X with flow $\{\varphi_t\}$ denote the vector field corresponding to X under the given action. Then $t \mapsto (\varphi_t^*\omega)_p$ is a smooth curve in $\Lambda(T_pM^*)$, and we define $(X.\omega)_p = \frac{d}{dt}(\varphi_t^*\omega)_p|_{t=0}$. There are of course some details left to prove to see that $X.\omega \in \Omega^*(M)$, but we will not go into this here. The point is that the Lie derivative is simply the action of $\mathfrak{X}(M)$ on $\Omega^*(M)$ induced by the trivial action of $\mathfrak{X}(M)$ on M, $\mathrm{id}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, $L_X(\omega) = X.\omega$. We have opted for the algebraic definition for the sake of clarity and less technicalities.

We go on to prove some properties of the Lie derivative.

Definition 2.2.8. Given $X \in \mathfrak{X}(M)$, define a map $i_X\colon \Omega^*(M) \to \Omega^{*-1}(M)$, called the *interior product* (with respect to X) by $i_X\omega(X_1, \dots, X_k) = \omega(X, X_1, \dots, X_k)$, for any $\omega \in \Omega^{k+1}(M)$, $X_i \in \mathfrak{X}(M)$, $k \geq 1$, and $i_X f = 0$, $f \in \Omega^0(M) = C^\infty(M)$.

Lemma 2.2.9. For $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$, $\eta \in \Omega^m(M)$, we have

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta).$$

Proof. For $X_1, \dots, X_{k+m-1} \in \mathfrak{X}(M)$, we have

$$\begin{aligned} i_X(\omega \wedge \eta)(X_1, \dots, X_{k+m-1}) &= (\omega \wedge \eta)(X, X_1, \dots, X_{k+m-1}) \\ &= \sum_{\sigma \in S_{k-1,m}} \operatorname{sign}(\sigma)\, \omega(X, X_{\sigma(1)}, \dots, X_{\sigma(k-1)}) \eta(X_{\sigma(k)}, \dots, X_{\sigma(k+m-1)}) \\ &+ \sum_{\sigma \in S_{k,m-1}} (-1)^k \operatorname{sign}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \eta(X, X_{\sigma(k+1)}, \dots, X_{\sigma(k+m-1)}) \\ &= ((i_X\omega) \wedge \eta)(X_1, \dots, X_{k+m-1}) + (-1)^k (\omega \wedge (i_X\eta))(X_1, \dots, X_{k+m-1}). \end{aligned}$$

□

Proposition 2.2.10 (Cartan's magic formula). We have the following identity for all $X \in \mathfrak{X}(M)$, $\omega \in \Omega^*(M)$

$$L_X\omega = i_X d\omega + d i_X \omega.$$

Proof. To prove the claim, we shall prove that $i_X d + d i_X\colon \Omega^*(M) \to \Omega^*(M)$ satisfies axioms 1-5 of Definition 2.2.6.

1. That $i_X d + d i_X$ is $\mathbb{R}$-linear is clear.

2. Let $f \in C^\infty(M)$. As $i_X f = 0$, we have
$$(i_X d + d i_X)(f) = i_X df = df(X) = X(f).$$

3. Let $\omega \in \Omega^k(M)$, $\eta \in \Omega^m(M)$. Then using Lemma 2.2.9 and the fact that the exterior derivative also has this relationship with the exterior product, we get:
$$\begin{aligned}(i_X d + d i_X)(\omega \wedge \eta) &= i_X(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) + d(i_X\omega \wedge \eta + (-1)^k \omega \wedge i_X \eta) \\ &= i_X d\omega \wedge \eta + (-1)^{k+1} d\omega \wedge i_X \eta + (-1)^k i_X \omega \wedge d\eta + (-1)^{2k} \omega \wedge i_X d\eta \\ &+ d i_X \omega \wedge \eta + (-1)^{k-1} i_X \omega \wedge d\eta + (-1)^k d\omega \wedge i_X \eta + (-1)^{2k} \omega \wedge d i_X \eta \\ &= (i_X d\omega + d i_X \omega) \wedge \eta + \omega \wedge (i_X d\eta + d i_X \eta).\end{aligned}$$

4. Now it suffices to prove the identity locally, and as we have already proved $\mathbb{R}$-linearity, we can consider $\eta \in \Omega^1(M)$, which satisfies $\eta = f dx$ on some chart U of M and some local coordinate x on U. Let $Y \in \mathfrak{X}(M)$. Then $d\eta = df \wedge dx$, and
$$\begin{aligned}(i_X d + d i_X)(\eta)(Y) + \eta([X,Y]) &= d\eta(X,Y) + d(\eta(X))(Y) + \eta([X,Y]) \\ &= (df \wedge dx)(X,Y) + Y(f dx(X)) + f dx([X,Y]) \\ &= df(X)dx(Y) - df(Y)dx(X) + Y(fX(x)) + f([X,Y](x)) \\ &= X(f)Y(x) - Y(f)X(x) + Y(fX(x)) + fX(Y(x)) - fY(X(x)),\end{aligned}$$
which at $p \in U$ gives
$$\begin{aligned}D_{p,X(p)} f D_{p,Y(p)} x - D_{p,Y(p)} f D_{p,X(p)} x + D_{p,Y(p)}(fX(x)) \\ + f(p) D_{p,X(p)} Y(x) - f(p) D_{p,Y(p)} X(x) \\ = D_{p,X(p)} f D_{p,Y(p)} x + f(p) D_{p,X(p)} Y(x) = D_{p,X(p)}(fY(x)),\end{aligned}$$
which is the value at p of $X(fY(x)) = X(\eta(Y)) = (i_X d + d i_X)(\eta(Y))$ by axiom 2.

5. For any $\omega \in \Omega^*(M)$,
$$(i_X d + d i_X)(d\omega) = i_X d^2 \omega + d i_X d\omega = d i_X d\omega + d^2 i_X \omega = d(i_X d + d i_X)(\omega).$$

□

Remark 2.2.11. Cartan's magic fomula implies that $L_X \colon \Omega^*(M) \to \Omega^*(M)$ is a chain map, and that it is null-homotopic.

LEMMA 2.2.12. For any $X, Y \in \mathfrak{X}(M)$, $i_X i_Y = -i_Y i_X$ and $i_{-X} = -i_X$.

PROOF. This is obvious. □

LEMMA 2.2.13. Given $X, Y \in \mathfrak{X}(M)$, we have the identity $L_X i_Y - i_Y L_X = i_{[X,Y]}$.

PROOF. Direct calculations using Lemma 2.2.9 and axiom 3 of Definition 2.2.6, show that for $\omega \in \Omega^k(M)$, $\eta \in \Omega^m(M)$, we have
$$(L_X i_Y - i_Y L_X)(\omega \wedge \eta) = ((L_X i_Y - i_Y L_X)\omega) \wedge \eta + (-1)^k \omega \wedge ((L_X i_Y - i_Y L_X)\eta).$$

We prove the claim by induction on the degree of the differential form. On $\Omega^0(M)$ the left- and right-hand side are both zero. For $\omega \in \Omega^1(M)$, axiom 4 of Definition 2.2.6 gives us

$$L_X i_Y \omega - i_Y L_X \omega = L_X(\omega(Y)) - (L_X\omega)(Y) = \omega([X,Y]) = i_{[X,Y]}\omega.$$

Now, assume the equality holds on $\Omega^{k-1}(M)$ for some $k > 1$. To show equality on $\Omega^k(M)$, it suffices to show it on an element of the form $\omega \wedge \eta \in \Omega^k(M)$, for $\omega \in \Omega^1(M)$, $\eta \in \Omega^{k-1}(M)$:

$$\begin{aligned}(L_X i_Y - i_Y L_X)(\omega \wedge \eta) &= ((L_X i_Y - i_Y L_X)\omega) \wedge \eta + (-1)^k \omega \wedge ((L_X i_Y - i_Y L_X)\eta) \\ &= (i_{[X,Y]}\omega) \wedge \eta + (-1)^k \omega \wedge (i_{[X,Y]}\eta) = i_{[X,Y]}(\omega \wedge \eta).\end{aligned}$$

□

PROPOSITION 2.2.14 (Invariant formula). For any $\omega \in \Omega^k(M)$, $X_0, \dots, X_k \in \mathfrak{X}(M)$, we have

$$\begin{aligned}d\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k).\end{aligned}$$

PROOF. First we prove that

$$L_{X_0}\omega(X_1, \dots, X_k) = X_0(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \dots, X_{i-1}, [X_0, X_i], X_{i+1}, \dots, X_k). \quad (2.2)$$

We do this by induction on k. For $k = 0$ and $k = 1$, this is axiom 2, respectively, axiom 4 of Definition 2.2.6. Now, assume that it holds for $k-1$ for some $k > 1$, and let $\omega \in \Omega^k(M)$. The induction hypothesis implies that

$$\begin{aligned}X_0(\omega(X_1, \dots, X_k)) &= X_0((i_{X_1}\omega)(X_2, \dots, X_k)) \\ &= (L_{X_0} i_{X_1}\omega)(X_2, \dots, X_k) + \sum_{i=2}^{k} (i_{X_1}\omega)(X_2, \dots, X_{i-1}, [X_0, X_i], X_{i+1}, \dots, X_k).\end{aligned}$$

Now, by Cartan's magic formula and Lemmas 2.2.12 and 2.2.13,

$$\begin{aligned}L_{X_0} i_{X_1} &= i_{X_0} d i_{X_1} + d i_{X_0} i_{X_1} = i_{X_0} d i_{X_1} - d i_{X_1} i_{X_0} = i_{X_0} L_{X_1} - i_{X_0} i_{X_1} d - L_{X_1} i_{X_0} + i_{X_1} d i_{X_0} \\ &= -(L_{X_1} i_{X_0} - i_{X_0} L_{X_1}) + i_{X_1}(i_{X_0} d + d i_{X_0}) = -i_{[X_1, X_0]} + i_{X_1} L_{X_0} = i_{[X_0, X_1]} + i_{X_1} L_{X_0}.\end{aligned}$$

Hence, as desired

$$\begin{aligned}X_0(\omega(X_1, \dots, X_k)) &= i_{[X_0, X_1]}\omega(X_1, \dots, X_k) + (i_{X_1} L_{X_0}\omega)(X_2, \dots, X_k) \\ &\qquad + \sum_{i=2}^{k} (i_{X_1}\omega)(X_2, \dots, X_{i-1}, [X_0, X_i], X_{i+1}, \dots, X_k) \\ &= L_{X_0}\omega(X_1, \dots, X_k) + \sum_{i=1}^{k} \omega(X_1, \dots, X_{i-1}, [X_0, X_i], X_{i+1}, \dots, X_k).\end{aligned}$$

Finally, we prove the invariant formula, also by induction on k. For $k = 0$, the equation reads $df(X) = X(f)$, which is true. Now assume that it holds for $k-1$ for some $k \geq 1$, and let $\omega \in \Omega^k(M)$. Then Cartan's magic formula, the induction hypothesis and equation 2.2 yield

$$\begin{aligned}
d\omega(X_0, \dots, X_k) &= i_{X_0} d\omega(X_1, \dots, X_k) = L_{X_0}\omega(X_1, \dots, X_k) - d i_{X_0}\omega(X_1, \dots, X_k) \\
&= X_0(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k} (-1)^{i-1} \omega([X_0, X_i], X_1, \dots, \widehat{X}_i, \dots, X_k) \\
&- \sum_{i=1}^{k} (-1)^{i+1} X_i(i_{X_0}\omega(X_1, .., \widehat{X}_i, .., X_k)) - \sum_{i<j} (-1)^{i+j} i_{X_0}\omega([X_i, X_j], X_1, .., \widehat{X}_i, .., \widehat{X}_j, .., X_k) \\
&= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k).
\end{aligned}$$

□

We immediately see the motivation for the differential in the Chevalley-Eilenberg complex in this formula. In the case of a Lie group G, its Lie algebra $\mathfrak{g}$ can be interpreted as the set of left-invariant vector fields on G. Then $C^\infty(G)$ is a $\mathfrak{g}$-module, and a differential k-form, $\omega \in \Omega^k(G)$, restricts to an alternating k-linear map $c := \omega|_{\mathfrak{g}^k} \colon \mathfrak{g}^k \to C^\infty(G)$, i.e. an element of $C^k(\mathfrak{g}, C^\infty(G))$. The derivative of c in the Chevalley-Eilenberg complex is, in view of the invariant formula, exactly the exterior derivative of ω restricted to $\mathfrak{g}^k$. This suggests an intricate relationship between the Lie algebra cohomology of $\mathfrak{g}$ and the de Rham cohomology of G, which we will see more explicitly in the following chapter.

3 | Historical Motivation II: De Rham Cohomology

In this chapter, we consider the relationship between Lie algebra cohomology and De Rham cohomology. As mentioned in the preface, the theory of Lie algebra cohomology was developed in an attempt to calculate the De Rham Cohomology of a compact Lie group. Corollary 3.2.6 below shows that the De Rham Cohomology of a compact connected Lie group can be computed purely in terms of its Lie algebra. To this end, we will prove an in itself important theorem about De Rham cohomology: If a compact connected Lie group acts on a manifold, then any element of the De Rham Cohomology of the manifold can be represented by an invariant differential form.

3.1 Lie Group Action

Let G be a Lie group acting smoothly on M, $G \times M \to M$, where we will also denote by g the smooth map $M \to M$ corresponding to $g \in G$ by the above action.

Definition 3.1.1. Let $\Omega^k(M)^G$ denote the set of invariant forms on M, i.e. the forms $\omega \in \Omega^k(M)$ satisfying $g^*\omega = \omega$. Note that $d\omega$ is invariant, if ω is invariant; hence, $\Omega^*(M)^G$ is a chain complex with differential the restriction of the exterior derivative.

Lemma 3.1.2. Let G be a compact Lie group with some fixed volume form ω_{vol} and associated measure μ, let M be a manifold, and $f\colon G \times M \to \mathbb{R}$ be a smooth map. Then the map $F\colon M \to \mathbb{R}$ defined as $F(p) = \int_G f(g,p)\,d\mu$ is smooth and $D_pF(v) = \int_G D_p(f \circ i_g)(v)\,d\mu$, where $i_g\colon M \to G \times M$ is the inclusion $p \mapsto (g,p)$.

Proof. Let $\sigma\colon \mathbb{R}^n \to U$ be an arbitrary chart on M, and let $\{\sigma_\alpha\colon \mathbb{R}^m \to V_\alpha\}$ be a finite open cover of G by charts such that $\sigma_\alpha^*\omega_{\text{vol}} = dx_1 \wedge \cdots \wedge dx_m$ is the standard Euclidean volume form, and let $\{\varphi_\alpha\}$ a smooth partition of unity subordinate to $\{V_\alpha\}$. Then

$$\begin{aligned} F \circ \sigma(x) &= \int_G f(g,\sigma(x))\,d\mu = \sum_\alpha \int_{V_\alpha} \varphi_\alpha(g) f(g,\sigma(x))\,d\mu \\ &= \sum_\alpha \int_{\mathbb{R}^m} \varphi_\alpha \circ \sigma_\alpha(y) f(\sigma_\alpha(y),\sigma(x))\,d\lambda \\ &= \sum_\alpha \int_{\operatorname{supp}\varphi_\alpha \circ \sigma_\alpha} \varphi_\alpha \circ \sigma_\alpha(y) f(\sigma_\alpha(y),\sigma(x))\,d\lambda, \end{aligned}$$

where λ denotes the usual Lebesgue measure on $\mathbb{R}^m$. Now, $(x,y) \mapsto \varphi_\alpha \circ \sigma_\alpha(y) f(\sigma_\alpha(y),\sigma(x))$ is a smooth map $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\operatorname{supp}\varphi_\alpha \circ \sigma_\alpha$ is compact. An application of Lebesgue's Dominated Convergence Theorem (see Corollary A.3.2) implies that $F \circ \sigma$ is smooth with

$$\frac{\partial}{\partial x_j}(F \circ \sigma)(x) = \int_G \frac{\partial}{\partial x_j} f(g,\sigma(x))\,d\mu.$$

It follows that F is smooth with $D_pF(v) = \int_G D_p(f \circ i_g)(v)\,d\mu$. □

Proposition 3.1.3. If M is a manifold and G is a connected compact Lie group acting on M by $\alpha\colon G\times M\to M$, then the inclusion $\iota\colon \Omega^*(M)^G\hookrightarrow\Omega^*(M)$ is a quasi-isomorphism.

Proof. Clearly, ι is a chain map. Suppose G is of dimension m, and let $\omega_{\mathrm{vol}}\in\Omega^m(G)$ be a right-invariant volume form on G satisfying $\int_G\omega_{\mathrm{vol}}=1$ with associated Haar measure μ. Let $\tau\colon G\times M\to G$ denote the projection onto the first coordinate, and $\pi\colon G\times M\to M$ the projection onto the second coordinate. We will for notational reasons denote tangent vectors on $G\times M$ by $v\times w\in T_{(g,p)}(G\times M)=T_gG\times T_pM$ for $v\in T_gG$, $w\in T_pM$. Define a map $\pi_*\colon\Omega^*(G\times M)\to\Omega^{*-m}(M)$ as follows: On $\Omega^k(G\times M)$, $k<m$, π_* is identically zero. For $k\geq m$, let $\omega\in\Omega^k(G\times M)$, $p\in M$, $v_1,\dots,v_{k-m}\in T_pM$ and define

$$(\pi_*\omega)_p(v_1,\dots,v_{k-m})=\int_G\widetilde{\omega}^{p,(v_i)}\ ,$$

where $\widetilde{\omega}^{p,(v_i)}\in\Omega^m(G)$ is given by

$$\widetilde{\omega}^{p,(v_i)}_g(w_1,\dots,w_m)=\omega_{(g,p)}(0\times v_1,\dots,0\times v_{k-m},w_1\times 0,\dots,w_m\times 0).$$

To see that $p\mapsto(\pi_*\omega)_p$ is smooth requires some technical work: Let U be a chart on M with local coordinates x_i, $\{V_\alpha\}$ a finite open cover of G by charts with local coordinates y_i^α, and let $\{\varphi_\alpha\}$ be a smooth partition of unity subordinate to this cover. We can extend $x_1,\dots,x_n,y_1^\alpha,\dots,y_m^\alpha$ to $U\times V_\alpha$ in an obvious way such that they become local coordinates on $U\times V_\alpha$, and therefore write

$$\omega=\sum_{r+s=k}\sum_{\substack{\sigma\in S_{r,n-r}\\ \tau\in S_{s,m-s}}}f^\alpha_{\sigma,\tau}\,dx_\sigma\wedge dy^\alpha_\tau,\quad\text{on } U\times V_\alpha,\ \text{for } f^\alpha_{\sigma,\tau}\in C^\infty(U\times V^\alpha).$$

Given $p\in U$, $v_1,\dots,v_{k-m}\in T_p(M)$, $g\in V_\alpha$, and $w_1,\dots,w_m\in T_gG$, we have

$$\begin{aligned}\widetilde{\omega}^{p,(v_i)}_g(w_1,..,w_m)&=\sum_{r+s=k}\sum_{\substack{\sigma\in S_{r,n-r}\\ \tau\in S_{s,m-s}}}f^\alpha_{\sigma,\tau}(g,p)\,dx_\sigma(p)\wedge dy^\alpha_\tau(g)(0\times v_1,..,0\times v_{k-m},w_1\times 0,..,w_m\times 0)\\ &=\sum_{\sigma\in S_{k-m,n+m-k}}f^\alpha_{\sigma,\mathrm{id}}(g,p)\,dx_\sigma(p)(v_1,\dots,v_{k-m})\,dy^\alpha_{\mathrm{id}}(g)(w_1,\dots,w_m).\end{aligned}$$

Then

$$(\pi_*\omega)_p(v_1,\dots,v_{k-m})=\sum_\alpha\int_{V_\alpha}\varphi_\alpha\widetilde{\omega}^{p,(v_i)}=\sum_{\alpha,\sigma}\left(\int_{V_\alpha}\varphi_\alpha(g)f^\alpha_{\sigma,\mathrm{id}}(g,p)\,dy^\alpha_{\mathrm{id}}\right)dx_\sigma(p)(v_1,\dots,v_{k-m}).$$

Now, for all choices of σ and α, $(g,p)\mapsto\varphi_\alpha(g)f^\alpha_{\sigma,\mathrm{id}}(g,p)$ extends smoothly to $G\times U\to\mathbb{R}$ by setting it to be zero outside of V_α. Then Lemma 3.1.2 implies that

$$f_\sigma:=\sum_\alpha\int_{V_\alpha}\varphi_\alpha(g)f^\alpha_{\sigma,\mathrm{id}}(g,p)\,dy^\alpha_{\mathrm{id}}$$

is a smooth map on U, using that dy^α_{id} is a volume form on G. We conclude that $\pi_*\omega$ is smooth. In addition, Lemma 3.1.2 tells us that

$$D_pf_\sigma(v)=\sum_\alpha\int_{V_\alpha}\varphi_\alpha(g)D_pf^\alpha_{\sigma,\mathrm{id}}(g,p)(v)\,dy^\alpha_{\mathrm{id}}.$$

Using this, we see that π_* commutes with d, as

$$\begin{aligned}(d\pi_*\omega)_p(v_1,\ldots,v_{k-m+1}) &= \sum_\sigma df_\sigma(p)\wedge dx_\sigma(p)(v_1,\ldots,v_{k-m+1})\\ &= \sum_\sigma\sum_{i=1}^{k-m+1}(-1)^{i-1}D_pf_\sigma(v_i)dx_\sigma(p)(v_1,\ldots,\widehat{v}_i,\ldots,v_{k-m+1})\end{aligned}$$

on U, and

$$\begin{aligned}(\widetilde{d\omega}^{p,(v_i)})_g(w_1,..,w_m) &= \sum_{\sigma,\tau} df^\alpha_{\sigma,\tau}(g,p)\wedge dx_\sigma(p)\wedge dy^\alpha_\tau(g)(0\times v_1,..,0\times v_{k-m+1},w_1\times 0,..,w_m\times 0)\\ &= \sum_\sigma\sum_{i=1}^{k-m+1}(-1)^{i-1}D_pf^\alpha_{\sigma,\mathrm{id}}(v_i)\,dx_\sigma(p)(v_1,\ldots,\widehat{v}_i,\ldots,v_{k-m+1})\,dy^\alpha_{\mathrm{id}}(g)(w_1,\ldots,w_m)\end{aligned}$$

on V_α, and thus

$$(\pi_*d\omega)_p(v_1,..,v_{k-m+1}) = \sum_{\sigma,i}(-1)^{i-1}\left(\sum_\alpha\int_{V_\alpha}\varphi_\alpha(g)D_pf^\alpha_{\sigma,\mathrm{id}}(g,p)(v_i)dy^\alpha_{\mathrm{id}}\right)dx_\sigma(p)(v_1,..,\widehat{v}_i,..,v_{k-m+1}).$$

Now, we define a map $I\colon\Omega^k(G\times M)\to\Omega^k(M)$ by $I(\omega)=\pi_*(\omega\wedge\tau^*\omega_{\mathrm{vol}})$. This is a chain map, as ω_{vol} is a top form on G, so $d\omega_{\mathrm{vol}}=0$, and thus

$$\begin{aligned}I(d\omega) &= \pi_*(d\omega\wedge\tau^*\omega_{\mathrm{vol}}) = \pi_*(d(\omega\wedge\tau^*\omega_{\mathrm{vol}})-(-1)^k\omega\wedge d\tau^*\omega_{\mathrm{vol}})\\ &= \pi_*(d(\omega\wedge\tau^*\omega_{\mathrm{vol}})-(-1)^k\omega\wedge\tau^*d\omega_{\mathrm{vol}}) = \pi_*(d(\omega\wedge\tau^*\omega_{\mathrm{vol}})) = d\pi_*(\omega\wedge\tau^*\omega_{\mathrm{vol}}) = dI(\omega).\end{aligned}$$

To ease the calculations to come, let $p\in M$, $v_1,\ldots,v_k\in T_pM$, $g\in G$, $w_1,\ldots,w_m\in T_gG$, and set $\beta:=(\omega\wedge\tau^*\omega_{\mathrm{vol}})^{\sim p,(v_i)}\in\Omega^m(G)$, $t_i:=0\times v_i$ for $i=1,\ldots k$, and $t_i:=w_{i-k}\times 0$ for $i=k+1,\ldots,k+m$. Note that $D_{(g,p)}\tau\colon T_gG\times T_pM\to T_gG$ is the projection onto the first coordinate. Then

$$\begin{aligned}\beta_g(w_1,\ldots,w_m) &= (\omega\wedge\tau^*\omega_{\mathrm{vol}})_{(g,p)}(t_1,\ldots,t_{k+m})\\ &= \sum_{\sigma\in S_{k,m}}\operatorname{sign}\sigma\,\omega_{(g,p)}(t_{\sigma(1)},\ldots,t_{\sigma(k)})\tau^*\omega_{\mathrm{vol}}(t_{\sigma(k+1)},\ldots,t_{\sigma(k+m)})\\ &= \omega_{(g,p)}(0\times v_1,\ldots,0\times v_k)(\omega_{\mathrm{vol}})_g(w_1,\ldots,w_m)\end{aligned}$$

so $\beta_g=\omega_{(g,p)}(0\times v_1,\ldots,0\times v_k)(\omega_{\mathrm{vol}})_g$, and thus

$$I(\omega)_p(v_1,\ldots,v_k)=\int_G\omega_{g,p}(0\times v_1,\ldots,0\times v_k)\,d\mu.$$

Aside: We have not defined I like this directly, as we will use the actual definition in some of the calculations below. Moreover, the construction above can be generalised: the map π_* is a fibre integral, which can be defined for any smooth vector bundle $E\to M$ with compact oriented fibres. However, as we are only interested in the trivial bundle $\pi\colon G\times M\to M$, it seems unnecessary to spend much time on it.

Define a chain map $\rho\colon\Omega^*(M)\to\Omega^*(M)^G$ as $\rho:=I\circ\alpha^*$. Note that ρ is explicitly given as "averaging the differential forms over G": For $\omega\in\Omega^*(M)$, $p\in M$, $v_1,\ldots,v_k\in T_pM$,

$$(\alpha^*\omega)_{(g,p)}(0\times v_1,\ldots,0\times v_k)=\omega_{g(p)}(D_pg(v_1),\ldots,D_pg(v_k))=(g^*\omega)_p(v_1,\ldots,v_k),$$

where we use that $D_{(g,p)}\alpha\colon T_gG\times T_pM \to T_gM$ is given by $D_{(g,p)}\alpha(v\times w) = D_g\mathrm{ev}_p(v)+D_pg(w)$, with $\mathrm{ev}_p\colon G\to M$ evaluation at p. Hence,

$$\begin{aligned}\rho(\omega)_p(v_1,\dots,v_k) &= I(\alpha^*\omega)_p(v_1,\dots,v_k)\\ &= \int_G(\alpha^*\omega)_{g,p}(0\times v_1,\dots,0\times v_k)\,d\mu = \int_G(g^*\omega)_p(v_1,\dots,v_k)\,d\mu.\end{aligned}$$

We also write $\rho(\omega) = \int_G g^*\omega\,d\mu$. To show that ρ is well-defined, we must show that $\rho(\omega)$ is G-invariant for any $\omega\in\Omega^*(M)$: For any $h\in G$, the above explicit expression of ρ and right-invariance of μ gives

$$h^*\rho(\omega) = h^*\left(\int_G g^*\omega\,d\mu\right) = \int_G h^*g^*\omega\,d\mu = \int_G(gh)^*\omega\,d\mu = \int_G g^*\omega\,d\mu = \rho(\omega).$$

We will show that on cohomology, $H(\rho) = H(\iota)^{-1}$. It is seen directly that $\rho\circ\iota = \mathrm{id}$. To show that $H(\iota)\circ H(\rho) = \mathrm{id}$ requires some more work.
We claim that in our definition of π_*, we may change the domain of integration to a neighbourhood U of $1\in G$ and obtain a map I_U, which is homotopic to I. To see this, let $U\subseteq G$ be a neighbourhood of 1 and let $\lambda\colon G\to\mathbb{R}$ be a bump function with support contained in U such that $\int_G\lambda\omega_{\mathrm{vol}} = 1$. Let $I_U\colon\Omega^k(G\times M)\to M$ denote the map arising from the above construction, replacing ω_{vol} by $\lambda\omega_{\mathrm{vol}}$. As ω_{vol} and $\lambda\omega_{\mathrm{vol}}$ integrate to the same, they differ by an exact form: Indeed, $\int_G\colon H^m(G)\to\mathbb{R}$ is an isomorphism as G is compact and connected (cf. Theorem 10.13 [8]). Let $\eta\in\Omega^{m-1}(G)$ such that $\omega_{\mathrm{vol}}-\lambda\omega_{\mathrm{vol}} = d\eta$, and consider the map

$$h\colon\Omega^*(G\times M)\to\Omega^{*-1}(M),\qquad h(\omega) = (-1)^k\pi_*(\omega\wedge\tau^*\eta),\quad\text{for }\omega\in\Omega^k(G\times M).$$

h is a chain homotopy from I to I_U: For any $\omega\in\Omega^k(G\times M)$,

$$\begin{aligned}h(d\omega)+dh(\omega) &= (-1)^{k+1}\pi_*(d\omega\wedge\tau^*\eta)+(-1)^kd\pi_*(\omega\wedge\tau^*\eta)\\ &= (-1)^{k+1}\Big(\pi_*(d(\omega\wedge\tau^*\eta)-(-1)^k\omega\wedge\tau^*d\eta)\Big)+(-1)^kd\pi_*(\omega\wedge\tau^*\eta)\\ &= \pi_*(\omega\wedge\tau^*d\eta) = \pi_*(\omega\wedge\tau^*(\omega_{\mathrm{vol}}-\lambda\omega_{\mathrm{vol}})) = I(\omega)-I_U(\omega).\end{aligned}$$

The advantage of this is that we can restrict I_U to $\Omega^k(U\times M)$, so to speak: Let $i\colon U\times M\to G\times M$ denote the inclusion, and define

$$\widetilde{I}_U\colon\Omega^k(U\times M)\to\Omega^k(M)\quad\text{as }\widetilde{I}_U(\omega)_p(v_1,\dots,v_k) = \int_G\lambda\omega_{(g,p)}(0\times v_1,\dots,0\times v_k)\,d\mu,$$

where λ is the bump function from above, and we extend ω (not necessarily continuously) to $G\times M$ by defining it to be zero outside of $U\times M$. Then $I_U = \widetilde{I}_U\circ i^*$. Moreover, $\widetilde{I}_U\circ\pi|_U^* = \mathrm{id}$, where $\pi|U\colon U\times M\to M$ is the restriction of π: Indeed, for any $\omega\in\Omega^k(U\times M)$, $p\in M$, $v_1,\dots,v_k\in T_pM$,

$$\begin{aligned}\widetilde{I}_U\circ\pi|_U^*(\omega)_p(v_1,\dots,v_k) &= \int_G\lambda(\pi|_U^*\omega)_{(g,p)}(0\times v_1,\dots,0\times v_k)\,d\mu\\ &= \int_G\lambda\omega_p(v_1,\dots,v_k)\,d\mu = \omega_p(v_1,\dots,v_k)\int_G\lambda d\mu = \omega_p(v_1,\dots,v_k).\end{aligned}$$

If we take U to be a contractible neighbourhood of 1, and let $j\colon M\to U\times M$ denote the inclusion $p\mapsto(1,p)$, then the composite $j\circ\pi|_U$ is homotopic to the identity on $U\times M$. Combining the above results, we see that

$$\begin{aligned}H(\iota)\circ H(\rho) &= H(I)\circ H(\alpha^*) = H(I_U)\circ H(\alpha^*) = H(\widetilde{I}_U)\circ H(i^*)\circ H(\alpha^*)\\ &= H(\widetilde{I}_U)\circ H(\pi|_U^*)\circ H(j^*)\circ H(i^*)\circ H(\alpha^*) = H((\alpha\circ i\circ j)^*) = H(\mathrm{id}) = \mathrm{id}.\end{aligned}$$

Thus, we finally conclude that $H(\rho) = H(\iota)^{-1}$. In other words, $\iota\colon \Omega^*(M)^G \to \Omega^*(M)$ is a quasi-isomorphism. □

3.2 Lie Groups and Lie Algebras

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let V be a vector space, and $\pi\colon G \to \mathrm{Aut}(V)$ a representation of G with derivative $\rho = D_1\pi\colon \mathfrak{g} \to \mathrm{End}(V)$. Recall that $x = \frac{d}{dt}\big|_{t=0} \exp(tx)$ for all $x \in \mathfrak{g}$, where $\exp\colon \mathfrak{g} = T_1G \to G$ is the exponential map, defined as $x \mapsto \gamma_x(1)$, where γ_x is the maximal integral curve of the left-invariant vector field defined by x with $\gamma_x(0) = 1$. Then by the chain law,

$$\rho(x) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tx)).$$

Definition 3.2.1. A vector $v \in V$ is G-invariant, if $\pi(g)(v) = v$ for all $g \in G$, and we denote by V^G the subspace of G-invariant elements. Likewise, $v \in V$ is $\mathfrak{g}$-invariant, if $\rho(x)(v) = 0$ for all $x \in \mathfrak{g}$, and we denote by $V^{\mathfrak{g}}$ the subspace of $\mathfrak{g}$-invariant elements.

Proposition 3.2.2. In the above situation, $V^G = V^{\mathfrak{g}}$.

Proof. Suppose $v \in V^G$. Then for any $x \in \mathfrak{g}$,

$$\rho(x)(v) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tx))(v) = \frac{d}{dt}\Big|_{t=0} v = 0,$$

so $v \in V^{\mathfrak{g}}$. Assume conversely that $v \in V^{\mathfrak{g}}$. Then for all $x \in \mathfrak{g}$,

$$D_1(\mathrm{ev}_v \circ \pi)(x) = \mathrm{ev}_v \circ D_1\pi(x) = \rho(x)(v) = 0.$$

As G is connected, this implies that $\mathrm{ev}_v \circ \pi$ is constant, and as $\mathrm{ev}_v \circ \pi(1) = v$, we conclude that $\pi(g)(v) = v$ for all $g \in G$. □

G acts on itself by left multiplication; let $\lambda_g \in \mathrm{Aut}(G)$ denote left multiplication by g.

Proposition 3.2.3. Evaluation at $1 \in G$, $\epsilon\colon \Omega^k(G)^G \to C^k(\mathfrak{g}, \mathbb{R})$, $\omega \mapsto \omega_1$ defines an isomorphism of chain complexes.

Remark 3.2.4. Here we identify $\mathfrak{g}$ with the tangent space at 1, T_1G, so

$$\epsilon(\omega) = \omega_1 \in \mathrm{Hom}_{\mathbb{R}}(\Lambda^k(T_1G), \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\Lambda^k\mathfrak{g}, \mathbb{R}) = C^k(\mathfrak{g}, \mathbb{R}).$$

Proof. Given $\omega \in \Omega^k(G)^G$, and $v_0, \ldots, v_k \in T_1G$, let X_i denote the left-invariant vector field on G with $X_i(1) = v_i$. Then

$$\begin{aligned}
(d\omega)_1(v_0, \ldots, v_k) &= d\omega(X_0, \ldots, X_k)(1) \\
&= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k))(1) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k)(1).
\end{aligned}$$

As ω and all the X_i are left-invariant, the function $\omega(X_0, \ldots, X_k)$ is left-invariant, and therefore constant. Hence, $\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k))(1) = 0$. As $\mathfrak{g}$ acts trivially on $\mathbb{R}$, we then directly see that $\epsilon d\omega = d\epsilon\omega$ for all $\omega \in \Omega^*(G)^G$, so ϵ is a chain map. We claim that invariant forms are completely determined by their value at 1: Let $\omega \in \Omega^k(G)^G$, $g \in G$ and $v_1, \dots, v_k \in T_gG$. Then

$$\omega_g(v_1, \dots, v_k) = (\lambda_{g^{-1}}^*\omega)_g(v_1, \dots, v_k) = \omega_1(D_g\lambda_{g^{-1}}(v_1), \dots, D_g\lambda_{g^{-1}}(v_k)).$$

It follows that ϵ is injective, and that $c \in C^k(\mathfrak{g}, \mathbb{R})$ defines a differential form $\omega \in \Omega^k(G)^G$ by setting

$$\omega_g(v_1, \dots, v_k) = c(D_g\lambda_{g^{-1}}(v_1), \dots, D_g\lambda_{g^{-1}}(v_k)), \quad \text{for all } g \in G,\ v_i \in T_gG,$$

so ϵ is also surjective. $\square$

G also acts on itself by right-multiplication: let $\mu_g \in \mathrm{Aut}(G)$ denote multiplication by g^{-1} on the right. Then $G \times G$ acts on G by left and right multiplication: $(g, h).x = gxh^{-1} = \mu_h\lambda_g x$. If $g = h$, this is conjugation by g, and we write $c_g = \mu_g\lambda_g$. G acts on $\mathfrak{g}$ by the adjoint action:

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{Ad}(g)(x) = T_1c_g(x).$$

This can be extended to an action on $\Lambda^k\mathfrak{g}$, which we also denote by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\Lambda^k\mathfrak{g})$,

$$\mathrm{Ad}(g)(x_1 \wedge \cdots \wedge x_k) := \mathrm{Ad}(g)(x_1) \wedge \cdots \wedge \mathrm{Ad}(g)(x_k),$$

which in turn dualises to an action on $C^k(\mathfrak{g}, \mathbb{R})$, $\pi\colon G \to \mathrm{Aut}(C^k(\mathfrak{g}, \mathbb{R}))$,

$$g \mapsto \mathrm{Ad}(g)^*, \quad \mathrm{Ad}(g)^*c(x_1, \cdots, x_k) := c(\mathrm{Ad}(g^{-1})(x_1), \dots, \mathrm{Ad}(g^{-1})(x_k)).$$

Note that if c is G-invariant with respect to the representation π, then so is dc:

$$\begin{aligned} dc(\mathrm{Ad}(g)(x_1), \dots, \mathrm{Ad}(g)(x_k)) &= \sum_{i<j}(-1)^{i+j}c([\mathrm{Ad}(g)(x_i), \mathrm{Ad}(g)(x_j)], \mathrm{Ad}(g)(x_0), \dots, \mathrm{Ad}(g)(x_k)) \\ &= \sum_{i<j}(-1)^{i+j}c(\mathrm{Ad}(g)[x_i, x_j], \mathrm{Ad}(g)(x_0), \dots, \mathrm{Ad}(g)(x_k)) \\ &= \sum_{i<j}(-1)^{i+j}c([x_i, x_j], x_0, \dots, \mathrm{Ad}(g)(x_k)) = dc(x_1, \dots, x_k). \end{aligned}$$

We denote by $(C^*(\mathfrak{g}, \mathbb{R})^G, d)$ the chain complex of G-invariant elements.
We also have the adjoint action of $\mathfrak{g}$ on itself, $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, which we can extend to an action of $\mathfrak{g}$ on $\Lambda^k(\mathfrak{g})$ by $\mathrm{ad}(x)(x_1 \wedge \cdots \wedge x_k) = \sum_{i=1}^k x_1 \wedge \cdots \wedge [x, x_i] \wedge \cdots \wedge x_k$. Again this dualises to an action on $C^k(\mathfrak{g}, \mathbb{R})$,

$$\mathrm{ad}(x)^*c(x_1, \dots, x_k) = \sum_{i=1}^k c(x_1, \dots, -[x, x_i], \dots, x_k) = \sum_{i=1}^k c(x_1, \dots, [x_i, x], \dots, x_k).$$

This is a representation $\rho\colon \mathfrak{g} \to \mathrm{End}(C^k(\mathfrak{g}, \mathbb{R}))$. Note that if c is $\mathfrak{g}$-invariant with respect to ρ, then so is dc (the omitted elements in the sums will be implicit in the following calculation):

$$\begin{aligned}\sum_{l=1}^{k} dc(x_1,\ldots,[x,x_l],\ldots,x_k) = \sum_{l=1}^{k}\Big(&\sum_{\substack{i<j\\ i,j\neq l}}(-1)^{i+j}c([x_i,x_j],x_1,\ldots,[x,x_l],\ldots,x_k)\\ &+\sum_{i<l}(-1)^{i+l}c([x_i,[x,x_l]],x_1,\ldots,x_k)\\ &+\sum_{l<j}(-1)^{l+j}c([[x,x_l],x_j],x_1,\ldots,x_k)\Big)\end{aligned}$$

$$\begin{aligned}&=\sum_{l=1}^{k}\Big(\sum_{\substack{i<j\\ i,j\neq l}}(-1)^{i+j}c([x_i,x_j],x_1,\ldots,[x,x_l],\ldots,x_k)+\sum_{i<j}(-1)^{i+j}c([x,[x_i,x_j]],x_1,\ldots,x_k)\Big)\\ &=\sum_{i<j}(-1)^{i+j}\mathrm{ad}(x)^*c([x_i,x_j],x_1,\ldots,x_k)=0.\end{aligned}$$

We denote by $(C^*(\mathfrak{g},\mathbb{R})^{\mathfrak{g}},d)$ the chain complex of $\mathfrak{g}$-invariant elements. We see that $\rho = D_1\pi$, as

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\pi(\exp(tx))(c)(x_1,\ldots,x_k) &= \frac{d}{dt}\Big|_{t=0}c(\mathrm{Ad}(\exp(-tx))(x_1),\ldots,\mathrm{Ad}(\exp(-tx))(x_k))\\ &=\sum_{i=1}^{k}c(x_1,\ldots,\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp(-tx))(x_1),\ldots,x_k)=\sum_{i=1}^{k}c(x_1,\ldots,[-x,x_i],\ldots,x_k)\end{aligned}$$

for all $c\in C^k(\mathfrak{g},\mathbb{R})$, $x,x_1,\ldots,x_k\in\mathfrak{g}$, using multilinearity of c. A direct consequence of Proposition 3.2.2 is then that $(C^*(\mathfrak{g},\mathbb{R})^G,d)=(C^*(\mathfrak{g},\mathbb{R})^{\mathfrak{g}},d)$.

Proposition 3.2.5. Evaluation at $1\in G$, $\kappa\colon \Omega^k(G)^{G\times G}\to C^k(\mathfrak{g},\mathbb{R})^G$, $\omega\mapsto\omega_1$ defines an isomorphism of chain complexes.

Proof. Let $\omega\in\Omega^k(G)^{G\times G}$; we must show that ω_1 is G-invariant. Since $c_g^*\omega=\omega$ for all g, we have

$$\omega_1(\mathrm{Ad}(g^{-1})(x_1),\ldots,\mathrm{Ad}(g^{-1})(x_k))=(c_{g^{-1}}^*\omega)_1(x_1,\ldots,x_k)=\omega_1(x_1,\ldots,x_k)$$

for all $g\in G$, $x_1,\ldots,x_k\in\mathfrak{g}$, where we use that $\mathrm{Ad}(g)=D_1c_g$.
It follows by the same arguments as in the proof of Proposition 3.2.3, that κ commutes with the exterior derivative. From that same proof we also deduce that κ is injective, as a $G\times G$-invariant differential form will in particular be left-invariant. So it only remains to show surjectivity of κ: Given $c\in C^k(\mathfrak{g},\mathbb{R})^G$, define $\omega\in\Omega^k(G)^G$ as in the proof of Proposition 3.2.3; that is,

$$\omega_g(v_1,\ldots,v_k)=c(D_g\lambda_{g^{-1}}(v_1),\ldots,D_g\lambda_{g^{-1}}(v_k)),\quad\text{for all } g\in G,\ v_i\in T_gG.$$

Now, let $x,y,g\in G$, and note that $\lambda_{yg^{-1}x^{-1}}\mu_y\lambda_x=c_y\lambda_{g^{-1}}$. Then for all $v_1,\ldots,v_k\in T_gG$,

$$\begin{aligned}((\mu_y\lambda_x)^*\omega)_g(v_1,\ldots,v_k)&=\omega_{xgy^{-1}}(D_g(\mu_y\lambda_x)(v_1),\ldots,D_g(\mu_y\lambda_x)(v_k))\\ &=c(D_{xgy^{-1}}\lambda_{yg^{-1}x^{-1}}\circ D_g(\mu_y\lambda_x)(v_1),\ldots,D_{xgy^{-1}}\lambda_{yg^{-1}x^{-1}}\circ D_g(\mu_y\lambda_x)(v_k))\\ &=c(D_g(c_y\lambda_{g^{-1}})(v_1),\ldots,D_g(c_y\lambda_{g^{-1}})(v_k))\\ &=c(\mathrm{Ad}(y)(D_g\lambda_{g^{-1}}(v_1)),\ldots,\mathrm{Ad}(y)(D_g\lambda_{g^{-1}}(v_k)))\\ &=c(D_g\lambda_{g^{-1}}(v_1),\ldots,D_g\lambda_{g^{-1}}(v_k))=\omega_g(v_1,\ldots,v_k).\end{aligned}$$

We conclude that $\omega \in \Omega^k(G)^{G\times G}$ with $\kappa(\omega) = c$. □

The following result exploits that G can act on itself in different ways, namely by left-multiplication and by left- *and* right-multiplication.

COROLLARY 3.2.6. If G is a compact connected Lie group with Lie algebra $\mathfrak{g}$, then

$$H^\bullet(\mathfrak{g}, \mathbb{R}) \cong H^\bullet_{dR}(G) \cong ((\Lambda^\bullet \mathfrak{g})^*)^{\mathfrak{g}},$$

where $\mathfrak{g}$ acts trivially on $\mathbb{R}$, and $((\Lambda^\bullet \mathfrak{g})^*)^{\mathfrak{g}} = \bigoplus_{k\in\mathbb{N}}((\Lambda^k \mathfrak{g})^*)^{\mathfrak{g}}$

PROOF. The first isomorphism follows directly from Propositions 3.1.3 and 3.2.3. Now, using Proposition 3.1.3 again, but this time combining with Proposition 3.2.5, we get that $H^\bullet_{dR}(G)$ is isomorphic to the homology of the chain complex

$$(C^\bullet(\mathfrak{g}, \mathbb{R})^G, d) = (C^\bullet(\mathfrak{g}, \mathbb{R})^{\mathfrak{g}}, d) = (((\Lambda^\bullet \mathfrak{g})^*)^{\mathfrak{g}}, d).$$

For any $c \in C^k(\mathfrak{g}, \mathbb{R})^{\mathfrak{g}}$, and $x_1, \ldots, x_{k+1} \in \mathfrak{g}$, we see that since $\mathfrak{g}$ acts trivially on $\mathbb{R}$ and c is $\mathfrak{g}$-invariant, we have

$$\begin{aligned}
2dc(x_1, \ldots, x_{k+1}) &= 2\sum_{i=1}^{k+1}(-1)^{i+1}x_i(c(x_1, \ldots, \hat{x}_i, \ldots, x_k) \\
&\quad + 2\sum_{i<j}(-1)^{i+j}c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k) \\
&= \sum_{i<j}(-1)^{i+j}c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k) \\
&\quad + \sum_{i<j}(-1)^{i+j-1}c([x_j, x_i], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k) \\
&= \sum_{i<j}(-1)^{j}c(x_1, \ldots, [x_i, x_j], \ldots, \hat{x}_j, \ldots, x_k) \\
&\quad + \sum_{j<i}(-1)^{j}c(x_1, \ldots, \hat{x}_j, \ldots, [x_i, x_j], \ldots, x_k) \\
&= \sum_{j=1}^{k+1}(-1)^{j}\mathrm{ad}(x_j)^*c(x_1, \ldots, \hat{x}_j, \ldots, x_k) = 0
\end{aligned}$$

Hence, the differentials in the complex $(C^*(\mathfrak{g}, \mathbb{R})^{\mathfrak{g}}, d)$ are identically zero, and the homology is equal to the chain complex itself. □

Remark 3.2.7. The above corollary reduces the computation of the De Rham cohomology of a compact connected Lie group to pure linear algebra.

4 Examples

In this chapter, we shall look at some examples. As we have not developed many calculational tools in this text, we are restricted to considering quite simple examples. One can exploit the structures on Lie algebras such as root systems, gradings and the like to compute the Lie algebra cohomology, but this is unfortunately outside the scope of this text.

Example 4.1.8. We begin with a very easy example: If $\mathfrak{g}$ is an abelian Lie algebra over a field K, then $H^*(\mathfrak{g}, K) = (\Lambda\mathfrak{g})^*$, where K is the trivial module. Indeed, as $\mathfrak{g}$ acts trivially on K, and the Lie bracket is trivial, the derivative in the Chevalley-Eilenberg complex is always zero, so the k'th cohomology group is just the k'th term of the complex. In particular, $H^k(\mathfrak{g}, K)$ is a K-vector space of dimension $\binom{n}{k}$, where n is the dimension of $\mathfrak{g}$.

Example 4.1.9. We will use Proposition 3.1.3 to compute the de Rham cohomology of S^2. This example may not give us a lot of new knowledge, but it illustrates very well how using invariant differential forms can simplify our lives immensely. Consider S^2 embedded in $\mathbb{R}^3$. Then

$$SO(3) = \{A \in M_3(\mathbb{R}) \mid A^tA = \mathrm{id}, \det A = 1\}$$

acts on S^2 by rotations. $SO(3)$ is a compact connected Lie group, so $\Omega^*(S^2)^{SO(3)} \hookrightarrow \Omega^*(S^2)$ is a quasi-isomorphism. As S^2 is of dimension 2, $H^n_{dR}(S^2) = 0$ for all $n > 2$. We compute $\Omega^n(S^2)^{SO(3)}$ for $n = 0, 1, 2$. An $f \in C^\infty(S^2)$ is invariant under rotations, if and only if it is constant; thus $\Omega^0(S^2)^{SO(3)} = \{M \to \mathbb{R} \text{ constant}\} \cong \mathbb{R}$. Now, take $\omega \in \Omega^1(S^2)^{SO(3)}$. As ω is invariant under rotation, we must have $\omega = 0$, as $\omega_a(v) = \omega_a(-v)$ for all $a \in S^2$, $v \in T_aS^2$ — this can be seen by letting L denote the "equator" with respect to which a is the north pole, and then rotating S^2 by π along L. We conclude that $\Omega^1(S^2) = 0$.
Now, let $\omega \in \Omega^2(S^2)^{SO(3)}$. Again, as ω is alternating, bilinear and invariant under rotation, we see that ω is completely determined by its value on orthonormal bases of the tangent spaces; that is, $\omega_a(v, v') = \omega_b(w, w')$ for any $a, b \in S^2$, and $v, v' \in T_aS^2$, $w, w' \in T_bS^2$ orthonormal bases of their respective tangent spaces. It is clear that setting $\omega_a(v, v') = r$ for some fixed $r \in \mathbb{R}$, $a \in S^2$ and $v, v' \in T_aS^2$ an orthonormal basis, defines an $SO(3)$-invariant differential form on S^2. Hence, $\Omega^2(S^2)^{SO(3)} \cong \mathbb{R}$.
Thus the De Rham complex looks like

$$\mathbb{R} \to 0 \to \mathbb{R} \to 0 \to 0 \to \cdots,$$

with zero differentials. We recover the well-known fact that

$$H^*_{dR}(S^2) \cong \begin{cases} \mathbb{R} & * = 0, 2 \\ 0 & \text{else} \end{cases}.$$

However, we also see that the rotation invariant differential forms uniquely represent the cohomology classes.

Example 4.1.10. We will now compute the De Rham cohomology of $SO(3)$ and the Lie algebra cohomology of its Lie algebra $\mathfrak{so}(3) = \{X \in M_3(\mathbb{R}) \mid X^t = -X\}$. Corollary 3.2.6 states that

$$H^\bullet_{dR}(SO(3)) \cong ((\Lambda^\bullet\mathfrak{so}(3))^*)^{\mathfrak{so}(3)}.$$

Note that $\mathfrak{so}(3)$ is spanned as an $\mathbb{R}$-vector space by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and note that $A = [C, B]$, $B = [A, C]$ and $C = [B, A]$. This implies that

$$H^1(SO(3)) \cong (\mathfrak{so}(3)^*)^{\mathfrak{so}(3)} = 0,$$

as any $\mathfrak{so}(3)$-invariant linear map $c\colon \mathfrak{so}(3) \to \mathbb{R}$ must satisfy

$$c(A) = c([C, B]) = \mathrm{ad}(B)^* c(C) = 0, \quad \text{and similarly } c(B) = c(C) = 0.$$

We also get that $H^2(SO(3)) \cong ((\Lambda^2\mathfrak{so}(3))^*)^{\mathfrak{so}(3)} = 0$: Indeed, let $c\colon \mathfrak{so}(3)^2 \to \mathbb{R}$ be a bilinear, alternating, $\mathfrak{so}(3)$-invariant map. Then

$$c(A, B) = c([C, B], B) = \mathrm{ad}(B)^* c(C, B) - c(C, [B, B]) = 0,$$

and analogously $c(A, C) = c(B, C) = 0$. Thus, $c = 0$.
Any 3-linear, alternating map $c\colon \mathfrak{so}(3)^3 \to \mathbb{R}$ is completely determined by its value on (A, B, C). Moreover, we see that for any $X = r_1A + r_2B + r_3C \in \mathfrak{so}(3)$, $r_i \in \mathbb{R}$,

$$[A, X] = -r_2C + r_3B, \quad [B, X] = r_1C - r_3A, \quad [C, X] = -r_1B + r_2A,$$

and thus

$$\mathrm{ad}(X)^* c(A, B, C) = c([A, X], B, C) + c(A, [B, X], C) + c(A, B, [C, X]) = 0.$$

Hence, $H^3_{dR}(SO(3)) \cong ((\Lambda^3\mathfrak{so}(3))^*)^{\mathfrak{so}(3)} = ((\Lambda^3\mathfrak{so}(3))^*) \cong \mathbb{R}$.
We have shown the following

$$H^*_{dR}(SO(3)) \cong H^*(\mathfrak{so}(3)) \cong \begin{cases} \mathbb{R} & * = 0, 3 \\ 0 & \text{else} \end{cases}$$

Example 4.1.11. We can show that $H^1(SO(n)) = 0$ for all n in that same way that we proved $H^1(SO(3)) = 0$ above. Indeed, $\mathfrak{so}(n)$ has $\mathbb{R}$-basis $\{E_{ij}\}_{1 \le i < j \le n}$, where $E_{ij} = (e_{lk})$ is the matrix with entries

$$e_{lk} = \begin{cases} 1 & l = i, k = j \\ -1 & l = j, k = i \\ 0 & \text{else} \end{cases}.$$

For any $1 \le i < j \le n$, we can write

$$E_{ij} = \begin{cases} [E_{1j}, E_{1i}] & \text{if } 1 < i \\ [E_{jn}, E_{in}] & \text{if } j < n \\ [E_{12}, E_{2n}] & \text{if } i = 1, j = n \end{cases}.$$

It follows that any $\mathfrak{so}(n)$-invariant linear map $\mathfrak{so}(n) \to \mathbb{R}$ is identically zero, and thus

$$H^1_{dR}(SO(n)) \cong H^1(\mathfrak{so}(n)) \cong (\mathfrak{so}(n)^*)^{\mathfrak{so}(n)} = 0.$$

Example 4.1.12 (Homogeneous Spaces). Let G be a compact connected Lie group and $H \subseteq G$ a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ denote the Lie algebras of G, respectively H. G acts by multiplication on the quotient G/H, and $\mathfrak{g}$ acts on $\mathfrak{g}/\mathfrak{h}$, the Lie algebra of G/H, by the adjoint action. This gives rise to action of G and $\mathfrak{g}$ on $C^*(\mathfrak{g}/\mathfrak{h}, \mathbb{R})$ as discussed in Chapter 3. One can show results analogous to Propositions 3.2.3 and 3.2.5, yielding chain complex isomorphisms

$$\Omega^*(G/H)^G \cong C^*(\mathfrak{g}/\mathfrak{h}, \mathbb{R}), \qquad \Omega^*(G/H)^{G\times G} \cong C^*(\mathfrak{g}(\mathfrak{h}, \mathbb{R})^{\mathfrak{g}},$$

and thus

$$H^\bullet_{dR}(G/H) \cong H^\bullet(\mathfrak{g}/\mathfrak{h}) \cong ((\Lambda^\bullet \mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{g}}.$$

A | Appendix

A.1 Chevalley-Eilenberg Derivative

Let K be a field, $\mathfrak{g}$ a Lie algebra over K, and Γ a $\mathfrak{g}$-module. Consider the Chevalley-Eilenberg complex $(C^*(\mathfrak{g}, \Gamma), d)$ as defined in Chapter 1.

Proposition A.1.1. $d^2 = 0$

Proof. Let $c \in C^k(\mathfrak{g}, \Gamma)$, $x_1, \ldots, x_k \in \mathfrak{g}$. For simplicity and space considerations, we write $(x_1, \hat{x}_j, x_k) := (x_1, \ldots, \hat{x}_j, \ldots, x_k)$. Then in the calculations on the following page, we see that

- Lines (2) and (3) cancel out line (7), as $[x_i, x_r] = x_i x_r - x_r x_i$ in $\mathrm{End}(\Gamma)$.
- Lines (4) and (10) cancel out.
- Lines (5) and (9) cancel out.
- Lines (6) and (8) cancel out.
- Lines (11) and (13) cancel out line (12), as

$$[[x_i, x_j], x_s] = [[x_i, x_s], x_j] - [[x_j, x_s], x_i] = [[x_i, x_s], x_j] + [[x_s, x_j], x_i].$$

- Lines (14) and (19) cancel out.
- Lines (15) and (18) cancel out.
- Lines (16) and (17) cancel out.

from which we conclude that $d^2 c(x_1, \ldots, x_n) = 0$, as desired.

$$
\begin{align}
d^2c(x_1,\dots,x_n) &= \sum_i (-1)^{i+1} x_i.dc(x_1,\hat{x}_i,x_k) + \sum_{i<j}(-1)^{i+j} dc([x_i,x_j],x_1,\hat{x}_i,\hat{x}_j,\dots,x_k) \tag{A.1}\\
&= \sum_{r<i}(-1)^{i+r}(x_i x_r).c(x_1,\hat{x}_r,\hat{x}_i,x_k) \tag{A.2}\\
&\quad + \sum_{i<r}(-1)^{i+r+1}(x_i x_r).c(x_1,\hat{x}_i,\hat{x}_r,x_k) \tag{A.3}\\
&\quad + \sum_{r<s<i}(-1)^{i+r+s+1} x_i.c([x_r,x_s]x_1,\hat{x}_r,\hat{x}_s,\hat{x}_i,x_k) \tag{A.4}\\
&\quad + \sum_{r<i<s}(-1)^{i+r+s} x_i.c([x_r,x_s]x_1,\hat{x}_r,\hat{x}_i,\hat{x}_s,x_k) \tag{A.5}\\
&\quad + \sum_{i<r<s}(-1)^{i+r+s+1} x_i.c([x_r,x_s]x_1,\hat{x}_i,\hat{x}_r,\hat{x}_s,x_k) \tag{A.6}\\
&\quad + \sum_{i<j}(-1)^{i+j+1}[x_i,x_j].c(x_1,\hat{x}_i,\hat{x}_j,x_k) \tag{A.7}\\
&\quad + \sum_{r<i<j}(-1)^{i+j+r} x_r.c([x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_j,x_k) \tag{A.8}\\
&\quad + \sum_{i<r<j}(-1)^{i+j+r+1} x_r.c([x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_j,x_k) \tag{A.9}\\
&\quad + \sum_{i<j<r}(-1)^{i+j+r} x_r.c([x_i,x_j],x_1,\hat{x}_i,\hat{x}_j,\hat{x}_r,x_k) \tag{A.10}\\
&\quad + \sum_{s<i<j}(-1)^{i+j+s} c([[x_i,x_j],x_s],x_1,\hat{x}_s,\hat{x}_i,\hat{x}_j,x_k) \tag{A.11}\\
&\quad + \sum_{i<s<j}(-1)^{i+j+s+1} c([[x_i,x_j],x_s],x_1,\hat{x}_i,\hat{x}_s,\hat{x}_j,x_k) \tag{A.12}\\
&\quad + \sum_{i<j<s}(-1)^{i+j+s} c([[x_i,x_j],x_s],x_1,\hat{x}_i,\hat{x}_j,\hat{x}_s,x_k) \tag{A.13}\\
&\quad + \sum_{r<s<i<j}(-1)^{i+j+r+s} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_s,\hat{x}_i,\hat{x}_j,x_k) \tag{A.14}\\
&\quad + \sum_{r<i<s<j}(-1)^{i+j+r+s+1} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_s,\hat{x}_j,x_k) \tag{A.15}\\
&\quad + \sum_{r<i<j<s}(-1)^{i+j+r+s} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_r,\hat{x}_i,\hat{x}_j,\hat{x}_s,x_k) \tag{A.16}\\
&\quad + \sum_{i<r<s<j}(-1)^{i+j+r+s} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_s,\hat{x}_j,x_k) \tag{A.17}\\
&\quad + \sum_{i<r<j<s}(-1)^{i+j+r+s+1} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_r,\hat{x}_j,\hat{x}_s,x_k) \tag{A.18}\\
&\quad + \sum_{i<j<r<s}(-1)^{i+j+r+s} c([x_r,x_s],[x_i,x_j],x_1,\hat{x}_i,\hat{x}_j,\hat{x}_r,\hat{x}_s,x_k). \tag{A.19}
\end{align}
$$

□

A.2 Homological Algebra Results

Let R be a commutative ring.

Proposition A.2.1. If P is a finitely generated projective R-module, then P is canonically isomorphic to its double dual, by the map $p \mapsto \varphi_p$, where $\varphi_p(f) = f(p)$, $f \in P^*$.

Proof. The claim obviously holds for finitely generated free R-modules. Let Q be an R-module such that $P \oplus Q \cong R^n$. Then

$$P \oplus Q \cong R^n \xrightarrow{\cong} (R^n)^{**} \cong (P \oplus Q)^{**} \cong P^{**} \oplus Q^{**}, \qquad p \oplus q \mapsto \varphi_p \oplus \varphi_q.$$

As the map respects the summands, we get the desired isomorphism $P \cong P^{**}$. □

Proposition A.2.2. If P and A are R-modules and P is finitely generated projective, then the map $P^* \otimes_R A^* \to (P \otimes_R A)^*$, $f \otimes g \mapsto f \times g$, where $f \times g(p \otimes a) = f(p)g(a)$, is an isomorphism.

Proof. The claim is easily shown for P finitely generated free. For P finitely generated projective, take Q such that $P \oplus Q \cong R^n$. Then

$$\begin{aligned}(P \otimes_R A)^* \oplus (Q \otimes_R A)^* \cong ((P \oplus Q) \otimes_R A)^* \cong (R^n \otimes_R A)^* &\xrightarrow{\cong} (R^n)^* \otimes_R A^* \\ &\cong (P \oplus Q)^* \otimes_R A^* \cong (P^* \otimes_R A^*) \oplus (Q^* \otimes_R A^*), \\ (f \otimes g) \oplus (f' \otimes g') &\mapsto (f \times g) \oplus (f' \times g').\end{aligned}$$

This map respects the summands, so we get the desired isomorphism $(P \otimes_R A)^* \cong P^* \otimes_R A^*$. □

A.3 Integration Preserves Smoothness

Proposition A.3.1. Let $f\colon \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be C^1 map. Let $K \subseteq \mathbb{R}^n$ be compact. Then $F\colon \mathbb{R}^k \to \mathbb{R}$ defined as $F(x) = \int_K f(x,y)\, d\lambda(y)$ has all partial derivatives, and

$$\frac{\partial F}{\partial x_j}(x) = \int_K \frac{\partial}{\partial x_j} f(x,y)\, d\lambda(y),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^n$.

Proof. Given $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $j \in \{1, \dots, k\}$, let

$$\gamma\colon \mathbb{R} \to \mathbb{R}^k, \qquad t \mapsto (x_1, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_k).$$

Now, $f \circ (\gamma \times \mathrm{id})\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies:

- $y \mapsto f(\gamma(t), y)$ is Lebesgue integrable on K for all fixed $t \in \mathbb{R}$.
- $t \mapsto f(\gamma(t), y)$ is differentiable for all fixed $y \in \mathbb{R}^n$.
- $(t, y) \mapsto \frac{\partial}{\partial t} f(\gamma(t), y)$ is continuous and therefore uniformly bounded on $[-1,1] \times K$.

Lebesgue's Dominated Convergence Theorem implies that $F \circ \gamma$ is differentiable on $(-1,1)$ with $(F \circ \gamma)'(t) = \int_K \frac{\partial}{\partial t} f(\gamma(t), y)\, d\lambda(y)$ (cf. Theorem 11.5 [10]). Thus,

$$\frac{\partial F}{\partial x_j}(x) = (F \circ \gamma)'(t) = \int_K \frac{\partial}{\partial t} f(\gamma(t), y)\, d\lambda(y) = \int_K \frac{\partial}{\partial x_j} f(x,y)\, d\lambda(y).$$

□

An immediate corollary of this proposition is:

Corollary A.3.2. Let $f\colon \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be a smooth map. Let $K \subseteq \mathbb{R}^n$ be compact. Then $F\colon \mathbb{R}^k \to \mathbb{R}$ defined as $F(x) = \int_K f(x,y)\,d\lambda(y)$ is smooth with

$$\frac{\partial^m F}{\partial x_{j_1} \cdots \partial x_{j_m}}(x) = \int_K \frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}} f(x,y)\,d\lambda(y).$$

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