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Master Thesis in Mathematics

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Group Actions on a Product of Spheres

Emphasising on simple rank two groups

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Abstract

In this thesis I show that the cohomology of a connected CW-complex is periodic if and only if it is the base space of an orientable spherical fibration with total space homotopy equivalent to a finite dimensional CW-complex. Moreover, I prove that most simple rank 2-groups(except $PSL_3(\mathbb{F}_p)$, where p an odd prime) act freely on a finite CW-complex $Y \simeq S^n \times S^m$. It partially proves the famous conjecture which states that a finite group G acts freely on a finite CW-complex Y with homotopy type of products of $r(G)$ spheres,$r(G)$ denoting the rank of G.

Contents

Introduction

This master thesis is concerned about the finite group actions on a product of spheres. This problem has more than 60 years history, and people have found many interesting and deep results about it. First let me give you two easy examples

Proposition 0.0.1. *The finite cyclic group is the only finite group can act freely on* S^1.

Proof. Let G be a finite group having free action on S^1. So we have a covering projection

$$p : S^1 \to S^1/G$$

Moreover, we know that S^1/G is homeomorphic to S^1. Hence

$$G \simeq \pi_1(S^1)/p_*(\pi_1(S^1)) \simeq \mathbb{Z}/|G|$$

□

Proposition 0.0.2. *The group* $\mathbb{Z}/p \times \mathbb{Z}/p$ *where* p *is a prime cannot act freely on any sphere.*

Proof. Suppose $G = \mathbb{Z}/p \times \mathbb{Z}/p$ can act freely on S^n. By the theory of covering spaces we know there is a covering projection $p : S^n \to S^n/G$. And according to the previous result I can assume that $n > 1$.

Now I can use this covering projection to create a model for the Eilenberg-Maclane space $K(\mathbb{Z}/p \times \mathbb{Z}/p; 1)$. First we have $\pi_1(S^n/G) \simeq \mathbb{Z}/p \times \mathbb{Z}/p$, moreover we know $\pi_n(S^n/G) = \pi_n(S^n) \simeq \mathbb{Z}$. So I can just use a $(n+1)$-cell to kill the n-th homotopy group and use higher cells to kill the higher homotopy groups so that we get a new space X has the homotopy type $K(\mathbb{Z}/p \times \mathbb{Z}/p; 1)$.

So the cellular structure of X has one $(n+1)$-cell. Then by the argument of cellular cohomology with $\mathbb{Z}/p$ as coefficients we know the $n+1$ degree cohomology group is either $\mathbb{Z}/p$ or 0. Moreover, since $K(\mathbb{Z}/p \times \mathbb{Z}/p; 1) = K(\mathbb{Z}/p; 1) \times K(\mathbb{Z}/p; 1)$ in the homotopy sense. So I apply the Künneth formula to the product of spaces

$$H^{n+1}(K(\mathbb{Z}/p \times \mathbb{Z}/p; 1), \mathbb{Z}/p) = \mathbb{Z}/p \oplus \mathbb{Z}/p$$

That is a contradiction. □

In this thesis I mainly followed the the paper by Alejandro Adam and Jeff Smith[4] which proved a result about simple rank 2-groups. In order to introduce the readers to the nature of this problem first let me state the big conjecture of this problem. To describe the conjecture more precisely, I define the p-rank of a finite group G as $r_p(G) = \max\{r|(\mathbb{Z}/p)^r \subset G\}$ and the rank of G as $r(G) = \max\{r_p(G)|p||G|\}$.

Conjecture 1. *G is a finite group. If G acts freely on $X = S^{n_1} \times \cdots \times S^{n_m}$ then $r(G) \leq m$.*

In fact it is more common to consider actions on the homotopy type of spheres in algebraic topology. And we have a homotopical version of this conjecture.

Conjecture 2. *A finite group G acts freely on a finite complex X which has the homotopy type of a product of $r(G)$ spheres.*

As for the simplest case of the conjecture 1, it was completely solved in 1976 by Madsen, Thomas and Wall[18]

Theorem 0.1. *A finite group G acts freely on a sphere if and only if every abelian subgroup is cyclic and every element of order 2 must be central.*

As for the general case, it has been studied intensely since 80's and it has been proven that the conjecture is true under some additional assumptions. In the following I give some known results about this conjecture.

Theorem 0.2. *Assume G acts freely on a finite complex $X \simeq (S^n)^m$. Then $r(G) \leq m$ if p is an odd prime. If $p = 2$, then it holds for $n \neq 1, 3, 7$.*

Klause [17] showed that

Theorem 0.3. *For p an odd prime, every finite p-group of rank 3 acts freely and cellularly on finite complex homotopical to the product of three spheres.*

As for my thesis, I am concerned about the result which is about the rank two groups shown by Adem and Smith.

Theorem 0.4. *Let G denote a rank two simple group except $PSL_3(\mathbb{F}_3)$ with p an odd prime. Then G acts freely on a finite CW-complex $X \simeq S^m \times S^n$.*

Besides, in 2009, Hanke [14] showed that this conjecture is true on the condition that p is relatively large. More precisely,

Theorem 0.5. *If $(\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times \cdots \times S^{n_k}$ and $p > 3 \dim X$, then $r \leq k$*

Structure of the Thesis

In order for this thesis to be cohesive it is separated into four parts: Cohomology of Groups, Periodic Complex, Effective Euler Classes and Local Euler Classes and Rank Two Groups.

The basic notions and results of cohomology of groups are necessary for the following discussion of the thesis. I first use both algebraic and topological ways to define what the group cohomology is. Both ways have their own advantages for describing the group cohomology. Then I introduce several important functors in cohomology of groups like restriction, induction, coinduction and transfer maps with some formulas to connect these functors, such as the famous Cartan-Eilenberg double cosets formula. Then I use the notion of complete resolution to define what Tate cohomology is and use it to describe the first important concept related to my thesis: periodic groups. Also I make use of the duality theorem to give some equivalent description of periodic groups. We will use that in Chapter 2.

Theorem 0.6. *The following conditions are equivalent*

1. *G has periodic cohomology*
2. *There exists integers n and d , with $d \neq 0$ such that $\hat{H}^n(G, M) \simeq \hat{H}^{n+d}(G, M)$ for all $\mathbb{Z}G$-modules M.*
3. *For some $d \neq 0$ we have $\hat{H}^d(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z}$.*
4. *For some $d \neq 0$, $\hat{H}^d(G, \mathbb{Z})$ contains an element u of order of group G.*

Finally I use double complexes to describe the equivariant cohomology of a space X and prove that it is equivalent to the singular cohomology of the Borel construction $X \times_G EG$.

The second and third parts of this thesis is about periodic complexes and Euler classes. The main result of these two part is the following theorem:

Theorem 0.7. *Let X be a connected CW-complex. The cohomology of X is periodic if and only if there is a spherical fibration $E \to X$ with a total space E that is homotopy equivalent to a finite dimensional CW-complex.*

The proof of this big theorem is based on partial Euler classes which is constructed by the Postnikov towers of spheres. Using partial Euler class to approximate the final Euler classes. In order to do it we need a technical lemma about partial Euler classes and which needs a long argument:

Lemma 0.0.1. *Let X be a connected CW-complex and $k \geq 0$ be an integer. For every cohomology class $\alpha \in H^*(X)$ of positive degree there is an integer $q \geq 1$ such that the cup power α^q is a k-partial Euler class.*

Then we can redefine the finite group G with periodic cohomology by saying its classifying space BG has periodic cohomology groups. And I prove that these two definitions about periodic groups are equivalent.

Moreover, as an application of periodic complexes there is a useful result which is helpful to construct the group actions on a product of spheres.

Theorem 0.8. *Let X denote a finite dimensional G-CW complex, where G is a finite group, such that all of its isotropy subgroups have periodic cohomology. Then there exists a finite dimensional CW-complex Y and a free G-action such that $Y \simeq S^N \times X$ for some positive integer N. Moreover, if X is simply connected and finitely dominated, then we can assume that Y is a finite complex.*

In the rest of this thesis I focus on proving the result for group actions of simple groups of rank two on a product of two spheres. The method used representation theory to construct the group actions we want. We all know that constructing free actions of groups or even simple groups is a particular difficult problem. In this chapter using character theory I construct free actions of some simple groups.

Theorem 0.9. *Each of the simple groups $A_5, SL_3(\mathbb{F}_2), SU(3,3)$ and $SU(3,4)$ acts on linear spheres with rank-one isotropy. Hence each acts freely on a finite complex that is homotopy equivalent to a product of spheres $S^n \times S^m$.*

However, this method doesn't work for many simple rank two groups. So in chapter 3 I develop a method to construct p-local Euler classes from local subgroups. Then I glue the p-local spheres to produce an action on a homotopy sphere having an integral Euler class.

Theorem 0.10. *Let G be a finite group, an integral cohomology class $\alpha \in H^{2n}(BG, \mathbb{Z})$ with $n \geq 2$ is an effective Euler class if and only if for each prime p such that $r(G) = r_p(G)$ its p-localization $\alpha_p \in H^{2n}(BG, \mathbb{Z}_{(p)})$, is an effective p-local Euler class.*

Then I concentrate on the prime 2 case and use amalgamations to prove that

Theorem 0.11. *The simple rank 2 group G acts on some $Y \simeq S^{2k-1}_{(2)}$ such that the associated Euler class is 2-effective.*

By using this result and the method of gluing the local Euler classes I get the goal of this thesis finally.

Theorem 0.12. *Let G denote a rank two simple group except $PSL_3(\mathbb{F}_p)$ with p an odd prime. Then G acts on a finite freely on a finite complex $Y \simeq S^n \times S^m$.*

For the convenience of some readers I add two appendices at the end. The appendix A is about homological algebra and the appendix B is about finite groups.

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Chapter 1

Cohomology of Groups

In this chapter, I will introduce basic notions of cohomology of groups that are necessary for our following discussions about group actions on spheres. I will first use two equivalent ways(algebraic and topological) to define group cohomology. Next, I will talk about the famous Cartan-Eilenberg double cosets formula which is important when computing group cohomology. Then I will introduce Tate cohomology and use it to discuss periodic groups. Finally I will give a spectral sequence related to group cohomology.

1.1 Classifying Spaces and The Topological Interpretation of Group Cohomology

Topological Groups and Principal Bundles

We will first introduce some basic definitions about topological groups and fiber bundles.

Definition 1.1.1. *A topological group G is a group with topology such that the multiplication map $G \times G \to G$ given by $(a, b) \to ab$ and the inverse map $G \to G$ given by $a \to a^{-1}$ are both continuous.*

Fiber bundles are ubiquitous in geometry and topology. People usually use properties of fiber bundles to reflect the properties of base spaces and many interesting geometric objects are fiber bundles.

Definition 1.1.2. *We assume the space B is connected and $b_0 \in B$. A continuous map $p : E \to B$ is a fiber bundle with fiber F if it satisfies the 3 following conditions:*

1. *$p^{-1}(b_0) = F$*
2. *$p : E \to B$ is surjective*
3. *For any point $b \in B$ there is an open neighborhood $U_b \subset B$ and a homeomorphism $\Psi : p^{-1}(U_b) \to U_b \times F$ which makes the following diagram commute.*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr] \arrow[dr] & & U_b \times F \arrow[dl] \\ & U_b & \end{tikzcd}$$

In this case we call the space B the base space and the space E the total space.

When the total space E is itself homeomorphic to the product $B \times F$ we call this fiber bundle as a trivial bundle.

The next important definition which called principal bundles will involve the two definitions about topological groups and fiber bundles above.

Definition 1.1.3. *Let G be a topological group, a principal G bundle is a fiber bundle $p : E \to B$ with fiber G such that*

1. *The total space E has a free right group action $\mu : E \times G \to E$ making the following diagram commute:*

$$\begin{tikzcd} E \times G \arrow[r, "\mu"] \arrow[d, "p\times\epsilon"] & E \arrow[d, "p"] \\ B \times 1 \arrow[r, "id"] & B \end{tikzcd}$$

2. *The induced action on fibers*

$$\mu : p^{-1}(b) \times G \to p^{-1}(b)$$

is free and transitive.

3. *For any point $b \in B$ there is an open subset U_b and a equivariant homeomorphism $\Psi : p^{-1}(U_b) \to p^{-1}(U_b) \times G$ such that the following diagram is commutative*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr] \arrow[dr] & & U_b \times G \arrow[dl] \\ & U_b & \end{tikzcd}$$

where the equivariant map Ψ means for any $g \in G, x \in p^{-1}(U_b)$ we have

$$\Psi(xg) = \Psi(x)g$$

Remark 1.1.1. *It is natural to ask if we have a free G action on a topological space E, is it the quotient map $\pi : E \to E/G$ a principal G bundle? The answer is no in general, however if we have an additional restriction that if the quotient map has local sections then it does. We omit the proof here, but people could find the detailed discussion about it in Steenrod's excellent book about fiber bundles[28]. In the following content, cases we meet are always like these.*

Definition 1.1.4. *A morphism between two principal G-bundles with same base space B is the equivariant maps $f : E_1 \to E_2$, where E_1, E_2 are two total spaces of these two principal G-bundles.*

Principal G-bundles are one of basic geometric objects in topology and geometry. For example, in differential geometry, many concepts like curvature are defined using connections on principal G-bundles. In topology, the classification of principal G bundles will induce the classifying spaces BG. Indeed, much of research in homotopy theory over last fifty years involves analyzing the homotopy type of BG for interesting group G. In next subsection we will discuss the definitions and properties of classifying spaces.

Classifying Spaces

In this subsection we want to introduce a nice space BG related to a topological group G which play a central role in the classification of principal G bundles. First we give a natural way to construct a new bundle out of an old.

Theorem 1.1. *Given a fiber bundle $p : E \to B$ with fiber F and a map $f : X \to B$, there is an induced bundle, denoted by $f^*(E)$ which is the pullback in the following sense:*

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

it is a new fiber bundle over X.

Now we give the famous classification theorem by Milnor:

Theorem 1.2 (Milnor)**.** *Given a group G there is a space BG and a principal G-bundle with total space EG*

$$G \to EG \to BG$$

so that if $G \to E \to X$ is any principal G bundle over X, there is a unique map up to homotopy $f : X \to BG$ such that $f^*(EG) = E$.

$$\begin{tikzcd} f^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Remark 1.1.2. *The space BG is called the classifying space for G, and EG is called the universal G-bundle. It is clear that the classifying spaces BG is unique up to homotopy. Since if there is another classifying space BG' then there are two maps $f : BG \to BG'$ and $g : BG' \to BG$. It is clear that $fg \sim id$ and $gf \sim id$.*

Remark 1.1.3. *If G is a discrete group, then the exact sequence of homotopy groups shows that the homotopy groups of BG is*

$$\pi_i(BG) = \begin{cases} 0, & i > 1 \\ G, & i = 1 \end{cases}$$

So it is just the Eilenberg-Maclane space $K(G, 1)$.

In the following I want to give a concise construction of EG and BG, people interested in detailed construction could refer the Milnor's famous paper [20]

Definition 1.1.5. *Let X and Y be two topological spaces then the* ***join*** *$X * Y$ is defined by*

$$X * Y = X \times I \times Y / \sim$$

Where the equivalence relations are:

$$(x, 0, y) \sim (x', 0, y)$$

$$(x, 1, y) \sim (x, 1, y')$$

Example 1.1.1. *(A model of classifying spaces)*
Now a model for EG is constructed as $EG = colimit_n G^n$, where $G^n = G * \cdots * G$ is the n-th fibre joins of G. So we can express EG as formal elements:

$$(t_1 g_1, t_2 g_2, \dots)$$

where $t_i \in [0, 1]$ with $\sum_{i=i}^{\infty} t_i = 1$ and $g_i \in G$. Clearly there is a natural free right G action defined by

$$(t_1 g_1, \cdots)g = (t_1 g_1 g, \cdots)$$

Then let BG be the set of orbits space of EG under the right G-action with quotient topology. In other words we have a G-bundle as follows:

$$G \to EG \to BG$$

The fact is that this bundle is the universal bundle we want [20]. Next I present several basic properties of EG.

Proposition 1.1.1. *The total space of the universal bundle EG is weekly contractible.*

Proof. We need to show that for any n, $\pi_n(EG) = 0$. So consider a map $f : S^n \to EG$. Since S^n is compact, there is a integer N such that $im(f) \subset G^N$. Then consider the natural inclusion of spaces $G^N \hookrightarrow G^{N+1}$. Let $g \in G$ be the base point of the topological space G. Then it is clear that $G^N \hookrightarrow G^N * g$. And by the construction of fiber join we know that $G^N * g$ is a cone in G^{N+1} so it is contractible in G^{N+1}. It implies the image of S^n is contractible in EG. In other words $\pi_n(EG) = 0$. Therefore EG is weakly contractible. □

In fact the property of being weakly contractible determines the universal bundle completely. More precisely [20]

Proposition 1.1.2. *Let G be a topological space. Suppose there is a principal G–bundle $p : E \to X$ where X is a CW-complex. Then this bundle is a universal G-bundle if and only if E is weakly contractible*

I have introduced the classification of principal G-bundles, however it is not sufficient for this thesis. So at the end of this subsection, I will generalize the classification theorem of principal G-bundles to general fibrations. For more details people could refer May's paper [19]

Let F be a CW-complex and $Aut(F)$ be the topological monoid of self homotopy equivalences of F. Moreover let $Aut_I(F)$ be the identity component of $Aut(F)$.

Theorem 1.3. *Let X be a CW-complex.Then the fibre homotopy equivalence of fibrations with X as base space and F as homotopy fiber are natural one-to-one correspondence with the homotopy classes between X and $Aut(F)$: $[X, BAut(F)]$.*

As for the oriented fibrations with X as base space and F as fiber, we replace $Aut(F)$ by $Aut_I(F)$: $[X, Aut_I(F)]$. I will define what an oriented fibration is in Chapter 2.

Cohomology of Groups

Let G be a topological group and A an abelian group, we define the homology and cohomology of group G with coefficient A as follows:

$$H_*(G; A) = H_*(BG; A)$$

$$H^*(G; A) = H^*(BG; A)$$

Remark 1.1.4. *It's clear that $H^*(-, A)$ is a contravariant functor from category of topological groups to category of abelian groups. Similarly $H_*(-; A)$ is a covariant functor.*

In the following discussion I will mainly focus on cohomology of groups. In many cases singular cohomology has more advantages than singular homology since it has a natural cup product structure. Based on this definition we know cohomology of groups has a natural cup product. I will give an alternative description about cohomology of groups using algebraic methods which are more computable and I will prove that these two definition coincide.

1.2 Algebraic Interpretation of Group Cohomology

Group Rings

We assume R is a ring with unit and G is a group. We define the group ring of G as

$$R[G] = \{\sum r_i g_i; r_i \in R, g_i \in G\}$$

Where the sum is a finite sum. It's clear that this new object is a ring with obvious addition and multiplication. Moreover we have an augmentation map:

$$R[G] \longrightarrow R$$

which is a R-map and maps every single element G to the unit of ring R. In the following we will always assume R is the integer $\mathbb{Z}$. The related group ring is $\mathbb{Z}G$.

Remark 1.2.1. *We can view group G as a subgroup of group ring $(\mathbb{Z}G)^*$ of units of $\mathbb{Z}G$. Generally, $\mathbb{Z}(-)$ is a functor from the category of groups to the category of ring and take unit of a ring is a functor from the category of rings to the category of groups. More precisely, we have the following adjunction formula:*

$$Hom_{Ring}(\mathbb{Z}G, R) \approx Hom_{Groups}(G, R^*)$$

Given a ring R, it is usual to consider modules over this ring. So a (left)$\mathbb{Z}G$-module, consists of an abelian group A and a (left)G-action on this group. For example, we can view $\mathbb{Z}$ as a trivial $\mathbb{Z}G$ module with trivial action of $\mathbb{Z}G$ on $\mathbb{Z}$.

Projective Resolutions and Group Cohomology

In this subsection, I will use projective resolutions of $\mathbb{Z}$ as $\mathbb{Z}G$-module to give a new definition of cohomology of groups.

Definition 1.2.1. *Let G be a group, A an abelian group(which we also can view it as a trivial $\mathbb{Z}G$-module) and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then we define the cohomology of G as follows:*

$$H^*(G; A) = H^*(Hom_{\mathbb{Z}G}(F; A))$$

Remark 1.2.2. *From homological algebra we know that any two projective resolution of $\mathbb{Z}$ are chain homotopy, so we will have a canonical isomorphism between the two cohomology we got. That means that it is well-defined. Moreover, by the uniqueness of projective resolutions up to chain homotopy we know that $H^*(-; A)$ is a contravariant functor for the category of groups to the category of abelian groups.*

Remark 1.2.3. *From homological algebra we know $H^*(G; A) \approx Ext^*_{\mathbb{Z}G}(\mathbb{Z}; A)$. So more general, given any $\mathbb{Z}G$-module M we can define the group cohomology of G with coefficient M as $H^*(G; M) = Ext^*_{\mathbb{Z}G}(\mathbb{Z}; M)$. And it's easy to see it is equivalent to $H^*(Hom_{\mathbb{Z}G}(F; M))$.*

Remark 1.2.4. *It is obvious that all constructions above also work for any ring R.*

Now I want to give some methods to compute the free(projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Example 1.2.1. *We say a CW-complex X is a G-complex if we have a G-action on it which permutes cells. So it induces a free G-action on the cellular chains $C_*(X)$. Which means that $C_*(X)$ are $\mathbb{Z}G$-modules. And from cellular homology we know that we can form a chain complex by these cellular chains.*

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \to \mathbb{Z}$$

Theorem 1.4. *If X is a contractible free G-complex, then the chain complex above is an exact sequence.*

So in this case we get a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

The method above is still too general to use.

The following example is a standard resolution of $\mathbb{Z}$ called the **bar resolution** which is much easier to handle.

Example 1.2.2 (Bar Resolution)**.** *The bar resolution is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules*

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}$$

where F_n is the free $\mathbb{Z}G$-module generated by the $(n+1)$-tuples $(g_0, \cdots, g_n)$ of elements in G. And there is a natural left G-action on it given by

$$g(g_0, \cdots, g_n) = (gg_0, \cdots, gg_n)$$

And the boundary map $\partial : F_n \to F_{n-1}$ is the alternating sums $\partial = \sum_{i=0}^{n}(-1)^i d_i$, where

$$d_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n)$$

More ovre we have an augmentation map $\epsilon : F_0 \to \mathbb{Z}$ defined by $\epsilon(g) = 1$ for any $g \in G$.

It is easily seen that it is a free resolution of $\mathbb{Z}$ as $\mathbb{Z}G$-modules.

Cup Products

We have a natural cup product structure of group cohomology using topological definition. However, in this subsection I will add a cup product structure in group cohomology defined via algebraic way. Here I just give the concise construction, people could refer Brown' book[8] for proof.

Before introducing cup products I first introduce cross product. Given group G, G' and its module M and M' respect. We want to construct a map in the following sense:

$$H^p(G; M) \otimes_{\mathbb{Z}} H^q(G', M') \longrightarrow H^{p+q}(G \times G'; M \otimes_{\mathbb{Z}} M')$$

So given two elements $u \in Hom_{\mathbb{Z}G}(F_n, M)$ and $u' \in Hom_{\mathbb{Z}G}(F'_m, M')$ we define the cross product $u \times u'$ in cochain level as

$$u \times u'(x \otimes x') = (-1)^{degu' \times degx} u(x) \otimes u'(x')$$

It is easy to see that this is a cochain map, so it induces on cohomology level which is want we want.

Next, we assume G and G' coincide and use the diagonal map $G \to G \times G$ given by $g \to (g, g)$. By functoriality, we get maps as follows:

$$H^p(G; M) \otimes_{\mathbb{Z}} H^q(G, M') \longrightarrow H^{p+q}(G \times G; M \otimes_{\mathbb{Z}} M') \longrightarrow H^{p+q}(G; M \otimes M')$$

The composition of these two maps is the cup product structure we want. In particular if two modules M and M' are both trivial module $\mathbb{Z}$. We have the cup product map in the following:

$$H^p(G; \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(G; \mathbb{Z}) \longrightarrow H^{p+q}(G; \mathbb{Z})$$

Equipped with the cup product, group cohomology $H^*(G; M)$ is in fact a graded commutative ring. More precisely

Theorem 1.5. *There is an element $1 in$ $H^0(G; \mathbb{Z})$ such that $u \cup 1 = 1 \cup u = u$ for any $u \in H^*(G; M)$. Moreover, cup product is associative and graded commutative. For graded commutativity we mean for any $u \in H^p(G; M), u' \in H^q(G; M')$ we have $u \cup u' = (-1)^{degu \times degu'} t_*(u' \cup u)$ where $t : M' \otimes M \to M \otimes M'$ is the canonical isomorphism.*

The Equivalence of Two Definitions of Group Cohomology

In this subsection I will prove that these two definition coincide when G is a discrete group. Since we defined a more general group cohomology in module coefficients we need to generalize the topological definition. The way we do is via local coefficients. Given a connected space X with a universal covering $\tilde{X}$. We denote G to be the fundamental group of X. According to covering spaces theory we know that G acts on $\tilde{X}$ via deck transformations. So it induces an action of G on its singular chain $C_*(\tilde{X})$. In this case, $C_*(\tilde{X})$ become a $\mathbb{Z}G$-module. Now given a $\mathbb{Z}G$-module M, we define the cohomology groups of X with coefficient M as follows:

$$H^*(X;M) = H^*(Hom_{\mathbb{Z}G}(C_*(\tilde{X}), M))$$

Remark 1.2.5. *This definition would back to the original definition of singular cohomology when M is just an abelian group. Since when M is an abelian group, it is a trivial $\mathbb{Z}G$-module. In this case we have*

$$Hom_{\mathbb{Z}G}(C_*(\tilde{X}), M) = Hom_{\mathbb{Z}}(C_*(X), M)$$

So it backs to the ordinary cohomology with abelian groups as coefficients.

Theorem 1.6. *Let M be a $\mathbb{Z}G$-module, then there is a natural isomorphism as abelian groups*

$$H^n(G, M) \simeq H^n(BG, M)$$

Proof. Let $EG \to BG$ be a universal cover of BG then by definition of singular cohomology with local coefficients

$$H^n(BG, M) = H^*(Hom_{\mathbb{Z}G}(C_*(EG), M))$$

On the other hand since $C_*(EG)$ are free $\mathbb{Z}G$-modules we know this expression is exactly the definition of group cohomology by algebraic way. □

Remark 1.2.6. *In fact this isomorphism is compatible with the ring structure, for the proof readers can see[8]*

1.3 Two Important Functors in Group Cohomology

Extension and Co-extension of Scalars

In group (co)homology it is natural to consider the change of coefficients M.

In general, let R, S be two rings with unit. And we have a ring map $\alpha : R \to S$.For any left S-module M, we can view it as a left R-module by $r \cdot m = \alpha(r) \cdot m$ So we have a functor from the category of left S-modules to the category of left R-modules:

$$Res : (S - Mod) \longrightarrow (R - Mod)$$

Next I will introduce two functors which are adjoint to these restriction functor.

Example 1.3.1. *(Tensor product)*
It is clear from the ring map α we can view S as a right R module, so we can use tensor product $S \otimes_R -$ to get a new functor which is called the extension of scalars

$$Ind : (R - mod) \xrightarrow{S \otimes_R -} (S - Mod)$$

And given a left R-module, we have a natural map

$$i : M \to S \otimes_R M$$

by sending m to $1 \otimes m$. It is clear that this map is an R-map. And given a S module N we have a universal property in the following:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & S \otimes_R M \arrow[dl, dotted, "g"] \\ N & \end{tikzcd}$$

That is we have a bijection as follows:

$$Hom_S(Ind(M), N) \cong Hom_R(M, Res(N))$$

In other words, the functor Ind is left adjoint to the functor Res.

$$\begin{tikzcd} R - Mod \arrow[r, shift left, "Ind"] & S - Mod \arrow[l, shift left, "Res"] \end{tikzcd}$$

Example 1.3.2. *(Hom functor)*
Dually, we can use Hom to form a new functor. More precisely, it is natural to view S as a left R-module via α. Given a left R-module M we can get a new left S module $Hom_R(S, M)$, the module structure is given by $sf(s') = f(ss')$. So we have a functor from R-modules to S-modules which is called the co-extension of scalars:

$$Coind : R - Mod \xrightarrow{Hom_R(S,-)} S - Mod$$

And we have a natural map

$$\pi : Hom_R(S, M) \to M$$

by sending f to $f(1)$. And given a S module N and an R map $f : N \to M$ we have a universal property in the following

$$\begin{tikzcd} & Hom_R(S, M) \arrow[d, "\pi"] \\ N \arrow[ur, dotted, "g"] \arrow[r, "f"] & M \end{tikzcd}$$

That is we have a bijection

$$Hom_S(N, Coind(M)) \cong Hom_R(Res(N), M)$$

In other words, the functor Res is left adjoint to the functor Coind.

$$\begin{tikzcd} S - Mod \arrow[r, shift left, "Res"] & R - Mod \arrow[l, shift left, "Coind"] \end{tikzcd}$$

Induced and Co-induced Modules

In this subsection we apply the general two functors in previous section to the group rings case. So let G be a group and H be a subgroup of G.

Definition 1.3.1. *We define two functors from the category of $\mathbb{Z}H$-modules to the category of $\mathbb{Z}G$-modules. Let M be a $\mathbb{Z}H$-module, and M could be viewed as a $\mathbb{Z}H$-module via the canonical inclusion $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$.*

$$Ind_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

$$Coind_H^G M = Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

In particular when H is trivial then $Ind_1^G M$ is called the induced module of M and $Coind_1^G M$ is called the co-induced module of M.

Proposition 1.3.1. *The $\mathbb{Z}G$-module $Ind_H^G M$ contains M as a $\mathbb{Z}H$-submodule and*

$$Ind_H^G M = \bigoplus_{g \in G/H} gM$$

Proof. By right multiplication of G we have a free action of H on G. So $\mathbb{Z}G$ can be viewed as a free $\mathbb{Z}H$-module and

$$\mathbb{Z}G = \bigoplus_{g \in G/H} \mathbb{Z}H$$

Then

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M = \Big(\bigoplus_{g \in G/H} \mathbb{Z}H \Big) \otimes_{\mathbb{Z}H} M = \bigoplus_{g \in G/H} g \otimes M$$

where $g \otimes M = \{g \otimes m : m \in M\}$. It is clear that $g \otimes M \simeq M$ by sending $g \otimes m$ to m. And we have a canonical $\mathbb{Z}H$ map i as follows:

$$M \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

$$m \longmapsto 1 \otimes m$$

So we can view $1 \otimes M$ as a $\mathbb{Z}H$-submodule of $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$. Moreover, the summand $g \otimes M$ is simply the transform of this submodule under the action of g. So we have

$$Ind_H^G M = \bigoplus_{g \in G/H} gM$$

$\square$

This proposition easily implies a useful formula for describing $Ind_H^G Res_H^G$:

Proposition 1.3.2. *Let $H \subset G$ as a subgroup and N be a $\mathbb{Z}G$-module then*

$$Ind_H^G Res_H^G N \simeq [G/H] \otimes N$$

where G acts diagonally on the tensor product

In the next subsection I will use the double cosets decomposition to give a formula about $Res_H^G Ind_H^G$.

Moreover, the description about the induction functor completely characterizes $\mathbb{Z}G$-modules with the form $Ind_H^G M$.

Proposition 1.3.3. *Let N be a $\mathbb{Z}G$-module whose underline abelian group is of the form $\bigoplus_{i \in I} M_i$. Assume that the G-action permutes the summands according to some actions of G on the index set I. Let G_i be the isotropy group of i and let E be a set of orbits I/G. Then M_i is a $\mathbb{Z}G_i$-module and there is a $\mathbb{Z}G$-isomorphism*

$$N \simeq \bigoplus_{i \in E} Ind_{G_i}^G M_i$$

Next example is an important application of the materials we discussed before and it will be used in the rest of this thesis.

Example 1.3.3. *Let X be a CW-complex with a G-action on it. So the cellular chains $C_*(X)$ are $\mathbb{Z}G$-modules. In fact the underline abelian group of $C_n(X)$ is $\oplus_i \mathbb{Z}$, one n-cell corresponds to one copy of $\mathbb{Z}$. Then by the previous proposition we know*

$$C_n(X) = \bigoplus_{\sigma \in E} Ind_{G_i}^{G} \mathbb{Z}$$

where E is the set of representatives for G-orbits of n-cells.

There is also a formula to describe $Res_H^G Ind_H^G$, I will give it in next section.

Applying the two functors to group cohomology we can get an important result which is very useful in proofs and calculations.

Theorem 1.7. *(Shapiro's lemma) Let G be a group and H be a subgroup of G. M is a $\mathbb{Z}H$-module, then we have*

$$H_*(H, M) \simeq H_*(G, Ind_H^G M)$$

$$H^*(H, M) \simeq H^*(G, Coind_H^G M)$$

Proof. Let F be a projective resolution of F of $\mathbb{Z}G$-modules. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. So we can also view F as a projective resolution of $\mathbb{Z}H$-modules. So by definition we have

$$H_*(H, M) = H_*(F \otimes_{\mathbb{Z}H} M)$$

On the other hand, we know that

$$F \otimes_{\mathbb{Z}H} M = F \otimes_{\mathbb{Z}G} \left(\mathbb{Z}G \otimes_{\mathbb{Z}H} M\right) = F \otimes_{\mathbb{Z}G} \left(Ind_H^G M\right)$$

So $H_*(H, M) \simeq H_*(G, Ind_H^G M)$.

The proof of the second equation is similar. $\square$

Finally, in the category of sets we know that the finite products coincides with the finite co-products. So we have

Proposition 1.3.4. *When $[G : H] < \infty$, then the two functors Ind_H^G and $Coind_H^G$ coincide.*

Proof. Given a $\mathbb{Z}G$-module M we have:

$$\begin{aligned} Hom_{\mathbb{Z}H}(\mathbb{Z}G, M) &= Hom_{\mathbb{Z}H}\Big(\bigoplus_{g \in G/H} \mathbb{Z}H, M\Big) \\ &= \prod_{g \in G/H} Hom_{\mathbb{Z}H}(\mathbb{Z}H, M) \\ &= \bigoplus_{g \in G/H} M \\ &= \mathbb{Z}G \otimes_{\mathbb{Z}H} M \end{aligned}$$

$\square$

1.4 The Cartan-Eilenberg Double Cosets Formula

In this section, I will introduce a powerful method for calculations of group cohomology called the **Cartan-Eilenberg double cosets formula**.

Double Cosets

Definition 1.4.1. *Let G be a group and H, K be its two subgroups. Let H acts on G by left multiplication and K acts on G by right multiplication. Then we say a (H, K)-double coset of X is*

$$HxK = \{hxk : h \in H, k \in K\}$$

When $H = K$ we call it H-double coset of x. Equivalently, we have a equivalence relation $\sim$ on G which is $x \sim y$ if there is elements $h \in H$ and $k \in K$ such that $hxk = y$. We use $H\backslash G/K$ denoting the quotient set of G by this equivalence relation, and it is also the set of double cosets of G.

After stating what are double cosets, I give some necessary properties of it in the following.

1. We have an other nice description about double cosets. Let $H \times K$ acts G by $(h, k)x = hxk^{-1}$ then it is clear that double cosets are orbits of this group action.
2. If H is a normal subgroup of G, then $H\backslash G$ is in fact a group. Then the right action of K on $H\backslash G$ factor through the right action of $H\backslash HK$ on it. Then by the isomorphism theorem we know $H\backslash G/K = HK\backslash G$.
3. $HxK = \coprod_{k \in K} Hxk = \coprod_{Hxk \in H\backslash xK} Hxk$
4. Since we can consider the double cosets HxK as the coset of right action of K on cosets Hx. And the stabilizer is $K \cap x^{-1}Hx$, so the number of cosets of $H\backslash G$ contained in HxK is $[K : K \cap x^{-1}Hx]$

The Transfer Map

In section 1.3 we introduced two important functors called induction functor and co-induction functor for any group G and its subgroup H. In this subsection, I focus on these functors on cohomology of groups and give an extremely important map called the **transfer map**.

Let G be a group and H be a subgroup of G, then we have a natural inclusion $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$ and it can induce a restriction map as follows:

$$Res_H^G : H^*(G, M) \to H^*(H, M)$$

And I have an alternating description of this map. Given a $\mathbb{Z}G$-module M, consider a natural inclusion

$$\alpha : M \hookrightarrow Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

Then apply the functor $H^*(G, -)$ and use the Shapiro's lemma we get:

$$\alpha^* : H^*(G, M) \to H^*(H, M)$$

And it is easily to see that α^* and Res_H^G coincide. Moreover, we also have a canonical epimorphism as follows:

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M$$

When $[G : H] < \infty$, by proposition 1.3.4, we know that $\mathbb{Z}G \otimes_{\mathbb{Z}H} M \cong Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$. So we have a canonical epimorphism in this case

$$Hom_{\mathbb{Z}H}(\mathbb{Z}G, M) \to M$$

Then apply the functor $H^*(G, -)$ and use the Shapiro's lemma again we have

$$Tr_H^G : H^*(H, M) \to H^*(G, M)$$

This map is called the transfer map from H to G.

However, this definition is some kind of formal and it is not very useful for calculations. In the following I will introduce a constructive definition of the transfer map and people could refer [8] for the equivalence of these two definitions.

Let F be a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules. Since $H^*(H, M) = H^*(Hom_{\mathbb{Z}H}(F, M))$, I first define the transfer map on chain level.

$$Tr : Hom_{\mathbb{Z}H}(F_i, M) \to Hom_{\mathbb{Z}G}(F_i, M)$$

For any element $\alpha \in Hom_{\mathbb{Z}H}(F_i, M)$ and $\lambda \in Hom_{\mathbb{Z}G}(F_i, M)$. I define

$$Tr(\alpha)(\lambda) = \sum_{i=1}^{[G:H]} g_i \alpha(g_i^{-1}\lambda)$$

where $g_1, \cdots, g_{[G:H]}$ range over all elements of left cosets G/H. It is easily to verify that this map is well-defined and $Tr(\alpha)$ is a $\mathbb{Z}G$-linear map. Moreover,

$$\begin{aligned} Tr(\delta\alpha)(b) &= \sum_{i=1}^{[G:H]} g_i \delta\alpha(g_i^{-1}b) \\ &= \sum_{i=1}^{[G:H]} g_i \alpha(g_i^{-1}\partial b) \\ &= Tr(\alpha)(\partial b) \end{aligned}$$

where δ, ∂ are two differentials of two cochain complexes above respectively. So Tr is a cochain map, it can pass to cohomology level and we get

$$Tr_H^G : H^*(H, M) \to H^*(G, M)$$

Proposition 1.4.1. *Given an extension of groups $K \subset H \subset G$, then*

1. *for the restriction functor, we have:*

$$Res_K^H Res_G^H = Res_K^G$$

2. *moreover, if $[G : K] < \infty$ then we have:*

$$Tr_H^G Tr_H^K = Tr_K^G$$

Proof. 1. Since we have natural inclusions of group rings

$$\mathbb{Z}K \hookrightarrow \mathbb{Z}H \hookrightarrow \mathbb{Z}G$$

So on cochain level we have a commutative diagram as follows

$$\begin{tikzcd} Hom_{\mathbb{Z}G}(F, M) \arrow[r] \arrow[dr] & Hom_{\mathbb{Z}H}(F, M) \arrow[d] \\ & Hom_{\mathbb{Z}K}(F, M) \end{tikzcd}$$

where F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules. Then pass to cohomology level we can get the desired result.

2. On cochain level, let $\alpha \in Hom_{\mathbb{Z}K}(F_i, M)$, $\lambda \in Hom_{\mathbb{Z}G}(F_i, M)$, and $\beta = Tr_K^H(\alpha)$. Then we have

$$\begin{aligned}
Tr_H^G Tr_K^H(\alpha)(\lambda) &= \sum_{i=1}^{[G:H]} g_i \beta(g_i^{-1}\lambda) \\
&= \sum_{i=1}^{[G:H]} g_i \sum_{j=1}^{[H:K]} h_j \alpha(h_j^{-1} g_i^{-1} \lambda) \\
&= \sum_{i=1}^{[G:H]} \sum_{j=1}^{[H:K]} g_i h_j \alpha((g_i h_j)^{-1}\lambda) \\
&= \sum_{i=1}^{[G:K]} m_i \alpha(m_i^{-1}\lambda)
\end{aligned}$$

Then pass to cohomology level we can get the result we want.

□

The Cartan-Eilenberg Double Cosets Formula

In last subsection we saw the formula about the composition of two restriction functors or two transfer maps. In this subsection I will investigate the formula about the composition of the restriction functor and the transfer map.

Proposition 1.4.2. *Let G be a group and H be a subgroup of G. Assume that $[G : H] < \infty$. Given any $\mathbb{Z}G$-module M. Then*

$$Tr_H^G Res_H^G z = [G : H]z$$

for any $z \in H^*(G, M)$

Proof. Let $\alpha \in Hom_{\mathbb{Z}G}(F_n, \mathbb{Z})$ represents the cohomology class $z \in H^n(G, \mathbb{Z})$. In other words α is a $\mathbb{Z}G$-homomorphism and it is automatically a $\mathbb{Z}H$-homomorphism. Then by the definition of the transfer map on cochain level we have

$$(Tr_H^G Res_H^G \alpha)(\lambda) = \sum_{i=1}^{[G:H]} g_i \alpha(g_i^{-1}\lambda) = [G : H]\alpha$$

Pass to the cohomology level we complete the proof.

□

Before stating the Cartan-Eilenberg double cosets formula I need to introduce a notation. For an element $x \in G$ Let:

$$c_x : H^*(H, M) \to H^*(xHx^{-1}, M)$$

denote the isomorphism induced by the homomorphism

$$Hom_{\mathbb{Z}H}(F, M) \to Hom_{\mathbb{Z}(xHx^{-1})}(F, M)$$

given by $c_x(f)(u) = xf(x^{-1}u)$. Where $H \subset G$ as a subgroup, and F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules.

Theorem 1.8. *(Cartan-Eilenberg)*
Let G be a group and H, K be two subgroups of G. Assume that $[G:H] < \infty$. Given any $\mathbb{Z}G$-module M. Then

$$Res_H^G Tr_K^G = \sum_{x_i \in H\backslash G/K} Tr^H_{H\cap x_iKx_i^{-1}} Res^{x_iKx_i^{-1}}_{H\cap x_iKx_i^{-1}} c_{x_i}$$

Proof. Let N be the number of double cosets respect to H and K. By the property 4 I listed before about double cosets we know

$$[G:K] = \sum_{i=1}^{N} [H : H \cap x_iKx_i^{-1}]$$

and for any i let z_{ji} be the representative of coset $H/H \cap x_iKx_i^{-1}$. Then for any $\alpha \in Hom_{\mathbb{Z}K}(F_n, \mathbb{Z})$ where F is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules we have

$$\begin{aligned} Res_H^G Tr_H^G(\alpha) &= Res_H^G\big(\sum_{g\in G/K} g\alpha g^{-1}\big) \\ &= \sum_{i}^{N}\big(\sum_{j} z_{ji}x_i\alpha x_i^{-1} z_{ji}^{-1}\big) \\ &= \sum_{i}^{N} Tr^H_{H\cap x_iKx_i^{-1}} Res^{x_iKx_i^{-1}}_{H\cap x_iKx_i^{-1}} \cdot c_{x_i}(\alpha) \end{aligned}$$

Pass to the cohomology level we complete the proof. □

Next are several applications of this powerful formula.

Corollary 1.4.1. *If H is a normal subgroup of G and $[G:H] < \infty$. Then for any cohomology class $\alpha \in H^*(H, M)$ We have*

$$Res_H^G Tr_H^G(\alpha) = \sum_{g_i \in G/H} c_{g_i}(\alpha)$$

Proof. Since H is a normal subgroup of G, for any $g \in G$ we know $gHg^{-1} = H$. Now apply the Cartan-Eilenberg double cosets formula

$$Res_H^G Tr_H^G(\alpha) = \sum_{g_i \in G/H} Tr_H^H Res_H^H c_{g_i}(\alpha) = \sum_{g_i \in G/H} c_{g_i}(\alpha)$$

□

Remark 1.4.1. *In fact there is a similar double cosets formula regarding $Res_H^G Ind_H^G$*

$$Res_H^G Ind_K^G M = \bigoplus_{g \in \backslash G/K} Ind^H_{H\cap gKg^{-1}} Res^{gKg^{-1}}_{H\cap gKg^{-1}} gM$$

where M is a $\mathbb{Z}G$-module and gM can be viewed as a gKg^{-1} module.

Definition 1.4.2. *A cohomology class $\alpha \in H^*(H, M)$ is called **stable** if*

$$Res^{xHx^{-1}}_{H\cap xHx^{-1}} \cdot c_x(a) = Res^H_{H\cap xHx^{-1}}(a)$$

for any $x \in G$. In particular, if H is a normal subgroup of G, it is equivalent to $c_x(a) = a$ for any $x \in G$.

Proposition 1.4.3. *If $a \in Im(Res^G_H)$, then a is stable.*

Proof. Since $a \in Im(Res^G_H)$ then there is a cohomology class $b \in H^*(G, M)$. It is clear that

$$c_x : H^*(G, M) \to H^*(G, M)$$

is the identity map. In other words, $c_x(b) = b$

Moreover,

$$\begin{aligned} c_x(a) &= c_x \cdot Res^G_H(b) \\ &= Res^G_{xHx^{-1}} \cdot c_x(b) \\ &= Res^G_{xHx^{-1}}(b) \end{aligned}$$

Then we have:

$$\begin{aligned} Res^{xHx^{-1}}_{H\cap xHx^{-1}} \cdot c_x(a) &= Res^{xHx^{-1}}_{H\cap xHx^{-1}} Res^G_{xHx^{-1}}(b) \\ &= Res^H_{H\cap xHx^{-1}} Res^G_H(b) \\ &= Res^H_{H\cap xHx^{-1}}(a) \end{aligned}$$

□

Proposition 1.4.4. *If $a \in H^*(H, M)$ is stable, then $Res^G_H Tr^G_H(a) = [G : H]a$.*

Proof. By the Cartan-Eilenberg double cosets formula

$$\begin{aligned} Res^G_H Tr^G_H(a) &= \sum_{x_i \in H\backslash G/H} Tr^H_{H\cap x_iHx_i^{-1}} \cdot Res^{x_iHx_i^{-1}}_{H\cap x_iHx_i^{-1}} c_{x_i}(a) \\ &= \sum_{x_i \in H\backslash G/H} Tr^H_{H\cap x_iHx_i^{-1}} \cdot Res^H_{H\cap x_iHx_i^{-1}}(a) \\ &= \sum_{x_i \in H\backslash G/H} [H : H \cap x_iHx_i^{-1}]a \\ &= [G : H]a \end{aligned}$$

□

Finally, I give an application of the definition of stable elements.

Lemma 1.4.1. *If G is finite, then $H^n(G, M)$ is annihilated by $|G|$ for all $n > 0$. If G is invertible in M, then $H^n(G, M) = 0$ for all $n > 0$.*

Proof. Apply proposition 1.4.2 when $H = 1$ completing the proof. □

Remark 1.4.2. *It follows that $H^n(G, M)$ admits a p-primary decomposition in the following:*

$$H^n(G, M) = \bigoplus_{p||G|} H^n(G, M)_{(p)}$$

where $H^n(G, M)_{(p)}$ denotes the p-primary component of $H^n(G, M)$. Then for a fixed prime number p. Select a p-Sylow subgroup H of G. Because $[G : H]$ is relative to p. By proposition 1.4.2 $Tr^G_H Res^G_H = id$ on $H^n(G, M)_{(p)}$. It implies there is an injection

$$H^n(G, M)_{(p)} \hookrightarrow H^n(H, M)$$

Next theorem will use the p-Sylow subgroups H to describe the p-primary component of $H^n(G, M)$.

Theorem 1.9. *Let G be a finite group and H be its p-Sylow subgroup. Then the restriction map Res_H^G will map $H^n(G, M)_{(p)}$ isomorphically onto the set of stable elements of $H^n(H, M)$. In particular if H is a normal subgroup, then*

$$H^n(G, M)_{(p)} \simeq H^n(H, M)^{G/H}$$

Proof. By proposition 1.4.3 and the previous remark Res_H^G maps $H^n(G, M)_{(p)}$ injective onto the stable elements in $H^n(G, M)$.

Conversely, let $z \in H^n(H, M)$ be a stable element and let $w = Tr_H^G z$. Then by proposition 1.4.4 I have

$$Res_H^G = [G : H]z$$

Then it follows that $z = Res_H^G w'$ where $w' = w/[G : H] \in H^n(G, M)_{(p)}$. □

In the rest of thesis thesis we will often use this double cosets formula for real calculations about group cohomology.

1.5 Tate Cohomology and its Duality

In this section, I will introduce a standard technique for studying properties of finite groups called **Tate Cohomology**. In this section I always assume G is a finite group.

Tate Cohomology

As we saw, the definition of group cohomology is using projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$-modules. To define Tate cohomology, we need a new resolution called **Complete Resolutions**

Definition 1.5.1. *A complete resolution is an acyclic chain complex $F_* = (F_i)_{i \in \mathbb{Z}}$ of projective $\mathbb{Z}G$-modules, together with a map $\epsilon : F_0 \to \mathbb{Z}$ such that $\epsilon F_+ \to \mathbb{Z}$ is a projective resolution in usual sense, where $F_+ = (F_i)_{i \geq o}$. It is clear that the map $F_0 \to F_{-1}$ has a factorization as $\partial = \eta\epsilon$ where $\eta : \mathbb{Z} \to F_{-1}$ is a monomorphism. So we have a diagram as follows:*

$$\begin{tikzcd} \dots \arrow[r] & F_0 \arrow[r, "\partial"] \arrow[d, "\epsilon"'] & F_{-1} \arrow[r] \arrow[dl, "\eta"] & \dots \\ & \mathbb{Z} & & \end{tikzcd}$$

Nest we state some properties about complete resolutions without proof, people can refer Brown's book[8]

Lemma 1.5.1. *If $\epsilon : F_* \to \mathbb{Z}$ and $\epsilon' : F'_* \to \mathbb{Z}$ are two complete resolutions over same $\mathbb{Z}G$-modules. Then exist a unique homotopy class of augmentation-preserving maps from $F_* \to F'_*$. And these maps are homotopy equivalences.*

Definition 1.5.2. *We say a $\mathbb{Z}G$-module $\mathbb{Z}$ admits a backwards resolution if we have a long exact sequence of $\mathbb{Z}G$-modules as follows:*

$$0 \to \mathbb{Z} \to Q^0 \to Q^1 \to \dots$$

Where every Q^i is finitely generated and projective.

Theorem 1.10. *If $\epsilon : P_* \to \mathbb{Z}$ is a finite type resolution(every P_i is finitely generated) of $\mathbb{Z}$ over $\mathbb{Z}G$-modules. Then $\epsilon^* : \mathbb{Z}^* \to \bar{F} = Hom_{\mathbb{Z}G}(P_*, \mathbb{Z}G)$ is a backwards projective resolution. Moreover, up to isomorphism every finite type backwards projective resolution arises in this way. Consequently, any finite type complete resolution is obtained by two finite type projective resolutions $P_* \to \mathbb{Z}$ and $P'_* \to \mathbb{Z}$ by gluing together P'_* and $\overline{\sum P_*}$ where $\sum P_*$ is the suspension complex of P_*.*

So now we can define what is Tate cohomology.

Definition 1.5.3. *The Tate cohomology of a finite group G with coefficients in a $\mathbb{Z}G$-module M is defined by:*

$$\hat{H}^i(G, M) = H^i(Hom_{\mathbb{Z}G}(F_*, M))$$

where ß $\in \mathbb{Z}$ *and* F_* *is a complete resolution of* G*. It is clear that Tate cohomology is well-defined up to a canonical isomorphism.*

After definitions I will discuss some basic properties about Tate cohomology. For simplicity let $\epsilon : F_+ \to \mathbb{Z}$ be a projective resolution of finite type and $C^+ = Hom_{\mathbb{Z}G}(F_+, M)$. So $\hat{H}^i(G, M) = H^i(C^+)$ for $i > 0$. And by Theorem 1.10 there is a projective resolution $P \to \mathbb{Z}$ of finite type such that $F_- = \overline{\sum P}$ and

$$C^- = Hom_{\mathbb{Z}G}(F_-, M) = Hom_{\mathbb{Z}G}(\overline{\sum P}, M) \simeq \sum P \otimes_{\mathbb{Z}G} M$$

So $\hat{H}^i(C^-) = H^i(C^-) \simeq H_{-i}(\sum P \otimes_{\mathbb{Z}G} M) = H_{-i-1}(G, M)$ for $i < -1$.

Moreover we have a short exact sequence like follows:

$$0 \to \hat{H}^{-1}(G, M) \to H_0(G, M) \xrightarrow{\alpha} H^0(G, M) \to \hat{H}^0(G, M) \to 0$$

From the definition of group homology we can easily get

$$H_0(G, M) \simeq M_G = M/m - gm, \forall m \in M, \forall g \in G$$

$$H^0(G, M) \simeq M^G = \{m \in M | gm = m \forall g \in G\}$$

Proposition 1.5.1. *Under the above identification the group homomorphism α can be described as follows:*

$$\alpha(x) = \big(\sum_{g \in G} g\big)x$$

Proof. For simplicity, I assume that the projective resolution has $\mathbb{Z}G$ in dimension 0 and with the canonical augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ given by $\epsilon(g) = 1$ for any $g \in G$. Then it is clear that $\epsilon^* : \mathbb{Z} \to \mathbb{Z}G$ is defined by $\epsilon^*(1) = \sum_{g \in G} g$. Under the natural identification $Hom_{\mathbb{Z}G}(\mathbb{Z}G, M)$ we have a commutative diagram as follows

$$\begin{tikzcd}
M \arrow[r, "\times N"] \arrow[d, two heads] & M \\
H_0(G, M) \arrow[r, "\alpha"] & H^0(G, M) \arrow[u, hook]
\end{tikzcd}$$

where $N = \sum_{g \in G} g$ and the vertical maps are $M \twoheadrightarrow M_G$ and $M^G \hookrightarrow M$.

By the commutativity we complete the proof. □

In particular $\hat{H}^{-1}(G, \mathbb{Z}) = 0$ and $\hat{H}^0(G, \mathbb{Z})$. I will use these two results later.

Since the definition of Tate cohomology also relies on resolutions, it is easy to show that many of the formal properties of H^* also hold for $\hat{H}^*$. The following are some important results we need.

Proposition 1.5.2. *Given a short exact sequence of* $\mathbb{Z}G$*-modules* $0 \to M' \to M \to M" \to 0$*. There is a long exact sequence in the following:*

$$\cdots \xrightarrow{\delta} \hat{H}^i(G, M') \to \hat{H}^i(G, M) \to \hat{H}^i(G, M") \xrightarrow{\delta} \hat{H}^{i+1}(G, M') \to \cdots$$

Proposition 1.5.3. *(Shapiro's lemma)*
Let $H \subset G$ be its subgroup and M be a $\mathbb{Z}H$-module, then

$$\hat{H}^*(H, M) \simeq \hat{H}^*(G, Ind_H^G M)$$

Proposition 1.5.4. *For any $H \subset G$ and any $\mathbb{Z}G$-module M, one has a restriction map*

$$Res_H^G : \hat{H}^*(G, M) \to \hat{H}^*(H, M)$$

and a transfer map

$$Tr_H^G : \hat{H}^*(H, M) \to \hat{H}^*(G, M)$$

These maps have the same formal properties analogues to these maps we defined in section 1.2

Remark 1.5.1. *The Cartan-Eilenberg double cosets formula also works in Tate cohomology.*

Proposition 1.5.5. *The Tate cohomology functor $\hat{H}^*$ satisfies the dimension shifting condition. Precisely, given a $\mathbb{Z}G$-module M there are $\mathbb{Z}G$-modules K and C such that*

$$\hat{H}^i(G, M) = \hat{H}^{i+1}(G, K)$$

$$\hat{H}^i(G, M) = \hat{H}^{i-1}(G, C)$$

in other words the Tate cohomology functor $\hat{H}^*$ is completely determined by any single one functor $\hat{H}^i$ for any $i \in \mathbb{Z}$.

Cup Products and the Duality Theorem

In this subsection I will give a concise construction of the cup products structure in Tate cohomology which is very similar with the cup products in ordinary group cohomology. Finally I will state the duality theorem concerned about cup products which gives a duality relation between the negative and positive parts of Tate cohomology.

More precisely, I need construct a map

$$\hat{H}^p(G, M) \otimes \hat{H}^q(G, N) \to \hat{H}^{p+q}(G, M \otimes N)$$

which satisfies the three properties listed in section 1.2. In the following I will give a construction of cup products concisely which is divided in 6 steps:

Step 1 I first review the construction of cup products in ordinary group cohomology. Let F be a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$-modules, note that the tensor product of resolutions is still a resolution. On the cochain level

$$Hom_{\mathbb{Z}G}(F, M) \otimes Hom_{\mathbb{Z}G}(F, N) \to Hom_{\mathbb{Z}G}(F \otimes F, M \otimes N)$$

the cup product is defined by tensor product of graded maps.

Step 2 Choose a diagonal approximation

$$\Delta : F \to F \otimes F$$

It induces a cochain map

$$Hom_{\mathbb{Z}G}(\Delta, M \otimes N) : Hom_{\mathbb{Z}G}(F \otimes F, M \otimes N) \to Hom_{\mathbb{Z}G}(F, M \otimes N)$$

The composition of the above two cochain maps will give the cup product in group cohomology

$$Hom_{\mathbb{Z}G}(F, M) \otimes Hom_{\mathbb{Z}G}(F, N) \to Hom_{\mathbb{Z}G}(F, M \otimes N)$$

Step 3 Now I assume that F is a complete resolution replace of projective resolution. However, the above construction for cup products in group cohomology does not work in Tate cohomology. One of reasons is that $F \otimes F$ may not be a complete resolution . In particular, $(F \otimes F)_+$ is not equal to the resolution $F_+ \otimes F_+$. The other reason is the the n-th degree of the tensor product of resolution $F \otimes F$ may have infinite non-zero elements. So the original definition of tensor product of chain complexes does not work for Tate cohomology.

Step 4 So I need a more suitable definition for tensor product. Let C and C' be two graded modules, I define their **completed tensor product** $C\hat{\otimes}C'$ by

$$(C\hat{\otimes}C')_n = \prod_{p+q=n} C_p \otimes C'_q$$

As for the morphisms of this construction, let maps $u : C \to D$ of degree r and $v : C' \to D'$ of degree s, there is a map $u\hat{\otimes}v : C\hat{\otimes}C' \to D\hat{\otimes}D'$ define by

$$(u\hat{\otimes}v)_n = \prod_{p+q=n} (-1)^{ps} u_p \otimes v_q$$

Step 5 Let d be the differential in F. Then it is clear that

$$\partial = d\hat{\otimes}id_F + id_F\hat{\otimes}d$$

satisfies $\partial^2 = 0$. So it makes $F\hat{\otimes}F$ as a chain complex. Moreover we have the augmentation map

$$\epsilon\hat{\otimes}\epsilon : F\hat{\otimes}F \to \mathbb{Z}\hat{\otimes}\mathbb{Z} \simeq \mathbb{Z}$$

Note that[8] there exists a new diagonal approximation for complete resolutions called the **complete diagonal approximation** which is an augmentation-preserving map

$$\hat{\Delta} : F \to F\hat{\otimes}F$$

Step 6 Finally the composition of the cochain maps induced by the complete tensor product and the complete diagonal approximation gives the true cup product structure in Tate cohomology. In other words, on cochain level

$$Hom_{\mathbb{Z}G}(F, M) \otimes Hom_{\mathbb{Z}G}(F, N) \to Hom_{\mathbb{Z}G}(F, M \otimes N)$$

the cup product is defined by $u \cup v = (u\hat{\otimes}v) \circ \hat{\Delta}$.

The second part of this subsection is the duality theorem of Tate cohomology. Roughly speaking there is a duality map in Tate cohomology

$$\hat{H}^i(G, \mathbb{Z}) \otimes \hat{H}^{-i}(G, \mathbb{Z}) \to \hat{H}^0(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z}$$

I first need to explain the precise meaning of duality in Tate cohomology. We know in linear algebra the definition of dual space of a vector space V is $V' = Hom(V, F)$ where F is the base field. In the abelian groups case I need find a replacement of F.

Proposition 1.5.6. *If A is an non-trivial abelian group with $0 \neq a \in A$. Then there is a group homomorphism*

$$f : A \to \mathbb{Q}/\mathbb{Z}$$

such that $f(a) \neq 0$

Proof. Consider the submodule B generated by the single element a. It is clear that there is a map

$$\bar{f} : B \to \mathbb{Q}/\mathbb{Z}$$

with $\bar{F}(a) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective module, $\bar{f}$ can be extended to a map $f : A \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ $\square$

So based on this proposition I can use $\mathbb{Q}/\mathbb{Z}$ to replace F in abelian groups case :$A' = Hom(A, \mathbb{Q}/\mathbb{Z})$. Moreover, since A is finite, there is a integer n such that $nA = 0$. Then we have

$$Hom(A, \mathbb{Q}/\mathbb{Z}) = Hom(A, n^{-1}\mathbb{Z}/\mathbb{Z}) \simeq Hom(A, \mathbb{Z}/n\mathbb{Z})$$

So it is clear that $nA' = 0$. In particular if A is a finite cyclic group of order n then so does A'. So A is isomorphic to A' but this isomorphism may not be natural. However, there is a natural isomorphism between A and $(A')' = A$", which is defined by $a \mapsto (f \mapsto f(a))$ where a is an element in A.

Definition 1.5.4. *A map $f : A \otimes B \to \mathbb{Q}/\mathbb{Z}$ is called a duality paring if its associated map*

$$\bar{f} : A \to B'$$

which defined by $\bar{f}(a)(b) = f(a, b)$ is an isomorphism,where $a \in A, b \in B$.

Now I can state the duality theorem in Tate cohomology[8]

Theorem 1.11. *The cup product $\hat{H}^i(G, \mathbb{Z}) \otimes \hat{H}^{-i}(G, \mathbb{Z}) \xrightarrow{-\cup-} \hat{H}^0(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is a duality paring.*

1.6 Groups with Periodic Cohomology

In this section I will use Tate cohomology I defined last section to define a group with periodic cohomology groups or periodic cohomology, which is a very important object in my thesis. In next chapter, I will use another approach to define this concept again.

Definition 1.6.1. *A finite group G is said to have* ***periodic cohomology*** *if for some $d \neq 0$ there is an element $u \in \hat{H}^d(G, \mathbb{Z})$ which is invertible in the ring $\hat{H}^*(G, \mathbb{Z})$. In that case it is easily seen that cup product with u gives an isomorphism like follows:*

$$u \cup - : \hat{H}^n(G, M) \xrightarrow{\simeq} \hat{H}^{n+d}(G, M)$$

For any integer n and any $\mathbb{Z}G$-modules M.

Remark 1.6.1. *If we take $n = 0, M = \mathbb{Z}$ we have $\hat{H}^d(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z}$, and u generates $\hat{H}^d(G, \mathbb{Z})$.*

Remark 1.6.2. *If G has periodic cohomology, so does its any subgroup H.*

The next theorem gives some equivalent description about groups with periodic cohomology.

Theorem 1.12. *The following conditions are equivalent*

1. *G has periodic cohomology*
2. *There exists integers n and d , with $d \neq 0$ such that $\hat{H}^n(G, M) \simeq \hat{H}^{n+d}(G, M)$ for all $\mathbb{Z}G$-modules M.*
3. *For some $d \neq 0$ we have $\hat{H}^d(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z}$.*

4. For some $d \neq 0$, $\hat{H}^d(G,\mathbb{Z})$ contains an element u of order of group G.

Proof.

- 1. $\Rightarrow$ 2. By definition.
- 2. $\Rightarrow$ 3. By dimension shifting we have $\hat{H}^n(G,M) \simeq H^{n+d}(G,M)$ for any n. Then taking $n=0, M=\mathbb{Z}$ we get the answer.
- 3. $\Rightarrow$ 4. Trivial.
- 4. $\Rightarrow$ 1. By the hypothesis there is a map $f : \hat{H}^d(G,\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ where $f(u) = \frac{1}{|G|}$ mod $\mathbb{Z}$. Then by the duality theorem 1.11 in Tate cohomology this map is given by

$$\hat{H}^d(G,\mathbb{Z}) \xrightarrow{-\cup v} \hat{H}^0(G,\mathbb{Z}) \simeq \mathbb{Z}/|G|\mathbb{Z} \simeq |G|^{-1}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

for some $v \in H^{-d}(G,\mathbb{Z})$. So $u \cup v = 1$, in other words u is an invertible elements.

$\square$

Let me give you some examples about periodic groups, before doing that we need a special case of **Lefschetz fixed-point theorem**

Theorem 1.13. *Let X be a finite CW-complex and $f : X \to Y$ a map such that, for every cell σ,we have*

$$f(\sigma) \subset \bigcup_{\tau \neq \sigma, dim\tau \leq dim\sigma} \tau$$

In particular, f has no fixed points. Then the Lefschetz number $\sum(-1)^i Tr(f_i) = 0$ where $Tr(f_i)$ means the trace of f_i and f_i is the endomorphism induced by f on the free abelian group $H_i(X)/torsion$.

Example 1.6.1. *Let G be a finite group and acts freely on odd dimension sphere $X = S^{2k-1}$. By Theorem 1.13 G acts freely on $H_{2k-1}(X) \simeq \mathbb{Z}$. Let $C_* = C_*(X)$ We have an exact sequence of $\mathbb{Z}G$-modules as follows:*

$$0 \to \mathbb{Z} \to C_{2k-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

Where each C_i is free. Use the same copy of this exact sequence we can get a infinite long exact sequence as follows:

$$\cdots \to C_1 \to C_0 \to C_{2k-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

Then by cellular cohomology argument we know G satisfies the condition 2 of previous theorem. So G has a periodic cohomology.

Remark 1.6.3. *Since any finite cyclic group G has rotation of a 2-dimensional plane which is a 2-dimensional fixed-point-free representation. It is a special case of previous example. So any finite cyclic group has periodic cohomology.*

Example 1.6.2. *Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be a quaternion algebra. If G is a finite subgroup of multiplication group $\mathbb{H}^*$, then it is clear that the multiplication action of G on $\mathbb{H}$ yields a 4-dimensional fixed-point -free representation of G, since $\mathbb{H}$ is a division algebra. Then by previous argument we did $\hat{H}^*(G)$ is periodic with period 4. The generalized quaternion group is a typical example. For $m \geq 2$, the generalized quaternion group Q_{4m} is a subgroup of $\mathbb{H}$ generated by $x = e^{\pi i/m}$ and $y = j$.*

Example 1.6.3. *If G is abelian but not cyclic, then G doesn't have periodic cohomology. The reason is that in this case G must contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p. Then by Künneth formula we know $dim\hat{H}^n(\mathbb{Z}_p \times \mathbb{Z}_p; \mathbb{Z}_p) = n+1$. By dimension argument, it does't have periodic cohomology.*

For finite groups ,we introduce an important definition called **rank**.

Definition 1.6.2. *The p-rank $r_p(G)$ of a finite group G is the maximal rank of elementary abelian p-subgroup of G, and the rank $r(G)$ of G is $\max_{p||G|} r_p(G)$.*

Based on these examples we have a nice description for p-groups with periodic cohomology.

Theorem 1.14. *Let G be a p-group for some prime p. Then these following conditions are equivalent.*

1. *G has periodic cohomology*
2. *Every abelian subgroup of G is cyclic*
3. *Every elementary abelian p-group of G has rank ≤ 1*
4. *G has a unique subgroup of order p.*
5. *G is cyclic or generalized quaternion group.*

Proof.
- 1. $\Rightarrow$ 2. By Example 1.6.3.
- 2. $\Rightarrow$ 3. Trivial
- 3. $\Rightarrow$ 4. According to a standard result about finite p-groups [13] G contains a central subgroup of order p.Suppose there is another abelian subgroup of order p, then these two groups will generate the elementary abelian subgroup of rank two. That is a contradiction.
- 4. $\Rightarrow$ 5. This is a standard theorem about finite p-groups [13] [8]
- 5. $\Rightarrow$ 1. By Example 1.6.2.

□

For general finite groups not just only p-groups we have:

Theorem 1.15. *A finite group G has periodic cohomology if and only if every Sylow subgroup has periodic cohomology.*

Proof. The "only if" part is trivial. We only need to consider the "if" part. So suppose every p-Sylow subgroup of G has periodic cohomology. For a fixed prime number p we select a p-Sylow subgroup H of G. By definition there is an invertible element $u \in \hat{H}^d(H, \mathbb{Z})$. I claim that u is a stable element.

First consider the ring of unit elements $(\mathbb{Z}/|H|\mathbb{Z})^*$. It is clear there is an integer e such that $a^e = 1$ for any $a \in (\mathbb{Z}/|H|\mathbb{Z})^*$. Then for any element $g \in G$ I set

$$v = Res^G_{H \cap gHg^{-1}} u$$

$$w = Res^G_{H \cap gHg^{-1}} c_g(u)$$

Then u, w are both invertible elements in the cyclic group $\hat{H}^d(H \cap gHg^{-1}, \mathbb{Z})$. So there is an element $a \in (\mathbb{Z}/|H|\mathbb{Z})^*$ such that $v = aw$. Both taking e-th power we get $v^e = w^e$. Therefore $u^e \in \hat{H}^{de}(H, \mathbb{Z})$ is stable, so is u.

I note the the lemma 1.4.1 for p-primary decomposition of group cohomology also works for Tate cohomology. In other words we have

$$\hat{H}^*(G, \mathbb{Z}) \simeq \bigoplus_{p||G|} \hat{H}^*(G, \mathbb{Z}_{(p)}$$

According to previous argument and Theorem 1.7 we know for any prime number p there is an invertible element $u(p)$ with degree $d(p)$. Since G is a finite group, we can choose a large enough number N to replace $u(p)$ by $u(p)^N$ such that the $d(p)$ is the same for any $p||G|$. Then we get an invertible element in $\hat{H}^*(G,\mathbb{Z})$. □

1.7 Equivariant Homology and Cohomology

The ordinary (co)homology theory is a functor from the category of topological spaces to the category of rings. It turns out to be a powerful tool for investigating the topological properties of a space. However, if I equip a G-action on a space X, there is a more natural cohomology theory to reflect the influence of the G-actions of these G-spaces called the **equivariant (co)homology**. In this section I will give two definitions of the equivariant (co)homology. One is by ordinary singular (co)homology and another is by (co)chain complexes as coefficients. Then I will prove that these two definitions coincide.

To use the singular (co)homology to define the equivariant (co)homology I need a good model to turn a G-spaces X to a space with the same homotopy type but with the free G-action on it. It turns out the **Borel construction** is a nice model to do it.

Definition 1.7.1. *(Borel)*
Given a G-complex X which is a CW-complex with G-action on it where G is a topological group. So we have a G-action on the space $X \times EG$ via the product action. Then the Borel construction $X \times_G EG$ is defined as the topological space of orbits space of this action.

Given a $\mathbb{Z}G$-module M we define the **equivariant homology and cohomology** of X as follows:

$$H_*^G(X,M) = H_*(X \times_G EG, M)$$

$$H_G^*(X,M) = H^*(X \times_G EG, M)$$

However, this definition is not so good to be handled. Next I will give a more algebraic way to redefine it.

Recall that the group homology $H_*(G,M)$ is defined to be $H_*(F \otimes_{\mathbb{Z}G} M)$ where F is a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$-modules. We can generalize this definition by allowing the coefficients to be a nonnegative chain complex C.

$$H_*(G,C) = H_*(F \otimes_{\mathbb{Z}G} C)$$

where $F \otimes_{\mathbb{Z}G} C$ is the product of two chain complexes.

It is clear that $F \otimes_{\mathbb{Z}G} C$ is the total complex of the double complex of abelian groups $(F_p \otimes_{\mathbb{Z}G} C_q)$ where $p,q \geq 0$. We know a double complex has a two natural way to derive two spectral sequences, so the first case has $E^1_{p,q} = H_q(F_p \otimes_{\mathbb{Z}G} C_*) = F_p \otimes_{\mathbb{Z}G} H_q(C)$. Then we can take the homology with respect to p. So we get the $E^2_{p,q} = H_p(G,H_q)$ and it converges to $H_{p+q}(G,C)$.

Similarly, as for the second spectral sequence we have $E^1_{p,q} = H_q(F_* \otimes_{\mathbb{Z}G} C_p) = H_q(G,C_p)$ which converges to $H_{p+q}(G,C)$.

Dually, I can also define the cohomology groups of G with coefficients of a cochain complex $C = (C^n)_{n\geq 0}$

$$H^*(G,C) = H^*(Hom_{\mathbb{Z}G}(F,C))$$

where F is a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}G$-modules.

After warming up, I use these settings to redefine the equivariant homology and cohomology and show that these two definitions are equivalent.

Definition 1.7.2. *Let X be a G-complex, then the cellular chain of X $C(X)$ is a chain complex of $\mathbb{Z}G$-modules. Given a $\mathbb{Z}G$-module M we let $C(X,M) = C(x) \otimes_{\mathbb{Z}G} M$ with diagonal G actions. Then we define the equivariant homology of X as*

$$H_*^G(X,M) = H_*(G, C_*(X,M))$$

Similarly, the equivariant cohomology is

$$H_G^*(X,M) = H^*(G, C^*(X,M))$$

where $C^*(X,M) = Hom_{\mathbb{Z}G}(C_*(X), M)$.

For simplicity, in the rest of this section we only discuss about homology cases, but every result holds for cohomology cases.

Since it is defined via coefficients of chain complexes we have a spectral sequence as follows:

$$E_{p,q}^2 = H_p(G, H_q(X,M)) \Rightarrow H_{p+q}^G(X,M)$$

As an application of this spectral sequence we consider a point pt as a trivial G-complex. Then it is clear that $H_*^G(pt,M) = H_*(G,M)$ and we have a canonical isomorphism induced by the canonical G-map from any G-complex X to this point.

$$H_*^G(X,M) \to H_*(G,M)$$

Using the spectral sequence argument we can get the following statement easily.

Proposition 1.7.1. *Let $f : X \to Y$ be a cellular map of G-complexes such that it induces an isomorphism in homology $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$. The for any $\mathbb{Z}G$-module M we have an induced isomorphism as follows:*

$$f_* : H_*^G(X,M) \xrightarrow{\simeq} H_*^G(Y,M)$$

In particular, if X is contractible then the canonical map above is an isomorphism

$$H_*^G(X,M) \xrightarrow{\simeq} H_*(G,M)$$

As for the second spectral sequence, let X_p the the set of p-cells of X and let $\sum_p$ be the representatives of orbits X_p/G. Then the underline abelian group structure of $C_p(X,M)$ is $\oplus_{\sigma \in X_p} M$. By proposition 1.3.3:

$$C_p(X,M) \simeq \bigoplus_{\sigma \in \sum_p} Ind_{G_\sigma}^G M$$

Then by Shapiro's lemma we have

$$H_q(G, C_p(X,M)) \simeq \bigoplus_{\sigma \in \sum_p} H_q(G_\sigma, M)$$

So the second spectral sequence is

$$E_{p,q}^1 = \bigoplus_{\sigma \in \sum_p} H_q(G_\sigma, M) \Rightarrow H_{p+q}^G(X,M)$$

In particular if the G-action if free that is G_σ is trivial for any $\sigma \in \sum_p$ and $M = \mathbb{Z}$ then this spectral sequence collapses on E^2 page. Therefore

$$H_*^G(X,\mathbb{Z}) \simeq H_*(C(X)_G) = H_*(X/G)$$

Combined these two spectral sequences we can conclude:

Theorem 1.16. *If X is a free G-CW-complex, then there is a spectral sequence as follows*

$$E^2_{p,q} = H_p(G, H_q(X)) \Rightarrow H_{p+q}(X/G)$$

So finally I can prove the equivalence of these two definitions about equivariant homology

Theorem 1.17. *Let $H_*(X, M)$ denote the equivariant homology defined via chain complexes then er have*

$$H_*(X, M) \simeq H_*(X \times_G EG, M)$$

Proof. Consider the natural equivariant map

$$\pi : X \times EG \to X$$

which is a homotopy equivalence. And we have two spectral sequences

$$E^2_{p,q} = H_p(G, H_q(X \times EG, M))$$

$$E'^2_{p,q} = H_p(G, H_q(X, M)))$$

Then by the comparison theorem of spectral sequences there is a natural isomorphism

$$H^G_*(X \times EG, M) \to H^G_*(X, M)$$

Since we have a free G-action on $X \times EG$, we have an isomorphism

$$H^G_*(X \times EG, M) \simeq H_*(X \times_G EG, M)$$

Therefore $H^G_*(X, M) \simeq H_*(X \times_G EG, M)$ $\square$

Chapter 2

Periodic Complexes and its Applications

In this chapter I generalize the definition of groups with periodic cohomology by defining a CW-complex with periodic cohomology after a certain degree. Using the transgression map in Serre spectral sequences generalizes Euler classes to partial Euler classes, which I can use to prove an important theorem about a CW-complex being a periodic complex. Then I give applications of periodic complexes, some in Wall's finiteness obstructions and some results about periodic discrete groups which generalize the classical results by Swan and Wall[30][31]. Finally, I give a useful proposition, which can be used to construct free actions of rank two groups on products of two spheres.

2.1 Periodic Complexes

We assume when coefficients do not appear in a (co)homology group that they are in $\mathbb{Z}$

Definition 2.1.1. *A CW-complex X has **Periodic Cohomology** if there is a cohomology class $\alpha \in H^*(X)$ with $|\alpha| > 0$, and an integer $d \geq 0$ such that the following map:*

$$\alpha \cup - : H^n(X; M) \longrightarrow H^{n+|\alpha|}(X; M)$$

is an isomorphism for any coefficients system and for any integer $n \geq d$.

Now we will recall the definitions of spherical fibration and its Euler class.

Definition 2.1.2. *A spherical fibration is a Serre fibration(has the homotopy extension property with CW-complexes) $E \to B$ for which the fiber is homotopy equivalent to a sphere S^m that is :*

$$S^m \to E \to B$$

We say this fibration is orientable if the action of $\pi_1(B)$ on $H^m(S^m) \simeq \mathbb{Z}$ is trivial. In this case we have the following Serre spectral sequence:

$$E_2^{pq} = H^p(B; H^q(S^m))$$

Then the Euler class of this fibration is the cohomology class in $H^{m+1}(B)$ which is the transgression of the generator $\mu \in H^m(S^m)$.

An important application of Euler classes is **Gysin sequences**.

Theorem 2.1. *(Gysin sequence) Let R be a commutative ring and we have an orientable spherical fibration like follows:*

$$S^n \to E \xrightarrow{\pi} X$$

*Then there is a long exact sequence called **Gysin sequence** in the following:*

$$\cdots \to H^k(X,R) \xrightarrow{\pi^*} H^k(E,R) \to H^{k-n}(X,R) \xrightarrow{c\cup -} H^{k+1}(X,R) \to \dots$$

Where $c \in H^{n+1}(X)$ is the Euler class of this fibration.

Proof. Since we have an orientable fibration, we get its Serre spectral sequence with coefficients R.

$$E_2^{p,q} = H^p(X, H^q(S^n;R))$$

which converges to $H^{p+q}(E,R)$. And let $\mu \in H^n(S^n;\mathbb{Z}) \simeq \mathbb{Z}$ be a generator and the Euler class $c \in H^{n+1}(X,\mathbb{Z})$ is the transgression of μ, that is $d_{n+1}(\mu) = c$. In E_2 page there are just two line non-trivial which are $E_2^{*,n}, E_2^{0,*}$. So $E_{n+1} \simeq E_2$. More over we have an isomorphism in the following sense:

$$\mu \cdot (-) : E_{n+1}^{*,0} \to E_{n+1}^{*,n}$$

Since this product derived from cup product in E_2 page. And according to multiplicative structure in cohomological spectral sequence we have:

$$d_{n+1}(\mu \cdot b) = d_{n+1}(\mu)\cdot b + (-1)^n \mu \cdot d_{n+1}(b) = c \cdot b$$

Since $d_{n+1}(b) = 0$ for any $b \in E_{n+1}^{*,0}$. Then we know the differential d_{n+1} acts like a cup product in the following sense:

$$\begin{tikzcd}
E_{n+1}^{s,n} \arrow[r, "\sim"] \arrow[d, "d_{n+1}"] & H^s(X,R) \arrow[d, "c\cup -"] \\
E_{n+1}^{s+n+1,0} \arrow[r, "\sim"] & H^{s+n+1}(X,R)
\end{tikzcd}$$

By 5-terms exact sequence in spectral sequence we have:

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{s,n} \arrow[r] & E_{n+1}^{s,t} \arrow[r] \arrow[d] & E_{n+1}^{s+n+1,0} \arrow[r] \arrow[d] & E_\infty^{s+n+1,0} \arrow[r] & 0 \\
& & H^s(X,R) \arrow[r, "c\cup -"'] & H^{s+n+1}(X,R) & &
\end{tikzcd}$$

And we also have a short exact sequence derived from spectral sequences:

$$0 \to E_\infty^{s,0} \to H^s(E,R) \to E_\infty^{s-n,n} \to 0$$

Combine previous two exact sequences we can get the Gysin sequence.

□

Now I recall two basic notions in homotopy theory, which are fiber join and Postnikov towers and in the following I will work in homotopy category. And all fibrations are assumed as Serre fibrations.

Definition 2.1.3. *Let $E_1 \to X$ and $E_2 \to X$ be two fibrations over the base space X, with fibers F_1, F_2 respectively, then we have $E_1 \times_X E_2$ as their pull-back(in the strict sense). The **fiber join** of these fibrations, denoted by $E_1 *_X E_2 \to X$, is defined by the homotopy push-out of the following diagram:*

$$\begin{tikzcd}
E_1 \times_X E_2 \arrow[r] \arrow[d] & E_2 \\
E_1 &
\end{tikzcd}$$

In the other words, it is the push-out in the homotopy category. By the properties of homotopy colimits we know the fiber of this new fibration is $F_1 * F_2$.See[7]

In what follows we will use the basic properties of Postnikov towers to give the definition of partial Euler class. So we need to recall what the Postnikov towers are.

Example 2.1.1 (Postnikov Towers). *A Postnikov tower for a path-connected space X is a commutative diagram in the following,*

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3X \arrow[d] \\
 & P_2X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & P_1X
\end{tikzcd}
$$

such that

1. *the map $X \to P_nX$ induces an isomorphism on π_i for $i \leq n$.*
2. *$\pi_i(P_nX) = 0$, for $i > n$*

Remark 2.1.1. *The basic idea to construct this Postnikov tower is by attaching more cells on X to vanish the generators of homotopy groups. As a result, the space P_nX consists of the space X and cells with dimension greater or equal to $n+2$. However, this construction depends on the choice of generators, so it is not canonical. Fortunately, we have a canonical construction of Postnikov towers so that P_n we be a functor on homotopy classes of maps for any n.*

Now we can introduce what partial Euler classes are:

Definition 2.1.4. *Let X be a path connected CW-complex and let $k \geq 0$ be an integer. A cohomology class $\alpha \in H^n(X)$ is a k-partial Euler class if there is an orientable fibration.*

$$P_{k+n-1}S^{n-1} \to E \to X$$

such that a generator of $H^{n-1}(P_{k+n-1}S^{n-1}) \simeq \mathbb{Z}$ transgresses to α.

In fact we can loose our restriction on orientation. More precisely, in this case, this definition also works if we only require that the action of $\pi_1(X)$ on $H^{n-1}(P_{k+n-1}S^{n-1}) \simeq \mathbb{Z}$ is trivial. Next lemma claims that this condition is equivalent to the orientation of the previous fibration.

Lemma 2.1.1. *For an integer $q \geq m$, let $P_qS^m \to E \to B$ be a fibration with the action of $\pi_1(B)$ on $H^m(P_qS^m) \simeq \mathbb{Z}$ is trivial then the action of $\pi_1(B)$ on all cohomology groups $H^*(P_qS^m)$ is also trivial.*

Proof. We consider a general Serre fibration $F \to E \to B$. We first recall the definition of the action of $\pi_1(B)$ on $H^*(F)$. By homotopy extension property we have a natural map $\pi_1(B) \to [F, F]$ where $[F, F]$ denote the homotopy class of the maps between F itself. Since cohomology is a contravariant functor we have a natural map $[F, F] \to Hom(H^*(F), H^*(F))$. Composite these two maps we can get the action of $\pi_1(B)$ on $H^*(F)$. Now back to our assumption, so $F = P_qS^m$. In this case, we have

$$[F, F] = [P_qS^m, P_qS^m] \simeq [S^m, S^m] \simeq \mathbb{Z}$$

Since the map of compositions is trivial

$$\pi_1(B) \to [F, F] \to Hom(H^m(F), H^m(F))$$

So the map $\pi_1(B) \to [F, F] \simeq \mathbb{Z}$ is trivial. So the composition map

$$\pi_1(B) \to [F, F] \to Hom(H^*(F), H^*(F))$$

is trivial for any integer $*$. Equivalently, the action of $\pi_1(B)$ on $H^*(P_q)$ is trivial.

□

Now we are going to prove one of the main theorems of this thesis which is the criterion of periodicity of CW-complexes. Since the proof need a long argument, I divid it into three steps.

The first step includes two lemmas about the partial Euler classes whose statements are relatively easy but need a long tedious argument.

We first refer a result about fiber joins of fibrations. This proposition could be viewed as the Whitney product formula for Euler classes[21]. People who interest detailed proof could refer Hall's paper[12]

Proposition 2.1.1. *Let $\pi_i : E_i \to X$ be two fibrations with fibers where $i = 1, 2$. And let $\alpha_i \in H^*(F_i)$ be transgressive and its transgression are represented by $\beta_i \in H^*(X)$. Then*

$$\alpha_1 * \alpha_2 \in H^*(F_1 * F_2)$$

is also transgressive and its transgression is $-\beta_1 \cup \beta_2$.

Here is the first lemma:

Lemma 2.1.2. *Suppose $\alpha \in H^n(X)$ is a k-partial Euler class, then for any positive integer q, $\alpha^2 \in H^{2n}(X)$ is also a k-partial Euler class.*

Proof. Without losing any gernerality, we can assume $q = 2$. Since we $\alpha \in H^n(X)$ is a k-partial class we have the associated orientable fibration like follows:

$$P_{n+k-1}S^{n-1} \to E \to X$$

Take 2-fold fiber joins we get:

$$*^2 P_{n+k-1}S^{n-1} \to *^2_X E \to X$$

Then we can take Postnokov sections for this fibration:

$$P_{k+2n-1}(*^2 P_{n+k-1}S^{n-1}) \to P_{k+2n-1}(*^2_X E) \to X$$

By cellular approximation we have a weak equivalence

$$P_{k+2n-1}(*^2 P_{n+k-1}S^{n-1}) \simeq P_{k+2n-1}S^{2n-1}$$

So we have a new orientable fibration:

$$P_{k+2n-1}S^{2n-1} \to P_{k+2n-1}(*^2_X E) \to X$$

That is the fibration with α^2 as its k-partial Euler class. The reason is that by Künneth formula for join we have

$$H^{2n-1}(P_{k+2n-1}S^{2n-1}) \simeq H^{n-1}(P_{k+n-1}S^{n-1}) \otimes H^{n-1}(P_{k+n-1}S^{n-1})$$

So one of generator of $H^{2n-1}(P_{k+2n-1}S^{2n-1})$ could be represented by $\nu = \mu * \mu$ where

$$\mu \in H^{n-1}(P_{k+n-1}S^{n-1}) \simeq \mathbb{Z}$$

is one generator. Then by previous proposition 2.1.2 We know that $-\nu$ is transgressive and its transgression is α^2. So in general, for any positive integer q, α^q is also a k-partial Euler class.

□

Here is the second lemma. The proof of this lemma uses three more pages, but the main idea is not so hard. I use the fiber joins and the finiteness of stable homotopy groups to kill the two obstructions in Serre spectral sequences.

Lemma 2.1.3. *Suppose X is a path-connected CW-complex and let $k \geq 0$ be an integer. Then for every cohomology class $\alpha \in H^*(X)$ of positive degree there is an integer $q \geq 1$ such that the cup power α^q is a k-partial Euler class.*

Proof. We prove it by induction on k. When $k = 0$, the cohomology class $\alpha \in H^n(X)$ could be represented by a map $f : X \to K(\mathbb{Z}, n)$. And we have a pullback square as follows:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & PK(\mathbb{Z},n) \arrow[d] \\ X \arrow[r, "f"] & K(\mathbb{Z},n) \end{tikzcd}$$

It is clear that these two fibrations have the same fiber $K(\mathbb{Z}, n)$. By naturality of Serre spectral sequences, the generator of $H^{n-1}(K(\mathbb{Z}, n-1) \simeq \mathbb{Z}$ transgresses to $\alpha \in H^n(X)$. That means every cohomology class is a 0-partial Euler class.

Next, we assume that $\alpha \in H^n(X)$ is a k-partial Euler class, we need to show that there is an integer q such that α^q is a $k+1$-partial Euler class. Since α is a k-partial Euler class we have a oriented fibration by the definition:

$$P_{k+n-1}S^{n-1} \to E_k \to X$$

for which the generator of $H^{n-1}(P_{k+n-1}S^{n-1})$ transgresses to α. We set a new Serre spectral sequences using coefficients $\pi_{n+k}(S^{n-1})$.

$$E_2^{p,q} = H^p(X, H^q(P_{n+k-1}S^{n-1}, \pi_{n+k}(S^{n-1}))).$$

We choose a k-invariant $g_k : P_{k+n-1}S^{n-1} \to K(\pi_{n+k}(S^{n-1}), k+n+1)$ of the Postnikov tower of S^{n-1} with the fiber $P_{k+n}S^{n-1}$. It corresponds a cohomology class $\gamma_k \in H^{n+k+1}(P_{n+k-1}S^{n-1}, \pi_{n+k}(S^{n-1})) = E_2^{0,n+k+1}$. By the construction of the Postnikov towers we know the cell structures of $P_{n+k-1}S^{n-1}$ has one 0-cell and one $(n-1)$-cell and many cells with dimension greater or equal to $n+k+1$. So the class γ_k survives to the $k+3$ page of this spectral sequences. If $d_{k+3}\gamma_k = 0$ then γ_k is transgressive, more over if this transgression is trivial, then this class γ_k survives to the infinite page E_∞. It is equivalent that the k-invariant map g_k could be extended in the following sense:

$$\begin{tikzcd} P_{k+n-1}S^{n-1} \arrow[r] \arrow[d, "g_k"'] & E_k \arrow[d] \\ K(\pi_{n+k}(S^{n-1}), k+n+1) \arrow[r] & K(\pi_{n+k}(S^{n-1}), k+n+1) \end{tikzcd}$$

Then we can take fibers of these two columns of fibrations to get a new fibration as follows:

$$\begin{tikzcd} P_{k+n}S^{n-1} \arrow[r] \arrow[d] & E_{k+1} \arrow[r] \arrow[d] & X \arrow[d, "id"] \\ P_{k+n-1}S^{n-1} \arrow[r] \arrow[d, "g_k"'] & E_k \arrow[r] \arrow[d] & X \arrow[d] \\ K(\pi_{n+k}(S^{n-1}), k+n+1) \arrow[r] & K(\pi_{n+k}(S^{n-1}), k+n+1) \arrow[r] & * \end{tikzcd}$$

Then by the naturality of Serre spectral sequences we know the generator of $H^{n-1}(P_{n+k}S^{n-1}) \simeq \mathbb{Z}$ transgresses to α which means that α is itself a $k+1$-partial Euler class. However, the assumptions we set before may not be the real case. So my idea is using fiber joins to make a new fibration which can satisfy these two assumptions. In general, consider the diagram as follows:

$$\begin{tikzcd}
*^q P_{k+n-1}S^{n-1} \arrow[r] \arrow[d] & *^q_X E_k \arrow[r] \arrow[d] & X \arrow[d] \\
P_{k+qn-1}(*^q P_{k+n-1}S^{n-1}) \arrow[r] & P^q_X E_k \arrow[r] & X
\end{tikzcd}$$

Where the first row is the q-th fiber join of the original fibration and the second row is taking $k+qn-1$-th Postnikov section of the total space and fiber space. Now we consider the natural map got from the Postnikov tower of S^{n-1}

$$*^q S^{n-1} \to *^q P_{n+k-1}S^{n-1}$$

Apply the functor P_{k+qn-1} and by cellular approximation we get a weak equivalence:

$$P_{k+qn-1}S^{qn-1} \to P_{k+qn-1}(*^q P_{k+n-1}S^{n-1})$$

Then we the k-invariant to the a fibration as follows:

$$P_{k+qn}S^{qn-1} \to P_{k+qn-1}S^{qn-1} \to K(\pi_{k+qn}(S^{qn-1}), k+qn+1)$$

Use the Serre spectral sequence(Homology version)of this fibration we can see that in total degree $k+qn+1$ there is only one group $\pi_{k+qn}(S^{qn-1})$ will survive. So we have:

$$H_{k+qn+1}(P_{k+qn-1}S^{qn-1}) = \pi_{k+qn}(S^{qn-1})$$

Moreover, by the Künneth formula we can get:

$$H_{k+qn+1}(*^q P_{n+k-1}S^{n-1}) = \oplus_q \pi_{n+k}(S^{n-1})$$

So the induced map in homology is the q-fold sum of maps $f_i : \pi_{k+n}(S^{n-1}) \to \pi_{k+qn}(S^{qn-1})$. By symmetry we know that $f_i = f_j$ for any $1 \leq i, j \geq q$. So we only need to figure out what is f_1. So we consider the composition of the following maps:

$$P_{k+n-1} * (*^{q-1}S^{n-1}) \to *^q P_{k+n-1}S^{n-1} \to P_{k+qn-1}S^{qn-1}$$

By adjunction of suspension functor and loop functor we an equivalent map

$$P_{k+n-1}S^{n-1} \to \Omega^{(q-1)n}P_{k+qn-1}S^{qn-1}$$

Moreover, I claim there is a weak equivalence $f : P_{n+k-1}\Omega^{(q-1)n}S^{qn-1} \to \Omega^{(q-1)n}P_{k+qn-1}S^{qn-1}$. To prove this claim, we first observe that there are two maps

$$\Omega^{(q-1)n}S^{qn-1} \hookrightarrow P_{k+n-1}\Omega^{(q-1)n}S^{qn-1}$$

$$\Omega^{(q-1)n}S^{qn-1} \to \Omega^{(q-1)n}P_{k+qn-1}S^{qn-1}$$

Where the first map is obtained by the functoriality of Postnikov towers, and the second map is obtained by the functoriality of Postnikov towers and loop spaces. So we have an extension problem in the following:

$$\begin{tikzcd}
\Omega^{(q-1)n}S^{qn-1} \arrow[r] \arrow[d] & \Omega^{(q-1)n}P_{k+qn-1}S^{qn-1} \\
P_{k+n-1}\Omega^{(q-1)n}S^{qn-1} \arrow[ur, dotted] &
\end{tikzcd}$$

And for any m, the homology group

$$H^{m+1}(P_{k+n-1}\Omega^{(q-1)n}S^{qn-1}, \Omega^{(q-1)n}S^{qn-1}; \pi_m(\Omega^{(q-1)n}P_{k+qn-1}S^{qn-1}) = 0$$

Since for $m > n+k-1$ we have $\pi_m(\Omega^{(q-1)n}P_{k+qn-1}S^{qn-1} = 0$, when $m \leq n+k-1$ we know $H^{m+1}(P_{k+n-1}\Omega^{(q-1)n}S^{qn-1}, \Omega^{(q-1)n}S^{qn-1}) = 0$ by cellular homology. Now apply the homology functor to the extension diagram we know the extension map is a weak equivalence, in other words it is a homotopy equivalence. So we can invert this map. Then we get a map

$$P_{n+k-1}S^{n-1} \to P_{k+n-1}\Omega^{(q-1)n}S^{qn-1}$$

which is determined by the bottom cell. So it is obtained by applying the functor P_{n+k-1} to the inclusion $S^{n-1} \to \Omega^{(q-1)n}S^{qn-1}$. It follows that the mapf_1 is the suspension homomorphism and therefore that the map $\oplus f_i$ is the fold map by the suspension homomorphism.

Let $g_k^{[q]}$ denote the k-invariant

$$P_{k+qn-1}S^{qn-1} \to K(\pi_{k+qn}(S^{qn-1}), k+qn+1)$$

and let $\gamma_k^{[q]}$ be the cohomology class which it represents. By universal coefficients theorem for cohomology we have an isomorphism

$$H^{k+qn+1}(*^q P_{k+n-1}S^{n-1}, \pi_{k+n}(S^{n-1}) \xrightarrow{\sim} Hom(H_{k+qn+1}(*^q P_{k+n-1}S^{n-1}), \pi_{k+n}(S^{n-1}))$$

And we have a commutative diagram as follows:

$$\begin{tikzcd}
H^{k+qn+1}(*^q P_{k+n-1}S^{n-1}, \pi_{k+n}(S^{n-1})) \arrow[r, "\sim"] & Hom(H_{k+qn+1}(*^q P_{k+n-1}S^{n-1}), \pi_{k+n}(S^{n-1})) \\
H^{k+qn+1}(P_{k+qn-1}S^{qn-1}, \pi_{k+n}(S^{n-1})) \arrow[r] \arrow[u] \arrow[d] & Hom(H_{k+qn+1}(P_{k+qn-1}S^{qn-1}), \pi_{k+n}(S^{n-1})) \arrow[u] \arrow[d] \\
H^{k+qn+1}(P_{k+qn-1}S^{qn-1}, \pi_{k+qn}(S^{qn-1})) \arrow[r, "\sim"] & Hom(H_{k+qn+1}(P_{k+qn-1}S^{qn-1}), \pi_{k+qn}(S^{qn-1}))
\end{tikzcd}$$

which is induced by the map

$$H_{k+qn+1}(*^q P_{k+n-1}S^{n-1}) \to H_{k+qn+1}(P_{k+qn-1}(S^{qn-1}))$$

which we discussed before.

And the suspension homomorphism

$$\pi_{k+n}(S^{n-1}) \to \pi_{k+qn}S^(qn-1)$$

And it is clear that if q is large enough this is also an isomorphism. Next we claim the cohomology class $\gamma_k^{[q]}$ corresponds to the identity map in $Hom(H_{k+qn+1}(P_{k+qn-1}S^{qn-1}), \pi_{k+qn}(S^{qn-1}))$ Let K denote the Eilenberg-Maclane space $K(\pi_{n+k}(S^{n-1}), k+n+1)$. Observe that we have the following commutative diagram by naturality:

$$\begin{tikzcd}
H^{k+n+1}(K, \pi_{k+n}(S^{n-1})) \arrow[r] \arrow[d, "\sim"] & H^{k+n+1}(P_{k+n-1}S^{n-1}, \pi_{n+k}(S^{n-1})) \arrow[d] \\
Hom(H_{k+n+1}(K, \pi_{n+k}(S^{n-1})) \arrow[r] & Hom(H_{k+n+1}(P_{n+k-1}S^{n-1}), \pi_{k+n}(S^{n-1}))
\end{tikzcd}$$

Moreover, consider the Serre spectral sequence of homological version of the fibration

$$P_{k+n}S^{n-1} \to P_{k+n-1}S^{n-1} \xrightarrow{g_k} K$$

we can get an identity map $H_{k+n+1}(P_{k+n-1}S^{n-1}) \to H_{k+n+1}(K)$. So by naturality we know the cohomology class $\gamma_k^{[q]}$ corresponds to the identity map in

$$Hom(H_{k+qn+1}(P_{k+qn-1}S^{qn-1}), \pi_{k+qn}(S^{qn-1}))$$

Back to the big commutative diagram we know that the identity map in

$$Hom(H_{k+qn+1}(P_{k+qn-1}S^{qn-1}), \pi_{k+qn}(S^{qn-1}))$$

correspondence to a series of identity maps $(id, \cdots, id)$ in

$$Hom(H_{k+qn+1}(*^q P_{k+n-1}S^{n-1}), \pi_{k+n}(S^{n-1})) \simeq \oplus_q \pi_{k+n}(S^{n-1})$$

Let $(id, 0, \cdots, 0)$ corresponds to a cohomology class β in

$$H^{k+qn+1}(*^q P_{k+n-1}S^{n-1}, \pi_{k+n}(S^{n-1}))$$

Then consider the three Serre spectral sequences in the following:

$$E_2^{m,n} = H^m(X, H^n(*^q P_{k+n-1}S^{n-1}, \pi_{n+k}(S^{n-1}))$$

$$F_2^{m,n} = H^m(X, H^n(P_{k+qn-1}(S^{qn-1}), \pi_{n+k}(S^{n-1}))$$
$$Q_2^{m,n} = H^m(X, H^n(P_{k+qn-1}S^{qn-1}), \pi_{k+qn}(S^{qn-1}))$$

Then by the naturality of Serre spectral sequences and the symmetry we have $d_{k+3}\gamma_k^{[q]}$ is the image of $qd_{k+3}\beta$ under the map:

$$H^{k+3}(X, \pi_{k+n}(S^{n-1})) \to H^{k+3}(X, \pi_{k+qn}(S^{qn-1})$$

Now the image of the suspension homomorphism is a finite group, so if q is large enough we get $d_{k+3}\gamma_k^{[q]} = 0$.

So with losing any generality we can assume that we have a fibration

$$P_{k+n-1}S^{n-1} \to E \to X$$

such that in its Serre spectral sequence γ_k is transgressive. Then similarly by taking q-th fiber join we can assume γ_k transgresses to 0. Now we back to the original case we discussed. So we complete the inductive step. □

The second step is a lemma which presents the existence of the Euler class which is the chosen partial Euler classes under the periodicity condition.

Lemma 2.1.4. *Let $P_{k+|\alpha|-1}S^{|\alpha|-1} \to E \to X$ be an orientable fibration sequence such that a generator of $H^{|\alpha|-1}(P_{k+|\alpha|-1}S^{|\alpha|-1})$ transgresses to $\alpha \in H^*(X)$. If $k \geq d-2$ and the map $\alpha \cup - : H^m(X) \to H^{m+|\alpha|}(X)$ is an isomorphism in integral cohomology for $m \geq d$ then there is an orientable spherical fibration $\bar{E} \to X$ with Euler class α. If in addition the map $\alpha \cup - : H^m(X, M) \to H^{m+|\alpha|}(X, M)$ is an isomorphism for every local coefficients M and every $m \geq d$ then the total space $\bar{E}$ is homotopy equivalent to a CW-complex of dimension less than $d + |\alpha|$.*

Proof. Let $n = |\alpha|$, we will follow the arguments of above lemma to construct a fibration

$$P_t S^{n-1} \to E_t \to X$$

by induction on the integer $t \geq n + k - 1$.

As for the beginning step, we are done by hypothesis.

As for the inductive step, first suppose we have a fibration

$$P_tS^{n-1} \to E_t \to X$$

Then let $g_t : P_tS^{n-1} \to K(\pi_{t+1}(S^{n-1}), t+2)$ be its k-invariant. This map extends to a map

$$E_t \to K(\pi_{t+1}(S^{n-1}), t+2)$$

if and only if the cohomology class γ_k which represents this k-invariant will survive to E_∞ page of the Serre spectral sequence of cohomology wit coefficients $\pi_{t+1}(S^{n-1})$. There are two obstructions in the spectral sequence. One is d_{t-n+4}, another is d_{t+3}.

First I claim that $\alpha \cup d_{t-n+4}\gamma_k = 0$. Let μ be a generator of $H^{n-1}(S^{n-1})$ and choose a representative β for the elements $d_{t-n+4}\gamma_K$. Without losing generality we assume the gap between $t+2$ and $n-1$ is large than $n-1$. Since $\beta = \beta \cup \mu$, by the multiplicative structure of spectral sequences we have

$$d_n(\beta) = d_n(\beta \cup \mu) = d_n(\beta) \cup \mu + \beta \cup d_n(\mu)$$

So we have

$$\beta \cup \alpha = 0$$

for any representative β for $d_{t-n+4}\gamma_k$ then passing to quotients we complete the proof of the statement. Therefore $d_{t-n+4}\gamma_k = 0$ since $\alpha \cup -$ is an isomorphism.

As for d_{t+3} since it takes values in $H^*(X, \pi_{t+1}(S^{n-1})/(\alpha)$. By hypothesis it is 0. So γ_k will survive to E_∞ page.

So we have the following diagram as we did in the proof of lemma 2.1.3.

$$\begin{tikzcd}
P_{t+1}S^{n-1} \arrow[r] \arrow[d] & P_tS^{n-1} \arrow[r] \arrow[d] & K(\pi_{t+1}(S^{n-1}), t+2) \arrow[d] \\
E_{t+1} \arrow[r] \arrow[d] & E_t \arrow[r] \arrow[d] & K(\pi_{t+1}(S^{n-1}), t+2) \arrow[d] \\
X \arrow[r] & X \arrow[r] & *
\end{tikzcd}$$

Now let $\overline{E} = holim_t E_t$, so the homotopy fiber of the map $\overline{E} \to X$ is weakly equivalent to S^{n-1}.

So we have an orientable fibration $\overline{E} \to X$ with α as its Euler class.

Furthermore if have the conditions for local coefficients M, then by Serre spectral sequence with local coefficients that $\overline{E}$ vanishes above dimension $d + |\alpha| - 1$. Then by a result of Wall[32], $\overline{E}$ is homotopy equivalent to a finite dimensional CW-complex.

□

Now the final step I give the criterion about periodicity of CW-complexes, which generalizes a result of Swan[30] concerning finite groups with periodic cohomology.

Theorem 2.2. *The cohomology of a connected CW-complex X is periodic if and only if there is a spherical fibration $E \to X$ with a total space E which is homotopy equivalent to a finite dimension CW-complex.*

Proof. First we note that by taking fiber joins if necessary we can always assume a spherical fibration is orientable. The reason is that if this fibration is not orientable, then there exists an element in $\pi_1(X)$ such that the action of this element on the fibre will induce a homomorphism on cohomology case:

$$H^n(S^n) \xrightarrow{-id} H^n(S^n)$$

So now if we take fiber joins then the action of this element will induce an action on $H^{2n-1}(S^n * S^n) \simeq H^n(S^n) \otimes H^n(S^n)$(by Künneth formula)which is $(-id)(-id) = id$. So the action of $\pi_1(X)$ on $H^{2n-1}(S^n * S^n)$ is trivial.

So we have a Gysin sequence with any local coefficients M like follows:

$$\cdots \to H^k(X, M) \to H^k(E, M) \to H^{k-n}(X, M) \xrightarrow{\alpha \cup -} H^{k+1}(X, M) \to \dots$$

Where α is the Euler class of this fibration. By assumption E is finite dimensional, there is an integer N such that

$$\alpha \cup - : H^{k-n}(X, M) \to H^{k+1}(X, M)$$

When $k \geq N$. So X has periodic cohomology. Conversely, if X has periodic cohomology. By definition, there is a cohomology class $\alpha \in H^*(X)$ and an integer $d \geq 0$such that:

$$\alpha \cup - : H^m(X) \to H^{m+|\alpha|}(X)$$

is an isomorphism for $m \geq d$. Then by lemma 2.1.3 there is a integer q and a fibration sequence as follows:

$$P_{d+q|\alpha|-1}S^{q|\alpha|-1} \to E \to X$$

such that a generator of $H^{q|\alpha|-1}(P_{d+q|\alpha|-1}S^{q|\alpha|-1})$ transgresses to α^q. Then by lemma 2.1.4 there is an orientable spherical fibration with Euler class α^q and the total space E is homotopy equivalent to a finite dimensional CW-complex. □

2.2 Applications of Periodic Complexes

Wall finiteness obstructions

In this subsection I will give some applications of periodic complexes on Wall finiteness obstruction theory. So we first need to review some basic notions in Wall finiteness obstruction theory.

Definition 2.2.1. *Let X be a CW complex, we say X is* ***finitely dominated*** *if it is a retract(in the homotopy category of CW-complexes) of a finite CW-complex Y. In other words, there is a finite CW-complex Y and maps*

$$i : X \to Y; r : Y \to X$$

such that $r \circ i \simeq id_X$.Which means $r \circ i$ is homotopy equivalent to the identity map between space X.

The central question of Wall finiteness obstruction theory is to judge in what conditions a finite dominated CW-complex has homotopy type of a finite CW-complex.

Definition 2.2.2. *(Functor K_0)*
Let R be a ring, we define $K_0(R)$ be the ***Grothendieck*** *group of projective R-modules. More precisely, it is the free Abelian group generated by classes $[p]$, where P is a projective R-module, modulo the relation*

$$[P] = [P'] + [P'']$$

if there is an isomorphism $P \simeq P' \oplus P''$.

It turns out this construction a functor from the category of rings to the category of abelian groups.

Moreover, we have a canonical map

$$\mathbb{Z} \to K_0(R)$$

which maps n to $n[R]$, where R is viewed a R-module by multiplications. We use $\tilde{K}_0(R)$ denote the co-kernel of this natural map and call it ***reduced*** *K-group of this ring R.*

Now we recall the definition of **Euler Characteristic** of a chain complex.

Definition 2.2.3. *If (C_*, d) is a chain complex of finite type of projective R-modules, we define its **Euler Characteristic** by*

$$\chi(C) = \sum_{-\infty}^{\infty} (-1)^j [C_j]$$

which is defined in $K_0(R)$. Note that this is really a finite sum and we can also define $\tilde{\chi}(C)$ in reduced K-group $K_0(R)$.

The Wall's finiteness obstruction theory concerns about when a finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex. Before give the precise statement we need a fact in the following[31]

Lemma 2.2.1. *Let X be a finitely dominated CW-complex, and $\tilde{X}$ is its universal cover. Then $C_*(\tilde{X})$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\pi_1(X)]$-module chain complex.*

Definition 2.2.4. *Suppose X is a finitely dominated CW-complex with $C_*(\tilde{X})$ as its universal cover. And let*

$$\mathcal{P} : \cdots \to P_2 \to P_1 \to P_0 \to 0$$

*be a finite projective $\mathbb{Z}[\pi_1(X)]$-module chain complex which is chain homotopy equivalent to $C_*(\tilde{X})$. Then define the **projective class** of X as*

$$[X] = \sum_{i=0}^{\infty} (-1)^i [p_i] \in \tilde{K_0}(\mathbb{Z}[\pi_1(X)])$$

Remark 2.2.1. *In fact this definition does not depend on the choice of the finite projective chain complex.*

Now I can state the important theorem by Wall [31]

Theorem 2.3. *(Wall) Suppose X is a finitely dominated CW-complex then X is homotopy equivalent to a finite CW-complex if and only if the projective class $[X] = 0$.*

Lemma 2.2.2. *Let X be a CW-complex and let $E \to X$ be a spherical orientable fibration with fibre an odd dimension sphere S^n other than S^1. By exact sequence of homotopy groups of fibrations we know that $\pi_1(E) = \pi_1(X)$. For every integer $q \geq 1$, the q-fold fiber joins $*^q_X E \to X$ is also a spherical fibration with fibre $*^q S^n$. If E is finitely dominated then so is the $*^q_X E$. Moreover, if $\mathcal{O}_E \in \tilde{K_0}(\mathbb{Z}\pi_1(X))$is the finiteness obstruction of E then $q\mathcal{O}_E$ is the finiteness obstruction of $*^q_X E$.*

Proof. Without losing any generality, we can assume that $q = 2$. And more general, let $E_1 \to X$ and $E_2 \to X$ be two orientable spherical fibrations. By definition we have push-out diagram in homotopy category like follows:

$$\begin{tikzcd} E_1 \times_X E_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & E_2 \arrow[d] \\ E_1 \arrow[r] & E_1 *_X E_2 \end{tikzcd}$$

The projections p_1 and p_2 are spherical fibrations with odd dimensional fibers. If E_1 and E_2 are finitely dominated, that is there are two finite CW complex K_1 and K_2 with maps $i_m : E_m \to K_m, r_m : K_m \to E_m$ such that $r_m \circ i_m = id$ in homotopy category where

$m = 1, 2$.Then I claim that the fibre product $E_1 \times_X E_2$ and fibre join $E_1 *_X E_2$ both are finitely dominated. Let $K_1 \times_X K_2$ be the pull pack of the following diagram

$$\begin{tikzcd} K_1 \times_X K_2 \arrow[r] \arrow[d] & K_2 \arrow[d] \\ K_1 \arrow[r] & X \end{tikzcd}$$

Where the maps $K_i \to X$ are the composition of $K_i \xrightarrow{r_i} E_i \to X$ where $i = 1, 2$.

Then we have a big commutative diagram as follows

$$\begin{tikzcd} K_1 \times_X K_2 \arrow[rr] \arrow[dd] \arrow[ddr] \arrow[dr, dotted] \arrow[drr] & & K_2 \arrow[d] \\ & E_1 \times_X E_2 \arrow[r] \arrow[d] & E_2 \arrow[d] \\ K_1 \arrow[r] & E_1 \arrow[r] & X \end{tikzcd}$$

By universal property of pull-back there exists a map

$$r : K_1 \times_X K_2 \to E_1 \times_X E_2$$

to make this big diagram commute. Moreover we have a natural inclusion

$$i : E_1 \times_X E_2 \to K_1 \times_X K_2$$

in homotopy category.

By diagram chasing we know $r \circ i = id$ in homotopy category. So $E_1 \times_X E_2$ is finitely dominated. Similar argument could show that the fiber join $E_1 * E_2$ is also finitely dominated. Moreover we have an equation in $\tilde{K}_0(\mathbb{Z}\pi_1(X))$

$$\mathcal{O}_{E_1} + \mathcal{O}_{E_2} = \mathcal{O}_{E_1 \times_X E_2} + \mathcal{O}_{E_1 *_X E_2}$$

Since $E_1 \times_X E_2 \to E_2$ is a spherical fibration with E_2 as a finitely dominated base space and an odd dimensional sphere as fiber, we conclude

$$\mathcal{O}_{E_1 *_X E_2} = \mathcal{O}_{E_1} + \mathcal{O}_{E_2}$$

In particular, repeat the above argument for $E_1 = E_2$ we know that $*_X^q E$ is finitely dominated and $q\mathcal{O}_E$ is the finiteness obstruction of $*_X^q E$.

The similar argument like Theorem 2.4 could show that if X is simply connected and finitely dominated then Y is homotopy equivalent to a finite complex.

□

Lemma 2.2.3. *Let $E \to X$ be a spherical fibration, where X is a CW-complex with k-skeleton $sk_k X$. Then for every integer $k \geq 0$ there is an integer q such that the fibration $E_k \to sk_k X$ in the pull-back square*

$$\begin{tikzcd} E_k \arrow[r] \arrow[d] & *_X^q E \arrow[d] \\ sk_k X \arrow[r] & X \end{tikzcd}$$

is a product fibration.

Proof. As we discussed in Theorem 2.2 We can always assume that the fibration is orientable. From Chapter 1 we know that if F is a CW-complex then the classifying spaces $BAut(F)$of the topological monoid $Aut(F)$ classifies the fibrations with fibre F

$$F \to E \to X$$

In other words we have a canonical bijection as follows:

$$[X, BAut(F)] \simeq Fib(F)$$

Where $Fib(F)$ denotes the isomorphism classes of fibrations with fibre F. Moreover, the identity component $Aut_I(F)$ classifies the orientable fibrations with fibre F.In our case, $BAut_I(S^m)$ classifies the orientable spherical fibrations. Use induction on k to prove this lemma. When $k = 0$ it is trivial. So noe we can assume that this lemma is true over the $(k-1)$-skeleton $sk_{k-1}X$. Without losing any generality we have the following pull-back diagram:

$$\begin{tikzcd} E_{k-1} \arrow[r] \arrow[d] & E \arrow[d] \\ sk_{k-1}X \arrow[r] & X \end{tikzcd}$$

where the two columns are both orientable spherical fibrations with same fibre $F \simeq S^m$, and E_{k-1} is the total space of the product fibration, in other words $E_{k-1} \simeq sk_{k-1}X \times F$. Then by the classification theorem of fibrations we know that the fibration

$$F \to E_{k-1} \to sk_{k-1}X$$

corresponds a map $f : sk_{k-1}X \to BAut_I(F)$ which is null-homotopy. And we have a diagram as follows:

$$\begin{tikzcd} sk_kX \arrow[dr] & \\ sk_{k-1}X \arrow[u] \arrow[r, "f"] & BAut_I(F) \end{tikzcd}$$

So for this extension problem being solved we need these obstruction classes $\mathcal{O}_\Delta \in \pi_k(BAut_I(F)) \simeq \pi_{k-1}(Aut_I(F))$ vanish where Δ ranges over all k-cells of X. Now we can take the 1-fold fibre joins to replace this obstructions with $q\Sigma_*(\mathcal{O}_\Delta)$,where

$$\Sigma : Aut_I(F) \to Aut_I(*^qF)$$

is the natural stabilization map defined by $f \mapsto f * id * \cdots * id$. By Suspension theorem we know for fixed r, if m is large enough the homotopy groups of spheres $\pi_r(S^{m+r})$ is stable, so $\pi_r(Aut_I(S^m)) \simeq \pi_r^{st}(S^0) = colimit_n\pi_r(S^{n+r})$. And we also know that these stable homotopy groups are finite groups. So if we choose the integer q which is larger than the order of the stable homotopy group, we have

$$q\pi_{k-1}(Aut_I(*^qF)) = 0$$

So we do not have the obstruction problem, that means now we have a null-homotopy map $\tilde{f} : sk_kX \to BAut_I(*^qF)$. By classification theorem of fibrations we now have a pull-back diagrams as follows which E_k is the total space of a trivial fibration.

$$\begin{tikzcd} E_k \arrow[r] \arrow[d] & *^q_X E \arrow[d] \\ sk_kX \arrow[r] & X \end{tikzcd}$$

□

After above prerequisite, now we can give the important theorem in this section which is similar to the big theorem of last section:

Theorem 2.4. *Let X be a CW-complex of finite type such that the reduced K-group $\tilde{K}_0(\mathbb{Z}\pi_1(X))$ is a torsion group. Then the cohomology of X is periodic if and only if there is a spherical fibration $E \to X$ such that the total space E is homotopy equivalent to a finite CW-complex.*

Proof. As for the only if condition, it has a similar argument as we proved theorem 2.2 By taking fiber joins, we can always assume that the fibration is orientable. Then using Gysin sequence we can get the cohomology of X is periodic.

Conversely, suppose X is a periodic complex. According to Theorem 2.2 we know that there is an orientable spherical fibration $E \to X$ with E homotopy equivalent to a finite dimensional CW-complex. By taking fiber joins of this fibration we can always assume that the fiber of this fibration is homotopy equivalent to a sphere with odd dimension≥ 3. So $\pi_1(X) \simeq \pi_1(E)$.

Moreover, since X is a CW-complex of finite type and E also has finite type. The it is clear that E is finitely dominated. And its finiteness obstruction class $\mathcal{O}_E$ lies in $\tilde{K}_0(\mathbb{Z}\pi_1(E)) = \tilde{K}_0(\mathbb{Z}\pi_1(X))$. Since $\tilde{K}_0(\mathbb{Z}\pi_1(X))$ is a torsion group, we can assume the order of $\mathcal{O}_E$ is q. So we can take q-fold fibre joins $*^q_X \to X$. By lemma2.2.3 $*^q_X E$ is also finitely dominated and its finiteness obstruction class is $q\mathcal{O}_E = 0$. BY Wall's finiteness obstruction theorem we know $*^q_X E$ is homotopy equivalent to a finite complex. □

Periodic groups

In this subsection, I will use the definition of periodic complexes to give another definition of periodic groups and I will prove that these two definition are equivalent.

Definition 2.2.5. *Let G denote a discrete group and let X be a model of Eilenberg-Maclane space $K(G,1)$. We say that G has periodic cohomology(or G is a periodic group) if X has periodic cohomology.*

Remark 2.2.2. *It is clear that if G is a periodic group then exist positive numbers N and d such that $H^{n+d}(BG,M) \simeq H^N(BG,M)$ for any $\mathbb{Z}G$-module M. Since when N is positive the N-th degree Tate cohomology coincides with the N-th degree ordinary cohomology, then by Theorem 1.12 we know the two definitions about periodic groups are equivalent when G is discrete group.*

Proposition 2.2.1. *A discrete group G has periodic cohomology if and only if G acts freely on a finite dimensional complex homotopy equivalent to a sphere.*

Proof. As for the "if" part, suppose there is a finite dimensional CW-complex F which is homotopy equivalent to sphere. And G acts freely on it. Then it is clear that the Borel construction of F is homotopy equivalent to F/G. So we have a spherical fibration

$$F \to F \times_G EG \to BG$$

where the total space is homotopy equivalent to a finite dimensional CW-complex. By Theorem 2.2 G is a periodic group.

Conversely, if G is a periodic group then by the Theorem 2.2 there is a spherical fibration:

$$S^N \to E \to BG$$

where E is homotopy equivalent to a finite dimensional CW-complex. First I claim we can always assume $N > 1$. For $N = 1$ by the Theorem 2.2 we know the periodicity of the cohomology groups of BG is 1 so 2 is also a periodicity for the space BG. Then using the Theorem 2.2 again we have a spherical fibration with fiber homotopy equivalent to S^2.

As for the case $N > 1$ we first apply the long exact sequence of homotopy groups of fibrations we know $\pi_1(E) = G$. Then let E' be the finite dimensional complex with the same homotopy type as E and let $\tilde{E}'$ be its universal cover. Since E' is a finite dimensional CW complex, its universal cover is also a finite dimensional CW complex. Then by the lifting property of covering spaces we have a commutative diagram as follows:

$$\begin{tikzcd} & & \tilde{E}' \arrow[d] \\ S^N \arrow[r] \arrow[urr] & E \arrow[r, "\sim"] & E' \end{tikzcd}$$

Apply the homotopy groups functor to this diagram we see that $\tilde{E}'$ has the same homotopy type as S^N. Moreover we know the group G can act $\tilde{E}'$ via Deck transformations. In other words G can act free on finite dimensional CW-complex which is homotopy equivalent to a sphere.

□

Moreover, we have:

Corollary 2.2.1. *A discrete group G acts freely and properly on $\mathbb{R}^n \times S^m$ for some $m, n > 0$ if and only if G is countable and has periodic cohomology.*

Proof. To prove this theorem we need Smale's famous h-cobordism theorem which beyond the scope of my thesis. People who interested in it could refer [4] □

Group actions with periodic isotropy

In this subsection, I will investigate the periodicity of cohomology of Borel's construction $X \times_G EG$. I will use a little knowledge about stable module categories in proofs. People who are not familiar with it can refer the appendix A of this thesis which has a concise introduction to it.

Lemma 2.2.4. *Let G be a finite group and X be a finite G-CW complex . Then the cohomology of $X \times_G EG$ is periodic if and only if all isotropy subgroups have periodic cohomology.*

Proof. For if condition, we first assume all isotropy subgroups have periodic cohomology. According to results in Adam's paper[1] we can choose a cohomology class $\alpha \in H^*(G, \mathbb{Z})$ such that for any isotropy subgroup, the restriction $Res^G_H(\alpha)$ is a periodic generator. By projective cover we can express the cohomology class α as a short exact sequence:

$$0 \to L_\alpha \to \Omega^{|\alpha|}(\mathbb{Z}) \to \mathbb{Z} \to 0$$

where $\Omega^{|\alpha|}(\mathbb{Z})$ denotes the $|\alpha|$-fold dimension shift of trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

In stable module category considering the restriction of this short exact sequence we get

$$0 \to Res^G_H L_\alpha \to \Omega^{|\alpha|}(\mathbb{Z}) \to \mathbb{Z} \to 0$$

By periodicity we have $\Omega^{|\alpha|}(\mathbb{Z}) \simeq \mathbb{Z}$. So in stable module category we have $Res^G_H L_\alpha = 0$. Pass it to ordinary module category it means the projectivity of $Res^G_H L_\alpha$. And it implies $L_\alpha \otimes \mathbb{Z}[G/H]$ is a projective $\mathbb{Z}G$-module.

Now Let $C^*(X)$ denote the cellular co-chains on X, since $C^*(X)$ are free abelian groups we have a short exact sequence:

$$0 \to C^*(X) \otimes L_\alpha \to C^*(X) \otimes \Omega^{|\alpha|}(\mathbb{Z}) \to C^*(X) \to 0$$

And since X is a finite G-CW complex, we know $C^*(X) \otimes L_\alpha$ is $\mathbb{Z}G$-projective. I claim it is a cohomologically trivial module which means that the group cohomology of G with it as coefficients is trivial. For any induced module which has the form $\mathbb{Z}G \times M$ it is cohomologically trivial since by Shapiro's lemma

$$H^i(G, \mathbb{Z} \otimes M) = H^i(*, M)$$

In particular if the module M is a $\mathbb{Z}G$-free module which means that $M \simeq \bigoplus_i \mathbb{Z}G$ and we have

$$\bigoplus_i \mathbb{Z}G \simeq \bigoplus_i \mathbb{Z}G \otimes \mathbb{Z}$$

So it is an induced module, therefore it is cohomologically trivial. Since any projective module is a direct summand of free module we know it is cohomologically trivial. Hence for any $C_n(X)$ we have $H^i(G, C^n(X) \otimes L_\alpha) = 0$. By the spectral sequences argument we have $H^i(G, C^*(X) \otimes L_\alpha) = 0$. Now apply the cohomology functor to the short exact sequence we get an isomorphism for any $\mathbb{Z}G$-module M as follows

$$H^i(X \times_G EG, M) \longrightarrow H^{i+|\alpha|}(X \times_G EG, M)$$

As for the converse, we observe that the periodicity condition could imply that for any prime p the cohomology ring $H^*(X \times_G EG, \mathbb{F}_p)$ have the Krull dimension one or zero. Then by a theorem due to Quillen[23], which stats that the Krull dimension of $H^*(G)$ is equal to the maximal rank of an elementary abelian p-subgroup of G. Therefore all isotropy subgroup must be of rank one, by Theorem 1.1.4 we know all isotropy subgroup have periodic cohomology.

□

Now I can state and prove a key lemma of my thesis which is very useful to construct some free group actions on products of spheres.

Theorem 2.5. *Let X denote a finite dimensional G-CW complex, where G is a finite group, such that all of its isotropy subgroups have periodic cohomology. Then there exists a finite dimensional CW-complex Y and a free G-action such that $Y \simeq S^N \times X$ for some positive integer N. Moreover, if X is simply connected and finitely dominated, then we can assume that Y is a finite complex.*

Proof. By lemma 2.2.4 we know $X \times_G EG$ is a periodic complex. And according to theorem 2.2 there is a spherical fibration $E \to X \times_G EG$. We have a pull back diagram as follows

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & X \times_G EG \end{tikzcd}$$

so we get a spherical fibration for X: $E' \to X$. And by lemma 2.2.3 using fiber join we have a pull back diagram

$$\begin{tikzcd} E'_k \arrow[r] \arrow[d] & *^q_X E' \arrow[d] \\ Sk_k(X) \arrow[r] & X \end{tikzcd}$$

such that E'_k is a product space. Since X is finite dimensional we can choose large k such that $Sk_k(X) = X$. So we can take E'_k as the space Y such that it is a finite dimensional CW-complex $Y \simeq S^N \times X$ with free G action.

Finally to prove that Y can be chosen as a finite complex, we first apply Lemma 2.2.2 to the above diagram then we get

$$\mathcal{O}_Y = q\mathcal{O}_{E'}$$

Without losing any generality we can assume the spherical fibration has the fiber homotopy equivalent to a sphere with dimension greater than 1. In that case by the log exact sequence of homotopy groups of fibrations we have $\pi_1(E') = \pi_1(X) = 0$ since X is a simply connected space. Therefore $\mathcal{O}_{E'} = 0$ which implies $\mathcal{O}_Y = 0$. By Wall finiteness obstruction theorem we know Y is a finite complex. $\square$

Chapter 3

Effective Euler Classes

As the main topic of this thesis is about the free group action on homotopy spheres, it is reasonable to consider the fixity of some group actions on spheres. Therefore, in this chapter, I first use the elementary abelian subgroups to define the effective Euler class which is a nice description about the stationarity of a finite group actions on homotopy spheres. A nice way to construct finite group actions on spheres is by the representations of finite groups. It is clear that a representation not containing the trivial representation gives a free action of G on the unit sphere of the vector space of representation. Hence in the second section of this chapter I will use the classical character theory(see appendix B) of complex representations to construct free actions on the product of two spheres for some specific simple rank two groups. This method however, is not working for all cases. The next chapter fills the gaps by making use of the concept of local Euler classes and the gluing method.

3.1 Euler Classes from Actions on Spheres

In this section, I will first define what the effective Euler classes are then use it to construct an important group action on product of two spheres.

Definition 3.1.1. *We say an Euler class $\beta \in H^N(G)$ related to the fibration of Borel's construction $X \to X \times_G EG \to BG$ where $X \simeq S^{N-1}$ is p-effective(p is a prime) if every maximal rank, elementary abelian p-subgroups of G acts without stationary points. Moreover, we say this Euler class is effective if it is p-effective for any prime $p||G|$.*

Remark 3.1.1. *By the fundamental properties of Euler classes we know it is equivalent to say that β is p-effective if $\beta|_E \neq 0$ for all $E \subset G$ elementary abelian p-subgroups.*

Remark 3.1.2. *We can also give the same definition of effective Euler class for* $\mod p$ *cohomology. And it is easily seen that it is equivalent to p-effective Euler class in integral cohomology.*

Representation of finite groups turns out a very powerful tool to construct the group action on spheres we want. So next I will investigate some properties of Euler classes arise from representations.

Theorem 3.1. *Let G denote a finite p-group acting on a finite complex X homotopic to a sphere. Then there exists a real representation V of G such that the Euler class associated to X is effective if and only if the Euler class for $S(V)$ is effective.*

Proof. Apply the main result in [10], which says that given a CW complex X as above there is a real representation V of group G with

$$\dim V^H = \dim X^H + 1$$

where V^H means the invariant subspace under the group action of subgroup H, H is any subgroup of G. By the definition of effectiveness we complete the proof. □

Proposition 3.1.1. *Let G be a finite group with center $Z(G) \simeq \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$ and corresponding to generators $x_1, \cdots, x_k$. Then there exist $[G : Z(G)]$-dimensional complex representations $V_1, V_2, \cdots, V_k$ of G such that $V_j^H = \{0\}$ for any subgroup $H \subset G$ containing x_j.*

Proof. Let $\chi_j : Z(G) \to \mathbb{C}$ denote the 1-dimensional complex representation of $Z(G)$ defined via $x_j \mapsto e^{2\pi i/n_j}$ and $x_h \mapsto 0$ when $h \neq j$. Then Let

$$V_j = Ind_{Z(G)}^G(\chi_j)$$

By the Cartan-Eilenberg double cosets formula we can get

$$V_j\big|_{Z(G)} = Res_{Z(G)}^G Ind_{Z(G)}^G(\chi_j) = \sum_{g \in E} Ind_{Z(G)\cap gZ(G)g^{-1}}^{Z(G)} Res_{Z(G)\cap gZ(G)g^{-1}}^{Z(G)}(\chi_j)$$

which is isomorphic to $\oplus^{[G:Z(G)]}(\chi_j)$. And since x_j is central, we have $V_j^{x_j} = \{0\}$. So for any subgroup H of G which containing x_j we have $V_j^H = \{0\}$.

□

Remark 3.1.3. *Based on a result in [11], we know(I can't give a complete argument here since it will use the notions about regular sequence in cohomology) if $X = S(V_1) \times \cdots S(V_k)$ with the diagonal G-action, then $Z(G)$ will act freely on X.*

Now use these representations we have the following result

Proposition 3.1.2. *Let G denote a finite p-group such that the p-rank of its center $Z(G)$ is equal one less than the rank $r(G)$ of G. Then G acts on $X = (S^{2[G:Z(G)]-1})^{r(G)-1}$ such that the isotropy subgroups are all p-rank at most equal to 1.*

Proof. Consider the same $X = S(V_1) \times \cdots \times S(V_k)$ in remark 3.1.3. In this case $k = r(G) - 1$. So G acts on $X = (S^{2[G:Z(G)]-1})^{r(G)-1}$ such that $Z(G)$ has a free action on it.

Without losing any generality we assume $E \simeq \mathbb{Z}/p \times \mathbb{Z}/p$ fixes a point in X. Then it is clear that $E \cap Z(G) = 0$ since $Z(G)$ acts freely on X. So the subgroup generated by $Z(G)$ and E has p-rank at least $R(G) + 1$, that is a contradiction

□

Since any finite p-group has a non-trivial center, so we get

Corollary 3.1.1. *Let G be a finite p-group of rank 2, then G acts on $X = S^{2[G:Z(G)]-1}$ with rank 1 isotropy subgroups.*

Combine Proposition 3.1.2 and Theorem 2.5 we have:

Theorem 3.2. *Let G be a finite p-group such that $r = r(G) \leq r(Z(G)) + 1$. Then there is an integer N such that G acts freely on a finite complex*

$$Y \simeq S^N \times (S^{2[G:Z(G)]-1})^{r-1}$$

In particular, we have

Theorem 3.3. *A finite p-group G acts freely on a finite complex $Y \simeq S^n \times S^m$ if and only if it doesn't have $(\mathbb{Z}/p)^3$ as its subgroup.*

Proof. As for "only if" part, observe that by hypothesis we know the rank of G is less or equal to 2, so it satisfies the conditions in Theorem 3.2 . So the action exists.

AS for "if" part, according to a result in [16] which states that $(\mathbb{Z}/p)^3$ does not act freely on $S^n \times S^n$. So if G has $(\mathbb{Z}/p)^3$ as its subgroup, that's a contradiction. □

3.2 Some Examples from Representations

In this section I will construct actions of simple rank 2-groups. There is a complete list of rank two simple groups in the following:

1. $PSL_2(\mathbb{F}_p)$, p odd, $p \geq 5$
2. $PSL_2(\mathbb{F}_{p^2})$, p odd
3. $PSL_3(\mathbb{F}_p)$, p odd
4. $PSU_3(\mathbb{F}_p)$, p odd
5. $PSU_3(\mathbb{F}_4)$
6. A_7
7. M_{11}

Remark 3.2.1. *There is a fact that these rank 2 simple groups have a copy of A_4, the alternating groups on 4 letters.*

Based on remark 3.2.1 we have the following interesting result about group actions on a products of 2 spheres.

Theorem 3.4. *If G is a simple rank 2 group, then it cannot act freely on a finite dimensional complex $Y \simeq S^{n_1} \times S^{n_2}$ via product actions, which means $g \cdot (s_1, s_2) = (g \cdot s_1, g \cdot s_2)$ for any $g \in G$ and $s_1 \in S^{n_1}, s_2 \in S^{n_2}$.*

Proof. Suppose we have a free action like it, then so does A_4 since A_4 could be viewed as a subgroup of a rank 2 simple group. By taking fibre joins A_4 can have a free actions on $S^{(n_1+1)(n_2+1)} \times S^{(n_1+1)(n_2+1)}$. But it is a contradiction according to a result by Oliver [22] $\square$

So we can't use product actions to construct free actions of rank 2 simple groups on products of 2 spheres.

Now consider a finite group G which contains $\mathbb{Z}/2 \times \mathbb{Z}/2$ as a subgroup. I want to construct actions of G on spheres with 2-effective Euler class. Use reduced complex regular representation(which means that modulo the trivial representation)of $\mathbb{Z}/2 \times \mathbb{Z}/2$ gives an action on S^5 with 2-effective Euler class. Using induced representation(see Appendix B) we get a new action on S^5, to ensure this new action has still 2-effective Euler class we need the following lemma:

Lemma 3.2.1. *Let χ be a character of G which takes constant values A on all involutions(order 2 elements). Then $\chi|_E$ is the character associated to a multiple of reduced regular representation of $E \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, for any E is a subgroup of G if and only if $\chi(1) = -3A$.*

Proof. The character table of irreducible representations of E is the following

And we have

$$\chi|_E = m_1\chi_1 + \cdots + m_4\chi_4$$

Evaluating this equation on elements a, b and ab based on the character table we have:

$$m_1 - m_2 + m_3 - m_4 = m_1 + m_2 - m_3 - m_4 = m_1 - m_2 - m_3 + m_4 = A$$

And by assumption we don't need the trivial representation which means that $m_1 = 0$. By solving these equations we have $\chi(1) = -3A$. $\square$

Theorem 3.5. *S_4, A_5, and $SL_3(F_2)$ acts freely on a finite complex $Y \simeq S^N \times S^5$ for some positive integer N.*

Proof. By checking finite group atlas[9] we know all these 3 groups satisfy the conditions of previous lemma. By theorem 2.5 if a G-CW complex with isotropy subgroups with periodic cohomology, then there is a free G-action on $Y \simeq S^N \times X$. So we need to show that every isotropy subgroups have periodic cohomology. According to Theorem 1.14 we judge if a group have periodic cohomology if and only if every its elementary abelian subgroup has rank at most 1. Let X be $S(V) = S^5$ where V is the representation associated to the character which satisfies the conditions in lemma 3.2.1. And since $|A_5| = 2^2 \times 5 \times 3$, $|S_4| = 2^3 \times 3$ and $|SL_3(\mathbb{F}_2)| = 2^3 \times 3 \times 7$, for odd primes there are only cyclic p-Sylow subgroups. According to Theorem 1.15 we only need to care about the prime 2. And by lemma 3.2.1 we know the elementary abelian subgroup in isotropy subgroups has rank at most 1. Therefore these 3 groups have free actions on $Y \simeq S^N \times S^5$ □

Next I give 2 little more complicated simple groups

Theorem 3.6. *The simple group $SU(3,3)$ acts freely on a finite complex $Y_1 \simeq S^{N_1} \times S^{11}$ and the simple group $SU(3,4)$ acts freely on $Y_2 \simeq S^{N_2} \times S^{23}$.*

Proof. First let's consider the simple group $SU(3,3)$. It has order

$$|SU(3,3)| = |U(3,3)|/(3+1) = 2^5 \times 3^3 \times 7$$

And $r_2(G) = r_3(G) = 2$.

Now let $P = Syl_3(SU(3,3))$ be an extra-special 3-group and we have a group extension as follows:

$$1 \to \mathbb{Z}/3 \to P \to \mathbb{Z}/3 \times \mathbb{Z}/3 \to 1$$

with $\mathbb{Z}/3 \subset P$ central. Let $3A$ denote the conjugacy class of this central order 3 element.

According to atlas of finite groups [9] there is a complex character χ with character table as follows

	1	$2A$	$3A$	$3B$
χ	6	-2	-3	0

where $2A$ denotes the conjugacy class of involutions and $3B$ denote the other conjugacy class of elements of order 3.

Observe this character table we know this character satisfies the conditions in Lemma 3.2.1 So the elementary abelian 2-subgroups in isotropy subgroups have rank at most one and as for elementary abelian 3-subgroups. We consider the subgroup generated by $3A$. Since the generator is central, the generated group is the cyclic group of order 3, C_3. Consider the representation of C_3 by restriction the representation of $SU(3,3)$ to C_3. Assume

$$\chi = n_1\chi_1 + n_2\chi_2 + n_3\chi_3$$

where χ_i, $i = 1, 2, 3$ are the 3 indecomposable characters of C_3. According to character atlas of C_3 we have the following equations

$$\begin{cases} n_1 + n_2 + n_3 = 6 \\ n_1 + \omega n_2 + \omega^2 n_3 = -3 \end{cases}$$

Solve these equations we get $n_1 = 0$ and $n_2 = n_3 = 3$. It means this representation does not contain the trivial representation. In other words this representation of C_3 has not fixed points, in other words C_3 acts freely on S^{11}. So $SU(3,3)$ acts on S^{11} with elementary abelian 3 subgroups with rank at most 1. By order decomposition of $SU(3,3)$ we deduce that $SU(3,3)$ acts on S^{11} with rank one isotropy and then by theorem 2.5 $SU(3,3)$ acts freely on a finite complex $Y_1 \simeq S^{N_1} \times S^{11}$.

Now we consider the other simple group $SU(3,4)$ with order $2^6 \times 3 \times 5^2 \times 13$. So we only need to consider the primes 2 and 5. By character atlas of $SU(3,4)$ there is a complex character χ with the following character table

	1	$2A$	$5A$
χ	12	-4	-3

Similar arguments could show that $SU(3,4)$ acts on S^{23} with isotropy subgroups having periodic cohomology. So it could act freely on $Y_2 \simeq S^{N_2} \times S^{23}$. $\square$

Chapter 4

Local Euler Classes and Rank Two Groups

The idea of using complex character to produce 2-effective Euler classes stated in chapter 3, is not enough for arguments of all rank two simple groups. In this chapter I introduce a local version of the Euler classes from the previous chapter called the **local Euler class** which is flexible enough to locally extend our collection of examples. Then I use a pullback diagram to glue these local pieces to an integral effective Euler class and complete the construction of free actions of most simple rank two groups on the product of two spheres.

4.1 Local Euler Classes

I first review some notions about localization of spaces.

Localization methods are very powerful in algebraic topology. For example, it is pretty hard to calculate homotopy groups of spheres. However, we can use localizations to get some partial information about homotopy groups. Next I will introduce how to construct a localization of spaces. We assume readers are familiar with the localization of a commutative ring.

Let K denote a set of primes.

Definition 4.1.1. *We denote $\mathbb{Z}_K$ for the ring of integers localized by inverting all primes not in K, that is the subring of $\mathbb{Q}$ whose denominators are primes not in K*

After definition I give some special cases as examples

Example 4.1.1. *1. If $K = \{p\}$ where p is a prime, then K-localization will be called localization at p or p-localization.*

2. $K = \emptyset$, K-localization will be called rationalization

Definition 4.1.2. *For a topological space Y we say it is a K-local space if $\pi_*(Y)$ is local that is $\pi_*(Y)$ is a $\mathbb{Z}_K$-module. Given a topological space X we say a map*

$$k : X \to X_K$$

is a K-localization of X if it satisfies the universal property as follows: given a map $f : X \to Y$ where Y is a K-local space there is a unique map $f : X_K \to Y$ makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & X_K \arrow[dl, dotted] \\ Y & \end{tikzcd}$$

We have a complete characterization for the localization of abelian spaces

Definition 4.1.3. *A connected topological space X is called* ***abelian*** *if the actions of $\pi_1(X)$ on $\pi_*(X)$ are trivial.*

Theorem 4.1. *1. For every abelian space X there is a K-localization X_K.*

2. A map $X \to X'$ of abelian spaces is a K-localization if and only if the reduced homology $\tilde{H}_*(X')$ is a $\mathbb{Z}_K$-module and the induced map

$$\tilde{H}_*(X) \otimes \mathbb{Z}_K \to \tilde{H}_*(X') \otimes \mathbb{Z}_K \simeq \tilde{H}_*(X')$$

3. K-localization is a functor between the categories of topological spaces. Moreover, it carries two homotopical maps to homotopical maps.

As for the proof of this theorem readers could refer[29]

Now let $S^m_{(p)}$ denote the p-local sphere. By previous argument we know that $S^1_{(p)}$ is a model of the classifying space $B\mathbb{Z}_{(p)}$ and $S^m_{(p)}$ where $m > 1$ is a simply connected space with $H_m(S^m_{(p)}; \mathbb{Z}) \cong \mathbb{Z}_{(p)}$.

Definition 4.1.4. *Let X be a connected CW-complex and let $k \geq 0$ be an integer. A cohomology class $\alpha \in H^n(X, \mathbb{Z}_{(p)})$ is a* ***p-local Euler class*** *if there is a orientable fibration*

$$S^{n-1}_{(p)} \to E \to X$$

such that a generator of $H^{n-1}(S^{n-1}_{(p)}, \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$ transgresses to α.

Similarly we have the definition of effectiveness of p-local Euler classes

Definition 4.1.5. *Given a finite group G. A p-local cohomology class $\alpha \in H^n(BG, \mathbb{Z}_{(p)})$ is an effective p-local Euler class if its restriction to every elementary abelian p-subgroup of maximal rank is nontrivial.*

Definition 4.1.6. *Let G and H denote discrete groups. We say they are p-equivalent $G \simeq_p H$ for a prime p if the localizations at p of their classifying spaces are homotopy equivalent, that is, $BG_{(p)} \simeq BH_{(p)}$*

Remark 4.1.1. *In fact, we don't need the homotopy equivalent condition , we just require that there is a map between classifying spaces of these two groups which induces isomorphisms on cohomology groups with $\mathbb{Z}_{(p)}$ coefficients.*

Based on this definition we can define what a p-effective Euler class of a discrete group is if it is p-equivalent to a finite group.

Definition 4.1.7. *A p-effective Euler class for a discrete group G is a cohomology element is a cohomology element $x \in H^{2n}(BG, \mathbb{Z}_{(p)})$ which is the of the form $f^*(y)$, where $f : BG_{(p)} \to BH_{(p)}$ is a cohomology $\mathbb{Z}_{(p)}$-equivalence and $y \in H^{2n}(BH_{(p)}, \mathbb{Z}_{(p)})$ is a p-effective Euler class for the finite group H.*

Like we are interested on spheres with p-effective Euler classes, there is a basic result about actions on local spheres with p-effective Euler classes.

Lemma 4.1.1. *Assume that two discrete groups G and H are p-equivalent, where one of them is a finite group. The there is an action of G on $S^{2k-1}_{(p)}$ with a p-effective Euler class if and only if the same holds for H.*

To prove this lemma first I need recall a standard result in homotopy theory[7]

Theorem 4.2. *Let K denote a set of primes, then $BAut_I S^n_K$ is a $\mathbb{Z}_K$-local space*

Now I can use this result to prove this Lemma

Proof. Suppose we have an action of G on $S^n_{(p)}$ with p-effective Euler class. So we have a fibration

$$S^n_{(p)} \to S^n_{(p)} \times_G EG \to BG$$

which is classified by a map in $[BG, BAut_I(S^n_{(p)})]$. Since $BAut_I(S^n_{(p)})$ is p-local, by universal property of localization we have a factorization

$$\begin{tikzcd} BG \arrow[r] \arrow[d] & BAut_I(S^n_{(p)}) \\ BG_{(p)} \arrow[ur, dotted] & \end{tikzcd}$$

Since G and H are p-equivalent, by naturality we can get a map in $[BH, BAut_I(S^n_{(p)})]$. And by naturality this map lifts to an action of H on $S^n_{(p)}$ with p-effective Euler class. □

Based on this lemma I can show some easy examples:

Example 4.1.2. *$\sum_5$ will act on a 2-local sphere with a 2-effective Euler class. The reason is in the following:*

Let us consider an inclusion $S_4 \to S_5$ then we have to maps on group cohomology with $\mathbb{Z}_{(2)}$ as follows:

$$Res^G_H : H^*(S_5, \mathbb{Z}_{(2)}) \to H^*(S_4, \mathbb{Z}_{(2)})$$

$$Tr^G_H : H^*(S_4, \mathbb{Z}_{(2)}) \to H^*(S_5, \mathbb{Z}_{(2)})$$

Since G is a finite group so for any cohomology class $\alpha \in H^*(S_5, \mathbb{Z}_{(2)})$ we have

$$Tr^G_H Res^G_H \alpha = [G : H]\alpha = 5\alpha = \alpha$$

The last equation holds since we are working on $\mathbb{Z}_{(2)}$ case.

Conversely, for any cohomology class $\beta \in H^*(S_4, \mathbb{Z}_{(2)})$ by the Cartan-Eilenberg double cosets formula we have

$$Res^G_H Tr^G_H \beta = \sum_{g \in G/H} Tr^H_{H \cap gHg^{-1}} Res^H_{H \cap gHg^{-1}} \beta = \sum_{g \in G/H} [H : H \cap gHg^{-1}]\beta = \beta$$

So the inclusion induces isomorphism on cohomology, then S_4 and S_5 are 2-equivalent. By the Theorem 3.5 we know S_4 has an action on $S^5_{(2)}$ with a effective 2-Euler class, so does S_5.

Example 4.1.3. *If G is a finite group has an abelian p-Sylow subgroup P, according to a theorem by Swan [3] the inclusion of the normalizer $N_G(P)$ induces isomorphisms on cohomology groups with $\mathbb{Z}_{(p)}$ coefficients.*

Theorem 4.3. *If a finite group G has a normal p-Sylow subgroup, then its cohomology has a p-effective Euler class. Hence if G has an abelian p-Sylow subgroup it has a p-effective Euler class.*

Proof. As for the first statement, let P denote the normal p-Sylow subgroup. use result in Proposition 4.8 we can construct a representation V such that it has an effective Euler class α for group P. Then we use induced representation to get a new representation for G

$$W = Ind^G_P V$$

Then by the Cartan-Eilenberg double coset formula with mod p coefficients we have:

$$Res_P^G Ind_P^G \alpha = \sum_{g \in G/P} g\alpha$$

which is clear an effective Euler class in mod p cohomology, that is, p-effective Euler class.

Next, for the second statement, since any p-Sylow subgroup P is s normal subgroup of its normalizer $N_G(P)$ in G. According to the first statement of this theorem we know that the cohomology of $N_G(P)$ has a p-effective Euler class. Moreover, since its p-Sylow subgroup is abelian, by the Example 4.1.3 we get G also has a p-effective Euler class. □

4.2 Simple Rank Two Groups

In this section I will construct the effective classes of other simple rank two groups which can be obtained by the complex presentations directly.

In order to do it, we need some prerequisites about amalgams. For more details people could refer[27].

Definition 4.2.1. *Suppose A, G_1 and G_3 are three groups. And we have two group homomorphism $\alpha_1 : A \to G_1$ and $\alpha_2 : A \to G_2$. Then we say an **amalgamated free product** is a a push out diagram as follows:*

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & G_1 \arrow[d] \\ G_2 \arrow[r] & G \end{tikzcd}$$

And it is denoted by $G = G_1 *_A G_2$.

Remark 4.2.1. *I recall the Seifert-van Kampen theorem about fundamental groups is precisely an amalgamated free product. More precise, let X be a CW-complex which is the union of two sub-complexes X_1 and X_2. And the intersection Y of X_1 and X_2 is nonempty. Then there is an amalgamated free product:*

$$\begin{tikzcd} \pi_1(Y) \arrow[r] \arrow[d] & \pi_1(X_1) \arrow[d] \\ \pi_1(X_2) \arrow[r] & \pi_1(X) \end{tikzcd}$$

In other words the functor π_1 preserves amalgamations.

In above remark we know the functor π_1 preserves amalgamations, moreover we have a statement about other direction with some limitations

Theorem 4.4. *(Whitehead) Any amalgamation diagram in Definition* 4.2.1 *with α_1 and α_2 injection could be realized by applying the functor π_1 to the following diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

of $K(\pi, 1)$-complexes with $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$.

Next I want to give a different topological interpretation of amalgamation, which I will use some basic notions about graphs.

Definition 4.2.2. *A **graph** is a 1-dimensional CW-complex and a **path** in a graph is sequence of* $e_1, \cdots, e_n$ *of oriented edges such that the final vertex of* e_i *is the initial vertex of* e_{i+1}*. In particular a path is called a **loop** if the final vertex of* e_n *is the initial vertex of* e_1*. Finally, a path is called reduced if* $e_{i+1} \neq \bar{e}_i$ *where* $\bar{e}_i$ *is the same edge with* e_i *but with the opposite orientation.*

Definition 4.2.3. *A graph* X *is called a **tree** if it satisfies the following conditions, which are equivalent*

1. *X is contractible*
2. *X is simply connected*
3. *X is acyclic*
4. *X is connected and contains no non-trivial loops.*

Now we consider an action of group G on a tree X. Select an edge e of X with vertexes v and w. We say e is a **fundamental domain** of this G-action if every edge is transitive to e by G-action. And every vertex of X is transitive to either v or w but v cannot be transitive to w. In other words the orbits of this action is following:

$$\begin{tikzcd} v \arrow[r, dash, "e"] & w \end{tikzcd}$$

Remark 4.2.2. *It is clear that the three isotropy subgroups* G_w, G_v *and* G_e *satisfies the realtion* $G_e = G_v \cap G_w$

Next theorem tells us the relation of the above construction and amalgamation [27].

Theorem 4.5. *(Serre)*
Let a group G *act on a tree* X*. Suppose* e *is an edge with vertices* v *and* w *such that it is a fundamental domain of this action. Then* $G = G_v *_{G_e} G_w$*. Conversely, given an amalgamation* $G = G_1 *_A G_2$ *where* A *is a common subgroup of* G_1 *and* G_2*, there exists a tree with a* G*-action on it, with* G_1, G_2 *and* A *as the isotropy subgroups* G_v, G_w *and* G_e*.*

The next corollary will be used in the rest

Corollary 4.2.1. *Let* $H \subset G_1 *_A G_2$ *be a finite subgroup. Then* H *is conjugate to a subgroup of* G_1 *or* G_2

Proof. Since isotropy subgroups of points in the same orbit are conjugate, We just need to show that H fixes some vertex of X. But according to a fact [27] which says that every finite group of automorphisms of a tree has a fixed point. □

I have introduced the notion of a tree of groups and its relation with amalgamation. Next I want to describe a generalization of amalgamation.

Definition 4.2.4. *We say a **graph of groups** is a connected graph* Y *with*

1. *Groups* G_e *and* G_v *where* v*(resp.* e*) ranges over all vertices(resp.* v*) of* Y
2. *monomorphisms* $\theta_0 : G_e \hookrightarrow G_v$ *and* $\theta_1 : G_e \hookrightarrow G_w$ *for any edge* e *and its vertices* v *and* w*.*

Remark 4.2.3. *In this definition it is possible to allow* $v = w$ *but* θ_0 *and* θ_1 *may still differ.*

Definition 4.2.5. *To define the notion* ***fundamental group*** *of a graph of groups we need use a suitable notion of path. First we add some loops to the graph by regarding an element of G_v as a loop from v to v. Then we modulo an equivalence relation that*

$$\theta_0(a)e = e\theta_1(a)$$

for any $a \in G_v$.

Next example shows how to use this new concept to represents amalgams.

Example 4.2.1. *The amalgam $H *_A K$ is the fundamental group of the graph*

$$\begin{tikzcd} H \arrow[r, dash, "A"] & K \end{tikzcd}$$

Now back to our discussion about local Euler classes. The following theorem is very useful for constructing Euler classes for rank two groups. Let G be a rank two finite group and $|A_p(G)|$ be the geometric realization of the poset(with order by inclusion of subgroups) of non-trivial p-elementary abelian subgroups in G [3] And it is clear that the conjugation by elements in G will induce a natural G-structure in $|A_p(G)|$.

Theorem 4.6. *Let G be a finite group with $|A_p(G)|$ path-connected. If Γ denotes the fundamental group of the graph of groups associated to $y = |A_p(G)|/G$, then the natural surjection $p : \Gamma \to G$ induces a $\mathbb{Z}_{(p)}$-cohomology isomorphism.*

Remark 4.2.4. *So we can replace the original group G by the fundamental group of the graph of groups Γ. Then use Γ we construct effective Euler classes.*

We know Γ is usually represented as amalgams. So to produce the effective Euler classed of Γ we can glue together characters on local subgroups to get a character of amalgams. Then use the associated localized spheres to produce effective Euler classes. Next lemma tells us that we need to assemble characters for generalized amalgams which arise form characters with p-effective Euler classes.

Lemma 4.2.1. *The Euler class associated to a Γ-sphere W will be effective if and only if it restricts to an effective Euler class on each of the generating subgroups.*

Proof. Then "only if" part is trivial, I only need to prove the "if" part.

Suppose $\Gamma = G_1 *_A G_2$, then we apply the classifying space functor getting a new push-out diagram

$$\begin{tikzcd} BA \arrow[r] \arrow[d] & BG_1 \arrow[d] \\ BG_2 \arrow[r] & B\Gamma \end{tikzcd}$$

By the classification theorem of orientable fibrations we know a spherical fibration over a space X corresponds a map in $[X, Aut_I(S^{n-1})]$. So in this case by the universal property of push-out diagram we have a commutative diagram like follows:

$$\begin{tikzcd} BA \arrow[r] \arrow[d] & BG_1 \arrow[d] \arrow[ddr] & \\ BG_2 \arrow[r] \arrow[drr] & B\Gamma \arrow[dr] & \\ & & Aut_I(S^{n-1}) \end{tikzcd}$$

So for any Euler class in $H^n(\Gamma)$ there are corresponding Euler classes in $H^n(G_1)$ and $H^n(G_2)$. Now let $H \subset \Gamma$ be a maximal rank elementary abelian p-subgroup. According to the Corollary 4.2.1 which is conjugate to a subgroup in either G_1 or G_2. Without losing generality I assume there is element $\gamma \in \Gamma$ such that $\gamma H \gamma^{-1} \in G_1$ which is a maximal rank elementary abelian p-subgroup in G_1. Since the conjugation induce an identity map on group cohomology there is a commutative diagram in the following:

$$\begin{array}{ccc} H^n(\Gamma) & \longrightarrow & H^n(H) \\ \downarrow & & \downarrow \\ H^n(\Gamma) & \longrightarrow & H^n(\gamma H \gamma^{-1}) \end{array}$$

So for the Euler class $\alpha \in H^n(\Gamma)$ I need to show the restriction of this Euler class on H is not zero. By the commutativity it is equivalent to show that the restriction of this Euler class on $\gamma H \gamma^{-1}$. However this map factors through $H^n(G_1)$. Since the Euler class of G_1 is effective, then the target is not zero.

Therefore the Euler class in $H^n(\Gamma)$ is effective. □

Now I apply these constructions to concrete rank two finite groups:

Example 4.2.2. *Let $G = M_{11}$, $p = 2$. In this case express (see [3]) M_{11} as a quotient of $\Gamma = GL_2(3) *_{D_8} S_4$ with $\Gamma \to G$ inducing a $\mathbb{Z}_2$ cohomology equivalence. According to the character table, we have a portion of character table with complete conjugate classes for $GL_2(3)$ as follows*

	1	$2A$	4	$2B$	3
χ_2	1	1	1	-1	1
χ_6	2	-2	0	0	-2
χ_8	4	-4	0	0	1

Then consider the character

$$\chi = 2\chi_2 + \chi_8$$

and the following is a character for S_4

	1	(12)	(12)(34)	(1234)	(123)
ν	3	-1	-1	1	0

It is clear that χ coincides with 2ν on elements of order 2 and 4, hence on the common subgroup D_8. Moreover, from character table we know for χ we have

$$\chi(1) = -3\chi(2) = 6$$

where $\chi(2)$ represents the character of any involution. It is well defined since in this case the character of any involution is equal. Similarly, it also works for character ν. By Lemma 3.2.1 we know there is a 6-dimensional complex representation for Γ which affords an effective 2-Euler class. So we obtain a 2-effective Euler class in $H^{12}(M_{11}, \mathbb{Z})$. Note a fact in group theory that $V = \mathbb{Z}/3 \times \mathbb{Z}/3 = Syl_3(M_{11})$, with normalizer of order 144. By Example 4.1.3 we can find a 3-local sphere of dimension 31 with 3-effective Euler class in $H^{32}(M_{11}, \mathbb{Z})$.

Example 4.2.3. *Let $G = PSL_2(\mathbb{F}_q)$, q odd and $q \equiv \pm 3 \mod 8$. In this case the 2-Sylow subgroup is elementary abelian. Then by Theorem 4.3 we have a 2-effective Euler class.*

Example 4.2.4. *Let $G = PSL_3(\mathbb{F}_q)$ where q is an odd number and $q \equiv -1 \mod 3$ Here the 2-local structure is as follows:*

$$GL_2(q) *_{\mathbb{Z}/q-1 \int \mathbb{Z}/2} (\mathbb{Z}/q-1)^2 \rtimes S_3$$

By Fermat's little theorem we always have

$$2^{q-1} \equiv 1 \mod q$$

So we can select two generators

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

in $GL_2(q)$, so we can view $\mathbb{Z}/q-1 \times \mathbb{Z}/q-1$ as a subgroup of $GL_2(q)$ and it is clear that we can view the group generated by

$$\begin{pmatrix} q-1 & 0 \\ 0 & q-1 \end{pmatrix}$$

as $\mathbb{Z}/2$ which is a subgroup in $\mathbb{Z}/q-1 \times \mathbb{Z}/q-1$

Now let $\chi : \mathbb{Z}/q-1 \hookrightarrow \mathbb{C}^{\times}$ be the 1-dimensional complex representation which sends the element

$$\begin{pmatrix} q-1 & 0 \\ 0 & 1 \end{pmatrix}$$

to -1. Then we composite

$$(\mathbb{Z}/q-1)^2 \to \mathbb{Z}/q-1 \to \mathbb{C}^{\times}$$

we get a 1-dimensional complex representation such that $\chi|_{\mathbb{Z}/2}$ is nontrivial.

Let

$$\nu = Ind_{\mathbb{Z}/q-1\times\mathbb{Z}/q-1}^{GL_2(q)}(\chi)$$

It is a $q(q+1)$-dimensional complex representation. And by the Cartan-Eilenberg double cosets formula we know the even order elements in $SL_2(q)$ act freely. Then identify

$$GL_2(q)/SL_2(q) \simeq \mathbb{Z}/q-1$$

It is clear the quotient classes are classified by their determinants. And we can select a 1-dimensional representation $\omega : \mathbb{Z}/q-1 \hookrightarrow S^1 \subset \mathbb{C}^{\times}$ such that every element of $GL_2(q)$ with even order acts freely on $S(\nu) \times S(\omega)$.

Now take

$$\mathcal{K} = 2\nu + q(q+1)\omega$$

Let $2A$ denote the central involution in $GL_2(q)$ and $2B$ denote the non-central one. As for $2A$, it is just the matrix

$$\begin{pmatrix} q-1 & 0 \\ 0 & q-1 \end{pmatrix}$$

So we get

$$\mathcal{K}(2A) = 2\nu(2A) + q(q+1)\omega(2A) = -2q(q+1) + q(q+1) = -q(q+1)$$

As for $2B$ we can select one representative

$$\begin{pmatrix} q-1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since its determinant is $q-1$ it corresponds to -1 in embedding $\mathbb{Z}/q-1 \hookrightarrow \mathbb{C}^\times$. So

$$\mathcal{K}(2B) = 2\nu(2B) + q(q+1)\omega(2B) = 0 - q(q+1) = -q(q+1)$$

By Lemma 3.2.1 $S(\mathcal{K})$ has an 2-effective Euler class.

Next let's consider

$$(\mathbb{Z}/q-1)^2 \rtimes S_3 \subset SL_3(\mathbb{F}_p)$$

which arises from the wreath product $\mathbb{Z}/q-1 \int S_3$. More precisely, it is the semi-direct product $(\mathbb{Z}/q-1)^3 \rtimes S_3$ where the action is the perumation of diagonal matrices

$$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

And the action in semi-direct product $(\mathbb{Z}/q-1)^2 \rtimes S_3$ is obtained by the permutation of S_3 on the diagonal matrices of determinant one.

Now let $\mathbb{Z}/q-1 \hookrightarrow U(1) \simeq S^1$ be the standard 1-dimensional complex representation by sending a generator to a primitive $(q-1)-$root of the unity. Then we have a composition of the following maps

$$(\mathbb{Z}/q-1)^2 \rtimes S_3 \hookrightarrow U(1) \int S_3 \hookrightarrow U(3)$$

which gives a 3-dimensional complex representation η. Moreover, it can be easily seen that the restriction $\eta|_{(\mathbb{Z})^2}$ is a copy of reduced regular representation for any elementary abelian 2-subgroups. So $S(\eta)$ has a 2-effective Euler class.

And we can verify directly that the character $q(q+1)\eta$ agrees with the character $\mathcal{K}$ on the common subgroup $\mathbb{Z}/q-1 \int \mathbb{Z}/2$. Hence there is a $3q(q+1)$-dimensional complex character for the amalgamation which provide a 2-effective Euler class.

So G has a 2 effective Euler class.

The other 4 examples such as $PSL_3(\mathbb{F}_q)$, q odd, $q \equiv 1 \mod 3$, $PSL_2(\mathbb{F}_q)$, q odd, $q \equiv \pm 1 \mod 8$; $PSU_3(\mathbb{F}_q)$,q odd and $PSU_3(\mathbb{F}_4)$ also have a 2-effective Euler class. But the constructions need more deep group theory. People interested could see [4]

Example 4.2.5. *Consider the inclusion $A_6 \hookrightarrow A_7$. Similar argument like Example 4.1.2 could show that it induces a mod 2 cohomology equivalence. And since $A_6 \cong PSL_2(\mathbb{F}_9)$ has a 2-effective Euler class. So does A_7.*

So we have completed arguments for all simple rank two groups, then we have the following result

Theorem 4.7. *Let G denote a finite rank two simple group. Then G acts some $Y \simeq S_2^{2k-1}$ such that its associated Euler class is 2-effective.*

After constructing the 2-effective Euler class for rank two groups, I turn direction to the existence of q-effective Euler class for q an odd primes. In most cases, odd primes are easier to deal with. So let's consider the list of rank two simple groups. As for groups A_7 and $PSU(3,4)$, by order formula we have

$$|A_7| = \frac{7!}{2} = 7 \times 5 \times 3 \times 2^3$$

$$|PSU(3,4)| = 2^4 \times 5$$

so for any odd primes p the p-Sylow subgroups are abelian. Which means there is a p-effective Euler class for each prime p such that $r(G) = r_p(G)$.

Note that if q is an odd primes, then the q-Sylow subgroups of $PSL_2(\mathbb{F}_p)$ and $PFL_2(\mathbb{F}_{p^2})$ are all cyclic subgroups, so they have periodic cohomology

Finally to summarize the results in this section we have

Theorem 4.8. *Given any rank two simple group G and any prime q which is dividing $|G|$(except possibly $PSL_3(\mathbb{F}_p)$, p an odd prime) then it has a q-effective Euler class in its cohomology.*

We will use this important result to prove the most important theorem in my thesis.

4.3 Homotopy Actions

In this section I show how to use local Euler class to glue a global effective Euler class. And I will combine this result and examples in section 3.2 to give a complete description about the action of simple rank 2 groups on a product of two spheres.

Let J and K be two sets of primes and X be a topological space. Then (see[7]) there is pull-back square of localization maps in the homotopy category

$$\begin{tikzcd} X_{J\cup K} \arrow[r] \arrow[d] & X_K \arrow[d] \\ X_J \arrow[r] & X_{J\cap K} \end{tikzcd}$$

It is easily seen that there is a pull-back diagram in the homotopy category

$$\begin{tikzcd} Map(X, X_{J\cup K}) \arrow[r] \arrow[d] & Map(X, X_K) \arrow[d] \\ Map(X, X_J) \arrow[r] & Map(X, X_{J\cap K}) \end{tikzcd}$$

By the universal property of localization we know these is a natural bijection

$$Map(X_J, X_J) \to Map(X, X_J)$$

In fact it is also a weak equivalence[7] Then by naturality we have a pull-back diagram in the homotopy category in the following

$$\begin{tikzcd} Map(X_{J\cup K}, X_{J\cup K}) \arrow[r] \arrow[d] & Map(X_K, X_K) \arrow[d] \\ Map(X_J, X_J) \arrow[r] & Map(X_{J\cap K}, X_{J\cap K}) \end{tikzcd}$$

Since the localization $f_{J\cup K}$ of a map f is a weak equivalence if and only if the maps f_K an d f_J both are weak equivalence, we can pass ordinary continuous maps to its homotopy equivalent classes. Hence we get a new pull-back diagram in the homotopy category:

$$\begin{tikzcd} AutX_{J\cup K} \arrow[r] \arrow[d] & AutX_K \arrow[d] \\ AutX_J \arrow[r] & AutX_{J\cap K} \end{tikzcd}$$

Moreover, if X is a finite nilpotent complex, then the localization $f_{J\cup K}$ of the map $f : X \to X$ is homotopic to the identity between $X_{J\cup K}$ if and only if maps f_J and f_K are homotopic to the identity map on X_J and X_K respectively. Therefore we have a new pull back diagrams in the homotopy category

$$\begin{tikzcd} Aut_I X_{J\cup K} \arrow[r] \arrow[d] & Aut_I X_K \arrow[d] \\ Aut_I X_J \arrow[r] & Aut_I X_{J\cap K} \end{tikzcd}$$

Since the geometric realization functor preserves pull back diagram, we get a new pull back diagram in the homotopy category in the following:

$$\begin{tikzcd} BAut_I X_{J\cup K} \arrow[r] \arrow[d] & BAut_I X_K \arrow[d] \\ BAut_I X_J \arrow[r] & BAut_I X_{J\cap K} \end{tikzcd}$$

After giving above prerequisite, I can use it to solve the problem of gluing together effective p-local Euler classes to a effective Euler class for a finite group G

Theorem 4.9. *Let G be a finite group, an integral cohomology class $\alpha \in H^{2n}(BG, \mathbb{Z})$ with $n \geq 2$ is an effective Euler class if and only if for each prime p such that $r(G) = r_p(G)$ its p-localization $\alpha_p \in H^{2n}(BG, \mathbb{Z}_{(p)})$, is an effective p-local Euler class.*

Proof. It is clear that the effectiveness of Euler classes and their p-localization for each p such that $r_p(G) = r(G)$. First if α is an effective Euler class, so we have a fibration

$$S^{2n-1} \to E \to BG$$

which is classified by a map in $[BG, Aut_I(S^{2n-1})]$. Moreover we have a natural map in the following which is induced by the natural localization map

$$[BG, Aut_I(S^{2n-1})] \to [BG, Aut_I(S^{2n-1}_{(p)})]$$

so $\alpha_{(p)}$ is a p-local Euler class.

For the converse, we assume the cohomology class $\alpha \in H^{2n}(BG, \mathbb{Z})$ has its p-localization $\alpha_{(p)}$ as p-local Euler class for each prime p such that $r_p(G) = r(G)$. In other words for each p with this property we have an orientable fibration

$$S^{2n-1}_{(p)} \to E_p \to BG$$

such that the associated transgression of the generator is the localization of α,$\alpha_{(p)}$. By the long exact sequence of homotopy groups of the above fibration we get $\pi_1(E_p) = G$ and the universal cover of E_p is a G-CW complex which is homotopy equivalent to $S^{2n-1}_{(p)}$.

Now let $p_1, \cdots, p_k$ be the primes for which $r(G) = r_p(G)$ and K be the set of all other primes. So we have the following diagram

$$S^{2n-1}_{(p_i)} \longrightarrow S^{2n-1}_{\mathbb{Q}} \longleftarrow S^{2n-1}_K$$

The limit of this diagram is clear S^{2m-1} by previous argument for homotopy pull-back of localizations.

Now consider the rationalization of S^{2n-1}. Note that [15] for odd dimension spheres we have

$$S^{2n-1}_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2n-1)$$

for $i = 1, \cdots, k$ And it is clear by cohomology argument that any map between $K(\mathbb{Q}, 2n-1)$ is homotopy equivalent to identity map. So we have

$$Aut_I S^{2n-1}_{\mathbb{Q}} \simeq *$$

Hence there is a unique homologically trivial action of G on rationalization $S^{2n-1}_{\mathbb{Q}}$.

Give the localization S_K^{2n-1} the trivial action of G we can get a commutative diagram as follows

$$\begin{tikzcd}
BG \arrow[dr, dotted] \arrow[drr] \arrow[ddr] & & \\
& Aut_I(S^{2n-1}) \arrow[r] \arrow[d] & Aut_I(S^{2n-1}_{(p_i)}) \arrow[d] \\
& Aut_I(S^{2n-1}_K) \arrow[r] & Aut_I(S^{2n-1}_{\mathbb{Q}})
\end{tikzcd}$$

By the universal property of pull back diagram we get a map in $[BG, Aut_I(S^{2n-1}]$. And by the classification of fibrations we know this map corresponds to the cohomology class α which can restrict to α_p for each prime p such that $r(G) = r_p(G)$. So the proof completes. $\square$

Now based on this gluing lemma we can prove the following two theorems finally,

Theorem 4.10. *Let G denote a rank two finite group such that all of its p-Sylow subgroups are either normal in G, abelian. Then G acts freely on a finite complex $Y \simeq S^n \times S^m$.*

Proof. By Theorem 4.3 if G has a normal p-Sylow subgroup or G has an abelian p-Sylow subgroup then it has a p-effective Euler class. Then by Theorem 4.8 there is a integral cohomology $\alpha \in H^{2n}(BG, \mathbb{Z})$ is an effective Euler class which corresponding to a group action of G on S^{2n-1} with the fibration in the following

$$S^{2n-1} \to S^{2n-1} \times_G EG \to BG$$

Since G is a rank two finite group then every isotropy subgroup except elementary abelian 2-groups has periodic cohomology. As for elementary abelian 2-groups, since α is an effective Euler class that means the restriction of action on this elementary abelian subgroups has not stationary points. So it is not the isotropy subgroup. I other words every isotropy subgroup in this group action has periodic cohomology. According to Theorem 2.5 we know G can act freely on a finite complex $Y \simeq S^{2n-1} \times S^m$. $\square$

Theorem 4.11. *Let G denote a rank two simple group except $PSL_3(\mathbb{F}_p)$, with p an odd prime. Then G acts freely on a finite CW-complex $Y \simeq S^n \times S^m$.*

Proof. According to Theorem 4.8 which states that for any rank two simple group G and any prime p dividing $|G|$(except $PSL_3(\mathbb{F}_p)$ at p), it has a p-effective Euler class in its cohomology. So for each prime p such that $r(G) = r_p(G)$ it has a p-effective Euler class. By Theorem 4.9 there is an effective Euler class α. And by similar argument in the proof above G acts freely on a finite complex $Y \simeq S^n \times S^m$. $\square$

Appendix A

Review of Homological Algebra

In this appendix I will review some knowledge about homological algebra which is necessary for my thesis.

Chain Complexes

I always assume that R denote an arbitrary ring with a unit.

Definition A.0.1. *A graded R-module is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of R-modules. And id $x \in C_n$ we say the degree of x is n.*

Moreover, a map of degree p between two R-modules C, C' is a series homomorphisms like follows:

$$f_n : C_n \to C'_{n+p}$$

Definition A.0.2. *We say a **chain complex** is a pair (C, d) where C is a graded R-module and d is a degree -1 map between C itself such that $d^2 = 0$. In this case we call d an **differential** or **boundary map**.*

It is clear that

$$Im(d) \subset Ker(d)$$

So we define the homology group of this chain complex as

$$H(C) = Ker(d)/Im(d)$$

And we can see that $H(C)$ is also a graded R-module.

Dually we can define a **cochain complex** as a pair (C, d) where the degree of d is 1. And we can also define its cohomology groups.

Definition A.0.3. *To make Chain complexes as a category we need morphisms between chain complexes. A morphism between chain complexes C, C' which is always called a **chain map** is a degree o map between two chain complexes and make the following diagram commute.*

$$\begin{tikzcd} C_n \arrow[r] \arrow[d] & C'_n \arrow[d] \\ C_{n-1} \arrow[r] & C'_{n-1} \end{tikzcd}$$

Again, dually we can define a **cochain map** between two cochain complexes.

We have the definition of homotopy of maps between topological spaces. We can also define **chain homotopy** between two chain complexes.

Definition A.0.4. *A chain homotopy between two chain maps $f, g : C \to C'$ is a map $s : C \to C'$ of degree 1 such that*

$$f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$$

for any n.

Next proposition is a fundamental property of chain complexes.

Proposition A.0.1. *A chain map $f : C \to C'$ will induce a map of two homology groups.*

$$H(f) : H(C) \to H(C')$$

And it is homotopy invariant. That is, if f is homotopy to g then their induced maps coincide $H(f) = H(g)$.

Homological algebra is a very powerful tool to study modules. And its basic idea is to study the exactness of sequences of modules.

Definition A.0.5. *An sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** at B if $Ker(g) = Im(f)$. We can a following sequence*

$$0 \to C' \to C \to C'' \to 0$$

is a short exact sequence if it is exact every where. Similarly we say a sequence like follows

$$\cdots \to A_n \to B_n \to C_n \to \cdots$$

a long exact sequence if it is exact everywhere.

Theorem A.1. ***(Fundamental Theorem of Homological Algebra)***
A short exact sequence

$$0 \to C' \xrightarrow{i} C \to C'' \xrightarrow{\pi} 0$$

will induce a long exact sequence like follows:

$$\cdots \to H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(\pi)} H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \to \cdots$$

*Where ∂ is called a **connecting homomorphism**.*

In topology we have a natural construction of spaces called the product. We have a similar construction for chain complexes.

Definition A.0.6. ***(Tensor Product of Chain Complexes)***
Let C, C' be two chain complexes, we define its tensor product which is a new chain complex as follows:

$$(C \otimes_R C')_n = \bigoplus_{p+q=n} C_p \otimes_R C'_q$$

Where the differential is

$$\partial(c \otimes c') = dc \otimes c' + (-1)^p c \otimes dc'$$

where $c \in C_p, c' \in C'_q$

Projective and Injective Modules

In this section I will review definitions of projective and injective modules which are basic buildings for homological algebra and many other mathematical areas.

Definition A.0.7. *A R-module P is called **projective** if given two R-modules M, M' and a surjective homomorphism $\mu : M \to M'$ and a homomorphism $\lambda : P \to M$ there is a map ν which makes the following diagram commute.*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\nu"'] \arrow[d, "\lambda"] & \\ M' \arrow[r, "\mu"'] & M \arrow[r] & 0 \end{tikzcd}$$

Dually, we have

Definition A.0.8. *A R-module I is called **injective** if given two R-modules M, M' and a monomorphism $\mu : M \to M'$ and a map $\lambda : M \to I$ there is a map $\nu : M' \to I$ makes the following diagram commute*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mu"] \arrow[d, "\lambda"'] & M' \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

We have some equivalent description about projective modules

Proposition A.0.2. *The following statements are equivalent.*

1. *A R-module P is projective.*
2. *Any epimorphism $\lambda : M \to P$ splits.*
3. *P is a direct summand of a free R module.*

Definition A.0.9. *A **projective cover** P_M of a finitely generated R-module M is a surjective map $P_M \to M$ such that P_M is minimal with respect to direct decomposition. And in this case we define $\Omega(M) = Ker(P_M \to M)$.*

Lemma A.0.1. ***(Schaunel)*** *Suppose we have two short exact sequences like follows:*

$$0 \to M_1 \to P_1 \to M \to 0$$

$$0 \to M_2 \to P_2 \to M \to 0$$

with P_1, P_2 two projective modules. Then we have $M_1 \oplus P_2 \simeq P_1 \oplus M_2$

Proof. (Sketch) Consider a pull-back diagram like follows:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P_2 \arrow[d] \\ P_1 \arrow[r] & M \end{tikzcd}$$

So we get a two new short exact sequences

$$0 \to M_2 \to X \to P_1 \to 0$$

$$0 \to M_1 \to X \to P_2 \to 0$$

By splitness we have

$$P_1 \oplus M_2 \simeq X \simeq P_2 \oplus M_1$$

□

Remark A.0.1. *So if we let $\tilde{\Omega}(M)$ be the kernel of some epimorphism $P \to M$ with P a projective module. Then $\tilde{\Omega}(M)$ is well-defined up to adding or removing projective summand.*

Corollary A.0.1. *If we have a homomorphism $\alpha : M \to M'$, then we have a map $\tilde{\alpha} : \tilde{\Omega}(M) \to \tilde{\Omega}(M')$ which is unique up to adding maps factoring through a projective module.*

Remark A.0.2. *After giving the definition of stable module categories we can see that Ω is a well-defined functor and it is very useful.*

Definition A.0.10. *The stable module category $Stmod(R)$ of finitely generated R-modules has objects same with the category of finite generated R-modules. Morphisms $\underline{Hom}_R(M, N)$ in $Stmod(R)$ are $Hom_R(M, N)/\sim$ where if two maps f, g from M to N such that $f - g$ factor through a projective module then $f \sim g$. It is clear in $Stmod(R)$ any projective R-module is isomorphic to 0.*

Remark A.0.3. *Although $\tilde{\Omega}$ is not a well-defined functor between the category of finitely generated R-modules $mod(R)$. But is is a well defined functor*

$$\tilde{\Omega} : Stmod(R) \to Stmod(R)$$

Dually, for injective modules we have similar definition called **injective hull** and for a injective hull $M \hookrightarrow I$ we define $\Omega^{-1}(M) = coker(M \hookrightarrow I)$.

Remark A.0.4. *In fact injective modules behave better than projective modules since for any ring R and any R-module M there is a unique injective module I minimal.*

Derived Functors

Definition A.0.11. *A **projective resolution** of a R-module M is a long exact sequence*

$$\cdots \to P_2 \to P_1 \to P_0$$

such that the following sequence is also exact

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

Projective resolutions are extremely important in homological algebra, since it is a basic construction for derived functors which describe failure of exactness of functors. In homological algebra, we usually investigate two natural functors $Hom(-, -)$ and $- \otimes -$. The next theorem gives uniqueness of projective resolution up to homotopy.

Theorem A.2. *The projective resolutions of a R-module M is unique up to chain homotopy.*

Based on this theorem we can define the derived functors of two functors we mentioned before.

Definition A.0.12. *Given two R-modules M, M' and a projective resolution of M*

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0$$

If we tensor it with M' we can get a chain complex as follows:

$$\cdots \to M' \otimes_R P_2 \xrightarrow{id \otimes \partial_2} M' \otimes_R P_1 \xrightarrow{id \otimes \partial_1} M' \otimes_R P_0 \to 0$$

*Then we define the **Tor** functor*

$$Tor_n^R(M', M) = H_n(M \otimes P_*, id \otimes \partial)$$

Next theorem will tell us why Tor could be called the derived functor of tensor products.

Theorem A.3. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules and given another R-modules M_0. We have a long exact sequence as follows*

$$\cdots \to Tor_1^R(M_0, M') \to M_0 \otimes_R M' \to M_0 \otimes_R M \to M_o \otimes_R M" \to 0$$

Dually, we can define the derived functor of $Hom(-,-)$.

Definition A.0.13. *With the same conditions as previous definition. Now we get a long exact sequence by taking Hom.*

$$0 \to Hom_R(P_0, M') \xrightarrow{\delta^*} Hom_R(P_1, M') \to \cdots$$

Then we define the Ext functor

$$Ext_R^n(M, M') = H^n(Hom_R(P_*, M'); \delta^*)$$

Remark A.0.5. *Similarly the Ext functor can be used to fill the lack of exactness of Hom functor like the Theorem A.3.*

It turns out we can use stable module categories to describe derived functors which is very useful in some argument.

Theorem A.4. *Let k be a field and G be a finite group. So we have a group ring $R = k[G]$. Given two left R-module we have natural isomorphisms as follows:*

$$Ext_R^n(M, N) = \underline{Hom}_R(\Omega^n(M), N) = \underline{Hom}_R(M, \Omega^{-n}(N))$$

for every positive integer n.

Now we have the derived functors for tensor products and Hom we can give some more complicated calculation about homology and cohomology. Given a chain complex C of left R-modules and a right R modules M we can use tensor product to form a new chain complex $C \otimes_R M$, the universal coefficients theorem will tell us how to calculate the homology groups of this new chain complex(denoted by $H_n(C, M)$)under some nice conditions.

Theorem A.5. ***(Universal Coefficients Theorem)*** *Suppose C is a chain complex of free left R-modules and a right R-module M. Then we have a natural short exact sequence which is split as follows:*

$$0 \to H_n(C) \otimes_R M \to H_n(C, M) \to Tor_1^R(H_{n-1}(C), M) \to 0$$

Moreover, we have the **Künneth formula** to calculate the homology groups of a new chain complex induced by taking the tensor product of two chain complexes. For simplicity I use the field k to replace general ring as coefficients.

Theorem A.6. ***(Künneth Formula)*** *Let k be a filed, and A and B are chain complexes as k-vector space. Then there is a short exact sequence to express the homology of tensor product of these two chain complexes using the homology of these chain complexes respectively.*

$$0 \to H_* \otimes_k H_*(B) \xrightarrow{\alpha} H_*(A \otimes_k B) \xrightarrow{\beta} Tor_1^k(H_*(A), H_*(B)) \to 0$$

where the degree of α is 0 and the degree of β is -1.

Appendix B

Finite Groups and its representations

The first part of this appendix I will introduce some basic notions about representations of finite groups. The second part I will list some properties of finite p-groups.

Representations of finite groups

In this section we give a concise description on representations of finite group. People interests details could refer Serre's excellent book[26]

Definition B.0.1. *Let G be a finite group. A* ***representation*** (π, V) *of G on a finite-dimensional vector space V over $\mathbb{C}$ is a group homomorphism.*

$$\pi : G \to GL(V)$$

Definition B.0.2. *A morphism between two representations (π, V) and (π', V') of G is an linear map T between V and V' such that*

$$\pi'(g)T(v) = T(\pi(g)v)$$

We also call this map as a G-module homomorphism. We say these two representations are ***equivalent*** *if there is an isomorphism between them.*

Remark B.0.1. *Using language of category theory we can say the representations of a finite group G form a category $Rep(G)$*

Definition B.0.3. *A subspace W of V is called G-invariant(or sub-representation) if $\pi(g)W \subset W$ for any $g \in G$. A representation (π, V) is called* ***irreducible*** *if there is no non-trivial G-invariant subspace.*

The following statement is a very useful.

Lemma B.0.1. *(Schur) Let (π, V) and (π', V') be two irreducible representations of G. Then any morphism between these two representations is either zero or an isomorphism.*

Moreover, if $V = V'$ then $T = \lambda id$ for some $\lambda \in \mathbb{C}$.

Definition B.0.4. *Let (π, V) and (π', V') be two representations of G. We can make the direct sum $V \oplus V'$ as a new representation of G defined by:*

$$(\pi \oplus \pi')(g)(v + v') = \pi(g)v + \pi'(g')v'$$

And we say a representation (π, V) is ***indecomposable*** *if it could not be expressed by the direct sum of two non-trivial representations*

Remark B.0.2. *A representation (π, V) is called a trivial representation if the homomorphism π is trivial. And it is clear that if a representation could not contain a trivial sub-representation then the action of G on the vector space G is fixed-point-free.*

Proposition B.0.1. *The definition of irreducible of a presentation is equivalent to the definition of indecomposable of a representation.*

Definition B.0.5. *The **character** χ of a representation (π, V) of G is a map $\chi : G \to \mathbb{C}$ defined by:*

$$\chi(g) = trace(\pi(g))$$

Lemma B.0.2. *The character χ is a **class function**(which is constant on each conjugacy class).*

Definition B.0.6. *We definite a Hermitian inner product on the set of all the class functions by*

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g')}$$

Now I list some properties about characters without proofs.

Theorem B.1. *Let $\pi_1, \pi_2, \cdots$ represent the distinct isomorphism classes of irreducible representations of G with vector spaces $V_1, V_2, \cdots$ and characters $\chi_1, \chi_2, \cdots$ respectively. Then we have*

1. *The sets of characters $\{\chi_1, \chi_2, \cdots\}$ is orthonormal basis for the vector space of class functions of G. Therefore, the number of isomorphism classes of irreducible representations of G is the same as the number of conjugacy classes of G*

2. *Put d_i for the dimension of V_i and r for the number of isomorphism classes of irreducible representations, then*

$$d_1^2 + \cdots + d_r^2 = |G|$$

Remark B.0.3. *The proof of this theorem is based on the following fact that the **regular** representation of G contains any irreducible representations. The regular representation is a representation π, V where $\dim V = |G|$ and we use elements of G as sub-index of a basis of V and then we define*

$$\pi(g)(e_{g'}) = e_{gg'}$$

Corollary B.0.1. *Any representation (π, V) of G is completely determined by its character χ. More precisely, if V is decomposed into a direct sum of irreducible representations*

$$V = n_1 V_1 \oplus \cdots \oplus n_r V_r$$

Then the multiplicity n_i of V_i in V is $\langle \chi, \chi_i \rangle$.

Corollary B.0.2. *Any representation (π, V) of G is irreducible if and only if its character χ is a unit vector which means $\langle \chi, \chi \rangle = 1$.*

Next is an easy example of calculations of characters of finite cyclic groups of order 3.

Example B.0.1. *Let $G =< a|a^3 = e >$. Let χ be a character of a representation of G. By Schur's lemma we know any irreducible representation of abelian groups has dimension 1. So any characters have value 1 at the identity element of groups. Since it is a cyclic group, we only need to know the value of this character at the generator a which is $\chi(a)$. Since $a^3 = 4$ we have $\chi(a)^3 = 1$. Solve this equation we have 3 solutions: $1, \omega, \omega^2$ where $\omega = \frac{-1+\sqrt{3}i}{2}$. So we have 3 irreducible characters with $\chi_1(a) = 1, \chi_2(a) = \omega$ and $\chi_3(a) = \omega^2$.*

Finally, I would like to introduce a very important concept in representation theory called **induced** representations. Let H be a subgroup of a group G. Then any representation of G can be restricted to a representation of H, denoted by $Res^G_H V$. Using language of category theory it means that we have a functor :

$$Res^G_H : Rep(G) \to Rep(H)$$

Now I want to find a functor goes another direction and there is an adjunction between these two functors.

Suppose V is a representation of G and a subspace W is H-invariant. Then for any $g \in G$, the subspace $g \cdot W = \{g \cdot w | w \in W\}$ depends only on the left coset gH. And it is clear that

$$V \simeq \bigoplus_{\sigma \in G/H} \sigma W$$

In this case we write $V = Ind^G_H W$.

Conversely, let (τ, W) be a representation of H. An induced representation π on the vector space $V = Ind^G_H W$ can be constructed as follows:

Let $V = \bigoplus_{i=1}^n g_i W$ where $\{g_i\}$ is a representative of left coset G/H. And each $g_i V$ consists of formal elements $\{g_i v\}$ where $v \in V$. For each $g \in G$ and each g_i there is an $h_i \in H$ and $j(i) in \{0, 1, \cdots, n\}$ such that:

$$g \cdot g_i = g_{j(i)} \cdot h_i$$

Then we can define the group action of G on V as follows:

$$g \cdot \sum_{i=1}^n g_i w_i = \sum_{i=1}^n g_{j(i)} \tau_{h_i} w_i$$

Now we have a functor goes another way:

$$Ind^G_H : Rep(H) \to Rep(G)$$

In fact next theorem says that these two functors form an adjunctions.

Theorem B.2. *Let W be a representation of H and V be a representation of G. Then any H- module homomorphism $\phi : W \to Res^G_H V$ can be extended uniquely to a G-module homomorphism $\tilde{\phi} : Ing^G_H W \to V$. That is we have an isomorphism as follows:*

$$Hom_H(W, Res^G_H V) \simeq Hom_G(Ind^G_H, V)$$

Wher $Hom_H(-, -)$ means H-module homomorphisms.

After introducing two natural functors it is natural to consider what is the interaction between these functor and characters of representations.

Theorem B.3. *(Frobenius Reciprocity) We have*

$$\langle \chi_W, \chi_{Res^G_H V} \rangle = \langle \chi_{Ind^G_H W}, \chi_V \rangle$$

Finite p-groups

In this section I list several basic properties about finite p-groups which can be found in any standard text about group theory.

Definition B.0.7. *Let p be a prime number, a p-group is a group in which every element has order a power of p.*

Next theorem duo to Cauchy gives a powerful tool to judge when a finite group contains order p elements.

Theorem B.4. *(Cauchy) If G is a finite group whose order is divisible by a prime p, then G contains an element of order p.*

As a corollary of this theorem, there is an alternative description about what is a p-group.

Proposition B.0.2. *A finite group G is a p-group if and only if $|G|$ is a power of p.*

In representation theory we often first construct a representation for the centre $Z(G)$ of a finite group G, then use induced functor to get a representation of G. So it is important to see wether the centre is trivial or not. Next theorem tells us that a non-trivial finite p-group G has a nontrivial centre.

Theorem B.5. *If $G \neq 1$ is a finite p group, then its center $Z(G) \neq 1$.*

In the progress of investigating structures of finite groups, researchers found a very important concept called **Sylow subgroups**

Definition B.0.8. *If p is a prime number, then a Sylow p-subgroup P of a group G is a maximal p-subgroup(with partial order by inclusion).*

Theorem B.6. *(Sylow)*

1. *All Sylow p-subgroups are conjugate to each other.*
2. *If there are r Sylow p-subgroup, then r is a divisor of $|G|$ and $r \equiv 1 \mod p$.*

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