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Homological Instability in Pre-Braided Homogeneous Categories

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Relatore

Prof. **Nathalie Wahl**

Candidato

Matteo Barucco

Corelatore

Dott. **Filippo Callegaro**

Controrelatore

Prof. **Mario Salvetti**

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To mom and dad

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Introduction

A sequence of groups $G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \dots$ is said to satisfy homological stability if the induced maps $H_i G_n \to H_i G_{n+1}$ are isomorphisms in a range $0 \leq i \leq f(n)$ increasing with n. In [18], the authors improve an important technique to study homological stability problems for sequences of groups admitting a braided monoidal structure. Namely, given such a family, with some extra assumptions, one can construct a pre-braided category locally homogeneous at a pair of objects (A, X). Moreover the groups G_n fit into the category as automorphism groups of the objects $A \otimes X^{\otimes n}$. Once the sequence fits into this categorical framework, one constructs a family of spaces whose connectivity yields stability for the sequence of groups.

In showing homological stability in this way, usually the hardest step is to prove connectivity. Indeed there are many examples of sequences of groups fitting the categorical framework, but for which it is not known yet whether or not the connectivity requirement is satisfied. What is more, connectivity for the constructed spaces is thought to be so strongly related to homological stability that it is conjectured to hold if and only if stability holds.

In this context we do not know many examples of sequences of groups fitting the categorical framework of pre-braided homogeneous categories, but for which we don't have connectivity, and homological stability fails. Peter Patzt in [13] constructed such an example with the symmetric groups on 2^n elements, endowed with a symmetric monoidal structure different from the classical one. The goal of this thesis is to present other examples of the same flavour. In particular we will present unstable sequences of groups fitting perfectly into pre-braided homogeneous categories, using general linear groups, and projective general linear groups. In all the examples we will show the failure at the level of H_1. In addition we will see also examples very similar to the previous ones, where H_1 stabilises and for which we do not know if we have or not failure at a higher level.

One interesting thing in the presented examples is that when stability fails dramatically (at the level of H_1) also the connectivity for the associated spaces fails dramatically (they are not even 0-connected). While in the examples where there is no failure at the level of H_1, the spaces are 0-connected. In showing 0-connectivity or not for the associated spaces we will make use of a quick criteria useful in our context.

The work is organised as follows. Chapter one provides all the algebraic tools needed later, as well as the necessary theory about homology of groups. Integral group rings, induced modules, standard resolution, homology with coefficients, Shapiro's Lemma, and a brief recap about spectral sequences.

Chapter two basically introduces homological stability in the categorical framework of pre-braided locally homogeneous categories. For pairs of objects in such categories we construct the associated sequence of spaces, and formulate the connectivity axiom. We give also a proof of an homological stability theorem (only for integral homology), which states that the connectivity implies stability for the sequence of automorphism groups associated to the pair. We also show how starting from a groupoid with a braided monoidal structure we can construct an associated pre-braided locally homogeneous category.

The last chapter is completely devoted in presenting the examples of instability that we have mentioned before. After the criteria for the 0-connectivity, we will see examples with symmetric groups, general linear groups and projective general linear groups.

Chapter 1

Homology of Groups

This Chapter is devoted to a brief summary about all the basic algebraic tools needed in this Thesis. We will begin with a brief recalling about integral group rings, and continue with an introduction about group homology. We will not prove everything in this chapter, so the main reference for these two parts is [3]. We will conclude with a brief summary about the properties of Spectral Sequences that are needed later, referring again to [3], but also to [12] for a complete development of the topic.

1.1 Integral group rings

In order to talk about homology of groups, first of all we need to talk about Integral Group Rings, which provide the algebraic base for group homology.

Definition 1.1. Let G be a group written multiplicatively. The integral group ring of G, denoted either $\mathbb{Z}G$ or $\mathbb{Z}[G]$ is the free $\mathbb{Z}$-module generated by the elements of G, with multiplication given by the unique extension of the multiplication of G to a $\mathbb{Z}$-bilinear product on $\mathbb{Z}G$.

Note that G is a subgroup of $(\mathbb{Z}G)^*$, and that we have the following universal mapping property:

UP of Integral Group Rings. Given a ring R and a group homomorphism from G to R^*, there is a unique extension of it to a ring homomorphism from $\mathbb{Z}G$ to R. Thus we have:

$$\mathrm{Hom}(\mathbb{Z}G, R) \cong \mathrm{Hom}(G, R^*).$$

In view of this, a (left) $\mathbb{Z}G$-module (also called a G-module), is simply an abelian group M together with a left action of G on M.

Example 1.2. The abelian group $\mathbb{Z}$ can always be considered a G-module, via the trivial action of G.

The simplest way to construct G-module is via permutation module.

Definition 1.3. Given a set X with a G-action. The permutation module $\mathbb{Z}X$ (also denoted $\mathbb{Z}[X]$), is the G-module obtained from the free abelian group generated by X, where we extends the action of G on X, to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$.

Example 1.4. Given a subgroup $H < G$ we have that G acts on ${}^{G}/_{H}$ (the set of cosets gH) by left translation. So we can obtain a permutation module $\mathbb{Z}\left[{}^{G}/_{H}\right]$

The operation of disjoint union in the category of G-sets corresponds to the direct sum operation in the category of G-modules:

$$\mathbb{Z}\left[\coprod X_i\right] = \bigoplus \mathbb{Z}X_i.$$

It follows that given a G-set X and a set of representatives E for the G-orbits, we can decompose $\mathbb{Z}X$:

$$\mathbb{Z}X \cong \bigoplus_{x\in E} \mathbb{Z}\left[{}^{G}/_{\mathrm{stab}(x)}\right].$$

If in addition the action of G on X is free, then every stabiliser contains only the identity, and $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E. So we have proved the following.

Proposition 1.5. If X is a free G-set and E a set of representatives for the G-orbits in X, then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.

1.1.1 Extension and restriction of scalars

Let $f : R \to S$ be a ring homomorphism. Recall that any S-module can be regarded as an R-module via f, and we obtain in this way a functor from S-module to R-module called restriction of scalars.

Conversely for any left R-module M, consider the tensor product $S \otimes_R M$, where S is regarded as a right R-module by $s \cdot r := s \cdot f(r)$. We can make $S \otimes_R M$

a left S-module by setting

$$s \cdot (s' \otimes m) := (ss') \otimes m. \tag{1.1}$$

This S-module is said to be obtained from M by extension of scalars from R to S. Note that there is a natural map $\iota : M \to S \otimes_R M$ given by $\iota(m) = 1 \otimes m$, which is an R-module homomorphism when $S \otimes_R M$ is regarded as an R-module by restriction of scalars. Moreover the following universal mapping property holds:

UP of extension of scalars. Given an S-module N and an R-module map $g : M \to N$ there is a unique S-module map $h : S \otimes_R M \to N$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & S \otimes_R M \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ N & & \end{array} \tag{1.2}$$

Thus we have

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N).$$

Remark 1.6. Recall that in order for a tensor product $M \otimes_R N$ to make sense, M must be a right R-module, and N a left R-module. In the case R is an integral group ring $\mathbb{Z}G$, we can avoid having to consider both left and right modules by using the anti-automorphism $g \mapsto g^{-1}$ of G. Thus we can regard any left G-module M as a right G-module by setting $mg = g^{-1}m$, and in this way we can make sense out of the tensor product $M \otimes_G N$ of two left modules. If M naturally admits both a left and a right G-action we will revert to the notation $M \otimes_{\mathbb{Z}G} N$ if we want to indicate that the tensor product is taken considering the given right action of G on M, rather than the right action obtained from the left action.

1.1.2 Induced Modules

We will see now how to apply the notation and theory just developed to inclusion of a subgroup $H < G$.

For every subgroup $H < G$, we have a natural inclusion of rings $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$, and we can apply the constructions just made to this ring homomorphism. In this case extension of scalars is called induction from H to G. For an H-module

M we will write

$$\mathrm{Ind}_H^G(M) := \mathbb{Z}G \otimes_{\mathbb{Z}H} M,$$

which is a left $\mathbb{Z}G$-module as shown in (1.1). Here $\mathbb{Z}G$ is a right $\mathbb{Z}H$-module, with H acting by right translation on G. since this action is free, $\mathbb{Z}G$ is a free right $\mathbb{Z}H$-module, and as basis we can take a set E of representatives for the left cosets gH. It follows that $\mathrm{Ind}_H^G(M)$, as abelian group, admits a decomposition

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M = \bigoplus_{g\in E} g \otimes M. \tag{1.3}$$

In particular the canonical H-map $\iota : M \hookrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ maps M isomorphically into its image $1 \otimes_{\mathbb{Z}H} M$. We can therefore use ι to regard M as an H-submodule of $\mathrm{Ind}_H^G(M)$. Moreover the summand $g \otimes M$ that occurs in (1.3) is simply the transform of this submodule under the action of g. Putting everything together in a proposition:

Proposition 1.7. The G-module $\mathrm{Ind}_H^G(M)$ contains M as an H-submodule, and is the direct sum of its transforms:

$$\mathrm{Ind}_H^G(M) = \bigoplus_{g\in G/H} gM$$

Remark 1.8. The H-submodule M is mapped onto itself by the action of H, so that the subgroup gM of $\mathrm{Ind}_H^G(M)$ depends only on the class of g in ${}^{G}/_{H}$.

The Proposition 1.7 completely characterizes induced G-modules. More precisely the following is true

Proposition 1.9. Let N be a G-module whose underlying abelian group is a direct sum $\oplus_{i\in I} M_i$. Assume also that the G-action transitively permutes the summands (i.e. there is a transitive action of G on I such that $gM_i = M_{gi}$). If M_{i_0} is one of the summands and $H = \mathrm{stab}(i_0)$, then M_{i_0} is an H-module and $N \cong \mathrm{Ind}_H^G(M_{i_0})$.

Proof. Let us show first that M_{i_0} is an H-submodule. If $g \in H$ we have $gM_{i_0} = M_{gi_0} = M_{i_0}$, so we have that the action of H restricted to M_{i_0}, gives an action of H on M_{i_0}.

Using the universal mapping property (1.2) we have that the inclusion $M_{i_0} \hookrightarrow N$ extends to a G-map $\varphi : \mathrm{Ind}_H^G(M_{i_0})$. Clearly $\forall g \in G$ we have

$$\varphi(gM_{i_0}) = g\varphi(M_{i_0}) = gM_{i_0}.$$

So φ maps the summand gM_{i_0} of $\mathrm{Ind}_H^G(M_{i_0})$ isomorphically onto the corresponding summand M_{gi_0} of N. □

Example 1.10. The permutation module $\mathbb{Z}\left[{}^{G}/_{H}\right]$ is direct sum of copies of $\mathbb{Z}$, with a transitive G-action on the summands, and H as stabilizer of every summand. So applying Proposition 1.9 we obtain $\mathbb{Z}\left[{}^{G}/_{H}\right] \cong \mathrm{Ind}_H^G(\mathbb{Z})$

Example 1.11. Let X be a CW-complex with a G-action which is transitive on simplices of each dimension. The G-module $C_n(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each n-simplex of X, and the summands are permuted by the G-action. Hence choosing a simplex $\sigma \in C_n(X)$ and applying Proposition 1.9 we have

$$C_n(X) \cong \mathrm{Ind}_{\mathrm{stab}(\sigma)}^G(\mathbb{Z}_\sigma). \tag{1.4}$$

Where $\mathbb{Z}_\sigma$ is the orientation module associated to σ: an infinite cyclic group whose two generators correspond to the two orientations of σ. Thus $g \in \mathrm{stab}(\sigma)$ acts on $\mathbb{Z}_\sigma$ as the identity if g preserves the orientation of σ, and $-$identity otherwise.

1.2 The Homology of a group

In this section we will consider $\mathbb{Z}$ as a G-module with trivial G-action.

Definition 1.12. A G-complex is simply a CW-complex with an action of G which permutes the cells.

When X is a G-complex, we have an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover the augmentation map $\varepsilon : C_0(X) \to \mathbb{Z}$ defined by sending every 0-cell in 1 is a G-modules homomorphism.

We will say that X is a free G-complex if the action of G freely permutes the cells of X. In this case each $C_n(X)$ admits a $\mathbb{Z}$-basis which is freely permuted by G, hence in view of Proposition 1.5 $C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. If in addition X is contractible, then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

1.2.1 The standard resolution

Let us now construct the so called standard resolution of $\mathbb{Z}$ in $\mathbb{Z}G$-modules. First of all let X be the "simplex" spanned by G. The complex X has vertices the elements of G with G acting by left translation, and every finite subset of G is a simplex of X. The action of G on X is free on vertices but not necessarily on higher dimensional simplices. We can bypass this problem considering the ordered chain complex $C'_*(X)$ instead of the usual chain complex $C(X)$ (see [15] Ch. 4, §3). Namely $C'_n(X)$ has a $\mathbb{Z}$-basis consisting of the ordered $(n+1)$-tuples $(V_0, \ldots, V_n)$ of vertices of X such that $\{V_0, \ldots, V_n\}$ is a simplex of X. In this way the action of G is free on these $n+1$-tuples. Combined with the fact that X is contractible by a straight-line homotopy we obtain a free resolution $F_* = C_*(X)$ of $\mathbb{Z}$ in $\mathbb{Z}G$-modules.

Explicitly F_n is the free $\mathbb{Z}$-module generated by the $(n+1)$-tuples $(g_0, \ldots, g_n)$ of elements of G, with the G-action given by $g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$. The boundary operator $\partial : F_n \to F_{n-1}$ is given by $\partial = \sum_{i=0}^{n} (-1)^i d_i$, where

$$d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n), \tag{1.5}$$

and the augmentation $\varepsilon : F_0 \to \mathbb{Z}$ by $\varepsilon(g_0) = 1$.

As basis for the free $\mathbb{Z}G$-module F_n we may take the $(n+1)$-tuples whose first element is 1, and write these elements in "bar" notation

$$[g_1|g_2|\ldots|g_n] := (1, g_1, g_1g_2, \ldots, g_1g_2\ldots g_n).$$

In this notation the boundary maps (1.5) become

$$d_i\,[g_1|\ldots|g_n] = \begin{cases} g_1[g_2|\ldots|g_n] & i = 0, \\ [g_1|\ldots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\ldots|g_n] & 0 < i < n, \\ [g_1|\ldots|g_{n-1}] & i = n. \end{cases} \tag{1.6}$$

In low dimensions the resolution has the form

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where

$$\partial_2([g|h]) = g[h] - [gh] + [g], \tag{1.7}$$

$$\partial_1([g]) = g[\quad] - [\quad] = g - 1, \tag{1.8}$$

$$\varepsilon(1) = 1. \tag{1.9}$$

1.2.2 Integral Homology

Definition 1.13. Given a group G and a G-module M, the group of co-invariants of M, denoted M_G, is the quotient of M by the additive subgroup generated by all the elements of the form $gm - m$ for $g \in G$ and $m \in M$.

Thus M_G is simply obtained from M by "dividing out" by the G-action. Regarding $\mathbb{Z}$ as a right G-module with trivial action, we can give another description of the co-invariants group

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

To justify this description simply note that tensoring with $\mathbb{Z}$ actually kills the G-action. Indeed in $\mathbb{Z} \otimes_{\mathbb{Z}G} M$ we have $1 \otimes gm = 1 \cdot g \otimes m = 1 \otimes m$.

Remark 1.14. If M is a free $\mathbb{Z}G$-module with basis (e_i), then M_G is a free $\mathbb{Z}$-module with basis $(\bar{e_i})$.

The co-invariants functor assigns to every G-module an abelian group. It is right-exact, but it is not an exact functor. Roughly speaking the homology groups of G measure the failure of this functor to be exact.

Definition 1.15. If $\varepsilon : F \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}G$-modules. Define the homology groups of G as

$$H_i G := H_i(F_G). \tag{1.10}$$

Where F_G is obtained from F applying the co-invariants functor to each G-module. The homology of F_G is independent from the choice of the resolution, up to canonical isomorphism.

For any group G we can always take F to be the standard resolution and write $C_*(G)$ for the chain complex F_G. Using the Remark 1.14 we can describe $C_*(G)$ explicitly. On the $(n+1)$-tuples of elements of G introduce the equivalence relation given by the action of G: $(g_0, \dots, g_n) \sim (gg_0, \dots, gg_n)$ for all $g \in G$. Then $C_n(G)$ has a $\mathbb{Z}$ basis consisting of the equivalence classes $[g_0, \dots, g_n]$ and boundary maps given by the alternating sum of the maps induced by (1.5) in the quotient.

We can also use the bar notation to describe $C_*(G)$. From this point of view $C_n(G)$ has a $\mathbb{Z}$ basis consisting of the classes of n-tuples $[g_1|\dots|g_n]$, where with abuse of notation we are omitting the square-brackets which denote the class of the element $[g_1|\dots|g_n] \in F_n$ in the quotient $(F_n)_G$. In this notation the boundary maps are given by the alternating sum of the maps (1.6) in the quotient:

$$d_i\,[g_1|\dots|g_n] = \begin{cases} [g_2|\dots|g_n] & i=0, \\ [g_1|\dots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\dots|g_n] & 0<i<n, \\ [g_1|\dots|g_{n-1}] & i=n. \end{cases}$$

In low dimension $C_*(G)$ has the form

$$C_2(G) \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} \mathbb{Z} \to 0,$$

where $\partial_1 = 0$ since the elements in the image of (1.8) are exactly the one we quotient. While (1.7) in the quotient becomes $\partial_2[g|h] = [h] - [gh] + [g]$. Consequently $H_0G = \mathbb{Z}$ and H_1G is isomorphic to the abelianization of G. More precisely the following holds

Lemma 1.16. Denoting with $\bar{g}$ the homology class of the cycle $[g]$ in $C_1(G)$, the map

$$H_1G \to {}^{G}/_{[G:G]}$$

sending $\bar{g}$ in the class of the element g in the quotient by $[G:G]$, is an isomorphism.

The homology H_*G is a covariant functor of G, since $C_*(G)$ is functorial in G. But we want to describe the induced map also in terms of arbitrary resolutions. Given an homomorphism $\alpha : G \to G'$ and projective resolutions F and F' of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively, think of F' as a complex of G-modules via α. Even though F' is not necessarily projective over $\mathbb{Z}G$, it is still acyclic. Combining this with the projectivity of F from a well known theorem of homological algebra (see for example [3] Ch.1, Theorem7.4) we obtain an augmentation preserving G-map $\tau : F \to F'$, unique up to homotopy. For τ the condition to be a G-map means that $\forall g \in G$, $\forall x \in F$ we have

$$\tau(gx) = \alpha(g)\tau(x). \tag{1.11}$$

The map τ induces a map $F_G \to F'_{G'}$, and so a well-defined map $\alpha_* : H_*G \to H_*G'$.

Lemma 1.17. Conjugation by an element $g_0 \in G$ is an automorphism of G which induces the identity on H_*G.

Proof. If F is a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}G$-modules we can take $\tau : F \to F$ to be $\tau(x) = g_0 \cdot x$. Indeed τ commutes with the boundary maps since these one are morphisms of $\mathbb{Z}G$-modules and moreover satisfies (1.11). But τ induces the identity on F_G, and so the identity also on H_*G. $\square$

1.2.3 Homology with coefficients

Definition 1.18. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and M a G-module. We define the homology of G with coefficients in M by

$$H_*(G, M) := H_*(F \otimes_G M). \tag{1.12}$$

Where $F \otimes_G M$ is the complex obtained from F applying the functor $\_\otimes_G M$. Then we can see that (1.12) is a true generalization of (1.10). Indeed we recover the latter by taking $M = \mathbb{Z}$, and for this reason it is also called integral homology.

The complex $F \otimes_G M$ can also be be thought as a tensor product of chain complexes, with M a complex concentrated in zero dimension. So with a bit of homological algebra we can also use a projective resolution in $\mathbb{Z}G$-modules $\eta : P \to M$ of M to compute the homology:

$$H_*(F \otimes_G M) \cong H_*(F \otimes_G P) \cong H_*(\mathbb{Z} \otimes_G P). \tag{1.13}$$

From this description, since $F \otimes_G \_$ is a covariant functor, we get that $H_*(G, \_)$ is also a covariant functor of the coefficients module. Furthermore $H_*(G, M)$ can be seen as a covariant functor of the couple (G, M). Let $\mathcal{C}$ be the following category: an object of $\mathcal{C}$ is a pair (G, M) where G is a group, and M is a G-module. A morphism in $\mathcal{C}$ from (G, M) to (G', M') is a pair (α, f) where $\alpha : G \to G'$ is a group homomorphism, and $f : M \to M'$ is a map of abelian groups such that $f(gm) = \alpha(g)f(m)$ for $g \in G$ and $m \in M$ (in other words f is a G-module morphism if M' is regarded as a G-module via α). We can now repeat the same construction we have made to show functoriality of integral homology. Consider F, F' projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively, exactly as before we have an augmentation preserving G-map satisfying (1.11) unique up to homotopy. Then there is a chain map

$$\tau \otimes f : F \otimes_G M \to F' \otimes_{G'} M',$$

which induces a well-defined map $(\alpha, f)_* : H_*(G, M) \to H_*(G', M')$. In this way H_* becomes a covariant functor on $\mathcal{C}$. If we have $M = M'$ and $f = \mathrm{Id}_M$ we simply write α_*. Since

$$\tau \otimes f = (\tau \otimes \mathrm{Id}_{M'}) \circ (\mathrm{Id}_F \otimes f),$$

we have that $(\alpha, f)_*$ can always be written as the composite

$$H_*(G, M) \xrightarrow{H_*(\mathrm{Id}_G, f)} H_*(G, M') \xrightarrow{\alpha_*} H_*(G', M'). \tag{1.14}$$

Proposition 1.19 (Shapiro's Lemma)**.** If $H \leq G$ is a subgroup, and M an H-module, then

$$(\alpha, \iota)_* : H_*(H, M) \xrightarrow{\sim} H_*(G, \mathrm{Ind}_H^G(M))$$

is an isomorphism, where $\alpha : H \hookrightarrow G$ is the inclusion of the subgroup, and $\iota : M \hookrightarrow \mathrm{Ind}_H^G(M)$ is the canonical H-map (see (1.3)).

Proof. If F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then it also can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. Moreover Id_F is an augmentation preserving H-map. This follows since $\alpha : H \hookrightarrow G$ is simply the inclusion of the subgroup, and because

$$\mathrm{Id}_F(hx) = \alpha(h)\,\mathrm{Id}_F(x)$$

holds for $h \in H$ and $x \in F$. This can be used to compute $(\alpha, \iota)_*$ which is induced by

$$\mathrm{Id}_F \otimes \iota : F \otimes_H M \to F \otimes_G \mathrm{Ind}_H^G(M). \tag{1.15}$$

Since

$$F \otimes_G \mathrm{Ind}_H^G(M) = F \otimes_G (\mathbb{Z}G \otimes_H M) \cong F \otimes_H M,$$

we have that (1.15) is an isomorphism. □

1.3 Review on spectral sequences

Definition 1.20. A differential bigraded module over a ring R is a collection of R-modules, $\{E_{p,q}\}$ where $p, q \in \mathbb{Z}$, together with an R-linear mapping $d : E_{**} \to E_{**}$, the differential, of bidegree $(-r, r-1)$, for some integer r, and satisfying $d \circ d = 0$.

One can easily imagine a differential bigraded module as an integral lattice in the Cartesian plane, where the R-module $E_{p,q}$ sits at the point (p,q). With the differential, we can consider also the homology of a differential bigraded module $H_{p,q}(E_{**}, d)$.

Definition 1.21. A Spectral sequence (of homological type) is a collection of differential bigraded R-modules $\{E^r_{*,*}, d^r\}$ where $r \in \mathbb{N}^*$, the differential d^r has bidegree $(-r, r-1)$, and for all p, q, r

$$E^{r+1}_{p,q} \cong H_{p,q}(E^r_{*,*}, d^r).$$

A common way of thinking about a spectral sequence is to imagine a book where in the page r there is a differential bigraded R-module $(E^r_{*,*}, d^r)$. Every differential bigraded module determines the bigraded module in the next page (given by simply taking the homology), but not necessarily the differential in the next page.

Although we have our spectral sequence indexed by $r = 1, 2, \dots$ it is clear that the indexing can begin at any integer. We want to define now the target of the spectral sequence. To identify this target we present a spectral sequence as a tower of submodules of a given module.

Let us begin with $E^1_{*,*}$. For the sake of clarity we suppress the bigrading. Let $Z^1 := \ker d^1$ and $B^1 := \operatorname{im} d^1$, we have $B^1 \subseteq Z^1 \subseteq E^1$ and by definition $E^2 \cong Z^1/B^1$. Denote $\bar{Z}^2 := \ker d^2 : E^2 \to E^2$. Since $\bar{Z}^2$ is a submodule of E^2 it can be written as Z^2/B^1 where Z^2 is a submodule of Z^1. Similarly $\bar{B}^2 = \operatorname{im} d^2$ is isomorphic to B^2/B^1 and so

$$E^3 \cong \bar{Z}^2/\bar{B}^2 \cong \left(Z^2/B^1\right)/\left(B^2/B^1\right) \cong Z^2/B^2.$$

These data can be presented as a tower of inclusions $B^1 \subseteq B^2 \subseteq Z^2 \subseteq Z^1 \subseteq E^1$. Iterating this process we present the spectral sequence as an infinite tower of submodules of E^1:

$$B^1 \subseteq B^2 \subseteq \cdots \subseteq B^n \subseteq \cdots \subseteq Z^n \subseteq \cdots \subseteq Z^2 \subseteq Z^1 \subseteq E^1$$

with the property that $E^{n+1} \cong Z^n/B^n$.

Now let $Z^\infty := \bigcap_n Z^n$ the submodules of elements that survives forever, and $B^\infty := \bigcup_n B^n$. From the tower of inclusions we know that $B^\infty \subseteq Z^\infty$, so $E^\infty := Z^\infty/B^\infty$ is the bigraded module that remains after the computation of the infinite sequence of homologies. The E^∞-term of a spectral sequence is the general goal of a computation.

In many cases our spectral sequence collapses at the N^{th}-term, which means that $d^r = 0$ for $r \geq N$. If this happens, the condition $d^r = 0$ forces $Z^r = Z^{r-1}$ and $B^r = B^{r-1}$, and then the tower of submodules becomes:

$$B^1 \subseteq \cdots \subseteq B^{N-1} = B^N = \cdots = B^\infty \subseteq Z^\infty = \cdots = Z^N = Z^{N-1} \subseteq \cdots \subseteq Z^1,$$

implying $E^\infty = E^N$.

1.3.1 The Spectral Sequence of a filtered complex

Definition 1.22. By an increasing filtration on an R-module M, we mean a family of submodules F_pM with $p \in \mathbb{Z}$, such that $F_pM \subseteq F_{p+1}M$. The filtration is said to be finite if $F_pM = 0$ for p sufficiently small and $F_pM = M$ for p sufficiently large.

Definition 1.23. Given a filtration on M the associated graded module $E^0_*(M)$ is defined by

$$E^0_p(M) := {F_pM}/{F_{p-1}M}.$$

One can think of M as being built up from the "pieces" $E^0_p(M)$.

Remark 1.24. If the filtered module M is graded (and each F_pM is a graded submodule), then we have for each $n \in \mathbb{Z}$ a filtration $\{F_pM_n\}$ on M_n, and hence there is an obvious way of associating a bigraded module to M, setting

$$E^0_{p,q}(M) := {F_pM_{p+q}}/{F_{p-1}M_{p+q}}.$$

To simplify the notation we will sometimes suppress the second subscript and simply write $E^0_pM = {F_pM}/{F_{p-1}M}$.

Definition 1.25. A spectral sequence $\{E^r_{*,*}, d^r\}$ is said to converge to the graded R-module M, if there is a filtration F on M such that

$$E^\infty_{p,q} \cong E^0_{p,q}(M, F)$$

Definition 1.26. An R-module C, is a filtered differential graded module if:

(1) C is the direct sum of submodules, $C = \bigoplus_{n=0}^{\infty} C_n$.

(2) There is an R-linear mapping $d : C \to C$ of degree -1 satisfying $d \circ d = 0$.

(3) C has a filtration F and the differential respects that filtration, that is $d : F_pC \to F_pC$.

Since the differential respects the filtration, also the homology $H(C)$ admits a filtration induced by the inclusion map $F_pC \hookrightarrow C$. Explicitly

$$F_pH(C) = {}^{F_pC \cap Z}/_{F_pC \cap B},$$

where Z and B are respectively the submodule of cycles and the submodule of boundaries of C. We will say that the filtration is dimension-wise finite, if $\{F_pC_n\}_{p\in\mathbb{Z}}$ is a finite filtration of C_n for each n.

We can now state the main theorem of this brief survey on spectral sequences. We refer to Theorem 2.6 of [12] for the proof.

Theorem 1.27. Each filtered differential graded module C determines a spectral sequence $\{E^r_{*,*}, d^r\}_{r=1}^{\infty}$ with d^r of bidegree $(-r, r-1)$, and

$$E^1_{p,q} \cong H_{p+q}({}^{F_pC}/_{F_{p-1}C}). \tag{1.16}$$

Moreover if the filtration of C is dimension-wise finite, then the spectral sequence converges to the homology $H(C)$, that is

$$E^{\infty}_{p,q} \cong {}^{F_pH_{p+q}(C)}/_{F_{p-1}H_{p+q}(C)}$$

1.3.2 Double complexes

Definition 1.28. A Double Complex is a bigraded R-module $C = (C_{pq})_{p,q\in\mathbb{Z}}$, with a horizontal differential d' of bidegree $(-1, 0)$ and a vertical differential d'' of bidegree $(0, -1)$, such that $d' \circ d'' = d'' \circ d'$.

A double complex C gives rise to an ordinary chain complex TC, called the total complex as follows:

$$(TC)_n := \bigoplus_{p+q=n} C_{pq},$$

with differential d given by

$$d_{|C_{pq}} = d' + (-1)^p d''.$$

The tensor product of two chain complexes C' and C'' provides a familiar example of this construction. Indeed we have a double complex C with $C_{pq} := C'_p \otimes C''_q$ and TC is simply the usual tensor product $C' \otimes C''$ of chain complexes.

We now filter TC by setting

$$F_p(TC)_n = \bigoplus_{i \leq p} C_{i,n-i}.$$

If the double complex C is a first quadrant double complex (i.e. if $C_{pq} = 0$ when $p < 0$ or $q < 0$) then TC is a filtered differential graded module, with a dimension-wise finite filtration. Thus we can apply Theorem 1.27 and we have a spectral sequence $\{E^r_{*,*}, d^r\}$ converging to $H_*(TC)$. Let us compute the E^1-page and the differential d^1 of this spectral sequence. First of all the bigraded module associated to the filtration of TC is given by:

$$E^0_{p,q}(TC) = \frac{F_p(TC_{p+q})}{F_{p-1}(TC_{p+q})} = \frac{\bigoplus_{i \leq p} C_{i,p+q-i}}{\bigoplus_{i \leq p-1} C_{i,p+q-i}} = C_{pq}.$$

The differential d of TC induces a differential in $E^0_{p,q}(TC)$, which simply is $\pm d''$ (with sign depending on p). Therefore using (1.16) we have that E^1 is the vertical homology of C (i.e. $E^1_{p,q} = H_q(C_{p,*})$), and the differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is the map induced by the chain map $d' : C_{p,q} \to C_{p-1,q}$. Thus E^2 can be described as the horizontal homology of the vertical homology of C.

One could as well filter TC "horizontally" instead of "vertically", which simply means

$$F_p(TC)_n = \bigoplus_{j \leq p} C_{n-j,j}.$$

We obtain then a second spectral sequence converging to $H_*(TC)$, but this time with $E^0_{p,q} = C_{q,p}$, $E^1_{p,q} = H_q(C_{*,p})$, and $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ equal up to sign to the map induced by $d'' : C_{*,p} \to C_{*,p-1}$.

Remark 1.29. Even though the two spectral sequences have the same abutment $H_*(TC)$ they do not in general have the same E^∞-term. Indeed we have to keep in mind that two different filtrations of TC leads possibly to two different filtrations on $H_*(TC)$, which means two different E^∞-terms.

Chapter 2

Homological Stability

Before approaching Instability, we must talk about Homological Stability. This chapter basically presents a simplified version of the machinery built in [18], focusing only on homology with constant coefficients $\mathbb{Z}$. Following the first three sections of [18], we will develop the categorical setting of (Locally) Homogeneous Categories, where certain sequences of groups arises naturally as automorphisms. Using Quillen's classical argument [14], we will prove also the main Homological Stability Theorem of [18] (only for constant coefficients $\mathbb{Z}$), which states that connectivity of certain spaces yields stability for the considered sequences. The last section is the most useful for our work. We will see how starting from a groupoid satisfying some mild extra assumptions, one can build a locally homogeneous category in which the groupoid fits naturally. Moreover in this category, it is possible to study the connectivity of the spaces associated to the sequence of groups, and possibly apply stability results.

2.1 Locally Homogeneous Categories

This first section provides the categorical setting where the main homological stability theorem is formulated.

Definition 2.1. A strict monoidal category $(\mathcal{C}, \otimes, e)$ is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative, and with an object e which is a left and right unit for $\otimes$.

Recall that being a bifunctor, means that $\otimes$ assigns to each pair of objects $A, B \in \mathcal{C}$ an object $A \otimes B$ of $\mathcal{C}$, and to each pair of arrows $f : A \to A'$ and

$g : B \to B'$ an arrow $f \otimes g : A \otimes B \to A' \otimes B'$, according to the composition laws

$$\mathbb{1}_A \otimes \mathbb{1}_B = \mathbb{1}_{A \otimes B}$$

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g),$$

where we denote by $\mathbb{1}_A$ the identity automorphism of the object A. Associativity means that $\otimes$ is associative both for objects $((A \otimes B) \otimes C) = (A \otimes (B \otimes C))$, and for arrows $((f \otimes g) \otimes h) = (f \otimes (g \otimes h))$. Similarly, being a left and right unit means that e is a unity for objects $e \otimes A = A = A \otimes e$, and $\mathbb{1}_e$ is a unity for arrows $\mathbb{1}_e \otimes f = f = f \otimes \mathbb{1}_e$.

Remark 2.2. From now on with monoidal category we will always mean strict monoidal category. We will also adopt the notation to indicate the identity isomorphism of an object $\mathbb{1}_X$, simply with the symbol X, (where it is clear that we are not talking about the object).

Consider now a monoidal category $(\mathcal{C}, \otimes, e)$ in which the unit e is initial (i.e. for each object $A \in \mathcal{C}$ there is exactly one arrow $e \to A$). For every pair of objects A and B in such a category, we have a preferred morphism

$$\iota_A \otimes B : B = e \otimes B \to A \otimes B$$

where ι_A denotes the unique morphism in $\mathcal{C}$ from the initial object e to A. Given a generic morphism $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, since we have that $\mathrm{Aut}_{\mathcal{C}}(Y)$ acts on this set under post-composition, we can indicate with $\mathrm{stab}(g)$ the subgroup of automorphisms that fix g under post-composition:

$$\mathrm{stab}(g) := \{\phi \in \mathrm{Aut}(Y) \mid \phi \circ g = g\}.$$

We are now ready to give the central definition.

Definition 2.3. A monoidal category $(\mathcal{C}, \otimes, e)$ is locally homogeneous at a pair of objects (A, X) if e is initial in $\mathcal{C}$ and if it satisfies the following two axioms:

LH1 For all $0 \le p < n$, $\mathrm{Hom}(X^{\otimes p+1}, A \otimes X^{\otimes n})$ is a transitive $\mathrm{Aut}(A \otimes X^{\otimes n})$-set under post-composition.

LH2 For all $0 \le p < n$, the map $\mathrm{Aut}(A \otimes X^{n-p-1}) \to \mathrm{Aut}(A \otimes X^{\otimes n})$ taking f to $f \otimes X^{\otimes p+1}$ is injective with image $\mathrm{stab}(\iota_{A \otimes X^{\otimes n-p-1}} \otimes X^{\otimes p+1})$.

At this point one can ask how these two conditions can possibly provide us a useful setting in studying homological stability problems. To see that, take a monoidal category locally homogeneous at a pair (A, X), and consider the groups

$$G_n := \mathrm{Aut}(A \otimes X^{\otimes n}).$$

We have a canonical map $\Sigma^X : G_n \to G_{n+1}$ (called stabilisation map) taking $f \in G_n$ to $f \otimes X \in G_{n+1}$. With this interpretation condition LH2 makes more sense now, because for example implies injectivity of Σ^X. As a consequence we obtain a sequence of groups: a family of groups and canonical injections between them

$$G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \ldots \tag{2.1}$$

Definition 2.4. Given a monoidal category locally homogeneous at a pair (A, X). We will call the sequence (2.1), the stabilisation sequence associated to (A, X).

Remark 2.5. Another parallel and important goal of Definition 2.3 is to simplify the study of the set $\mathrm{Hom}(X^{\otimes p+1}, A \otimes X^{\otimes n})$. Condition LH1 says that for $0 \leq p < n$ this set is a transitive G_n-set under post-composition, and moreover we have already pointed out a special element $\sigma_p \in \mathrm{Hom}(X^{\otimes p+1}, A \otimes X^{\otimes n})$ defined as

$$\sigma_p := \iota_{A \otimes X^{\otimes n-p-1}} \otimes X^{\otimes p+1}. \tag{2.2}$$

As a consequence we can write every other element $f \in \mathrm{Hom}(X^{\otimes p+1}, A \otimes X^{\otimes n})$ as

$$f = \phi \circ \sigma_p \quad \text{with} \quad \phi \in G_n. \tag{2.3}$$

In these terms we can reformulate condition LH2 which says that in the same range of p the map $\_ \otimes X^{\otimes p+1}$ sends G_{n-p-1} isomorphically onto $\mathrm{stab}(\sigma_p)$. Then the subgroup of G_n that fixes our preferred morphism is $G_{n-p-1} \otimes X^{\otimes p+1}$, which is the immersion of G_{n-p-1} inside G_n under composition of stabilisation maps (with abuse of notation we often will indicate it simply as G_{n-p-1}). With this in mind we have

$$\mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n}) \cong G_n/G_{n-p-1} \tag{2.4}$$

where the isomorphism is obtained sending the element $f \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$ in the corresponding class σG_{n-p-1} accordingly with (2.3).

In many cases the axioms LH1 and LH2 holds for all objects A and X of $\mathcal{C}$, because the category satisfies the following stronger and global version of these axioms.

Definition 2.6. A monoidal category $(\mathcal{C}, \otimes, e)$ is homogeneous if e is initial in $\mathcal{C}$ and if it satisfies the following two axioms:

H1 For all objects A, B in $\mathcal{C}$, $\mathrm{Hom}(A, B)$ is a transitive $\mathrm{Aut}(B)$-set under post-composition.

H2 For all objects A, B in $\mathcal{C}$, the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \otimes B)$ taking f to $f \otimes B$ is injective with image $\mathrm{stab}(\iota_A \otimes B)$.

2.1.1 Introducing a Braiding

Definition 2.7. A Braiding for a monoidal category $(\mathcal{C}, \otimes, e)$ consists of a family of isomorphisms

$$b_{A,B} : A \otimes B \to B \otimes A$$

natural in A and $B \in \mathcal{C}$, which satisfy for e the commutativity $b_{A,e} = b_{e,A} = \mathbb{1}_A$, and which, with the associativity, make both the following diagrams commute

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\quad b_{A \otimes B, C} \quad} & C \otimes A \otimes B \\ & {\scriptstyle \mathbb{1}_A \otimes b_{B,C}} \searrow \quad \nearrow {\scriptstyle b^{-1}_{C,A} \otimes \mathbb{1}_B} & \\ & A \otimes C \otimes B & \end{array} \tag{2.5}$$

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\quad b_{A, B \otimes C} \quad} & B \otimes C \otimes A \\ & {\scriptstyle b_{A,B} \otimes \mathbb{1}_C} \searrow \quad \nearrow {\scriptstyle \mathbb{1}_B \otimes b^{-1}_{C,A}} & \\ & B \otimes A \otimes C & \end{array} . \tag{2.6}$$

Definition 2.8. A symmetric monoidal category, is a monoidal category $(\mathcal{C}, \otimes, e)$, with a braiding b, such that $b_{B,A} \circ b_{A,B} = \mathbb{1}_{A \otimes B}$ for all objects $A, B \in \mathcal{C}$.

Definition 2.9. Let $(\mathcal{C}, \otimes, e)$ be a monoidal category with e initial. We say that $\mathcal{C}$ is pre-braided if its underlying groupoid is braided and for each pair of objects A and B in $\mathcal{C}$, the groupoid braiding $b_{A,B}$ satisfies

$$b_{A,B} \circ (A \otimes \iota_B) = \iota_B \otimes A : A \to B \otimes A \tag{2.7}$$

Recall that a groupoid is simply a category in which every arrow is invertible. In this way every category $\mathcal{C}$ has an underlying groupoid with objects the same objects of $\mathcal{C}$, and with morphisms simply the invertible arrows of $\mathcal{C}$. We will denote this category with $\mathrm{Iso}(\mathcal{C})$.

Remark 2.10. A pre-braided monoidal category is not necessarily a braided monoidal category (Naturality can fail).

2.1.2 The Semi-Simplicial Set associated to a pair

We have seen how the condition to be locally homogeneous at a couple, gives better insight about the action of $\mathrm{Aut}(A \otimes X^{\otimes n})$ on the set $\mathrm{Hom}(X^{\otimes p+1}, A \otimes X^{\otimes n})$. We want now to build a semi-simplicial set where we can use this action to gather informations about the groups $\mathrm{Aut}(A \otimes X^{\otimes n})$. To fix the notation we recall the definition of a semi-simplicial set

Definition 2.11. A semi-simplicial set consists of a set X_n $\forall n \geq 0$ (called the set of n-simplices), and $\forall i : 0 \leq i \leq n+1$, a face map $d_i : X_{n+1} \to X_n$ such that $d_i d_j = d_{j-1} d_i$ whenever $i < j$.

Remark 2.12. While in a simplicial complex a p-simplex is determined by an unordered set of $p+1$ distinct vertices, in a semi-simplicial set, every p-simplex has an ordered set of $(p+1)$ vertices not necessarily distinct (obtained by applying repeatedly the boundary maps). Moreover in a semi-simplicial set two distinct simplices may have the same set of vertices, while in a simplicial complex the list of vertices determines the simplex.

Definition 2.13. Let $(\mathcal{C}, \otimes, e)$ be a monoidal category with e initial and (A, X) a pair of objects in $\mathcal{C}$. Define $W_n(A, X)$ to be the semi-simplicial set with set of p-simplices

$$W_n(A, X)_p := \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$$

and with face map

$$d_i : \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p}, A \otimes X^{\otimes n}) \tag{2.8}$$

defined by pre-composing with $X^{\otimes i} \otimes \iota_X \otimes X^{\otimes p-i}$.

Remark 2.14. Given a p-simplex $f \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$, from the definition of the boundary maps we can obtain easily the $p+1$ vertices of the simplex. Define $\iota_j : X \to X^{\otimes p+1}$ for $0 \leq j \leq p$ to be the inclusion into the j-factor:

$$\iota_j := \iota_{X^{\otimes j}} \otimes X \otimes \iota_{X^{\otimes p-j}} : X \to X^{\otimes p+1}.$$

Then the j-vertex of the simplex f is simply given by the composition $f \circ \iota_j : X \to A \otimes X^{\otimes n}$.

Post-composition in $\mathcal{C}$ defines a simplicial action of the group $\mathrm{Aut}(A \otimes X^{\otimes n})$ on $W_n(A, X)$, and condition LH1 simply says that this action is transitive on p-simplices for $0 \leq p < n$. In this context we will call our preferred morphism σ_p (defined in (2.2)) the standard p-simplex of $W_n(A, X)$. The importance of the standard p-simplex in our work will be clear later, and for the moment we will only make some observations.

Remark 2.15. Consider the stabilisation sequence associated to a pair (A, X) (see Definition 2.4). The composition of the stabilisation maps $G_{n-p-1} \hookrightarrow G_n$ gives the map of condition LH2, which takes $f \in G_{n-p-1}$ to $f \otimes X^{\otimes p+1}$. In this way condition LH2 says exactly that this map takes G_{n-p-1} isomorphically into $\mathrm{stab}(\sigma_p) \subseteq G_n$ (for $0 \leq p < n$), considering the action of G_n just defined.

Remark 2.16. Let us analyse the boundary of the standard p-simplex for $0 < p < n$. From the definition of the face maps (2.8) we obtain

$$\begin{aligned} d_i(\sigma_p) &= (\iota_{A \otimes X^{\otimes n-p-1}} \otimes X^{\otimes p+1}) \circ (X^{\otimes i} \otimes \iota_X \otimes X^{\otimes p-i}) \\ &= \iota_{A \otimes X^{\otimes n-p-1}} \otimes X^{\otimes i} \otimes \iota_X \otimes X^{\otimes p-i}, \end{aligned} \tag{2.9}$$

for $0 \leq i \leq p$. In particular $d_0(\sigma_p) = \sigma_{p-1}$, but also the other faces are not so different from σ_{p-1}: they simply differ from σ_{p-1} for the position of the term ι_X in (2.9), which in any case "sits" in one of the last $p+1$ spots. If in addition $\mathcal{C}$ is pre-braided we have an easy way to switch the position of ι_X. If $0 < i \leq p$, the condition (2.7) ensures that

$$b_{X^{\otimes i}, X} \circ (X^{\otimes i} \otimes \iota_X) = \iota_X \otimes X^{\otimes i}.$$

Hence if we set

$$h_i := A \otimes X^{\otimes n-p-1} \otimes b_{X^{\otimes i}, X} \otimes X^{\otimes p-i} \in \mathrm{Aut}_{\mathcal{C}}(A \otimes X^{\otimes n}) \tag{2.10}$$

we have $h_i \circ d_i(\sigma_p) = \sigma_{p-1}$. Another feature of these elements h_i arises if $\mathcal{C}$ is locally homogeneous at the pair (A, X). From the Remark 2.15 we know that every element in $\mathrm{stab}(\sigma_p)$ can be written in the form $f \otimes X^{\otimes p+1}$ with $f \in \mathrm{Aut}_{\mathcal{C}}(A \otimes X^{\otimes n-p-1})$, as a consequence they all commute with these h_i, since they act on disjoint part:

$$(f \otimes X^{\otimes p+1}) \circ h_i = f \otimes b_{X^{\otimes i},X} \otimes X^{\otimes p-i} = h_i \circ (f \otimes X^{\otimes p+1}).$$

Remark 2.17. In general the stabiliser of a simplex fixes the simplex pointwise. Indeed suppose $g \in \mathrm{Aut}(A \otimes X^{\otimes n})$ fixes the simplex f, then $g \circ f = f$, and we have that $g \circ f \circ \iota_j = f \circ \iota_j$. Fixing the vertices of f, g fixes f pointwise.

We can now formulate a connectivity condition depending on a parameter $k \in \mathbb{N}$ that this semi-simplicial sets can satisfy. We will call it "connectivity axiom", and we will see later how strongly it is related to the homological stability problem for the groups $\mathrm{Aut}(A \otimes X^{\otimes n})$.

Definition 2.18. Let $(\mathcal{C}, \otimes, e)$ be a monoidal category and (A, X) a pair of objects in $\mathcal{C}$. We say that $\mathcal{C}$ satisfies LH3 at (A, X) with slope $k \in \mathbb{N}$ if

LH3 For all $n \geq 1$, $|W_n(A, X)|$ is $\left(\frac{n-2}{k}\right)$-connected.

Remark 2.19. For the sake of clarity we fix some conventions about connectivity. A non-empty space X is said to be m-connected if $\pi_i(X) = 0$ for all integers i such that $0 \leq i \leq m$, and all basepoints. A non-empty space is always at least (-1)-connected, while we will consider the empty-space (-2)-connected.

2.2 The Main Homological Stability Theorem

In this section we use Quillen's classical argument [14] in the case of general linear groups to show that in a locally homogeneous category in which the semi-simplicial sets $W_n(A, X)$ are highly-connected, the stabilisation sequence associated to the pair (A, X) satisfy homological stability with constant coefficients $\mathbb{Z}$. The precise statement is the following.

Theorem 2.20. Let $(\mathcal{C}, \otimes, e)$ be a pre-braided category locally homogeneous at a pair of objects (A, X). Suppose that $\mathcal{C}$ satisfies LH3 at (A, X) with slope $k \geq 2$. Then the map

$$H_i(\mathrm{Aut}(A \otimes X^{\otimes n})) \xrightarrow{H_i(\Sigma^X)} H_i(\mathrm{Aut}(A \otimes X^{\otimes n+1})) \tag{2.11}$$

is an epimorphism if $n \geq ik$ and an isomorphism if $n \geq ik + 1$.

Proof. Let us denote $G_n := \mathrm{Aut}(A \otimes X^{\otimes n})$ and $W_n := W_n(A, X)$, we will work on the general index $n + 1$ because admits an easier development of the computation. We begin recalling and making a number of observations about the set-up of the theorem:

(1) The semi-simplicial set W_{n+1} is $\left(\frac{n-1}{k}\right)$-connected. This holds by LH3.

(2) The action of G_{n+1} on W_{n+1} is transitive on p-simplices for each $p \leq n$ and the stabiliser of each simplex fixes the simplex pointwise (Remark 2.17).

(3) The map $\_\otimes X^{\otimes p+1} : G_{n-p} \to G_{n+1}$ is an isomorphism onto the stabiliser of the standard p-simplex σ_p of W_{n+1} (Remark 2.15).

We will construct a double complex and we will gather informations about the two spectral sequences associated to it. Then we will conclude proving homological stability by induction on the homological dimension i.

Step1: The two spectral sequences argument. For $G = G_{n+1}$ let F_* be a free resolution of $\mathbb{Z}$ in $\mathbb{Z}G$-modules, and let

$$\cdots \to C_p \to C_{p-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

be the augmented simplicial chain complex of $W = W_{n+1}$. The action of G on W makes C_* a complex of $\mathbb{Z}G$-modules, so we can consider the double complex

$$\tilde{C}_{**} = C_* \otimes_G F_*. \tag{2.12}$$

First of all $\tilde{C}$ is a first quadrant double complex, since $C_p \otimes_G F_q = 0$ if $p < -1$ or $q < 0$. As a consequence we can apply the machinery developed in Subsection 1.3.2. We will filter the double complex "horizontally" and then "vertically", obtaining two different spectral sequences both converging to $H_*(T\tilde{C})$.

Let us begin filtering the double complex "horizontally". As stated in the recalled Subsection we obtain a spectral sequence $\{\bar{E}^r_{*,*}, d^r\}$ converging to $H_*(T\tilde{C})$ with

$$\begin{aligned}\bar{E}^0_{p,q} &= \tilde{C}_{q,p} = C_q \otimes_G F_p,\\ \bar{E}^1_{p,q} &= H_q(\tilde{C}_{*,p}) = H_q(C_* \otimes_G F_p).\end{aligned}$$

If we assume W is $c(W)$-connected (In our case $c(W) = \frac{n-1}{k}$), then the complex C_* is exact through dimension $c(W)$. Since F_p is free, $C_* \otimes_G F_p$ is exact in the same range so $\bar{E}^1_{p,q}$ for $q \leq c(W)$. Combining this information with the fact that $\bar{E}^0_{p,q} = 0$ for $p < 0$ we obtain that $\bar{E}^\infty_{p,q} = 0$ for $p + q \leq c(W)$. With this computation we can say something about the abutment of the spectral sequence: $H_*(T\tilde{C})$. For $p + q \leq c(W)$ we have

$$0 \cong \bar{E}^\infty_{p,q} \cong F_p H_{p+q}(T\tilde{C}) / F_{p-1} H_{p+q}(T\tilde{C}),$$

so $F_p H_{p+q}(T\tilde{C}) = F_{p-1} H_{p+q}(T\tilde{C})$ whenever $p + q \leq c(W)$, which implies $H_{p+q}(T\tilde{C}) = 0$ for $p + q \leq c(W)$.

Let now filter the double complex $\tilde{C}$ "vertically". We obtain a second spectral sequence $\{E^r_{*,*}, d^r\}$ with the same abutment as the first one. Since we know from the first one that $H_{p+q}(T\tilde{C}) = 0$ for $p + q \leq c(W)$, we must have that $E^\infty_{p,q} = 0$ in the same range of degrees. From Subsection 1.3.2 we have

$$\begin{aligned} E^0_{p,q} &= \tilde{C}_{p,q} = C_p \otimes_G F_q, \\ E^1_{p,q} &= H_q(\tilde{C}_{p,*}) = H_q(C_p \otimes_G F_*) = H_q(G; C_p), \end{aligned}$$

and the differential $d^1 = (\mathrm{Id}_G, \partial)_*$ (see Subsection 1.2.3) is the map induced in homology by the boundary map ∂ of the complex C_*. Observe now that since the action of $G = G_{n+1}$ on $W = W_{n+1}$ is transitive on p-simplices for each $0 \leq p \leq n$, we are exactly in the situation described in the Example 1.11. Then choosing the standard p-simplex σ_p as representative of its orbit, the equation (1.4) becomes

$$C_p \cong \mathrm{Ind}^G_{\mathrm{stab}(\sigma_p)}(\mathbb{Z}_{\sigma_p}).$$

Moreover as observed in (2), the stabiliser of a simplex fixes the simplex pointwise. This implies that every element in $\mathrm{stab}(\sigma_p)$ preserves the orientation of σ_p, and so $\mathbb{Z}_{\sigma_p}$ as $\mathrm{stab}(\sigma_p)$-module is simply $\mathbb{Z}$ with trivial action. In this way as we stated in Example 1.10, C_p is simply a permutation module

$$C_p \cong \mathrm{Ind}^G_{\mathrm{stab}(\sigma_p)} \mathbb{Z} \cong \mathbb{Z}\left[G / \mathrm{stab}(\sigma_p) \right]. \tag{2.13}$$

Combining this observation with Shapiro's Lemma (Proposition 1.19), for $0 \leq p \leq n$ we obtain that the natural inclusion

$$(\mathrm{stab}(\sigma_p), \mathbb{Z}) \hookrightarrow (G, \mathrm{Ind}^G_{\mathrm{stab}(\sigma_p)} \mathbb{Z})$$

induces an isomorphism in homology

$$H_q(\mathrm{stab}(\sigma_p);\mathbb{Z}) \cong H_q(G;\mathrm{Ind}^G_{\mathrm{stab}(\sigma_p)}\mathbb{Z}) \cong H_q(G;C_p).$$

Then Considering σ_{-1} to be the empty set, with stabiliser the whole group G_{n+1} the E^1-page takes the form:

$$E^1_{p,q} = H_q(\mathrm{stab}(\sigma_p)) \quad \text{for} \quad -1 \leq p \leq n.$$

We will be interested only in degrees $p+q \leq c(W) = \frac{n-1}{k}$(for which $E^\infty_{p,q} = 0$), so shall ignore the fact that the above holds only for $p \leq n$.

By observation (3) we have a preferred isomorphism of $\mathrm{stab}(\sigma_p)$ with G_{n-p}, but we want also to keep track of the d^1 differential after all these isomorphisms of $E^1_{p,q}$. To do so, recall that it is induced by the boundary map $\partial_p : C_p \to C_{p-1}$, which is the alternating sum of the maps d_i as defined in (2.8). Using the last form in (2.13), we need only to choose an element $h_i \in G_{n+1}$ which takes $d_i\sigma_p$ to σ_{p-1}. Then the d^1 differential is induced by the alternating sum of the maps

$$H_q(\mathrm{stab}(\sigma_p)) \xrightarrow{(\alpha_i)_*} H_q(\mathrm{stab}(d_i\sigma_p)) \xrightarrow{(c_{h_i})_*} H_q(\mathrm{stab}(\sigma_{p-1})), \tag{2.14}$$

where $\alpha_i : \mathrm{stab}(\sigma_p) \hookrightarrow \mathrm{stab}(d_i\sigma_p)$ is the inclusion , and c_{h_i} is the conjugation by the element h_i: $c_{h_i}(g) = h_i g h_i^{-1}$.

For $p = 0$ we don't have the map induced by conjugation, so

$$d = d^1 : E^1_{0,i} = H_i(G_n) \to E^1_{-1,i} = H_i(G_{n+1})$$

is simply the map in homology induced by the inclusion of a vertex stabiliser into the whole group, and it identifies with the map in the statement of the theorem by observation (3).

For $p \geq 1$ we can choose the elements h_i as stated in Remark 2.16 (remember to use $n+1$ instead of n). In this way the map

$$c_{h_i} \circ \alpha_i : \mathrm{stab}(\sigma_p) \to \mathrm{stab}(\sigma_{p-1})$$

takes the element $g \in \mathrm{stab}(\sigma_p)$ to the element $h_i g h_i^{-1} = g$, since h_i commutes with every element in $\mathrm{stab}(\sigma_p)$. As a consequence the map 2.14 is independent from i and always induced by the inclusion $\iota : \mathrm{stab}(\sigma_p) \hookrightarrow \mathrm{stab}(\sigma_{p-1})$. Then

$$d^1 = \sum_{i=0}^{p} (-1)^i \iota_* : H_*(\mathrm{stab}(\sigma_p)) \to H_*(\mathrm{stab}(\sigma_{p-1})) \tag{2.15}$$

is the zero map when p is odd, and can be identified with the stabilisation map $H_q G_{n-p} \to H_q G_{n-p+1}$ when p is even.

Step2: Inductive argument for surjectivity. In the situation we are considering, the E^1-page of the spectral sequence has the following form:

$$\begin{array}{c|cccccccc}
i & H_i(G_{n+1}) & \xleftarrow{d} & H_i(G_n) & \xleftarrow{0} & H_i(G_{n-1}) & \longleftarrow & \cdots \\
i-1 & H_{i-1}(G_{n+1}) & \leftarrow & H_{i-1}(G_n) & \xleftarrow{0} & H_{i-1}(G_{n-1}) & \leftarrow & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \\
q=0 & H_0(G_{n+1}) & \longleftarrow & H_0(G_n) & \xleftarrow{0} & H_0(G_{n-1}) & \longleftarrow & \cdots \\
\hline
 & p=-1 & & 0 & & 1 & & \cdots
\end{array}$$

We want to show that the map d is surjective when $n \geq ki$ and injective when $n \geq ki+1$. We prove this by induction on the homological dimension i, so consider the statements

($\mathbf{S}_I$) The map d is surjective for $i \leq I$ and $n \geq ki$.

($\mathbf{I}_I$) The map d is an isomorphism for $i \leq I$ and $n \geq ki+1$.

The statement (S_0) holds trivially. Indeed we have $i = 0$ and $n \geq 0$, so $\frac{n-1}{k} \geq -1$ and we must have that $E^\infty_{-1,0} = 0$. But the only differential that can kill it is d^1, so d must be surjective.

To prove (I_0) simply notice that for $i = 0$ and $n \geq 1$ we have $\frac{n-1}{k} \geq 0$, so $E^\infty_{-1,0} = E^\infty_{0,0} = 0$. This implies surjectivity of d, and exactness in $E^1_{0,0}$, since the d^1 differential is the only one capable of killing that term. Now since $d^1 : E^1_{1,0} \to E^1_{0,0}$ is the zero map as stated at the end of the previous step, then d must be also injective.

We start by showing the implication $(\mathrm{S}_{I-1}) + (\mathrm{I}_{I-1}) \Rightarrow (\mathrm{S}_I)$. So let $i \leq I$ and $n \geq ki$. Surjectivity of d follows from:

(S$_I$1) $E^{\infty}_{-1,i} = 0$.

(S$_I$2) $E^2_{p,q} = 0$ for $p+q=i$ and $q<i$.

Indeed (S$_I$1) says that $E^1_{-1,i}$ has to be killed before E^∞, and condition (S$_I$2) says that the sources of all possible differentials to that term after d^1 are 0, and therefore d^1 is the only differential that can kill it, so d must be surjective.

Condition (S$_I$1) holds because $E^{\infty}_{p,q} = 0$ when $p+q \leq \frac{n-1}{k}$, and $i-1 \leq \frac{n}{k} - 1 \leq \frac{n-1}{k}$ when $n \geq ki$ and $k \geq 1$.

In order to prove (S$_I$2) we begin by showing that for $p+q=i$, with $q<i$ the map induced by inclusion of stabilisers

$$H_q(\mathrm{stab}(\sigma_j)) \xrightarrow{\iota_*} H_q(G_{n+1}) \tag{2.16}$$

is an isomorphism when $j \leq p$, and an epimorphism if $j = p+1$. This map can be written as a composition of stabilisation maps

$$H_q(G_{n-j}) \to H_q(G_{n-j+1}) \to \cdots \to H_q(G_{n+1}) \tag{2.17}$$

as a consequence of observation (3). Since $q \leq I-1$, each one of these maps is an epimorphism when $n-j \geq kq$, and an isomorphism when $n-j \geq kq+1$, due to (S$_{I-1}$) and (I$_{I-1}$) respectively. For $j=-1$ there is nothing to prove, and for $j=0$ we have $n \geq ki \geq kq+1$, so (2.17) reduces to a single map, which is an isomorphism by (I$_{I-1}$). Otherwise we can assume $j \geq 2$, and since $k \geq 2$ by hypothesis, we have the inequalities

$$n \geq ki \geq kp + kq \geq kj + kq \geq j + kq + 1.$$

Bringing j to the other side gives $n-j \geq kq+1$, so the maps (2.17) are all isomorphisms. The argument for the case $j=p+1$ is similar. Indeed

$$j + q = i + 1 \leq \frac{n}{k} + 1 = \frac{n+k}{k},$$

which implies $n+k \geq kj+kq$, and so $n + k(1-j) \geq kq$. Observe also that $j \geq 2$, because $q<i$ and $j+q=i+1$, so we have $\frac{j}{(j-1)} \leq 2 \leq k$, which implies $-j \geq k(1-j)$. Combining these two computations we get

$$n - j \geq n + k(1-j) \geq kq,$$

which shows that the maps (2.17) are all epimorphisms.

Let us consider now the diagram

$$\begin{array}{ccccc} H_q(\mathrm{stab}(\sigma_{p-1})) & \xleftarrow{d^1} & H_q(\mathrm{stab}(\sigma_p)) & \xleftarrow{d^1} & H_q(\mathrm{stab}(\sigma_{p+1})) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ H_q(G_{n+1}) & \xleftarrow{0} & H_q(G_{n+1}) & \xleftarrow{\mathrm{Id}} & H_q(G_{n+1}) \end{array} \tag{2.18}$$

where the vertical maps are all induced by inclusion of stabilisers (2.16), and the top row is a "piece" taken from the E^1-page. As stated at the end of the previous step, the top horizontal maps are alternately the zero map and the map induced by inclusion of stabilisers, which on the bottom line correspond to the zero map and the identity map. In (2.18) is illustrated the case when p is odd. The left square commutes because we have the zero map on the top and on the bottom line. The right square commutes because all the maps involved are induced by subgroup inclusions. The case p even is identical, simply switching the two squares in (2.18). In both cases the bottom sequence is exact in the middle since alternates identities and zero maps. By the previous paragraph the vertical map on the left and the one in the middle are isomorphisms, while the right one is an epimorphism. This implies that also the top line is exact in the middle, and in particular we have that $E^2_{p,q} = 0$ when $p + q = i$ and $q < i$.

Step3: Inductive argument for injectivity. To conclude the proof of the Theorem it remains only to show that $(\mathrm{S}_I) + (\mathrm{I}_{I-1}) \Rightarrow (\mathrm{I}_I)$. So assume $i \leq I$ and $n \geq ki + 1$. As before we will prove the two sentences

($\mathbf{I}_I$1) The term $E^\infty_{0,i} = 0$;

($\mathbf{I}_I$2) The term $E^2_{p,q} = 0$ when $p + q = i + 1$ and $q < i$;

since together imply (I_I). Indeed condition $(\mathrm{I}_I 1)$ says that $E^1_{0,i}$ has to be killed before E^∞, and condition $(\mathrm{I}_I 2)$ says that the sources of all possible differentials to that term after d^1 are 0. Therefore d^1 is the only differential that can kill it. But from the end of the first step we know that $d^1 : E^1_{1,i} \to E^1_{0,i}$ is the zero map. So necessarily d must be injective to kill $E^1_{0,i}$.

To prove $(\mathrm{I}_I 1)$ and $(\mathrm{I}_I 2)$ the same argument as in Step2 works. Indeed for $(\mathrm{I}_I 1)$ we need only $i \leq \frac{n-1}{k}$ which we have assumed. For $(\mathrm{I}_I 2)$ the vertical map on the left and the one in the middle of (2.18) are isomorphisms when $n - p \geq kq + 1$, which holds if $p + q = i + 1$ with $q < i$ and $n \geq ki + 1$. While the right vertical map is an epimorphism when $n - p - 1 \geq kq$, which holds under the same conditions. □

2.3 Starting from a groupoid

Now that we have developed all these techniques in the setting of locally homogeneous categories, one can ask how to apply them if we start with a family of groups which doesn't fit already into a locally homogeneous category. So this section is devoted to the construction of a locally homogeneous category starting from a monoidal groupoid. The main tool here will be Quillen's construction of a category $\langle \mathcal{G}, \mathcal{G} \rangle$ (see [7]), which we will indicate $U\mathcal{G}$, following the notation of [18].

So let $(\mathcal{G}, \otimes, e)$ be a monoidal Groupoid, and define $U\mathcal{G}$ to be the category with the same objects of $\mathcal{G}$, but with different morphisms. A morphism in $U\mathcal{G}$ from A to B is an equivalence class of pairs (X, f) where X is an object of $\mathcal{G}$ and $f : X \otimes A \to B$ is a morphism in $\mathcal{G}$, and where $(X, f) \sim (X', f')$ if there exists an isomorphism $g : X \to X'$ in $\mathcal{G}$ making the diagram

$$\begin{array}{ccc} X \otimes A & \xrightarrow{f} & B \\ {\scriptstyle g \otimes A} \downarrow & \nearrow {\scriptstyle f'} & \\ X' \otimes A & & \end{array} \tag{2.19}$$

commute. We will use the notation $[X, f]$ for such an equivalence class. The composition of two morphisms $[X, f] \in \mathrm{Hom}_{U\mathcal{G}}(A, B)$, and $[Y, g] \in \mathrm{Hom}_{U\mathcal{G}}(B, C)$ is defined as

$$[Y, g] \circ [X, f] = [Y \otimes X, g \circ (Y \otimes f)].$$

2.3.1 Preserving the automorphisms

When using the construction for $U\mathcal{G}$, we will be interested in the relationship between the automorphism groups in $\mathcal{G}$ we start with, and those in $U\mathcal{G}$. With particular attention to the cases where they remain unchanged. First of all notice that we have a functor $F : \mathcal{G} \to \mathrm{Iso}(\mathcal{G})$ taking an isomorphism f in $\mathcal{G}$ to the class $[e, f]$. The following proposition gives us conditions that ensures F to be full and faithful.

Remark 2.21. Recall that having no zero divisors for a monoidal category $\mathcal{G}$

means that the following implication is true for objects in $\mathcal{G}$

$$U \otimes V \cong e \Longrightarrow U \cong V \cong e.$$

Proposition 2.22. If $(\mathcal{G}, \otimes, e)$ is a monoidal groupoid, then the following holds:

(i) If $\mathrm{Aut}_{\mathcal{G}}(e) = \{\mathbb{1}_e\}$, the functor F is faithful.

(ii) If $\mathcal{G}$ has no zero divisors, the functor F is full.

in particular if both conditions hold, $\mathcal{G}$ is the underlying groupoid of $U\mathcal{G}$.

Proof. For (i), if $[e, f] = [e, g]$ in $U\mathcal{G}$, we have that there is an isomorphism $\phi : e \to e$ in $\mathcal{G}$ such that $g = f \circ (\phi \otimes A)$. Since $\mathrm{Aut}_{\mathcal{G}}(e) = \{\mathbb{1}_e\}$, we must have that $\phi = \mathbb{1}_e$ and hence that $f = g$.

For (ii) suppose that $[X, f] \in \mathrm{Hom}_{U\mathcal{G}}(A, B)$ is an isomorphism, with inverse $[Y, g] \in \mathrm{Hom}_{U\mathcal{G}}(B, A)$. This means that

$$[Y, g] \circ [X, f] = [Y \otimes X, g \circ (Y \otimes f)] = [e, \mathbb{1}_e].$$

So in particular $Y \otimes X \cong e$ in $\mathcal{G}$, which implies $Y \cong X \cong e$ in $\mathcal{G}$, because there are no zero divisors in $\mathcal{G}$. Choosing an isomorphism $\phi : e \to X$, we get $[X, f] = [e, f \circ \phi]$, which shows that $[X, f]$ is in the image of the functor. □

2.3.2 Extending the monoidal structure

Starting with a monoidal groupoid, we want to extend this structure at least to $U\mathcal{G}$, preserving the properties we have seen useful in the previous sections. This is a proposition in this direction.

Proposition 2.23. Let $(\mathcal{G}, \otimes, e)$ be a monoidal groupoid, then:

(i) The object e is initial in $U\mathcal{G}$

(ii) If $\mathcal{G}$ is braided monoidal with no zero divisors then $U\mathcal{G}$ is a pre-braided monoidal category.

(iii) If $\mathcal{G}$ is symmetric monoidal then $U\mathcal{G}$ is a symmetric monoidal category.

Proof. We first show that the unit e is initial in $U\mathcal{G}$. If $[X, f]$ and $[Y, g]$ are two elements of $\mathrm{Hom}_{U\mathcal{G}}(e, A)$, then $g^{-1} \circ f : X \to Y$ is an isomorphism such that the following diagram

$$\begin{array}{ccc} X \otimes e & \xrightarrow{f} & A \\ {\scriptstyle g^{-1}\circ f}\downarrow & \nearrow_{g} & \\ Y \otimes e & & \end{array} \tag{2.20}$$

commutes, which implies that $(X, f) \sim (Y, g)$ represent the same element of $\mathrm{Hom}_{U\mathcal{G}}(e, A)$.

Assuming now that $\mathcal{G}$ is braided monoidal, we define the monoidal structure on $U\mathcal{G}$ as follows: we can use on objects exactly the same monoidal structure of $\mathcal{G}$, while given $[X, f] \in \mathrm{Hom}(A, B)$ and $[Y, g] \in \mathrm{Hom}(C, D)$, we define

$$[X, f] \otimes [Y, g] := [X \otimes Y, (f \otimes g) \circ (X \otimes b_{A,Y}^{-1} \otimes C)] \in \mathrm{Hom}(A \otimes C, B \otimes D).$$

We have that $\mathcal{G}$ can be seen in $U\mathcal{G}$ through the functor F already defined, and in this sense we have that the monoidal structure on $\mathcal{G}$ is compatible with the one defined on $U\mathcal{G}$. Indeed if $[e, f] \in \mathrm{Hom}(A, B)$ and $[e, g] \in \mathrm{Hom}(C, D)$, then

$$[e, f] \otimes [e, g] = [e \otimes e, (f \otimes g) \circ (\mathbb{1}_e \otimes b_{A,e}^{-1} \otimes C)] = [e, f \otimes g]$$

because in a braided monoidal category we have $b_{A,e} = A$. With this monoidal product, $(U\mathcal{G}, \otimes, e)$ is a monoidal category: the only nontrivial condition to check is the associativity for arrows, which follows from the compatibility conditions (2.5), (2.6) on $\mathcal{G}$. Indeed given $[X, f] \in \mathrm{Hom}(A, B)$, $[Y, g] \in \mathrm{Hom}(C, D)$ and $[W, h] \in \mathrm{Hom}(E, F)$, let's check that

$$([X, f] \otimes [Y, g]) \otimes [W, h] = [X, f] \otimes ([Y, g] \otimes [W, h])\,. \tag{2.21}$$

The left hand side is equal to

$$\Big[X \otimes Y \otimes W, (f \otimes g \otimes h) \circ (X \otimes A \otimes Y \otimes b_{C,W}^{-1} \otimes E) \circ (X \otimes b_{A,(Y\otimes W)}^{-1} \otimes C \otimes E)\Big]$$

which we can write in a more concise form omitting the symbol $\otimes$ in the monoidal products between objects, and also omitting completely the monoidal products with identity morphisms:

$$\Big[XYW, (f \otimes g \otimes h) \circ b_{C,W}^{-1} \circ b_{A,(Y\otimes W)}^{-1}\Big]\,.$$

In the same way the right hand side is equal to

$$\left[XYW, (f \otimes g \otimes h) \circ b^{-1}_{A,Y} \circ b^{-1}_{(A\otimes C),W}\right],$$

so (keeping this notation with identity morphisms omitted) we have only to check that

$$b^{-1}_{C,W} \circ b^{-1}_{A,(Y\otimes W)} = b^{-1}_{A,Y} \circ b^{-1}_{(A\otimes C),W}$$

as elements in $\mathrm{Hom}_{\mathcal{G}}(XYWACE, XAYCWE)$, but now simply using (2.5), and (2.6) we have

$$b_{A,(Y\otimes W)} = b^{-1}_{W,A} \circ b_{A,Y}$$
$$b_{(A\otimes C),W} = b^{-1}_{W,A} \circ b_{C,W}.$$

The functor $F : \mathcal{G} \to \mathrm{Iso}(U\mathcal{G})$ is a bijection on objects, and since $\mathcal{G}$ has no zero divisors, applying Proposition 2.22 we get also that the functor is full. In view of the fact that in this way we can reach every isomorphism of $U\mathcal{G}$, we get that the braided monoidal structure on $\mathcal{G}$ induces one on $\mathrm{Iso}(U\mathcal{G})$. In order to prove that $U\mathcal{G}$ is pre-braided, remains only to check that

$$b_{A,B} \circ (A \otimes \iota_B) = \iota_B \otimes A,$$

but this follows from the computation

$$\begin{aligned} b_{A,B} \circ (A \otimes \iota_B) &= [e, b_{A,B}] \circ ([e, \mathbb{1}_A] \otimes [B, \mathbb{1}_B]) \\ &= [e, b_{A,B}] \circ [B, \mathbb{1}_{A\otimes B} \circ b^{-1}_{A,B}] = [B, \mathbb{1}_{A\otimes B}] \end{aligned}$$

$$\iota_B \otimes A = [B, \mathbb{1}_B] \otimes [e, \mathbb{1}_A] = [B, \mathbb{1}_{A\otimes B} \circ b^{-1}_{e,e}] = [B, \mathbb{1}_{A\otimes B}].$$

Suppose now that $\mathcal{G}$ is symmetric monoidal. On $U\mathcal{G}$ we can use the same braiding as $\mathcal{G}$ since they have the same objects. Moreover the only nontrivial condition to check in order to prove that $U\mathcal{G}$ is symmetric monoidal is naturality in both arguments of the braiding. Namely given $[X, f] \in \mathrm{Hom}(A, B)$ and $[Y, g] \in \mathrm{Hom}(C, D)$ we have to show

$$([Y, g] \otimes [X, f]) \circ [e, b_{A,C}] = [e, b_{B,D}] \circ ([X, f] \otimes [Y, g]).$$

The left hand side is

$$[Y \otimes X, (g \otimes f) \circ (Y \otimes b_{C,X}^{-1} \otimes A) \circ (Y \otimes X \otimes b_{A,C})]$$

and the right hand side is

$$[X \otimes Y, b_{B,D} \circ (f \otimes g) \circ (X \otimes b_{A,Y}^{-1} \otimes C)].$$

Notice now that $b_{X,Y} : X \otimes Y \to Y \otimes X$ defines an isomorphism between the complements of these two morphisms. The fact that they represent the same morphism corresponds to the commutativity of the following diagram:

$$\begin{array}{ccc}
XYAC & \xrightarrow{b_{X,Y}} & YXAC \\
{\scriptstyle b_{A,Y}^{-1}}\downarrow & & \downarrow{\scriptstyle b_{C,X}^{-1} \circ b_{A,C}} \\
XAYC & \xrightarrow{b_{X\otimes A, Y\otimes C}} & YCXA \\
{\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\
BD & \xrightarrow{b_{B,D}} & DB
\end{array} .$$

The bottom square commutes for naturality of the braiding b in $\mathcal{G}$. Let us prove that the top square commutes. Using (2.5) and (2.6) we have that

$$b_{C,X}^{-1} \circ b_{A,C} = b_{X\otimes A,C},$$
$$b_{X\otimes A,Y\otimes C} = b_{C,X\otimes A}^{-1} \circ b_{X\otimes A,Y} = b_{X\otimes A,C} \circ b_{X\otimes A,Y}.$$

Where in the last one we use that b is a symmetry, so $b_{C,X\otimes A}^{-1} = b_{X\otimes A,C}$. After substituting these two arrows we are left to show only that $b_{X,Y} = b_{X\otimes A,Y} \circ b_{A,Y}^{-1}$. But using again that b is a symmetry we get exactly (2.5). □

2.3.3 Getting homogeneity

The aim of this construction is to obtain local homogeneity at couples of objects (A, X) in $U\mathcal{G}$, or more generally global homogeneity for the category $U\mathcal{G}$. So let us look for conditions on $\mathcal{G}$ which provides local or global homogeneity.

Definition 2.24. For a pair of objects (A, X) in a monoidal groupoid $(\mathcal{G}, \otimes, e)$ we say that $\mathcal{G}$ satisfies local cancellation at (A, X) if it satisfies

LC For all $0 \leq p < n$, if $Y \in \mathcal{G}$ is such that $Y \otimes X^{\otimes p+1} \cong A \otimes X^{\otimes n}$, then

$Y \cong A \otimes X^{\otimes n-p-1}$

Remark 2.25. The axiom LC implies in particular that morphisms $X^{\otimes p+1} \to A \otimes X^n$ in $U\mathcal{G}$ have unique complements, and this is one of the main way in which we shall use it.

Sometimes LC holds because our groupoid satisfies a stronger global condition which implies LC at each pair of objects.

Definition 2.26. We say that a monoidal groupoid $(\mathcal{G}, \otimes, e)$ satisfies cancellation if the following holds

C For all objects A, B, C in $\mathcal{G}$, if $A \otimes C \cong B \otimes C$ then $A \cong B$.

The following Theorem states that these two conditions are exactly what we are looking for.

Theorem 2.27. Let $(\mathcal{G}, \otimes, e)$ be a braided monoidal groupoid with no zero divisors, and $U\mathcal{G}$ its associated pre-braided category. Then

(a) The category $U\mathcal{G}$ satisfies LH1 at (A, X) if and only if $\mathcal{G}$ satisfies LC at (A, X).

(b) If the map $\mathrm{Aut}_{\mathcal{G}}(A \otimes X^{\otimes n-p-1}) \to \mathrm{Aut}_{\mathcal{G}}(A \otimes X^{\otimes n})$ taking f to $f \otimes X^{\otimes p+1}$ is injective for all $0 \leq p < n$, then $U\mathcal{G}$ satisfies LH2 at (A, X).

In particular if (a) and (b) are both satisfied then $U\mathcal{G}$ is locally homogeneous at (A, X). The corresponding global results are:

(c) The category $U\mathcal{G}$ satisfies H1 if and only if $\mathcal{G}$ satisfies C.

(d) If for all objects A,B in $\mathcal{G}$ the map $\mathrm{Aut}_{\mathcal{G}}(A) \to \mathrm{Aut}_{\mathcal{G}}(A \otimes B)$ taking f to $f \otimes B$ is injective, then $U\mathcal{G}$ satisfies H2.

In particular if (c) and (d) are both satisfied then $U\mathcal{G}$ is homogeneous.

Proof. Let's start with (a). Suppose that $\mathcal{G}$ satisfies LC at (A, X) and let $[U, f], [V, g] \in \mathrm{Hom}_{U\mathcal{G}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$. As we noticed earlier in the Remark 2.25, since f and g are isomorphisms, by two applications of LC we have that U and V are isomorphic. Choose $\phi : U \xrightarrow{\sim} V$, then $[V, g] = [U, g \circ (\phi \otimes X^{\otimes p+1})]$, thanks to the same isomorphism ϕ, and moreover from the last one we can obtain $[U, f]$ simply post-composing with $[e, f \circ (g \circ (\phi \otimes X^{\otimes p+1})^{-1})] \in \mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n})$:

$$[e, f \circ (g \circ (\phi \otimes X^{\otimes p+1}))^{-1}] \circ [U, g \circ (\phi \otimes X^{\otimes p+1})] = [U, f],$$

which proves LH1.

Conversely assume that $U\mathcal{G}$ satisfies LH1 and that we have an isomorphism $\phi : Y \otimes X^{\otimes p+1} \xrightarrow{\sim} A \otimes X^{\otimes n}$. We can consider $[Y, \phi]$ as an element of $\mathrm{Hom}_{U\mathcal{G}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$, but also $[A \otimes X^{\otimes n-p-1}, \mathbb{1}_{A\otimes X^{\otimes n}}]$ is an element in that set. By LH1, there exists an automorphism $[U, \psi]$ of $A \otimes X^{\otimes n}$ such that

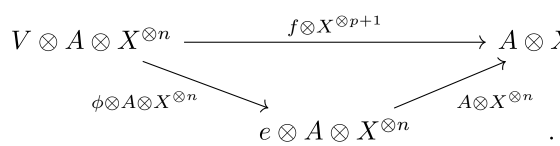
$$[U, \psi] \circ [Y, \phi] = [A \otimes X^{\otimes n-p-1}, \mathbb{1}_{A\otimes X^{\otimes n}}].$$

Since $\mathcal{G}$ has no zero divisors we can apply Proposition 2.22. Then exists $\psi' \in \mathrm{Aut}_{\mathcal{G}}(A \otimes X^{\otimes n})$ such that $[U, \psi] = [e, \psi']$. It follows that $[Y, \psi' \circ \phi] = [A \otimes X^{\otimes n-p-1}, \mathbb{1}_{A\otimes X^{\otimes n}}]$. But this implies that $Y \cong A \otimes X^{\otimes n-p-1}$, proving LC and finishing the proof of (a).

For (b) we must show that the map

$$\_ \otimes X^{\otimes p+1} : \mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n-p-1}) \to \mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n}) \tag{2.22}$$

is injective and identify its image. The map is a group homomorphism, then to check injectivity we can look at the kernel. Suppose that $[V, f] \in \mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n-p-1})$, is such that $[V, f] \otimes X^{\otimes p+1} = [V, f \otimes X^{\otimes p+1}]$ is the identity in $\mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n})$. This means that there exists an isomorphism $\phi : V \to e$ in $\mathcal{G}$ such that the following diagram commute



In other words $f \otimes X^{\otimes p+1} = \phi \otimes A \otimes X^{\otimes n-p-1} \otimes X^{\otimes p+1}$ in $\mathcal{G}$, where the hypothesis of injectivity implies $f = \phi \otimes A \otimes X^{\otimes n-p-1}$. But this means that $[V, f] = [e, A \otimes X^{\otimes n-p-1}]$ in $\mathrm{Aut}_{U\mathcal{G}}(A \otimes X^{\otimes n-p-1})$, which proves injectivity. It remains only to show that the image of the map (2.22) is $\mathrm{stab}(\sigma_p)$. For simplicity let's denote $U = A \otimes X^{\otimes n-p-1}$, $V = X^{\otimes p+1}$. We have

$$\iota_U \otimes V = [U, \mathbb{1}_U] \otimes [e, \mathbb{1}_V] = [U, \mathbb{1}_{U\otimes V}].$$

Moreover the usual functor F is full since we have no zero divisors, so we obtain

$$\mathrm{stab}(\iota_U \otimes V) = \{[e, \phi] \in \mathrm{Aut}_{U\mathcal{G}}(U \otimes V) \mid [e, \phi] \circ [U, \mathbb{1}_{U\otimes V}] = [U, \mathbb{1}_{U\otimes V}]\}$$

$$= \{[e, \phi] \in \mathrm{Aut}_{U\mathcal{G}}(U \otimes V) \mid [U, \phi] = [U, \mathbb{1}_{U \otimes V}]\}.$$

The last equality is equivalent to saying that $\mathrm{stab}(\iota_U \otimes V)$ consists of the morphisms $[e, \phi]$ such that there is an isomorphism $\psi : U \to U$ in $\mathcal{G}$ satisfying $\phi = \psi \otimes V$, which is exactly saying that $[e, \phi]$ is the image of $[e, \psi] \in \mathrm{Aut}_{U\mathcal{G}}(U)$.

For (c) and (d) the proof is identical. □

Chapter 3

Homological Instability

In this chapter we present some sequences of groups showing the following interesting pattern. We start with a braided monoidal groupoid which satisfies cancellation and everything is needed to apply Theorems 2.22, 2.23, 2.27, and get an associated pre-braided homogeneous category. Choosing a pair (A, X) in this category, we have seen in the previous chapter that homological stability for the associated stabilisation sequence follows if the $|W_n(A, X)|$ are highly connected. Here instead we will present a pair in the category for which homological stability of the associated stabilisation sequence fails. In this way we obtain a family of groups not stable even if it fits perfectly into the framework of pre-braided homogeneous category. In all the examples the failure is already at the level of H_1, but we present also examples very similar where the H_1 stabilises and for which we do not know if we have or not failure at a higher level. The interesting thing in these type of examples is that when stability fails dramatically (at the level of H_1) also the high-connectivity of the $|W_n(A, X)|$ fails dramatically (they are not even connected). While in the examples where there is no failure at the level of H_1, the $|W_n(A, X)|$ are connected.

3.1 Showing connectivity

In this section we will present a quick and nice criteria useful in many of our examples to check connectivity of $|W_n(A, X)|$.

Let $(\mathcal{C}, \otimes, e)$ be a pre-braided monoidal category, locally homogeneous at a pair of objects (A, X). Recall the nice description of the set of p-simplices we gave in Remark 2.5: we fix our preferred p-simplex $\sigma_p := \iota_{A\otimes X^{\otimes n-p-1}} \otimes X^{\otimes p+1}$ and write every other element $f \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n})$ as $f = \sigma \circ \sigma_p$ with $\sigma \in G_n$. In this way we have

$$\mathrm{Hom}_{\mathcal{C}}(X^{\otimes p+1}, A \otimes X^{\otimes n}) \cong G_n/G_{n-p-1}$$

where the isomorphism is obtained sending the element f in the corresponding class σG_{n-p-1}.

We now want to interpret in these terms the boundary maps of $W_n(A, X)$, at least for 1-simplices. Define then $t_n := A \otimes X^{n-2} \otimes b^{-1}_{X,X} \in G_n$ for $n \geq 2$, and consider a 1-simplex $\sigma G_{n-2} \in G_n/G_{n-2}$; from the definition of the first boundary map we have:

$$d_0(\sigma G_{n-2}) = \sigma \circ (\iota_{A\otimes X^{\otimes n-2}} \otimes X^{\otimes 2}) \circ (\iota_x \otimes X) = \sigma G_{n-1} \in G_n/G_{n-1}.$$

While for the second boundary map we have:

$$\begin{aligned} d_1(\sigma G_{n-2}) &= \sigma \circ (\iota_{A\otimes X^{\otimes n-2}} \otimes X^{\otimes 2}) \circ (X \otimes \iota_X) \\ &= \sigma \circ t_n \circ (\iota_{A\otimes X^{\otimes n-2}} \otimes X^{\otimes 2}) \circ (\iota_x \otimes X) = \sigma t_n G_{n-1} \in G_n/G_{n-1}, \end{aligned}$$

since from the definition of pre-braiding we have $X \otimes \iota_X = b^{-1}_{X,X} \circ (\iota_X \otimes X)$.

Denoting with $\langle \quad \rangle_G$ the generated subgroup in G, we can now easily prove the following

Lemma 3.1. For $n \geq 2$, $\langle t_n, G_{n-1} \otimes X\rangle_{G_n} = G_n$ if and only if $|W_n(A, X)|$ is connected.

Proof. Suppose first that $\langle t_n, G_{n-1} \otimes X\rangle = G_n$. Denoting with e the unity of G_n we have to show that from every vertex σG_{n-1} we can reach the vertex eG_{n-1} through a path of 1-simplices. Choose a representation $\sigma = g_1 t_n g_2 \dots g_k t_n$ with $g_i \in G_{n-1} \otimes X$ (the value $k = 0$ corresponds to $\sigma = e$), and work by induction on k. For $k = 0$ there is nothing to prove, since we already are in the vertex eG_{n-1}. For $k \geq 1$ consider the 1-simplex $g_1 t_n \dots t_n g_k G_{n-2}$. Since

$$\begin{aligned} d_0(g_1 t_n \dots t_n g_k G_{n-2}) &= g_1 t_n \dots g_{k-1} t_n G_{n-1} \\ d_1(g_1 t_n \dots t_n g_k G_{n-2}) &= g_1 t_n \dots g_k t_n G_{n-1} = \sigma G_{n-1} \end{aligned}$$

we can move from σG_{n-1} to a vertex with a lower value of k, and use the inductive hypothesis.

Conversely suppose that $|W_n(A,X)|$ is connected, and pick an element $\sigma \in G_n$. We have a minimal path of vertices $\sigma G_{n-1} = V_0, V_1, \dots, V_k = eG_{n-1}$, with the property that each vertex V_i is connected to V_{i+1} through a 1-simplex. Again we can prove that $\sigma \in \langle t_n, G_{n-1}\rangle$ by induction on k. For $k = 0$ we have $\sigma \in G_{n-1}$. For $k \geq 1$ if τG_{n-2} is the 1-simplex connecting σG_{n-1} and V_1 (represented by $\sigma' G_{n-1}$) we have two possibilities. If $d_0(\tau G_{n-2}) = \sigma G_{n-1}$ and $d_1(\tau G_{n-2}) = \sigma' G_{n-1}$, then $\exists g, g' \in G_{n-1}$ such that $\sigma = \sigma' g' t_n^{-1} g$. Otherwise $d_0(\tau G_{n-2}) = \sigma' G_{n-1}$ and $d_1(\tau G_{n-2}) = \sigma G_{n-1}$, so $\exists g, g' \in G_{n-1}$ such that $\sigma = \sigma' g' t_n g$. In both cases we have only to prove that $\sigma' \in \langle t_n, G_{n-1}\rangle$, which holds for inductive hypothesis, since $\sigma' G_{n-1} = V_1$ admits a lower value of k. $\square$

Remark 3.2. If we are in the case $A = e$ the unit object, and we already know that $|W_{n-1}(e,X)|$ is connected, we can take advantage of it in showing connectivity of $|W_n(e,X)|$ via the previous lemma. Indeed suppose to be in this case, with $n \geq 3$; we have that $G_{n-1} \otimes X$ contains $X \otimes G_{n-2} \otimes X$ as a subgroup, and also we can write $t_n = X \otimes X^{n-3} \otimes b_{X,X}^{-1} = X \otimes t_{n-1}$. Now applying the previous lemma, from connectivity of $|W_{n-1}(e,X)|$ we have that $\langle t_{n-1}, G_{n-2} \otimes X\rangle_{G_{n-1}} = G_{n-1}$, so $\langle X \otimes t_{n-1}, X \otimes G_{n-2} \otimes X\rangle_{G_n} = X \otimes G_{n-1}$. Then the condition $\langle t_n, G_{n-1} \otimes X\rangle_{G_n} = G_n$ is equivalent to $\langle X \otimes G_{n-1}, G_{n-1} \otimes X\rangle_{G_n} = G_n$, which can be more easy to verify.

3.2 Symmetric Groups

Consider the following groupoid $\mathcal{G}$: for every nonzero natural number n the set $n := \{1, \dots, n\}$ is an object of $\mathcal{G}$, and morphisms are simply the self-bijections of these sets. In this way $\mathrm{Aut}_{\mathcal{G}}(n) = \mathfrak{S}_n$ is the symmetric group on n elements. The usual cartesian product of sets, induces a well defined symmetric monoidal structure $(\mathcal{G}, \otimes, 1)$ on this groupoid. Indeed we have only to fix a bijection between $\{1, \dots, n\} \times \{1, \dots, m\}$ and $\{1, \dots, nm\}$, which means to choose an order for the nm elements of $\{1, \dots, n\} \times \{1, \dots, m\}$. To order this set think about it as a matrix with m rows and n columns with the element $(i,j) \in \{1, \dots, n\} \times \{1, \dots, m\}$ in position (i,j) in the matrix. Now simply start ordering the matrix from left to right starting from the first row and so on.

$$\begin{matrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ mn-n+1 & \cdots & \cdots & mn \end{matrix}$$

On morphisms, given two self-bijections $\sigma \in \mathrm{Aut}_{\mathcal{G}}(n)$ and $\tau \in \mathrm{Aut}_{\mathcal{G}}(m)$, we have a natural well defined self-bijection $\sigma \times \tau$ of the cartesian product, given by $(\sigma \times \tau)(i,j) := (\sigma(i), \tau(j))$, and so through the ordering just presented a well defined self bijection $\sigma \otimes \tau \in \mathfrak{S}_{nm}$. In the matrix representation, the permutation $\sigma \otimes \tau$ acts on these elements permuting the m rows with τ, and the n columns with σ.

The usual symmetry $T : n \times m \cong m \times n$ of the cartesian product induces a symmetry $b_{n,m} : n \otimes m \to m \otimes n$ in our monoidal groupoid. Namely in the matrix representation is simply the transposition of the matrix, so as a permutation $b_{n,m} \in \mathfrak{S}_{nm}$ is simply this transposition seen under the two isomorphisms with $\{1, \dots, nm\}$:

$$\begin{array}{ccc} n \times m & \xrightarrow{T} & m \times n \\ \cong \downarrow & & \downarrow \cong \\ nm & \xrightarrow{b_{n,m}} & nm \end{array}$$

Example 3.3. Let us see the computation of $b_{2,3} \in \mathfrak{S}_6$ to fix the idea of the procedure. We start with our six elements disposed in a 3×2 matrix (left matrix in (3.1)), and transpose it:

$$\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{matrix} \quad \curvearrowright \quad \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix} \tag{3.1}$$

In the matrix on the right 3 is in the second place, 5 in the third, 2 in the fourth and 4 in the fifth. So the desired permutation is $b_{2,3} = (3, 2, 4, 5) \in \mathfrak{S}_6$.

Since $(\mathcal{G}, \otimes, e)$ is a symmetric monoidal groupoid, we can apply Proposition 2.22 and get that 1 is initial in the symmetric monoidal category $U\mathcal{G}$.

As stated in the outline of these examples we consider now the category $U\mathcal{G}$. Our groupoid $\mathcal{G}$ has no zero divisors ($mn = 1 \Rightarrow m = n = 1$), and

$\mathrm{Aut}_{\mathcal{G}}(1) = \{\mathbb{1}_1\}$. Then applying Proposition 2.22 and Proposition 2.23 we have that $U\mathcal{G}$ is a symmetric monoidal category and $\mathcal{G}$ its underlying groupoid.

The groupoid $\mathcal{G}$ satisfies cancellation ($mn = ln$ implies $m = l$ since $n \neq 0$), and $\forall n, m$ in $\mathcal{G}$ the map $\mathrm{Aut}_{\mathcal{G}}(n) \to \mathrm{Aut}_{\mathcal{G}}(n \otimes m)$ taking σ to $\sigma \otimes m$ is injective. Then applying Theorem 2.27 we get that $U\mathcal{G}$ is a homogeneous category.

3.2.1 Powers of 2

The first example of Homological instability that we want to analyse was pointed out by Peter Patzt in a brief note [13]. It's the simplest example that we present here, and also the inspiration for further generalizations that we will see later in this chapter.

In the symmetric homogeneous category $U\mathcal{G}$ just constructed, consider the stabilisation sequence associated to the couple $(1, 2)$:

$$\ldots \xrightarrow{-\otimes 2} \mathfrak{S}_{2^n} \xrightarrow{-\otimes 2} \mathfrak{S}_{2^{n+1}} \xrightarrow{-\otimes 2} \ldots \tag{3.2}$$

Explicitly the stabilisation map $\Sigma^2 : \mathfrak{S}_{2^n} \to \mathfrak{S}_{2^{n+1}}$ takes the permutation $\sigma \in \mathfrak{S}_{2^n}$ into the permutation on 2^{n+1} elements which performs σ on the first 2^n elements and σ on the remaining 2^n elements (let's indicate it by $\sigma \oplus \sigma$). It's clear from this description that an element in the image of Σ^2 must be an even permutation, because an arbitrary decomposition of σ in transpositions, induces a decomposition of $\sigma \oplus \sigma$ with double number of transpositions. For this reason the map induced on the H_1 is the zero map. Indeed as stated in Lemma 1.16, to obtain the induced maps on the H_1, we have only to take the quotient by the commutator subgroup, and the commutator subgroup of $\mathfrak{S}_{2^n}$ is A_{2^n}. So the chain (3.2) doesn't satisfy homological stability, even if it arises as a stabilisation sequence in a homogeneous category.

Let also look at $|W_n(1, 2)|$. We have $b_{2,2}^{-1} = (2, 3) \in \mathfrak{S}_4$, as shown in 3.3

$$b_{2,2}^{-1} = b_{2,2} = \begin{array}{ccc} 1 & & 2 \\ & \swarrow\!\!\!\nearrow & \\ 3 & & 4 \end{array} \tag{3.3}$$

so we can compute $t_n = 2^{n-2} \otimes (2, 3) \in \mathfrak{S}_{2^n}$. Using the matrix representation we have four rows of 2^{n-2} elements and t_n simply switches the second and the third row:

$$\begin{array}{cccc} 1 & 2 & \cdots & 2^{n-2} \\ 2^{n-2}+1 & 2^{n-2}+2 & \cdots & 2^{n-1} \\ 2^{n-1}+1 & 2^{n-1}+1 & \cdots & 2^{n-1}+2^{n-2} \\ 2^{n-1}+2^{n-2}+1 & 2^{n-1}+2^{n-2}+1 & \cdots & 2^{n} \end{array}$$

Explicitly

$$t_n = (2^{n-2}+1, 2^{n-1}+1)\dots(2^{n-1}, 2^{n-1}+2^{n-2}) \in \mathfrak{S}_{2^n}.$$

This element can be decomposed into $2^{n-2}(2^{n-2}-1)$ transpositions, so for $n \geq 3$ is an even permutation, and we have already noticed that $\mathfrak{S}_{2^{n-1}} \otimes 2$ is also made by even permutations. Therefore for $n \geq 3$, $\langle t_n, \mathfrak{S}_{2^{n-1}} \otimes 2\rangle_{\mathfrak{S}_{2^n}} \neq \mathfrak{S}_{2^n}$ because is contained in A_{2^n}, then applying Lemma 3.1, $|W_n(1,2)|$ is not connected for $n \geq 3$.

3.2.2 Powers of 3

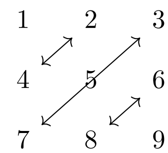
In the same category $U\mathcal{G}$ we can consider the stabilisation sequence associated to the pair $(1,3)$

$$\dots \xrightarrow{\_\otimes 3} \mathfrak{S}_{3^n} \xrightarrow{\_\otimes 3} \mathfrak{S}_{3^{n+1}} \xrightarrow{\_\otimes 3} \dots \tag{3.4}$$

where exactly as the previous example the map $\Sigma^3 : \mathfrak{S}_{3^n} \to \mathfrak{S}_{3^{n+1}}$ takes the permutation $\sigma \in \mathfrak{S}_{3^n}$ into the permutation on 3^{n+1} elements which performs σ separately on the first 3^n elements, the second 3^n elements, and also on the remaining 3^n elements: $\Sigma^3(\sigma) = \sigma \oplus \sigma \oplus \sigma$.

In this case the map induced on the H_1 is an isomorphism. We have $H_1(\mathfrak{S}_{3^n}) = \mathfrak{S}_{3^n}/A_{3^n} = \mathbb{Z}/2\mathbb{Z}$, and in the image of Σ^3 we have also odd permutations (pick σ to be for example a single transposition).

We are interested in understanding the behaviour of $|W_n(1,3)|$ in this case where homological stability doesn't fail at the level of H_1. We will use the lemma 3.1, and the Remark immediately after, so it is useful to describe explicitly t_n and the subgroup $3 \otimes \mathfrak{S}_{3^{n-1}}$. First of all we have that $b_{3,3}^{-1} = (2,4)(3,7)(6,8) \in \mathfrak{S}_9$, since

$b_{3,3}^{-1} = b_{3,3} =$ 

Then $t_n = 3^{\otimes n-2} \otimes b_{3,3} \in \mathfrak{S}_{3^n}$ can be described in this way as a permutation acting on 3^n ordered elements: first of all divide the 3^n elements in 9 consecutive blocks of 3^{n-2} elements, and then do the permutation $b_{3,3}$ using these 9 blocks as elements. The Picture 3.1 represent the permutation t_n, where we omit the identities, and the unlabelled segments means that the elements of the up block are sent in the elements of the bottom block with the same order.

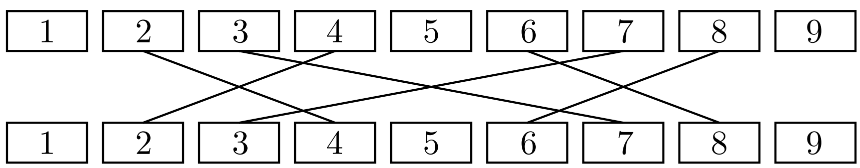


Figure 3.1: The permutation t_n

In the same way we can also describe the subgroup $3 \otimes \mathfrak{S}_{3^{n-1}}$; given $\sigma \in \mathfrak{S}_{3^{n-1}}$, the element $3 \otimes \sigma$ acts on 3^n ordered elements in this way: divide the 3^n elements in 3^{n-1} consecutive blocks of 3 elements, and then do the permutation σ using these 3^{n-1} blocks as elements.

Lemma 3.4. For $n \geq 3$ $|W_n(1,3)|$ is connected.

Proof. We will prove it by induction on n. For the base case ($n = 3$) we can simply use GAP: a system for computational discrete algebra available at `https://www.gap-system.org/`. As generators fo the group $\mathfrak{S}_m$, we can consider the single transposition $y_m := (1,2) \in \mathfrak{S}_m$ and the cycle $c_m := (1, 2, \ldots, m) \in \mathfrak{S}_m$. Using GAP we know that $\langle t_3, y_9 \otimes 3, c_9 \otimes 3 \rangle_{\mathfrak{S}_{27}} = \mathfrak{S}_{27}$, and by Lemma 3.1 we obtain that $|W_3(1,3)|$ is connected.

For the inductive step we have that $n \geq 4$ and that $\langle \mathfrak{S}_{3^{n-2}} \otimes 3, t_{n-1} \rangle = \mathfrak{S}_{3^{n-1}}$. Using the Lemma 3.1, and the Remark immediately after we have only to show that $\langle 3 \otimes \mathfrak{S}_{3^{n-1}}, \mathfrak{S}_{3^{n-1}} \otimes 3 \rangle = \mathfrak{S}_{3^n}$. Let us show first that $\langle 3 \otimes \mathfrak{S}_{3^{n-1}}, \mathfrak{S}_{3^{n-1}} \otimes 3 \rangle$ contains the subgroup $(\mathfrak{S}_{3^{n-1}} \oplus 3^{n-1} \oplus 3^{n-1})$. Given $\sigma \in \mathfrak{S}_{3^{n-2}}$, define

$$\tau := \sigma \oplus 3^{n-2} \oplus 3^{n-2} \in \mathfrak{S}_{3^{n-1}}. \tag{3.5}$$

Dividing the 3^n elements in 9 blocks of 3^{n-2} elements we have that $\tau_1 \otimes 3 \in$

$\mathfrak{S}_{3^{n-1}} \otimes 3$ (shown in the middle row of Figure 3.2) simply performs σ in the first, fourth, and seventh block, and is the identity on the other elements. The computation

$$\begin{aligned} t_n \circ (\tau_1 \otimes 3) \circ t_n &= (\sigma \oplus \sigma \oplus \sigma) \oplus 3^{n-1} \oplus 3^{n-1} \\ &= (\sigma \otimes 3) \oplus 3^{n-1} \oplus 3^{n-1}, \end{aligned} \tag{3.6}$$

shown in Figure 3.2 implies that we can generate $(\mathfrak{S}_{3^{n-2}} \otimes 3) \oplus 3^{n-1} \oplus 3^{n-1}$. Moreover since $n \geq 4$, we also have the element

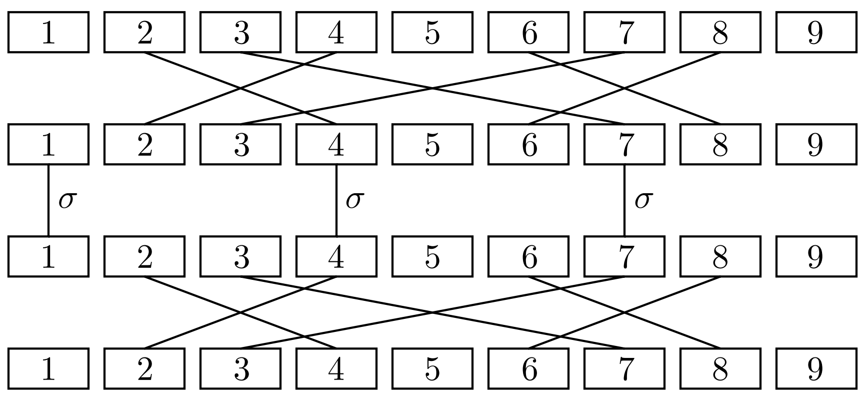


Figure 3.2: Computation of $t_n \circ (\tau_1 \otimes 3) \circ t_n$.

$$\begin{aligned} t_{n-1} \oplus 3^{n-1} \oplus 3^{n-1} &= (3^{n-3} \otimes b_{3,3}) \oplus 3^{n-1} \oplus 3^{n-1} \\ &= (3^{n-3} \otimes b_{3,3}) \oplus (3^{n-3} \otimes 3^2) \oplus (3^{n-3} \otimes 3^2) \\ &= 3^{n-3} \otimes (b_{3,3} \oplus 3^2 \oplus 3^2) \in 3 \otimes \mathfrak{S}_{3^{n-1}}, \end{aligned}$$

so we can use the inductive hypothesis and get

$$\langle (\mathfrak{S}_{3^{n-2}} \otimes 3) \oplus 3^{n-1} \oplus 3^{n-1}, t_{n-1} \oplus 3^{n-1} \oplus 3^{n-1} \rangle = \mathfrak{S}_{3^{n-1}} \oplus 3^{n-1} \oplus 3^{n-1}.$$

As a consequence we obtain the single transposition y_{3^n}.

To conclude now, we have only to show that we can obtain also the cycle c_{3^n}. To do so introduce this notation: dividing the 3^n blocks into 9 blocks as in Figure 3.2, indicate with m_l the m^{th}-element in the l-block. In this way for example $(1_2, 1_3)$ is the element in $\mathfrak{S}_{3^n}$ which switches the first element in the second block with the first element in the third block. Using this notation, we are allowed to do the permutation $(1_1, 1_2, 1_3)$, and we have already seen in Figure

3.2 the effect of conjugating by t_n:

$$t_n \circ (1_1, 1_2, 1_3) \circ t_n = (1_1, 1_4, 1_7).$$

So we can obtain the desired cycle with

$$(1_1, 1_4, 1_7) \circ (c_{3^{n-1}} \oplus c_{3^{n-1}} \oplus c_{3^{n-1}}) = c_{3^n}.$$

□

3.3 Matrix Groups

Since the notation and results in this brief introduction to the section are standard matters of linear algebra, we refer to an on-line resource [4].

Given a field K, denote $M_n(K)$ the set of all $n \times n$ matrices with coefficients in K. We will denote with I_n the identity matrix of $M_n(K)$(omitting the n when it is clear), and e_{ij} the matrix whose (i, j)-entry is 1 and all other entries zero.

Definition 3.5. For every $1 \leq i \neq j \leq n$ and $\alpha \in K$, we will call $E_{ij}(\alpha) := I + \alpha e_{ij}$ an elementary matrix.

We have that $\det E_{ij}(\alpha) = 1$, and $E_{i,j}(\alpha)^{-1} = E_{ij}(-\alpha)$.

Remark 3.6. If $A \in M_n(K)$, then multiplying A on the left by $E_{ij}(\alpha)$ takes A and adds on α times the j^{th} row of A to the i^{th} row of A. Similarly multiplying A on the right by $E_{ij}(\alpha)$ takes A and adds on α times the i^{th} column of A to the j^{th} column of A.

Denote $\mathrm{GL}_n(K)$ the group of invertible matrices in $M_n(K)$, and $\mathrm{SL}_n(K)$ the subgroup of matrices with determinant 1.

Lemma 3.7. If $n \geq 3$, $\mathrm{SL}_n(K)$ is the commutator subgroup of $\mathrm{GL}_n(K)$.

Proof. First of all $[\mathrm{GL}_n(K) : \mathrm{GL}_n(K)] \subseteq \mathrm{SL}_n(K)$, since for any $A, B \in \mathrm{GL}_n(K)$ we have $\det(ABA^{-1}B^{-1}) = 1$, and then $[A, B] \in \mathrm{SL}_n(K)$.

To prove $\mathrm{SL}_n(K) \subseteq [\mathrm{GL}_n(K) : \mathrm{GL}_n(K)]$, it suffices to show that every elementary matrix is a commutator, since $\mathrm{SL}_n(K)$ is generated by all elementary matrices. To do so we will show that for every i, j, r distinct, and $\alpha, \beta \in K$ the following holds:

$$E_{ij}(\alpha\beta) = [E_{ir}(\alpha), E_{rj}(\beta)].$$

With a simple computation

$$\begin{aligned}[E_{ir}(\alpha), E_{rj}(\beta)] &= (I + \alpha e_{ir})(I + \beta e_{rj})(I - \alpha e_{ir})(I - \beta e_{rj}) \\ &= (I + \alpha e_{ir} + \beta e_{rj} + \alpha\beta e_{ij})(I - \alpha e_{ir} - \beta e_{rj} + \alpha\beta e_{ij}) \\ &= I + 2\alpha\beta e_{ij} - \alpha\beta e_{ij} = E_{ij}(\alpha\beta).\end{aligned}$$

As a consequence given an elementary matrix $E_{ij}(\alpha)$ the condition $n \geq 3$ ensures that we can choose $r \neq i, j$ and write it as a commutator:

$$E_{ij}(\alpha) = [E_{ir}(\alpha), E_{rj}(1)].$$

□

3.3.1 General Linear Groups

Given a field K, consider the following groupoid $\mathcal{G}$: for every nonzero natural number n, the vector space K^n is an object of $\mathcal{G}$, and morphisms $\mathrm{Aut}_{\mathcal{G}}(K^n) = \mathrm{GL}_n(K)$ if $n > 1$, simply the identity of K in the case $n = 1$ (we will often indicate the object K^m simply with m).

The usual tensor product over K: $\otimes_K$ endows our groupoid with a symmetric monoidal structure $(\mathcal{G}, \otimes, K)$. Indeed we only have to choose a natural isomorphism between $K^n \otimes K^m$ and K^{nm}, and we do this choosing an order for the standard base of $K^n \otimes K^m$. We choose the isomorphism which sends $e_i \otimes e_j \mapsto e_{(i-1)m+j}$, $(1 \leq i \leq n$ and $1 \leq j \leq m)$ where we denote with e_i the i-element of the canonical base of K^n. Again this is not as strange as it might seems: simply put the element $e_i \otimes e_j$ in position (i, j) in a $n \times m$ matrix, and now start no enumerate the elements from left to right starting from the first row.

Remark 3.8. Here we are using the ordering which is opposite with respect to the one made for the cartesian product $n \times m$ in the symmetric groups examples, but this one is more natural in this context, as we will see later.

Under this identification between $K^n \otimes K^m$ and K^{nm}, if $A \in \mathrm{GL}_n(K)$ and $B \in \mathrm{GL}_m(K)$, we have an explicit element $A \otimes B \in \mathrm{GL}_{nm}(K)$, which is simply a blocks matrix: with $n \times n$ blocks $A_{ij}B$, each one of dimension $m \times m$:

$$A \otimes B = \begin{pmatrix} A_{1,1}B & \dots & A_{1,n}B \\ \vdots & \ddots & \vdots \\ A_{n,1}B & \dots & A_{n,n}B \end{pmatrix} \in \mathrm{GL}_{nm}(K)$$

Possibly one can take this as the definition of the bifunctor $\otimes$ in our groupoid $\mathcal{G}$ forgetting that this is the usual tensor product of vector spaces, and then verify that this bifunctor is symmetric monoidal with unity the object K (the symmetry given by the usual symmetry $K^n \otimes K^m \cong K^m \otimes K^n$ seen under the two isomorphisms with K^{nm}, which gives simply a permutation $b_{n,m} \in \mathrm{GL}_{nm}(K)$ of the variables of K^{nm}). Again thinking about the base elements $e_i \otimes e_j$ of $K^n \otimes K^m$ as disposed in a $n \times m$ matrix, we have an easy interpretation of the symmetry $b_{n,m}$. Simply transpose the matrix and renumber the base elements according to this new matrix (from left to right starting with the first row), to get the desired permutation of the variables $b_{n,m} \in \mathrm{GL}_{nm}(K)$.

The groupoid $\mathcal{G}$ has no zero divisors: if $K^a \otimes K^b \cong K$ then the dimensions must be equal: $ab = 1$, and so $a = b = 1$. Therefore we have forced $\mathrm{Aut}_{\mathcal{G}}(K) = \{id\}$ in order to apply Proposition 2.23 and Proposition 2.22 and obtain that $\mathcal{G}$ is the underlying groupoid of the symmetric monoidal category $U\mathcal{G}$.

Exactly as in the previous examples the groupoid $(\mathcal{G}, \otimes, K)$ satisfy also cancellation. Indeed if $K^a \otimes K^c = K^b \otimes K^c$, then the dimensions must be equal: $ac = bc$, and so $a = b$, since $c \neq 0$. Moreover $\forall K^a, K^b$ in $\mathcal{G}$ the map $\mathrm{Aut}_{\mathcal{G}}(K^a) \to \mathrm{Aut}_{\mathcal{G}}(K^b \otimes K^a)$ taking $A \in \mathrm{GL}_a(K)$ to $I_b \otimes A$ is injective, since $I_b \otimes A$ has only b blocks equal A on the diagonal, and zeros elsewhere:

$$A \xmapsto{I_b \otimes} \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \in \mathrm{GL}_{ab}(K).$$

Then applying Theorem 2.27 we obtain that $U\mathcal{G}$ is also homogeneous.

Remark 3.9. In this category we will work with homogeneous conditions and stabilisation maps tensoring with the stabilisation object X to the left and not to the right as usual. This means for example that the stabilisation map will be

$$\Sigma^X : \mathrm{Aut}_{U\mathcal{G}}(X^{\otimes n} \otimes A) \xrightarrow{X \otimes} \mathrm{Aut}_{U\mathcal{G}}(X^{\otimes n+1} \otimes A),$$

which is more easy to visualize.

Consider now the stabilisation sequence associated to the pair (K, K^2):

$$\dots \xrightarrow{I_2\otimes\_} \mathrm{GL}_{2^n}(K) \xrightarrow{I_2\otimes\_} \mathrm{GL}_{2^{n+1}}(K) \xrightarrow{I_2\otimes\_} \dots \tag{3.7}$$

The stabilisation map $\Sigma^{K^2} : \mathrm{GL}_{2^n}(K) \to \mathrm{GL}_{2^{n+1}}(K)$ takes the matrix A into a matrix with two copies of A on the diagonal, and zeros elsewhere

$$A \xmapsto{I_2\otimes\_} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

so its determinant is $(\det A)^2$.

Remark 3.10. If $A \in M_a(K)$ and $B \in M_b(K)$, let us denote with $A \oplus B \in M_{a+b}(K)$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

In this way $I_2 \otimes A = A \oplus A$.

Recall from Lemma 3.7 that for $n \geq 2$, $\mathrm{SL}_{2^n}(K)$ is the commutator subgroup of $\mathrm{GL}_{2^n}(K)$. Then if we quotient by the commutator subgroup, in the same range of n, we have that the class of a matrix is obtained by taking the determinant. If the field K contains an element $x \in K^*$ which is not a square, we can easily see that the map induced on the H_1 cannot be surjective. Indeed it cannot reach matrices with determinant which is not a square. This implies that the family of groups (3.7) doesn't satisfy homological stability, even if it arises as a stabilisation sequence into a symmetric homogeneous category.

Let also look at $|W_n(K, K^2)|$. Since

$$b_{2,2}^{-1} = b_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(K), \tag{3.8}$$

we can compute $t_n = b_{2,2}^{-1} \otimes I_{2^{n-2}} \in \mathrm{GL}_{2^n}(K)$. For $n \geq 3$ this element has determinant 1 and we also already have noticed that $I_2 \otimes \mathrm{GL}_{2^{n-1}}(K)$ is made by matrices with a square determinant. Therefore for $n \geq 3$, and with the assumption of the existence of a non square element $x \in K^*$, $\langle t_n, I_2 \otimes \mathrm{GL}_{2^{n-1}}(K)\rangle_{\mathrm{GL}_{2^n}(K)} \neq$

$\mathrm{GL}_{2^n}(K)$ because it cannot reach matrices with a non square determinant. Then applying Lemma 3.1, $|W_n(K, K^2)|$ is not connected for $n \geq 3$.

Remark 3.11. The matrix b_{K^2,K^2} induces exactly the permutation $b_{2,2}$ (the symmetry of the examples with symmetric groups) on the coordinates. And also the corresponding t_n are related in the same way. This follows because in these two examples the braiding is constructed exactly in the same way in these two examples, and is given by switching the two dimensions.

3.3.2 The $\mathbb{F}_2$ case

In the previous example the presence in the field K of a non-square element yields to homological instability for the stabilisation sequence associated to the pair (K, K^2), and the non connectivity for the spaces $|W_n(K, K^2)|$ with $n \geq 3$. A natural question that arises now is what happens if K does not admit a non square element. The simplest case that we can consider is $K = \mathbb{F}_2$.

We remain in the same symmetric homogeneous category of the previous example with $K = \mathbb{F}_2$: $(U\mathcal{G}, \otimes, \mathbb{F}_2)$. Exactly as in (3.7) consider the stabilisation sequence associated to the pair $(\mathbb{F}_2, \mathbb{F}_2^2)$:

$$\ldots \xrightarrow{I_2 \otimes -} \mathrm{GL}_{2^n}(\mathbb{F}_2) \xrightarrow{I_2 \otimes -} \mathrm{GL}_{2^{n+1}}(\mathbb{F}_2) \xrightarrow{I_2 \otimes -} \ldots \tag{3.9}$$

where as before the stabilisation map $\Sigma^{\mathbb{F}_2^2} : \mathrm{GL}_{2^n}(\mathbb{F}_2) \to \mathrm{GL}_{2^{n+1}}(\mathbb{F}_2)$ takes the matrix A into a matrix with two copies of A on the diagonal, and zeros elsewhere.

As stated in Lemma 1.16, to obtain the induced maps on the H_1, we have only to take the quotient by the commutator subgroup. But for Lemma 3.7 we have $[\mathrm{GL}_{2^n}(\mathbb{F}_2) : \mathrm{GL}_{2^n}(\mathbb{F}_2)] = \mathrm{SL}_{2^n}(\mathbb{F}_2)$ for $n \geq 2$, and also $\mathrm{GL}_{2^n}(\mathbb{F}_2) = \mathrm{SL}_{2^n}(\mathbb{F}_2)$, since the only element in $\mathbb{F}_2$ different from zero is 1. As a consequence $H_1 \, \mathrm{GL}_{2^n}(\mathbb{F}_2)$ is trivial for $n \geq 2$, and so the maps induced at the level of H_1 are isomorphisms in the same range.

Looking at the spaces $|W_n(\mathbb{F}_2, \mathbb{F}_2^2)|$, we can compute t_n from (3.8), which is simply a block matrix with blocks $2^{n-2} \times 2^{n-2}$:

$$t_n = b_{2,2}^{-1} \otimes I_{2^{n-2}} = \begin{pmatrix} I_{2^{n-2}} & 0 & 0 & 0 \\ 0 & 0 & I_{2^{n-2}} & 0 \\ 0 & I_{2^{n-2}} & 0 & 0 \\ 0 & 0 & 0 & I_{2^{n-2}} \end{pmatrix} \in \mathrm{GL}_{2^n}(\mathbb{F}_2). \tag{3.10}$$

Remark 3.12. If $A \in \mathrm{GL}_{2^n}(\mathbb{F}_2)$, thinking about A as a 4×4 blocks matrix, with blocks $2^{n-2} \times 2^{n-2}$, the matrix $t_n A$ is obtained from A switching the second and the third block rows. While At_n is obtained from A switching the second an the third block columns.

Lemma 3.13. For $n \geq 3$ $|W_n(\mathbb{F}_2, \mathbb{F}_2^2)|$ is connected.

Proof. We will prove it by induction on n. For the base case ($n = 3$) we can use again GAP: a system for computational discrete algebra available at `https://www.gap-system.org/`. As stated in [16] we can use as generators for $\mathrm{GL}_m(\mathbb{F}_2)$ the two matrices

$$Y_m := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix}, \quad Z_m := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & & 1 & 0 \end{pmatrix}.$$

Using GAP we know that $\langle t_3, I_2 \otimes Y_4, I_2 \otimes Z_4 \rangle_{\mathrm{GL}_8(\mathbb{F}_2)} = \mathrm{GL}_8(\mathbb{F}_2)$, and by Lemma 3.1 we obtain that $|W_3(\mathbb{F}_2, \mathbb{F}_2^2)|$ is connected.

For the inductive step we have that $n \geq 4$ and that $\langle I_2 \otimes \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2), t_{n-1} \rangle = \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2)$. Using the Lemma 3.1, and the Remark immediately after we have only to show that $\langle I_2 \otimes \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2), \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \otimes I_2 \rangle = \mathrm{GL}_{2^n}(\mathbb{F}_2)$.

Let us show first that $\langle I_2 \otimes \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2), \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \otimes I_2 \rangle$ contains the subgroup $(\mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \oplus I_{2^{n-1}})$. Given $A \in \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)$, and recalling Remark 3.12, from the computation

$$\begin{aligned} t_n(I_2 \otimes (A \oplus I_{2^{n-2}}))t_n^{-1} &= t_n(A \oplus I_{2^{n-2}} \oplus A \oplus I_{2^{n-2}})t_n \\ &= A \oplus A \oplus I_{2^{n-2}} \oplus I_{2^{n-2}} \\ &= (I_2 \otimes A) \oplus I_{2^{n-1}}, \end{aligned}$$

we get that

$$t_n(I_2 \otimes (\mathrm{GL}_{2^{n-2}}(\mathbb{F}_2) \oplus I_{2^{n-2}}))t_n^{-1} = (I_2 \otimes \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)) \oplus I_{2^{n-1}}. \tag{3.11}$$

So we can generate $(I_2 \otimes \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)) \oplus I_{2^{n-1}}$. Moreover since $n \geq 4$ we also have the element

$$t_{n-1} \oplus I_{2^{n-1}} = (b_{2,2}^{-1} \otimes I_{2^{n-3}}) \oplus (I_{2^{n-2}} \otimes I_2)$$

$$= ((b_{2,2}^{-1} \otimes I_{2^{n-4}}) \oplus I_{2^{n-2}}) \otimes I_2 \in (\mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \otimes I_2).$$

Combining these two observations and using the inductive hypothesis we get the desired subgroup

$$\langle (I_2 \otimes \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)) \oplus I_{2^{n-1}}, t_{n-1} \oplus I_{2^{n-1}} \rangle = \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \oplus I_{2^{n-1}}.$$

As a consequence we get the element Y_{2^n}, since it can be written as $Y_{2^n} = Y_{2^{n-1}} \oplus I_{2^{n-1}} \in (\mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \oplus I_{2^{n-1}})$. In the same way we can also obtain the subgroup $I_{2^{n-1}} \oplus \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2)$: in (3.11) use the subgroup $I_2 \otimes (I_{2^{n-2}} \oplus \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2))$ to obtain

$$t_n(I_2 \otimes (I_{2^{n-2}} \oplus \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)))t_n^{-1} = I_{2^{n-1}} \oplus (I_2 \otimes \mathrm{GL}_{2^{n-2}}(\mathbb{F}_2)).$$

Since we also have the element $I_{2^{n-1}} \oplus t_{n-1}$, we obtain $I_{2^{n-1}} \oplus \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2)$.

To conclude now we have only to show that we can get also Z_{2^n}. The matrix Z_{2^n} performs the cycle c_{2^n} on the elements of the standard base of $(\mathbb{F}_2)^{2^n}$. Dividing the 2^n vectors of the standard base in two blocks of 2^{n-1} vectors, and given $\sigma \in \mathfrak{S}_{2^{n-1}}$, the subgroup $\mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \oplus I_{2^{n-1}}$ contains the matrix that performs σ into the first block of 2^{n-1} elements, and keeps fixed the elements in the second block. Analogously the subgroup $I_{2^{n-1}} \oplus \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2)$ contains the matrix that performs σ into the second block, and keeps fixed the elements in the first block. As a consequence we have only to show that in $\langle I_2 \otimes \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2), \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2) \otimes I_2 \rangle$ we have a matrix that only switches a single base vector from the first block, with a single base vector from the second, since in this way we obtain every permutation of the 2^n elements. But if we consider

$$A = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ \hline & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) \in \mathrm{GL}_{2^{n-1}}(\mathbb{F}_2),$$

recalling Remark 3.12, the matrix $t_n(A \oplus I_{2^{n-1}})t_n$ provides the desired permutation. □

3.3.3 Projective General Linear Groups

In the groupoid that we have constructed with general linear groups, we have forced the automorphism group of the unity K to be the trivial group, instead of $\mathrm{GL}_1(K)$. In this way we were able to apply Proposition 2.23 which guarantees that the automorphisms groups remain unchanged after applying Quillen's construction. A natural question that arises now is what happens if we do not change the automorphisms of the unity, and so $\mathrm{Aut}_{\mathcal{G}}(K) = \mathrm{GL}_1(K) = K^*$.

First of all if $[K^x, \phi] \in \mathrm{Aut}_{U\mathcal{G}}(K^n)$ we must have $K^x \otimes K^n = K^n$, so $x = 1$ and every automorphism of the generic object K^n can be written in the form $[K, A]$ with the unit object K as complement, and $A \in \mathrm{GL}_n(K)$. Moreover recall that we have $[K, A] = [K, B] \in \mathrm{Aut}_{U\mathcal{G}}(K^n)$ if and only if $\exists \lambda \in K^* = \mathrm{GL}_1(K)$ such that $A = B \circ (\lambda \otimes I_a) \in \mathrm{Hom}_{\mathcal{G}}(K \otimes K^a, K^a)$. Since $\lambda \otimes I_a$ is the scalar matrix in $\mathrm{GL}_a(K)$ with λ on the diagonal and zeros elsewhere we obtain that $\mathrm{Aut}_{U\mathcal{G}}(K^n)$ is the quotient of $\mathrm{GL}_n(K)$ by the subgroup of the scalar matrices (isomorphic to K^*), so

$$\mathrm{Aut}_{U\mathcal{G}}(K^n) = \mathbb{P}(\mathrm{GL}_n(K)) = \mathrm{GL}_n(K)/_{K^*}.$$

Therefore studying homological stability in this groupoid is exactly the same as studying it in the following groupoid $\mathcal{G}$, which has as objects the nonzero natural numbers, and morphisms $\mathrm{Aut}_{\mathcal{G}}(n) = \mathbb{P}(\mathrm{GL}_n(K))$.

Using the tensor product pointed out in the previous example of general linear groups, we have a well defined symmetric monoidal structure $(\mathcal{G}, \otimes, 1)$. On the objects this is simply the multiplication of natural numbers, while given $[A] \in \mathbb{P}(\mathrm{GL}_n(K))$ and $[B] \in \mathbb{P}(\mathrm{GL}_m(K))$,

$$[A] \otimes [B] := [A \otimes B] \in \mathbb{P}(\mathrm{GL}_{nm}(K)).$$

Everything we have shown for the groupoid with $\mathrm{GL}_n(K)$ works equally well in the quotient $\mathbb{P}(\mathrm{GL}_n(K))$. We can take as symmetries the classes in the quotient of the symmetries $[b_{n,m}]$. We can apply Theorem 2.27 and Propositions 2.23, 2.22 and get that $U\mathcal{G}$ is a symmetric homogeneous category and that $\mathcal{G}$ is its underlying groupoid. The only difference here is that now we have only the identity as automorphism of the unity in a more natural way.

Consider as before the chain of stabilisation maps for the couple $(1, 2)$

$$\dots \xrightarrow{[I_2]\otimes\_} \mathbb{P}(\mathrm{GL}_{2^n}(K)) \xrightarrow{[I_2]\otimes\_} \mathbb{P}(\mathrm{GL}_{2^{n+1}}(K)) \xrightarrow{[I_2]\otimes\_} \dots \tag{3.12}$$

The stabilisation map $\Sigma^2 : \mathbb{P}(\mathrm{GL}_{2^n}(K)) \to \mathbb{P}(\mathrm{GL}_{2^{n+1}}(K))$ takes the class $[A]$ into the class of the matrix with two copies of A on the diagonal, and zeros elsewhere

$$[A] \xmapsto{[I_2]\otimes\_} \left[\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\right]. \tag{3.13}$$

Let us look first at the map induced on the H_1. A commutator in $\mathbb{P}(\mathrm{GL}_m(K))$ is given by the class of a commutator in $\mathrm{GL}_m(K)$, since $[A][B][A]^{-1}[B]^{-1} = [ABA^{-1}B^{-1}]$. For $m \geq 3$ we have seen in Lemma 3.7 that the derived subgroup of $\mathrm{GL}_m(K)$ is given by the matrices with determinant the unity of K ($\mathrm{SL}_m(K)$). Then $[A]$ is a commutator in $\mathbb{P}(\mathrm{GL}_m(K))$ if and only if $\exists \lambda \in K^*$ such that $\lambda A \in \mathrm{SL}_m(K)$. Since $\det(\lambda A) = \lambda^m \det(A)$, this last condition is equivalent to require that $\exists y \in K^*$ such that $y^m = \det(A)$. Indeed if this is true we can set $\lambda = y^{-1}$. Denoting $(K^*)^m := \{x^m \mid x \in K^*\}$ we have

$$H_1(\mathbb{P}(\mathrm{GL}_m(K))) = \mathbb{P}(\mathrm{GL}_m(K))/[\mathbb{P}(\mathrm{GL}_m(K)); \mathbb{P}(\mathrm{GL}_m(K))] \cong K^*/(K^*)^m,$$

where the isomorphism is given by sending $[A] \mapsto [\det(A)] \in K^*/(K^*)^m$. At the level of H_1 the chain (3.12) induces

$$\dots \xrightarrow{H_1(\Sigma^2)} K^*/(K^*)^{2^n} \xrightarrow{H_1(\Sigma^2)} K^*/(K^*)^{2^{n+1}} \xrightarrow{H_1(\Sigma^2)} \dots$$

and remembering the form of the stabilisation map (3.13), we have that $H_1(\Sigma^2)$ sends $[x] \in K^*/(K^*)^{2^n}$ to $[x^2] \in K^*/(K^*)^{2^{n+1}}$. We can also notice that $H_1(\Sigma^2)$ is injective, since the map $x \mapsto x^2$ in K^* sends $(K^*)^{2^n}$ into $(K^*)^{2^{n+1}}$. Now simply requiring again that our field K contains an element $x \in K^*$ which is not a square, we can easily see that the map induced on the H_1 cannot be surjective. Indeed it cannot reach the element $[x] \in K^*/(K^*)^{2^{n+1}}$. This implies that the family of groups 3.12 doesn't satisfy homological stability, even if it arises as a chain of stabilisation maps into a locally homogeneous category.

Let also analyse $|W_n(1,2)|$. In the previous example we have already computed $b_{2,2}^{-1} \in \mathrm{GL}_4(K)$ (3.8), and so $[t_n] = [b_{2,2}^{-1}] \otimes [I_{2^{n-2}}] \in \mathbb{P}(\mathrm{GL}_{2^n}(K))$. For $n \geq 3$ t_n has determinant 1 and we also know that $I_2 \otimes \mathrm{GL}_{2^{n-1}}(K)$ is made by matrices with a square determinant. The determinant in not well defined for the classes in $\mathbb{P}(\mathrm{GL}_{2^n}(K))$, but as we have already seen in a computation in the previous paragraph, it is well defined its class in $K^*/(K^*)^{2^n}$, and so the property to have or not a square determinant. Therefore for $n \geq 3$, and with the assumption of the existence of a non square element $x \in K^*$,

$\langle [t_n], [I_2] \otimes \mathbb{P}(\mathrm{GL}_{2^{n-1}}(K)) \rangle_{\mathbb{P}(\mathrm{GL}_{2^n}(K))} \neq \mathbb{P}(\mathrm{GL}_{2^n}(K))$ because it cannot reach classes of matrices with a non square determinant. Then applying Lemma 3.1, $|W_n(1,2)|$ is not connected for $n \geq 3$.

3.3.4 Changing the field in the chain

In all the previous examples about general linear groups the field was always fixed and we obtained a chain of stabilisation maps increasing the dimension. We present now an example where both dimension and field increase in the chain.

Take a sequence of fields $(K_i)_{i=1}^{\infty}$:

$$K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_i \subsetneq \dots$$

where for every $i \in \mathbb{N}^*$, K_i is a subfield of K_{i+1} and $K_i \neq K_{i+1}$.

Consider the groupoid $\mathcal{G}$ with objects the natural numbers, and morphisms $\mathrm{Aut}_{\mathcal{G}}(n) = \mathrm{GL}_n(K_n)$, with the convention to have only $\mathbb{1}_0$ as automorphism of zero. Define a monoidal structure $\otimes$ on this groupoid, which on the elements $n \otimes m := n + m$ is simply the sum in $\mathbb{N}$, while on morphisms is simply the block sum operation. If $A \in \mathrm{GL}_n(K_n)$ and $B \in \mathrm{GL}_m(K_m)$

$$A \otimes B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}_{n+m}(K_{n+m}),$$

where everything is well defined since K_n and K_m are both subfields of K_{n+m}. In this way A and B can be thought with coefficients in K_{n+m}: $A \in \mathrm{GL}_n(K_{n+m})$ and $B \in \mathrm{GL}_m(K_{n+m})$. As a consequence $\det(A \otimes B) = \det(A)\det(B) \in K^*_{n+m}$. We also adopt the convention $\mathbb{1}_0 \otimes A = A \otimes \mathbb{1}_0 = A$ for the unity 0. We have that $\otimes$ is a well defined bifunctor, since

$$I_n \otimes I_m = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m} \in \mathrm{GL}_{n+m}(K_{n+m}),$$

$$\begin{aligned}(A' \otimes B') \circ (A \otimes B) &= \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ &= \begin{pmatrix} A'A & 0 \\ 0 & B'B \end{pmatrix} = (A' \circ A) \otimes (B' \circ B),\end{aligned}$$

for $A, A' \in \mathrm{GL}_n(K_n)$ and $B, B' \in \mathrm{GL}_m(K_m)$. Associativity for objects is simply associativity of the sum, while on morphisms

$$((A \otimes B) \otimes C) = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} = (A \otimes (B \otimes C)). \tag{3.14}$$

Define the braiding $b_{n,m} : n \otimes m \to m \otimes n$ as

$$b_{n,m} := \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} \in \mathrm{GL}_{n+m}(K_{n+m}),$$

which simply takes the first n vectors of the canonical base of $(K_{n+m})^{n+m}$ and put them at the end of the base. In this way naturality and compatibility with the associativity are quick to verify, and also the symmetric condition

$$b_{m,n} \circ b_{n,m} = I_{n+m}.$$

So we have a well defined symmetric monoidal groupoid $(\mathcal{G}, \otimes, 0)$.

As usual we consider now the category $U\mathcal{G}$. The groupoid $\mathcal{G}$ has no zero divisors ($n+m = 0 \Rightarrow n = m = 0$), and $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathbb{1}_0\}$ so applying Proposition 2.22 and Proposition 2.23 we have that $U\mathcal{G}$ is a symmetric monoidal category and $\mathcal{G}$ its underlying groupoid.

The groupoid $\mathcal{G}$ satisfies cancellation ($m + n = l + n$ implies $m = l$), and the map $\mathrm{GL}_n(K_n) \to \mathrm{GL}_{n+m}(K_{n+m})$ taking A to $A \otimes I_m$ is injective then applying Theorem 2.27 we get that $U\mathcal{G}$ is also homogeneous.

Consider now the stabilisation sequence associated to the pair $(0, 1)$

$$\dots \xrightarrow{-\otimes 1} \mathrm{GL}_n(K_n) \xrightarrow{-\otimes 1} \mathrm{GL}_{n+1}(K_{n+1}) \xrightarrow{-\otimes 1} \dots \tag{3.15}$$

The stabilisation map $\Sigma^1 : \mathrm{GL}_n(K_n) \to \mathrm{GL}_{n+1}(K_{n+1})$ takes the matrix A into the matrix

$$A \xmapsto{\Sigma^1} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall from Lemma 3.7 that for $n \geq 3$, $\mathrm{SL}_n(K_n)$ is the commutator subgroup of $\mathrm{GL}_n(K_n)$. Then $H_1(\mathrm{GL}_n(K_n)) \cong K_n^*$ in the same range of n, where the isomorphism is induced by the determinant. Since $\det(\Sigma^1(A)) = \det(A)$ we have that $H_1(\Sigma^1) : K_n^* \to K_{n+1}^*$ is the inclusion $K_n^* \subset K_{n+1}^*$ so it cannot be surjective. This implies that the family of groups (3.15) doesn't satisfy homological stability, even if it arises as a stabilisation sequence into a symmetric homogeneous category.

As usual let's analyse also $|W_n(0,1)|$, and prove that it is not connected using 3.1. From the definition of the braiding

$$b_{1,1}^{-1} = b_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(K_2),$$

$$t_n = I_{n-2} \otimes b_{1,1}^{-1} = \left(\begin{array}{c|cc} I_{n-2} & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right) \in \mathrm{GL}_n(K_n).$$

The matrix t_n has coefficients in K_1, while every element in $\mathrm{GL}_{n-1}(K_{n-1}) \otimes 1$ has coefficients in K_{n-1}. As a consequence

$$\langle t_n ; \mathrm{GL}_{n-1}(K_{n-1}) \otimes 1 \rangle_{\mathrm{GL}_n(K_n)} \subseteq \mathrm{GL}_n(K_{n-1}) \neq \mathrm{GL}_n(K_n)$$

and applying Lemma 3.1 we get non connectivity of the semi-simplicial set for $n \geq 2$.

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