Elmendorf's Theorem via Model Categories

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1 Introduction

In [2], working in the category of compactly generated spaces $\mathcal{U}$, Elmendorf relates the equivariant homotopy theory of G-spaces to a homotopy theory of diagrams using fixed point sets. The diagrams are indexed by a topological category $\mathcal{O}_G$ with objects the orbit spaces $\{G/H\}_H$ for the closed subgroups $H \subset G$. Although, his general assumption there is that G is a compact Lie group, a formulation of Elmendorf's Theorem can be found on page 44 in [10] for any topological group in $\mathcal{U}$. A more modern approach has been given by Piacenza in [11] using model categories. For any topological group G in $\mathcal{U}$, he equips the category $\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}}$ of continuous contravariant functors from $\mathcal{O}_G$ to $\mathcal{U}$ with a model category structure, where the weak equivalences are the objectwise weak equivalences. Concerning equivariant homotopy theory, a morphism f in the category of G-spaces $\mathcal{U}^G$ is defined to be a weak equivalence, if for all closed subgroups $H \subset G$, the map $(f)^H$ is a weak equivalence between spaces, where $(-)^H : \mathcal{U}^G \to U$ is the H-fixed point functor. That is, for a G-space X one has

$$X^H = \{x \in X; hx = x \text{ for all } h \in H\}.$$

Piacenza's formulation of Elmendorf's Theorem states in particular that the homotopy categories $\mathrm{Ho}(\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}})$ and $\mathrm{Ho}(\mathcal{U}^G)$, which are obtained by formally inverting the weak equivalences, are equivalent. For the proof, he generalizes the cellular theory of the category of spaces $\mathcal{U}$ to any category $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$ of continuous contravariant diagrams from a topological category $\mathcal{J}$ to $\mathcal{U}$.

We will formulate and prove a generalization of Elmendorf's Theorem and reach the following two goals. First, the model category theoretical approach will be emphasized by equipping the category of G-spaces $\mathcal{U}^G$ with a model category structure and by proving that there is a pair of Quillen equivalences

$$\Theta : \mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}} \rightleftarrows \mathcal{U}^G : \Phi.$$

Thus, by the theory of model categories, the equivalence of the homotopy categories comes for free. Secondly, we generalize this result to the case, where one considers not all closed subgroups of the topological group G, but only a subset, which contains the trivial subgroup $\{e\}$.

These two results might be well known to the experts. For instance, the generalization for a family of closed subgroups, which is closed under conjugation, is stated in Remark 1.3 of [5]. Though, the author did not find a published proof. In the case, where the group G is discrete, the first result is outlined in [3] and generalized to model categories different from $\mathcal{U}$.

Many terms used up to now will be introduced in the article. A necessary prerequisite is familiarity with some category theory and algebraic topology, in particular with CW-complexes and homotopy groups. Some knowledge about model categories might be an advantage, but we introduce the necessary definitions and results in section 2 and show a useful argument to equip a category with a model category structure in section 3. In the next section, we list the basics of several subcategories of the category of topological spaces, for instance of the category of compactly generated spaces, and equip these subcategories with a model category structure. In section 5, a theorem to lift a model category structure to another category will be proved. It will be applied to the category of G-spaces in section 6 and to an example, which will be specialized to the category $\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}}$ in the following section. There, we state and prove the main theorem.

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2 Model Categories

Model categories as we will define them, have been introduced by Quillen in [12] under the name closed model categories. Following [1], we recall their basics. For stating the definition of a model category and its homotopy category, we introduce the following terms.

Definition 2.1. A morphism $f\colon X \to X'$ in a category $\mathcal{C}$ is called a *retract* of a morphism $g\colon Y \to Y'$ of $\mathcal{C}$, if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}, \tag{1}$$

where the compositions in the top and bottow row are the corresponding identity morphisms.

Lemma 2.2. *Let g be an isomorphism in a category $\mathcal{C}$ and f a retract of g. Then f is also an isomorphism.*

Proof. By definition, there exists a commutative diagram as in (1) with $ri = \mathrm{id}_X$ and $r'i' = \mathrm{id}_{X'}$. By assumption, g has an inverse g^{-1}. One checks that $rg^{-1}i'$ is the inverse of f. □

Definition 2.3. Let $i\colon A \to B$, $p\colon X \to Y$ be morphisms in a category $\mathcal{C}$. We say that i has the *left lifting property* (LLP) with respect to p and that p has the *right lifting property* (RLP) with respect to i, if there exists a lift in any commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \quad . \tag{2}$$

Definition 2.4. Let W be a class of morphisms of a category $\mathcal{C}$. A category $\mathcal{D}$ together with a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called the *localization of C with respect to W*, if F maps morphisms of W to isomorphisms and satisfies the following universal property: Given any functor G from $\mathcal{C}$ to a category $\mathcal{D}'$, which maps morphism of W to isomorphisms, there exists a unique functor $G'\colon \mathcal{D} \to \mathcal{D}'$ with $G'F = G$.

Definition 2.5. A *model category* is a category $\mathcal{C}$ together with three classes of morphisms of $\mathcal{C}$, the class of *weak equivalences*, of *fibrations* and of *cofibrations*, each of which is closed under composition and contains all identity morphisms of $\mathcal{C}$, such that the five axioms **MC1-MC5** below hold. A morphism of $\mathcal{C}$ is called an *acyclic fibration* if it is both a weak equivalence and a fibration, it is called an *acyclic cofibration* if it is both a weak equivalence and a cofibration.

MC1: Every functor from a finite category to $\mathcal{C}$ has a limit and a colimit.

MC2: If f and g are morphisms of $\mathcal{C}$ such that gf is defined and if two out of the three morphisms f, g and gf are weak equivalences, then so is the third.

MC3: If f is a retract of a morphism g of $\mathcal{C}$ and g is a weak equivalence, a fibration or a cofibration, then so is f.

MC4:

i) Every cofibration has the LLP with respect to all acyclic fibrations.

ii) Every fibration has the RLP with respect to all acyclic cofibrations.

MC5: Any morphism f of $\mathcal{C}$ can be factored as

i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and as

ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

In any model category, the class of weak equivalences together with another class of the model category structure determines the third one. More precisely, by Proposition 3.13 of [1], the following holds.

Proposition 2.6. *Let $\mathcal{C}$ be a model category.*

i) *The cofibrations in $\mathcal{C}$ are the morphisms that have the LLP with respect to the acyclic fibrations.*

ii) *The acyclic cofibrations in $\mathcal{C}$ are the morphisms that have the LLP with respect to the fibrations.*

iii) *The fibrations in $\mathcal{C}$ are the morphisms that have the RLP with respect to the acyclic cofibrations.*

iv) *The acyclic fibrations in $\mathcal{C}$ are the morphisms that have the RLP with respect to the cofibrations.*

Remark 2.7. Since any isomorphism in category is a retract of an identity morphism, it follows that in a model category, every isomorphism is a weak equivalence, a fibration and a cofibration.

The following objects of a model category are of particular interest.

Definition 2.8. An object X of a model category $\mathcal{C}$ is called *fibrant*, if the unique morphism from X to the terminal object is a fibration. It is called *cofibrant*, if the unique morphism from the initial object to X is a cofibration.

For instance in section 5 and 6 of [1], these objects are used to show that the localization of a model category with respect to the weak equivalences exists. This enables us to make the following definition.

Definition 2.9. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the localization of $\mathcal{C}$ with respect to the class of weak equivalences.

To compare model categories, the following notion is used.

Definition 2.10. Let $\mathcal{C}$, $\mathcal{D}$ be model categories and $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ a pair of adjoint functors. The pair (F, G) is called a *pair of Quillen equivalences*, if it satisfies the following two conditions:

i) G preserves fibrations and acyclic fibrations,

ii) for each cofibrant object A of $\mathcal{C}$ and each fibrant object X of $\mathcal{D}$, a morphism $A \to G(X)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $F(A) \to X$ is a weak equivalence in $\mathcal{D}$.

Remark 9.8 of [1] states the following.

Remark 2.11. For a pair (F, G) of adjoint functors between model categories, the following three conditions are equivalent.

i) G preserves fibrations and acyclic fibrations,

ii) F preserves cofibrations and acyclic cofibrations and

iii) F preserves cofibrations and G preserves fibrations.

The term Quillen equivalence is justified by the following theorem, which is proved in section 9 of [1].

Theorem 2.12. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a pair of Quillen equivalences between two model categories $\mathcal{C}$ and $\mathcal{D}$. Then the homotopy categories* $\mathrm{Ho}(\mathcal{C})$ *and* $\mathrm{Ho}(\mathcal{D})$ *are equivalent.*

The proof of this theorem uses Lemma 9.9 of [1], i.e. the following.

Lemma 2.13 (K. Brown)**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories, that takes acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

3 The small object argument

Our main tool in proving that a category together with three chosen classes of morphisms is a model category, will be the small object argument.

Definition 3.1. Let $\mathcal{C}$ be a category and

$$X^0 \to X^1 \to X^2 \to \cdots \to X^n \to \cdots$$

a diagram in $\mathcal{C}$, which has a colimit X with natural map $f_k\colon X^k \to X$, $k \geq 0$. An object A of the category $\mathcal{C}$ is called *small relatively to the sequential colimit X* if the map

$$\mathrm{colim}_k \mathrm{Hom}_{\mathcal{C}}(A, X^k) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

induced by the maps $\{\mathrm{Hom}_{\mathcal{C}}(A, f_k)\}_k$ is an isomorphism.

Roughly speaking, depending on a given set I of morphisms of the category, whose domains are small enough, the small object argument will provide a factorization $f = p_\infty i_\infty$ of any morphism f of the category, such that p_∞ and i_∞ satisfy some lifting properties.

Definition 3.2. Let $\mathcal{C}$ be a category and let I be a set of morphisms of $\mathcal{C}$. The class of morphisms of $\mathcal{C}$, which have the RLP with respect to all morphisms in I, is called the *class of I-Injectives of $\mathcal{C}$* and denoted by I-Inj.

Let $\mathcal{C}$ be a category, which has all small colimits, and let $I = \{f_i\colon A_i \to B_i\}_i$ be a set of morphisms in $\mathcal{C}$. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Set $G^0(I, f) := X$ and $p_0 := f$. Inductively, for $k \geq 1$, we construct an object $G^k(I, f)$ and morphisms $p_k\colon G^k(I, f) \to Y$, $i_k\colon G^{k-1}(I, f) \to G^k(I, f)$ in $\mathcal{C}$.

For all i, let $S(i) = S_k(i)$ be the set of pairs (g, h) of morphisms in $\mathcal{C}$, which make the diagram

$$\begin{CD} A_i @>{g}>> G^{k-1}(I,f) \\ @V{f_i}VV @VV{p_{k-1}}V \\ B_i @>{h}>> Y \end{CD} \tag{3}$$

commute. Define the object $G^k(I, f)$ and the morphism i_k via the pushout diagram

$$\begin{CD} \coprod_i \coprod_{(g,h)\in S(i)} A_i @>{+_i +_{(g,h)} g}>> G^{k-1}(I,f) \\ @V{\coprod f_i}VV @VV{i_k}V \\ \coprod_i \coprod_{(g,h)\in S(i)} B_i @>>> G^k(I,f) \end{CD}$$

in $\mathcal{C}$. One checks that i_k has the LLP with respect to all morphisms in I-Inj. Recall that for each i and each pair $(g, h) \in S(i)$, the square (3) commutes. So, these squares induce a morphism p_k from the pushout $G^k(I, f)$ to Y, which satisfies $p_k i_k = p_{k-1}$.

Define the *Infinite Gluing Constrution* $G^\infty(I, f)$ to be the sequential colimit $G^\infty(I, f) := \mathrm{colim}_{k\geq 0} G^k(I, f)$. Denote the natural map $G^0(I, f) \to G^\infty(I, f)$ by i_∞. Let $p_\infty : G^\infty(I, f) \to Y$ be the morphism induced by the natural transformation $\{p_k : G^k(I, f) \to Y\}_{k\geq 0}$. So, we have factored f as $f = p_\infty i_\infty$, where the morphism i_∞ has the following lifting property.

Lemma 3.3. *The morphism i_∞ has the LLP with respect to every morphism of I-Inj.*

Proof. Let a commutative square

$$\begin{CD} X @>>> X' \\ @V{i_\infty}VV @VV{p'}V \\ G^\infty(I,f) @>>> Y' \end{CD}$$

with p' in I-Inj be given. To construct a lift $h : G^\infty(I, f) \to X'$, let $h_0 : G^0(I, f) \to X'$ be given by the morphism in the top row of the square. Inductively, for $k \geq 1$, use that i_k has the LLP with respect to p' to construct a lift h_k in

$$\begin{CD} G^{k-1}(I,f) @>{h_{k-1}}>> @. X' \\ @V{i_k}VV @. @VV{p'}V \\ G^k(I,f) @>>> G^\infty(I,f) @>>> Y' \end{CD}\ .$$

Then the morphisms $\{h_k\}_{k\geq 0}$ induce the desired lift h. $\square$

The morphism p_∞ also has a lifting property, assuming that some smallness condition for the domains of the morphisms in I holds.

Lemma 3.4. *Suppose that for all i the domain A_i of f_i is small relatively to $G^\infty(I, f)$, then the map $p_\infty \colon G^\infty(I, f) \to Y$ is in I-Inj.*

Proof. Let a lifting problem

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^\infty(I,f) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y \end{array}$$

be given. By the smallness assumption for A_i, there exists a $k \geq 0$ and a map $g' \colon A_i \to G^k(I, f)$ such that g equals the composite $A_i \xrightarrow{g'} G^k(I, f) \to G^\infty(I, f)$. Hence, the diagram

$$\begin{array}{ccccccc} A_i & \xrightarrow{g'} & G^k(I,f) & \xrightarrow{i_{k+1}} & G^{k+1}(I,f) & \longrightarrow & G^\infty(I,f) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k+1}} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y & \longrightarrow & Y & \longrightarrow & Y \end{array}$$

commutes and so, the pair (g', h) is in $S_{k+1}(i)$. It follows from the construction of $G^{k+1}(I, f)$ and p_{k+1}, that the desired lift is given by the composite

$$B_i \to \coprod_i \coprod_{S_{k+1}(i)} B_i \to G^{k+1}(I, f) \to G^\infty(I, f).$$

□

Summarizing, we have proved the following.

Proposition 3.5 (Small object argument)**.** *Let $\mathcal{C}$ be a category, which has all small colimits. Let I be a set of morphisms of $\mathcal{C}$. Suppose f is a morphism in $\mathcal{C}$ such that the domains of the morphisms in I are small relatively to $G^\infty(I, f)$. Then f can be factored as $f = p_\infty i_\infty$, where i_∞ has the LLP with respect to every map in I-Inj and p_∞ is in I-Inj.*

4 Spaces

Several subcategories of the category of topological spaces **Top** will be given a model category structure. Each of them will contain the following spaces and maps. The unit interval I, the n-disk D^n and its boundary S^{n-1}, $n \geq 0$. Furthermore, the set of bottom inclusions $\{D^n \to D^n \times I\}_{n\geq 0}$, which we denote by J, and the set of inclusions $\{S^{n-1} \to D^n\}_{n\geq 0}$, which by abuse of notation is also denoted by I.

4.1 Topological spaces

To equip the category **Top** with a model category structure, we will use the following terms and results.

Definition 4.1. A map f in **Top** is called a *Serre fibration* if it is in J-Inj.

Remark 4.2. Using that for any $n \geq 0$, the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic, one deduces that a map f is a Serre fibration if and only if it has the RLP with respect to any inclusion $X \times \{0\} \cup A \times I \to X \times I$, where (X, A) is a CW-pair.

Definition 4.3. A map $f\colon X \to Y$ in **Top** is called a *weak homotopy equivalence* if either both X and Y are empty or if both are non-empty and for every point $x \in X$ and every $n \geq 0$, the morphism $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ between homotopy groups is an isomorphism.

The following characterization holds.

Lemma 4.4. *A map $p\colon X \to Y$ in **Top** is both a Serre fibration and a weak homotopy equivalence if and only if it is in I-Inj.*

Proof. For one direction, suppose that p is a Serre fibration and a weak homotopy equivalence. Consider a lifting problem

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ D^{n+1} & \xrightarrow{g} & Y \end{array} .$$

If $n = -1$, then the disk D^0 is mapped to a point $y \in Y$. Since p is a weak homotopy equivalence, there exists a point $x \in X$ such that $p(x)$ can be connected to y by a path H. Since p is a Serre fibration, there exists a lift $\tilde{H}$ in

$$\begin{array}{ccc} D^0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ D^0 \times I & \xrightarrow{H} & Y \end{array} .$$

The restriction of $\tilde{H}$ to $D^0 \times \{1\}$ solves our lifting problem.

If $n \geq 0$, factor the inclusion of the sphere over the cylinder and the cone as $S^n \to S^n \times I \to CS^n \cong D^{n+1}$ and consider the commutative diagram

$$\begin{array}{ccccc} S^n & & \xrightarrow{\quad f \quad} & & X \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ S^n \times I & \longrightarrow & D^{n+1} & \xrightarrow{g} & Y \end{array} . \tag{4}$$

By Remark 4.2 it has a lift h since p is a Serre fibration. But since h is not necessarily constant on the top of the cylinder, it does not directly induce a solution of our lifting problem. We will deform it slightly. The composite ph maps $S^n \times \{1\}$ to some point $y \in Y$. Hence the image of the restricted map $h| : S^n \times \{1\} \to X$ lies in the fiber $F := p^{-1}\{y\}$. Denote by $x \in F$ the point to which $h|$ maps the basepoint of the sphere. Since p is a weak homotopy equivalence, it follows by the long exact sequence in homotopy of a Serre fibration, that $\pi_n(F, x)$ is trivial. Therefore, there exists a homotopy $H\colon S^n \times \{1\} \times I \to X$ from $h|$ to the constant map c_x. This map H and the composites $S^n \times \{0\} \times I \to S^n \xrightarrow{f} X$, $S^n \times I \times \{0\} \to S^n \times I \xrightarrow{h} X$ and $S^n \times I \times I \to S^n \times I \to D^n \xrightarrow{g} Y$ provide a commutative square

$$\begin{array}{ccc} (S^n \times \{1\} \cup S^n \times \{0\}) \times I \cup S^n \times I \times \{0\} & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ S^n \times I \times I & \longrightarrow & Y \end{array} .$$

In this diagram, there exists a lift $\tilde{H}$ by Remark 4.2 applied to the CW-pair $(S^n \times I, S^n \times \{1\} \cup S^n \times \{0\})$. The restricted map $\tilde{H}|_{S^n \times I \times \{1\}}$ is another lift in (4), but which satisfies $\tilde{H}|_{S^n \times \{1\} \times \{1\}} = c_x$. It therefore induces the solution $D^{n+1} \to X$ of the original lifting problem.

For the other direction, assume that p is in I-Inj. First we show, that p has the RLP with respect to every inclusion $A \to B$, where (B, A) is a relative CW-complex. Recall that in this case, the topological space B is a sequential colimit $B = \mathrm{colim}_{k \geq -1} B^k$, where B^{-1} is the given space A and inductively, for $k \geq 0$, the space B^k is obtained from B^{k-1} by adjoining k-cells. Given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} ,$$

set $h_{-1} := f\colon B^{-1} \to X$ and inductively, for $k \geq 0$, use that p is in I-Inj and the construction of B^k to find a lift h_k in

$$\begin{array}{ccccc} B^{k-1} & & \xrightarrow{h_{k-1}} & & X \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ B^k & \longrightarrow & B & \longrightarrow & Y \end{array} .$$

Then the maps $\{h_k\}_k$ induce the desired lift.

Since for any $n \geq 0$ the pair $(D^n \times I, D^n)$ is a relative CW-complex, it follows that p is a Serre fibration. We show that it is also a weak homotopy equivalence. Let $n \geq 0$, $x \in X$. For the surjectivity of $\pi_n(p)$, let

$f\colon (D^n, S^{n-1}) \to (Y, p(x))$ be a representative of a homotopy class $[f] \in \pi_n(Y, p(x))$. Choose a lift h in

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c_x} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ D^n & \xrightarrow{f} & Y \end{array} .$$

Then h is a representative of a homotopy class in $\pi_n(X, x)$, which $\pi_n(p)$ maps to $[f]$. For the injectivity, consider two homotopy classes $[f], [g] \in \pi_n(X, x)$ such that there exists a homotopy of pairs $H\colon (D^n, S^{n-1}) \times I \to (Y, p(x))$ between pf and pg. Since $(D^n \times I, S^{n-1} \times I \cup D^n \times \{0\} \cup D^n \times \{1\})$ is a relative CW-complex, there exists a lift $\tilde{H}$ in the commutative square

$$\begin{array}{ccc} S^{n-1} \times I \cup D^n \times \{0\} \cup D^n \times \{1\} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ D^n \times I & \xrightarrow{\quad H \quad} & Y \end{array} ,$$

where $S^{n-1} \times I \to X$ is the constant map c_x. The homotopy $\tilde{H}$ shows $[f] = [g] \in \pi_n(X, x)$. □

Furthermore, we will need a smallness result.

Lemma 4.5. *Let $\{A_n \to B_n\}_{n \geq 1}$ be a family of closed embeddings in* ***Top*** *such that each B_n is a T_1 space. Let X^0 be a topological space. Suppose that inductively for $n \geq 1$, the space X^n is given by the pushout $B_n \sqcup_{\varphi_n} X^{n-1}$ for some attaching map $\varphi_n\colon A_n \to X^{n-1}$. Set $X := \mathrm{colim}_{n \geq 0} X^n$ with natural map $f_n\colon X^n \to X$, $n \geq 0$.*

Then any compact space K is small relatively to X, that is

$$\mathrm{colim}_n Hom_{\boldsymbol{Top}}(K, X^n) \to Hom_{\boldsymbol{Top}}(K, X)$$

is an isomorphism.

Proof. Since $A_n \to B_n$ is a closed embedding, also the natural map $X^{n-1} \to X^n$ is a closed embedding, $n \geq 1$. Hence, each map f_n is a closed embedding.

For the surjectivity, let $g \in \mathrm{Hom}_{\mathbf{Top}}(K, X)$ be given. It's enough to show that there exists an $n \geq 0$ such that $g(K) \subset f_n(X^n)$ holds. Conversely, suppose that there exists a set $S = \{x_n\}_{n \geq 0}$ of points in X such that $x_n \in g(K) \cap X \backslash f_n(X^n)$. We show that $S \subset g(K)$ is a closed discrete subspace of $g(K)$, which contradicts the compactness of $g(K)$. Let $T \subset S$ be a subset. Using that each B_n is T_1, one deduces that every point $x \in X \backslash f_0(X^0)$ is closed in X. For all $n \geq 0$, it follows that the finite set $T \cap f_n(X^n)$ is closed in $f_n(X^n)$. Hence, T is closed in X and therefore also in $g(K)$.

One shows the injectivity by using the surjectivity and that the maps $\{f_n\}_n$ are embeddings. □

Corollary 4.6. *Suppose that in the situation of Lemma 4.5 each space B_n deformation retracts onto the image of the embedding $A_n \to B_n$. Then $f_0 \colon X^0 \to X$ is a weak homotopy equivalence.*

Proof. Since $- \times I$ preserves pushouts, it follows that for all $n \geq 1$, the space X^n deformation retracts onto the image of the embedding $X^{n-1} \to X^n$. Therefore, this embedding is a weak homotopy equivalence. To show that for $k \geq 0$ and $x \in X^0$, the morphism $\pi_k(f_0)$ is an isomorphism, use the surjectivity of the morphism in Lemma 4.5 with $K = S^k$ for the surjectivity and with $K = S^k \times I$ for the injectivity. □

Using Lemma 4.5 and its corollary, the following factorization is constructed.

Lemma 4.7. *Every map $f \colon X \to Y$ in **Top** can be factored as $f = p_\infty i_\infty$, where i_∞ is a weak homotopy equivalence, which has the LLP with respect to all maps in J-Inj, and p_∞ is in J-Inj.*

Proof. Construct the factorization $f = p_\infty i_\infty$ using the Infinite Gluing Construction $G^\infty(J, f)$. By Corollary 4.6, it follows that i_∞ is a weak homotopy equivalence. By Lemma 4.5, the domains of the morphisms in J are small enough to apply the small object argument, which shows the remaining claims. □

Now, we are ready to equip **Top** with a model category structure.

Proposition 4.8. *Call a map in **Top***

i) a weak equivalence if it is a weak homotopy equivalence,

ii) a fibration if it is in J-Inj and

iii) a cofibration if it has the LLP with respect to every map in I-Inj.

*Then with these choices **Top** is a model category. Furthermore, every object in **Top** is fibrant.*

Proof. One checks, that each of these three classes is closed unter composition and contains all identity morphisms.

To prove **MC1**, let a functor F from a small category $\mathcal{D}$ to **Top** be given. The colimit of F is given by the quotient of the disjoint sum $\coprod_{d \in \mathcal{D}} F(d)$ with respect to the equivalence relation generated by $(d, x) \sim (d', F(f)(x))$, where $f \colon d \to d'$ is a morphism of $\mathcal{D}$ and $x \in F(d)$. The limit of F is given by the subspace

$$\{(x_d)_d; x_{d'} = F(f)(x_d) \text{ for every morphism } f \colon d \to d' \text{ in } \mathcal{D}\}$$

of the product space $\prod_{d \in \mathcal{D}} F(d)$.

We show the interesting case of **MC2**. That is when $f\colon X \to Y$, $g\colon Y \to Z$ are maps in **Top** such that f and gf are weak equivalences. Let $n \geq 0$, $y \in Y$. We want to show that $\pi_n(g)\colon \pi_n(Y,y) \to \pi_n(Z,g(z))$ is an isomorphism. Since by assumption $\pi_0(f)$ is surjective, there exists a point $x \in X$ such that y can be connected to $f(x)$ via a path γ. The path γ induces via concatenation an isomorphism $\pi_n(Y,f(x)) \to \pi_n(Y,y)$ and the path $g \circ \gamma$ induces an isomorphism $\pi_n(Z,gf(x)) \to \pi_n(Z,g(y))$, which make the diagram

$$\begin{array}{ccccc}
\pi_n(Y,f(x)) & \xleftarrow[\cong]{\pi_n(f)} & \pi_n(X,x) & \xrightarrow[\cong]{\pi_n(gf)} & \pi_n(Z,gf(x)) \\
\downarrow{\cong} & & & & \downarrow{\cong} \\
\pi_n(Y,y) & & \xrightarrow{\pi_n(g)} & & \pi_n(Z,g(y))
\end{array}$$

commute. It follows that $\pi_n(g)$ is an isomorphism.

To show **MC3**, let a commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{i} & X' & \xrightarrow{r} & X \\
\downarrow{f} & & \downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{i'} & Y' & \xrightarrow{r'} & Y
\end{array} \tag{5}$$

in **Top** with $ri = \mathrm{id}_X$ and $r'i' = \mathrm{id}_Y$ be given.

First, assume that f' is a weak equivalence. We want to prove that f is a weak equivalence. Let $n \geq 0$, $x \in X$. Applying the functor π_n to the diagram (5) shows that $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,f(x))$ is a retract of the isomorphism $\pi_n(f')$ and hence by Lemma 2.2 an isomorphism itself. Thus f is a weak equivalence.

Secondly, assume that f' is fibration. Given a lifting problem

$$\begin{array}{ccc}
D^n & \longrightarrow & X \\
\downarrow & & \downarrow{f} \\
D^n \times I & \longrightarrow & Y
\end{array}$$

in **Top**, one constructs the lifting problem

$$\begin{array}{ccccc}
D^n & \longrightarrow & X & \longrightarrow & X' \\
\downarrow & & & & \downarrow{f'} \\
D^n \times I & \longrightarrow & Y & \longrightarrow & Y'
\end{array} ,$$

which has a solution h, since f' is a fibration. The composite rh gives the desired lift. It follows, that f is a fibration.

The third case, where f' is a cofibration is similar.

Axiom **MC4**i) holds, since by Lemma 4.4, a map, which is both a fibration and a weak equivalence, is in I-Inj. Concerning **MC4**ii), we show that any acyclic cofibration $f\colon A \to B$ has the LLP with respect to every fibration. Let $A \overset{i_\infty}{\to} A' \overset{p_\infty}{\to} B$ be a factorization as in Lemma 4.7. Since f and i_∞ are weak equivalences, it follows that also p_∞ is a weak equivalence by **MC2**. Thus p_∞ is in I-Inj by Lemma 4.4. By the definition of the cofibrations, there exists a lift $g : B \to A'$ in

$$\begin{array}{ccc} A & \xrightarrow{i_\infty} & A' \\ \downarrow f & & \downarrow p_\infty \\ B & \xrightarrow{=} & B \end{array} .$$

The diagram

$$\begin{array}{ccccc} A & \xrightarrow{=} & A & \xrightarrow{=} & A \\ \downarrow f & & \downarrow i_\infty & & \downarrow f \\ B & \xrightarrow{g} & A' & \xrightarrow{p_\infty} & B \end{array}$$

shows that f is a retract of i_∞. Using that i_∞ has the LLP with respect to fibrations, one deduces that f shares the same property.

For **MC5**i), let a map $f\colon X \to Y$ in **Top** be given. By Lemma 4.5, the domains of the maps in I are small enough to apply the small object argument, providing a factorization $f = p_\infty i_\infty$, where i_∞ has the desired lifting property to be a cofibration, and where p_∞ is in I-Inj and hence by Lemma 4.4 is an acyclic fibration.

Note that by Lemma 4.4 every map in I-Inj is a fibration and hence is in J-Inj. In particular, a map in **Top** which has the LLP with respect to every map in J-Inj, has also the LLP with respect to the I-Injectives. So, Lemma 4.7 provides the desired factorization of a given map in **Top** to show **MC5**ii).

We have shown, that **Top** is a model category. Since for any $n \geq 0$, the cylinder $D^n \times I$ retracts onto D^n, it follows that every topological space is fibrant. $\square$

4.2 k-spaces and compactly generated spaces

Concerning Elmendorf's Theorem, we will not work in the category **Top**, but in a full subcategory, called the category of compactly generated spaces, which is better behaved. For instance, it is cartesian closed. Compactly generated spaces will be defined as weak Hausdorff k-spaces. They form a subcategory of another categorically well-behaved subcategory of **Top**, the category of k-spaces. But they have the advantage of fulfilling a separation axiom, the weak Hausdorff condition, which is stronger than T_1 but weaker than the Hausdorff condition T_2. We summarize the for the working knowledge necessary definitions and facts concering these two subcategories. For the proofs the reader is referred to Appendix A of [8].

Definition 4.9. Let X be a topological space.

a) The space X is called *weak Hausdorff* if for all maps $g\colon K \to X$ in **Top** with K compact Hausdorff, the image $g(K)$ is closed in X.

b) A subset $A \subset X$ is called *compactly closed* if for all maps $g\colon K \to X$ in **Top** with K compact Hausdorff, the preimage $g^{-1}(A)$ is closed.

c) The space X is a *k-space* if every compactly closed subset is closed. Let $\mathcal{K}$ denote the full subcategory of **Top** consisting of the k-spaces.

d) The space X is called *compactly generated* if it is both weak Hausdorff and a k-space. Let $\mathcal{U}$ denote the full subcategory of $\mathcal{K}$ consisting of the compactly generated spaces.

e) The *k-space topology* on X is the topology, where the closed sets are precisely the compactly closed sets.

Let $\mathcal{C}$ be either the category of k-spaces $\mathcal{K}$ or the category of compactly generated spaces $\mathcal{U}$.

Example 4.10. The category $\mathcal{C}$ contains

a) locally compact Hausdorff spaces,

b) metric spaces,

c) closed and open subsets of spaces in $\mathcal{C}$.

Proposition 4.11 (Limits, colimits, quotient maps)**.** *a) The inclusion functor $\mathcal{K} \to$ **Top** has a right adjoint and left inverse $k\colon$ **Top** $\to \mathcal{K}$, which takes a topological space to its underlying set equipped with the k-space topology.*

*b) The category $\mathcal{K}$ has all small limits and colimits. Colimits are inherited from **Top** and limits are obtained by applying k to the limit in **Top**.*

*c) If X and Y are k-spaces and Y is locally compact Hausdorff, then the product $X \times Y$ in **Top** is a k-space and hence, it is also the product in $\mathcal{K}$.*

*d) Let $p : X \to Y$ be a quotient map in **Top**, where X is a k-space. Then Y is a k-space. It is weak Hausdorff and thus in $\mathcal{U}$ if and only if the preimage $(p \times p)^{-1}(\Delta)$ of the diagonal $\Delta \subset Y \times Y$ is closed in the product $X \times X$ in $\mathcal{K}$. In particular, X is in $\mathcal{U}$ if and only if the diagonal in the product $X \times X$ in $\mathcal{K}$ is closed.*

e) The inclusion functor $\mathcal{U} \to \mathcal{K}$ has a left adjoint and left inverse $wH : \mathcal{K} \to \mathcal{U}$, which takes a k-space X to the quotient X/R, where $R \subset X \times X$ is the smallest closed equivalence relation.

f) The category $\mathcal{U}$ has all small limits and colimits. Limits are inherited from $\mathcal{K}$ and colimits are obtained by applying wH to the colimit in $\mathcal{K}$.

Proposition 4.12 (Function spaces)**.** *Let X, Y be in $\mathcal{K}$. For any map $h\colon K \to U$ from a compact Hausdorff space K to an open subset U of Y, denote the set of maps $f\colon X \to Y$ with $f(h(K)) \subset U$ by $N(h,U)$. Let $C(X,Y)$ be the set $Hom_{\mathcal{K}}(X,Y)$ with the topology generated by the subbasis $\{N(h,U)\}$. Let Y^X be the $k-space$ $kC(X,Y)$.*

a) For all X, Y, Z in $\mathcal{K}$, there is a natural isomorphism $Hom_{\mathcal{K}}(X \times Y, Z) \to Hom_{\mathcal{K}}(X, Z^Y)$, which sends a map f to $\tilde{f}$ given by $\tilde{f}(x)(y) = f(x,y)$. In particular, $\mathcal{K}$ is cartesian closed.

b) For all X, Y in $\mathcal{K}$, evaluation $Y^X \times X \to Y$ is the counit of the adjunction above.

c) If X is in $\mathcal{K}$ and Y is in $\mathcal{U}$, then Y^X is weak Hausdorff. Hence, for all X, Y, Z in $\mathcal{U}$, there is a natural isomorphism $Hom_{\mathcal{U}}(X \times Y, Z) \to Hom_{\mathcal{U}}(X, Z^Y)$, which sends a map f to $\tilde{f}$ given by $\tilde{f}(x)(y) = f(x,y)$. In particular, $\mathcal{U}$ is cartesian closed.

Remark 4.13 (Subspace)**.** Let Y be a space in $\mathcal{C}$ and $A \subset Y$ a subset of Y. We call A equipped with the topology obtained by applying k to the subspace $A \subset Y$ in **Top** a subspace of Y. Note, that if X is space in $\mathcal{C}$ and $f\colon X \to Y$ is a set function with $f(X) \subset A$, then $f\colon X \to Y$ is in $\mathcal{C}$ if and only if $f\colon X \to A$ is in $\mathcal{C}$. Most often, the subspace $A \subset Y$ in **Top** will be closed and hence we won't have to apply k to get a subspace in $\mathcal{C}$.

Proposition 4.14 (Miscellaneous)**.** *We call a map f in $\mathcal{C}$ a closed embedding, if it is a closed embedding in **Top**. We call f a quotient map, if it is a qoutient map in **Top**.*

a) An arbitrary product of closed embeddings in $\mathcal{C}$ is a closed embedding.

b) An arbitrary coprodut of closed embeddings in $\mathcal{C}$ is a closed embedding.

*c) An arbitrary coproduct taken in **Top** of spaces in $\mathcal{C}$ is also the coproduct in $\mathcal{C}$.*

*d) The pushout taken in **Top** of a diagram $X \leftarrow A \to Y$ in $\mathcal{C}$, where one arrow is a closed embedding, is also the pushout in $\mathcal{C}$*

*e) The sequential colimit taken in **Top** of a diagram*

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots,$$

where each arrow is an injective map, is also the colimit in $\mathcal{C}$.

f) Let X, X', Y, Y' be spaces in $\mathcal{C}$ and $p\colon X \to Y$, $q\colon X' \to Y'$ quotient maps. Then $p \times q\colon X \times X' \to Y \times Y'$ in $\mathcal{C}$ is a quotient map.

We give $\mathcal{C}$ a model category structure.

Proposition 4.15. *Call a morphism f in $\mathcal{C}$*

*i) a weak equivalence if f is a weak equivalence in **Top**,*

*ii) a fibration if it is a fibration in **Top**.*

*iii) a cofibration if it is a cofibration in **Top**.*

Then $\mathcal{C}$ together with these choices is a model category.

Proof. Each of the three classes is closed under composition and contains all identity morphism by the same result in **Top**. Axiom **MC1** holds by Proposition 4.11. Since **MC2**, **MC3** and **MC4** hold in **Top**, they also hold in $\mathcal{C}$.

For **MC5**, we have to factor a given map f in $\mathcal{C}$ in two ways. The factorization proving **MC5**i) can be constructed by using the Infinite Gluing Construction $G^{\infty}(I, f)$ in **Top**, since by Proposition 4.14, the space $G^{\infty}(I, f)$ lies in $\mathcal{C}$. Similar for **MC5**ii) with $G^{\infty}(J, f)$ in **Top**. $\square$

Proposition 4.16. *In the model category $\mathcal{C}$, the fibrations are precisely the J-Injectives of $\mathcal{C}$ and the cofibrations are precisely the maps which have the LLP with respect to the I-Injectives of $\mathcal{C}$.*

Proof. The first assertion holds by definition of the fibrations in **Top**. For the second assertion, it's enough to show that the acyclic fibrations in $\mathcal{C}$ are the I-Injectives of $\mathcal{C}$ by Proposition 2.6. But this follows from Proposition 4.4. $\square$

5 Lifting the model category structure of spaces

Let $\mathcal{C}$ be either the model category **Top**, $\mathcal{K}$ or $\mathcal{U}$. Let $\mathcal{D}$ be a category, which has all small limits and all small colimits. Depending on a set

$$\{F_\iota\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G_\iota\}_{\iota \in \mathcal{F}}$$

of adjoint functors, we show how under certain assumptions, the model category structure on $\mathcal{C}$ can be lifted to the category $\mathcal{D}$. Set $FJ := \bigcup_\iota \{F_\iota(j); j \in J\}$ and $FI := \bigcup_\iota \{F_\iota(i); i \in I\}$.

Theorem 5.1. *Call a morphism f in $\mathcal{D}$*

i) a weak equivalence if $G_\iota(f)$ is a weak equivalence in $\mathcal{C}$ for all ι, *ii) a fibration if it is in FJ-Inj.*

iii) a cofibration if it has the LLP with respect to every map in FI-Inj.

For all $\iota \in \mathcal{F}$, $n \geq 0$ and any morphism $f\colon X \to Y$ in $\mathcal{D}$, suppose that the object $F_\iota(S^{n-1})$ is small relative to the Infinite Gluing Construction $G^\infty(FI, f)$, that the object $F_\iota(D^n)$ is small relative to $G^\infty(FJ, f)$ and that the factor $i_\infty\colon X \to G^\infty(FJ, f)$ is a weak equivalence in $\mathcal{D}$.

Then with the choices made, the category $\mathcal{D}$ is a model category.

Proof. By functoriality, it follows that the class of weak equivalences of $\mathcal{D}$ is closed under composition and contains all identity morphisms. One checks that these two conditions are also satisfied by the class of fibrations and the class of cofibrations.

By assumption, $\mathcal{D}$ satisfies **MC1**. Axiom **MC2** holds in $\mathcal{C}$ and by functoriality also in $\mathcal{D}$. The case of **MC3** in $\mathcal{D}$, where a morphism is a retract of a weak equivalence, follows by functoriality and the same axiom in $\mathcal{C}$. The other two cases are shown as for **Top**.

For the remaining part, we show three results, where the first two are based on the next remark. For any $\iota \in \mathcal{F}$, any morphism $i\colon A \to B$ in $\mathcal{C}$ and morphism $p\colon X \to Y$ in $\mathcal{D}$, one checks that by adjointness of the functors F_ι and G_ι, the lifting problems

$$\begin{array}{ccc} F_\iota(A) & \longrightarrow & X \\ \downarrow{\scriptstyle F_\iota(i)} & & \downarrow{\scriptstyle p} \\ F_\iota(B) & \longrightarrow & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \longrightarrow & G_\iota(X) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle G_\iota(p)} \\ B & \longrightarrow & G_\iota(Y) \end{array}$$

are equivalent. It follows the first result:

a) A morphism p in $\mathcal{D}$ is a fibration if and only if $G_\iota(p)$ is a fibration in $\mathcal{C}$ for every $\iota \in \mathcal{F}$.

Using a) and that a map in $\mathcal{C}$ is both a fibration and a weak equivalence if and only if it is in J-Inj one deduces the second result:

b) A morphism p in $\mathcal{D}$ is both a fibration and a weak equivalence if and only if it is in FJ-Inj.

By the assumptions, one can use the Infinite Gluing Construction depending on the set FJ and apply the small object argument to conclude the third result:

c) Any morphism f in $\mathcal{D}$ can be factored as $f = p_\infty i_\infty$, where i_∞ is a weak equivalence which has the LLP with respect to all maps in FJ-Inj and, p_∞ is in FJ-Inj.

Using the results b), c) and the Infinite Gluing Construction depending on the set FI for **MC5**i), one shows the remaining axioms as for **Top**. $\square$

Remark 5.2. With regard to the smallness assumptions in Theorem 5.1, it's worth to mention that for any $\iota \in \mathcal{F}$, any sequential colimit $G^\infty = \mathrm{colim}_k G^k$ in $\mathcal{D}$ and any object A of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} \mathrm{colim}_k \mathrm{Hom}_{\mathcal{D}}(F_\iota(A), G^k) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F_\iota(A), G^\infty) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{colim}_k \mathrm{Hom}_{\mathcal{C}}(A, G_\iota(G^k)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, G_\iota(G^\infty)) \end{array}$$

commutes, following from adjointness.

6 Lifting model category structures: Examples

From now on, concerning spaces, we will work in the category of compactly generated spaces $\mathcal{U}$. We give two examples of lifting the model category structure from $\mathcal{U}$ to some category of what will be called continuous functors. They will provide the model category structures involved in Elemendorf's Theorem.

Definition 6.1. A *topological category* (with discrete object space) is a category $\mathcal{C}$, where each hom-set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is topologized as a space $\mathcal{C}(X,Y)$ in $\mathcal{U}$, such that composition $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ is continuous.

Definition 6.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between topological categories is said to be *continuous* if each function $F\colon \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is continuous.

Let $\mathcal{J}$ be a small topological category until the end of this section. We give examples of topological categories.

Example 6.3. a) Since the category of compactly generated spaces $\mathcal{U}$ is cartesian closed, it is a topological category with $\mathcal{U}(X,Y) = Y^X$ for spaces X, Y in $\mathcal{U}$.

b) The category of continuous functors from $\mathcal{J}$ to $\mathcal{U}$ is a topological category, which we denote by $\mathcal{U}^{\mathcal{J}}$. The space $\mathcal{U}^{\mathcal{J}}(X,Y)$ is a subspace of $\prod_{j \in \mathcal{J}} \mathcal{U}(X(j), Y(j))$ in $\mathcal{U}$ for any diagrams X, Y in $\mathcal{U}^{\mathcal{J}}$.

c) Call a group G, which is topologized as a space in $\mathcal{U}$ such that composition and taking inverses is continuous, a *topological group*. Then a topological group G is a topological category with only one object.

d) Let G be a topological group and X a space in $\mathcal{U}$. Call a map $G \times X \to X$, $(g, x) \mapsto gx$, in $\mathcal{U}$ with $g(h(x)) = (gh)x$ and $ex = x$ for all $g, h \in G$, $x \in X$ and the neutral element $e \in G$, an *action of G on X* or a *G-action.* Call a space in $\mathcal{U}$ together with a G-action a G-space and a map in $\mathcal{U}$ between G-spaces, which commutes with the actions, an *equivariant map.* Then the category of G-spaces together with the equivariant maps is a topological category, which can be identified with the category $\mathcal{U}^G$ since $\mathcal{U}$ is cartesian closed.

Axiom **MC1** of a model category in mind, the following result will be used in the two examples of lifting the model category structure from $\mathcal{U}$.

Proposition 6.4. *The category $\mathcal{U}^{\mathcal{J}}$ has all small limits and colimits.*

Proof. Let $\mathcal{D}$ be a small category, let $S\colon \mathcal{D} \to \mathcal{U}^{\mathcal{J}}$ be a functor. The category $\mathcal{U}$ has all small limits. So as objectwise constructed in [9], the composite functor $\mathcal{D} \xrightarrow{S} \mathcal{U}^{\mathcal{J}^{\mathrm{op}}} \to \mathrm{Fun}(\mathcal{J},\mathcal{U})$ has a limit L. Using that $\mathcal{U}$ is cartesian closed, one checks that L is a continuous functor and thus in $\mathcal{U}^{\mathcal{J}}$. Similarly, one shows that $\mathcal{U}^{\mathcal{J}}$ has all small colimits. □

6.1 Lifting along a family of contravariant Hom-functors

Since $\mathcal{J}$ is a small topological category, its opposite category $\mathcal{J}^{\mathrm{op}}$ shares the same property. We will use Theorem 5.1 to equip the topological category of continuous contravariant functors $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$ with a model category structure.

For any object j of $\mathcal{J}^{\mathrm{op}}$, denote by $\underline{j}$ the continuous contravariant functor $\mathcal{U}(-, j)$. Let $\mathrm{ev}_j\colon \mathcal{U}^{\mathcal{J}^{\mathrm{op}}} \to \mathcal{U}$ be the functor given by evaluation in the object j. We will show that $\underline{j} \times (-)\colon \mathcal{U} \rightleftarrows \mathcal{U}^{\mathcal{J}^{\mathrm{op}}}\colon \mathrm{ev}_j$ is an adjunction, where for Y in $\mathcal{U}$, we write again Y for the corresponding constant diagram in $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$. For any Z in $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$, we denote the composite $\mathcal{J}^{\mathrm{op}} \xrightarrow{Z} \mathcal{U} \xrightarrow{(-)^Y} \mathcal{U}$ by Z^Y. The next lemma shows in particular that $(-) \times Y\colon \mathcal{U}^{\mathcal{J}^{\mathrm{op}}} \rightleftarrows \mathcal{U}\colon (-)^Y$ is an adjunction.

Lemma 6.5. *For all X, Z in $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$ and all Y in $\mathcal{U}$, there is a natural isomorphism*

$$\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}(X \times Y, Z) \xrightarrow{\varphi} \mathcal{U}^{\mathcal{J}^{\mathrm{op}}}(X, Z^Y)$$

in $\mathcal{U}$.

Proof. Using the isomorphism from Proposition 4.12, which shows that $\mathcal{U}$ is cartesian closed, define φ by $\varphi(f)_j := \tilde{f}_j\colon X(j) \to Z(j)^Y$ in an object j of $\mathcal{J}^{\mathrm{op}}$ and check that it satisfies the desired properties. □

Another natural isomorphism comes from the Yoneda Lemma.

Lemma 6.6. *Let Z be a diagram in $\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}$ and let j by an object of $\mathcal{J}^{\mathrm{op}}$. Then there is a natural isomorphism*

$$\mathcal{U}^{\mathcal{J}^{\mathrm{op}}}(\underline{j}, Z) \to Z(j)$$

in $\mathcal{U}$.

Proof. Regarding $\underline{j}$ and Z as functors from $\mathcal{J}^{\text{op}}$ to the category of sets, the Yoneda Lemma states that the morphism $\psi\colon \mathrm{Nat}(\underline{j}, Z) \to Z(j)$ given by $f \mapsto f_j(\mathrm{id}_j)$ is a natural bijection of sets. Using Proposition 4.12, one checks that its inverse takes a point in $Z(j)$ to a natural transformation, which lies in $\mathrm{Hom}_{\mathcal{U}^{\mathcal{J}^{\text{op}}}}(\underline{j}, Z)$. It follows that $\mathrm{Nat}(\underline{j}, Z) = \mathrm{Hom}_{\mathcal{U}^{\mathcal{J}^{\text{op}}}}(\underline{j}, Z)$. Using Proposition 4.12, one deduces that ψ is the desired natural isomorphism in $\mathcal{U}$. □

Combining Lemma 6.5 and 6.6, one gets in particular the following.

Corollary 6.7. *For any object j of $\mathcal{J}^{\text{op}}$, there is an adjunction*

$$\underline{j} \times (-)\colon \mathcal{U} \rightleftarrows \mathcal{U}^{\mathcal{J}^{\text{op}}} : \mathrm{ev}_j.$$

Using Theorem 5.1, we define a model category structure on $\mathcal{U}^{\mathcal{J}^{\text{op}}}$.

Proposition 6.8. *Let $\mathcal{F}$ be a non-empty subset of the set of objects of $\mathcal{J}^{\text{op}}$. Then $\mathcal{U}^{\mathcal{J}^{\text{op}}}$ together with the adjunctions $\{\underline{j} \times (-)\colon \mathcal{U} \rightleftarrows \mathcal{U}^{\mathcal{J}^{\text{op}}} : \mathrm{ev}_j\}_{j\in\mathcal{F}}$ satisfies the assumptions of Theorem 5.1, which therefore defines a model category structure on $\mathcal{U}^{\mathcal{J}^{\text{op}}}$.*

Proof. The category $\mathcal{U}^{\mathcal{J}^{\text{op}}}$ has all small limits and colimits by Proposition 6.4.

For the smallness assumptions, let $G^\infty = \mathrm{colim}_k G^k$ be one of the Infinite Gluing Constructions in question and corresponding to it, let the space A be either D^n or S^{n-1}, $n \geq 0$. By Corollary 5.2, it's enough to show that

$$\mathrm{colim}_k \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{ev}_j(G^k)) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{ev}_j(G^\infty))$$

is an isomorphism for any $j \in \mathcal{F}$. Recall that colimits of diagrams are computed objectwise. Hence, for $k \geq 1$, the space $(G^k)(j)$ is a pushout in $\mathcal{U}$, which by Proposition 4.14 in fact is the pushout in **Top**. Furthermore, it follows that $\mathrm{ev}_j(G^\infty) = \mathrm{colim}_k(G^k(j))$ in $\mathcal{U}$, which in fact is also the sequential colimit in **Top** by Proposition 4.14. The smallness result follows now from Lemma 4.5 since any space in $\mathcal{U}$ is T_1.

To show that the factor i_∞ in question is a weak equivalence in $\mathcal{U}^{\mathcal{J}^{\text{op}}}$, i.e. that $\mathrm{ev}_j(i_\infty)$ is a weak equivalence in $\mathcal{U}$ for any $j \in \mathcal{F}$, one uses Corollary 4.6.

□

6.2 Lifting to the category of G-spaces

Let G be a topological group in $\mathcal{U}$. Let $H \subset G$ be a closed subgroup. Deduce from Proposition 4.11d) that the quotient space G/H of cosets gH with $g \in G$ in **Top** is in $\mathcal{U}$. We call it an *orbit space*. With the action

$G \times G/H \to G$, $(g, g'H) \mapsto gg'H$, the orbit space G/H becomes a G-space. Thus, one gets a functor

$$G/H \times (-) \colon \mathcal{U} \to \mathcal{U}^G.$$

Let X be a G-space. Let X^H be the subset $\{x \in X; hx = x \text{ for all } h \in H\}$ of X. We call it the *H-fixed point set*. Using that the diagonal in $X \times X$ is closed, deduce that the fixed point set X^H is closed in X and hence it is in $\mathcal{U}$. One obtains a functor

$$(-)^H \colon \mathcal{U}^G \to U,$$

called *fixed point functor*, which by the following proposition in particular is right adjoint to $G/H \times (-)$.

Proposition 6.9. *For all spaces Y in $\mathcal{U}$ and G-spaces Z in $\mathcal{U}^G$, there is a natural isomorphism*

$$\mathcal{U}^G(G/H \times Y, Z) \xrightarrow{\varphi} \mathcal{U}(Y, Z^H)$$

in $\mathcal{U}$.

Proof. Check that φ defined by $\varphi(f)(y) = f(H, y)$ for $y \in Y$ is the desired natural isomorphism. □

As a right adjoint, the functor $(-)^H$ preserves limits. It also preserves some colimits.

Lemma 6.10. *The functor $(-)^H$ preserves*

a) arbitrary coproducts in $\mathcal{U}^G$,

b) pushouts of diagrams $X \leftarrow A \to Y$ in $\mathcal{U}^G$, where one arrow is a closed embedding as a map in $\mathcal{U}$, and

c) sequential colimits of diagrams $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ in $\mathcal{U}^G$, where each arrow is injective.

Proof. Recall that the colimit in $\mathcal{U}^G$ is calculated objectwise, hence calculated in $\mathcal{U}$. To check the claims, use Proposition 4.14 and for b) and c), note that $(-)^H$ preserves closed embeddings. □

We are ready to lift the model category structure from $\mathcal{U}$ to $\mathcal{U}^G$.

Proposition 6.11. *Let $\mathcal{F}$ be a non-empty subset of the closed subgroups of G. Then $\mathcal{U}^G = \mathcal{U}^G(\mathcal{F})$ together with the adjunctions $\{G/H \times (-) \colon \mathcal{U} \rightleftarrows \mathcal{U}^G \colon (-)^H\}_{H \in \mathcal{F}}$ satisfies the assumptions of Theorem 5.1, which therefore defines a model category structure on $\mathcal{U}^G = \mathcal{U}^G(\mathcal{F})$.*

Proof. The category $\mathcal{U}^G$ has all small limits and colimits by Proposition 6.4.

For the smallness assumptions, let $G^\infty = \mathrm{colim}_k G^k$ be one of the Infinite Gluing Constructions in question and corresponding to it, let the space A be either D^n or S^{n-1}, $n \geq 0$. By Corollary 5.2, it's enough to show that

$$\mathrm{colim}_k \mathrm{Hom}_{\mathcal{C}}(A, (G^k)^H) \to \mathrm{Hom}_{\mathcal{C}}(A, (G^\infty)^H)$$

is an isomorphism for any $H \in \mathcal{F}$. Using Lemma 6.10 and Proposition 4.14, one deduces the smallness result from Lemma 4.5 since any space in $\mathcal{U}$ is T_1.

To show that the factor i_∞ in question is a weak equivalence in $\mathcal{U}^G$, i.e. that $(i_\infty)^H$ is a weak equivalence in $\mathcal{U}$ for any $H \in \mathcal{F}$, one uses Corollary 4.6. □

7 Elmendorf's Theorem

We will state and prove the main theorem. Let G be a topological group in $\mathcal{U}$. Let H, K be closed subgroups of G.

Definition 7.1. The subgroup H is called *subconjugate* to K, if there exists an element $a \in G$ such that $a^{-1}Ha \subset K$.

If $a \in G$ satisfies $a^{-1}Ha \subset K$, then $G/H \to G/K$, $gH \mapsto gaK$, is an equivariant map, which we denote by R_a. Conversely, given an equivariant map $f\colon G/H \to G/K$, choose $a \in G$ such that $f(H) = aK$. Then $a^{-1}Ha$ is contained in K. Thus, we have proved the next lemma.

Lemma 7.2. *There exists an equivariant map $G/H \to G/K$ if and only if H is subconjugate to K.*

Furthermore, the following characterization holds.

Lemma 7.3. *In $\mathcal{U}$, there is an isomorphism*

$$\mathcal{U}^G(G/H, G/K) \xrightarrow{\varphi} (G/K)^H. \tag{6}$$

Proof. The isomorphism φ is given by $\varphi(f) = f(H)$ for $f \in \mathcal{U}^G(G/H, G/K)$ and its inverse sends $aK \in (G/K)^H$ to the equivariant map R_a. To check continuity issues, use Proposition 4.12, Remark 4.13 and Proposition 4.14. □

Let $\mathcal{F}$ be a set of closed subgroups of G containing the trivial subgroup $\{e\}$. Depending on $\mathcal{F}$, we will show how the homotopy theory of G-spaces relates to the homotopy theory of diagrams indexed by the following category.

Definition 7.4. The *orbit category $\mathcal{O}_\mathcal{F}$ of G with respect to $\mathcal{F}$* is the full subcategory of $\mathcal{U}^G$ with objects the orbit spaces $\{G/H\}_{H\in\mathcal{F}}$.

The orbit category is small and a topological category, as a subcategory of a topological category. Setting $J := \mathcal{O}_{\mathcal{F}}$ in Proposition 6.8, the category $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ becomes a model category. It will turn out, that there is a pair of Quillen equivalences between the model categories $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and $\mathcal{U}^G(\mathcal{F})$.

For any G-space X, let the object X^* of $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ be defined by $X^*(G/H) = X^H$ on objects and $X^*(f)\colon X^K \to X^H$, $x \mapsto ax$ for any representative a of $f(H)$, on morphisms. Define the functor

$$\Phi\colon \mathcal{U}^G \to \mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$$

by $X \mapsto X^*$ on objects and $f \mapsto \Phi(f)$, given by f^H in an object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, on morphisms.

Recall that $\mathcal{F}$ contains the trivial subgroup $\{e\}$ by assumption. If T is an object of $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$, then $T(G/\{e\})$ becomes a G-space with the action $G \times T(G/\{e\}) \to T(G/\{e\})$ defined by $(g, x) \mapsto T(R_g)(x)$ using that $\{e\}$ is subconjugate to $\{e\}$. Using this action on T evaluated in $G/\{e\}$, we construct a left adjoint of Φ.

Lemma 7.5. *Define the functor* $\Theta\colon \mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \to \mathcal{U}^G$ *by evaluation in* $G/\{e\}$*. Then* Θ *is left inverse and left adjoint to* Φ*.*

Proof. We show that $\Theta\Phi = \mathrm{id}_{\mathcal{U}^G}$. For any G-space X, one has $\Theta\Phi(X) = \Theta(X^*) = X^*(G/\{e\}) = X^{\{e\}}$ and checks that the G-action agrees with the one on X. If f is a map in $\mathcal{U}^G$, then $\Theta\Phi(f) = f^{\{e\}}$. Thus, Θ is left inverse to Φ.

For the adjunction, let T be a diagram in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and X a G-space. Note that for $R_e\colon G/\{e\} \to G/H$ in $\mathcal{O}_{\mathcal{F}}$, the image of $T(R_e)\colon T(G/H) \to T(G/\{e\})$ lies in the H-fixed point set of $\Theta(T) = T(G/\{e\})$, thus in $\Phi(\Theta(T))$. Let the morphism

$$\eta_T\colon T \to \Phi(\Theta(T))$$

of $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ be given by $T(R_e)$ in the object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$. One checks that η_T is a natural transformation by using the commutativity of the diagram

$$\begin{array}{ccc} G/\{e\} & \xrightarrow{R_e} & G/H \\ \downarrow{\scriptstyle R_a} & & \downarrow{\scriptstyle R_a} \\ G/\{e\} & \xrightarrow{R_e} & G/K \end{array}$$

for $a \in G$, $H, K \in \mathcal{F}$ with $a^{-1}Ha \subset K$. If $f\colon T \to S$ is a morphism in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$, then in particular the diagram

$$\begin{array}{ccc} T(G/H) & \xrightarrow{T(R_e)} & T(G/\{e\})^H \\ \downarrow{\scriptstyle f_{G/H}} & & \downarrow{\scriptstyle f^H_{G/\{e\}}} \\ S(G/H) & \xrightarrow{S(R_e)} & S(G/\{e\})^H \end{array}$$

commutes for any $H \in \mathcal{F}$. It follows that $\eta = \{\eta_T\}_T \colon \mathrm{id} \to \Phi\Theta$ is a natural transformation. It will turn out to be the unit of the adjunction. Note that $\Theta(\eta_T) = \mathrm{id}_{\Theta(T)}$ and $\eta_{\Phi(X)} = \mathrm{id}_{\Phi(X)}$. Letting $\varepsilon \colon \Theta\Phi \to \mathrm{id}$ be the identity, it follows that η and ϵ satisfy the necessary equations to determine the adjunction and hence are the unit and counit, respectively. $\square$

Remark 7.6. For any diagram T in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and G-space X, the natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}^G}(\Theta T, X) \to \mathrm{Hom}_{\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}}(T, \Phi(X))$$

of the adjunction is given by $f \mapsto \Phi(f)\eta_T$ with inverse $g \mapsto \Theta(g)$ by the description of the unit and counit above.

As defined, the counit ε of the adjunction (Θ, Φ) is an isomorphism in every object of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$. The unit is an isomorphism in some objects of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$. Recall that for $K \in \mathcal{F}$, we denote the contravariant functor $\mathcal{O}_{\mathcal{F}}(-, G/K)$ in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ by $\underline{G/K}$.

Lemma 7.7. *Let Y be a space in $\mathcal{U}$ and $K \in \mathcal{F}$. Then*

$$\eta_{\underline{G/K} \times Y} \colon \underline{G/K} \times Y \to \Phi\Theta(\underline{G/K} \times Y)$$

is an isomorphism in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$.

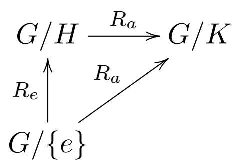
Proof. In an object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, the right-hand side is evaluated

$$(\Phi\Theta(\underline{G/K} \times Y))(G/H) = (\underline{G/K}(G/\{e\}) \times Y)^H = (\underline{G/K}(G/\{e\}))^H \times Y$$

and $\eta_{\underline{G/K} \times Y}$ is given by

$$(\underline{G/K} \times \mathrm{id}_Y)(R_e) = \underline{G/K}(R_e) \times \mathrm{id}_Y.$$

Hence, it's enough to show that $\eta_{\underline{G/K}} \colon \underline{G/K} \to \Phi\Theta(\underline{G/K})$, is an isomorphism in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$. In the object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, this morphism sends $R_a \in \underline{G/K}(G/H)$ to $R_a R_e \in (\underline{G/K}(G/\{e\}))^H$ for $a \in G$ with $a^{-1}Ha \subset K$. One concludes that it is an isomorphism by noting that



commutes and that $R_a \in \underline{G/K}(G/\{e\})$ is an H-fixed point if and only if $a^{-1}Ha \subset K$. $\square$

We are ready for the main theorem of this paper.

Theorem 7.8 (Elmendorf)**.** *The adjunction*

$$\Theta \colon \mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{U}^G(\mathcal{F}) \colon \Phi$$

is a pair of Quillen equivalences. In particular, the homotopy categories $\mathrm{Ho}(\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}})$ *and* $\mathrm{Ho}(\mathcal{U}^G(\mathcal{F}))$ *are equivalent.*

Proof. By the result a) in the proof of Theorem 5.1, it follows that a morphism f in $\mathcal{U}^G$ is an (acyclic) fibration if and only if f is an (acyclic) fibration for all $H \in \mathcal{F}$ and that $\Phi(f)$ is an (acyclic) fibration if and only if $\mathrm{ev}_{G/H}(\Phi(f)) = f^H$ is an (acyclic) fibration. Thus, Φ preserves (acyclic) fibrations.

Next, given a cofibrant object T of $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and a fibrant object X of $\mathcal{U}^G$, we have to show that a map $f' \colon T \to \Phi(X)$ in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ is a weak equivalence if and only if its adjoint $\Theta(f') \colon \Theta(T) \to X$ is a weak equivalence in $\mathcal{U}^G$. In fact, since in **Top** every object is fibrant, the category $\mathcal{U}^G$ shares the same property. So, X can be any G-space. Set

$$FI := \bigcup_{H \in \mathcal{F}} \{\underline{G/H} \times S^{n-1} \to \underline{G/H} \times D^n\}_{n \geq 0}.$$

Factor the unique map $i \colon \emptyset \to T$ from the initial object to T in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ as a cofibration i_∞ folllowed by an acyclic fibration p_∞ using the Infinite Gluing Construction $G^\infty(FI, i)$. Note that by Remark 2.11, the functor Θ preserves in particular acyclic cofibrations between cofibrant objects. Hence by Lemma 2.13, the weak equivalence p_∞ is taken to a weak equivalence by Θ. By **MC2**, the map f' is a weak equivalence if and only if $f'p_\infty$ is a weak equivalence and $\Theta(f')$ is a weak equivalence if and only if $\Theta(f')\Theta(p_\infty)$ is a weak equivalence. Thus, it's enough to show that a map $f \colon G^\infty \to \Phi(X)$ in $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ is a weak equivalence if and only if its adjoint $\Theta(f) \colon \Theta(G^\infty) \to X$ is a weak equivalence in $\mathcal{U}^G$, where we abbreviated $G^\infty := G^\infty(FI, i)$. Similarly, we write $G^k := G^k(FI, i)$, $k \geq 0$. Recall that by definition, f is a weak equivalence if $f_{G/H}$ is a weak equivalence in $\mathcal{U}$ for all $H \in \mathcal{F}$ and that $\Theta(f)$ is a weak equivalence if $(\Theta(f))^H = (f_{G/\{e\}})^H$ is a weak equivalence in $\mathcal{U}$ for all $H \in \mathcal{F}$. Fix $H \in \mathcal{F}$. Since f is a natural transformation, the diagram

$$\begin{array}{ccc} G^\infty(G/H) & \xrightarrow{f_{G/H}} & X^H \\ \downarrow{\scriptstyle G^\infty(R_e)} & & \downarrow{\scriptstyle X^*(R_e)} \\ G^\infty(G/\{e\}) & \xrightarrow{f_{G/\{e\}}} & X^{\{e\}} \end{array}$$

commutes. It follows that $f_{G/H} = (f_{G/\{e\}})^H(\eta_{G^\infty})_{G/H}$. Thus, by **MC2** it suffices to show that

$$(\eta_{G^\infty})_{G/H} \colon G^\infty(G/H) \to (G^\infty(G/\{e\}))^H$$

is a weak equivalence in $\mathcal{U}$ or more generally, that it is an isomorphism. Using that $G^0 = \emptyset$, one deduces that $(\eta_{G^0})_{G/H}$ is an isomorphism in $\mathcal{U}$. Recall that as a left adjoint, the functor Θ preserves colimits. Inductively, for $k \geq 1$, by using Proposition 6.10 and applying Lemma 7.7, it follows that $(\eta_{G^k})_{G/H}$ is an isomorphism in $\mathcal{U}$. Finally, by Proposition 6.10c), we deduce that $(\eta_{G^\infty})_{G/H}$ indeed is an isomorphism.

The second assertion holds by Theorem 2.12. □

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