DUALITY AND AMBIDEXTERITY IN $K(n)$-LOCAL SPECTRA

JONAS MCCANDLESS

Contents

1. Introduction

Let k be a field and let G be a finite group. In this situation we may form the category $\mathrm{Rep}_k(G)$ of representations of G. An object of this category is a pair (V,ρ) consisting of a vector space V over k and a representation ρ of G on V. A morphism from (V,ρ) to (W,σ) is a k-linear map $\varphi\colon V \to W$ which satisfy that $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$ for each element g of G. We have the following classical result due to Maschke.

Theorem 1.1 (Maschke)**.** *Let G be a finite group. If k is a field whose characteristic does not divide the order of G, then the category $\mathrm{Rep}_k(G)$ of representations of G is semi-simple.*

To put this in more concrete terms, if k is a field whose characteristic does not divide the order of G, then every representation of G is a direct sum of irreducible representations. On the other hand, if k is a field of positive characteristic which divides the order of G, then the category of representations of G is extremely complicated. Let us try to rephrase Maschke's result in a different way. First observe that the category of representations of G is isomorphic to the category of left $k[G]$-modules. More concretely, a representation of G is a vector space

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V over k equipped with a left action of G. There is a k-linear endomorphism of V given by $v \mapsto \sum_{g \in G} gv$ which factors as follows

$$V \twoheadrightarrow V_G \xrightarrow{\mathrm{Nm}_G} V^G \hookrightarrow V.$$

The k-linear map $\mathrm{Nm}_G : V_G \to V^G$ from the coinvariants to the invariants of the action of G on V is called the norm map. Note that $V_G \simeq H_0(G;V)$ and $V^G \simeq H^0(G;V)$. In [36], Tate defined what is now called the Tate cohomology groups. These are defined by splicing together group homology and group cohomology. More precisely, the Tate cohomology groups are defined by

$$\widehat{H}^n(G;V) = \begin{cases} H^n(G;V) & n \geq 1 \\ \operatorname{coker} \mathrm{Nm}_G & n = 0 \\ \ker \mathrm{Nm}_G & n = -1 \\ H_{-(n+1)}(G;V) & n \geq -2 \end{cases}$$

and there is an exact sequence

$$0 \to \widehat{H}^{-1}(G;V) \to H_0(G;V) \xrightarrow{\mathrm{Nm}_G} H^0(G;V) \to \widehat{H}^0(G;V) \to 0$$

of groups. In particular, the Tate cohomology groups measure the failure of the norm map being an isomorphism. The Tate cohomology groups are most naturally thought of as corepresentable functors on the stable module category $\mathrm{StMod}_{k[G]}$ of left $k[G]$-modules. Recall that an object of the stable module category is a left $k[G]$-module. The set of morphisms from M to N in the stable module category is defined as $\mathrm{Hom}_{k[G]}(M,N)$ modulo the relation that $\varphi \sim \psi$ if the difference $\varphi - \psi$ factors through a projective $k[G]$-module. If M is a $k[G]$-module, then we may choose a projective $k[G]$-module P and a surjective $\varphi : P \twoheadrightarrow M$ homomorphism of $k[G]$-modules. The assignment $M \mapsto \ker \varphi$ defines an endofunctor Ω on the stable module category of $k[G]$-modules and there is an isomorphism

$$\widehat{H}^n(G;V) \simeq \mathrm{Hom}_{\mathrm{StMod}_{k[G]}}(k, \Omega^n V)$$

where k is equipped with the trivial G-action. If the characteristic of k does not divide the order of k, then it follows from Maschke's theorem that $k[G]$ is semi-simple. Consequently, every module over $k[G]$ is projective. We conclude that the Tate cohomology groups $\widehat{H}^n(G;V)$ vanish for every integer n or equivalently that the norm map $\mathrm{Nm}_G : V_G \to V^G$ is an isomorphism. On the other hand, if the characteristic of k divides the order of G, then the norm map is in general not an isomorphism. In conclusion, the perspective that we have tried to advocate here is the following. If V is a representation of G, then we should ask whether the norm map is an isomorphism or equivalently, whether the Tate cohomology groups vanish. The goal of this project is twofold. First of all, we refine this story to the ∞-category of spectra and various chromatic localizations of the ∞-category of spectra. More precisely, we will see that there is a functor $(-)^{tG}$ from the ∞-category of spectra with a G-action to the ∞-category of spectra such that for every spectrum X with an action of G, there is a cofiber sequence

$$X_{hG} \xrightarrow{\mathrm{Nm}_G} X^{hG} \to X^{tG}$$

in the ∞-category of spectra, where X_{hG} and X^{hG} denote the homotopy coinvariants and homotopy invariants respectively (see Remark 2.8). The functor $(-)^{tG}$ is called the Tate construction and the map $\mathrm{Nm}_G : X_{hG} \to X^{hG}$ is called the norm map. The Tate construction is a refinement of the Tate cohomology groups to the ∞-category of spectra. More precisely, if A is abelian group equipped with an action of G, then

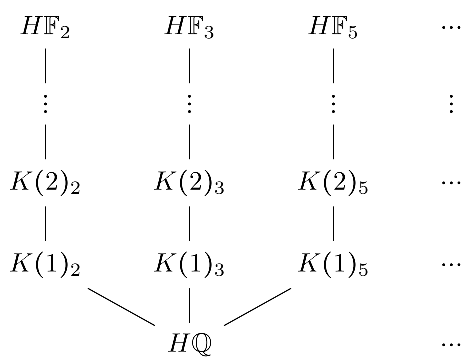
$$\pi_n HA^{tG} \simeq \widehat{H}^{-n}(G;A)$$

where the Eilenberg–Mac Lane spectrum HA is equipped with the trivial G-action. Moreover, the long exact sequence associated to the fiber sequence

$$(HA)_{hG} \xrightarrow{\mathrm{Nm}_G} (HA)^{hG} \to (HA)^{tG}$$

recovers the exact sequence above (see Example 2.14). Just as we may think of a representation of G as a functor $BG \to \mathbf{Vect}_k$ from the group G considered as a groupoid with a single object to the category of vector spaces over k, we define a spectrum with an action of G as a functor from a classifying space of G considered as a Kan complex to the ∞-category of spectra. The second goal of this project is to replace the classifying space of G with more general Kan complexes which satisfy certain finiteness conditions.

In general, if X is a spectrum equipped with an action of G, then we may ask if the norm map is an equivalence of spectra or equivalently if the Tate construction vanishes. Let us fix a prime number p. If X is the Eilenberg–Mac Lane spectrum of the rational numbers equipped with the trivial G-action, then the norm map $\mathrm{Nm}_G\colon (H\mathbb{Q})_{hG} \to (H\mathbb{Q})^{hG}$ is an equivalence. On the other hand, if X is the Eilenberg–Mac Lane spectrum of $\mathbb{F}_p$ equipped with the trivial G-action, then the norm map $\mathrm{Nm}_G\colon (H\mathbb{F}_p)_{hG} \to (H\mathbb{F}_p)^{hG}$ is in general not an equivalence. In fact, it is an equivalence precisely if G is the trivial group. This is analogous to the situation in ordinary representation theory as described above. However, in higher algebra the situation is more complicated. Besides the Eilenberg–Mac Lane spectrum of the rational numbers and the Eilenberg–Mac Lane spectrum of $\mathbb{F}_p$ there are fields of intermediate characteristic which interpolate between $H\mathbb{Q}$ and $H\mathbb{F}_p$. More precisely, for each natural number n there is a spectrum $K(n)$ called the nth Morava K-theory spectrum at the prime p. For example, the first Morava K-theory $K(1)$ is equivalent to mod p complex K-theory KU/p. The nth Morava K-theory $K(n)$ admits a complex orientation and the associated formal group have height exactly n. In fact, it is consequence of the thick subcategory theorem of Devinatz–Hopkins–Smith [13] that the Morava K-theories are the prime fields in the ∞-category of spectra. In the language of Balmer (see [1]), the prime spectrum of the ∞-category of p-local finite spectra is labeled by the Morava K-theories. Impressionistically, we picture the prime fields of the ∞-category of spectra as follows



where $K(n)_p$ denotes the nth Morava K-theory at the prime p. In conclusion, the Morava K-theories provide deep information about the global structure of the ∞-category of spectra. For example, we might be interested in computing the homotopy groups $\pi_* S^0$ of the sphere spectrum. Let $L_{K(n)}$ denote the Bousfield localization of the ∞-category of spectra with respect to the nth Morava K-theory. We can try to compute the homotopy groups $\pi_* L_{K(n)} S^0$ of the $K(n)$-local sphere spectrum for every integer n and every prime number p. However, even though we might understand the local pieces $\pi_* L_{K(n)} S^0$ it is not evident how to actually assemble these pieces into a global computation of $\pi_* S^0$. This is were the Morava E-theories enter the story[1]. For each natural number there is a spectrum E_n called the nth Morava E-theory spectrum (see Section 5). Let L_n denote the Bousfield localization of the ∞-category of spectra with respect to the nth Morava E-theory spectrum. If X is a spectrum, then the Bousfield localizations L_n assemble into the chromatic tower

$$\cdots \to L_2 X \to L_1 X \to L_0 X$$

and if X is a p-local finite spectrum, then the tower converges. This is the content of the chromatic convergence theorem due to Hopkins and Ravenel (see [23, Lecture 32]). More precisely, if X is a p-local finite spectrum, then the canonical map

$$X \xrightarrow{\simeq} \lim_n L_n X$$

from X to the limit of the chromatic tower is an equivalence in the ∞-category of spectra. Moreover, for every natural number n there is a chromatic fracture square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

in the ∞-category of spectra (see Remark 6.7). Consequently, to compute $\pi_* S^0$ it suffices to compute $\pi_* L_{K(n)} S^0$ for every integer n and prime number p together with the gluing data expressed by the chromatic fracture squares. This is by no means an easy task. However, the ∞-category of $K(n)$-local spectra exhibits some surprising duality phenomena that we will investigate in this paper. In [9], Greenlees and Sadofsky establish an interesting result in this direction.

Theorem 1.2 (Greenlees–Sadofsky)**.** *Let G be a finite group and let $K(n)$ denote the nth Morava K-theory spectrum equipped with the trivial G-action. Then the norm map*

$$\mathrm{Nm}_G : K(n)_{hG} \to K(n)^{hG}$$

is an equivalence of spectra.

Intuitively, this means that representation theory over the intermediate characteristics in higher algebra behave more like representation theory in characteristic zero than representation theory in characteristic p. We will provide a proof of this theorem in Section 4. More generally, Hovey and Sadofsky [14] prove a globalization of this result.

Theorem 1.3 (Hovey–Sadofsky)**.** *Let G be a finite group and let X be a $K(n)$-local spectrum equipped with an action of G. Then the $K(n)$-localized norm map*

$$\mathrm{Nm}_G : X_{hG} \to X^{hG}$$

is an equivalence in the ∞-category of $K(n)$-local spectra.

The first part of this paper will culminate with a proof of Hovey and Sadofsky's result (see Theorem 6.1). In the second part of this paper we will place Hovey and Sadofsky's result into a more general categorical framework following Hopkins and Lurie's paper [12]. Let X be a $K(n)$-local spectrum equipped with an action of a finite group G classified by a functor $X : BG \to \mathrm{Sp}_{K(n)}$ of ∞-categories. The unique map $BG \to *$ induces a functor $\mathrm{Sp}_{K(n)} \to \mathrm{Sp}_{K(n)}^{BG}$ from the ∞-category of $K(n)$-local spectra to the ∞-category of $K(n)$-local spectra equipped

[1]We will later refer to Morava E-theory as Lubin–Tate spectra.

with an action of G and this functor is informally given by endowing the $K(n)$-local spectrum X with the trivial G-action. This functor admits a left adjoint determined by $X \mapsto X_{hG}$ and a right adjoint determined by $X \mapsto X^{hG}$. An equivalent way to formulate Hovey and Sadofsky's result is that the natural transformation $\mathrm{Nm}_G : (-)_{hG} \to (-)^{hG}$ determined by the $K(n)$-localized norm map is an equivalence of functors. More generally, we will study the following situation. Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits and let $\rho : X \to \mathcal{C}$ be a diagram in $\mathcal{C}$ indexed by a Kan complex X. The unique map $f : X \to *$ induces a functor $f^* : \mathcal{C} \to \mathcal{C}^X$ and this functor admits a left adjoint $f_!$ and a right adjoint f_*. In Section 7 we will define a natural transformation $\mathrm{Nm}_X : f_! \to f_*$ which recovers the usual norm map in the case where X is the classifying space of a finite group (see Example 8.10). We can ask when this natural transformation is an equivalence of functors. The main theorem of [12] is the following (see Theorem 11.1).

Theorem 1.4 (Hopkins–Lurie). *Let X be a Kan complex and suppose that for every vertex x of X the sets $\pi_n(X, x)$ are finite for every integer n and trivial for $n \gg 0$. Let $\rho : X \to \mathrm{Sp}_{K(n)}$ be a diagram of $K(n)$-local spectra indexed by X. Then the natural transformation Nm_X induces a natural equivalence*

$$\mathrm{Nm}_X : \mathrm{colim}_X \, \rho \xrightarrow{\simeq} \lim{}_X \rho$$

in the ∞-category of $K(n)$-local spectra.

If X is the classifying space of a finite group, then Hopkins and Lurie's result recovers Hovey and Sadofsky's result above. The proof of Hopkins and Lurie's theorem depends crucially on the Ravenel–Wilson calculation [32] which we state in Section 10. Before each section of this paper we will give a more detailed outline of the contents which will appear.

Terminology. We will freely use the language of ∞-categories as developed by Lurie in [22] and [24]. Let Sp denote the ∞-category of spectra and let S^0 denote the sphere spectrum. Let $\mathcal{S}$ denote the ∞-category of spaces. Concretely, an object of the ∞-category of spaces is a Kan complex. If A is an $\mathbb{E}_1$-ring, then we let LMod_A denote the ∞-category of left modules over A in the ∞-category of spectra. Similarly, we let RMod_A denote the ∞-category of right modules over A. If A is an $\mathbb{E}_\infty$-ring, then we write Mod_A instead of LMod_A or RMod_A.

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Part 1. Vanishing of the $K(n)$-local Tate construction

Let G be a finite group. The goal of the first part of this paper is to prove Hovey and Sadofsky's result that the Tate construction vanishes $K(n)$-locally. In Section 2 we introduce the Tate construction following [27] and obtain a universal characterization of the Tate construction due to Klein [16]. Moreover, we discuss that the Tate construction admits a lax symmetric monoidal structure which will be important later (see Theorem 2.15). In Section 3 we show that the $K(n)$-cohomology of the classifying space of G is finitely generated as a left module over the $K(n)$-cohomology of a point (see Proposition 3.6). This is the crucial input for the proof of Greenlees and Sadofsky's result that the Tate construction of the trivial G-action on the nth Morava K-theory vanishes which we prove in Section 4 (see Theorem 4.1). To prove Greenlees and Sadofsky's result we present an argument due to Kuhn (see Proposition 4.3) which allows us to reduce to the case where G is a cyclic group of order p. In Section 5 we recall some material from stable homotopy theory that we will need. More precisely, we briefly discuss

Bousfield localizations, Lubin–Tate spectra, and a 2-periodic version of Morava K-theory. In Section 6 we employ the vanishing result due to Greenlees and Sadofsky to prove Hovey and Sadofsky's theorem that the Tate construction vanishes $K(n)$-locally (see Theorem 6.1).

2. The Tate construction

We will start by defining the ∞-category of spectra with an action of a group following [27].

Definition 2.1. Let G be a group and let $\mathcal{C}$ be an ∞-category. A G-equivariant object of $\mathcal{C}$ is a functor from the classifying space BG of G regarded as a Kan complex to $\mathcal{C}$. The ∞-category of G-equivariant objects of $\mathcal{C}$ is the ∞-category $\mathrm{Fun}(BG, \mathcal{C})$ of functors from BG to $\mathcal{C}$.

Remark 2.2. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. If $f : X \to Y$ is a map of Kan complexes, then the induced functor $f^* : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ given by composition with f admits a left adjoint $f_!$ and a right adjoint f_* as depicted in the following diagram

$$\begin{array}{c} \mathrm{Fun}(Y, \mathcal{C}) \\ f_! \uparrow \quad \downarrow f^* \quad \uparrow f_* \\ \mathrm{Fun}(X, \mathcal{C}) \end{array}$$

The left adjoint $f_!$ of f^* is given by sending a functor $F : X \to \mathcal{C}$ to a left Kan extension of F along f (see [22, Proposition 4.3.3.7]). Similarly, the right adjoint f_* of f^* is given by sending a functor $F : X \to \mathcal{C}$ to a right Kan extension of F along f. In particular, if $f : X \to *$ is the unique map, then a left adjoint of f^* is given by $F \mapsto \mathrm{colim}_X F$ and a right adjoint of f^* is given by $F \mapsto \lim_X F$.

Definition 2.3. Let G be a group and let $p : BG \to *$ denote the unique map. Let $\mathcal{C}$ be an ∞-category which admits limits and colimits indexed by BG.

(1) The homotopy orbit functor $(-)_{hG} : \mathcal{C}^{BG} \to \mathcal{C}$ is a left adjoint of p^*.
(2) The homotopy fixed point functor $(-)^{hG} : \mathcal{C}^{BG} \to \mathcal{C}$ is a right adjoint of p^*.

Remark 2.4. We will primarily be interested in the case where $\mathcal{C}$ is the ∞-category of spectra. Recall that Sp is a stable and presentable ∞-category (see [24, Proposition 1.4.3.6] and [24, Proposition 1.4.4.4]) which implies that the ∞-category Sp^{BG} of G-equivariant objects of Sp inherits the structure of a stable presentable ∞-category ([24, Proposition 1.1.3.1] and [22, Proposition 5.5.3.6]). Instead of saying that X is a G-equivariant object of the ∞-category of spectra we will often say that X is a spectrum with a G-action.

Let G be a topological group. We want to construct a natural transformation

$$\mathrm{Nm}_G : (-)_{hG} \to (-)^{hG}$$

of functors from the ∞-category of spectra with a G-action to the ∞-category of spectra. We follow [27, Section I.3 and I.4].

Theorem 2.5 ([27, Theorem I.3.3])**.** *Let $\mathcal{C}$ be a small stable ∞-category and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ which is stable and satisfies that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is exact.*

(1) Let W be the set of morphisms in $\mathcal{C}$ whose cofiber is an object of $\mathcal{D}$. Then the Dwyer-Kan localization $\mathcal{C}/\mathcal{D} := \mathcal{C}[W^{-1}]$ is a stable ∞-category and the canonical functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is exact.

(2) *Let $\mathcal{E}$ be a stable ∞-category and let $\mathrm{Fun}_0^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})$ denote the full subcategory of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})$ spanned by those functors which carry an object of $\mathcal{D}$ to the zero object of $\mathcal{E}$. Then composition with $\pi\colon\mathcal{C}\to\mathcal{C}/\mathcal{D}$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}/\mathcal{D},\mathcal{E})\to\mathrm{Fun}_0^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})$$

of ∞-categories.

(3) *Let $\mathcal{E}$ be a presentable stable ∞-category. The functor $i\colon\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}/\mathcal{D},\mathcal{E})\hookrightarrow\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})$ admits a left adjoint $L\colon\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})\to\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}/\mathcal{D},\mathcal{E})$ which is a localization functor.*

Proof. See [27, Theorem I.3.3]. □

Theorem 2.5 above allows us to construct a norm map $\mathrm{Nm}_G\colon(-)_{hG}\to(-)^{hG}$ and show that it is uniquely characterized by a few simple properties completely bypassing the use of genuine equivariant homotopy theory. The characterization of the norm map is due to Klein (see [16]) but the proof that we will give is due to Nikolaus and Scholze.

Theorem 2.6 (Klein [16], Nikolaus–Scholze [27, Theorem I.4.1])**.** *Let S be a Kan complex, and let $p\colon S\to *$ denote the unique map.*

(1) *There exists an essentially unique initial functor $p_*^t\colon\mathrm{Sp}^S\to\mathrm{Sp}$ together with a natural transformation $p_*\to p_*^t$ which satisfies that p_*^t vanishes on compact objects of Sp^S and that the fiber of the natural transformation $p_*\to p_*^t$ preserves small colimits.*

(2) *There exists an essentially unique object D_S of Sp^S such that the fiber of the natural transformation $p_*\to p_*^t$ is given by the construction*

$$X\mapsto p_!(D_S\otimes X).$$

The object D_S of Sp^S is given by the composite

$$S\to\mathcal{S}^S\xrightarrow{\Sigma^\infty_+}\mathrm{Sp}^S\xrightarrow{p_*}\mathrm{Sp}$$

where the first functor is induced by the functor $\mathrm{Map}_S\colon S\times S\to\mathcal{S}$ given by sending a pair (s,t) of vertices of S to the space of paths from s to t.

Proof. We will show part (1) by applying Theorem 2.5 with $\mathcal{C}=\mathrm{Sp}^S$ and $\mathcal{D}=(\mathrm{Sp}^S)_\omega$ the compact objects of Sp^S. We cannot directly apply this theorem since $\mathcal{C}$ is not a small ∞-category. However, we may choose a cardinal κ large enough such that p_* and p_*^t are κ-accessible. Then we prove the theorem in the full subcategory of κ-compact objects of $\mathcal{C}$ and $\mathcal{D}$ and then pass to Ind_κ-categories. Consequently, we will carry out the proof working in $\mathcal{C}$ with the understanding that a formal arguement takes care of the set-theretical issues. It follows from Theorem 2.5 that the canonical functor $\pi\colon\mathrm{Sp}^S\to\mathrm{Sp}^S/(\mathrm{Sp}^S)_\omega$ is an exact functor of stable ∞-categories and the full inclusion $i\colon\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}/\mathcal{D},\mathcal{E})\hookrightarrow\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})$ admits a left adjoint $L\colon\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{E})\to\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}/\mathcal{D},\mathcal{E})$ which is a localization functor. We have a diagram

$$\begin{array}{ccc}\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}^S/(\mathrm{Sp}^S)_\omega,\mathrm{Sp}) & \xrightarrow{\simeq} & \mathrm{Fun}_0^{\mathrm{Ex}}(\mathrm{Sp}^S,\mathrm{Sp})\\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle i} & & \\ \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}^S,\mathrm{Sp}) & & \end{array}$$

of ∞-categories. Note that $p_*\colon\mathrm{Sp}^S\to\mathrm{Sp}$ is an exact functor. Consequently, we define the functor $p_*^t\colon\mathrm{Sp}^S\to\mathrm{Sp}$ by $(i\circ L)(p_*)$. It follows from the horizontal equivalence in the diagram above that the functor p_*^t vanishes on compact objects of Sp^S. The unit $\mathrm{id}\to i\circ L$ of the adjunction which exhibits L as a left adjoint of i provides a natural transformation $p_*\to p_*^t$ and

p^t_* is initial with these properties. The argument that the fiber of the natural transformation $p_* \to p^t_*$ preserves small colimits is completely formal (see [27, Theorem I.3.3]). To end the proof of (1) it remains to show that the functor p^t_* is the essentially unique functor with these properties. It follows from [22, Proposition 5.5.1.9] that composition with the inclusion $(\mathrm{Sp}^S)_\omega \hookrightarrow \mathrm{Sp}^S$ induces an equivalence

$$\mathrm{LFun}(\mathrm{Sp}^S, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{Ex}}((\mathrm{Sp}^S)_\omega, \mathrm{Sp})$$

of ∞-categories, where $\mathrm{LFun}(\mathrm{Sp}^S, \mathrm{Sp})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Sp}^S, \mathrm{Sp})$ spanned by those functors that preserves small colimits. The restriction of p_* to the full subcategory $(\mathrm{Sp}^S)_\omega$ of compact objects of Sp^S is exact which proves the wanted. See [27, Theorem I.4.1] for the argument of (2). □

Remark 2.7. Let G be a topological group. In [17, Theorem 10.1], Klein shows that $D_{BG} \simeq S^{\mathrm{Ad}G}$ where $S^{\mathrm{Ad}G}$ denotes the suspension spectrum of the one-point compactification of the adjoint representation of G. It follows from Theorem 2.6 that there exists an essentially unique functor $(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ together with a natural transformation $(-)^{hG} \to (-)^{tG}$ which satisfies that $(-)^{tG}$ vanishes on compact objects of Sp^{BG}. Furthermore, for every spectrum X with an action of G, there exists a fiber sequence

$$(S^{\mathrm{Ad}G} \otimes X)_{hG} \longrightarrow X^{hG} \longrightarrow X^{tG}$$

in the ∞-category of spectra. The functor $(-)^{tG}$ is called the Tate construction and the natural transformation $(S^{\mathrm{Ad}G} \otimes -) \to (-)^{hG}$ is called the norm map which we denote Nm_G. For example if X is a spectrum equipped with an action of the circle group $\mathbb{T}$, then $D_{B\mathbb{T}} \simeq S^1$ so there is a fiber sequence

$$\Sigma X_{h\mathbb{T}} \xrightarrow{\mathrm{Nm}_{\mathbb{T}}} X^{h\mathbb{T}} \longrightarrow X^{t\mathbb{T}}$$

in the ∞-category of spectra (see [27, Corollary I.4.3]).

Remark 2.8. We will primarily be interested in the case where G is a finite group. In this case there is a natural equivalence $D_{BG} \simeq S^0$ of spectra. The Tate construction $(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ is the essentially unique functor which vanishes on compact objects of Sp^{BG} and sits in a fiber sequence

$$X_{hG} \xrightarrow{\mathrm{Nm}_G} X^{hG} \longrightarrow X^{tG}$$

in the ∞-category of spectra.

Let G be a finite group and let $F : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ be a functor. We need a way to recognize when F is equivalent to the Tate construction $(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$. It follows from the universal characterization of the Tate construction that if F is equipped with a natural transformation $(-)^{hG} \to F$ and F vanishes on compact objects of the ∞-category of spectra with a G-action, then there is an equivalence $(-)^{tG} \simeq F$ of functors. The following result identifies the compact generators of Sp^S. If $s : * \to BG$ is a map of Kan complexes specifying a basepoint of BG, then the pullback functor $s^* : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ admits a left adjoint $s_! : \mathrm{Sp} \to \mathrm{Sp}^{BG}$.

Lemma 2.9. *Let G be a finite group.*

1. *The functor $s_!$ preserves compact objects and $s_! S^0$ is a compact generator of Sp^{BG}.*
2. *There is an equivalence $s^* s_! S^0 \simeq \Sigma^\infty_+ G$ of spectra.*

Proof. The functor s^* preserves small colimits. It follows that the left adjoint $s_!$ of s^* carries compact objects of Sp to compact objects of Sp^{BG}. Let $f : X \to Y$ be a map in the ∞-category

Sp^{BG} of spectra with a G-action. The map f is an equivalence if the induced map $\pi_n s^* f$ is an isomorphism for all integers n since the functor s^* is conservative. We have that

$$\pi_n s^* X \simeq \mathrm{Hom}_{\mathrm{h\,Sp}}(S^n, s^* X) \simeq \mathrm{Hom}_{\mathrm{h\,Sp}^{BG}}(s_! S^n, X).$$

It follows that $s_! S^0$ is a compact generator of Sp^{BG} which proves (1). There is a pullback diagram

$$\begin{array}{ccc} \Omega BG & \xrightarrow{q} & * \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle s} \\ * & \xrightarrow{s} & BG \end{array}$$

in the ∞-category of spaces. Invoking [24, Lemma 6.1.6.3] we find that the natural transformation $s^* s_! \to q_! q^*$ is an equivalence in $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$. Since $q_! q^* S^0 \simeq \Sigma^\infty_+ G$ we conclude that $s^* s_! S^0 \simeq \Sigma^\infty_+ G$. □

Remark 2.10. Let G be a finite group and let $s : * \to BG$ specify a basepoint of BG. Suppose that $F : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ is a functor which is equipped with a natural transformation $(-)^{hG} \to F$ and satisfies that $F s_! S^0 \simeq 0$. It follows from Theorem 2.6 and Lemma 2.9 that there exists a unique, up to a contractible space of choice, natural transformation $(-)^{tG} \to F$ which is an equivalence.

Remark 2.11. In the second part of this paper we will provide a different construction of the norm map due to Hopkins and Lurie in [12]. We can use Remark 2.10 above to show that the two constructions agree in certain cases.

Remark 2.12. Let G be a finite group. In this paper it will suffice to work in the ∞-category Sp^{BG} of spectra with a G-action. However, in this remark we will say a few things about the ∞-category Sp_G of genuine G-equivariant spectra. The reader is refered to Section II.2 of [27] for a precise definition of Sp_G. If X is a genuine G-equivariant spectrum, then we may form the categorical fixed points X^H for every subgroup H of G and we obtain a functor $(-)^H : \mathrm{Sp}_G \to \mathrm{Sp}$. There is natural map $X^H \to X^{hH}$ for every subgroup H of G and X is Borel-complete if this map is an equivalence for all subgroups H of G. The ∞-category Sp^{BG} of spectra with a G-action is equivalent to the full subcategory of Sp_G spanned by those genuine G-equivariant spectra which are Borel-complete. The Tate construction of a genuine G-equivariant spectrum was originally introduced by Greenless and May in [8] and we recall their construction. Let X be a genuine G-equivariant spectrum and let $\widetilde{EG}$ denote the cofiber of the canonical map $\Sigma^\infty_+ EG \to S^0$ in Sp_G. There is a commutative diagram

$$\begin{array}{ccccc} \Sigma^\infty_+ EG \otimes X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \otimes X \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty_+ EG \otimes F(\Sigma^\infty_+ EG, X) & \longrightarrow & F(\Sigma^\infty_+ EG, X) & \longrightarrow & \widetilde{EG} \otimes F(\Sigma^\infty_+ EG, X) \end{array}$$

in Sp_G. It follows from the Adams equivalence [20, Theorem II.7.1] that

$$\Sigma^\infty_+ EG \otimes F(\Sigma^\infty_+ EG, X) \simeq \Sigma^\infty_+ EG \otimes X$$

in Sp_G. Applying the functor $(-)^G$ we obtain a commutative diagram

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & \Phi^G X \\ \downarrow{\scriptstyle \simeq} & & \downarrow & & \downarrow \\ X_{hG} & \xrightarrow{\mathrm{Nm}_G} & X^{hG} & \longrightarrow & X^{tG} \end{array}$$

of spectra. The spectrum $\Phi^G X$ is called the geometric fixed point spectrum of X. By construction

$$X^{tG} \simeq (\widetilde{EG} \otimes F(\Sigma^\infty_+ EG, X))^G.$$

Alternatively, it follows from [8, Proposition 2.6] that the Tate construction can be computed by

$$X^{tG} \simeq F(\widetilde{EG}, \Sigma(X \otimes \Sigma^\infty_+ EG))^G.$$

Remark 2.13. Let G be a finite group and let X be a spectrum equipped with a G-action. There are spectral sequences computing the homotopy groups of X_{hG}, X^{hG}, and X^{tG} respectively. We will describe these spectral sequences very briefly here. For a more complete reference see [8, Section 10] or [2, Section 2]. There is a spectral sequence

$$E^2_{s,t} \simeq H_s(G; \pi_t X) \Rightarrow \pi_{t+s} X_{hG}$$

with differentials $d_r : E^2_{s,t} \to E^2_{s-r,t+r-1}$ called the homotopy orbits spectral sequence. Similarly, there are two spectral sequences with signature

$$E_2^{s,t} \simeq H^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{hG}$$
$$E_2^{s,t} \simeq \widehat{H}^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{tG}$$

and differentials $d_r : E_2^{s,t} \to E_2^{s+r,t+r-1}$. The first one is called the homotopy fixed-point spectral sequence and the second one is called the Tate spectral sequence. If X is the Eilenberg–Mac Lane spectrum HA of an abelian group A equipped with an action of G, then all of these spectral sequences converge strongly in the sense of [4, Definition 5.2].

Example 2.14. Let A be an abelian and let G be a finite group and equip the Eilenberg–Mac Lane spectrum HA with the trivial G-action. The E_2-page of the homotopy fixed-point spectral sequence is given by

$$E_2^{s,t} \simeq H^s(G; \pi_t HA) \simeq \begin{cases} H^s(G; A) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Consequently, there are no non-trivial differentials on the E_2-page. It follows that $\pi_n(HA^{hG}) \simeq H^{-n}(G; A)$ for every integer n. A completely similar argument reveals that $\pi_n(HA^{tG}) \simeq \widehat{H}^{-n}(G; A)$. The E_2-page of the homotopy orbits spectral sequence is given by

$$E^2_{s,t} \simeq H_s(G; \pi_t HA) \simeq \begin{cases} H_s(G; A) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

We conclude that $\pi_n(HA_{hG}) \simeq H_n(G; A)$ for every integer n. The fiber sequence

$$HA_{hG} \to HA^{hG} \to HA^{tG}$$

induces a long exact sequence in homotopy

$$\cdots \to \widehat{H}^{-1}(G, A) \to H_0(G; A) \xrightarrow{\mathrm{Nm}_G} H^0(G; A) \to \widehat{H}^0(G; A) \ \to \cdots$$

We will end this section by discussing multiplicative structures on the Tate construction. Let us fix a compact Lie group G. Recall that the ∞-category of spectra admits the structure of a symmetric monoidal ∞-category [24, Corollary 4.8.2.19]. It follows that the ∞-category Sp^{BG} of spectra with a G-action aquires the structure of a symmetric monoidal ∞-category. Moreover, the pullback functor $p^* : \mathrm{Sp} \to \mathrm{Sp}^{BG}$ is symmetric monoidal. It follows from [24, Corollary 7.3.2.7] that the homotopy fixed point functor $(-)^{hG}$ aquires a lax symmetric monoidal structure. It turns out that the Tate construction admits a lax symmetric monoidal structure as the following theorem makes precise.

Theorem 2.15. *Let G be a compact Lie group. The Tate construction $(-)^{tG}\colon \mathrm{Sp}^{BG} \to \mathrm{Sp}$ admits an essentially unique lax symmetric monoidal structure such that the natural transformation $(-)^{hG} \to (-)^{tG}$ refines to a lax symmetric monoidal transformation.*

Proof. See [27, Theorem I.3.1]. □

Remark 2.16. Theorem 2.15 states that the Tate construction admits an essentially unique lax symmetric monoidal structure. More precisely, this means that the space consisting of pairs of a lax symmetric monoidal structure on $(-)^{tG}$ and a refinement of the natural transformation $(-)^{hG} \to (-)^{tG}$ to a lax symmetric monoidal transformation is contractible.

Remark 2.17. Let R be an $\mathbb{E}_\infty$-ring spectrum equipped with an action of a finite group G. It follows from Theorem 2.15 above that the Tate construction R^{tG} aquires the structure of an $\mathbb{E}_\infty$-ring spectrum. Moreover, if M is an R-module spectrum, then M^{tG} is an R^{tG}-module spectrum.

Remark 2.18. Theorem 2.15 is originally due to Greenlees and May in [8]. Recall from Remark 2.12 that if X is a genuine G-equivariant spectrum, then the Tate construction is given by

$$X^{tG} \simeq (\widetilde{EG} \otimes F(\Sigma^\infty_+ EG, X))^G.$$

It follows that the Tate construction is a composite of functors which admit lax symmetric monoidal structures. However, the uniqueness part of Theorem 2.15 is not obvious from this perspective.

Remark 2.19. In this project we will mainly be interested in the case where G is a finite group. However, the case where G is a compact Lie group is very important. For example if A is an $\mathbb{E}_1$-ring, then we can construct a spectrum $\mathrm{THH}(A)$ called topological Hochschild homology of A. It turns out that this spectrum admits a cyclotomic structure in the sense of [27]. More precisely, the spectrum $\mathrm{THH}(A)$ admits a $\mathbb{T}$-action and there is a $\mathbb{T}$-equivariant map $\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ of spectra for every prime p, where the target is equipped with the residual $\mathbb{T}/C_p \simeq \mathbb{T}$-action. See [27] for details on this construction. In [10], Hesselholt introduced topological periodic cyclic homology as the Tate construction of the $\mathbb{T}$-action on $\mathrm{THH}(A)$, that is $\mathrm{TP}(A) := \mathrm{THH}(A)^{t\mathbb{T}}$. Topological periodic cyclic homology has significant arithmetic interest. For example if A is the Eilenberg-MacLane spectrum of a commutative algebra over a perfect field, then Bhatt–Morrow–Scholze construct a filtration on topological periodic cyclic homology whose graded pieces are related to crystaline cohomology. See [3] for details.

3. Morava K-theory and classifying spaces of finite groups

Let p be a prime number and let $K(n)$ be the nth Morava K-theory spectrum with $\pi_* K(n) \simeq \mathbb{F}_p[v_n^{\pm 1}]$, where v_n is in degree $2(p^n - 1)$. Recall that $K(n)$ admits the structure of an $\mathbb{E}_1$-ring and there is a map $MU_{(p)} \to K(n)$ of spectra which equips $K(n)$ with a complex orientation. The associated formal group has height exactly n and the p-series of this formal group is given by $[p](t) = v_n t^{p^n}$. In fact, these properties characterize $K(n)$ up to equivalence in the ∞-category of spectra. More precisely, we have the following result (see [23, Lecture 24]).

Proposition 3.1. *Let R be an $\mathbb{E}_1$-ring equipped with a complex orientation whose formal group has height exactly n, and whose homotopy groups of the underlying spectrum are given by $\pi_* R \simeq \mathbb{F}_p[v_n^{\pm 1}]$. Then there exists an equivalence $K(n) \simeq R$ of spectra.*

The reader is refered to [31] or [23] for details on Morava K-theory.

Remark 3.2. In the second part of this project we will be interested in a 2-periodic version of Morava K-theory and some finer structural properties of the ∞-category of $K(n)$-local spectra. In section 5 we will recall all the material that we will need.

The goal of this section is to show that $K(n)^*BG$ is finitely generated as a left module over $K(n)^*$ when G is a finite group. This argument is due to Ravenel in [30] which does not seem to be available. I learned the argument from Tobias Barthel.

Lemma 3.3. *If C_{p^k} denotes the cyclic group of order p^k for some natural number k, then*

$$K(n)^*(BC_{p^k}) \simeq K(n)^*[[x]]/(x^{p^{kn}}),$$

where $x \in K(n)^2(\mathbb{C}P^\infty)$.

Proof. The degree p^k map on $\mathbb{C}P^\infty$ induces a fiber sequence $BC_{p^k} \to \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ which deeloops to a fiber sequence $S^1 \to BC_{p^k} \xrightarrow{\pi} \mathbb{C}P^\infty$. Since $K(n)$ admits a complex orientation, there is a Gysin sequence

$$\cdots \longrightarrow K(n)^*(BC_{p^k}) \xrightarrow{\partial} K(n)^*(\mathbb{C}P^\infty) \longrightarrow K(n)^*(\mathbb{C}P^\infty) \xrightarrow{\pi^*} K(n)^*(BC_{p^k}) \longrightarrow \cdots$$

where the map $K(n)^*(\mathbb{C}P^\infty) \to K(n)^*(\mathbb{C}P^\infty)$ is multiplication by the p^k-series $x^{p^{kn}}$. It follows that ∂ is surjective since multiplication by $x^{p^{kn}}$ is injective on $K(n)^*(\mathbb{C}P^\infty)$. Recall that $K(n)^*(\mathbb{C}P^\infty) \simeq K(n)^*[[x]]$ where $x \in K(n)^2(\mathbb{C}P^\infty)$ since $K(n)$ admits a complex orientation. Consequently, we find that

$$K(n)^*(BC_{p^k}) \simeq K(n)^*(\mathbb{C}P^\infty)/(x^{p^{kn}}) \simeq K(n)^*[[x]]/(x^{p^{kn}})$$

where $x \in K(n)^2(\mathbb{C}P^\infty)$ as wanted. □

Remark 3.4. The Künneth formula (see [31]) combined with Lemma 3.3 above allow us to compute $K(n)^*(BA)$ for a finite abelian group A.

Proposition 3.5. *If G is a p-group, then $K(n)^*(BG)$ is a finitely generated left $K(n)^*$-module.*

Proof. We proceed by induction on the order of G. If G is the trivial group, then there is nothing to prove. Suppose that G is not the trivial group and let H be a normal subgroup of G of index p and assume that $K(n)^*(BH)$ is a finitely generated left $K(n)^*$-module. There exists a fiber sequence $BH \to BG \to BC_p$ since H is a subgroup of G of index p. Let K be a group defined by the pullback of $G \to C_p$ along the quotient map $\mathbb{Z} \to C_p$, and consider the resulting commutative diagram

$$\begin{array}{ccccc} BH & \longrightarrow & BK & \longrightarrow & S^1 \\ \| & & \downarrow & & \downarrow \\ BH & \longrightarrow & BG & \longrightarrow & BC_p \end{array}$$

We first show that $K(n)^*(BK)$ is a finitely generated left $K(n)^*$-module. The Atiyah–Hirzebruch spectral sequence associated to the fiber sequence $BH \to BK \to S^1$ collapses on the second page since there are no non-trivial differentials. It follows that $K(n)^*(BK)$ is finitely generated over $K(n)^*$ since $K(n)^*(BH)$ is finitely generated over $K(n)^*$ by assumption. Note that there is a fiber sequence $S^1 \to BC_p \to \mathbb{C}P^\infty$ and consider the associated Atiyah–Hirzebruch spectral sequence

$$E_2^{s,t} \simeq H^s(\mathbb{C}P^\infty; K(n)^t(S^1)) \Rightarrow K(n)^{s+t}(BC_p)$$

We show that there exists an element y on some page of E with satisfies that $d_r(y) = x^{p^n}$, where $x \in K(n)^*(BC_p) \simeq K(n)^*[[x]]/(x^{p^n})$ and d_r denotes the differential on the r^{th}-page of E. Consider the Atiyah–Hirzebruch spectral sequence

$$\tilde{E}_2^{s,t} \simeq H^s(\mathbb{C}P^\infty; K(n)^t(*)) \Rightarrow K(n)^{s+t}(\mathbb{C}P^\infty)$$

associated to the fiber sequence $* \to \mathbb{C}P^\infty \xrightarrow{\text{id}} \mathbb{C}P^\infty$. The element $x \in \tilde{E}_2^{2,0}$ survives to the $\tilde{E}_\infty$-page which means that $\tilde{d}_r(x) = 0$ for all $r \geq 2$. There is a commutative diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & BC_p & \longrightarrow & \mathbb{C}P^\infty \\ \downarrow & & \downarrow & & \| \\ * & \longrightarrow & \mathbb{C}P^\infty & = & \mathbb{C}P^\infty \end{array}$$

of fiber sequences which induces a map $\varphi : \tilde{E}_r^{s,t} \to E_r^{s,t}$ of spectral sequences. Let $\varphi(x)$ denote the image of x in the spectral sequence E. By naturality of the Atiyah–Hirzebruch spectral sequence we conclude that $d_r(\varphi(x)) = 0$ for all $r \geq 2$ since $\tilde{d}_r(x) = 0$ for all $r \geq 2$. The element x^{p^n} cannot survive to the E_∞-page since $x^{p^n} = 0$ in $K(n)^*(BC_p)$. Thus, there exists an element y on some page E_r which satisfies that $d_r(y) = x^{p^n}$. This proves the claim. Note that there is a fiber sequence $BK \to BG \to \mathbb{C}P^\infty$ and consider the associated Atiyah–Hirzebruch spectral sequence

$$\widehat{E}_2^{s,t} \simeq H^s(\mathbb{C}P^\infty; K(n)^t(BK)) \Rightarrow K(n)^{s+t}(BG)$$

There is a commutative diagram

$$\begin{array}{ccccc} BK & \longrightarrow & BG & \longrightarrow & \mathbb{C}P^\infty \\ \downarrow & & \downarrow & & \| \\ S^1 & \longrightarrow & BC_p & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

of fiber sequences which induces a map $\psi : E_r^{s,t} \to \widehat{E}_r^{s,t}$ of spectral sequences. There exists an element y on some page E_r of E with $d_r(y) = x^{p^n}$ by the argument above. It follows that the image of x^{p^n} under the map ψ of spectral sequences must die in $\widehat{E}$ by naturality of the Atiyah–Hirzebruch spectral sequence. We conclude that $K(n)^*(BG)$ is a subquotient of $K(n)^*(BK)[[x]]/(x^{p^n})$ which is finitely generated since $K(n)^*(BK)$ is finitely generated over $K(n)^*$. It follows that $K(n)^*(BG)$ is finitely generated over $K(n)^*$ since $K(n)^*$ is a graded Noetherian ring. □

Proposition 3.6. *If G is a finite group, then $K(n)^*(BG)$ is a finitely generated left $K(n)^*$-module.*

Proof. As before we may assume that G is not the trivial group. Let H be a Sylow p-subgroup of G and let $i : H \hookrightarrow G$ denote the inclusion homomorphism. There is a transfer map $\mathrm{Tr} : \Sigma^\infty_+ BG \to \Sigma^\infty_+ BH$ of spectra (see [15]) such that the composite

$$\Sigma^\infty_+ BG \xrightarrow{\mathrm{Tr}} \Sigma^\infty_+ BH \xrightarrow{\Sigma^\infty_+ Bi} \Sigma^\infty_+ BG$$

induces multiplication by the index $[G : H]$ on $H^*(-;\mathbb{Z})$. We conclude that the composite

$$(\Sigma^\infty_+ BG)_{(p)} \xrightarrow{\mathrm{Tr}_{(p)}} (\Sigma^\infty_+ BH)_{(p)} \xrightarrow{(\Sigma^\infty_+ Bi)_{(p)}} (\Sigma^\infty_+ BG)_{(p)}$$

is an equivalence in the ∞-category of spectra since p is coprime to the index $[G : H]$. It follows that the composite

$$K(n)^*(BG) \xrightarrow{(\Sigma^\infty_+ Bi)^*} K(n)^*(BH) \xrightarrow{\mathrm{Tr}^*} K(n)^*(BG)$$

on $K(n)$-cohomology is an isomorphism since $K(n)^*(X_{(p)}) \simeq K(n)^*(X)$ for all spectra X. We conclude that $K(n)^*(BG)$ is a retract of $K(n)^*(BH)$ as left $K(n)^*$-modules. This proves the wanted since $K(n)^*(BH)$ is finitely generated as a left $K(n)^*$-module by Proposition 3.5. □

Remark 3.7. Let X be a spectrum. If $K(n)^*(X)$ is a finitely generated left $K(n)^*$-module, then $K(n)_*(X)$ is also a finitely generated left $K(n)^*$-module. This follows from the fact that the canonical map

$$K(n)^*(X) \to \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*(X))$$

is an isomorphism (See [31]).

4. Vanishing of the Tate construction on $K(n)$

Fix a prime number p. The goal of this section is to prove the following result due to Greenlees and Sadofsky.

Theorem 4.1 (Greenlees–Sadofsky [9, Theorem 1.1])**.** *Let G be a finite group. If $K(n)$ denotes the nth Morava K-theory spectrum at the prime p equipped with the trivial G-action, then $K(n)^{tG} \simeq 0$.*

Remark 4.2. The proof of this result is elementary in the sense that it does not rely on the Periodicity Theorem of Hopkins and Smith [13]. The proof will crucially use the characterization of the Tate construction we obtained previously, the complex orientation of Morava K-theory, and the computation that the $K(n)$-cohomology of the classifying space of a finite group is finitely generated as a left $K(n)^*$-module. In Section 6 we will show that the Tate construction vanishes $K(n)$-locally which is due to Hovey and Sadofsky [14]. Theorem 4.1 above is perhaps a first indication that such a result is true. Indeed, Hovey and Sadofsky's argument uses Greenlees and Sadofsky's result, the Periodicity Theorem, and that the Bousfield localization with respect to Morava E-theory is smashing due to Hopkins and Ravenel (see [23, Lecture 31]).

We will first present an argument due to Kuhn [19] which allows us to reduce to the case where G is a cyclic group of order p. This relies crucially on the characterization of the Tate construction that we obtained previously.

Proposition 4.3 (Kuhn [19, Proposition 1.10])**.** *Let R be an $\mathbb{E}_1$-ring. If $R^{tC_p} \simeq 0$ for all prime numbers p where R is equipped with the trivial C_p-action, then $M^{tG} \simeq 0$ for all left R-module spectra M and all finite groups G.*

Remark 4.4. In the setting of Proposition 4.3 above, if R is the nth Morava K-theory spectrum at the prime p, then it suffices to show that $K(n)^{tC_p}$ vanishes at the single prime p.

We will prove Propositon 4.3 using the following two lemmas.

Lemma 4.5 (Kuhn [19, Lemma 2.7])**.** *Let G be a p-group and let X be a spectrum with a G-action. If $X^{tG_p} \simeq 0$ for every p-Sylow subgroup G_p of G where p is a prime number dividing the order of G, then $X^{tG} \simeq 0$.*

Proof. See [19, Lemma 2.7]. □

Lemma 4.6 (Kuhn [19, Lemma 2.8])**.** *Let H be a normal subgroup of a finite group G and let R be an $\mathbb{E}_1$-ring spectrum equipped with a G-action. If $R^{tH} \simeq 0$ and $R^{tG/H} \simeq 0$, then $R^{tG} \simeq 0$.*

Proof. We wish to show that the norm map $\mathrm{Nm}_G(R)\colon R_{hG} \to R^{hG}$ is an equivalence. It follows from Remark 2.10 that if the composite

$$\begin{array}{ccc} \Sigma^\infty_+ G_{hG} \simeq (\Sigma^\infty_+ G_{hH})_{hG/H} & \xrightarrow{\mathrm{Nm}_H(\Sigma^\infty_+ G)_{hG/H}} & (\Sigma^\infty_+ G^{hH})_{hG/H} \\ & & \Big\downarrow {\scriptstyle \mathrm{Nm}_{G/H}(\Sigma^\infty_+ G^{hH})} \\ & & (\Sigma^\infty_+ G^{hH})^{hG/H} \simeq \Sigma^\infty_+ G^{hG} \end{array}$$

is an equivalence in Sp, then $\mathrm{Nm}_G(X)$ is an equivalence precisely if both $\mathrm{Nm}_H(X)_{hG/H}$ and $\mathrm{Nm}_{G/H}(X^{hH})$ are equivalences in Sp for all X in Sp^{BG}. We first show that $\mathrm{Nm}_H(\Sigma^\infty_+ G)_{hG/H}$ is an equivalence. There is an equivalence

$$\Sigma^\infty_+ G \simeq \bigoplus_{gH \in G/H} \Sigma^\infty_+ H$$

of spectra from which it follows that $\mathrm{Nm}_H(\Sigma^\infty_+ G)$ is an equivalence. Consequently, we conclude that $\mathrm{Nm}_H(\Sigma^\infty_+ G)_{hG/H}$ is an equivalence. Next, we show that $\mathrm{Nm}_{G/H}(\Sigma^\infty_+ G^{hH})$ is an equivalence. Observe that there are equivalences $\Sigma^\infty_+ G^{hH} \simeq \Sigma^\infty_+ G_{hH} \simeq \Sigma^\infty_+ G/H$. This gives rise to a commutative diagram

$$\begin{array}{ccc} (\Sigma^\infty_+ G^{hH})_{hG/H} & \xrightarrow{\mathrm{Nm}_{G/H}(\Sigma^\infty_+ G^{hH})} & (\Sigma^\infty_+ G^{hH})^{hG/H} \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ (\Sigma^\infty_+ G/H)_{hG/H} & \xrightarrow[\mathrm{Nm}_{G/H}(\Sigma^\infty_+ G/H)]{\simeq} & (\Sigma^\infty_+ G/H)^{hG/H} \end{array}$$

where $\mathrm{Nm}_{G/H}(\Sigma^\infty_+ G/H)$ is an equivalence by Remark 2.10. We conclude that $\mathrm{Nm}_{G/H}(\Sigma^\infty_+ G^{hH})$ is an equivalence. To end the proof we need to show that both $\mathrm{Nm}_H(R)_{hG/H}$ and $\mathrm{Nm}_{G/H}(R^{hH})$ are equivalences in Sp. We have that $R^{tH} \simeq 0$ by assumption which implies that $\mathrm{Nm}_K(R)$ is an equivalence. It follows that $\mathrm{Nm}_H(R)_{hG/H}$ is an equivalence. Note that R^{hH} admits the structure of a left R-module by Remark 2.17. It follows that $(R^{hH})^{tG/H}$ admits the structure of a left $R^{tG/H}$-module. But $R^{tG/H}$ vanishes by assumption so we conclude that $(R^{hH})^{tG/H} \simeq 0$ which implies that $\mathrm{Nm}_{G/H}(R^{hH})$ is an equivalence. This ends the proof. □

Proof of Proposition 4.3. By Lemma 4.5 we may assume that G is a p-group. If M is a left R-module, then M^{tG} admits the structure of a left R^{tG}-module by Remark 2.17. Consequently, it suffices to show that R^{tG} vanishes. Since G is a p-group it is in particular a solvable group so we may choose a series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{n-1} = G_n = G$$

such that each G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is a cyclic group of prime order for all $i \in \{0, 1, \ldots, n-1\}$. Note that $R^{tG_1} \simeq 0$ since G_1 is a cyclic group of prime order. Since G_2/G_1 is a cyclic group of prime order then $R^{tG_2/G_1} \simeq 0$. It follows from Lemma 4.6 that $R^{tG_2} \simeq 0$. If we continue in this fashion we ultimately find that $R^{tG_{n-1}} \simeq 0$. As before we find that $R^{tG/G_{n-1}} \simeq 0$ since G/G_{n-1} is a cyclic group of prime order so Lemma 4.6 gives $R^{tG} \simeq 0$ as wanted. □

We have reduced proving Theorem 4.1 to the case where G is a cyclic group of order p. The following argument is due to Greenlees and Sadofsky in [9].

Remark 4.7. Recall that if $\xi : E \to X$ is a complex vector bundle over a space X, then we can form the associated Thom spectrum X^{ξ} of ξ as the suspension spectrum $\Sigma^{\infty}\operatorname{Th}\xi$ of the Thom space of ξ. If ζ is a complex vector bundle over a compact space X, then there exists a complex vector bundle η over X such that $\xi \oplus \eta \simeq \varepsilon_n$ for some integer n. In this case we define $X^{-\zeta} := \Sigma^{-n} X^{\eta} \simeq \Sigma^{\infty - n}\operatorname{Th}\eta$.

Lemma 4.8 (Greenlees–Sadofsky [9, Lemma 2.1])**.** *Let G be a finite group and let ζ be a finite dimensional complex vector bundle over BG. If R is an $\mathbb{E}_1$-ring equipped with a complex orientation satisfying that R_*BG is a finitely generated left R_*-module, then*

$$\varprojlim_s (R \otimes BG^{-s\zeta}) \simeq 0$$

in the ∞-category Sp^{BG} of spectra with a G-action.

Proof. Let $BG_{(k)}$ denote the k-skeleton in a CW-filtration on the classifying space BG of G. There exists a sufficiently large natural number r such that $R_*(BG_{(r)}) \to R_*(BG)$ is surjective since R_*BG is a finitely generated left R_*-module. Let λ denote the rank of ζ and choose an integer j such that $r - (s+j)\lambda < -s\lambda$. There is a commutative diagram

$$\begin{array}{ccc} R_*BG_{(r)} & \twoheadrightarrow & R_*BG \\ \downarrow \simeq & & \downarrow \simeq \\ R_{*-\lambda}BG_{(r)}^{-s\zeta} & \longrightarrow & R_{*-\lambda}BG^{-s\zeta} \end{array}$$

for every integer s by the Thom isomorphism. It follows that the map $R \otimes BG_{(r)}^{-s\zeta} \to R \otimes BG^{-s\zeta}$ of spectra is surjective on π_* for every integer s. Note that the composite

$$BG_{(r)}^{-(s+j)\zeta} \to BG^{-(s+j)\zeta} \to BG^{-s\zeta}$$

is nullhomotopic in Sp^{BG} since the top dimensional cell of $BG_{(r)}^{-(s+j)\zeta}$ is in a lower dimensional than the bottom dimensional cell of $BG^{-s\zeta}$ by the choice of j. This means that the composite

$$R \otimes BG_{(r)}^{-(s+j)\zeta} \to R \otimes BG^{-(s+j)\zeta} \to R \otimes BG^{-s\zeta}$$

is nullhomotopic in Sp^{BG}. Since $R \otimes BG_{(r)}^{-(s+j)\zeta} \to R \otimes BG^{-(s+j)\zeta}$ is surjective on π_* we must have that $R \otimes BG^{-(s+j)\zeta} \to R \otimes BG^{-s\zeta}$ is the zero homomorphism on π_*. It follows that $\varprojlim_s (R \otimes BG^{-s\zeta}) \simeq 0$ in Sp^{BG} as wanted. □

Proposition 4.9 (Greenlees–Sadofsky [9, Lemma 2.2])**.** *Let G be a finite group and let V be a complex representation of G. If R is an $\mathbb{E}_1$-ring equipped with a complex orientation satisfying that R_*BH is a finitely generated left R_*-module for every subgroup H of G, then*

$$F(S^{\infty V}, R \otimes \Sigma^{\infty}_{+} EG) \simeq 0$$

in the ∞-category Sp^{BG} of spectra with a G-action, where R is endowed with the trivial G-action.

Proof. First note that

$$\begin{aligned} F(S^{\infty V}, R \otimes \Sigma^\infty_+ EG) &\simeq F(\varinjlim_r S^{rV}, R \otimes \Sigma^\infty_+ EG) \\ &\simeq \varinjlim_r F(S^{rV}, R \otimes \Sigma^\infty_+ EG) \\ &\simeq \varinjlim_r R \otimes \Sigma^\infty_+ EG \otimes S^{-rV}. \end{aligned}$$

If H is a subgroup of G, then the Adams equivalence [20, Theorem II.7.1] provides a natural equivalence

$$(R \otimes \Sigma^\infty_+ EG \otimes S^{-rV})^H \simeq \varinjlim_r R \otimes BH^{-r\zeta}$$

in the ∞-category Sp^{BG} of spectra with a G-action, where ζ is the complex vector bundle over BH induced by restricting the complex representation of G to the subgroup H. Consequently, we find that

$$F(S^{\infty V}, R \otimes \Sigma^\infty_+ EG)^H \simeq \varinjlim_r R \otimes BH^{-r\zeta} \simeq 0$$

where the last equivalence follows from Lemma 4.8 above. We conclude that

$$F(S^{\infty V}, R \otimes \Sigma^\infty_+ EG) \simeq 0$$

in Sp^{BG} by the G-Whitehead theorem. □

The nth Morava K-theory spectrum at a prime p is equipped with a complex orientation and Proposition 3.6 combined with Remark 3.7 assert that $K(n)_*(BG)$ is a finitely generated left $K(n)_*$-module when G is a finite group. With this in mind we can now finally prove Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.3 and Remark 4.4 it suffices to show that $K(n)^{tC_p}$ vanishes where C_p denotes the cyclic group of order p. Let V be the cyclic complex representation of C_p, and let ζ be the associated complex line bundle over BC_p. Observe that $\Sigma^\infty_+ EC_p \simeq \Sigma^\infty_+ S(\infty V)$, and the cofiber $\widetilde{EC_p}$ of $\Sigma^\infty_+ EC_p \to S^0$ is equivalent to $\Sigma^\infty S^{\infty V}$. It follows from Remark 2.12 and Proposition 4.9 that

$$\begin{aligned} K(n)^{tC_p} &\simeq (\Sigma F(\widetilde{EC_p}, K(n) \otimes \Sigma^\infty_+ EC_p))^{C_p} \\ &\simeq (\Sigma F(S^{\infty V}, K(n) \otimes \Sigma^\infty_+ EC_p))^{C_p} \simeq 0 \end{aligned}$$

which ends the proof. □

5. Lubin–Tate spectra and the $K(n)$-local category

In this section we briefly review some terminology from stable homotopy theory following [12] and [23] that we will need later. To keep the exposition brisk we refer the interested reader to [12] or [23] for a more complete reference. We will start by briefly discussing the Bousfield localization of the ∞-category of spectra. Let E be a spectrum. Recall that a spectrum X is E-acyclic if $X \otimes E \simeq 0$. A spectrum Y is E-local if every map $X \to Y$ out of an E-acyclic spectrum X is nullhomotopic. A map $X \to Y$ of spectra is an E-equivalence if the induced map $X \otimes E \to Y \otimes E$ is an equivalence.

Proposition 5.1 (Bousfield [5])**.** *Let E be a spectrum. If X is a spectrum, then there exists an essentially unique cofiber sequence*

$$G_E X \to X \to L_E X$$

in the ∞-category of spectra satisfying the following:

(1) The spectrum $G_E X$ is E-acyclic.
(2) The spectrum $L_E X$ is E-local.

We obtain a functor $L_E : \mathrm{Sp} \to \mathrm{Sp}$ called the Bousfield localization of the ∞-category of spectra with respect to E. The map $X \to L_E X$ is characterized up to equivalence by the following properties:

(1) The spectrum $L_E X$ is E-local.
(2) The map $X \to L_E X$ is an E-equivalence.

Let $L_{K(n)}$ denote a Bousfield localization of the ∞-category of spectra with respect to the nth Morava K-theory $K(n)$. Similarly, let L_n denote a Bousfield localization of the ∞-category of spectra with respect to the nth Morava E-theory. We let $\mathrm{Sp}_{K(n)}$ denote the ∞-category of $K(n)$-local spectra, that is $\mathrm{Sp}_{K(n)} := L_{K(n)}\mathrm{Sp}$ is the essential image of the Bousfield localization of the ∞-category of spectra with respect to the nth Morava K-theory.

Let us briefly recall the construction of Lubin–Tate spectra[2] following [12, Section 2.1].

Definition 5.2. Let κ be a perfect field of characteristic $p > 0$. An infinitesimal thickening of κ is a pair (A, ρ_A) where A is a complete local Noetherian ring with maximal ideal $\mathfrak{m}_A$ and ρ_A is a ring homomorphism $\rho_A : A \to \kappa$ which induces an isomorphism $A/\mathfrak{m}_A \simeq \kappa$.

Definition 5.3. Let κ be a perfect field of characteristic $p > 0$. Let $\mathbb{G}_0$ be a formal group over κ and let (A, ρ_A) be an infinitesimal thickening of κ. A deformation of $\mathbb{G}_0$ along ρ_A is a pair $(\mathbb{G}, \alpha)$ where $\mathbb{G}$ is a formal group over A and α is an isomorphism $\mathbb{G}_0 \simeq (\rho_A)_* \mathbb{G}$ of formal groups.

Remark 5.4. The collection of deformations of $\mathbb{G}_0$ along ρ_A can be organized into a groupoid $\mathrm{Def}_{\mathbb{G}_0}(A, \rho_A)$ and this groupoid is always discrete. See [23, Lecture 21, Remark 3] for the argument.

Let $\mathbb{G}_0$ be a formal group over κ and let (A, ρ_A) be a infinitesimal thickening of κ. Lubin and Tate [21] show that there exists a universal deformation of $\mathbb{G}_0$. We formulate the theorem as in [25, Theorem 3.0.1].

Theorem 5.5 (Lubin–Tate)**.** *Let κ be a perfect field of characteristic $p > 0$, and let $\mathbb{G}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Then there exists an infinitesimal thickening $(R_{\mathrm{LT}}, \rho_{\mathrm{LT}})$ of κ and a deformation $(\mathbb{G}, \alpha)$ of $\mathbb{G}_0$ along ρ_{LT} which satisfies the following universal property: if (A, ρ_A) is any other infinitesimal thickening of κ, then extension of scalars induces an equivalence*

$$\mathrm{Hom}_{/\kappa}(R_{\mathrm{LT}}, A) \simeq \mathrm{Def}_{\mathbb{G}_0}(A, \rho_A)$$

of categories, where $\mathrm{Hom}_{/\kappa}(R_{\mathrm{LT}}, A)$ denotes the set of ring homomorphisms over κ considered as a category with only identity morphisms.

Proof. See [23, Lecture 21] for a proof. □

Definition 5.6. Define a category $\mathcal{FG}_{\mathrm{pf}}$ as follows:

(1) An object of $\mathcal{FG}_{\mathrm{pf}}$ is a pair $(\kappa, \mathbb{G})$, where κ is a perfect field of characteristic $p > 0$ and $\mathbb{G}$ is a 1-dimensional formal group of finite height over κ.
(2) A morphism from $(\kappa, \mathbb{G})$ to $(\kappa', \mathbb{G}')$ in $\mathcal{FG}_{\mathrm{pf}}$ is a pair (f, α), where $f : \kappa \to \kappa'$ is a ring homomorphism and α is an isomorphism $\mathbb{G}' \simeq f_* \mathbb{G}$ of formal groups over κ'.

[2] In the introduction we called these Morava E-theories.

We will think of the category $\mathcal{FG}_{\mathrm{pf}}$ as an ∞-category by applying the nerve construction N.

Definition 5.7 ([12, Definition 2.1.8])**.** A Lubin–Tate spectrum is an $\mathbb{E}_\infty$-ring E which satisfies the following conditions:

(1) The $\mathbb{E}_\infty$-ring E is even-periodic. This means that if n is odd, then $\pi_n E \simeq 0$ and there exists an element $\beta \in \pi_2 E$ such that multiplication by β induces an isomorphism $\pi_n E \to \pi_{n+2} E$ for every integer n.

(2) The underlying commutative ring $\pi_0 E$ is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field $\kappa(E) \coloneqq \pi_0 E/\mathfrak{m}$ is perfect of characteristic $p > 0$.

(3) Let $\mathbb{G}$ denote the formal group over $\pi_0 E$ induced from the complex orientation of E, and let $\mathbb{G}_0$ denote the formal group over $\kappa(E)$. Then $\mathbb{G}_0$ has finite height and $\mathbb{G}$ is a universal deformation of $\mathbb{G}_0$ in the sense of Theorem 5.5.

Let $\mathrm{CAlg}_{\mathrm{LT}}$ denote the full subcategory of the ∞-category CAlg of $\mathbb{E}_\infty$-rings spanned by the Lubin–Tate spectra. Define the height of E to be the height of the formal group $\mathbb{G}_0$ over the residue field $\kappa(E)$.

Theorem 5.8 (Goerss–Hopkins–Miller)**.** *The construction $E \mapsto (\kappa(E), \mathbb{G}_0)$ refines to an equivalence*

$$\mathrm{CAlg}_{\mathrm{LT}} \xrightarrow{\simeq} N\mathcal{FG}_{\mathrm{pf}}$$

from the ∞-category of Lubin–Tate spectra to the ∞-category of 1-dimensional formal groups of finite height over perfect fields of characteristic $p > 0$.

Remark 5.9. Let us provide some background on the Goerss–Hopkins–Miller theorem. Let $\mathbb{G}_0$ be a 1-dimensional formal group of finite height over a perfect field κ of characteristic $p > 0$. It follows from Theorem 5.5 that there exists an infinitesimal thickening $(R_{\mathrm{LT}}, \rho_{\mathrm{LT}})$ of κ and a universal deformation $(\mathbb{G}, \alpha)$ of $\mathbb{G}_0$ along ρ_{LT}. The complete local Noetherian ring R_{LT} is called the Lubin–Tate ring. Morava observed that the universal deformation $\mathbb{G}$ over the Lubin–Tate ring R_{LT} was Landweber exact. Thus we obtain an even-periodic cohomology theory which is represented by an even-periodic spectrum $E_{(\mathbb{G}_0,\kappa)}$ by Brown representability. Note that the spectrum $E_{(\kappa,\mathbb{G}_0)}$ is functorial in the pair $(\kappa, \mathbb{G}_0)$ up to homotopy. Hopkins and Miller show that the spectrum $E_{(\kappa,\mathbb{G}_0)}$ admits an essentially unique $\mathbb{E}_1$-ring structure using obstruction theory. Goerss and Hopkins obtained a refinement of this result namely that the spectrum $E_{(\kappa,\mathbb{G}_0)}$ admits an essentially unique $\mathbb{E}_\infty$-ring structure and that $E_{(\kappa,\mathbb{G}_0)}$ depends functorially on the pair $(\kappa, \mathbb{G}_0)$ (see [7]). Recently, Lurie has given a new construction of Lubin–Tate spectra in [25] which bypasses the use of Landweber's theorem. Very roughly, the idea is to realize $E_{(\kappa,\mathbb{G}_0)}$ as a solution to a moduli problem in the ∞-category of $\mathbb{E}_\infty$-rings.

Example 5.10. Let κ be a perfect field of characteristic $p > 0$, and let $\mathbb{G}_0$ be a 1-dimensional formal group of finite height over κ. The content of the Goerss–Hopkins–Miller theorem above is that there exists a unique, up to contractible choice, $\mathbb{E}_\infty$-ring E which is admits the structure of a Lubin–Tate spectrum in the sense of Definition 5.7. In fact, there is a non-canonical identification $\pi_0 E \simeq W(\kappa)[[u_1, \dots, u_n]]$ of the complete local Noetherian ring $\pi_0 E$ with the ring of formal power series over the ring of Witt vectors $W(\kappa)$ of κ. It will be convenient to set $u_0 \coloneqq p$. For each $0 \le i \le n$ we obtain an E-module homomorphism $u_i : E \to E$. For each i we let M_i denote the cofiber of $u_i : E \to E$ in the ∞-category Mod_E of E-modules. For each integer $n \ge 1$ define the 2-periodic Morava K-theory by

$$K(n) \coloneqq \bigotimes_{0 \le i < n} M_i$$

where the tensor product is formed in the symmetric monoidal ∞-category Mod_E of E-modules. The 2-periodic Morava K-theory spectrum $K(n)$ is a E-module and the homotopy groups are given by

$$\pi_i K(n) \simeq \begin{cases} \kappa & i = 2j \\ 0 & i = 2j+1. \end{cases}$$

In the second part of this paper we will only be concerned with 2-periodic Morava K-theory in contrast to the $2(p^n-1)$-periodic Morava K-theory considered previously. Observe that the $2(p^n-1)$-periodic Morava K-theory is a direct summand of 2-periodic Morava K-theory.

The following theorem characterizes the ∞-category of $K(n)$-local spectra as the essential image of an accesible and exact localization functor.

Proposition 5.11. *Let E be a Lubin-Tate spectrum of height n and let $K(n)$ denote the Morava K-theory spectrum asssociated to E. Then the following are satisfied:*

1. *(1) The inclusion $\iota : \mathrm{Sp}_{K(n)} \hookrightarrow \mathrm{Sp}$ admits a left adjoint F which is accesible and exact, and the ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra is the essential image of $L = \iota \circ F : \mathrm{Sp} \to \mathrm{Sp}$.*
2. *(2) The symmetric monoidal structure on Sp given by the smash product of spectra induces a symmetric monoidal structure on the ∞-category of $K(n)$-local spectra. The inclusion $\iota : \mathrm{Sp}_{K(n)} \hookrightarrow \mathrm{Sp}$ aquires a lax symmetric monoidal structure and its left adjoint $F : \mathrm{Sp} \to \mathrm{Sp}_{K(n)}$ aquires a symmetric monoidal structure.*

Proof. See [12, Proposition 2.1.15] and [12, Proposition 2.1.3]. □

Remark 5.12. Recall that the symmetric monoidal structure on the ∞-category of spectra given by the smash product of spectra satisfies that the tensor product $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves small colimits seperately in each variable and that the sphere spectrum S^0 is the unit [24, Corollary 4.8.2.19]. Let

$$\hat{\otimes} : \mathrm{Sp}_{K(n)} \times \mathrm{Sp}_{K(n)} \to \mathrm{Sp}_{K(n)}$$

denote the localized tensor product on the ∞-category of $K(n)$-local spectra induced by the usual symmetric monoidal structure on Sp. Explicitly, if X and Y are $K(n)$-local spectra, then $X \hat{\otimes} Y \simeq L(X \otimes Y)$. It follows that the functor $\hat{\otimes}$ preserves small colimits seperately in each variable and that the $K(n)$-local sphere $L_{K(n)} S^0$ is the unit of $\hat{\otimes}$.

Remark 5.13. Let R be an $\mathbb{E}_\infty$-ring and let A be an $\mathbb{E}_1$-algebra object of the ∞-category Mod_R of R-modules. Let ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$ denote the ∞-category of A-A-bimodule objects of Mod_R. The ∞-category of A-A-bimodules is presentable [24, Corollary 4.3.3.10] and the relative tensor product $\otimes_A$ endows ${}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$ with a monoidal structure and $\otimes_A$ preserves small colimits seperately in each variable [24, Corollary 4.4.2.15]. Endow the ∞-category of spaces with its Cartesian monoidal structure. It follows that there is an essentially unique monoidal functor

$$A[-] : \mathcal{S} \to {}_A\mathrm{BMod}_A(\mathrm{Mod}_R)$$

which preserves small colimits. If X is a space, then the underlying spectrum of $A[X]$ can be identified with $A \otimes \Sigma^\infty_+ X$. We will be interested in the case where R is a Lubin–Tate spectrum. Let E be a Lubin–Tate spectrum associated to the pair $(\kappa, \mathbb{G}_0)$ where κ is a perfect field of characteristic $p > 0$ and $\mathbb{G}_0$ is a formal group of height n over κ, and let $K(n)$ be the associated Morava K-theory spectrum. If X is a space, then we define $E^\wedge_*(X) = \pi_* L_{K(n)} E[X]$.

Proposition 5.14. *The functor $\hat{\otimes}$ determines a fully faithful embedding*

$$\alpha : \mathrm{Sp}_{K(n)} \to \mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$$

whose essential image is the full subcategory of $\mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$ *spanned by those functors which preserve small colimits. Moreover, the embedding* α *aquires a monoidal structure if we endow* $\mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$ *with the monoidal structure given by composition of functors.*

Proof. Combine [12, Proposition 2.1.5] and [12, Remark 2.1.6]. □

We will end this section with a result which shows that every $K(n)$-local spectrum can be constructed from $K(n)$-local E-module spectra where E is a Lubin–Tate spectrum of height n. This will be crucial later on.

Proposition 5.15. *Let* E *be a Lubin–Tate spectrum of height* n *and let* $K(n)$ *be the Morava* K*-theory spectrum associated to* E*. If* $\mathcal{C}$ *is a stable subcategory of* $\mathrm{Sp}_{K(n)}$ *which is closed under retracts and contains the essential image of the forgetful functor* $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}) \to \mathrm{Sp}_{K(n)}$*, then* $\mathcal{C} = \mathrm{Sp}_{K(n)}$*.*

Proof. See [12, Propositon 5.2.6]. □

6. Vanishing of the $K(n)$-local Tate construction

Let us once more fix a prime number p. In section 4 we arrived at the following result due to Greenlees and Sadofsky: if the nth Morava K-theory spectrum $K(n)$ is equipped with a trivial action of a finite group G, then the Tate construction of the trivial action of G on $K(n)$ vanishes (see Theorem 4.1). The goal of this section is to explain a globalization of this result namely that the Tate construction vanishes $K(n)$-locally.

Theorem 6.1 (Hovey–Sadofsky [14])**.** *If* G *is a finite group, then* $L_{K(n)}(L_{K(n)}S^0)^{tG} \simeq 0$*, where* $L_{K(n)}S^0$ *is equipped with the trivial* G*-action.*

Remark 6.2. If X is an object of $\mathrm{Sp}_{K(n)}$ equipped with an action of a finite group G, then X^{tG} admits the structure of a module over $(L_{K(n)}S^0)^{tG}$ since both $L_{K(n)}$ and $(-)^{tG}$ admit a lax symmetric monoidal structures. It follows from Theorem 6.1 above that $X^{tG} \simeq 0$ in $\mathrm{Sp}_{K(n)}$ or equivalently that the $K(n)$-local norm map $\mathrm{Nm}_G\colon X_{hG} \to X^{hG}$ is an equivalence in $\mathrm{Sp}_{K(n)}$.

If X is a spectrum, then we let $\langle X\rangle$ denote the Bousfield class of X as defined in [5]. We need the following lemma due to Ravenel.

Lemma 6.3 (Ravenel [29])**.** *If* X *is a spectrum equipped with a self-map* $f\colon \Sigma^k X \to X$*, then*

$$\langle X\rangle = \langle X/f \oplus f^{-1}X\rangle.$$

Proof. If Y is a spectrum in the Bousfield class of X, then it is easy to see that Y is X/f-acyclic and $f^{-1}X$-acyclic. Conversely, if Y is contained in the Bousfield class of $\langle X/f \oplus f^{-1}X\rangle$, then we in particular have that $X/f \otimes Y \simeq 0$. It follows that $f\colon \Sigma^k(X\otimes Y) \simeq \Sigma^k X \otimes Y \to X \otimes Y$ is an equivalence. We conclude that $X \otimes Y \simeq f^{-1}(X\otimes Y) \simeq 0$ as wanted. □

If X is a spectrum, then we let $\mathrm{Thick}^{\otimes} X$ denote the full subcategory of the ∞-category of spectra spanned by those spectra that can be obtained in finitely many steps from spectra on the form $X \otimes Y$ by taking cofibers and retracts. The proof of Theorem 6.1 is a consequence of the following three results due to Hovey and Sadofsky.

Lemma 6.4 (Hovey–Sadofsky [14, Lemma 2.1])**.** *There exists a finite spectrum* F *of type* n *such that* L_nF *is an object of* $\mathrm{Thick}^{\otimes}K(n)$*.*

Proof. See [14, Lemma 2.1]. □

Lemma 6.5 (Hovey–Sadofsky [14, Lemma 3.1])**.** *Let G be a finite group. If X is a spectrum contained in* $\mathrm{Thick}^{\otimes}K(n)$*, then $X^{tG} \simeq 0$ where X is equipped with the trivial G-action.*

Proof. It suffices to show that $(K(n)\otimes Y)^{tG} \simeq 0$ for every spectrum Y which we equip with the trivial G-action. Recall that $K(n)$ admits the structure of an $\mathbb{E}_1$-ring and $K(n)\otimes Y$ aquires the structure of a left $K(n)$-module spectrum. It follows from Remark 2.17 that $(K(n)\otimes Y)^{tG}$ admits the structure of a left $K(n)^{tG}$-module. We conclude that $(K(n)\otimes Y)^{tG} \simeq 0$ since $K(n)^{tG} \simeq 0$ by Theorem 4.1. □

Proposition 6.6 (Hovey–Sadofsky [14, Lemma 3.2])**.** *Let G be a finite group. If F is a finite spectrum of type n equipped with the trivial G-action, then $L_n(X\otimes F)^{tG} \simeq 0$ for every spectrum X with a G-action.*

Proof. Let $\mathcal{C}$ denote the full subcategory of Sp spanned by those finite spectra F which satisfy that $L_n(X\otimes F)^{tG} \simeq 0$ for every spectrum X with a G-action, where F is equipped with the trivial G-action. The full subcategory $\mathcal{C}$ of Sp is a thick subcategory. It follows from the Thick Subcategory Theorem [13, Theorem 7] that $\mathcal{C}$ is contained in the full subcategory $\mathcal{C}_m$ of Sp spanned by those spectra which are $K(m-1)$-acyclic for some m. We show that $\mathcal{C}$ is contained in $\mathcal{C}_n$. It suffices to show that the intersection of $\mathcal{C}$ and $\mathcal{C}_n$ is non-zero. There exists a finite spectrum F' of type n which satisfies that L_nF' is an object of $\mathrm{Thick}^{\otimes}K(n)$ by Lemma 6.4. Let X be a spectrum equipped with a G-action. It follows that $X\otimes L_nF'$ is an object of $\mathrm{Thick}^{\otimes}K(n)$. We conclude that $L_n(X\otimes F')$ is contained in $\mathrm{Thick}^{\otimes}K(n)$ since L_n is smashing (see [23, Lecture 31]). It follows from Lemma 6.5 that $L_n(X\otimes F')^{tG} \simeq 0$ where F' is equipped with the trivial G-action. This means that F' is an object of $\mathcal{C}$. We certainly also have that F' is an object of $\mathcal{C}_n$ since F' has type n. In conclusion, we have shown that $\mathcal{C}_n \subseteq \mathcal{C}$. Consequently, if F is a finite spectrum of type n equipped with the trivial G-action, then the underlying spectrum of F is contained in $\mathcal{C}_n$ hence in $\mathcal{C}$ which yields the wanted. □

Remark 6.7. The previous results were stated in terms of the Bousfield localization of the ∞-category of spectra with respect to Lubin–Tate spectra. If X is a spectrum, then for every integer $n \geq 1$ there is a pullback sqaure

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array}$$

in the ∞-category of spectra called the chromatic fracture square (see [23, Lecture 23]). In particular, if F is a finite spectrum of type n, then

$$L_{n-1}L_{K(n)}F \simeq L_{n-1}(F\otimes L_{K(n)}S^0) \simeq L_{K(n)}S^0 \otimes L_{n-1}F.$$

since F is a compact object of Sp and the Bousfield localization L_n is smashing. It follows that the horizontal map $L_nF \to L_{K(n)}F$ in the chromatic fracture square above is an equivalence since $L_{n-1}F$ vanishes.

Corollary 6.8. *Let G be a finite group. If F is a finite spectrum of type n, then $(L_{K(n)}F)^{tG} \simeq 0$ where F is equipped with the trivial G-action.*

Proof. Follows from Proposition 6.6 and Remark 6.7 above. □

Proof of Theorem 6.1. It suffices to show that $K(n)\otimes(L_{K(n)}S^0)^{tG}$ vanishes. Using Lemma 6.3 inductively we may write

$$\langle S^0\rangle = \langle F(n) \oplus \bigoplus_{i=0}^{n-1} T(i)\rangle,$$

where $F(n)$ is a finite spectrum of type n and $T(i)$ is a telescope of a v_i self-map on a finite spectrum of type i. Note that $K(n) \otimes T(i) \simeq 0$ for every integer $0 \le i \le n-1$. Using this observation and that $F(n)$ is a compact object we find that

$$\begin{aligned}\langle K(n) \otimes (L_{K(n)}S^0)^{tG} \rangle &= \langle K(n) \otimes (L_{K(n)}S^0)^{tG} \otimes F(n) \rangle \\ &= \langle K(n) \otimes (L_{K(n)}F(n))^{tG} \rangle.\end{aligned}$$

Corollary 6.8 gives that $(L_{K(n)}F(n))^{tG} \simeq 0$ so $K(n) \otimes (L_{K(n)}S^0)^{tG} \simeq 0$ as wanted. □

Example 6.9. If G is not a finite group, then we cannot expect Theorem 6.1 to be true anymore. For example if $G = \mathbb{T}$ is the circle group and the nth Morava K-theory spectrum $K(n)$ is equipped with the trivial $\mathbb{T}$-action, then Greenlees and Sadofsky show that

$$K(n)^{t\mathbb{T}} \simeq \bigoplus_{k \ge a} \Sigma^{2k} K(n) \oplus \prod_{k<a} \Sigma^{2k} K(n)$$

for every integer a (see [9, page 8]).

Remark 6.10. We have established that the Tate construction vanishes $K(n)$-locally. In [19], Kuhn shows that the Tate construction also vanishes $T(n)$-locally. In practice the $T(n)$-cohomology is computationally inaccessible so Kuhn's proof proceeds in a fundamentally different way than the proof of Greenlees, Hovey, and Sadofsky. Kuhn's proof is based on the Bousfield–Kuhn functor and the Kahn–Priddy splitting. Recall that the Bousfield–Kuhn functor is a functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$ from the ∞-category of pointed spaces to the ∞-category of $T(n)$-local spectra such that there is a natural equivalence $\Phi\Omega^\infty \simeq L_{T(n)}$. A good reference on the Bousfield–Kuhn functor is [18]. Recently, Clausen and Mathew gave a very short and elegant proof of Tate vanishing $T(n)$-locally which also relies on the Kahn–Priddy theorem but in a simpler way than Kuhn's argument (see [6]).

Remark 6.11. The Tate construction vanishing $T(n)$-locally implies that the Tate construction vanishes $K(n)$-locally as well. If the Telescope Conjecture is true, then the Tate construction vanishing $K(n)$-locally would imply the Tate construction vanishing $T(n)$-locally. The Telescope conjecture is not known to be true but it is not known to be false either.

Remark 6.12. There are many surprising consequences of the Tate construction vanishing $T(n)$-locally. A map $f : X \to Y$ of pointed spaces is a v_n-periodic equivalence if $\Phi(f)$ is an equivalence, where Φ denotes the Bousfield–Kuhn functor. In [11], Heuts describes the ∞-category $\mathcal{M}_n^f$ obtained from the ∞-category of pointed spaces by inverting the v_n-periodic equivalences. The main result of Heuts' paper is an equivalence $\mathcal{M}_n^f \simeq \mathrm{Lie}(\mathrm{Sp}_{T(n)})$ betweeen the ∞-category of pointed spaces with the v_n-periodic equivalences inverted and the ∞-category of Lie algebras in $T(n)$-local spectra. Recall that a symmetric sequence of $T(n)$-local spectra is a collection of $T(n)$-local spectra $C := \{C_k\}_{k \ge 0}$ where C_k is equipped with an action of the symmetric group Σ_k for every $k \ge 0$. The construction

$$X \mapsto \bigoplus_{k \ge 0} (C_k \otimes X^{\otimes k})_{h\Sigma_k}$$

defines an endofunctor $F_C : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ on the ∞-category of $T(n)$-local spectra. Such a functor is called a coanalytic functor. It turns out that the construction defined above induces an equivalence from the ∞-category of symmetric sequences of $T(n)$-local spectra to the ∞-category of coanalytic functors [11, Proposition 4.8]. This is a special feature of the $T(n)$-local category and it is a shadow of the Tate construction vanishing $T(n)$-locally. This result is the crucial input for Heuts' result described above. We refer the reader to the recent paper [11] of Heuts for more details.

Remark 6.13. Let G a finite group. In [34], Strickland exploits that the Tate construction vanishes $K(n)$-locally to show that the spectrum $L_{K(n)}\Sigma^\infty_+ BG$ admits the structure of a Frobenius object in the stable homotopy category of $K(n)$-local spectra.

Part 2. Ambidexterity

In the second part of this project we place Hovey and Sadofsky's result on the Tate construction vanishing $K(n)$-locally into a categorical framework following [12]. More concretely, we will be interested in the following situation. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits and let $f: X \to Y$ be a map of Kan complexes. The induced functor $f^*: \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ admits a left adjoint $f_!$ and a right adjoint f_*. We picture the situation as follows

$$\begin{array}{c} \mathrm{Fun}(Y,\mathcal{C}) \\ f_! \left(\uparrow \; f^* \downarrow \; \uparrow \right) f_* \\ \mathrm{Fun}(X,\mathcal{C}) \end{array}$$

In Section 7 we will inductively construct a natural transformation $f_! \to f_*$ of functors and we will be interested in the situation where the natural transformation $f_! \to f_*$ is an equivalence of functors. More generally, we will specify the following data:

(1) A collection of morphisms in the ∞-category of spaces called ambidextrous.

(2) For each ambidextrous morphism $f: X \to Y$ in the ∞-category of spaces a natural transformation
$$\mu_f : \mathrm{id}_{\mathrm{Fun}(Y,\mathcal{C})} \to f_! f^*$$
which exhibits $f_!$ as a right adjoint of f^*.

The collection of ambidextrous maps in the ∞-category of spaces captures precisely when the natural transformation $f_! \to f_*$ is an equivalence of functors. We will work with the slightly more flexible notion of Beck–Chevalley fibrations which we introduce in Section 7. However, we will mostly apply this machinery to the situation where X is a Kan complex and f is the unique map $X \to *$ from X to a point. In this case we say that X is $\mathcal{C}$-ambidextrous if the unique map $X \to *$ is ambidextrous. In Section 8 we impose certain finiteness conditions on the Kan complex X and examine what kind of structure or property of the ∞-category $\mathcal{C}$ we should impose to force X is to be $\mathcal{C}$-ambidextrous. The results of Section 8 are completely formal. However, the ultimate goal of this project is to prove a result due to Hopkins and Lurie which generalizes Hovey and Sadofsky's result which occupied us throughout the first part of this paper. Hopkins and Lurie prove the following result.

Theorem 6.14 (Hopkins–Lurie)**.** *Let $K(n)$ denote the nth Morava K-theory spectrum. Let X be a Kan complex and suppose that for every vertex x of X the sets $\pi_m(X, x)$ are finite for every integer m and trivial for $m \gg 0$. Then X is $\mathrm{Sp}_{K(n)}$-ambidextrous.*

The remaining sections of this paper are devoted to the proof of this result. In contrast to the first part of this paper we will work with a 2-periodic version of Morava K-theory. The 2-periodic Morava K-theories naturally admit the structure of a module over certain $\mathbb{E}_\infty$-rings called Lubin–Tate spectra as discussed in Section 5 or more specifically in Example 5.10. In general it is hard to determine whether a Kan complex X is $\mathcal{C}$-ambidextrous. In Section 9 we define a trace form on X and we will see that this trace form is a perfect pairing precisely if X is $\mathcal{C}$-ambidextrous (see Proposition 9.8). In Section 10 we state the Ravenel–Wilson calculation

[32] as rephrased by Hopkins and Lurie in [12] (see Theorem 10.12). Finally, in Section 11 we give a proof of Hopkins and Lurie's result (see Theorem 11.1).

7. Beck–Chevalley fibrations and norm maps

We will start by introducing some terminology following [12, Section 4]. Let $\mathcal{X}$ be an ∞-category and let $q:\mathcal{C}\to\mathcal{X}$ be a map of simplicial sets which is both a Cartesian fibration and a coCartesian fibration. If X is an object of $\mathcal{X}$, then the fiber $\mathcal{C}_X := \mathcal{C}\times_{\mathcal{X}} X$ is an ∞-category. If $f:X\to Y$ is a morphism in $\mathcal{X}$, then f gives rise to an adjunction

$$\mathcal{C}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{C}_Y$$

of ∞-categories. If the reader is not familiar with adjunctions between ∞-categories we refer to [22, Section 5.2]. Let $\eta_f:\mathrm{id}_{\mathcal{C}_X}\to f^*f_!$ denote the unit and let $\varepsilon_f:f_!f^*\to\mathrm{id}_{\mathcal{C}_Y}$ denote the counit of this adjunction. If $\sigma:\Delta^1\times\Delta^1\to\mathcal{X}$ is a commutative diagram in $\mathcal{X}$ depicted as follows

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

then there is a canonical equivalence $(g')^*f^* \xrightarrow{\simeq} (f')^*g^*$ of functors from $\mathcal{C}_Y$ from $\mathcal{C}_{X'}$.

Definition 7.1. Let $\mathcal{X}$ be an ∞-category and let $q:\mathcal{C}\to\mathcal{X}$ be a map of simplicial sets which is both a Cartesian fibration and a coCartesian fibration. The Beck–Chevalley transformation associated to a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{X}$ is the natural transformation $\mathrm{BC}[\sigma]:f'_!(g')^*\to g^*f_!$ given by the composition

$$f'_!(g')^* \xrightarrow{\eta_f} f'_!(g')^*f^*f_! \xrightarrow{\simeq} f'_!(f')^*g^*f_! \xrightarrow{\varepsilon_{f'}} g^*f_!.$$

Definition 7.2. Let $\mathcal{X}$ be an ∞-category which admits pullbacks. A map $q:\mathcal{C}\to\mathcal{X}$ of simplicial sets which is both a Cartesian fibration and a coCartesian fibration is a Beck–Chevalley fibration if for every pullback diagram σ

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{X}$, the associated Beck–Chevalley transformation $\mathrm{BC}[\sigma]:f'_!(g')^*\to g^*f_!$ is an equivalence of functors from $\mathcal{C}_X$ to $\mathcal{C}_{Y'}$.

Let $\mathcal{X}$ be an ∞-category which admits pullbacks, and let $q:\mathcal{C}\to\mathcal{X}$ be a Beck–Chevalley fibration. As mentioned in the introduction we will define a class of morphisms in $\mathcal{X}$ called ambidextrous which satisfy that if f is ambidextrous, then the left adjoint $f_!$ of f^* is also a right adjoint of f^*. In fact, we will construct a natural transformation $\mathrm{Nm}_f:f_!\to f_*$ called the norm map and show that if f is ambidextrous, then the norm map is an equivalence. More precisely, for every integer $n\geq -2$ we will specify the following data:

(1) A collection of morphisms in $\mathcal{X}$ called n-ambidextrous.
(2) For each n-ambidextrous morphism $f\colon X \to Y$ in $\mathcal{X}$ a natural transformation

$$\mu_f^{(n)}\colon \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$$

which exhibits $f_!$ as a right adjoint of f^*.

Construction 7.3. We will achieve this by induction on n. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q\colon \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley fibration. If $n = -2$, then a morphism $f\colon X \to Y$ in $\mathcal{X}$ is n-ambidextrous precisely if f is an equivalence. In this case the functor f^* is an equivalence of ∞-categories which means that the counit $\varepsilon_f\colon f_! f^* \to \mathrm{id}_{\mathcal{C}_Y}$ is an equivalence of functors. We let $\mu_f^{(n)}\colon \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ be a homotopy inverse of ε_f. For the induction step assume that the collection of n-ambidextrous morphisms in $\mathcal{X}$ have been defined for some integer $n \geq -2$ and that the natural transformation $\mu_g^{(n)}\colon \mathrm{id}_{\mathcal{C}_Y} \to g_! g^*$ has been specified for every n-ambidextrous morphism $g\colon X \to Y$ in $\mathcal{X}$. Let $f\colon X \to Y$ be a morphism in $\mathcal{X}$, and let $\delta\colon X \to X \times_Y X$ denote the diagonal map determined by the commutative diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{\delta} & & \overset{\mathrm{id}_X}{\searrow} & \\ & & X \times_Y X & \xrightarrow{\pi_1} & X \\ & {\scriptstyle \mathrm{id}_X}\searrow & \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle f} \\ & & X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{X}$. The Beck–Chevalley transformation $\mathrm{BC}[\sigma]\colon (\pi_1)_! \pi_2^* \xrightarrow{\simeq} f^* f_!$ associated to the pullback square above is an equivalence since $q\colon \mathcal{C} \to \mathcal{X}$ is a Beck–Chevalley fibration. Consequently, we let $\mathrm{BC}[\sigma]^{-1}\colon f^* f_! \xrightarrow{\simeq} (\pi_1)_! \pi_2^*$ denote a homotopy inverse of $\mathrm{BC}[\sigma]$. The morphism f is weakly $(n+1)$-ambidextrous if the diagonal map δ is n-ambidextrous. If f is weakly $(n+1)$-ambidextrous, then we define a natural transformation $\nu_f^{(n+1)}\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ by the composition

$$f^* f_! \xrightarrow{\mathrm{BC}[\sigma]^{-1}} (\pi_1)_! \pi_2^* \xrightarrow{\mu_\delta^{(n)}} (\pi_1)_! \delta_! \delta^* \pi_2^* \simeq (\pi_1 \delta)_! (\pi_2 \delta)^* = \mathrm{id}_{\mathcal{C}_X}\,.$$

The morphism $f\colon X \to Y$ is $(n+1)$-ambidextrous if for every pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{X}$, the morphism f' is weakly $(n+1)$-ambidextrous and the associated natural transformation

$$\nu_{f'}^{(n+1)}\colon (f')^* f'_! \to \mathrm{id}_{\mathcal{C}_{X'}}$$

exhibits $(f')^*$ as a left adjoint of $f'_!$. In other words the natural transformation $\nu_{f'}^{(n+1)}$ is the counit of an adjunction between $(f')^*$ and $f'_!$. If f is $(n+1)$-ambidextrous, then f is in particular weakly $(n+1)$-ambidextrous and we let $\mu_f^{(n+1)}\colon \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ be a unit of the adjunction between f^* and $f_!$ compatible with the counit $\nu_f^{(n+1)}\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$.

Remark 7.4. In the setting of Construction 7.3 above, if f is a weakly m-ambidextrous morphism in $\mathcal{X}$ for some integer $m \geq -1$, then one can show that the natural transformation $\nu_f^{(m)}\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ is independent of m. Let us specify exactly what this means. Let m and n

be integers with $-1 \le m \le n$. If f is a weakly m-ambidextrous morphism in $\mathcal{X}$, then f is also n-ambidextrous and the natural transformations $\nu_f^{(m)}$ and $\nu_f^{(n)}$ agree up to homotopy. Similarly, let m and n be integers with $-2 \le m \le n$. If f is an m-ambidextrous morphism in $\mathcal{X}$, then f is also n-ambidextrous and the natural transformations $\mu_f^{(m)}$ and $\mu_f^{(m)}$ agree up to homotopy. This is proved in [12, Proposition 4.1.10].

Definition 7.5. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q: \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley transformation. Let $f: X \to Y$ be a morphism of $\mathcal{X}$.

(1) The morphism f is weakly ambidextrous if f is weakly n-ambidextrous for some integer $n \ge -1$. Set $\nu_f := \nu_f^{(n)}$ where $\nu_f^{(n)}$ is the natural transformation defined in Construction 7.3.
(2) The morphism f is ambidextrous if f is n-ambidextrous for some integer $n \ge -2$ that is, if every pullback f' of f is weakly ambidextrous and the natural transformation $\nu_{f'}$ exhibits $(f')^*$ as a left adjoint of $f'_!$. We let $\mu_f : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ denote a compatible unit of this adjunction.

Definition 7.6. Let $q: \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley fibration and let $f: X \to Y$ be a morphism in $\mathcal{X}$. Suppose that $f^*: \mathcal{C}_Y \to \mathcal{C}_X$ admits a right adjoint $f_*: \mathcal{C}_X \to \mathcal{C}_Y$. If f is weakly ambidextrous, then we let $\mathrm{Nm}_f : f_! \to f_*$ denote the image of the natural transformation ν_f under the equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_X)}(f^* f_!, \mathrm{id}_{\mathcal{C}_X}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_Y)}(f_!, f_*)$$

of spaces. The natural transformation $\mathrm{Nm}_f : f_! \to f_*$ is called the norm map associated to f.

Remark 7.7. Let $q: \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley fibration. A morphism $f: X \to Y$ in $\mathcal{X}$ is ambidextrous precisely if for every pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{X}$ the following three conditions are satisfied:

(1) The morphism f' is weakly ambidextrous.
(2) The functor $(f')^*$ admits a right adjoint f'_*.
(3) The norm map $\mathrm{Nm}_{f'} : f'_! \to f'_*$ associated to f' is an equivalence.

Let $\mathcal{X}$ be an ∞-category which admits pullbacks. If $q: \mathcal{C} \to \mathcal{X}$ is a Beck–Chevalley fibration, then we have introduced a collection of morphisms in $\mathcal{X}$ called ambidextrous. Now we want to specialize to the case where $\mathcal{X}$ is the ∞-category $\mathcal{S}$ of spaces. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. The construction $X \mapsto \mathrm{Fun}(X, \mathcal{C})$ refines to a functor $\mathrm{Fun}(-, \mathcal{C}): \mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ of ∞-categories which is classified by a Cartesian fibration $q: \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ by unstraightening (see [22, Section 3.3.2]). If $f: X \to Y$ is a morphism of spaces, then the functor $f^*: \mathrm{LocSys}(\mathcal{C})_Y \to \mathrm{LocSys}(\mathcal{C})_X$ can be canonically identified with the functor $f^*: \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ given by composition with f. Since $\mathcal{C}$ admits small limits and colimits we conclude that the functor f^* admits a left adjoint given by a left Kan extension of f, and a right adjoint f_* given by a right Kan extension of f. It follows from [22, Corollary 5.2.2.5] that $q: \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ is also a coCartesian fibration.

Proposition 7.8. *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. The functor $q\colon \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ classifying the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ determined by $X \mapsto \mathrm{Fun}(X, \mathcal{C})$ is a Beck–Chevalley fibration.*

Proof. See [12, Proposition 4.3.3]. □

Definition 7.9. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits, and let $q\colon \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ be the Beck–Chevalley fibration of Proposition 7.8.

(1) A Kan complex X is weakly $\mathcal{C}$-ambidextrous if the unique map $X \to *$ is weakly ambidextrous.
(2) A Kan complex X is $\mathcal{C}$-ambidextrous if X is weakly $\mathcal{C}$-ambidextrous and the natural transformation $\nu_f\colon f^* f_! \to \mathrm{id}_{\mathrm{Fun}(X,\mathcal{C})}$ exhibits f^* as a left adjoint of $f_!$.

Remark 7.10. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. Let X be a Kan complex and let $f\colon X \to *$ denote the unique map. The Kan complex X is $\mathcal{C}$-ambidextrous precisely if the norm map $\mathrm{Nm}_X\colon f_! \to f_*$ is an equivalence.

Example 7.11. Let $\mathcal{C}$ be an ∞-category which admits small colimits. The empty Kan complex $\varnothing$ is weakly $\mathcal{C}$-ambidextrous. This follows from the observation that the diagonal map $\delta\colon \varnothing \to \varnothing \times_* \varnothing$ is an equivalence.

Ambidexterity of a map $f\colon X \to Y$ of Kan complexes is a condition on the fibers of f as the following Proposition makes precise. We will say that a map $f\colon X \to Y$ is n-truncated for some integer n if for every object Z of the ∞-category of spaces, the induced map $\mathrm{Map}_{\mathcal{S}}(Z, X) \to \mathrm{Map}_{\mathcal{S}}(Z, Y)$ has n-truncated fibers which means that the homotopy groups of the fibers vanish for every point of the fibers and every $i > n$.

Proposition 7.12. *Let $\mathcal{C}$ be an ∞-category which admits small colimits.*

(1) A Kan fibration $f\colon X \to Y$ between Kan complexes is weakly ambidextrous if and only if f is n-truncated for some integer n and each fiber of f is weakly $\mathcal{C}$-ambidextrous.
(2) A Kan fibration $f\colon X \to Y$ between Kan complexes is ambidextrous if and only if f is n-truncated for some integer n and each fiber of f is $\mathcal{C}$-ambidextrous.

Proof. See [12, Proposition 4.3.5]. □

8. First examples of ambidexterity

Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. In the previous section we defined what it means for a Kan complex X to be $\mathcal{C}$-ambidextrous (see Definition 7.9). The goal of this section is to provide a host of examples of $\mathcal{C}$-ambidextrous spaces. We follow [12, Section 4.4] closely. One might expect that ambidexterity is a property of the chosen Kan complex X but we will see that it also relies crucially on the ∞-category $\mathcal{C}$. However, we will need to impose certain finiteness conditions on the Kan complex X which are captured in the following definition.

Definition 8.1. Let n be an integer. A Kan complex X is a finite n-type if the following are satisfied.

(1) For every vertex x of X, the homotopy groups $\pi_m(X, x)$ vanish for $m > n$.
(2) For every vertex x of X, the sets $\pi_m(X, x)$ are finite for every integer m.

Definition 8.2. Let $n \geq -2$ be an integer. An ∞-category $\mathcal{C}$ which admits small colimits is n-semiadditive if every finite n-type is $\mathcal{C}$-ambidextrous.

Remark 8.3. Let $\mathcal{C}$ be an ∞-category which admits small colimits and let X be a Kan complex. For every pair of vertices x and y in X we can form the space of paths $P_{x,y}$ from x to y in X by

$$P_{x,y} = \{x\} \times_{\mathrm{Fun}(\{0\},X)} \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\{1\},X)} \{y\}.$$

It follows from Proposition 7.12 that X is weakly $\mathcal{C}$-ambidextrous if and only if for all vertices x and y in X the space of paths in X from x to y is $\mathcal{C}$-ambidextrous. This observation will be important. For example if $\mathcal{C}$ is the ∞-category Sp of spectra and X is the classifying space BG of a finite group, then BG is weakly Sp-ambidextrous precisely if the loop space ΩBG is Sp-ambidextrous.

Lemma 8.4. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. The empty Kan complex is $\mathcal{C}$-ambidextrous if and only if $\mathcal{C}$ is pointed.*

Proof. Let E denote the unique vertex of $\mathrm{Fun}(\varnothing, \mathcal{C}) \simeq \Delta^0$ and let $f : \varnothing \to *$ denote the unique map. The functor $f^* : \mathcal{C} \to \Delta^0$ is the constant functor with value E. It follows that a left adjoint $f_! : \Delta^0 \to \mathcal{C}$ of f^* carries E to an initial object of $\mathcal{C}$. By definition the empty Kan complex is $\mathcal{C}$-ambidextrous precisely if the natural transformation $\nu_f : f^* f_! \to \mathrm{id}$ is the counit of an adjunction which exhibits f^* as a left adjoint of $f_!$. This is equivalent to the following: for every object C of $\mathcal{C}$ the canonical map

$$\mathrm{Map}_{\mathcal{C}}(C, f_! E) \to \mathrm{Map}_{\mathrm{Fun}(\varnothing,\mathcal{C})}(f^* C, f^* f_! E) \xrightarrow{\nu_f} \mathrm{Map}_{\mathrm{Fun}(\varnothing,\mathcal{C})}(f^* C, E)$$

of spaces is an equivalence. Since $\mathrm{Map}_{\mathrm{Fun}(\varnothing,\mathcal{C})}(f^* C, E) \simeq *$ this is equivalent to $f_! E$ being a final object. This ends the proof. □

Remark 8.5. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. In this case we can give a more conceptually pleasing proof of Lemma 8.4. A left adjoint $f_!$ of f^* carries E to an initial object of $\mathcal{C}$, and a right adjoint f_* carries E to a final object of $\mathcal{C}$. Consequently, the norm map $\mathrm{Nm}_f : f_! E \to f_* E$ is an equivalence if and only if $\mathcal{C}$ is pointed.

Proposition 8.6. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. Then $\mathcal{C}$ is (-1)-semiadditive if and only if $\mathcal{C}$ is pointed.*

Proof. Assume that $\mathcal{C}$ is (-1)-semiadditive. The empty Kan complex $\varnothing$ is a finite (-1)-type so $\varnothing$ is $\mathcal{C}$-ambidextrous by assumption. It follows from Lemma 8.4 that $\mathcal{C}$ is pointed. Conversely, assume that $\mathcal{C}$ is pointed and let X be a finite (-1)-type. If X is empty, then X is $\mathcal{C}$-ambidextrous by Lemma 8.4. If X is contractible, then X is $\mathcal{C}$-ambidextrous since $\mathcal{C}$ admits small colimits. □

Proposition 8.6 above gives a complete characterization of those ∞-categories which are (-1)-semiadditive. Our next goal is to obtain a similar characterization for 0-semiadditive ∞-categories. The following remark is crucial.

Remark 8.7. Let $\mathcal{C}$ be a pointed ∞-category which admits small limits and colimits. Let X be a set which we regard as a discrete Kan complex. An object of the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ of functors from X to $\mathcal{C}$ can be identified with a sequence $(C_x)_{x\in X}$ of objects in $\mathcal{C}$ indexed by X. Let $f : X \to *$ denote the unique map. The functor $f^* : \mathcal{C} \to \mathrm{Fun}(X, \mathcal{C})$ admits both a left adjoint $f_!$ and a right adjoint f_*. The left adjoint $f_!$ is given by

$$(C_x)_{x\in X} \mapsto \coprod_{x\in X} C_x$$

while the right adjoint f_* is given by

$$(C_x)_{x\in X} \mapsto \prod_{x\in X} C_x.$$

Since $\mathcal{C}$ is pointed it follows that $\mathcal{C}$ is (-1)-ambidextrous by Proposition 8.6. We conclude that X is weakly $\mathcal{C}$-ambidextrous by Remark 8.3. In particular, there is a norm map $\mathrm{Nm}_f : f_! \to f_*$ which assigns to each sequence $(C_x)_{x\in X}$ of objects in $\mathcal{C}$ the morphism

$$\theta : \coprod_{x\in X} C_x \longrightarrow \prod_{y\in X} C_y$$

whose (x,y)-component is id_{C_x} if $x = y$ and the zero morphism otherwise. Consequently, we find that X is $\mathcal{C}$-ambidextrous if and only if the morphism θ is an equivalence for every sequence $(C_x)_{x\in X}$ of objects of $\mathcal{C}$.

Proposition 8.8. *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. Then the following are equivalent:*

(1) The ∞-category $\mathcal{C}$ is 0-semiadditive
(2) The ∞-category $\mathcal{C}$ is pointed and for every pair of objects C and D in $\mathcal{C}$ the canonical map $C \amalg D \to C \times D$ given by the matrix $\begin{pmatrix} \mathrm{id}_C & 0 \\ 0 & \mathrm{id}_D \end{pmatrix}$ is an equivalence.

Proof. Assume that $\mathcal{C}$ is 0-semiadditive. Apply Remark 8.7 above with X a discrete space consisting of two points. For the converse, let X be a finite 0-type. We want to show that X is $\mathcal{C}$-ambidextrous. We will prove this by induction on the cardinality of X. If X is empty, then X is $\mathcal{C}$-ambidextrous since $\mathcal{C}$ is pointed. Assume that X is non-empty and choose a point x in X. Let $Y = \{x, y\}$ be a discrete space consisting of two points. Define a function $f : X \to Y$ by

$$f(t) = \begin{cases} x & \text{if } t = x \\ y & \text{if } t \neq x \end{cases}$$

By Proposition 7.12 it suffices to show that the fibers of f are $\mathcal{C}$-ambidextrous and that Y is $\mathcal{C}$-ambidextrous. First note that the fiber $f^{-1}\{x\}$ is $\mathcal{C}$-ambidextrous since it is contractible. The cardinality of $f^{-1}\{y\}$ is strictly less than the cardinality of X so $f^{-1}\{y\}$ is $\mathcal{C}$-ambidextrous by the inductive hypothesis. It follows from Remark 8.7 that Y is $\mathcal{C}$-ambidextrous. □

Corollary 8.9. *If $\mathcal{C}$ is a stable ∞-category which admits small limits and colimits, then $\mathcal{C}$ is 0-semiadditive.*

Example 8.10. Let G be a finite group and let $f : BG \to *$ denote the unique map. It follows from Corollary 8.9 that the stable ∞-category Sp of spectra is 0-semiadditive. Since the classifying space BG is a finite 1-type it follows from Remark 8.3 that BG is weakly Sp-ambidextrous. This means that we have a natural transformation $\mathrm{Nm}_G : (-)_{hG} \to (-)^{hG}$ and we define the Tate construction $(-)^{tG}$ as the cofiber of Nm_G. We claim that this norm map is equivalent to the norm map we defined in Section 2 (see Remark 2.8). Let $s : * \to BG$ classify a basepoint of BG. By Remark 2.10 it suffices to show that $f_! s_! S^0 \to f_* s_! S^0$ is an equivalence of spectra. First observe that $f_! s_! \simeq (f \circ s)_! S^0 \simeq S^0$. Since the classifying space BG of G is weakly Sp-ambidextrous it follows from Proposition 7.12 that the map $s : * \to BG$ is ambidextrous. It follows that the norm map $\mathrm{Nm}_s : s_! \to s_*$ is a natural equivalence. Consequently, we find that $f_* s_! S^0 \simeq f_* s_* S^0 \simeq S^0$ and the norm map $\mathrm{Nm}_{s_! S^0}$ is equivalent to the identity on S^0.

Example 8.11. Let G be a finite group. It follows from the Tate construction vanishing $K(n)$-locally (see Theorem 6.1) that the classifying space BG of G is $\mathrm{Sp}_{K(n)}$-ambidextrous. The Tate construction vanishes $T(n)$-locally (see Remark 6.10) so the classifying space BG of G is also $\mathrm{Sp}_{T(n)}$-ambidextrous.

Remark 8.12. Let G be a finite group and let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. In section 2 we were solely interested in the ∞-category of spectra with a G-action. However, we can define a natural transformation $(-)_{hG} \to (-)^{hG}$ of functors from $\mathcal{C}^{BG}$ to $\mathcal{C}$ using Corollary 8.9. Consequently, we can define the Tate construction on $\mathcal{C}$. An example of particular interest is the derived ∞-category $\mathcal{D}(R)$ of modules over a commutative ring R.

Finally, we examine what kind of structure on the underlying ∞-category we need to impose to go from 0-semiadditivity to 1-semiadditive and more generally from n-semiadditivity to $(n+1)$-semiadditivity. We will first introduce some terminology.

Notation 8.13. Let $\mathcal{C}$ be an n-semiadditive ∞-category for some integer $n \geq -2$. Let X be a finite n-type and let $p : X \to *$ denote the unique map. There is a natural transformation $\mu_p : \mathrm{id}_{\mathcal{C}} \to p_! p^*$ which exhibits $p_!$ as a right adjoint of p^*. Let C and D be a pair of objects of $\mathcal{C}$ and let $f : X \to \mathrm{Map}_{\mathcal{C}}(C, D)$ be a map of Kan complexes. The natural transformation μ_p determines a natural transformation $u : p^* C \to p^* D$. The construction that sends a map $f : X \to \mathrm{Map}_{\mathcal{C}}(C, D)$ to the composite

$$C \xrightarrow{\mu_p} p_! p^* C \xrightarrow{p_! u} p_! p^* D \xrightarrow{\varepsilon_p} D$$

determines a map

$$\mathrm{d}\mu_X : \mathrm{Fun}(X, \mathrm{Map}_{\mathcal{C}}(C, D)) \longrightarrow \mathrm{Map}_{\mathcal{C}}(C, D)$$

of spaces.

Notation 8.14. Let $\mathcal{C}$ be a 0-semiadditive ∞-category and let X be a set consisting of two elements. For every pair of objects C and D in $\mathcal{C}$ we ontain an addition map

$$+ : \mathrm{Map}_{\mathcal{C}}(C, D) \times \mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D)$$

by specializing the map $\mathrm{d}\mu_X$ described in Notation 8.13 above. This addition map is associative and commutative up to homotopy with unit the zero map $0 : C \to D$ in $\mathcal{C}$. Let $n \geq 0$ be an integer, and let $[n] : C \to C$ denote the n-fold sum of id_C with itself under the addition map on $\mathrm{Map}_{\mathcal{C}}(C, C)$. The morphism $[n] : C \to C$ can be identified with the following composite

$$C \xrightarrow{\delta} \prod_{1 \leq i \leq n} C \simeq \coprod_{1 \leq j \leq n} C \xrightarrow{\delta'} C$$

where δ is the canonical diagonal map, δ' is the codiagonal, and the equivalence follows from Remark 8.7.

Example 8.15. Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let $n \geq 1$ be an integer. For every object C of $\mathcal{C}$ we can form the (discrete) ring $\mathrm{End}_{\mathcal{C}}(C) := \mathrm{Ext}^0_{\mathcal{C}}(C, C)$ of endomorphisms of C in $\mathcal{C}$. If the endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$ admits the structure of a $\mathbb{Q}$-algebra, then the morphism $[n] : C \to C$ in $\mathrm{End}_{\mathcal{C}}(C)$ is an equivalence. An inverse is given by the morphism $\frac{1}{n}\mathrm{id}_C : C \to C$ in $\mathrm{End}_{\mathcal{C}}(C)$.

Example 8.16. Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let C be an object of $\mathcal{C}$. Let p be a prime number and let $n \geq 1$ be an integer which is relatively prime to p. If the endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$ admits the structure of a $\mathbb{Z}_{(p)}$-module, then the morphism $[n] : C \to C$ is an equivalence. Note that $\frac{1}{n}$ is an element of $\mathbb{Z}_{(p)}$ since n is relatively prime to p. It follows that an inverse of $[n]$ is given by $\frac{1}{n}\mathrm{id}_C : C \to C$.

Proposition 8.17. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category which admits small limits and colimits. If there exists a prime number p which satisfies that for every integer $n \geq 1$ which is relatively prime to p, the morphism $[n]: C \to C$ is an equivalence for every object C of $\mathcal{C}$, then $\mathcal{C}$ is 1-semiadditive if and only if the Eilenberg–Mac Lane space $K(\mathbb{Z}/p, 1)$ is $\mathcal{C}$-ambidextrous.*

Proof. If $\mathcal{C}$ is 1-semiadditive, then the Eilenberg–Mac Lane space $K(\mathbb{Z}/p, 1)$ is $\mathcal{C}$-ambidextrous since $K(\mathbb{Z}/p, 1)$ is a finite 1-type. Conversely, suppose that the Eilenberg–Mac Lane space $K(\mathbb{Z}/p, 1)$ is $\mathcal{C}$-ambidextrous, and let X be a finite 1-type. We may assume that X is connected by applying Proposition 7.12 to the map $X \to \pi_0 X$. In other words, we may assume that X is homotopy equivalent to the classifying space BG of a finite group G. We first assume that G is a p-group and proceed by induction on the order of G. If G is the trivial group, then we are done. Assume that G is not the trivial group and choose a normal subgroup H of G of index p. By the inductive hypothesis we conclude that $B(G/H)$ is $\mathcal{C}$-ambidextrous. For every $x \in B(G/H)$ the fiber of the map $BG \to B(G/H)$ is equivalent to $BH \simeq K(\mathbb{Z}/p, 1)$ which is $\mathcal{C}$-ambidextrous by assumption. It follows that BG is $\mathcal{C}$-ambidextrous from Proposition 7.12. It remains to show that it suffices to handle the case where G is a p-group. Choose a p-Sylow subgroup of the finite group G, and let $g: BP \to BG$ denote the map induced by the inclusion $P \hookrightarrow G$, and let $f: BG \to *$ denote the unique map. It follows from Proposition 7.12 that the map g is ambidextrous since g is equivalent to a covering map with finite fibers. Let $\mathcal{L}: BG \to \mathcal{C}$ be a local system and define a natural transformation $\alpha: \mathcal{L} \to \mathcal{L}$ by the composite

$$\mathcal{L} \to g_* g^* \mathcal{L} \simeq g_! g^* \mathcal{L} \to \mathcal{L}.$$

We show that α is an equivalence. It suffices to show that the induced map $x^*\alpha: x^*\mathcal{L} \to x^*\mathcal{L}$ is an equivalence for every point $x: * \to BG$ of BG. Let n be the cardinality of G/P and note that n is relatively prime to p since P is a p-Sylow subgroup. Now, the map $x^*\alpha$ is homotopic to $[n]: x^*\mathcal{L} \to x^*\mathcal{L}$ which is an equivalence by assumption. We conclude that $\mathcal{L}$ is a retract of $g_! g^* \mathcal{L}$ and we may therefore assume that $\mathcal{L}$ is on the form $g_! \mathcal{L}'$ for some $\mathcal{L}' \in \operatorname{Fun}(BP, \mathcal{C})$. Ultimately, we want to show that $\operatorname{Nm}_f : f_! \to f_*$ is an equivalence since BG is weakly $\mathcal{C}$-ambidextrous. Thus, it suffices to show that

$$f_! g_! \mathcal{L}' \xrightarrow{\operatorname{Nm}_f} f_* g_! \mathcal{L}'$$

is an equivalence. Since g is ambidextrous it suffices to show that the composite

$$f_! g_! \mathcal{L}' \xrightarrow{\operatorname{Nm}_f} f_* g_! \mathcal{L}' \xrightarrow[\operatorname{Nm}_g]{\simeq} f_* g_* \mathcal{L}'$$

is an equivalence. There is an equivalence $\operatorname{Nm}_g \circ \operatorname{Nm}_f \simeq \operatorname{Nm}_{fg}$ by [12, Remark 4.2.4]. Consequently, we can replace G by the p-group P. This ends the proof. ☐

Proposition 8.18. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category which admits small limits and colimits, and let p be a prime number. If the morphism $[p]: C \to C$ is an equivalence for every object C of $\mathcal{C}$, then the classifying space BG of every finite p-group is $\mathcal{C}$-ambidextrous.*

Proof. The proof is very similar to the proof of Proposition 8.17. See [12, Proposition 4.4.17] for details. ☐

Proposition 8.19. *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. If $n \geq 2$ is an integer, then $\mathcal{C}$ is n-semiadditive if and only if $\mathcal{C}$ is $(n-1)$-semiadditive and the Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p, n)$ are $\mathcal{C}$-ambidextrous for every prime p.*

Proof. Let $n \geq 2$ be an integer. The only if direction is immediate. Suppose that $\mathcal{C}$ is $(n-1)$-semiadditive and that the Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p, n)$ are $\mathcal{C}$-ambidextrous for every prime p. Let X be a finite n-type. The truncation $\tau_{\leq n-1} X$ is a finite $(n-1)$-type hence $\mathcal{C}$-ambidextrous by assumption. Consequently, it suffices to show that the homotopy fibers of the map $X \to \tau_{\leq n-1}X$ are $\mathcal{C}$-ambidextrous. We may therefore assume that X is homotopy equivalent to $K(A,n)$ for some finite abelian group A. We proceed by induction on the cardinality of A. If A is trivial, then there is nothing to prove. If $A \simeq \mathbb{Z}/p$, then $X \simeq K(\mathbb{Z}/p,n)$ is $\mathcal{C}$-ambidextrous by assumption. Otherwise, choose a short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups with $|A'| < |A|$ and $|A''| < |A|$. This induces a fiber sequence

$$K(A',n) \to X \to K(A'',n)$$

from which we conclude that the homotopy fibers of the map $X \to K(A'',n)$ are homotopy equivalent to $K(A',n)$. By the induction hypothesis we conclude that both $K(A',n)$ and $K(A'',n)$ are $\mathcal{C}$-ambidextrous. Invoking Proposition 7.12, we conclude that X is $\mathcal{C}$-ambidextrous. □

Proposition 8.20. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. If p is a prime number which satisfies that $[p]: C \to C$ is an equivalence for every object C of $\mathcal{C}$, then $K(\mathbb{Z}/p,m)$ is $\mathcal{C}$-ambidextrous for every integer $m \geq 1$.*

Proof. The proof will proceed by induction on m. If $m = 1$, then $K(\mathbb{Z}/p,1)$ is $\mathcal{C}$-ambidextrous by Propositon 8.18 above. Assume that $m \geq 2$ and let $f: K(\mathbb{Z}/p,m) \to *$ denote the unique map. It follows from the induction hypothesis that $K(\mathbb{Z}/p,m)$ is weakly $\mathcal{C}$-ambidextrous since $\Omega K(\mathbb{Z}/p,m-1) \simeq K(\mathbb{Z}/p,m)$. Consequently, it suffices to show that the right adjoint $f_*: \mathrm{Fun}(K(\mathbb{Z}/p,m),\mathcal{C}) \to \mathcal{C}$ preserves small colimits. We will show that f_* is an equivalence of ∞-categories. It suffices to show that the diagonal map $\mathcal{C} \to \mathrm{Fun}(K(\mathbb{Z}/p,m),\mathcal{C})$ is an equivalence of ∞-categories. By [22, Lemma 3.1.3.2] it suffices to show that for every simplicial set K the map

$$\mathrm{Fun}(K,\mathcal{C})^{\simeq} \to \mathrm{Fun}(K,\mathrm{Fun}(K(\mathbb{Z}/p,m),\mathcal{C}))^{\simeq}$$

induced by the diagonal is a homotopy equivalence. We may replace $\mathcal{C}$ by $\mathrm{Fun}(K,\mathcal{C})$ and reduce to showing that the diagonal map

$$\mathcal{C}^{\simeq} \to \mathrm{Fun}(K(\mathbb{Z}/p,m),\mathcal{C})^{\simeq} \simeq \mathrm{Fun}(K(\mathbb{Z}/p,m),\mathcal{C}^{\simeq})$$

is a homotopy equivalence. For this it suffices to show that for every integer $n \geq 1$ the map

$$\delta_n: \tau_{\leq n}\mathcal{C}^{\simeq} \to \mathrm{Fun}(K(\mathbb{Z}/p,m),\tau_{\leq n}\mathcal{C}^{\simeq})$$

is a homotopy equivalence. We proceed by induction on n. Since $m \geq 2$ we conclude that $K(\mathbb{Z}/p,m)$ is 2-connective. The case $n = 1$ follows immediately. Assume that δ_n is a homotopy equivalence. Consider the following commutative diagram

$$\begin{array}{ccc} \tau_{\leq n+1}\mathcal{C}^{\simeq} & \xrightarrow{\delta_{n+1}} & \mathrm{Fun}(K(\mathbb{Z}/p,m),\tau_{\leq n+1}\mathcal{C}^{\simeq}) \\ \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \mathrm{Fun}(K(\mathbb{Z}/p,m),\varphi_{n+1})} \\ \tau_{\leq n}\mathcal{C}^{\simeq} & \xrightarrow{\delta_n} & \mathrm{Fun}(K(\mathbb{Z}/p,m),\tau_{\leq n}\mathcal{C}^{\simeq}) \end{array}$$

in the ∞-category of spaces. We want to show that δ_{n+1} is a homotopy equivalence. It suffices to show that δ_{n+1} induces a homotopy equivalence between the homotopy fiber of φ_{n+1} and $\mathrm{Fun}(K(\mathbb{Z}/p,m),\varphi_{n+1})$ respectively over every vertex of $\tau_{\leq n}\mathcal{C}^{\simeq}$. The fiber of φ_{n+1} over C is homotopy equivalent to the Eilenberg–Mac Lane space $K(\mathrm{Ext}_{\mathcal{C}}^{-n}(C,C),n+1)$ where C is an object of $\mathcal{C}$ which depends on the choice of vertex of $\tau_{\leq n}\mathcal{C}^{\simeq}$. We conclude that it suffices to show that the map

$$K(\mathrm{Ext}_{\mathcal{C}}^{-n}(C,C),n+1) \to \mathrm{Fun}(K(\mathbb{Z}/p,m),K(\mathrm{Ext}_{\mathcal{C}}^{-n}(C,C),n+1))$$

is a homotopy equivalence or equivalently that the fiber of the map above vanishes. The abelian group $\mathrm{Ext}_{\mathcal{C}}^{-n}(C,C)$ admits the structure of a $\mathbb{Z}[\frac{1}{p}]$-module from which it follows that $\mathrm{H}^k_{\mathrm{red}}(K(\mathbb{Z}/p,m);\mathrm{Ext}_{\mathcal{C}}^{-n}(C,C))$ vanishes for all k. This ends the proof. □

We end this section by pointing out some nice consequences of the results above.

Corollary 8.21. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category which admits small limits and colimits. If the morphism $[n]\colon C \to C$ is an equivalence for every integer $n \geq 1$ and every object C of $\mathcal{C}$, then $\mathcal{C}$ is 1-semiadditive.*

Proof. It suffices to show that $K(\mathbb{Z}/p,1)$ is $\mathcal{C}$-ambidextrous by Proposition 8.17. Since $\mathbb{Z}/p$ is a p-group, and $B\mathbb{Z}/p \simeq K(\mathbb{Z}/p,1)$ this follows from Proposition 8.18. □

Corollary 8.22. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. If the endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$ admits the structure of a $\mathbb{Q}$-algebra for every object C of $\mathcal{C}$, then $\mathcal{C}$ is n-semiadditive for every integer $n \geq -2$.*

Proof. We may assume that $n \geq 1$ by Corollary 8.9. It follows from Example 8.15 that the morphism $[n]\colon C \to C$ is an equivalence for every object C of $\mathcal{C}$. We conclude that $\mathcal{C}$ is 1-semiadditive using Corollary 8.21. By Proposition 8.20 we see that $K(\mathbb{Z}/p,m)$ is $\mathcal{C}$-ambidextrous for every prime number p and every integer $m \geq 1$. Proposition 8.19 implies that $\mathcal{C}$ is n-semiadditive as wanted. □

Corollary 8.23. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let p be a prime. If the endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$ admits the structure of a $\mathbb{Z}_{(p)}$-module for every object C of $\mathcal{C}$, then $\mathcal{C}$ is n-semiadditive if and only if the Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p,m)$ are $\mathcal{C}$-ambidextrous for $1 \leq m \leq n$.*

Proof. The only if direction is immediate. Suppose that the Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p,m)$ are $\mathcal{C}$-ambidextrous for $1 \leq m \leq n$. It suffices to show that $\mathcal{C}$ is 1-semiadditive by Proposition 8.19. By Example 8.16 the condidition in Proposition 8.17 is satisfied, thus it suffices to show that $K(\mathbb{Z}/p,1)$ is $\mathcal{C}$-ambidextrous which follows by assumption. □

Example 8.24. Let $n \geq 1$ be an integer and let X be spectrum. The endomorphism ring $\mathrm{End}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}X)$ admits the structure of a $\mathbb{Z}_{(p)}$-module. Let l be an integer which is not divisible by p and let $l\colon K(n) \to K(n)$ denote multiplication by l on $K(n)$. Since l is not divisible by p we conclude that l is an equivalence which in turn means that

$$l\colon X \otimes K(n) \to X \otimes K(n)$$

is an equivalence. It follows that $l\colon L_{K(n)}X \to L_{K(n)}X$ is an equivalence. Consequently, we obtain a ring homomorphism $\mathbb{Z}_{(p)} \to \mathrm{End}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}X)$ which endows the endomorphism ring $\mathrm{End}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}X)$ with the structure of a $\mathbb{Z}_{(p)}$-module. Since $\mathrm{Sp}_{K(n)}$ is the essential image of the Bousfield localization $L_{K(n)}$ (see Propositon 5.11) we conclude that the endomorphism ring $\mathrm{End}_{\mathrm{Sp}_{K(n)}}(Y)$ admits the structure of a $\mathbb{Z}_{(p)}$-module for every $K(n)$-local spectrum Y.

9. Duality and trace forms

Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits and let X be a Kan complex. In general it is difficult to determine whether X is $\mathcal{C}$-ambidextrous. For example if G is a finite group, then the classifying space of G is $\mathrm{Sp}_{K(n)}$-ambidextrous precisely if the Tate construction vanishes $K(n)$-locally. The goal of this section is to provide some tools in determining whether X is $\mathcal{C}$-ambidextrous. Assume that X is weakly $\mathcal{C}$-ambidextrous. In section 7 we constructed a norm map on X which should be thought of as a map from the cohomology of X to the homology of X, that is a bilinear pairing on the homology of X. The idea is that this pairing is perfect precisely if X is $\mathcal{C}$-ambidextrous. The notion of a pairing is encoded by the notion of duality in a monoidal ∞-category which we now recall following [12, Section 5.1].

Definition 9.1. Let $\mathbf{C}$ be a category equipped with a monoidal structure $\otimes$ and let 1 denote the unit object. Let X and Y be objects of $\mathbf{C}$. A map $e\colon X \otimes Y \to 1$ in $\mathbf{C}$ is a duality datum if there exists a map $c\colon 1 \to Y \otimes X$ such that the composites

$$X \xrightarrow{\mathrm{id}\otimes c} X \otimes Y \otimes X \xrightarrow{e\otimes \mathrm{id}} X$$

$$Y \xrightarrow{c\otimes \mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes e} Y$$

equal the identities on X and Y respectively. If $\mathcal{C}$ is a monoidal ∞-category with unit 1, then a map $e\colon X \otimes Y \to 1$ in $\mathcal{C}$ is a duality datum if it is a duality datum in the homotopy category of $\mathcal{C}$.

Definition 9.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit 1. An object X of $\mathcal{C}$ is dualizable if there exists another object Y of $\mathcal{C}$ and a duality datum $e\colon X \otimes Y \to 1$. Let $\dim(X) \in \pi_0 \operatorname{Map}_{\mathcal{C}}(1,1)$ be the morphism given by the composite

$$1 \xrightarrow{c} Y \otimes X \simeq X \otimes Y \xrightarrow{e} 1$$

where $c\colon 1 \to Y \otimes X$ is compatible with e.

Notation 9.3. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q\colon \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley fibration. If $f\colon X \to Y$ be a morphism in $\mathcal{X}$, then the functor $f^*\colon \mathcal{C}_Y \to \mathcal{C}_X$ admits a left adjoint $f_!$. Let $[X/Y]$ denote the functor $[X/Y]\colon \mathcal{C}_Y \to \mathcal{C}_Y$ given by $[X/Y] \coloneqq f_! f^*$. If f is weakly $\mathcal{C}$-ambidextrous, then we have a natural transformation $\nu_f\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$. In the case where f is weakly $\mathcal{C}$-ambidextrous we define a natural transformation

$$\mathrm{TrFm}_f\colon [X/Y] \circ [X/Y] \to \mathrm{id}_{\mathcal{C}_Y}$$

by

$$(f_! f^*)(f_! f^*) \simeq f_!(f^* f_!) f^* \xrightarrow{\nu_f} f_! f^* \xrightarrow{\varepsilon_f} \mathrm{id}_{\mathcal{C}_Y}\,.$$

The natural transformation TrFm_f is called the trace form of f.

Remark 9.4. Assume that the ∞-category $\mathcal{C}$ admits small colimits and let X be a Kan complex. The unique map $f\colon X \to *$ defines a functor $[X/*] \coloneqq [X]\colon \mathcal{C} \to \mathcal{C}$ as above. The ∞-category $\mathcal{C}$ is naturally tensored over the ∞-category $\mathcal{S}$ of spaces (see [22, Section 4.4.4]) and the functor $[X]\colon \mathcal{C} \to \mathcal{C}$ is given by tensoring with X. If X is weakly $\mathcal{C}$-ambidextrous, then we denote the trace form of f by TrFm_X. If Y is a Kan complex and $g\colon Y \to X$ is a map of Kan complexes, then the counit $g_! g^* \to \mathrm{id}$ induces a natural transformation

$$[Y] = f_! g_! g^* f^* \to f_! f^* = [X]$$

which we denote α_g.

Notation 9.5. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Let X be a pointed Kan complex and let $e\colon * \to X$ denote a basepoint of X, and let $f\colon X \to *$ denote the unique map. Assume that e is ambidextrous and let $\mu_e\colon \mathrm{id}_{\mathcal{C}} \to e_! e^*$ denote the natural transformation which exhibits $e_!$ as a right adjoint of e^*. Let $\mathrm{Tr}_e\colon [X] \to \mathrm{id}_{\mathcal{C}}$ denote the natural transformation given by

$$[X] = f_! f^* \xrightarrow{\mu_e} f_! e_! e^* f^* \simeq \mathrm{id}_{\mathcal{C}}\,.$$

Notation 9.6. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Let X be a pointed Kan complex and let $e\colon * \to X$ denote a point of X, and let $f\colon X \to *$ denote the unique map. If $\alpha\colon [X] \to \mathrm{id}_{\mathcal{C}}$ is a natural transformation, then we let $\alpha(e)\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be the natural transformation given by the composite

$$\mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{\eta_e} f_! f^* \xrightarrow{\alpha} \mathrm{id}_{\mathcal{C}}.$$

In the case where e is ambidextrous and $\alpha = \mathrm{Tr}_e$, then $\mathrm{Tr}_e(e)$ is given by

$$\mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{\eta_e} f_! f^* \xrightarrow{\mathrm{Tr}_e} \mathrm{id}_{\mathcal{C}}.$$

Remark 9.7. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Let X be a Kan complex and let $f\colon X \to *$ denote the unique map. If $e\colon * \to X$ is an ambidextrous map, then the following two natural transformations

$$f_! f^* \xrightarrow{\mathrm{Tr}_e} \mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{f_! \eta_e f^*} f_! f^*$$

$$f_! f^* \xrightarrow{f_! \varepsilon_f f^*} f_! f^* f_! f^* \xrightarrow{\mathrm{Tr}_e f_! f^*} f_! f^*$$

are homotopic. The assertion is consequence of the following commutative diagram

$$\begin{array}{ccc}
f_! f^* & \xrightarrow{\varepsilon_f} & f_! f^* f_! f^* \\
\downarrow{\scriptstyle \mu_e} & & \downarrow{\scriptstyle \mu_e} \\
f_! e_! e^* f^* & \xrightarrow{\varepsilon_f} & f_! e_! e^* f^* f_! f^* \\
\downarrow{\scriptstyle \eta_e} & & \downarrow{\scriptstyle \eta_e} \\
f_! f^* & \xrightarrow{\varepsilon_f} & f_! f^* f_! f^* \\
& \searrow{=} & \downarrow{\scriptstyle \eta_f} \\
& & f_! f^*
\end{array}$$

The following result makes precise what we tried to explain in the introduction of this section.

Proposition 9.8 (Hopkins–Lurie [12, Proposition 5.1.8])**.** *Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q\colon \mathcal{C} \to \mathcal{X}$ be a Beck–Chevalley fibration. Let $f\colon X \to Y$ be a weakly ambidextrous morphism in $\mathcal{X}$. The following are equivalent:*

(1) The natural transformation $\nu_f\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ exhibits $f_!$ as a right adjoint of f^*.

(2) The trace form $\mathrm{TrFm}_f\colon [X/Y] \circ [X/Y] \to \mathrm{id}_{\mathcal{C}_Y}$ of f is a duality datum in the ∞-category $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$ of endofunctors on $\mathcal{C}_Y$ equipped with the monoidal structure given by composition of functors.

Proof. We first show that (1) implies (2). By assumption the natural transformation $\nu_f\colon f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ exhibits $f_!$ as a right adjoint of f^*. We may choose a compatible unit $\mu_f\colon \mathrm{id}_{\mathcal{C}_X} \to f_! f^*$ for this adjunction such that the following two composites

$$f^* \xrightarrow{f^* \mu_f} f^* f_! f^* \xrightarrow{\nu_f f^*} f^*$$

$$f_! \xrightarrow{\mu_f f_!} f_! f^* f_! \xrightarrow{f_! \nu_f} f_!$$

are homotopic to the identity. Define a natural transformation $c\colon \mathrm{id}_{\mathcal{C}_Y} \to [X/Y] \circ [X/Y]$ by

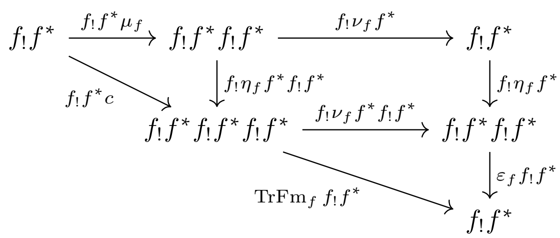
$$\mathrm{id}_{\mathcal{C}_Y} \xrightarrow{\mu_f} f_! f^* \xrightarrow{f_! \eta_f f^*} f_! f^* f_! f^*,$$

where $\eta_f : \mathrm{id}_{\mathcal{C}_X} \to f^* f_!$ is the unit of the adjunction which exhibits $f_!$ as a left adjoint of f^*. We show that the pair (c, TrFm_f) exhibits the functor $[X/Y]$ as a self-dual object of $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$. We must show that the following two composites

$$[X/Y] \xrightarrow{f_! f^* c} [X/Y] \circ [X/Y] \circ [X/Y] \xrightarrow{\mathrm{TrFm}_f\, f_! f^*} [X/Y]$$

$$[X/Y] \xrightarrow{c f_! f^*} [X/Y] \circ [X/Y] \circ [X/Y] \xrightarrow{f_! f^*\, \mathrm{TrFm}_f} [X/Y]$$

are homotopic to the identity. We will show that the first composite is homotopic to the identity. The remaining case follows from a completely similar argument. There is a commutative diagram



which implies the wanted. It remains to show that (2) implies (1). Suppose that the trace form TrFm_f exhibits the functor $[X/Y]$ as self-dual object of $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$ and choose a natural transformation $c : \mathrm{id}_{\mathcal{C}_Y} \to [X/Y] \circ [X/Y]$ compatible with TrFm_f. We want to show that the natural transformation $\nu_f : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ exhibits $f_!$ as a right adjoint of f^*. We need to specify a compatible unit of ν_f. Define two natural transformations

$$\mu : \mathrm{id}_{\mathcal{C}_Y} \xrightarrow{c} f_! f^* f_! f^* \xrightarrow{\varepsilon_f f_! f^*} f_! f^*$$

$$\mu' : \mathrm{id}_{\mathcal{C}_Y} \xrightarrow{c} f_! f^* f_! f^* \xrightarrow{f_! f^* \varepsilon_f} f_! f^*.$$

We show that the following natural transformations

$$\alpha : f_! \xrightarrow{\mu f_!} f_! f^* f_! \xrightarrow{f_! \nu_f} f_!$$

$$\alpha' : f^* \xrightarrow{f^* \mu'} f^* f_! f^* \xrightarrow{\nu_f f^*} f^*$$

are homotopic to the identity. This would give that $\mu \simeq \mu'$ is the unit for the adjunction specified by the counit ν_f as wanted. We show that α is homotopic to the identity. Let $\varepsilon_f : f_! f^* \to \mathrm{id}_{\mathcal{C}_Y}$ denote a counit of the adjunction between $f_!$ and f^*. It follows that the composite

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_Y)}(f_!, f_!) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_! f^*, f_! f^*) \xrightarrow{\varepsilon_f} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_! f^*, \mathrm{id}_{\mathcal{C}_Y})$$

is a homotopy equivalence of spaces. Under this homotopy equivalence the natural transformation μ is carried to

$$f_! f^* \xrightarrow{\mu f_! f^*} f_! f^* f_! f^* \xrightarrow{f_! \nu_f f^*} f_! f^* \xrightarrow{\varepsilon_f} \mathrm{id}_{\mathcal{C}_Y}$$

thus it suffices to show that this natural transformation is homotopic to the counit ε_f. This natural transformation is equivalent to the following natural transformation

$$f_! f^* \xrightarrow{c f_! f^*} f_! f^* f_! f^* f_! f^* \xrightarrow{\varepsilon_f f_! f^* f_! f^*} f_! f^* f_! f^* \xrightarrow{\mathrm{TrFm}_f} \mathrm{id}_{\mathcal{C}_Y}.$$

Moreover, there is a commutative diagram

$$\begin{array}{ccccc} f_!f^* & \xrightarrow{c f_! f^*} & f_!f^*f_!f^*f_!f^* & \xrightarrow{\varepsilon_f f_! f^* f_! f^*} & f_!f^*f_!f^* \\ & \overset{\mathrm{id}}{\searrow} & \downarrow{\scriptstyle f_!f^*\,\mathrm{TrFm}_f} & & \downarrow{\scriptstyle \mathrm{TrFm}_f} \\ & & f_!f^* & \xrightarrow{\varepsilon_f} & \mathrm{id}_{\mathcal{C}_Y} \end{array}$$

which shows that the composite above is homotopic to the counit ε_f. This ends the proof. □

Remark 9.9. Let $n \geq 1$ be an integer and let X be a Kan complex. Recall from Remark 5.12 that the ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra admits a symmetric monoidal structure $\hat{\otimes}$ and the functor $\hat{\otimes}$ determines a fully faithful embedding

$$\alpha : \mathrm{Sp}_{K(n)} \to \mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$$

whose essential image is the full subcategory of $\mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$ spanned by those functors that preserve small colimits (see Propositon 5.14). Furthermore, if $\mathrm{Fun}(\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{K(n)})$ is equipped with the monoidal structure given by composition, then α admits a monoidal structure. The functor $[X] : \mathcal{C} \to \mathcal{C}$ preserves small colimits since it is given by tensoring with X (see Remark 9.4) and the functor $[X]$ corresponds to the $K(n)$-local spectrum $[X](L_{K(n)}S^0) \simeq L_{K(n)}\Sigma^\infty_+ X$ under the equivalence α.

Corollary 9.10. *Let $n \geq 1$ be an integer and let X be a Kan complex. If X is weakly $\mathrm{Sp}_{K(n)}$-ambidextrous, then the map*

$$\mathrm{TrFm}_X : L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} L_{K(n)}\Sigma^\infty_+ X \to L_{K(n)}S^0$$

is a duality datum in $\mathrm{Sp}_{K(n)}$ if and only if X is $\mathrm{Sp}_{K(n)}$-ambidextrous.

Proof. Follows from Remark 9.9 and Proposition 9.8. □

We will need the following two result from [12].

Proposition 9.11 (Hopkins–Lurie [12, Proposition 5.1.13])**.** *Let $\mathcal{C}$ be an ∞-category which admits small colimits and let X be a simplicial set equipped with the structure of a simplicial group. Let $f : X \to *$ denote the unique map, $e : * \to X$ denote the identity element of X, and let $s : X \times X \to X$ denote the subtraction map on X given by $(x, y) \mapsto x^{-1}y$ on the simplices of X. If X is weakly $\mathcal{C}$-ambidextrous, then the trace form TrFm_X on X is given by the composite*

$$[X] \circ [X] \simeq [X \times X] \xrightarrow{\alpha_s} [X] \xrightarrow{\mathrm{Tr}_e} \mathrm{id}_{\mathcal{C}}\,.$$

Proof. See Hopkins–Lurie [12, Proposition 5.1.13]. □

Proposition 9.12 (Hopkins–Lurie [12, Proposition 5.1.18])**.** *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. Let G be a simplicial group and assume that G is $\mathcal{C}$-ambidextrous. Let $e : * \to G$ denote the inclusion of the identity element and let $E : * \to BG$ denote the inclusion of the basepoint. Then there exists a canonical homotopy*

$$\dim(G) \simeq \mathrm{Tr}_e(e) \circ \mathrm{Tr}_E(E)$$

of natural transformations from the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself.

Proof. See Hopkins–Lurie [12, Propostion 5.1.18]. □

10. On p-divisible groups and the Ravenel–Wilson calculation

The goal of this section is to state part of the Ravenel–Wilson calculation [32] as rephrased by Hopkins and Lurie in [12] that we will need. First our primary concern will be to define the notion of a truncated p-divisible group over a commutative ring which will require us to recall some terminology from algebraic geometry. We follow [28], [33, Tag 03NV], and [35].

Definition 10.1. Let $\mathbf{C}$ be a category. A Grothendieck topology on $\mathbf{C}$ is a set $\mathrm{Cov}(\mathbf{C})$ of collections $\{\varphi_i : U_i \to U \mid U_i, U \in \mathbf{C}\}_{i\in I}$ of morphisms in $\mathbf{C}$ called coverings which satisfy the following conditions:

(1) If $\varphi : V \to U$ is an isomorphism, then $\{\varphi : V \to U\}$ is a covering in $\mathbf{C}$.
(2) If $\{\varphi_i : U_i \to U\}_{i\in I}$ is a covering in $\mathbf{C}$ and $\{\psi_j : V_{ij} \to U_i\}_{j\in J}$ is a covering in $\mathbf{C}$ for every $i \in I$, then $\{\varphi_i \circ \psi_j : V_{ij} \to U\}_{i\in I, j\in J}$ is a covering in $\mathbf{C}$.
(3) If $\{\varphi_i : U_i \to U\}_{i\in I}$ is a covering in $\mathbf{C}$ and $\varphi : V \to U$ is a morphism in $\mathbf{C}$, then the pullback

$$\begin{array}{ccc} U_i \times_U V & \xrightarrow{f_i} & V \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle \varphi} \\ U_i & \xrightarrow{\varphi_i} & U \end{array}$$

exists in $\mathbf{C}$ for every $i \in I$ and $\{f_i : U_i \times_U V \to V\}_{i\in I}$ is a covering in $\mathbf{C}$.

A site is a category equipped with a Grothendieck topology.

We will primarily be interested in a Grothendieck topology on the category of schemes determined by the fpqc coverings which we now define.

Definition 10.2. Let S be a scheme. A collection of morphisms $\{U_i \to U\}_{i\in I}$ in the category $\mathbf{Sch}/S$ of schemes over S is a fpqc covering if the induced morphism

$$\coprod_{i\in I} U_i \to U$$

is faithfully flat and quasi-compact.

Remark 10.3. The collection of fpqc coverings endows the category $\mathbf{Sch}/S$ of schemes over S with the structure of a site. See [33, Tag 022D] for the argument.

Let S be a scheme and let $\{\varphi_i : U_i \to U\}_{i\in I}$ be a fpqc covering of schemes over S. If $F : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Grp}$ is a functor, then there is a diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \times_U U_j)$$

in the category of groups where the group homomorphism $F(U) \to \prod_{i\in I} F(U_i)$ is determined by sending $x \in F(U)$ to $(F\varphi_i)(x) \in F(U_i)$. Let $i \in I$ and $j \in J$ and form the pullback

$$\begin{array}{ccc} U_i \times_U U_j & \xrightarrow{f_j} & U_j \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \varphi_j} \\ U_i & \xrightarrow{\varphi_i} & U \end{array}$$

in the category of schemes over S. The group homomorphism α is determined by sending $s_i \in F(U_i)$ to $(Ff_i)(s_i) \in F(U_i \times_U U_j)$ while β is determined by sending $s_i \in F(U_i)$ to $(Ff_j)(s_i) \in F(U_i \times_U U_j)$.

Definition 10.4. Let S be a scheme and let $\mathbf{Sch}/S$ denote the category of schemes over S. A functor $F\colon(\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Grp}$ satisfies the sheaf condition for the fpqc topology if for every fpqc covering $\{U_i \to U\}_{i\in I}$ of schemes over S the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is an equalizer in the category $\mathbf{Grp}$ of groups.

Let S be a scheme. Recall that a group scheme $\mathbb{G}$ over S is a scheme $\mathbb{G} \to S$ over S which admits the structure of a group object in the category $\mathbf{Sch}/S$ of schemes over S. Equivalently, a group scheme over S is a functor $\mathbb{G}\colon(\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Grp}$. If $\mathbb{G}$ takes values in the category of abelian groups $\mathbf{Ab}$, then $\mathbb{G}$ is a commutative group scheme over S. Let $\mathbf{GrSch}/S$ denote the category of group schemes over S. Note that a group scheme over S satisfies the sheaf condition for the fpqc topology.

Definition 10.5. Let S be a scheme. A finite flat group scheme over S is a commutative group scheme $\varphi\colon G \to S$ over S which satisfies that

(1) The morphism φ is a finite flat morphism of schemes over S.
(2) The sheaf $\varphi_*(\mathcal{O}_G)$ is a locally free $\mathcal{O}_S$-module of locally constant rank $r > 0$.

Definition 10.6 (Tate [35])**.** Let S be a scheme and let p be a prime number. Let $h \geq 0$ be an integer. A p-divisible group of height h over S is an inductive system $G = (G_v, i_v)_{v\geq 0}$ satisfying the following conditions:

(1) Each G_v is a finite flat group scheme over S of order p^{vh}.
(2) The sequence
$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}$$
is exact where $[p^v]$ denotes the homomorphism given by multiplication by p^v on the group scheme G_v.

Example 10.7 ([35, Section 2.1])**.** Let R be a commutative ring and let p be a prime number. Let X be an abelian scheme over R of dimension d. For each integer $n \geq 0$ let $[p^v]\colon X \to X$ denote the homomorphism of abelian schemes over R given by multiplication by p^v and let $X[p^v]$ denote the kernel of $[p^v]$. For each integer $v \geq 0$ there is a canonical map $i_v\colon X[p^v] \to X[p^{v+1}]$ and the inductive system $(X[p^v], i_v)$ forms a p-divisible group over R of height $2d$. One could for example consider an elliptic curve E over R which is an abelian scheme of dimension 1.

Example 10.8 ([35, Section 1.2])**.** Let G be a finite group scheme over a scheme S. The Cartier dual G^* of G is defined by

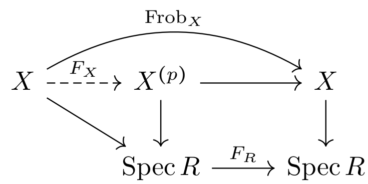
$$G^* = \mathrm{Hom}_{\mathbf{GrSch}/S}(G, \mathbb{G}_m).$$

The construction $G \mapsto G^*$ defines an endofunctor on the category of finite group schemes over S. There is a canonical isomorphism $G \xrightarrow{\simeq} (G^*)^*$ of group schemes over S.

We need to introduce the relative Frobenius homomorphism on scheme (see [33, Tag 0CC6]). Let R be a commutative ring and let p be a prime number. Assume that $p = 0$ in R. Let X be a scheme over $\operatorname{Spec} R$ and let $\mathrm{Frob}_X\colon X \to X$ denote the absolute Frobenius map on X. Form a pullback

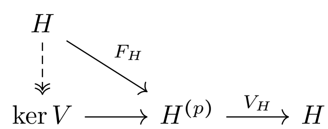
$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{\mathrm{Frob}_R} & \operatorname{Spec} R \end{array}$$

in the category of schemes over $\operatorname{Spec} R$. The absolute Frobenius homomorphism on X induces a unique homomorphism $F_X : X \to X^{(p)}$ of schemes over $\operatorname{Spec} R$ such that the following diagram



commutes. The homomorphism $F_X : X \to X^{(p)}$ is called the relative Frobenius on X. Let G be a finite flat group scheme over R and let $F_G : G \to G^{(p)}$ denote the relative Frobenius. Let $F_{G^*} : G^* \to (G^*)^{(p)} \simeq (G^{(p)})^*$ denote the relative Frobenius on the Cartier dual of G. Define the relative Verschiebung homomorphism $V_G : G^{(p)} \to G$ as the Cartier dual of F_{G^*}, that is $V_G := (F_{G^*})^*$.

Definition 10.9. Let R be a commutative ring and let G be a finite flat group scheme of rank p^n. Suppose that G is annihilated by p. Set $H = G \times_{\operatorname{Spec} R} \operatorname{Spec} R/p$ and let $H^{(p)}$ denote the pullback of G along the Frobenius map on R/p. The group scheme G is a truncated p-divisible group of height n and level 1 over R if the relative Frobenius homomorphism $F_H : H \to H^{(p)}$ induces an epimorphism



of fpqc sheaves where $V_H : H^{(p)} \to H$ denotes the relative Verschiebung homomorphism.

Remark 10.10. There are different ways of defining a truncated p-divisible group of level 1. Definition 10.9 above is taken from [12, Proposition 3.1.5]. See also [12, Definition 3.1.3].

Remark 10.11 ([12, Remark 3.1.9])**.** Let R be a commutative ring and let G be a truncated p-divisible group of level 1 over R. Set $G_0 := \operatorname{Spec} R/p \times_{\operatorname{Spec} R} G$ and let $G_0^{(p)}$ denote the pullback along the Frobenius $\varphi : R/p \to R/p$. Let $F : G_0 \to G_0^{(p)}$ denote the relative Frobenius. It follows from the definition that the kernel $\ker F$ of F is a finite flat group scheme over $\operatorname{Spec} R/p$. We say that the truncated p-divisible group G has dimension d if $\ker F$ has rank p^d over R/p.

Finally, we state the Ravenel–Wilson calculation as rephrased by Hopkins and Lurie in [12].

Theorem 10.12 (Ravenel–Wilson, Johnson–Wilson, Hopkins–Lurie)**.** *Let E be a Lubin–Tate spectrum associated to the pair $(\kappa, \mathbb{G}_0)$ where κ is a perfect field of characteristic p and $\mathbb{G}_0$ is a formal group of height n over κ. Let $m \geq 1$ be an integer.*

1. *Both $E^0 K(\mathbb{Z}/p, m)$ and $E_0^\wedge K(\mathbb{Z}/p, m)$ are free modules of rank $p^{\binom{n}{m}}$ over $\pi_0 E$.*
2. *The scheme $\operatorname{Spec} E_0^\wedge K(\mathbb{Z}/p, m)$ is a truncated p-divisible group of height $\binom{n}{m}$, dimension $\binom{n-1}{m}$, and level 1 over $\pi_0 E$.*

Proof. Combine [12, Theorem 3.4.1] and [12, Theorem 3.5.1]. □

Remark 10.13. It follows from Theorem 10.12 above that $E_0^\wedge K(\mathbb{Z}/p, m)$ is a free module of rank $p^{\binom{n}{m}}$ over the Lubin–Tate ring $\pi_0 E$. In particular, we conclude that $E_0^\wedge K(\mathbb{Z}/p, m)$ is a projective module over $\pi_0 E$. Furthermore, there is a equivalence

$$L_{K(n)} E[K(\mathbb{Z}/p, m)] \simeq E^{p^{\binom{n}{m}}}$$

of E-modules [12, Proposition 3.4.3].

Let X be a Kan complex and suppose that X is weakly $\mathrm{Sp}_{K(n)}$-ambidextrous. In this situation we constructed a trace form TrFm_X on X and it follows from Corollary 9.10 that X is $\mathrm{Sp}_{K(n)}$-ambidextrous precisely if the trace form on X is a duality datum in the ∞-category of $K(n)$-local spectra. The trace form on X induces a homomorphism

$$\beta_0 : E_0^\wedge(X) \otimes_{\pi_0 E} E_0^\wedge(X) \to \pi_0 E$$

of $\pi_0 E$-modules and we will see that it suffices to show that β_0 is a duality datum in the category of $\pi_0 E$-modules. If X is the Eilenberg–MacLane space $K(\mathbb{Z}/p, m)$ for some $m \geq 1$, then it follows from Theorem 10.12 that $E_0^\wedge X$ admits the structure of a finite flat $\pi_0 E$-algebra. In this case there is an algebraicly defined trace form on $E_0^\wedge X$. Invoking the following general result due to Tate we conclude that this trace form on $E_0^\wedge X$ is a duality datum.

Proposition 10.14 (Tate)**.** *Let R be a commutative ring and let p be a prime number which is not a zero-divisor in R. Let G be a truncated p-divisible group over R of height h, dimension d, and level 1 and write $G = \operatorname{Spec} A$ where A is a finite flat R-algebra of rank p^h. Let $\sigma : A \to A$ dentote the antipodal map and let $\lambda : A \to R$ be an R-module homomorphism with the following properties:*

1. *The R-module homomorphism λ satisfies that $\lambda(1) = p^{h-d}$.*
2. *If $\Delta : A \to A \otimes_R A$ is the ring homomorphism classifying the multiplication on G, then the composite*
$$A \xrightarrow{\Delta} A \otimes_R A \xrightarrow{\lambda \otimes \mathrm{id}} A$$
is given by $a \mapsto \lambda(a)$.

Then the construction $(a, b) \mapsto \lambda((\sigma a)b)$ determines a duality datum $A \otimes_R A \to R$ in the category of R-modules.

Proof. See [12, Corollary 5.2.4]. □

11. Proof of the main result

In this section we will be occupied with the proof of the following result due to Hopkins and Lurie in [12].

Theorem 11.1 (Hopkins–Lurie [12, Theorem 5.2.1])**.** *Let $K(n)$ be the nth Morava K-theory spectrum and let X be a Kan complex. If X is a finite m-type for some integer $m \geq -2$, then X is $\mathrm{Sp}_{K(n)}$-ambidextrous.*

The proof will be carried out in several steps using most of the material that we have presented so far. In [12], Hopkins and Lurie provide an outline of their strategy which we repeat here.

1. Using Corollary 8.23 we may assume that X is the Eilenberg–MacLane space $K(\mathbb{Z}/p, m)$.
2. It follows from Corollary 9.10 that it suffices to show that the trace form
$$\mathrm{TrFm}_X : L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} L_{K(n)}\Sigma^\infty_+ X \to L_{K(n)} S^0$$
is a duality datum in the symmetric monoidal ∞-category of $K(n)$-local spectra.
3. Let E denote a Lubin–Tate spectrum associated to the Morava K-theory spectrum $K(n)$. Using Proposition 11.9, we reduce to showing that the trace form
$$L_{K(n)}E[X] \otimes_E L_{K(n)}E[X] \to E$$

is a duality datum in the symmetric monoidal ∞-category $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$ of E-modules in $\mathrm{Sp}_{K(n)}$.

(4) A consequence of the Ravenel–Wilson calculation is that $L_{K(n)}E[X]$ is a projective E-module of finite rank (see Example 11.6). Using Proposition 11.8 it suffices to show the trace form
$$\beta_0 : E_0^\wedge(X) \otimes_{\pi_0 E} E_0^\wedge(X) \to \pi_0 E$$
is a duality datum in the category of $\pi_0 E$-modules which is a purely algebraic claim.

(5) The pairing β_0 on $E_0^\wedge(X)$ can be identified with a multiple of an algebraicly defined trace pairing on $E_0^\wedge(X)$. Invoking a general result due to Tate (see Proposition 10.14) we conclude that β_0 is a duality datum.

We will start by reviewing some facts about projective and flat modules over $\mathbb{E}_1$-rings following [24, Chapter 7] and [12, Section 5.1].

Definition 11.2. Let R be an $\mathbb{E}_1$-ring.

(1) A left R-module M is π-projective if $\pi_0 M$ is projective as a left module over $\pi_0 R$ and the canonical map
$$\pi_m R \otimes_{\pi_0 R} \pi_0 M \to \pi_m M$$
is an isomorphism of left $\pi_0 R$-modules for every integer m. Let $\mathrm{LMod}_R^{\mathrm{proj}}$ denote the full subcategory of the ∞-category LMod_R of left modules over R spanned by the π-projective left R-modules.

(2) A left R-module M is flat if $\pi_0 M$ is flat as a left module over $\pi_0 R$ and the canonical map
$$\pi_m R \otimes_{\pi_0 R} \pi_0 M \to \pi_m M$$
is an isomorphism of left $\pi_0 R$-modules for every integer m. Let $\mathrm{LMod}_R^\flat$ denote the full subcategory of the ∞-category LMod_R of left modules over R spanned by the flat left R-modules.

Remark 11.3. Let R be a connective $\mathbb{E}_1$-ring. In [24, Section 7.2.2], a left module over R is said to be projective if M is a projective object of the ∞-category LMod_R of modules over R in the sense of [22, Definition 5.5.8.18]. Let LProj_R denote the full subcategory of LMod_R spanned by the left modules over R which are projective. In this project we will need to work over nonconnective $\mathbb{E}_\infty$-rings so we will need the weaker notion of being π-projective. It follows from [24, Lemma 7.2.2.14] and [24, Proposition 7.2.2.18] that if R is a connective $\mathbb{E}_1$-ring, then a left module M over R is projective precisely if M is π-projective as a left module over R.

Remark 11.4. Let R be an $\mathbb{E}_1$-ring and let M be a left module over R. If M admits the structure of a π-projective left module over R, then M is also a flat left module over R.

Example 11.5. Let $m \geq 1$ be an integer, and let R be an $\mathbb{E}_1$-ring. The m-fold product R^m of R with itself admits the structure of a left R-module. We have that $\pi_0 R^m \simeq (\pi_0 R)^m$ is a projective $\pi_0 R$-module and for every integer n the canonical map
$$\pi_n R \otimes_{\pi_0 R} \pi_0 R^m \to \pi_n R^m$$
is an isomorphism of $\pi_0 R$-modules. It follows that R^m is a π-projective R-module.

Example 11.6. Let E be a Lubin–Tate spectrum associated to the pair $(\kappa, \mathbb{G}_0)$ where κ is a perfect field of characteristic p and $\mathbb{G}_0$ is a formal group of height n over κ. Let $K(n)$ be the associated Morava K-theory spectrum. It follows from Remark 10.13 and Example 11.5 that $L_{K(n)}E[K(\mathbb{Z}/p, m)]$ is a π-projective E-module of rank $p^{\binom{n}{m}}$ for every integer $m \geq 1$.

We need the following variant of [24, Proposition 7.2.2.16]. The proof is completely analogous to the proof of [24, Proposition 7.2.2.16].

Proposition 11.7. *Let $f : R \to S$ be a map of $\mathbb{E}_1$-rings and suppose that f induces an isomorphism $\pi_n f : \pi_n R \to \pi_n S$ for every integer $n \geq 0$. Let $G : \mathrm{LMod}_S \to \mathrm{LMod}_R$ denote the forgetful functor and let $F : \mathrm{LMod}_R \to \mathrm{LMod}_S$ denote a left adjoint of G determined by the construction $M \mapsto S \otimes_R M$. Then the functor F induces an equivalence*

$$F : \mathrm{LMod}_R^{\mathrm{proj}} \xrightarrow{\simeq} \mathrm{LMod}_S^{\mathrm{proj}}$$

of ∞-categories.

Proof. Let $\tau_{\geq 0} R \to R$ denote a connective cover of R which exists by [24, Proposition 7.1.3.13]. We obtain a commutative diagram

$$\begin{tikzcd} & \mathrm{LMod}_R \arrow[dr] & \\ \mathrm{LMod}_{\tau_{\geq 0} R} \arrow[ur] \arrow[rr] & & \mathrm{LMod}_S \end{tikzcd}$$

of ∞-categories. We may therefore assume that R is connective. Let $\mathrm{LMod}_R^{\geq 0}$ denote the full subcategory of LMod_R spanned by those left modules over R which satisfy that $\pi_n M \simeq 0$ for $n < 0$. Let $\mathrm{LMod}_R^{\leq 0}$ denote the full subcategory of LMod_R spanned by those left modules over R which satisfy that $\pi_n M \simeq 0$ for $n > 0$. Let F' denote the compsosite

$$\mathrm{LMod}_R^{\geq 0} \hookrightarrow \mathrm{LMod}_R \xrightarrow{F} \mathrm{LMod}_S.$$

The pair $(\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\leq 0})$ determines an accesible t-structure on LMod_R since R is connective ([24, Proposition 7.1.1.13]). It follows that the inclusion $\mathrm{LMod}_R^{\geq 0} \hookrightarrow \mathrm{LMod}_R$ admits a right adjoint $\tau_{\geq 0}$. Consequently, the functor F' admits a right adjoint G' given by the composite

$$\mathrm{LMod}_S \xrightarrow{G} \mathrm{LMod}_R \xrightarrow{\tau_{\geq 0}} \mathrm{LMod}_R^{\geq 0}.$$

Using [24, Propositon 7.2.2.13] it is obvious that both F' and G' preserve flatness. Similarly, one finds that F' and G' also carry π-projectives to π-projectives. It follows that the functors F' and G' restrict to an adjunction

$$\mathrm{LMod}_R^{\mathrm{proj}} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{LMod}_S^{\mathrm{proj}}$$

of ∞-categories. To show that F' is an equivalence of ∞-categories it suffices to show that the unit and the counit of the adjunction above are equivalences.

(1) Let M be a left module over R which is π-projective. We want to show that the unit map $M \to \tau_{\geq 0}(S \otimes_R M)$ is an equivalence. If $n < 0$, then $\pi_n M \simeq 0$ and $\pi_n \tau_{\geq 0}(S \otimes_R M) \simeq 0$. If $n \geq 0$, then it follows from [24, Proposition 7.2.2.13] that

$$\pi_n \tau_{\geq 0}(S \otimes_R M) \simeq \pi_n(S \otimes_R M) \simeq \pi_n S \otimes_{\pi_0 R} \pi_0 M.$$

Invoking the assumption that $\pi_n S \simeq \pi_n R$ and that M is π-projective we conclude that

$$\pi_n S \otimes_{\pi_0 R} \pi_0 M \simeq \pi_n R \otimes_{\pi_0 R} \pi_0 M \simeq \pi_n M$$

as wanted.

(2) Let N be a left S-module which is π-projective. We want to show that the counit $S \otimes_R G'(N) \to N$ is an equivalence. Since $G'(N) \simeq \tau_{\geq 0} G(N)$ is π-projective we also have that $G'(N)$ is flat. As before we find that

$$\pi_n(S \otimes_R G'(N)) \simeq \pi_n R \otimes_{\pi_0 R} \pi_0 N \simeq \pi_n N$$

for every integer n.

□

Proposition 11.8. *Let M and N be π-projective R-modules over an $\mathbb{E}_\infty$-ring R. A morphism $e\colon M \otimes_R N \to R$ is a duality datum in the ∞-category Mod_R of R-modules if and only if the induced map*

$$\pi_0 M \otimes_{\pi_0 R} \pi_0 N \to \pi_0 R$$

is a duality datum in the category $\mathbf{Mod}_{\pi_0 R}$ of $\pi_0 R$-modules.

Proof. Let R be an $\mathbb{E}_\infty$-ring and consider the connective cover $f\colon \tau_{\geq 0} R \to R$ of R. It follows from [24, Proposition 7.1.3.13] that R admits a connective cover and the same result also gives that the map f induces an isomorphism $\pi_n(\tau_{\geq 0} R) \to \pi_n R$ for every integer $n \geq 0$. Proposition 11.7 above supplies an equivalence

$$\mathrm{LMod}^{\mathrm{proj}}_{\tau_{\geq 0} R} \xrightarrow{\simeq} \mathrm{LMod}^{\mathrm{proj}}_{R}$$

of ∞-categories given by the construction $M \mapsto R \otimes_{\tau_{\geq 0} R} M$. Consequently, we may assume that R is connective. In this case we may apply [24, Corollary 7.2.2.19]. Since R is connective we have that $\mathrm{Proj}_R \simeq \mathrm{Mod}^{\mathrm{proj}}_R$ (see Remark 11.3). We conclude that the construction $M \mapsto \pi_0 M$ determines an equivalence

$$\mathrm{hMod}^{\mathrm{proj}}_R \xrightarrow{\simeq} \mathbf{Mod}^{\mathrm{proj}}_{\pi_0 R}$$

of categories, where $\mathbf{Mod}^{\mathrm{proj}}_{\pi_0 R}$ denotes the category of modules over π_0 which are projective. This ends the proof. □

Theorem 11.1 will be a consequence of the following two results.

Proposition 11.9 (Hopkins–Lurie [12, Corollary 5.2.5])**.** *Let E be a Lubin–Tate spectrum associated to the pair $(\kappa, \mathbb{G}_0)$ where κ is a perfect field of characteristic p and $\mathbb{G}_0$ is a formal group of height n over κ. Let $K(n)$ denote the associated Morava K-theory spectrum. Let $m \geq 1$ be an integer and assume that $K(\mathbb{Z}/p, m-1)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous. Then the map*

$$\beta\colon L_{K(n)} E[K(\mathbb{Z}/p, m)] \otimes_E L_{K(n)} E[K(\mathbb{Z}/p, m)] \to E$$

of E-module spectra in $\mathrm{Sp}_{K(n)}$ determined by the trace form of $K(\mathbb{Z}/p, m)$ is a duality datum.

Proof. Let $m \geq 1$ be an integer and set $X = K(\mathbb{Z}/p, m)$ to ease notation. Recall that $L_{K(n)} E[X]$ is a projective E-module of finite rank (see Example 11.6). Invoking Proposition 11.8, it suffices to show that the map β induces a duality datum

$$\beta_0\colon E^\wedge_0(X) \otimes_{\pi_0 E} E^\wedge_0(X) \to \pi_0 E$$

in the category of $\pi_0 E$-modules. If $e\colon * \to X$ classifies the identity element of X, then the natural transformation $\mathrm{Tr}_e\colon [X] \to \mathrm{id}_{\mathrm{Sp}_{K(n)}}$ induces a homomorphism $\lambda\colon E^\wedge_0(X) \to \pi_0 E$ of $\pi_0 E$-modules. Let $\sigma\colon E^\wedge_0(X) \to E^\wedge_0(X)$ denote the antipode of $E^\wedge_0(X)$. It follows from Proposition

9.11 that β admits a factorization

$$\begin{array}{ccc} E_0^{\wedge}(X)\otimes_{\pi_0 E} E_0^{\wedge}(X) & \xrightarrow{\beta} & \pi_0 E \\ \downarrow & \nearrow_{\lambda} & \\ E_0^{\wedge}(X) & & \end{array}$$

where the vertical $\pi_0 E$-module homomorphism is determined by the construction $(x,y)\mapsto(\sigma x)y$. Consequently, it suffices to show that λ satisfies (1) and (2) of Proposition 10.14.

(1) We wish to show that
$$\lambda(1) = p^{\binom{n}{m}-\binom{n-1}{m}} = p^{\binom{n-1}{m-1}}.$$
We proceed by induction on m. Let $e'\colon * \to K(\mathbb{Z}/p, m-1)$ classify a basepoint of $K(\mathbb{Z}/p, m-1)$. The map e' induces a homomorphism $\lambda'\colon E_0^{\wedge}K(\mathbb{Z}/p,m-1)\to\pi_0 E$ of $\pi_0 E$-modules as above. Let $\eta'\colon\pi_0 E\to E_0^{\wedge}K(\mathbb{Z}/p,m-1)$ denote the unit of $E_0^{\wedge}K(\mathbb{Z}/p,m-1)$. Unwinding the definition of $\mathrm{Tr}_{e'}(e')$ we obtain a commutative diagram

$$\begin{array}{ccc} \pi_0 E & \xrightarrow{\mathrm{Tr}_{e'}(e')} & \pi_0 E \\ \downarrow{\scriptstyle \eta'} & \nearrow_{\lambda'} & \\ E_0^{\wedge}K(\mathbb{Z}/p,m-1) & & \end{array}$$

of $\pi_0 E$-modules which means that $\lambda'(1) = \mathrm{Tr}_{e'}(e')(1)$. Similarly, we conclude that $\lambda(1) = \mathrm{Tr}_e(e)(1)$. Recall that $E_0^{\wedge}K(\mathbb{Z}/p,m-1)$ is a projective $\pi_0 E$-module of rank $p^{\binom{n}{m-1}}$. It follows that $\dim E_0^{\wedge}K(\mathbb{Z}/p,m-1) = p^{\binom{n}{m-1}}$. Using Proposition 9.12 we find that
$$\lambda(1)\lambda'(1) = p^{\binom{n}{m-1}}.$$
Consequently, it suffices to show that
$$\lambda'(1) = p^{\binom{n}{m-1}-\binom{n-1}{m-1}} = \begin{cases} p^{\binom{n-1}{m-2}} & m\geq 2 \\ 1 & m = 1 \end{cases}$$
This follows from the inductive hypothesis if $m\geq 2$. If $m=1$, then the identity clearly holds. This proves (1).

(2) Let $\Delta\colon E_0^{\wedge}X\to E_0^{\wedge}X\otimes_{\pi_0 E}E_0^{\wedge}X$ denote the comultiplication on $E_0^{\wedge}X$. It follows from Remark 9.7 that the composite
$$E_0^{\wedge}X \xrightarrow{\Delta} E_0^{\wedge}X\otimes_{\pi_0 E}E_0^{\wedge}X \xrightarrow{\lambda\otimes\mathrm{id}} E_0^{\wedge}X$$
coincides with the composite
$$E_0^{\wedge}X \xrightarrow{\lambda} \pi_0 E \xrightarrow{\eta} E_0^{\wedge}X.$$
This proves (2).

We conclude that the homomorphism $\beta_0\colon E_0^{\wedge}(X)\otimes_{\pi_0 E}E_0^{\wedge}(X)\to\pi_0 E$ of $\pi_0 E$-modules is a duality datum in the category of $\pi_0 E$-modules. It follows that the map
$$\beta\colon L_{K(n)}E[K(\mathbb{Z}/p,m)]\otimes_E L_{K(n)}E[K(\mathbb{Z}/p,m)]\to E$$
of E-module spectra in $\mathrm{Sp}_{K(n)}$ determined by the trace form of $K(\mathbb{Z}/p,m)$ is a duality datum as wanted. $\square$

Proposition 11.10 (Hopkins–Lurie [12, Corollary 5.2.7]). *Let E be a Lubin–Tate spectrum associated to the pair $(\kappa, \mathbb{G}_0)$ where κ is a perfect field of characteristic p and $\mathbb{G}_0$ is a formal group of height n over κ. Let $K(n)$ denote the associated Morava K-theory spectrum. Let $m \geq 1$ be an integer and assume that $K(\mathbb{Z}/p, m-1)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous. Then $K(\mathbb{Z}/p, m)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous.*

Proof. Let $m \geq 1$ be an integer and let $X = K(\mathbb{Z}/p, m)$ to ease notation. We have that X is weakly $\mathrm{Sp}_{K(n)}$-ambidextrous since $K(\mathbb{Z}/p, m-1)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous by assumption. Using Corollary 9.10, it suffices to show that the trace form

$$\mathrm{TrFm}_X : L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} L_{K(n)}\Sigma^\infty_+ X \to L_{K(n)} S^0$$

is a dualtity datum in $\mathrm{Sp}_{K(n)}$. Equivalently, for every pair Y and Z of $K(n)$-local spectra the composite $\theta_{Y,Z}$ given by

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(Y, L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} Z) & \longrightarrow & \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} Y, L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} Z) \\ & & \Big\downarrow {\scriptstyle -\circ(\mathrm{TrFm}_X \hat{\otimes} \mathrm{id}_Z)} \\ & & \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}\Sigma^\infty_+ X \hat{\otimes} Y, Z) \end{array}$$

is a homotopy equivalence, where the first map is given by tensoring with $L_{K(n)}\Sigma^\infty_+ X$ in $\mathrm{Sp}_{K(n)}$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Sp}_{K(n)}$ spanned by those Z which satisfy that $\theta_{Y,Z}$ is a homotopy equivalence for all Y in $\mathrm{Sp}_{K(n)}$. Note that $\mathcal{C}$ is a stable subcategory of $\mathrm{Sp}_{K(n)}$ closed under retracts. We use Proposition 5.15 to show that $\mathcal{C} = \mathrm{Sp}_{K(n)}$. Suppose that Z admits the structure of an E-module. Recall that the underlying spectrum of $E[X]$ is equivalent to $E \hat{\otimes} L_{K(n)}\Sigma^\infty_+ X$. Consequently, we can identity $\theta_{Y,Z}$ with the following map

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}(E\hat{\otimes} Y, L_{K(n)}E[X] \otimes_E Z) \\ \Big\downarrow \\ \mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}(L_{K(n)}E[X] \otimes_E (E\hat{\otimes} Y), L_{K(n)}E[X] \otimes_E L_{K(n)}E[X] \otimes_E Z) \\ \Big\downarrow {\scriptstyle -\circ(\beta \otimes \mathrm{id}_Z)} \\ \mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}(L_{K(n)}E[X] \otimes_E (E\hat{\otimes} Y), Z) \end{array}$$

which is a homotopy equivalence since β is a duality datum by Proposition 11.9. □

Proof of Theorem 11.1. We may assume that $m \geq 1$ since the ∞-category of $K(n)$-local spectra is a stable ∞-category which admits small limits and colimits. We want to show that the ∞-category of $K(n)$-local spectra is m-semiadditive. It follows from Corollary 11.10 that the Eilenberg–Mac Lane space $K(\mathbb{Z}/p, m)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous for every $m \geq 1$. Using Example 8.24 and Corollary 8.23 we conclude that the ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra is m-semiadditive as wanted. □

Remark 11.11. Recall from Remark 6.10 that the Tate construction vanishes $T(n)$-locally. However, the analogue of Theorem 11.1 for the ∞-category of $T(n)$-local spectra is not known.

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