Exotic fusion systems and Euler characteristics

Eske Sparsø

Advisor: Jesper M. Møller

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Department of Mathematical Sciences, University of Copenhagen

Abstract

We construct and examine some specific exotic fusion systems: The Ruiz-Viruel examples defined over over the extraspecial group of order 7^3 and exponent 7, and the Solomon fusion systems defined over certain 2-groups. The Euler characteristics (as defined by Leinster) of these fusion systems are computed.
A theorem of Jacobsen and Møller states that a group fusion system and its exterior quotient have a common coweighting, and in particular that they have the same Euler characteristic. We extend this theorem to all fusion systems.

Resume

Vi konstruerer og studerer en række specifikke eksotiske fusionssystemer: Ruiz-Viruel eksemplerne defineret på den ekstraspecielle gruppe af orden 7^3 og eksponent 7, samt Solomon-fusionssystemerne defineret på visse 2-grupper. Vi udregner Eulerkarakteristikkerne af disse fusionssystemer, jvf. Leinsters definition.
En sætning af Jacobsen og Møller, siger, at et fusionssystem og dets ydre kvotient har samme kovægtning, og specielt samme Eulerkarakteristik. Vi udvider denne sætning til at omhandle alle fusionssystemer.
Specialet er skrevet på engelsk.

Contents

Introduction

The Euler characteristic is one of the oldest topological invariants. Despite its simplicity, being just a number, it has remarkable properties and uses. For instance, it behaves in the same way with respect to disjoint unions and products as cardinality of sets does, and may be viewed as a generalization of cardinality. Leinster puts it very well in [Le]:
"We first learn of Euler characteristic as 'vertices minus edges plus faces', and later as an alternating sum of ranks of homology groups. But Euler characteristic is much more fundamental than these definitions make apparent, as has been made increasingly explicit over the last fifty years; it is something akin to cardinality or measure. More precisely, it is the fundamental dimensionless quantity associated with an object."
Euler characteristics can be defined for certain finite categories by introducing so-called weightings and coweightings of the category. Important examples of categories with Euler characteristic are fusion systems and the partially ordered set of p-subgroups of some fixed group. The definition is consistent with the definition of the Euler characteristic of a topological space, in the sense that a category and its geometric realization have the same Euler characteristic if the latter has only finitely many non-degenerate simplices.
While Euler characteristics cannot establish homotopy equivalences, they can indicate the existence of such equivalences between spaces with the same Euler characteristic. An example is the partial order of all nontrivial p-subgroups of a given finite group G. The partial order may be thought of as the category $\mathcal{S}_G^*$ whose objects are the nontrivial p-subgroups, with a single morphism from P to Q if $P \leqslant Q$. It is known that if G contains a nontrivial, normal p-subgroup, then $\mathcal{S}_G^*$ is contractible (Quillen, 1978). This, of course, implies that the Euler characteristic of the category is $\chi(\mathcal{S}_G^*) = 1$, but that result is certainly much easier to obtain and motivates a search for such a homotopy equivalence. In the same manner, if we let $\mathcal{S}_G^{ea}$ denote the full subcategory of $\mathcal{S}_G^*$ generated by the non-trivial, elementary abelian p-subgroups of G, then $\chi(\mathcal{S}_G^*) = \chi(\mathcal{S}_G^{ea})$. This result can be proved by showing that the coweighting of $\mathcal{S}_G^*$ is concentrated on the elementary abelian subgroups. It also follows from the fact that the inclusion $\mathcal{S}_G^{ea} \to \mathcal{S}_G^*$ is a homotopy equivalence (Quillen).
But the nature of Euler characteristics is not only comparative. For instance, $|G|_p \mid (1-\chi(\mathcal{S}_G^*))$, where $|G|_p$ denotes the p-part of $|G|$, i.e. the highest power of p that divides $|G|$. If $\mathcal{F}_p(G)$ denotes the fusion system of the group G at the prime p, and $\mathcal{F}_p(G)^*$ its full subcategory obtained by removing the trivial group, one has that $|G|_{p'} \cdot \chi(\mathcal{F}_p(G)^*) \in \mathbb{Z}$. This puts restrictions on groups that may realize the fusion system $\mathcal{F}_p(G)$, which is interesting if G itself is unknown. It also leads to the question of existence of $|G|_{p'}$-fold covering maps $E \to B\mathcal{F}_p(G)^*$, such that E has Euler characteristic $|G|_{p'}$. Another property is that all known Euler characteristics of fusion systems are positive. Most of them are also less than 1; the smallest group G that gives rise to a fusion system of Euler characteristic > 1 has order 288.
The coweighting of $\mathcal{F}_p(G)^*$ turns out to be concentrated on the elementary abelian p-subgroups, just like it is for $\mathcal{S}_G^*$. But whether or not $\mathcal{F}_p(G)^*$ and its full subcategory of the elementary abelian p-subgroups are homotopy equivalent or related in some other way is unknown.
If one looks at weightings instead, the weighting of $\mathcal{S}_G^*$ is concentrated on the p-radical subgroups of G, and in fact, $\mathcal{S}_G^* \simeq \mathcal{S}_G^r$, where $\mathcal{S}_G^r$ is the full subcategory of p-radical subgroups. The analogue in the setting of fusion systems is that the weighting of $\mathcal{F}_G^c$ (the full subcategory of so-called $\mathcal{F}_G$-centric subgroups) is concentrated on those subgroups that in addition are $\mathcal{F}_G$-radical, i.e. $\chi(\mathcal{F}_p(G)^c) = \chi(\mathcal{F}_p(G)^{cr})$. But whether their geometric realizations are homotopy equivalent is unknown.

In [JM] the notion of the exterior quotient $\widetilde{\mathcal{F}}^*$ of a fusion system $\mathcal{F}$ is defined, and it is proved that $\chi(\mathcal{F}^*) = \chi(\widetilde{\mathcal{F}}^*)$ whenever $\mathcal{F}$ is a group fusion system. This is a corollary to a stronger result, namely that $\widetilde{\mathcal{F}}_p(G)^*$ and $\mathcal{F}_p(G)^*$ have the same coweighting for all finite groups G. In this thesis, we generalize these two results to include all fusion systems. This is done near the end of section 5. The thesis focuses on some examples of exotic fusion systems as Euler characteristics related to group fusion systems are treated in detail in [JM]. Section 4 describes three examples of exotic fusion systems that come from the classification of all p-local finite groups over the extraspecial group of order p^3 and exponent p ([RV]). The computations of the Euler characteristics of these fusion systems and their exterior quotients is what led to the generalizations of the mentioned theorems of [JM]. Section 1 provides the definition and most basic properties of fusion systems, as well as some technical lemmas. Finally, we compute the Euler characteristics of the Solomon fusion systems, a well-known class of examples of exotic fusion systems at the prime $p = 2$. The fusion systems are constructed and studied in the sections 7 through 10. It turns out that they all have the same Euler characteristic, which is rather interesting.

1 Fusion systems

Let G be a group, and let $g \in G$. g induces a homomorphism $c_g\colon G \to G$ by conjugation by g, that is $c_g(x) = {}^g x = gxg^{-1}$ for all $x \in G$. If H and K are subgroups of G such that ${}^g H := gHg^{-1} \leqslant K$, then conjugation by g induces a homomorphism $H \to K$ which we denote $c_g|_{H,K}$. (In general, if ϕ is some homomorphism, A a subgroup of its source, and $\phi(A) \leqslant B$, we write $\phi|_{A,B}$ to denote the homomorphism obtained by restricting the source of ϕ to A and its target to B. We may write $\phi|_A$ or just ϕ when the meaning is clear from the context). Define $\mathrm{Hom}_G(H,K) = \{c_g|_{H,K} \mid {}^g H \leqslant K\}$. Notice that any homomorphism given by conjugation by some element is injective.

Definition. *A fusion system $\mathcal{F}$ is a category whose objects are the set of all subgroups of some finite p-group S, and whose morphisms $P \to Q$ are a set of injective group homomorphisms $P \to Q$, such that $\mathrm{Hom}_S(P,Q) \subseteq \mathcal{F}(P,Q)$, for every pair of objects $P,Q \leqslant S$.*
Futhermore, each morphism of $\mathcal{F}$ must decompose as an $\mathcal{F}$-isomorphism (in the categorial sense) followed by an inclusion (as a homomorphism of groups).

We say that $\mathcal{F}$ is a fusion system *over* S. Since all $\mathcal{F}$-morphisms are group homomorphisms we write $\mathrm{Hom}_\mathcal{F}(P,Q) = \mathcal{F}(P,Q)$ and may refer to them as $\mathcal{F}$-homomorphisms. Whenever $P \leqslant Q$, conjugation by the identity element of S induces the inclusion homomorphism $\iota_{P,Q}\colon P \hookrightarrow Q$, i.e. $\iota_{P,Q} \in \mathrm{Hom}_S(P,Q)$ for all $P \leqslant Q \leqslant S$. If we combine this fact with the 'decomposition property' of $\mathcal{F}$-homomorphisms from the definition, we see that we may restrict any given $\mathcal{F}$-homomorphism to any subgroup of its source, and we may restrict its target to any subgroup that contains its image.
We say that P and Q are $\mathcal{F}$-conjugate when they are isomorphic in $\mathcal{F}$, and write $\mathrm{Iso}_\mathcal{F}(P,Q) = \mathrm{Hom}_\mathcal{F}(P,Q)$. The set of all $\mathcal{F}$-conjugates to P will be denoted $P^\mathcal{F}$. We also write $\mathrm{Aut}_\mathcal{F}(P) = \mathrm{Hom}_\mathcal{F}(P,P)$ (an injective endomorphism of a finite group is necessarily an automorphism). Similarly, $\mathrm{Aut}_S(P) = \mathrm{Hom}_S(P,P)$, and so on. With this notation, the inner automorphisms of a subgroup, $P \leqslant S$, is $\mathrm{Inn}(P) = \mathrm{Hom}_P(P,P)$. The group of outer $\mathcal{F}$-automorphisms of P is defined as $\mathrm{Out}_\mathcal{F}(P) = \mathrm{Aut}_\mathcal{F}(P)/\mathrm{Inn}(P)$. $\mathrm{Out}(P)$ and $\mathrm{Out}_S(P)$ are defined analogously.
Two elements $x, x' \in S$ are said to be $\mathcal{F}$-conjugate if there is an $\mathcal{F}$-homomorphism with $\phi(x) = x'$. In that case ϕ restricts to an $\mathcal{F}$-isomorphism $\langle x\rangle \to \langle x'\rangle$ by the above comments. By $x^\mathcal{F}$ we shall denote the set of all $\mathcal{F}$-conjugates to x.

A very important class of fusion systems are group fusion systems.

Definition. *Let G be a finite group, p a prime, and $S \in \mathrm{Syl}_p(G)$. The fusion system of G over S, denoted $\mathcal{F}_S(G)$, is the fusion system over S with morphisms $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P,Q \leqslant S$.*

If $\mathcal{F}$ is a fusion system over S, $\mathcal{F}'$ is a fusion system over S', and $\phi\colon S \to S'$ is an isomorphism which induces a bijection $\mathrm{Hom}_\mathcal{F}(P,Q) \to \mathrm{Hom}_{\mathcal{F}'}(\phi(P),\phi(Q))$ for each pair $P,Q \leqslant S$, we say that $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic fusion systems. If S and T are two Sylow-p-subgroups of a group G, then $\mathcal{F}_S(G)$ and $\mathcal{F}_T(G)$ are isomorphic. We may therefore just speak of 'the fusion of G at p' without specifying a Sylow-p-subgroup of G.

Definition. *Let S be a finite p-group, and let $\mathcal{F}$ be a fusion system over S. A subgroup $P \leqslant S$ is said to be*

- *fully $\mathcal{F}$-centralized if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^\mathcal{F}$.*
- *fully $\mathcal{F}$-normalized if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^\mathcal{F}$.*
- *fully $\mathcal{F}$-automized if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(P))$.*

- *receptive if for all $Q \in P^{\mathcal{F}}$, each isomorphism $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$ extends to an $\mathcal{F}$-homomorphism $N_\phi \to S$, where*

$$N_\phi \overset{def}{=} \{g \in N_S(Q) \mid \phi \circ c_g|_{Q,Q} \circ \phi^{-1} \in \mathrm{Aut}_S(P)\}$$

$\mathcal{F}$ is saturated if every $\mathcal{F}$-conjugacy class contains an element which is both fully automized and receptive.

The definition of receptivity might at first sight look somewhat technical, but it is not hard to see that N_ϕ is the largest subgroup of $N_S(Q)$ to which ϕ could possibly extend (within $\mathcal{F}$). ϕ induces a homomorphism $\mathrm{Aut}_S(Q)$ to $\mathrm{Aut}_{\mathcal{F}}(P)$ by conjugation; one can also think of N_ϕ as as all elements that induce automorphisms in the preimage of $\mathrm{Aut}_S(P)$. Note that N_ϕ always contains Q as well as $C_S(Q)$.
We may write 'fully centralized in $\mathcal{F}$' instead of 'fully $\mathcal{F}$-centralized' or even drop the $\mathcal{F}$ altogether. Likewise for the other properties.

Every group fusion system is saturated. The proof is basic and relies just on the explicit structure of the morphisms.
A saturated fusion system that is not a group fusion system is called *exotic*. Such fusion systems exist; we will study some examples later.
The definition of saturation varies in the literature. We will mention another one which will be important to us. As we have already stated one definition, this other definition will have the form of a proposition.

Proposition 1. *[AKO, 2.5]. Let S be a p-group, and $\mathcal{F}$ a fusion system over S. Then $\mathcal{F}$ is saturated if and only if*

i. every fully $\mathcal{F}$-normalized subgroup is fully $\mathcal{F}$-centralized and $\mathcal{F}$-automized.

ii. every fully $\mathcal{F}$-centralized subgroup is receptive in $\mathcal{F}$.

Definition. *Let S be a p-group, and $\mathcal{F}$ a fusion system over S. To each $P \leqslant S$ we define a fusion subsystem $C_{\mathcal{F}}(P)$, the centralizer fusion system of P, as the fusion system over $C_S(P)$ whose morphisms are*

$$\begin{aligned}\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q') = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,Q') \mid \exists \widetilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PQ') :& \\ \widetilde{\phi}(Q) = Q',\ \widetilde{\phi}|_{Q,Q'} = \phi;& \\ \widetilde{\phi}(P) = P,\ \widetilde{\phi}|_{P,P} = \mathrm{id}_P\}&\end{aligned}$$

for all pairs $Q, Q' \leqslant C_S(P)$.

Centralizer fusion systems are a special case of a larger class of fusion subsystems which also include the so-called normalizer fusion systems. When $x \in S$, we will write $C_{\mathcal{F}}(x)$ instead of $C_{\mathcal{F}}(\langle x \rangle)$ for brevity.
Having defined a fusion system, we would like it to be saturated. The following proposition provides a sufficient condition.

Proposition 2. *Let S be a finite p-group, $\mathcal{F}$ a saturated fusion system over S, and $P \leqslant S$. The centralizer fusion system of P, $C_{\mathcal{F}}(P)$ is saturated if P is fully $\mathcal{F}$-centralized.*

Proof. (see [AKO, pp. 20-21]). Let $R \leqslant C_S(P)$ be given. We will show that $R^{C_{\mathcal{F}}(P)}$ contains an element R_0 which is both fully automized and receptive.
For each $\mathcal{F}$-homomorphism ϕ with source PR, consider the quantity $|N_S(\phi(R)) \cap C_S(\phi(P))|$; in case ϕ is the extension of an $C_{\mathcal{F}}(P)$-homomorphism, $\phi(P) = P$. We claim that

$|N_S(\phi(R)) \cap C_S(\phi(P))|$ is maximal for some such homomorphism. Choose $\phi \in \mathrm{Hom}_{\mathcal{F}}(PR, S)$ such that $|N_S(\phi(R)) \cap C_S(\phi(P))|$ is maximal, and set $P_1 = \phi(P)$. Then $(\phi|_{P,P_1})^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P_1, P)$, and since P is fully $\mathcal{F}$-centralized it is receptive, so we may extend $(\phi|_{P,P_1})^{-1}$ to an $\mathcal{F}$-homomorphism $N_{(\phi|_{P,P_1})^{-1}} \to S$. Since P_1 as well as $C_S(P_1)$ are contained in $N_{(\phi|_{P,P_1})^{-1}}$ we can restrict to $P_1 \cdot C_S(P_1)$, and we obtain an $\mathcal{F}$-homomorphism $\alpha\colon P_1 \cdot C_S(P_1) \to S$. As $R \leqslant C_S(P)$, we have that $\phi(R) \leqslant C_S(\phi(P)) = C_S(P_1)$. Therefore, $\phi(PR) \leqslant P_1 \cdot C_S(P_1)$, so we may form the composition $\alpha \circ \phi \in \mathrm{Hom}_{\mathcal{F}}(PR, S)$. It satisfies $\alpha \circ \phi|_P = \mathrm{id}_P$, and since $R \leqslant C_S(P)$, $\alpha \circ \phi$ maps R into $C_S(P)$. By definition of $C_{\mathcal{F}}(P)$, its restriction to R defines a $C_{\mathcal{F}}(P)$-homomorphism. Set $R_0 = \alpha \circ \phi(R)$, we see that

$$\alpha(N_S(\phi(R)) \cap C_S(\phi(P))) \leqslant N_S(R_0) \cap C_S(P).$$

This proves the claim by choice of ϕ. Explicitly:

$$|N_S(R_0) \cap C_S(P)| \geq |N_S(\psi(R_0)) \cap C_S(\psi(P))| \tag{1}$$

for all $\psi \in \mathrm{Hom}_{\mathcal{F}}(PR_0, S)$.

Now set $I = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(PR_0) \mid \alpha(P) = P,\ \alpha|_{P,P} = \mathrm{id}_P,\ \alpha(R_0) = R_0\}$, i.e. I consists of all $\mathcal{F}$-automorphisms of PR_0 that restrict to $C_{\mathcal{F}}(P)$-automorphisms of R_0. Note that restriction in this manner actually defines an isomorphism of groups. If $x \in S$ is such that $c_x|_{PR_0,PR_0} \in \mathrm{Aut}_S(PR_0) \cap I$, then $x \in N_S(R_0)$ (since $c_x(R_0) = R_0$) and $x \in C_S(P)$ (since $c_x|_{P,P} = \mathrm{id}_P$), i.e. $x \in N_{C_S(P)}(R_0)$ which means that $c_x|_{R_0,R_0} \in \mathrm{Aut}_{C_S(P)}(R_0)$. Conversely, any such automorphism is the restriction of an element of $\mathrm{Aut}_S(PR_0) \cap I$. It follows that R_0 is fully $C_{\mathcal{F}}(P)$-automized if $\mathrm{Aut}_S(PR_0) \cap I \in \mathrm{Syl}_p(I)$. As $\mathcal{F}$ is saturated, there is $Q \in (PR_0)^{\mathcal{F}}$ which is fully automized and receptive. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(PR_0, Q)$, and let $T \in \mathrm{Syl}_p(I)$ such that $\mathrm{Aut}_S(PR_0) \cap I \leqslant T$. $\psi T \psi^{-1}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, so we can find $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\alpha\psi T \psi^{-1}\alpha^{-1} \leqslant \mathrm{Aut}_S(Q)$. Now

$$\mathrm{Aut}_S(Q) \cap \alpha\psi I \psi^{-1}\alpha^{-1} \geqslant \alpha\psi T \psi^{-1}\alpha^{-1} \in \mathrm{Syl}_p(\alpha\psi I \psi^{-1}\alpha^{-1})$$

The left-hand side is a p-group, so equality must hold, i.e.

$$\mathrm{Aut}_S(Q) \cap \alpha\psi I \psi^{-1}\alpha^{-1} \in \mathrm{Syl}_p(\alpha\psi I \psi^{-1}\alpha^{-1})$$

In particular

$$|\mathrm{Aut}_S(PR_0) \cap I| \leq \left|\mathrm{Aut}_S(Q) \cap \alpha\psi I \psi^{-1}\alpha^{-1}\right|. \tag{2}$$

Since Q is fully $\mathcal{F}$-centralized we also have that

$$|C_S(PR_0)| \leq |C_S(Q)| \tag{3}$$

On the other hand we have established that

$$\mathrm{Aut}_S(PR_0) \cap I \cong (N_S(R_0) \cap C_S(P))/C_S(PR_0)$$

and likewise

$$\mathrm{Aut}_S(Q) \cap \alpha\psi I \psi^{-1}\alpha^{-1} \cong (N_S(\alpha\psi R_0) \cap C_S(\alpha\psi P))/C_S(Q)$$

Using the property (1) of R_0 we get that

$$|C_S(PR_0)|\,|\mathrm{Aut}_S(PR_0) \cap I| \geq \left|\mathrm{Aut}_S(Q) \cap \alpha\psi I \psi^{-1}\alpha^{-1}\right|\,|C_S(Q)| \tag{4}$$

Comparing the inequalities of (2), (3), and (4), we see that equality must hold in all of them. Then (2) tells us that $\mathrm{Aut}_S(PR_0) \cap I$ is a Sylow-p-subgroup of I, and (3) tells us that PR_0 is fully centralized in $\mathcal{F}$. These results allow us to prove that R_0 is receptive in $C_{\mathcal{F}}(P)$.

Let $R \in R_0^{C_{\mathcal{F}}(P)}$, and let $\phi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(R, R_0)$ be an isomorphism. By definition of the centralizer fusion system, there is an extension $\widetilde{\phi} \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(PR, PR_0)$ of ϕ which restricts to the identity on P. Now let $J \leqslant I$ be the subgroup $\{\alpha \in I \mid \alpha|_{R_0,R_0} \in \mathrm{Aut}_{C_S(P)}(R_0, R_0)\}$. If $g \in N_S(PR_0)$ induces an element of $\mathrm{Aut}_S(PR_0) \cap I$, then $g \in N_S(R_0)$ and $g \in C_S(P)$, that is $g \in N_{C_S(P)}(R_0)$, which means that $c_g|_{PR_0,PR_0} \in J$. Thus

$$\mathrm{Aut}_S(PR_0) \cap J = \mathrm{Aut}_S(PR_0) \cap I$$

and since $\mathrm{Aut}_S(PR_0) \cap I \in \mathrm{Syl}_p(I)$ anj $J \leqslant I$, we see that $\mathrm{Aut}_S(PR_0) \cap J \in \mathrm{Syl}_p(J)$. $\widetilde{\phi}\mathrm{Aut}_S(PR)\widetilde{\phi}^{-1} \cap J$ is a p-subgroup of J; let $\alpha \in J$ such that

$$\alpha\widetilde{\phi}(\mathrm{Aut}_S(PR) \cap \widetilde{\phi}^{-1}J\widetilde{\phi})\widetilde{\phi}^{-1}\alpha^{-1} \leqslant \mathrm{Aut}_S(PR_0) \cap J \tag{5}$$

$\alpha\widetilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(PR, PR_0)$ extends to a homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_{\alpha\widetilde{\phi}}, S)$ since PR_0 is receptive in $\mathcal{F}$ (it is fully $\mathcal{F}$-centralized). We wish to see that $N_\phi \leqslant N_{\alpha\widetilde{\phi}}$. Let $g \in N_\phi$, i.e. $g \in N_{C_S(P)}(R)$ and $\phi c_g|_{R,R}\phi^{-1} \in \mathrm{Aut}_{C_S(P)}(R_0)$. The first property implies that $g \in C_S(P)N_S(R) \leqslant N_S(PR)$, and then the second one implies that $\widetilde{\phi}c_g|_{PR,PR}\widetilde{\phi}^{-1} \in J$. We conclude that $c_g|_{PR,PR} \in \mathrm{Aut}_S(PR) \cap \widetilde{\phi}^{-1}J\widetilde{\phi}$, but then (5) says that $g \in N_{\alpha\widetilde{\phi}}$, as desired. By definition of J and I, α is conjugation by some element $x \in N_{C_S(P)}(R_0)$. This allows us to form the composition $c_x^{-1} \circ \psi \in \mathrm{Hom}_{\mathcal{F}}(N_{\alpha\widetilde{\phi}}, S)$. Both c_x^{-1} and ψ restrict to the identity on P; if an element of $N_{\alpha\widetilde{\phi}}$ centralizes P, then so does its image under $c_x^{-1} \circ \psi$. In particular $c_x^{-1} \circ \psi$ maps N_ϕ to $C_S(P)$. Furthermore, $c_x^{-1} \circ \psi$ maps R to R_0. Combining these results we see that restriction of $c_x^{-1} \circ \psi$ defines a $C_{\mathcal{F}}(P)$-homomorphism $N_\phi \to C_S(P)$, which restricted to R equals ϕ. This shows that R_0 is receptive. □

The next proposition, although more technical in its presentation, also relates saturation of a fusion system with saturation of some centralizer subsystems.

Proposition 3. *[LO, 1.1]. Let S be a p-group and $\mathcal{F}$ a fusion system over S. $\mathcal{F}$ is saturated if and only if there is a set $\mathcal{C} \subset S$ of elements of order p with the following three properties:*

1. *Each element of S of order p is $\mathcal{F}$-conjugate to an element of $\mathcal{C}$.*
2. *If $x \in S$ has order p and is $\mathcal{F}$-conjugate to $x' \in \mathcal{C}$, then there exists an $\mathcal{F}$-homomorphism $\psi\colon C_S(x) \to C_S(x')$ with $\psi(x) = x'$.*
3. *The centralizer fusion system $C_{\mathcal{F}}(x)$ is saturated for all $x \in \mathcal{C}$.*

Proof. The 'only if' part is rather straightforward: Take $\mathcal{C} = \{x \in S \mid |x| = p,\ \langle x \rangle \text{ fully centralized}\}$. If $x \in S$ is of order p but $\langle x \rangle$ is not fully centralized there is an $\mathcal{F}$-isomrphism $\psi\colon \langle x \rangle \to P$ with P fully centralized. P has order p and is generated by $\psi(x)$, and $\psi(x)$ must then belong to $\mathcal{C}$, i.e. 1. holds. N_ψ clearly contains $C_S(x)$, since each element of $C_S(x)$ induces the identity on $\langle x \rangle$. As $\langle \psi(x) \rangle$ is fully centralized and $\mathcal{F}$ is saturated, $\langle \psi(x) \rangle$ is receptive, and so ψ extends to an $\mathcal{F}$-homomorphism $\widetilde{\psi}\colon C_S(x) \to S$. But the image is $\widetilde{\psi}(C_S(x)) \leqslant (C_S(\psi(x)))$, and 2. is satisfied. $\mathcal{C}$ also satisfies condition 3. by Proposition 2.

Now assume that $\mathcal{C}$ is a set of elements of S of order p, and that $\mathcal{C}$ has the three properties listed above. To prove that $\mathcal{F}$ is saturated, we show that it satisfies the conditions of Proposition 1.

Given subgroups $P \leqslant S$ and $A \leqslant \mathrm{Aut}_{\mathcal{F}}(P)$, A acts on $Z(P)$, and we will let $Z(P)^A$ denote the subgroup of elements fixed by this action. Define a set

$$\mathcal{U} = \{(P, x) \mid P \leqslant S,\ x \in S,\ |x| = p;\ \exists T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))\colon \mathrm{Aut}_S(P) \leqslant T,\ x \in Z(P)^T\}$$

and define a subset $\mathcal{U}_0 = \{(P,x) \in \mathcal{U} \mid x \in \mathcal{C}\}$. If $P \neq 1$ and $\mathrm{Aut}_S(P) \leqslant T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, then $|Z(P)^T|$ is divisible by p and non-empty; in particular it contains an element of order p. In other words, for every $P \leqslant S$, $P \neq 1$, there is an $x \in S$ such that $(P,x) \in \mathcal{U}$.
Now let $(P,x) \in \mathcal{U}_0$ and assume that P is not fully $\mathcal{F}$-centralized. Let $P' \in P^{\mathcal{F}}$ be fully $\mathcal{F}$-centralized and let $\phi \in \mathrm{Iso}_{\mathcal{F}}(P,P')$, then $\phi(x) \in Z(P')$. By property 2., there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(\phi(x)), C_S(x))$ such that $\psi(\phi(x)) = x$. Let $P'' = \psi(\phi(P))$, then $\psi \circ \phi$ is an $\mathcal{F}$-isomorphism $P \to P''$ which maps x to x. Since $x \in Z(P)$ we also have $x \in Z(P'')$, and so $P, P'' \leqslant C_S(x)$. As such, $\psi \circ \phi$ is also a $C_{\mathcal{F}}(x)$-isomorphism $P \to P''$. $C_S(P') \leqslant C_S(\phi(x))$ so we may apply ψ to $C_S(P')$ and we have $\psi(C_S(P')) \leqslant C_S(P'')$. We now have inequalities $|C_S(P)| < |C_S(P')| \leq |C_S(P'')|$. Both $C_S(P)$ and $C_S(P'')$ are contained in $C_S(x)$, hence $\left|C_{C_S(x)}(P)\right| < \left|C_{C_S(x)}(P'')\right|$. We have thus proved

$$\forall (P,x) \in \mathcal{U}_0 : P \text{ is fully } C_{\mathcal{F}}(x)\text{-centralized } \Rightarrow P \text{ is fully } \mathcal{F}\text{-centralized} \tag{6}$$

Given $(P,x) \in \mathcal{U}$, we have that $N_S(P) \leqslant C_S(x)$ since $x \in Z(P)$ is fixed by each element of $\mathrm{Aut}_S(P)$. Therefore $\mathrm{Aut}_S(P) = \mathrm{Aut}_{C_S(x)}(P)$. The $C_{\mathcal{F}}(x)$-automorphisms of P are exactly the $\mathcal{F}$-automorphisms of P that restrict to the identity on $\langle x \rangle$. Every element of the $T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ associated to (P,x) has this property, i.e. $T \leqslant \mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)$. But then T is also a Sylow-p-subgroup of $\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P) \leqslant \mathrm{Aut}_{\mathcal{F}}(P)$. As $\mathrm{Aut}_S(P) = \mathrm{Aut}_{C_S(x)}(P)$ we conlude that $\mathrm{Aut}_{C_S(x)}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P))$ if and only if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, i.e. we have proven that

$$\forall (P,x) \in \mathcal{U} : P \text{ is fully } C_{\mathcal{F}}(x)\text{-automized } \Leftrightarrow P \text{ is fully } \mathcal{F}\text{-automized} \tag{7}$$

Let $P \leqslant S$ be given and assume that P is fully $\mathcal{F}$-normalized. We will show that P is fully centralized and fully automized in $\mathcal{F}$. Pick $x \in S$ such that $(P,x) \in \mathcal{U}$ (such an x exists) and choose $x' \in \mathcal{C}$ and a homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(x'))$ that maps x to x'. Like above, $N_S(P) \leqslant C_S(x)$ and $\psi(N_S(P)) \leqslant N_S(\psi(P))$. But since P is fully $\mathcal{F}$-normalized, we must have $\psi(N_S(P)) = N_S(\psi(P))$. Let $T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ be as in the definition of $\mathcal{U}$. Then $\psi T \psi^{-1} \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(\psi(P)))$, and since $\mathrm{Aut}_S(P) \leqslant T$ and $N_S(\psi(P)) = \psi(N_S(P))$ we get that $\mathrm{Aut}_S(\psi(P)) \leqslant \psi T \psi^{-1}$. In addition, given any $\alpha \in T$ we have that $\psi\alpha\psi^{-1}(x') = x'$ since $x' = \psi(x)$ and each $\alpha \in T$ fixes x. This shows that $x' \in Z(\psi(P))^{\psi T \psi^{-1}}$, and we conclude that $(\psi(P), x') \in \mathcal{U}_0$. As P is fully $\mathcal{F}$-normalized and $\psi(N_S(P)) = N_S(\psi(P))$, we see that $\psi(P)$ is fully $\mathcal{F}$-normalized as well. Furthermore, $N_S(\psi(P)) \leqslant C_S(x')$, and so $N_S(\psi(P)) = N_{C_S(x')}$. In general, given $Q \leqslant C_S(x')$ we clearly have the inequality $|N_S(Q)| \geq \left|N_{C_S(x')}(Q)\right|$. We conclude that $\psi(P)$ is fully $C_{\mathcal{F}}(x')$-normalized, and using Proposition 1 we get that $\psi(P)$ is fully centralized and fully automized in $C_{\mathcal{F}}(x')$. By (6) and (7), $\psi(P)$ is fully automized and fully centralized in $\mathcal{F}$. But since P is also fully $\mathcal{F}$-normalized, $\mathcal{F}$-conjugate to $\psi(P)$, and since

$$\mathrm{Aut}_S(P) \cong N_S(P)/C_S(P)$$

we see that P is fully automized and fully centralized in $\mathcal{F}$.
Next we show that $\mathcal{F}$ satisfies the second condition of Proposition 1 which will finish the proof. Let $\phi \colon P \to P'$ be an $\mathcal{F}$-isomorphism, and assume that P' is fully $\mathcal{F}$-centralized. Let $x' \in S$ be such that $(P', x') \in \mathcal{U}$, and set $x = \phi^{-1}(x') \in Z(P)$. By definition of $\mathcal{F}$, x' is fixed under the action of each element of $\mathrm{Aut}_S(P')$. Recall that

$$N_\phi = \{g \in N_S(P) \mid \phi c_g|_{P,P} \phi^{-1} \in \mathrm{Aut}_S(P')\}$$

We see that $c_g|_{P,P}(x) = x$ for all $g \in N_\phi$, i.e. $N_\phi \leqslant C_S(x)$. By arguments presented earlier $(P', x') \in \mathcal{U}$ implies that $N_S(P') \leqslant C_S(x')$. Now pick $y \in \mathcal{C}$ such that y is $\mathcal{F}$-conjugate to x, and thereby also to x'. Pick homomorphisms $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ and

$\psi' \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(y))$ such that $\psi(x) = \psi'(x') = y$, and set $Q = \psi(P)$, $Q' = \psi'(P')$. From the relations established so far, we see that

$$\psi'(C_{C_S(x')}(P')) = \psi'(C_S(P')) = C_S(Q') = C_{C_S(y)}(Q')$$

where the second equality follows from the fact that P' is fully $\mathcal{F}$-centralized. Now define $\tau = \psi'|_{P,Q} \circ \phi \circ (\psi|_{P',Q'})^{-1}$. τ is an $\mathcal{F}$-isomorphism $Q \to Q'$ which fixes y. So τ is even an $C_{\mathcal{F}}(y)$-isomorphism. Since

$$|C_S(P')| = |C_S(Q')| = |C_{C_S(y)}(Q')|$$

and P' is fully $\mathcal{F}$-centralized, Q' is fully $C_{\mathcal{F}}(y)$-centralized. As $C_{\mathcal{F}}(y)$ is saturated, Q' is receptive in $C_{\mathcal{F}}(y)$ by Proposition 1. Hence τ extends to some $\widetilde{\tau} \in \mathrm{Hom}_{C_{\mathcal{F}}(y)}(N_\tau, C_S(y))$. Given $q' \in Q'$ and $g \in N_\phi$ we have

$$(\widetilde{\tau}\psi(g))q'(\widetilde{\tau}\psi(g))^{-1} = \tau(\psi(g)\tau^{-1}(q')\psi(g)^{-1})$$

The argument of the right-hand side is in P since $\psi(g) \in \psi(N_\phi) \leqslant N_S(Q)$, and we can apply τ to it. As $\tau^{-1}(q')$ is in the domain of ψ we can continue the rewriting:

$$\tau(\psi(g)\tau^{-1}(q')\psi(g)^{-1}) = \tau\psi(g(\psi^{-1}\tau^{-1}(q'))g^{-1}) = \psi'\phi(g(\psi'\phi)^{-1}(q')g^{-1}) = \psi'\phi c_g \phi^{-1}(\psi')^{-1}(q')$$

(with proper restrictions of some of the homomorphisms; these have been omitted in order not to make the notation too cumbersome). By choice of g we can find $h \in N_S(P')$ such that $\phi c_g \phi^{-1} = c_h$, and then

$$\psi'\phi c_g \phi^{-1}(\psi')^{-1}(q') = \psi' c_h (\psi')^{-1}(q') = \psi'(h)q'\psi'(h)^{-1} = c_{\psi'(h)}(q')$$

These calculations show that $\tau c_{\psi(g)}\tau^{-1} = c_{\psi'(h)}$ as automorphisms $Q' \to Q'$. As $c_{\psi'(h)} \in \mathrm{Aut}_{C_S(y)}(Q')$ (since $h \in N_S(P') \leqslant C_S(x')$ and ψ maps $C_S(x')$ to $C_S(y)$) we have that $\psi(g) \in N_\tau$ by definition. Thus $\psi(N_\phi) \leqslant N_\tau$. The calculations also show that $\widetilde{\tau}(\psi(g))$ and $\psi'(h)$ induce the same element of $\mathrm{Aut}_{C_S(y)}(Q')$, and so $\widetilde{\tau}(\psi(g))\psi'(h)^{-1} \in C_S(Q')$. But we have already seen that $C_S(Q') = \psi(C_S(P'))$, hence we get that $\widetilde{\tau}(\psi(N_\phi)) \leqslant \mathrm{Im}(\psi')$. This allows us to form the composition

$$\widetilde{\phi} := (\psi')^{-1} \circ \widetilde{\tau} \circ \psi|_{N_\phi}$$

Then $\widetilde{\phi}$ is an $\mathcal{F}$-homomorphism $N_\phi \to S$, and $\widetilde{\phi}|_P = \phi$, by construction. Hence $\widetilde{\phi}$ is our desired extension of ϕ, and we conclude that P' is receptive in $\mathcal{F}$. □

We wish to state Alperin's Fusion Theorem. To do so, we must first define a few more properties that subgroups of a fusion system may have.

Definition. *Let S be a finite p-group and $\mathcal{F}$ a fusion system over S. A subgroup $P \leqslant S$ is said to be*

- *$\mathcal{F}$-centric if $C_S(Q) \leqslant Q$ (equivalently $C_S(Q) = Z(Q)$) for all $Q \in P^{\mathcal{F}}$.*
- *$\mathcal{F}$-radical if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. (Recall that for a group G, $O_p(G)$ is the smallest normal p-subgroup of G).*

Notice that if $P \leqslant S$ is $\mathcal{F}$-centric then so is each member of $P^{\mathcal{F}}$, and in particular they are all fully $\mathcal{F}$-centralized. The $\mathcal{F}$-centric subgroups are closed under taking overgroups, i.e. if $P \leqslant Q \leqslant S$ and P is $\mathcal{F}$-centric, then so is Q: Every $Q' \in Q^{\mathcal{F}}$ contains a subgroup $P' \in P^{\mathcal{F}}$. But then $C_S(Q') \leqslant C_S(P') \leqslant P' \leqslant Q'$.

The set of $\mathcal{F}$-radical subgroups are also closed under $\mathcal{F}$-conjugation: If $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$, then $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{Out}_{\mathcal{F}}(P)$ since conjugation by ϕ defines automorphisms $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Inn}(Q) \cong \mathrm{Inn}(P)$.
When P is a p-subgoup of a group G, one says that P is *p-centric* (in G) if $Z(P) \in \mathrm{Syl}_p(C_G(P))$, and P is *p-radical* if $O_P(N_G(P)/P) = 1$. If $S \in \mathrm{Syl}_p(G)$ and $P \leqslant S$, it is not hard to see that P is p-centric if and only if it is $\mathcal{F}_S(G)$-centric. It can be shown using basic manipulations and the explicit structure of the morphisms of $\mathcal{F}_S(G)$. There is no immediate connection between being p-racical in G and being $\mathcal{F}_S(G)$-radical; one does not imply the other. However, if P is $\mathcal{F}_S(G)$-centric and $\mathcal{F}_S(G)$-radical, then P is p-radical in G.

We can now state Alperin's Fusion Theorem, which essentially concerns uniqueness of fusion systems.

Theorem 4 (Alperin's Fusion Theorem)**.** *Let S be a finite p-group, and $\mathcal{F}$ a saturated fusion system over S. Set*

$$\mathcal{F}^{crn} = \{S\} \cup \{P \leqslant S \mid P \text{ is centric, radical, and fully normalized in } \mathcal{F}\}$$

Given any $\mathcal{F}$-isomorphism $\phi \in \mathrm{Iso}_{\mathcal{F}}(P,P')$ there exist subgroups

$$P = P_0, P_1, \ldots, P_n = P' \leqslant S, \quad \text{and} \quad Q_1, \ldots, Q_n \in \mathcal{F}^{crn}$$

and automorphisms $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

- *$P_{i-1}, P_i \leqslant Q_i$ and $\alpha_i(P_{i-1}) = P_i$, for $i = 1, \ldots, n$.*
- *$\phi = \alpha_n|_{P_{n-1},P_n} \circ \ldots \circ \alpha_1|_{P_0,P_1}$.* □

The set $\mathcal{F}^{crn}$ is an example of a so-called conjugation family, which is just a collection $\mathcal{C}$ of subgroups of S, for which the statement of the theorem is true with $\mathcal{C}$ in place of $\mathcal{F}^{crn}$.
Alperin's Fusion Theorem is a version of the more general Alperin-Goldschmidt Fusion Theorem, which has the exact same formulation, except that $\mathcal{F}^{crn}$ is replaced with a smaller conjugation family. One can take this idea further and determine all minimal (with respect to inclusion) conjugation families. However, for our purposes, $\mathcal{F}^{crn}$ will suffice.
Loosely speaking, Alperin's Fusion Theorem states that $\langle \mathrm{Aut}_{\mathcal{F}}(P) \mid P \in \mathcal{F}^{crn}\rangle = \mathcal{F}$. It is enough to know the isomorphisms of a fusion system, since every morphism is an isomorphism followed by an inclusion.

2 The extraspecial group of order p^3 and exponent p

The examples of exotic fusion systems provided by Ruiz and Viruel is [RV] are defined on the extraspecial group of order p^3 and exponent p for the prime $p = 7$. The group is defined for all odd primes and has the presentation

$$p_+^{1+2} \overset{def}{=} \langle a, b, c \mid a^p = b^p = c^p = 1,\ ac = ca,\ bc = cb,\ ab = bac\rangle$$

We will need to study this group and its group of automorphisms in detail. This section, and the next, follow the [RV].
We first note that a and b generate p_+^{1+2} and that every element of p_+^{1+2} can be written in the form $a^r b^s c^t$ with $a, b, c \in \{0, \ldots, p-1\}$, in particular the order of p_+^{1+2} is at most p^3. It is indeed p^3: c is central so $\langle c\rangle \leqslant Z(p_+^{1+2})$ and $\langle c\rangle \trianglelefteq p_+^{1+2}$. The quotient $p_+^{1+2}/\langle c\rangle$ is abelian; it is generated by the images of a and b under the canonical epimorphism $p_+^{1+2} \to p_+^{1+2}/\langle c\rangle$. Elementary calculations show that $p_+^{1+2}/\langle c\rangle$ has order p^2, since otherwise a or b would be central in p_+^{1+2}. As such, $|p_+^{1+2}| = p^3$ and every element can be written uniquely in the form $a^r b^s c^t$ mentioned above.

We also note that the fact that $\langle c\rangle$ has order p and is normal in p_+^{1+2} with abelian quotient implies that $\langle c\rangle = [p_+^{1+2}, p_+^{1+2}]$, the commutator subgroup of p_+^{1+2}. It also holds that $\langle c\rangle = Z(p_+^{1+2})$: c is central and if $a^r b^s c^t \in Z(p_+^{1+2})$ for some $r, s, t \in \{0, \dots, p-1\}$ then

$$a^{r+1}b^s c^t = aa^r b^s c^t = a^r b^s c^t a = a^{r+1}b^s c^{t-s}$$

and thus $s = 0$ (by the uniqueness of such presentations of elements of p_+^{1+2}). A similar argument shows that $r = 0$.
As a and b generate p_+^{1+2}, any endomorphism of p_+^{1+2} is defined by its values on these two elements. The next lemma tells us explicitly what the endomorphisms and automorphisms of p_+^{1+2} look like.

Proposition 5. *Any assigment $a \mapsto a^{r'}b^{s'}c^{t'}$, $b \mapsto a^r b^s c^t$ with $r', s', t', r, s, t \in \{0, \dots, p-1\}$ defines an endomorphism of p_+^{1+2}.*
The endomorphism is an automorphism if and only if $p \nmid r's - rs'$.

Proof. Let $\phi\colon \langle a, b\rangle \to p_+^{1+2}$ be the homomorphism defined by $\phi(a) = a^{r'}b^{s'}c^{t'}$ and $\phi(b) = a^r b^s c^t$ (here $\langle a, b\rangle$ denotes the free group on the symbols a and b). If ϕ respects the relations in our presentation of p_+^{1+2}, it induces an endomorphism of p_+^{1+2}. To be precise, if we let c denote the element $a^{-1}b^{-1}ab$ of $\langle a, b\rangle$ we have to check that $\phi(a)^p = \phi(b)^p = \phi(c)^p = 1$, that $\phi(a)\phi(c) = \phi(c)\phi(a)$, and that $\phi(b)\phi(c) = \phi(c)\phi(b)$.
Computations using only the definition of ϕ and the relations of p_+^{1+2} show that $\phi(c) = c^{r's-rs'}$. The conditions $\phi(c)^p = 1$, $\phi(a)\phi(c) = \phi(c)\phi(a)$ and $\phi(b)\phi(c) = \phi(c)\phi(b)$ are therefore satisfied. That $\phi(b)^p = 1$ can be shown as follows:

$$\begin{aligned}\phi(b)^p &= (a^r b^s c^t)^p = (a^r b^s)^p = a^r b^s a^r b^s (a^r b^s)^{p-2} = c^{-rs}a^{2r}b^{2r}(a^r b^s)^{p-2} \\ &= \dots = c^{-rs\sum_{i=1}^{p-1} i} a^{pr} b^{pr} = c^{-rs\frac{p(p-1)}{2}} = 1\end{aligned}$$

where the last equality follows from the fact that $p \mid \frac{p(p-1)}{2}$, since p is an odd prime. Analogously, one can show that $\phi(a)^p = 1$. In the following, ϕ will denote the induced endomorphism of p_+^{1+2}.

Assume that $p \nmid r's - rs'$. Let $u, v, w \in \{0, \dots, p-1\}$, and assume that $\phi(a^u b^v c^w) = 1$. I.e.

$$\begin{aligned}1 &= \phi(a)^u \phi(b)^v \phi(c)^w = (a^{r'}b^{s'}c^{t'})^u (a^r b^s c^t)^v (c^{r's-rs'})^w \\ &= a^{ur'+vr} b^{us'+vs} c^*\end{aligned}$$

(the exponent of c is irrelevant to the argument, so we write $*$ in its place for brevity). Modulo p we have $ur' \equiv -vr$ and $-us' \equiv vs$. In particular $uvr's \equiv uvrs' \pmod p$, and so $p \mid uv(r's - rs')$. By assumption we must have $p \mid u$ or $p \mid v$. If $p \mid u$ but $p \nmid v$, the first pair of congruences imply that $p \mid r$ and $p \mid s$, which contradicts $p \nmid r's - rs'$. Likewise $p \mid \Rightarrow p \mid u$, hence p divides both u and v. Now

$$1 = \phi(a^u b^v c^w) = \phi(c)^w = c^{(r's-rs')w}$$

so $p \mid w$. We conclude that ϕ is injective if $p \nmid r's - rs'$. On the other hand we know that $\phi(c) = c^{r's-rs'}$, so the condition is necessary. □

Note that the proposition accounts for all automorphisms of p_+^{1+2}. We continue with a description of the structure of the group of automorphisms of p_+^{1+2}.

Proposition 6. *There is a surjective homomorphism $\mathrm{Aut}(p_+^{1+2}) \to \mathrm{GL}_2(\mathbb{F}_p)$ given by*

$$\phi \mapsto \begin{pmatrix} r' & r \\ s' & s \end{pmatrix} =: M_\phi,$$

where r', s', r, s are given by $\phi(a) = a^{r'}b^{s'}c^{t'}$, $\phi(b) = a^r b^s c^t$.
The homomorphism induces an isomorphism $\mathrm{Out}(p_+^{1+2}) \to \mathrm{GL}_2(\mathbb{F}_p)$.

Proof. Using Proposition 5, it is clear that the assignment $\phi \mapsto M_\phi$ defines a surjective map $\mathrm{Aut}(p_+^{1+2}) \to \mathrm{GL}_2(\mathbb{F}_p)$. To see that it is a homomorphism, let $\psi \in \mathrm{Aut}(p_+^{1+2})$ be given by $\psi(a) = a^{u'}b^{v'}c^{w'}$ and $\psi(b) = a^u b^v c^w$. Then we have

$$\psi(\phi(a)) = a^{u'r'+us'}b^{v'r'+vs'}c^*$$
$$\psi(\phi(b)) = a^{u'r+us}b^{v'r+vs}c^*$$

and so

$$M_{\psi\phi} = \begin{pmatrix} u'r' + us' & u'r + us \\ v'r' + vs' & v'r + vs \end{pmatrix} = \begin{pmatrix} u' & u \\ v' & v \end{pmatrix}\begin{pmatrix} r' & r \\ s' & s \end{pmatrix} = M_\psi M_\phi$$

To prove the last claim of the lemma, we need to consider inner automorphisms of p_+^{1+2}. Let $x, y, z \in \{0, \ldots, p-1\}$, then

$$(a^x b^y c^z)a(a^x b^y c^z)^{-1} = ac^{-y}$$
$$(a^x b^y c^z)b(a^x b^y c^z)^{-1} = bc^x$$

which shows that all inner automorphisms of p_+^{1+2} map to the identity matrix of $\mathrm{GL}_2(\mathbb{F}_p)$. Conversely, let $\phi \in \mathrm{Aut}(p_+^{1+2})$ and assume that $M_\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then there are $t', t \in \{0, \ldots, p-1\}$ such that $\phi(a) = ac^{t'}$ and $\phi(b) = bc^t$. But then ϕ is the automorphism given by conjugation by $a^t b^{-t'}$. Hence $\mathrm{Inn}(p_+^{1+2})$ is the kernel of the epimorphism $\mathrm{Aut}(p_+^{1+2}) \to \mathrm{GL}_2(\mathbb{F}_p)$. □

The identification $\mathrm{Out}(p_+^{1+2}) \cong \mathrm{GL}_2(\mathbb{F}_p)$ will be very important in the analysis of fusion systems defined over p_+^{1+2}. Another ingredient will be Alperin's Fusion Theorem. In order to apply it to a given fusion system $\mathcal{F}$ over p_+^{1+2}, we need to study the subgroups of p_+^{1+2} and determine which of them are $\mathcal{F}$-centric and $\mathcal{F}$-radical.

Lemma 7. *Every proper subgroup of p_+^{1+2} is abelian. They can be described as follows:*

1. *There are $p+1$ elementary abelian subgroups of p_+^{1+2} of rank 2. These are $V_i := \langle c, ab^i \rangle$, for $i = 0, \ldots, p-1$, and $V_p := \langle c, b \rangle$. They are all normal in p_+^{1+2}.*
2. *There are $1 + p + p^2$ subgroups of p_+^{1+2} of order p.*

Proof. Let H be a non-abelian subgroup of p_+^{1+2}, and let $x, y \in H$ be two non-commuting elements. By Proposition 5, the assignment $a \mapsto x$, $b \mapsto y$ defines a homomorphism $\phi\colon p_+^{1+2} \to p_+^{1+2}$, whose image is $\langle x, y \rangle$. This image is non-abelian and isomorphic to $p_+^{1+2}/\ker\phi$, hence $\ker\phi$ does not contain the commutator subgroup of p_+^{1+2} which is $\langle c \rangle$. In particular $\phi(c) \neq 1$, but from the proof of Proposition 5, this is the same as saying that ϕ is an automorphism of p_+^{1+2}. Hence $H = p_+^{1+2}$.
ad 1. Let $V \leqslant p_+^{1+2}$ be elementary abelian of rank 2. As $V \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, we may regard V as an $\mathbb{F}_p$-vector space, hence the choice of the letter 'V'. (Explicitly, the action of $\mathbb{F}_p$ on V is given by $z.x = x^z$, which is well-defined since every element (save the neutral one) of p_+^{1+2} has order p).
We claim that $c \in V$: Assume this is not the case and let $x, y \in V$ be generators of V. Then $V = \langle x, y \rangle \lneqq \langle x, y, c \rangle \leqslant p_+^{1+2}$. But as c is central in p_+^{1+2}, we get that $\langle x, y, c \rangle$ is an abelian subgroup of p_+^{1+2} of order $> p^2$, a clear contradiction. Therefore, $V = \langle c, x \rangle$ for some $x \in V$. Write $x = a^r b^s c^t$, $r, s, t \in \{0, \ldots, p-1\}$. If $r = 0$, then $V = V_p$. If $r \neq 0$, we may assume that $r = 1$ by replacing x by some suitable power of x, and so $V = V_s$.
That each V_i, $i = 0, \ldots, p$, is normal in p_+^{1+2} is immediate once we recall that c is central in p_+^{1+2}, and that the exponents r, s of every element $a^r b^s c^t \in p_+^{1+2}$ are unchanged under conjugation by all elements of p_+^{1+2}.

ad 2. This result is a matter of counting: Every element of p_+^{1+2}, save the neutral element, has order p, and lies in excatly one subgroup of p_+^{1+2} of order p. Consequently, there are $(p^3-1)/(p-1) = 1+p+p^2$ such subgroups. □

We will often need to refer to the set of elementary abelian subgroups of p_+^{1+2} of rank 2; let $\mathcal{V}$ denote this set, i.e. $\mathcal{V} = \{V_i \mid i = 0, \dots, p\}$ with the notation of the lemma.

3 Fusion systems over p_+^{1+2}

We now turn our attention to fusion systems defined over p_+^{1+2}. Alperin's Fusion Theorem play a key role in the classification of saturated fusion systems over p_+^{1+2} done by A. Ruiz and A. Viruel. To make use of it, we need to understand the $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of p_+^{1+2} for a given fusion system $\mathcal{F}$ over p_+^{1+2}. If $V \leqslant p_+^{1+2}$ is elementary abelian of rank 2, its group of automorphisms can be identified canonically with $\mathrm{GL}_2(\mathbb{F}_p)$ (though this identification is unlike the one established in the previous section; $\mathrm{Out}(p_+^{1+2}) \cong \mathrm{GL}_2(\mathbb{F}_p)$). And since V is abelian, the group of automorphisms of V is the same as the group of outer automorphisms of V; $\mathrm{Aut}(V) = \mathrm{Out}(V)$, in particular $\mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{Out}_{\mathcal{F}}(V)$ for any given fusion system $\mathcal{F}$ over p_+^{1+2}.

Lemma 8. *Let $\mathcal{F}$ be a fusion system over p_+^{1+2}.*

1. *The $\mathcal{F}$-centric subgroups are exactly the elementary abelian subgroups of rank 2 and p_+^{1+2} itself.*

2. *Each $V \in \mathcal{V}$ is $\mathcal{F}$-radical if and only if $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$. ([RV, 4.1]).*

Proof. ad 1. It is trivial that p_+^{1+2} is $\mathcal{F}$-centric.
We know that $\langle c \rangle = Z(p_+^{1+2})$, which implies that $C_{p_+^{1+2}}(\langle c \rangle) = p_+^{1+2}$, and so $\langle c \rangle$ cannot be $\mathcal{F}$-centric. But $Z(p_+^{1+2}) \leqslant C_{p_+^{1+2}}(P)$ for any subgroup $P \leqslant p_+^{1+2}$, so any $\mathcal{F}$-centric subgroup must contain $\langle c \rangle$. The rank 2 elementary abelian subgroups are therefore the only remaining candidates, and we have to show that every one of them is $\mathcal{F}$-centric. Given $V \in \mathcal{V}$, we have $V \leqslant C_{p_+^{1+2}}(V)$, but equality must hold since otherwise $C_{p_+^{1+2}}(V) = p_+^{1+2}$ and V would be central. Hence every rank two elementary abelian subgroup is $\mathcal{F}$-centric.

ad 2. Let $V \in \mathcal{V}$ be given. Assume first that $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$. The order of $\mathrm{SL}_2(\mathbb{F}_p)$ is $(p-1)p(p+1)$ while the order of $\mathrm{GL}_2(\mathbb{F}_p)$ is $(p-1)^2p(p+1)$. Any non-trivial p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$ must therefore have order p and be a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$. The groups

$$\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{F}_p \right\}, \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{F}_p \right\},$$

are two distinct Sylow-p-subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ and of $\mathrm{Aut}_{\mathcal{F}}(V)$ as well. But then $\mathrm{Aut}_{\mathcal{F}}(V)$ has no normal subgroup of order p, and we conclude that $O_p(\mathrm{Aut}_{\mathcal{F}}(V)) = 1$, i.e. that V is $\mathcal{F}$-centric.

Now assume that V is $\mathcal{F}$-radical. V is one of the groups described in Lemma 7. Assume for convenience that $V = V_0 = \langle c, a \rangle$; the argument is the same for all the V_i. The elements of V are exactly the elements $c^i a^j$, $i, j \in \{0, \dots, p-1\}$. $c^i a^j \mapsto (i, j)$ defines an isomorphism $V \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and this isomorphism induces the isomorphism $\mathrm{Aut}(V) \cong \mathrm{GL}_2(\mathbb{F}_p)$ mentioned earlier, though we will just write $\mathrm{Aut}(V) = \mathrm{GL}_2(\mathbb{F}_p)$. The matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p)$ corresponds to the automorphism of V given by $c \mapsto c$, $a \mapsto ca$. But that is the restriction to V of the inner automorphism of p_+^{1+2} given by conjugation by b^{-1}. Therefore $T \in \mathrm{Aut}_{\mathcal{F}}(V)$.

T has order p, so $\langle T\rangle$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$. But then $\mathrm{Aut}_{\mathcal{F}}(V)$ must contain more than one Sylow-p-subgroup, since otherwise $\langle T\rangle$ would be normal in $\mathrm{Aut}_{\mathcal{F}}(V)$. We claim that the number of Sylow-p-subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ is $p+1$; using Sylow's theorems, this would imply that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains all of them. The number of Sylow-p-subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ equals $[\mathrm{GL}_2(\mathbb{F}_p) : N_{\mathrm{GL}_2(\mathbb{F}_p)}(\langle T\rangle)]$. Simple computations show that an element of $\mathrm{GL}_2(\mathbb{F}_p)$ normalizes $\langle T\rangle$ if and only if it is an upper triangular matrix, i.e. $\left|N_{\mathrm{GL}_2(\mathbb{F}_p)}(\langle T\rangle)\right| = (p-1)^2p$.
In particular, we now know that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains all elements of $\mathrm{GL}_2(\mathbb{F}_p)$ of order p. We will use this to show that $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$. We first show that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains all diagonal matrices of $\mathrm{SL}_2(\mathbb{F}_p)$. Let $x \in \mathbb{F}_p$, $x \neq 0$, then

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}_{\mathcal{F}}(V)$$

Now let $M = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p)$. Note that

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}_{\mathcal{F}}(V)$$

If $q = 0$, then $r, s \neq 0$ and $s = -r^{-1}$ and we have that

$$M = \begin{pmatrix} 1 & 0 \\ -ts & 1 \end{pmatrix}\begin{pmatrix} -r & 0 \\ 0 & s \end{pmatrix} S^{-1} \in \mathrm{Aut}_{\mathcal{F}}(V)$$

If $q \neq 0$, we have that

$$M = \begin{pmatrix} 1 & 0 \\ q^{-1}s & 1 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & t-q^{-1}rs \end{pmatrix}\begin{pmatrix} 1 & q^{-1}r \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}_{\mathcal{F}}(V)$$

Which finishes the proof of the 'only if'-part.

□

Since all rank 2 elementary abelian subgroups are normal in p_+^{1+2}, they are all fully normalized. As a consequence of the lemma, we therefore have that such a subgroup V is an element of $\mathcal{F}^{crn}$ if and only if V is $\mathcal{F}$-radical. Let $\mathcal{F}^{er}$ denote the set of all rank 2 elementary abelian subgroups of p_+^{1+2} that are $\mathcal{F}$-radical.
We wish to condense the information needed to uniquely define any saturated fusion system over p_+^{1+2} to a minimum. The next lemma takes another step towards that goal.

Lemma 9. *[RV, 4.4]. Let $\mathcal{F}$ be a saturated fusion system over p_+^{1+2}, and let $V \leqslant p_+^{1+2}$ be elementary abelian of rank 2. Then $\mathrm{Aut}_{\mathcal{F}}(V)$ can be determined explicitly from the following knowledge: $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$; and whether or not V is $\mathcal{F}$-radical.*

Proof. First we deal with the case in which V is not $\mathcal{F}$-radical. Applying Alperin's Fusion Theorem (theorem 4) we see that every $\mathcal{F}$-automorphism of V is the restriction of an $\mathcal{F}$-automorphism of p_+^{1+2}. (And any such restriction defines an $\mathcal{F}$-automorphism of V by the definition of fusion systems).

Now assume that V is $\mathcal{F}$-radical. V has the form $V = \langle C, A^iB^j\rangle$ for some $i, j \in \{0, 1, \dots, p-1\}$. We have our usual identification $\mathrm{Aut}_{\mathcal{F}}(V) \leqslant \mathrm{GL}_2(\mathbb{F}_p)$ with respect to this basis.
From Lemma 8 we know that $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$. Furthermore $\mathrm{Aut}_{\mathcal{F}}(V)$ must contain all possible restrictions of elements of $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$. These restrictions form a subgroup of $\mathrm{Aut}_{\mathcal{F}}(V) \leqslant \mathrm{GL}_2(\mathbb{F}_p)$ so their determinants form a subgroup of $\mathbb{F}_p^*$. Let R denote this subgroup. Then $\mathrm{Aut}_{\mathcal{F}}(V)$ contains every element of $\mathrm{GL}_2(\mathbb{F}_p)$ whose determinant is an element of R. (We can identify R with the subgroup $\{\mathrm{diag}\,(r, 1) \mid r \in R\} \leqslant \mathrm{GL}_2(\mathbb{F}_p)$. Then R acts on $\mathrm{SL}_2(\mathbb{F}_p)$ by

conjugation. Our statement can now be rephrased to say that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains $\mathrm{SL}_2(\mathbb{F}_p) \rtimes R$; the semidirect product of $\mathrm{SL}_2(\mathbb{F}_p)$ and R with respect to the acion of R just defined).
We claim that in fact $\mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes R$. Let $\phi \in \mathrm{Aut}_{\mathcal{F}}(V)$ and set $d = \det(\phi)$. As $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$, every matrix with determinant d is contained in $\mathrm{Aut}_{\mathcal{F}}(V)$; let $\psi := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$, then $\psi \in \mathrm{Aut}_{\mathcal{F}}(V)$. V is $\mathcal{F}$-centric (by Lemma 8); in particular, V is fully $\mathcal{F}$-centralized and thereby receptive (by Proposition 1). Thus ψ extends to a $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, p_+^{1+2})$. We claim that $N_\psi = p_+^{1+2}$. Let $g = a^r b^s c^t \in N_{p_+^{1+2}}(V) = p_+^{1+2}$, then $c_g|_{V,V}$ is the $\mathcal{F}$-automorphism of V with matrix $\begin{pmatrix} 1 & si - rj \\ 0 & 1 \end{pmatrix}$. Now

$$\psi c_g|_{V,V} \psi^{-1} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & si - rj \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (ds)i - (dr)j \\ 0 & 1 \end{pmatrix}$$

i.e. $\psi c_g|_{V,V} \psi^{-1} = c_{a^{dr} b^{ds}}|_{V,V} \in \mathrm{Aut}_{p_+^{1+2}}(V)$, hence $N_\psi = p_+^{1+2}$. So ψ is the restriction of $\widetilde{\psi} \in \mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$. Therefore $d = \det(\psi) \in R$, and we conclude that $\phi \in \mathrm{SL}_2(\mathbb{F}_p) : R$. ☐

Proposition 10. *Let* $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{F}_p) = \mathrm{Out}(p_+^{1+2})$ *be given, and let* $\phi_g \in \mathrm{Aut}(p_+^{1+2})$ *be the automorphism given by*

$$\phi_g(a) = a^\alpha b^\gamma c^{-\frac{1}{2}\alpha\gamma}, \quad \text{and} \quad \phi_g(b) = a^\beta b^\delta c^{-\frac{1}{2}\beta\delta}$$

Then we have the following series of results:

1. *The map* $\Phi\colon \mathrm{Out}(p_+^{1+2}) \to \mathrm{Aut}(p_+^{1+2})$ *given by* $g \mapsto \phi_g$ *is a homomorphism which splits the canonical homomorphism* $\mathrm{Aut}(p_+^{1+2}) \to \mathrm{Out}(p_+^{1+2})$.
2. *The action of* $\mathrm{Out}(p_+^{1+2})$ *on* p_+^{1+2} *defined by* $g.x = \phi_g(x)$ *induces an action of* $\mathrm{Out}(p_+^{1+2})$ *on* $\mathrm{Inn}(p_+^{1+2})$.
3. *There is an isomorphism* $\Psi\colon \mathrm{Inn}(p_+^{1+2}) \rtimes \mathrm{Out}(p_+^{1+2}) \to \mathrm{Aut}(p_+^{1+2})$ *defined by* $(c_x, g) \mapsto c_x \circ \phi_g$. *(The semidirect product is formed with respect to the action given in 2.).*
4. *If* $\mathcal{F}$ *is a fusion system over* p_+^{1+2}, *the above results all hold with* $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ *and* $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ *in place of* $\mathrm{Out}(p_+^{1+2})$ *and* $\mathrm{Aut}(p_+^{1+2})$, *respectively.*

Proof. First of all we note that ϕ_g really is an automorphism of p_+^{1+2}, see Proposition 5. We will now prove the statements 1.–4. one by one:
ad 1. We allow ourselves to write $\mathrm{Out}(p_+^{1+2}) = \mathrm{GL}_2(\mathbb{F}_p)$, by use of Proposition 6. With the notation of that proposition, it is clear that $M_{\phi_g} = g$ for all $g \in \mathrm{Out}(p_+^{1+2})$, hence Φ is a splitting. The more difficult part is to prove that Φ is a homomorphism. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \; h = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in \mathrm{Out}(p_+^{1+2})$$

be given. Then

$$\begin{aligned} \phi_g(\phi_h(a)) &= \phi_g(a^q b^s c^{-\frac{1}{2}qs}) = (a^\alpha b^\gamma c^{-\frac{1}{2}\alpha\gamma})^q (a^\beta b^\delta c^{-\frac{1}{2}\beta\delta})^s (c^{\alpha\delta-\beta\gamma})^{-\frac{1}{2}qs} \\ &= \left(a^{\alpha q} b^{\gamma q} c^{-\frac{1}{2}(\alpha\gamma q(q-1)+\alpha\gamma q)}\right) \left(a^{\beta s} b^{\delta s} c^{-\frac{1}{2}(\beta\delta s(s-1)+\beta\delta s)}\right) c^{-\frac{1}{2}(\alpha\delta-\beta\gamma)qs} \\ &= a^{\alpha q+\beta s} b^{\gamma q+\delta s} c^{-\frac{1}{2}(q^2\alpha\gamma+s^2\beta\delta+\alpha\delta qs+\beta\gamma qs)} \end{aligned}$$

But since

$$q^2\alpha\gamma + s^2\beta\delta + \alpha\delta qs + \beta\gamma qs = (\alpha q + \beta s)(\gamma q + \delta s)$$

and

$$gh = \begin{pmatrix} \alpha q + \beta s & \alpha r + \beta t \\ \gamma q + \delta s & \gamma r + \delta t \end{pmatrix}$$

We see that $\phi_g(\phi_h(a)) = \phi_{gh}(a)$. Similar computations show that $\phi_g(\phi_h(b)) = \phi_{gh}(b)$ as well, hence $\phi_g \circ \phi_h = \phi_{gh}$, i.e. Φ is a homomorphism.

ad 2. With $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Out}(p_+^{1+2})$, we have that $\phi_g(c) = c^{\det(g)}$ (we have already used this in the proof of the first part). As $\mathrm{Inn}(p_+^{1+2}) \cong p_+^{1+2}/Z(p_+^{1+2})$ and $Z(p_+^{1+2}) = \langle c \rangle$ the claim follows. Note that the action is $g.c_x = c_{\phi_g(x)}$ for all $g \in \mathrm{Out}(p_+^{1+2})$, $x \in p_+^{1+2}$. This will be used below.

ad 3. From 1. it follows that Ψ as defined is bijective. To see that Ψ is a homomorphism, let (c_x, g), $(c_y, h) \in \mathrm{Inn}(p_+^{1+2}) \rtimes \mathrm{Out}(p_+^{1+2})$ be given. Then

$$\begin{aligned} \Psi((c_x, g) \cdot (c_y, h)) &= \Psi(c_x \circ c_{\phi_g(y)}, gh) \\ &= c_x \circ (\phi_g \circ c_y \circ {\phi_g}^{-1}) \circ \phi_{gh} \\ &= c_x \circ \phi_g \circ c_y \circ \phi_h \\ &= \Psi(c_x, g) \circ \Psi(c_y, h) \end{aligned}$$

ad 4. Note that for each $\phi \in \mathrm{Aut}(p_+^{1+2})$, all elements of its coset in $\mathrm{Aut}(p_+^{1+2})/\mathrm{Inn}(p_+^{1+2}) = \mathrm{Out}(p_+^{1+2})$ are $\mathcal{F}$-morphisms if and only if ϕ is an $\mathcal{F}$-morphism, since all inner automorphisms of p_+^{1+2} are $\mathcal{F}$-morphisms. This allows us to form the needed restrictions of the maps presented in 1.–3. □

The key result of the proposition is that we can recover $\mathrm{Aut}_\mathcal{F}(p_+^{1+2})$ from $\mathrm{Out}_\mathcal{F}(p_+^{1+2})$ for any fusion system $\mathcal{F}$ over p_+^{1+2}. Combining this result with Alperin's Fusion Theorem (theorem 4), we see that $\mathrm{Out}_\mathcal{F}(p_+^{1+2})$ and $\mathcal{F}^{er}$ (the rank 2 elementary abelian subgroups of p_+^{1+2} that are $\mathcal{F}$-radical) uniquely determine $\mathcal{F}$. This is what the uniqueness half of the classification of the saturated fusion systems over p_+^{1+2} is based on. However, the existence half is barely touched upon, and how to construct such fusion systems is only stated implicitly. One chooses candidates for $\mathrm{Out}_\mathcal{F}(p_+^{1+2})$ and $\mathcal{F}^{er}$, and then constructs a fusion system that matches these choices. Most possible choices lead to group fusion systems, but since our interest lies in the exotic examples we need to describe the process of constructing fusion systems over p_+^{1+2} in much greater detail. If $\mathcal{F}$ is a saturated fusion system over p_+^{1+2}, then $\mathrm{Out}_\mathcal{F}(p_+^{1+2})$ has order prime to p, since p_+^{1+2} is fully $\mathcal{F}$-automized. We also note that $\mathrm{Aut}_\mathcal{F}(p_+^{1+2})$ determines the $\mathcal{F}$-conjugacy classes of the rank 2 elementary abelian subgroups of p_+^{1+2}; we just apply Alperin's Fusion Theorem to an $\mathcal{F}$-isomorphism of two distinct elements of $\mathcal{V}$. We can rephrase this and say that the $\mathcal{F}$-conjugacy classes are the orbits under the obvious action of $\mathrm{Aut}_\mathcal{F}(p_+^{1+2})$ on $\mathcal{V}$. In fact, $\mathrm{Out}_\mathcal{F}(p_+^{1+2})$ also acts on $\mathcal{V}$ by $g.V = \phi_g(V)$, and since each $V \in \mathcal{V}$ is normal in p_+^{1+2}, the orbits of these two actions are the same.
We can now describe how every saturated fusion system over p_+^{1+2} can be constructed:

Proposition 11. *Let $O \leqslant \mathrm{GL}_2(\mathbb{F}_p)$ be a subgroup of order prime to p. Let $\mathcal{V}^r \subseteq \mathcal{V}$ be the union of a subset of the orbits of the action of O described in Proposition 10; that is $g.V = \phi_g(V)$, $g \in O$.*
Then one can construct a fusion system $\mathcal{F}_{O,\mathcal{V}^r}$ over p_+^{1+2} which satisfies $\mathrm{Out}_{\mathcal{F}_{O,\mathcal{V}^r}}(p_+^{1+2}) = O$, and that $V \in \mathcal{V}$ is $\mathcal{F}_{O,\mathcal{V}^r}$-radical if and only if $V \in \mathcal{V}^r$.

Proof. Define $A = \mathrm{Inn}(p_+^{1+2}) \rtimes O$ with the action of O on $\mathrm{Inn}(p_+^{1+2})$ being the restriction of the action described in Proposition 10. For each $V \in \mathcal{V}^r$, set $A|_V = \{\alpha|_{V,V} \mid \alpha \in A,\ \alpha(V) = V\}$.

Then define $A_V \leqslant \mathrm{Aut}(V) = GL_2(\mathbb{F}_p)$ as the subgroup $\mathrm{SL}_2(\mathbb{F}_p) \rtimes R$, where $R \leqslant \mathbb{F}_p^*$ is the group of determinants of the elements of $A|_V$. R only depends on the orbit of V. By an A-automorphism we shall mean an element of $A \cup \left(\bigcup_{V\in\mathcal{V}^r} A|_V\right)$.
We now define $\mathcal{F} = \mathcal{F}_{O,\mathcal{V}^r}$ as follows: Given subgroups $P, P' \leqslant p_+^{1+2}$ that are isomorphic as groups, let $\mathrm{Iso}_\mathcal{F}(P, P')$ be the set of all isomorphisms $P \to P'$ of the form

$$\alpha_n|_{P_{n-1},P_n} \circ \ldots \circ \alpha_1|_{P_0,P_1}$$

where $P_0 = P, P_1, \ldots, P_n = P' \leqslant p_+^{1+2}$, and where α_i is an A-automorphism of some $Q_i \in \{p_+^{1+2}\} \cup \mathcal{V}^r$ such that $P_{i-1}, P_i \leqslant Q_i$, and $\alpha_i(P_{i-1}) = P_i$ for $i = 1, \ldots, n$. (I.e. we mimic the morphism structure from the formulation of Alperin's Fusion Theorem). Note that $\mathrm{Iso}_\mathcal{F}(P, P')$ may very well be empty. Next, for arbitrary $P, Q \leqslant p_+^{1+2}$, define

$$\mathrm{Hom}_\mathcal{F}(P, Q) = \bigcup_{P'\leqslant Q} \{\iota_{P',Q} \circ \phi \mid \phi \in \mathrm{Iso}_\mathcal{F}(P, P')\}$$

where $\iota_{P',Q}$ denotes the inclusion of P' into Q. We claim that with these morphisms, $\mathcal{F}$ is a fusion system with $\mathrm{Aut}_\mathcal{F}(p_+^{1+2}) = A$, and $\mathrm{Aut}_\mathcal{F}(V) = A_V$ for all $V \in \mathcal{V}^r$. To prove this claim we need to verify several properties. First we show that $\mathcal{F}$ is closed under composition of morphisms: Let $P, Q, R \leqslant p_+^{1+2}$ and let $\phi \in \mathrm{Hom}_\mathcal{F}(P, Q)$ and $\psi \in \mathrm{Hom}_\mathcal{F}(Q, R)$. There are subgroups

$$P = P_0, P_1, \ldots, P_n = \phi(P), \hat{P}_0 = Q, \hat{P}_1, \ldots, \hat{P}_m = \psi(Q) \leqslant p_+^{1+2}$$

and A-automorphisms $\alpha_1, \ldots, \alpha_n, \hat{\alpha}_1, \ldots, \hat{\alpha}_m$ such that

$$\psi = \iota_{\psi(Q),R} \circ (\hat{\alpha}_m|_{\hat{P}_{m-1},\hat{P}_m} \circ \ldots \circ \hat{\alpha}_1|_{\hat{P}_0,\hat{P}_1}), \quad \text{and} \quad \phi = \iota_{\phi(P),Q} \circ (\alpha_n|_{P_{n-1},P_n} \circ \ldots \circ \alpha_1|_{P_0,P_1})$$

Set $P_{n+1} = \hat{\alpha}_1(P_n)$, and define $P_{n+2}, \ldots, P_{n+m}$ recursively by $P_{n+i} = \hat{\alpha}_i(P_{n+i-1})$. Then

$$\psi\phi = \iota_{\psi\phi(P),R} \circ \hat{\alpha}_m|_{P_{n+m-1},P_{n+m}} \circ \ldots \circ \hat{\alpha}_1|_{P_n,P_{n+1}} \circ \alpha_n|_{P_{n-1},P_n} \circ \ldots \circ \alpha_1|_{P_0,P_1}$$

and so $\psi\phi$ is also an $\mathcal{F}$-homomorphism.
Next we need to show that $\mathrm{Hom}_{p_+^{1+2}}(P, Q) \subseteq \mathrm{Hom}_\mathcal{F}(P, Q)$ for all $P, Q \leqslant p_+^{1+2}$. It is immediate from our definition of the $\mathcal{F}$-homomorphisms that $\mathrm{Aut}_\mathcal{F}(p_+^{1+2}) = A$. In particular $\mathrm{Inn}(p_+^{1+2}) \subseteq \mathrm{Aut}_\mathcal{F}(p_+^{1+2})$. An element of $\mathrm{Hom}_{p_+^{1+2}}(P, Q)$ has the form $\iota_{c_g(P),Q} \circ c_g|_{P,c_g(P)}$, but that is an $\mathcal{F}$-homomorphism, since $c_g \in A$.
To prove the claim, it remains to show that $\mathrm{Aut}_\mathcal{F}(V) = A_V$ for all $V \in \mathcal{V}^r$. We obviously have $\mathrm{Aut}_\mathcal{F}(V) \geqslant A_V$. Now let $\phi \in \mathrm{Aut}_\mathcal{F}(V)$; there are A-automorphisms α_i of subgroups $Q_i \in \{p_+^{1+2}\} \cup \mathcal{V}^r$ for $i = 1, \ldots, n$, and subgroups $P_0, \ldots, P_n \leqslant p_+^{1+2}$ such that $P_{i-1}, P_i \leqslant Q_i$ and $\alpha_i(P_{i-1}) = P_i$, and

$$\phi = \alpha_n|_{P_{n-1},P_n} \circ \ldots \circ \alpha_1|_{P_0,P_1} \tag{8}$$

We prove that $\phi \in A_V$ by induction on n. We know that $\alpha(V) \in \mathcal{V}^r$ for all $\alpha \in A$ and all $V \in \mathcal{V}^r$. Therefore, $P_i \in \mathcal{V}^r$ for all i. If $n = 1$, then $P_0 = P_1 = V$ and either $Q_1 = V$ or $Q_1 = p_+^{1+2}$. In either case, $\alpha_1|_{V,V} \in A_V$.
Now assume $n > 1$. If $\alpha_n|_{P_{n-1},P_n} \in A_V$, then $(\alpha_n|_{P_{n-1},P_n})^{-1} \circ \phi \in A_V$ by induction, hence $\phi \in A_V$. If $\alpha_n|_{P_{n-1},P_n} \notin A_V$, then $\alpha_n \in A$. If also $\alpha_{n-1} \in A$, then

$$\phi = (\alpha_n\alpha_{n-1})|_{P_{n-2},P_n} \circ \alpha_{n-2}|_{P_{n-3},P_{n-2}} \circ \ldots \circ \alpha_1|_{P_0,P_1}$$

and inductively $\phi \in A_V$. If $\alpha_{n-1} \notin A$, then $P_{n-2} = P_{n-1} =: V' \neq V$ and $\alpha_{n-1} \in A_{V'}$. We may then rewrite (8) as follows:

$$\phi = (\alpha_n|_{V',V} \circ \alpha_{n-1} \circ (\alpha_n|_{V',V})^{-1}) \circ \alpha_n|_{V',V} \circ \alpha_{n-2}|_{P_{n-3},V'} \circ \ldots \circ \alpha_1|_{P_0,P_1} \tag{9}$$

Both A_V and $A_{V'}$ are isomorphic to $\mathrm{SL}_2(\mathbb{F}_p) \rtimes R$, and since V and V' are rank two elementary abelian groups we may view $\alpha_n|_{V',V}$ as an element of $\mathrm{GL}_2(\mathbb{F}_p)$. But then $\alpha_n|_{V',V} \circ \alpha_{n-1} \circ (\alpha_n|_{V',V})^{-1}$ is an element of $\mathrm{Aut}_{\mathcal{F}}(V) \leqslant \mathrm{GL}_2(\mathbb{F}_p)$ with the same determinant as α_{n-1}. Hence $\alpha_n|_{V',V} \circ \alpha_{n-1} \circ (\alpha_n|_{V',V})^{-1} \in A_V$. From (9) we conclude inductively that $\phi \in A_V$. We have now covered all possible cases.

To finish the proof of the proposition we have to show that $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) = O$, and that any given $V \in \mathcal{V}$ is $\mathcal{F}$-radical if and only if $V \in \mathcal{V}^r$. The former property clearly holds since $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2}) = A$. Furthermore, each $V \in \mathcal{V}^r$ is $\mathcal{F}$-radical by Lemma 8. Now let $V \in \mathcal{V} \setminus \mathcal{V}^r$ and assume that V is $\mathcal{F}$-radical, i.e. that $\mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$. From our definition of the $\mathcal{F}$-homomorphisms we see that $\mathrm{Aut}_{\mathcal{F}}(V) = \{\alpha|_{V,V} \mid \alpha \in A,\ \alpha(V) = V\}$, which means that every $\mathcal{F}$-automorphism of V extends to an $\mathcal{F}$-automorphism of p_+^{1+2}. If we write $V = \langle c, x\rangle$, the automorphism corresponding to the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p) \leqslant \mathrm{Aut}_{\mathcal{F}}(V)$ is the one given by $c \mapsto cx$, $x \mapsto x$. This automorphism of V extends to an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$. $\alpha^k(c) = c$ if and only if $p \mid k$, hence if α^k is an inner automorphism of p_+^{1+2}, k must be a multiple of p, i.e. the order of the class of α in $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ is a multiple of p. But this contradicts the fact that the order of $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ is prime to p. □

The proof is constructive and provides a procedure to construct fusion systems over p_+^{1+2}. Note that if we start out with a saturated fusion system $\mathcal{F}$ and determine $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and $\mathcal{F}^{er}$ the construction presented above recovers $\mathcal{F}$, i.e. $\mathcal{F} = \mathcal{F}_{\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}),\mathcal{F}^{er}}$; Alperin's Fusion Theorem provides the inclusion '$\subseteq$' while the axioms of the definition of fusion systems provides the other one.
We should also make a warning concerning the uniqueness of the construction: Let $O, O' \leqslant \mathrm{GL}_2(\mathbb{F}_p)$ be isomorphic and of order prime to p, and let $\mathcal{V}^r$ be chosen (w.r.t. the action of O on $\mathcal{V}$). The isomorphism $O \cong O'$ need not provide a way to meaningully choose a subset of $\mathcal{V}$ with respect to the action of O' that should correspond to $\mathcal{V}^r$. Even if we can make a meaningful choice (e.g. if $\mathcal{V}^r = \mathcal{V}$) we don't know whether the resulting fusion systems are isomorphic. However, if O and O' are conjugate in $\mathrm{GL}_2(\mathbb{F}_p)$ via some element g, then we have an element $\phi_g \in \mathrm{Aut}(p_+^{1+2})$ that relates the resulting fusion systems. In fact, ϕ_g induces an isomorphism of these fusion systems, which can easily be checked.
Note also that the proposition makes no mention of saturation.

The classification is performed by inspecting the possible choices of O and $\mathcal{V}^r$. Our interest does not lie in the entire classification, but rather in the examples of exotic fusion systems it provides. We shall therefore just mention the choices which lead to these exotic fusion systems. There are three of them and they all occur at the prime $p = 7$.

4 Exotic fusion systems over p_+^{1+2}

$\mathrm{GL}_2(\mathbb{F}_7)$ contains the element $y = \begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix}$ of order 48 ($= 7^2 - 1$). The twisting $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts on $\langle y\rangle$ by conjugation (explicitly, $xyx = y^7$). The semidirect product $\langle y\rangle \rtimes \langle x\rangle$ with respect to this action is the subgroup $\langle y, x\rangle$. So $\mathrm{GL}_2(\mathbb{F}_7)$ contains a subgroup isomorphic to 48:2 (using the notation of [At]). Such a subgroup is unique up to conjugation in $\mathrm{GL}_2(\mathbb{F}_p)$. $\mathrm{GL}_2(\mathbb{F}_7)$ also contains a subgroup isomorphic to 6^2:2 (i.e. $(6 \times 6) : 2$); e.g. the subgroup consisting of the 36 diagonal matrices and the 36 anti-diagonal matrices (x acts on either by conjugation).
Every maximal subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$ of order prime to p is isomorphic to either 48:2, 6^2:2 or $6S_4$. (One can provide explicit generators of a subgroup isomorphic to $6S_4$). Furthermore any such subgroup is unique up to conjugation in $\mathrm{GL}_2(\mathbb{F}_7)$. The subgroup $\langle y^2, x\rangle$ is isomorphic to 24:2.

A subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$ which is isomorphic to 24:2 must be contained in a maximal subgroup of the form 48:2 since it contains an element of order 24 which neither $6S_4$ or 6^2:2 does. In addition 48:2 contains just a single subgroup which is isomorphic to 24:2 (since the cyclic group of order 48 contains a unique subgroup of order 24). Therefore a subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$ which is isomorphic to 24:2 is unique up to conjugation.
From now on we will use 48:2, 24:2 and 6^2:2 to refer to fixed subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$. 48:2 will be the subgroup $\langle y, x\rangle$ with y and x as defined above, 24:2 will be the subgroup $\langle y^2, x\rangle$, and 6^2:2 will be the subgroup consisting of all diagonal and anti-diagonal matrices.
We now define three fusion systems $\mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$, and $\mathcal{F}_{6^2:2}$ over 7_+^{1+2}, corresponding to the choices $O = 48{:}2,\ 24{:}2,\ 6^2{:}2$ respectively, with $\mathcal{V}^r = \mathcal{V}$ in all three cases.

Proposition 12. *The fusion systems $\mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$ and $\mathcal{F}_{6^2:2}$ are saturated and exotic.*

Proof. We will use Proposition 3 with $\mathcal{C} = \{c\}$ to show that the fusion systems are saturated. Let $\mathcal{F}$ be one of them. We first show that $\mathcal{F}$ satisfies the combination of the first two properties of the proposition: We need to check every element of $\mathcal{F}$ (save the identity) since they all have order 7. Each of the subgroups 6^2:2, 48:2, 24:2 $\leqslant \mathrm{GL}_2(\mathbb{F}_7)$ contain an element of determinant 3: 6^2:2 contains $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, 48:2 contains $y = \begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix}$, and 24:2 $= \langle y^2, x\rangle$ contains y^4x. In each case, we obtain an $\mathcal{F}$-automorphism of 7_+^{1+2} which maps c to c^3. But then c can be mapped to every non-identity element of $\langle c\rangle$ by an $\mathcal{F}$-automorphism of 7_+^{1+2}, since 3 generates $\mathbb{F}_7^*$. As $C_{7_+^{1+2}}(c) = 7_+^{1+2}$, this argument covers all elements of order 7 that are central.
Now let $x \in 7_+^{1+2}$ and assume that x is not central. Then $\langle c, x\rangle \in \mathcal{V}$, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_7) \leqslant \mathrm{Aut}_{\mathcal{F}}(\langle c, x\rangle)$ is an $\mathcal{F}$-automorphism of $\langle c, x\rangle$ which maps x to c. Since $\langle c, x\rangle$ is abelian, we have $\langle c, x\rangle \leqslant C_{7_+^{1+2}}(\langle c, x\rangle)$, but if the inclusion were strict we would have $C_{7_+^{1+2}}(\langle c, x\rangle) = 7_+^{1+2}$, and x would be central. Hence $C_{7_+^{1+2}}(\langle c, x\rangle) = \langle c, x\rangle$, and after a composition with the inclusion homomorphism $\langle c, x\rangle \hookrightarrow 7_+^{1+2}$, we obtain our desired $\mathcal{F}$-homomorphism $C_{7_+^{1+2}}(\langle c, x\rangle) \to 7_+^{1+2}$ which maps x to c.
Next we need to see that the centralizer fusion system $C_{\mathcal{F}}(c)$ is saturated. We will use Proposition 1 to do so. $\langle c\rangle$ must clearly constitute its own $C_{\mathcal{F}}(c)$-conjugacy class, and there is just one $C_{\mathcal{F}}(c)$-automorphism of $\langle c\rangle$. So $\langle c\rangle$ is trivially fully normalized, fully centralized, fully automized and receptive in $C_{\mathcal{F}}(c)$.
If $x \in 7_+^{1+2}$ is non-central, then we just saw that $C_{7_+^{1+2}}(\langle x\rangle) = \langle c, x\rangle$. Furthermore $N_{7_+^{1+2}}(\langle x\rangle) = \langle c, x\rangle$. So $\langle x\rangle$ is fully centralized and fully normalized in $C_{\mathcal{F}}(c)$. $\mathrm{Aut}_{C_{\mathcal{F}}(c)}(\langle x\rangle)$ is a subgroup of $\mathrm{Aut}(\langle x\rangle)$ which is cyclic of order $p-1$. Therefore, $\langle x\rangle$ is also fully $C_{\mathcal{F}}(c)$-automized. Now let $y \in 7_+^{1+2}$ be another non-central element, and assume ϕ is a $C_{\mathcal{F}}(c)$-isomorphism $\langle y\rangle \to \langle x\rangle$. By definition of $C_{\mathcal{F}}(c)$, there is an $\mathcal{F}$-isomorphism $\widetilde{\phi} : \langle c, y\rangle \to \langle c, x\rangle$ which extends ϕ and maps c to c. In particular $\widetilde{\phi}$ is a $C_{\mathcal{F}}(c)$-morphism, and its source is $N_{7_+^{1+2}}(\langle y\rangle)$. We may therefore restrict it to N_ϕ, and we conclude that $\langle x\rangle$ is receptive.
Now let $V \in \mathcal{V}$. We know that $N_{7_+^{1+2}}(V) = 7_+^{1+2}$ and that $C_{7_+^{1+2}}(V) = V$. Therefore every element of $\mathcal{V}$ is fully normalized and fully centralized in $C_{\mathcal{F}}(c)$. The $C_{\mathcal{F}}(c)$-automorphisms of V are all $\mathcal{F}$-automorphisms of V that map c to c. We know that $\mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{SL}_2(\mathbb{F}_7) \rtimes R$ for some subgroup $R \leqslant \mathbb{F}_7^*$, and so $\mathrm{Aut}_{C_{\mathcal{F}}(c)}(V)$ consists of all matrices of the form $\begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}$ with $r \in R$, $s \in \mathbb{F}_7$. There are $7\,|R|$ such matrices, and since $\mathrm{Aut}_{7_+^{1+2}}(V) \cong 7_+^{1+2}/V$ has order 7, we conclude that V is fully automized in $C_{\mathcal{F}}(c)$.
Next we need to show that V is receptive in $C_{\mathcal{F}}(c)$. We do that by showing that each $C_{\mathcal{F}}(c)$-isomorphism with V as target extends to an $C_{\mathcal{F}}(c)$-automorphism of 7_+^{1+2}. In order to do so, we claim that it is enough to show that:

1. Every element of $\mathrm{Aut}_{C_{\mathcal{F}}(c)}(V)$ extends to an element of $\mathrm{Aut}_{C_{\mathcal{F}}(c)}(7_+^{1+2})$.
2. For each $V' \in V^{C_{\mathcal{F}}(c)}$ there is an isomorphism $\psi \in \mathrm{Iso}_{C_{\mathcal{F}}(c)}(V', V)$ which extends to an element of $\mathrm{Aut}_{C_{\mathcal{F}}(c)}(7_+^{1+2})$.

The claim is true because every $C_{\mathcal{F}}(c)$-isomorphism $\phi\colon V' \to V$ can be written as $\phi = \alpha \circ \psi$ for some suitable $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}}(c)}(V)$. And if α and ψ can be extended then the composite of their extension will be an extension of ϕ.
Additionally, if V satisfies the two properties listed above, then so does every $C_{\mathcal{F}}(c)$-conjugate to V; every isomorphism between elements of $V^{C_{\mathcal{F}}(c)}$ factors through V.
Unfortunately, showing that V satisfies the two conditions requires a separate study of each of the fusion systems $\mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$ and $\mathcal{F}_{6^2:2}$. Lemma 13 below contains the information needed, and we use it to handle each of the three centralizer fusion systems in question:

$C_{\mathcal{F}_{6^2:2}}(c)$**:** First we need to see what the $C_{\mathcal{F}_{6^2:2}}(c)$-conjugacy classes of the rank 2 elementary abelian subgroups are. Let $r \in \mathbb{F}_7^*$ and set $x' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}$ and $h = \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}$. Computations show that

$$\begin{aligned} &\phi_{x'}(c) = \phi_g(c) = \phi_h(c) = c, \\ &\phi_{x'}(a) = b, \\ &\phi_g(ab) = (ab^{-r^{-2}})^r c^*, \quad \phi_h(ab) = (ab^{r^2})^{r^{-1}} c^* \end{aligned}$$

The first line shows that $\phi_{x'}, \phi_g, \phi_h \in \mathrm{Aut}_{C_{\mathcal{F}_{6^2:2}}(c)}(7_+^{1+2})$. The second line shows that V_0 and V_7 are $C_{\mathcal{F}_{6^2:2}}(c)$-conjugate. By letting r assume all values of $\mathbb{F}_7^*$, the last line shows that $V_1, V_2, \ldots, V_6$ are $C_{\mathcal{F}_{6^2:2}}(c)$-conjugate. Since $C_{\mathcal{F}_{6^2:2}}(c)$ is a subsystem of $\mathcal{F}_{6^2:2}$ there can be no further relations. These arguments also show that every $V \in \mathcal{V}$ satisfies property 2. listed above, since we have shown the conjugacy relations via restrictions of $C_{\mathcal{F}_{6^2:2}}(c)$-automorphisms of 7_+^{1+2}.

Now let $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}_{6^2:2}}(c)}(V_0)$ be given. Then $\alpha = \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}$ for some $s \in \mathbb{F}_7$ and $r \in \mathbb{F}_7^*$ (since $\mathrm{Aut}_{\mathcal{F}_{6^2:2}}(V_0) = \mathrm{GL}_2(\mathbb{F}_7)$). It is easy to check that $(c_{b^{-s}} \circ \phi_{h^{-1}})|_{V_0,V_0} = \alpha$.

Finally let $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}_{6^2:2}}(c)}(V_1)$ be given. Then $\alpha = \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}$ for some $s \in \mathbb{F}_7$ and $r \in \{\pm 1\}$. Define $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\det(g) = 1$, so $\phi_g \in \mathrm{Aut}_{C_{\mathcal{F}_{6^2:2}}(c)}(7_+^{1+2})$. One checks that $c_{b^{-s}}|_{V_1,V_1} = \alpha$ if $r = 1$, and that $(c_{b^{-(s-1)}} \circ \phi_g)|_{V_1,V_1} = \alpha$ if $r = -1$.

$C_{\mathcal{F}_{48:2}}(c)$**:** First we determine the elements of 48:2 that have determinant 1, (they are exactly the ones that define $C_{\mathcal{F}_{48:2}}(c)$-automorphisms of 7_+^{1+2}). y has determinant 3, which generates $\mathbb{F}_7^*$ and x has determinant -1, so the matrices we are looking for are exactly y^{6k} and $y^{3+6k}x$

for $k = 0, 1, \ldots, 7$. $y^6 = \begin{pmatrix} 0 & 6 \\ 1 & 4 \end{pmatrix}$ and we determine the orbits of ϕ_{y^6} on $\mathcal{V}$ as follows

$$\begin{aligned}
&\phi_{y^6}(a) = b, \\
&\phi_{y^6}(b) = a^6b^4c^2 = (ab^3)^6c^*, \\
&\phi_{y^6}(ab^3) = b(a^6b^4c^2)^3 = a^4b^6c^* = (ab^5)^4c^*, \\
&\phi_{y^6}(ab^5) = b(a^6b^4c^2)^5 = a^2c^*; \\
&\phi_{y^6}(ab) = b(a^6b^4c^2) = a^6b^5c^2 = (ab^2)^6c^*, \\
&\phi_{y^6}(ab^2) = b(a^6b^4c^2)^2 = a^5b^2c^* = (ab^6)^5c^*, \\
&\phi_{y^6}(ab^6) = b(a^6b^4c^2)^6 = ab^4c^*
\end{aligned}$$

So there are two orbits of ϕ_{y^6}: $\{V_0, V_7, V_3, V_5\}$ and $\{V_1, V_2, V_6, V_4\}$.

$y^3x = \begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}$ and we see that $\phi_{y^3x}(a) = a^2bc^6 = (ab^4)^2c^*$, which shows that V_0 and V_4 are $C_{\mathcal{F}_{48:2}}(c)$-conjugate. Therefore all rank 2 elementary abelian subgroups of 7_+^{1+2} are $C_{\mathcal{F}_{48:2}}(c)$-conjugate, and they all have property 2. by the same argument as above.

The exact same argument as in the case of $C_{\mathcal{F}_{6^2:2}}(c)$ shows that V_1 has property 1.

$C_{\mathcal{F}_{24:2}}(c)$**:** We no longer have the element y^3x at our disposal, since 24:2 is the subgroup of 48:2 generated by y^2 and x. We still have y^6, so the $C_{\mathcal{F}_{24:2}}(c)$-conjugacy classes of $\mathcal{V}$ are $\{V_0, V_7, V_3, V_5\}$ and $\{V_1, V_2, V_6, V_4\}$. In the arguments above we only used the matrix $-I = y^{24}$ and elements of $\mathrm{Inn}(7_+^{1+2})$ to show that V_1 had property 2. as an object of $C_{\mathcal{F}_{48:2}}(c)$, so since $\mathrm{Aut}_{C_{\mathcal{F}_{24:2}}(c)}(V_1) = \mathrm{Aut}_{C_{\mathcal{F}_{48:2}}}(c)(V_1)$, V_1 still has this property.

Now let $\alpha = \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}$ be a $C_{\mathcal{F}_{24:2}}(c)$-automorphism of V_0. We see that $\alpha = c_{b^{-s}}|_{V_0,V_0}$ if $r = 1$, and $\alpha = (c_{b^s} \circ \phi_{y^{24}})|_{V_0,V_0}$ if $r = -1$. Hence in $C_{\mathcal{F}_{24:2}}(c)$ every automorphism of V_0 extends to an automorphism of 7_+^{1+2}.

Finally we note that 7_+^{1+2} is fully normalized, fully centralized and receptive in $C_{\mathcal{F}}(c)$ in all three cases (this is trivial). That 7_+^{1+2} is fully automized in $C_{\mathcal{F}_{24:2}}(c)$ means that $\mathrm{Aut}_{C_{\mathcal{F}_{24:2}}(c)}/\mathrm{Inn}(7_+^{1+2})$ has order prime to p. But $\mathrm{Aut}_{C_{\mathcal{F}_{24:2}}(c)}(7_+^{1+2})$ is a subgroup of $\mathrm{Aut}_{\mathcal{F}}(7_+^{1+2})$, and $\mathrm{Aut}_{\mathcal{F}}(7_+^{1+2})/\mathrm{Inn}(7_+^{1+2}) = \mathrm{Out}_{\mathcal{F}}(7_+^{1+2})$ which was chosen to have order prime to p.
Having now been through all subgroups of 7_+^{1+2} we conclude that $C_{\mathcal{F}}(c)$ is saturated for $\mathcal{F} = \mathcal{F}_{6^2:2}, \mathcal{F}_{48:2}, \mathcal{F}_{24:2}$, by use of Proposition 1. This finishes the proof of the saturation of the three fusion systems.

That they are exotic fusion systems requires a proof of non-existence of finite groups G with 7_+^{1+2} as Sylow-7-subgroup such that the group fusion system $\mathcal{F}_{7_+^{1+2}}(G)$ is isomorphic as a fusion system to $\mathcal{F}_{6^2:2}, \mathcal{F}_{48:2}$ or $\mathcal{F}_{24:2}$. The proof relies on the classification of the finite simple groups and will be omitted. See [RV, 4.17] for details. □

We have already used the following lemma in the proof above. It will also be used to compute the Euler characteristics of the fusion systems $\mathcal{F}_{6^2:2}, \mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$, and their outer quotients.

Lemma 13. *In the fusion systems $\mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$ and $\mathcal{F}_{6^2:2}$ the (fusion) conjugacy classes of the rank 2 elementary abelian subgroups of 7_+^{1+2} and their automorphism groups are as presented in the table below.*

$\mathcal{F}$	*$\mathcal{F}$-conjugacy classes of $\mathcal{V}$*	*$\mathcal{F}$-automorphisms*
$\mathcal{F}_{48:2}$	$\{V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$	$\mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$
$\mathcal{F}_{24:2}$	$\{V_0, V_3, V_5, V_7\}$, $\{V_1, V_2, V_4, V_6\}$	$\mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$, $\mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$ *resp.*
$\mathcal{F}_{6^2:2}$	$\{V_0, V_7\}$, $\{V_1, V_2, V_3, V_4, V_5, V_6\}$	$\mathrm{GL}_2(\mathbb{F}_7)$, $\mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$ *resp.*

Proof. Recall that the action of $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2})$ determines the $\mathcal{F}$-conjugacy classes of the rank 2 elementary abelian subgroups. The automorphism groups are determined as in the proof of Lemma 9. Conjugate subgroups obviously have the same automorphisms.

$\mathcal{F}_{48:2}$**:** Computations show that the lowest positive power of y for which $\phi_y(V_0) = V_0$ is $y^8 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (the criterion is that the lower left entry should be 0). ϕ_y defines a permutation of the set $\mathcal{V} = \{V_0, \ldots, V_7\}$, and this permutation must have order 8, so it is an 8-cycle. This means that all the elements of $\mathcal{V}$ are $\mathcal{F}_{48:2}$-conjugate. To see that $\mathrm{Aut}_{\mathcal{F}_{48:2}}(V_0) = \mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$ we need to consider those elements of 48:2 whose induced automorphisms of 7_+^{1+2} map V_0 to itself. The elements of 48:2 are y^i and $y^i x$ for $i = 0, \ldots, 47$. The remarks made so far show that $\phi_{y^i}(V_0) = V_0$ exactly when $8|i$. We have that $\phi_{y^8}(c) = c^2$ and $\phi_{y^8}(a) = a^3$ which means that $\phi_{y^8}|_{V_0,V_0}$ has the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ with determinant -1. The other automorphisms of the form ϕ_{y^i} we need to consider are just powers of ϕ_{y^8}, so their determinants when restricted to automorphisms of V_0 are just powers of -1.

Next we need to consider the automorphisms of the form $\phi_{y^i x}$ that map V_0 to V_0. Since ϕ_x maps a to b, the requirement is that ϕ_{y^i} should map b to a power of a. ϕ_{y^2} has this property, hence $\phi_{y^i x}(V_0) = V_0$ exactly if $i \equiv 2 \pmod 8$. We have that $\phi_{y^2 x}(c) = c^5$ and $\phi_{y^2 x}(a) = a^3$, so $\phi_{y^2 x}|_{V_0,V_0} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ with determinant 1. The other automorphisms we need to consider have the form $\phi_{y^{2+8k}x}$, $k = 1, \ldots, 5$. The decomposition $\phi_{y^{2+8k}x} = \phi_{y^8}^k \circ \phi_{y^2 x}$ allow us to compute the remaining determinants. They are $(-1)^k$, $k = 1, \ldots, 5$.

These arguments show that $R = \{\pm 1\}$ in the identification $\mathrm{Aut}_{\mathcal{F}_{48:2}} = \mathrm{SL}_2(\mathbb{F}_7) \rtimes R$.

$\mathcal{F}_{24:2}$: The analysis will be very similar to the one above, since 24:2 is a subgroup of 48:2. The action of ϕ_{y^2} on $\mathcal{V}$ has orbits $\{V_0, V_3, V_5, V_7\}$ and $\{V_1, V_2, V_4, V_6\}$, while ϕ_x has orbits $\{V_0, V_7\}$, $\{V_1\}$, $\{V_2, V_4\}$, $\{V_3, V_5\}$ and $\{V_6\}$. So the $\mathcal{F}_{24:2}$-conjugacy classes are as in the table.

$\mathrm{Aut}_{\mathcal{F}_{24:2}}(V_0) = \mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$ by the exact same arguments as above. $\mathrm{Aut}_{\mathcal{F}_{24:2}}(V_1)$ can be determined by a similar analysis. The elements of 24:2 whose induced automorphisms of 7_+^{1+2} map V_1 to V_1 are y^{8k} and $y^{8k}x$. $\phi_{y^8}(c) = c^2$ and $\phi_{y^8}(ab) = (ab)^3 c^3$ meaning that $\phi_{y^8}|_{V_1,V_1} = \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}$, which has determinant -1. $\phi_x(c) = c^{-1}$ while $\phi_x(ab) = ab$, so $\phi_x|_{V_1,V_1} = \begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}$ with determinant -1. We conclude that $\mathrm{Aut}_{\mathcal{F}_{24:2}}(V_1) = \mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$ as well.

$\mathcal{F}_{6^2:2}$: The matrix $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in 6^2{:}2$, defines the automorphism $\phi_x \in \mathrm{Aut}_{\mathcal{F}_{6^2:2}}$ which interchanges a and b, i.e. $\phi_x(a) = b$, $\phi_x(b) = a$. So V_0 and V_7 are $\mathcal{F}_{6^2:2}$-conjugate; we write $V_0 \sim V_7$. Every diagonal matrix will fix V_0 and V_7 while every anti-diagonal matrix interchanges V_0 and V_7, hence $\{V_0, V_7\}$ is an $\mathcal{F}_{6^2:2}$-conjugacy class.

The matrix $g = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in 6^2 : 2$, $i \neq 0$, gives $\phi_g(ab) = ab^i$, which shows that $V_1 \sim V_2 \sim V_3 \sim V_4 \sim V_5 \sim V_6$. Hence the $\mathcal{F}_{6^2:2}$-conjugacy classes of the elementary abelian subgroups of 7_+^{1+2} of rank 2 are as in the table.

To see that $\mathrm{Aut}_{\mathcal{F}_{6^2:2}}(V_0) = \mathrm{GL}_2(\mathbb{F}_7)$ we just need to find a $g \in 6^2{:}2$ such that ϕ_g restricts to an automorphism of V_0 with determinant 3 or 5, the generators of $\mathbb{F}_7^*$. Take $g = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Then $\phi_g(c) = c^3$ and $\phi_g(a) = a$, and so $\phi_g|_{V_0,V_0} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

To se that $\mathrm{Aut}_{\mathcal{F}_{6^2:2}}(V_1) = \mathrm{SL}_2(\mathbb{F}_7) \rtimes \{\pm 1\}$, let $g = \begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix}$, $h = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix} \in 6^2 : 2$ be given. We see that

$$\begin{aligned} \phi_g(ab) = a^q b^r, \quad \phi_g(c) = c^{qr}, \\ \phi_h(ab) = a^s b^t, \quad \phi_g(c) = c^{-st} \end{aligned}$$

so these automorphisms map V_1 to V_1 exactly when $q = r$ and $s = t$. Assume now that $q = r$ and $s = t$. Computations show that $a^q b^q = (ab)^q c^{4q(q-1)}$, and $a^s b^s = (ab)^s c^{4s(s-1)}$, and we get that

$$\phi_g|_{V_1,V_1} = \begin{pmatrix} q^2 & 4q(q-1) \\ 0 & q \end{pmatrix} \quad \text{and} \quad \phi_g|_{V_1,V_1} = \begin{pmatrix} -s^2 & 4s(s-1) \\ 0 & s \end{pmatrix}$$

with determinants q^3 and $-s^3 = (-s)^3$. This means that the determinants of the elements of $\mathrm{Aut}_{\mathcal{F}_{6^2:2}}(V_1) \leqslant \mathrm{GL}_2(\mathbb{F}_7)$ are exactly the cubes in $\mathbb{F}_7^*$, that is $\{\pm 1\}$.

□

5 Euler characteristics of finite categories

Definition. *Let $\mathcal{C}$ be a finite category (i.e. a category with a finite number of objects), and define a square matrix $\zeta(\mathcal{C})$ whose rows and columns are indexed by $\mathrm{Ob}(\mathcal{C})$ by setting $\zeta(\mathcal{C})_{a,b} = |\mathcal{C}(a,b)|$ for all pairs of objects $a, b \in \mathrm{Ob}(\mathcal{C})$.*
A weighting for $\mathcal{C}$ is a map $k^\bullet \colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{Q}$ which for all $a \in \mathrm{Ob}(\mathcal{C})$ satisfies

$$\sum_{b \in \mathrm{Ob}(\mathcal{C})} |\mathcal{C}(a,b)|\, k^\bullet(b) = 1$$

A coweighting for $\mathcal{C}$ is a map $k_\bullet \colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{Q}$ which for all $b \in \mathrm{Ob}(\mathcal{C})$ satisfies

$$\sum_{a \in \mathrm{Ob}(\mathcal{C})} k_\bullet(a)\, |\mathcal{C}(a,b)| = 1 \tag{10}$$

We will write k^b instead of $k(b)$ when $k^\bullet$ is a weighting of $\mathcal{C}$, and k_a instead of $k(a)$ when $k_\bullet$ is a coweighting of $\mathcal{C}$, and consider $(k^b)_{b \in \mathrm{Ob}(\mathcal{C})}$ as a column vector and $(k_a)_{a \in \mathrm{Ob}(\mathcal{C})}$ as a row vector. With this notation the requirements of the definition can be rewritten to say that

$$\zeta(\mathcal{C})(k^b) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \text{and} \quad (k_a)\zeta(\mathcal{C}) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \tag{11}$$

If $k^\bullet$ is weighting of $\mathcal{C}$ and $k_\bullet$ is a coweighting of $\mathcal{C}$ we see that

$$\begin{aligned} \sum_{b \in \mathrm{Ob}(\mathcal{C})} k^b &= \sum_{b \in \mathrm{Ob}(\mathcal{C})} \sum_{a \in \mathrm{Ob}(\mathcal{C})} k_a\, |\mathcal{C}(a,b)|\, k^b \\ &= \sum_{a \in \mathrm{Ob}(\mathcal{C})} \sum_{b \in \mathrm{Ob}(\mathcal{C})} k_a\, |\mathcal{C}(a,b)|\, k^b = \sum_{a \in \mathrm{Ob}(\mathcal{C})} k_a \sum_{b \in \mathrm{Ob}(\mathcal{C})} |\mathcal{C}(a,b)|\, k^b = \sum_{a \in \mathrm{Ob}(\mathcal{C})} k_a \end{aligned}$$

Hence the sum of the values of every weighting and coweighting of $\mathcal{C}$ is the same. This leads to the following definition:

Definition. *If a finite category $\mathcal{C}$ has both a weighting $k^\bullet$ and a coweighting $k_\bullet$, $\mathcal{C}$ is said to have Euler characteristic, in which case its Euler characteristic is defined as the sum*

$$\chi(\mathcal{C}) = \sum_{b \in \mathrm{Ob}(\mathcal{C})} k^b = \sum_{a \in \mathrm{Ob}(\mathcal{C})} k_a$$

Note that a finite category need not have Euler characteristic, let alone weightings or coweightings. If $\zeta(\mathcal{C})$ is invertible (11) defines a unique weighting and coweighting of $\mathcal{C}$, so $\mathcal{C}$ has Euler characteristic, and the Euler characteristic equals the sum of all entries of $\zeta(\mathcal{C})^{-1}$. (In our definition of $\zeta(\mathcal{C})$ we implicitly assumed that we were given a total order of the objects of $\mathcal{C}$. However, the definitions of weightings, coweightings, and (consequently) Euler characteritics are defined independently os such an order).

When computing the Euler characteristic of a category $\mathcal{C}$, it may be easier to work with the category 'up to of isomorphism objects': Define $[\mathcal{C}]$ as the category whose objects are the isomorphism classes of objects of $\mathcal{C}$. Then fix representatives $c_1, \ldots, c_n$ of the isomorphism classes and set $[\mathcal{C}]([c_i],[c_j]) = \mathcal{C}(c_i, c_j)$. One can think of $[\mathcal{C}]$ as the full subcategory of $\mathcal{C}$ generated by $c_1, \ldots, c_n$; that category is isomorphic to $[\mathcal{C}]$. The next proposition relates weightings, coweightins and Euler characteristics of $\mathcal{C}$ and $[\mathcal{C}]$.

Proposition 14. *[JM, 2.13]. Let $\mathcal{C}$ be a finite category for which $\zeta([\mathcal{C}])$ is invertible, and let $k^{[\bullet]}$ and $k_{[\bullet]}$ denote the unique weighting and coweighting of $[\mathcal{C}]$. Then $\mathcal{C}$ has a weighting $k^\bullet$ and a coweighting $k_\bullet$ given by*

$$k^b = \frac{1}{|[b]|} k^{[b]}, \quad \textit{respectively} \quad k_a = \frac{1}{|[a]|} k_{[a]}$$

Furthermore $\mathcal{C}$ has Euler characteristic and $\chi(\mathcal{C}) = \chi([\mathcal{C}])$.

Proof. Let $a \in \mathrm{Ob}(\mathcal{C})$ be given. The computation

$$1 = \sum_{[b] \in \mathrm{Ob}([\mathcal{C}])} |[\mathcal{C}]([a],[b])|\, k^{[b]} = \sum_{b \in \mathrm{Ob}(\mathcal{C})} \frac{1}{|[b]|} |\mathcal{C}(a,b)|\, k^{[b]} = \sum_{b \in \mathrm{Ob}(\mathcal{C})} |\mathcal{C}(a,b)|\, k^b$$

shows that $k^\bullet$ is a weighting of $\mathcal{C}$. A similar computation shows that $k_\bullet$ is a coweighting of $\mathcal{C}$. Hence $\mathcal{C}$ has Euler characteristic and clearly $\chi(\mathcal{C}) = \chi([\mathcal{C}])$. □

Proposition 15. *Every fusion system has Euler characteristic.*

Proof. Let S be a finite p-group, and $\mathcal{F}$ a fusion system over S. We prove that $\zeta([\mathcal{F}])$ is invertible. Let $\{1\} = P_1, P_2, \ldots, P_n = S$ be representatives of the $\mathcal{F}$-conjugacy classes such that $|P_1| \leqslant |P_2| \leqslant \ldots \leqslant |P_n|$. Consider $\zeta([\mathcal{F}])$ with respect to this order of the objects of $[\mathcal{F}]$. There are clearly no $\mathcal{F}$-morphisms $P_j \to P_i$ when $j > i$, which means that all entries of $\zeta([\mathcal{F}])$ below the diagonal are 0. The diagonal entries are $|\mathrm{Aut}_{\mathcal{F}}(P_i)|$, $i = 1, \ldots, n$, none of which are 0. Hence $\det(\zeta([\mathcal{F}])) \neq 0$. □

If $\mathcal{C}$ has an initial object 1, the Kroenecker delta $\delta(\cdot, 1)$ defines a coweighting of $\mathcal{C}$, since the sum 10 reduces to the single term $|\mathcal{C}(1,b)|$ which equals 1 for all $b \in \mathrm{ob}(\mathcal{C})$. In particular, $\chi(\mathcal{F}) = 1$ for a fusion system $\mathcal{F}$, since the trivial group is an initial object. We therefore define $\mathcal{F}^*$ as the full subcategory generated by all subgroups of S except the trivial group. Our interest lies in the Euler characteristic of $\mathcal{F}^*$. In the proof of Proposition 15, ignoring the trivial group just eliminates the first row and column of $\zeta([\mathcal{F}])$, so $\zeta([\mathcal{F}^*])$ is also invertible, and then $\chi(\mathcal{F}^*) = \chi([\mathcal{F}^*])$ by Proposition 14. With a slight abuse of language we shall from now on refer to $\chi(\mathcal{F}^*)$ when speaking of the Euler characteristic of $\mathcal{F}$.

From $\mathcal{F}$ we also derive another category.

Definition. *Let S be a finite p-group and let $\mathcal{F}$ be a fusion system over S. The exterior quotient of $\mathcal{F}^*$, denoted $\widetilde{\mathcal{F}}^*$, is the category with objects* $\mathrm{Ob}(\widetilde{\mathcal{F}}^*) = \mathrm{Ob}(\mathcal{F}^*)$, *and whose morphisms are defined as follows: Given a pair of objects $1 \neq P, Q \leqslant S$, Q acts on $\mathcal{F}^*(P,Q)$ by composing with c_q, $q \in Q$, on the left. We define $\widetilde{\mathcal{F}}^*(P,Q)$ as the orbits under this action, i.e.*

$$\widetilde{\mathcal{F}}^*(P,Q) = Q\backslash \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

Composition of morphisms in $\mathcal{F}$ induces the composition of morphisms in $\widetilde{\mathcal{F}}^*$.

It is straightforward to check that composition of morphisms in $\widetilde{\mathcal{F}}^*$ is well-defined. If $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $\psi_1, \psi_2 \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ such that $[\phi_1] = [\phi_2]$ and $[\psi_1] = [\psi_2]$ in $\widetilde{\mathcal{F}}^*$, there is $q \in Q$ and $r \in R$ such that $c_q\phi_1\phi_2$ and $c_r\psi_1 = \psi_2$. But then

$$\psi_2\psi_1 = c_r\phi_2 c_q\phi_1 = c_{r\phi_2(q)}\phi_2\phi_1,$$

hence $[\psi_2\psi_1] = [\phi_2\phi_1]$. The argument in the proof of Proposition 15 works for $[\widetilde{\mathcal{F}}^*]$ as well, showing that $\widetilde{\mathcal{F}}^*$ has Euler characteristic.

Theorem 16. *[JM, 3.6]. Let G be a finite group and S a Sylow-p-subgroup of G. Then*

$$\chi(\mathcal{F}_S(G)^*) = \chi(\widetilde{\mathcal{F}}_S(G)^*)$$

The proof makes use of the underlying group G. In general it was not known whether $\chi(\mathcal{F}^*) = \chi(\widetilde{\mathcal{F}}^*)$ for all fusion systems. Using the method provided by Proposition 14 to compute $\chi(\mathcal{F}^*)$ for a concrete example of a fusion system can be somewhat tedious. [JM, 7.3] provides a formula for computing $\chi(\mathcal{F}^*)$ once one knows the $\mathcal{F}$-morphisms of the elementary abelian subgroups of S. The formula contains the Möbius function μ defined on all finite groups by setting $\mu(1) = 1$ and recursively $0 \sum_{1 \leqslant H \leqslant G} \mu(H) = 0$ for every finite group $G \neq 1$. The key property of μ that we need is the following: If P is a p-group, then $\mu(P) = 0$ unless P is elementary abelian; if $P \cong (\mathbb{Z}/p\mathbb{Z})^n$, then $\mu(P) = (-1)^n p^{\binom{n}{2}}$.

Theorem 17. *[JM, 7.3]. Let S be a finite p-group and $\mathcal{F}$ a fusion system over S. $\mathcal{F}^*$ has a coweighting $k_\bullet$ defined by*

$$k_P = \frac{-\mu(P)}{|\mathcal{F}^*(P,S)|} = \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|\,|P^{\mathcal{F}}|}, \quad \textit{for all } 1 \neq P \leqslant S.$$

The Euler characteristic of $\mathcal{F}^*$ is therefore

$$\chi(\mathcal{F}^*) = \sum_{1 \neq P \leqslant S} \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|\,|P^{\mathcal{F}}|} = \sum_{[P] \in \mathrm{Ob}([\mathcal{F}^*])} \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|}$$

Proof. Every $\mathcal{F}$-homomorphism can be factored as an isomorphism followed by an inclusion. Consequently

$$|\mathrm{Hom}_{\mathcal{F}}(P,Q)| = \sum_{P' \in P^{\mathcal{F}},\, P' \leqslant Q} \left|\mathrm{Iso}_{\mathcal{F}}(P,P')\right| = |\mathrm{Aut}_{\mathcal{F}}(P)|\,\left|\{P' \in P^{\mathcal{F}} \mid P' \leqslant Q\}\right|$$

for all pairs $P, Q \in \mathcal{F}^*$. We can now verify that $k_\bullet$ is a coweighting of $\mathcal{F}^*$; given $Q \in \mathcal{F}^*$ we see that

$$\begin{aligned}\sum_{P \in \mathcal{F}^*} \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|\,|P^{\mathcal{F}}|}\,|\mathrm{Hom}_{\mathcal{F}}(P,Q)| &= \sum_{P \in \mathcal{F}^*} -\mu(P)\frac{\left|\{P' \in P^{\mathcal{F}} \mid P' \leqslant Q\}\right|}{|P^{\mathcal{F}}|} \\ &= \sum_{[P] \in [\mathcal{F}^*]} -\mu(P)\left|\{P' \in P^{\mathcal{F}} \mid P' \leqslant Q\}\right| \\ &= \sum_{1 \neq P \leqslant Q} -\mu(P) = 1\end{aligned}$$

where the last equality follows from the definition of μ. □

It turns out that $k_\bullet$ is a coweighting for the exterior quotient $\widetilde{\mathcal{F}}^*$ as well. To prove this we need a few lemmas

Lemma 18. *Let $\mathcal{C}$ and $\mathcal{D}$ are categories with Euler characteristics. If there is an adjunction $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$, then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

Proof. Choose a coweighting $k_\bullet$ of $\mathcal{C}$ and a weighting $k^\bullet$ of $\mathcal{D}$. Then

$$\begin{aligned}\sum_{a\in\mathcal{C}} k_a &= \sum_{a\in\mathcal{C}} k_a \left(\sum_{b\in\mathcal{D}} |\mathcal{D}(R(a),b)|\, k^b\right) = \sum_{a\in\mathcal{C}}\sum_{b\in\mathcal{D}} k_a\, |\mathcal{D}(R(a),b)|\, k^b \\ &= \sum_{b\in\mathcal{D}} \left(\sum_{a\in\mathcal{C}} k_a\, |\mathcal{C}(a, L(b))|\right) k^b = \sum_{b\in\mathcal{D}} k^b\end{aligned}$$

□

We need a little more terminology for the next lemma. If H and K are subgroups of a group G, we define the *transporter set* as $N_G(H,K) = \{g \in G \mid {}^gH \leqslant K\}$. The elements of $N_G(H,K)$ are exactly the elements of G that induce the homomorphisms $\mathrm{Hom}_G(H,K)$. $C_G(H)$ acts on $N_G(H,K)$ by right multiplication, and it is easily seen that two elements of $N_G(H,K)$ are in the same orbit exactly if they induce the same homomorphism $H \to K$. This provides a bijection $\mathrm{Hom}_G(H,K) \leftrightarrow N_G(H,K)/C_G(H)$, and we write $\mathrm{Hom}_G(H,K) \approx N_G(H,K)/C_G(H)$

Lemma 19. *[JM, 5.1]. Suppose S is a non-trivial finite p-group. Then $\chi(\mathcal{F}_S(S)^*) = 1$.*

Proof. Let Z^+ denote the full subcateogry of $\mathcal{F}_S(S)^*$ generated by all nonidentity subgroups of S that contain $Z(S)$. Since S is a non-trivial p-group, its center is not the trivial group, i.e. $Z(S) \in Z^+$. Let $R\colon Z^+ \to \mathcal{F}_S(S)^*$ denote the inclusion functor.
If P and Q are subgroups of S, then so are $PZ(S)$ and $QZ(S)$, and every $\mathcal{F}_S(S)$-homomorphism $\phi\colon P \to Q$ has the form $c_s|_{P,Q}$ for some $s \in S$. Define a functor $L\colon \mathcal{F}_S(S)^* \to Z^+$ by $L(P) = PZ(S)$, and $L(c_s|_{P,Q}) = c_s|_{PZ(S),QZ(S)}$, for all $P, Q \in \mathcal{F}_S(S)^*$ and all $s \in N_S(P,Q)$. We claim that L is left adjoint to R. Let $P, Q \leqslant S$ be such that $Q \neq 1$, and $Z(S) \leqslant P$. Then

$$\begin{aligned}Z^+(L(Q),P) &= Z^+(QZ(S),P) = \mathcal{F}_S(S)^*(QZ(S),P) \\ &\approx N_S(QZ(S),P)/C_S(QZ(S)) = N_S(Q,R(P))/C_S(Q) \\ &\approx \mathcal{F}_S(S)^*(Q,R(P))\end{aligned}$$

It is not hard to see that the bijection is natural in both P and Q; all morphisms are restrictions of inner automorphisms of S. (And strictly speaking, we do not need naturality to apply the above lemma). Now $\chi(\mathcal{F}_S(S)^*) = \chi(Z^+)$ by Lemma 19. But $Z(S)$ is an initial object of Z^+, so $\chi(Z^+) = 1$. □

The next lemma is purely group theoretical in its appearance, but its proof make use of Theorem 17.

Lemma 20. *Let S be a non-trivial finite p-group. Then*

$$\sum_{1\neq P\leqslant S} -\mu(P)\, |C_S(P)| = |S|$$

Proof. Theorem 17 tells us that

$$k_P = \frac{-\mu(P)}{\left|\mathrm{Aut}_{\mathcal{F}_S(S)}(P)\right| \left|P^{\mathcal{F}_S(S)}\right|}, \quad P \in \mathcal{F}_S(S)^*$$

is a coweighting of $\mathcal{F}_S(S)^*$. In general $\left|\mathrm{Aut}_{\mathcal{F}_S(S)}(P)\right| \left|P^{\mathcal{F}_S(S)}\right| = \left|\mathrm{Hom}_{\mathcal{F}_S(S)}(P,S)\right|$, and since $\mathcal{F}_S(S)$ is a group fusion system

$$\mathrm{Hom}_{\mathcal{F}_S(S)}(P,S) \approx N_S(P,S)/C_S(P) = S/C_S(P)$$

We can therefore compute the Euler characteristic of $\mathcal{F}_S(S)^*$ as

$$\chi(\mathcal{F}_S(S)^*) = \sum_{1 \neq P \leqslant S} \frac{-\mu(P)\,|C_S(P)|}{|S|}$$

Applying Lemma 19 yields the desired result. □

We are now able to prove that $\widetilde{\mathcal{F}}^*$ has the same coweighting as $\mathcal{F}^*$ defined in Theorem 17.

Theorem 21. *Let S be a finite p-group, and $\mathcal{F}$ a fusion system over S. $\widetilde{\mathcal{F}}^*$ has a coweighting $k_\bullet$ defined by*

$$k_P = \frac{-\mu(P)}{|\mathcal{F}^*(P,S)|}, \quad \text{for all } 1 \neq P \leqslant S.$$

The Euler characteristic of $\widetilde{\mathcal{F}}^*$ is therefore

$$\chi(\widetilde{\mathcal{F}}^*) = \sum_{1 \neq P \leqslant S} \frac{-\mu(P)}{|\mathcal{F}^*(P,S)|} = \sum_{[P] \in \mathrm{Ob}([\mathcal{F}^*])} \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|}$$

Proof. To verify that $k_\bullet$ is a coweighting we need to determine the order of $\widetilde{\mathcal{F}}^*(P,Q) = Q\backslash\mathcal{F}^*(P,Q)$. We can use Burnside's counting lemma to do so:

$$|\widetilde{\mathcal{F}}^*(P,Q)| = \frac{1}{|Q|} \sum_{\phi \in \mathcal{F}^*(P,Q)} {}_\phi Q \tag{12}$$

where ${}_\phi Q$ is the isotropy subgroup of ϕ for the action of Q. We see that

$${}_\phi Q = \{q \in Q \mid c_q \circ \phi = \phi\} = C_Q(\phi(P))$$

Therefore, the terms in the sum above only depend on $\phi(P)$. The possibilities for $\phi(P)$ are exactly the elements of $X := \{P' \in [P] \mid P' \leqslant Q\}$, and if $P' \in X$ there are exactly $|\mathcal{F}^*(P,P')| = |\mathrm{Iso}_{\mathcal{F}}(P,P')| = |\mathrm{Aut}_{\mathcal{F}}(P)|$ $\mathcal{F}^*$-morphisms $\phi\colon P \to Q$ with $\phi(P) = P'$. Combining these observations with (12) we get

$$|\widetilde{\mathcal{F}}^*(P,Q)| = \frac{1}{|Q|} \sum_{P' \in [P],\, P' \leqslant Q} |\mathrm{Aut}_{\mathcal{F}}(P)|\,\left|C_Q(P')\right|$$

We now verify that $k_\bullet$ is a coweighting. Let $Q \in \widetilde{\mathcal{F}}^*$ be given. Then

$$\begin{aligned}
\sum_{P \in \widetilde{\mathcal{F}}^*} k_P |\widetilde{\mathcal{F}}^*(P,S)| &= \sum_{1 \neq P \leqslant S} \frac{-\mu(P)}{\mathcal{F}^*(P,S)} \frac{1}{|Q|} \sum_{P' \in [P],\, P' \leqslant Q} |\mathrm{Aut}_{\mathcal{F}}(P)|\,\left|C_Q(P')\right| \\
&= \sum_{1 \neq P \leqslant S} \frac{-\mu(P)}{|[P]|} \frac{1}{|Q|} \sum_{P' \in [P],\, P' \leqslant Q} \left|C_Q(P')\right| \\
&= \frac{1}{|Q|} \sum_{[P] \in [\mathcal{F}^*]} \sum_{P' \in [P],\, P' \leqslant Q} -\mu(P)\,\left|C_Q(P')\right| \\
&= \frac{1}{|Q|} \sum_{1 \neq P' \leqslant Q} -\mu(P')\,\left|C_Q(P')\right| = 1
\end{aligned}$$

The last equality follows from Lemma 20. □

As a corollary to Theorem 17 and Theorem 21 we get a generalization of Theorem 16:

Corollary 22. *Let $\mathcal{F}$ be a fusion system. Then $\chi(\mathcal{F}^*) = \chi(\widetilde{\mathcal{F}}^*)$.*

Example. We compute the Euler characteristics of the three exotic fusion systems $\mathcal{F}_{48:2}$, $\mathcal{F}_{24:2}$, and $\mathcal{F}_{6^2:2}$ over 7_+^{1+2}. These fusion systems are described in section 4. In the proof of Proposition 12 we saw that every non-identity element of 7_+^{1+2} is $\mathcal{F}$-conjugate to c, for $\mathcal{F} = \mathcal{F}_{48:2}, \mathcal{F}_{24:2}, \mathcal{F}_{6^2:2}$. This implies that all subgroups of 7_+^{1+2} of order 7 are $\mathcal{F}$-conjugate and that every possible isomorphism between such groups is an $\mathcal{F}$-morphism, i.e. $\mathrm{Iso}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle) = \mathrm{Iso}(\langle x\rangle, \langle y\rangle)$, for all $1 \neq x, y \in 7_+^{1+2}$.
The Euler characteristic is

$$\chi(\mathcal{F}^*) = \sum_{[P]\in \mathrm{Ob}([\mathcal{F}^*])} \frac{-\mu(P)}{|\mathrm{Aut}_{\mathcal{F}}(P)|}$$

We now know that $\mathrm{Aut}_{\mathcal{F}}(P) = 6$ and $\mu(P) = -1$ when $|P| = 7$. There are $1+7+7^2 = 57$ distinct subgroups of 7_+^{1+2} of order 7. Because 7_+^{1+2} is not elementary abelian, the term corresponding to 7_+^{1+2} is 0. If $P \in \mathcal{V}$, i.e. elementary abelian of rank two, then $\mu(P) = 7$. (Recall that $\mu((\mathbb{Z}/p\mathbb{Z})^n) = (-1)^n p^{\binom{n}{2}}$). The remaining data needed to compute $\chi(\mathcal{F}^*)$ is provided in the table of Lemma 13. We get that

$$\begin{aligned}
\chi(\mathcal{F}^*_{48:2}) &= \frac{1}{6} + \frac{-7}{2\cdot 6\cdot 7\cdot 8} && = \frac{5}{32}\\
\chi(\mathcal{F}^*_{24:2}) &= \frac{1}{6} + \frac{-7}{2\cdot 6\cdot 7\cdot 8} + \frac{-7}{2\cdot 6\cdot 7\cdot 8} && = \frac{7}{48}\\
\chi(\mathcal{F}^*_{6^2:2}) &= \frac{1}{6} + \frac{-7}{6^2\cdot 7\cdot 8} + \frac{-7}{2\cdot 6\cdot 7\cdot 8} && = \frac{11}{72}
\end{aligned}$$

Prior to proving Theorem 21, the Euler characteristics of $\widetilde{\mathcal{F}}^*_{48:2}$, $\widetilde{\mathcal{F}}^*_{24:2}$, and $\widetilde{\mathcal{F}}^*_{6^2:2}$ had been computed separately from the matrices of $[\widetilde{\mathcal{F}}^*_{48:2}]$, $[\widetilde{\mathcal{F}}^*_{24:2}]$, and $[\widetilde{\mathcal{F}}^*_{6^2:2}]$. They turned out to have the same coweightings as $[\mathcal{F}^*_{48:2}]$, $[\mathcal{F}^*_{24:2}]$, and $[\mathcal{F}^*_{6^2:2}]$, which motivated the search for a proof the generalization of Theorem 16.

6 Quadratic spaces

In this section, we describe the concept of quadratic spaces and some of their most basic properties. Quadratic spaces play a fundamental role in the construction of the Solomon fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$.
Throughtout this section fix a field F of characteristic $\neq 2$ and a finite dimensional vector space V over F.

Definition. *A quadratic form (or map) on V is a map $Q\colon V \to F$ which satisfies the following two conditions:*

1. *$Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$ and all $\lambda \in F$.*
2. *The map $B_Q\colon V \times V \to F$ given by $B_Q(v,w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$ is symmetric and bilinear.*

When Q is a quadratic form on V, we call the pair (V, Q) a quadratic space.

Note that $B_Q(v,v) = Q(v)$ for all $v \in V$, when (V,Q) is a quadratic space. One can think of B_Q as an inner product and Q as the sqaure of the associated norm. There are similarities, but for instance, $Q(v) = 0$ does not imply $v = 0$.
Given a basis $\{v_1, \ldots, v_n\}$, we define an $n \times n$-matrix $\underline{\underline{B}}_Q$, by $\underline{\underline{B}}_Q(i,j) = B_Q(v_i, v_j)$ for $1 \le i, j \le n$. If $\det(\underline{\underline{B}}_Q) \neq 0$, Q and (V,Q) are called *nonsingular* or *regular*. The determinant of $\underline{\underline{B}}_Q$ is called the discriminant of Q. It is only defined modulo squares of $\mathbb{F}$.

Definition. *Let (V,Q) be a quadratic space.*

1. *Two vectors $v, w \in V$ are said to be orthogonal if $B_Q(v,w) = 0$, in which case we write $v \perp w$. v is called isotropic if $B_Q(v) = 0$, and nonisotropic otherwise.*

2. *A subspace $U \subseteq V$ is called isotropic if all its vectors are isotropic.*

3. *An orthogonal basis of V is a basis $\{v_1, \ldots, v_k\}$ which satisfies $v_i \perp v_j$ for all $i \neq j$.*

4. *Two subspaces $W_1, W_2 \subseteq V$ are said to be orthogonal, written $W_1 \perp W_2$, if $w_1 \perp w_2$ for all $w_1 \in W_1$ and all $w_2 \in W_2$.*

5. *To a subset $W \subseteq V$, we define the orthogonal complement of W as $W^\perp = \{v \in V \mid \forall w \in W \colon v \perp w\}$. We write $v^\perp$ instead of $\{w\}^\perp$ for $v \in V$.*

It is not hard to see that (V,Q) is nonsingular if and only if $V^\perp = \{0\}$.

Proposition 23. *Let (V,Q) be a nonsingular quadratic space. Then V has an orthogonal basis of nonisotropic vectors.*

Proof. If $\dim V = 1$ there is nothing to show. In general we can find $v \in V$ such that $Q(v) \neq 0$; if not, then $B_Q(v,w) = 0$ for all $v, w \in V$ by definition of B_Q and (V,Q) would be singular. The subspace $v^\perp$ of V is the kernel of the the surjective linear map $B(v,\cdot)\colon V \to F$. Therefore $\dim(v^\perp) = \dim(V) - 1$. Since $v \notin v^\perp$ (as $B_Q(v,v) = Q(v) \neq 0$), V decomposes as $V = v^\perp \oplus Fv$. We claim that the quadratic space $(v^\perp, Q|_{v^\perp})$ is nonsingular. Assume that it is singular. Then there is a $w \in v^\perp$, $w \neq 0$, such that $B_Q(u,w) = 0$ for all $u \in v^\perp$. But since $w \perp v$ and $V = v^\perp \oplus Fv$, we would have $w \perp u$ for all $u \in V$ by linearity of B_Q. Thus $(v^\perp, Q|_{v^\perp})$ is nonsingular.
Inductively, $v^\perp$ has an orthogonal basis $\{v_1, \ldots, v_{n-1}\}$ of nonisotropic elements, and then $\{v_1, \ldots, v_{n-1}, v\}$ is an orthogonal basis of V consisting of nonisotropic elements. □

If $g \in \mathrm{GL}(V)$ satisfies $Q(g(v)) = Q(v)$ for all $v \in V$, we say that g is an isometry of the space (V,Q). The set of all isometries is a subgroup of $\mathrm{GL}(V)$ denoted $\mathrm{O}(V,Q)$. The subgroup of isometries of determinant 1 is denoted $\mathrm{SO}(V,Q)$. We have a special interest in the so-called hyperplane reflections.

Definition. *Let (V,Q) be a quadratic space and suppose $v \in V$ is nonisotropic; as in the proof of Proposition 23, $V = v^\perp \oplus Fv$. The map linear map $V \to V$ given by $v \mapsto -v$ and $w \mapsto w$, for all $w \in V^\perp$, is an isometry of (V,Q). We refer to this isometry as the reflection in the hyperplane $v^\perp$, and in general we refer to such isometries as hyperplane reflections. Clearly, any hyperplane reflection has determinant -1.*

Theorem 24 (Cartan–Dieudonné)**.** *[Ar, p. 129]. Every isometry of a nonsingular quadratic space (V,Q) is a composition of at most* $\dim V$ *hyperplane reflections.* □

In particular, an isometry of a nonsingular quadratic space has determinant ± 1.
We also need a description of $\mathrm{SO}(V,Q)$. Let $U \subseteq V$ be a subspace of dimension 2 and assume it has a basis of nonisotropic vectors $\{v_2, v_2\}$. We wish to see that we can even find an orthogonal basis of nonisotropic vectors. If U has no isotropic elements save 0, then $(U, Q|_U)$ is nonsingular

and we just apply Proposition 23. If U has an isotropic element $v \neq 0$, one can find another isotropic element $w \in U$ such that $\{v, w\}$ is a basis of U and $B_Q(v, w) = 1$ (see [Ar, p. 118]). But then $\{v + w, v - w\}$ is an orthogonal basis and $v + w$ and $v - w$ are both nonisotropic. As in the proof of 23 we get that $V = U \oplus U^\perp$ (by considering the linear map $V \to F \times F$ given by $u \mapsto (B_Q(v + w, u), B_Q(v - w, u))$ which is surjective and has kernel $(v + w)^\perp \cap (v - w)^\perp = U^\perp$). This allows us to define an isometry of V by $u \mapsto -u$ for all $u \in U$, and $w \mapsto w$ for all $w \in U^\perp$. We call this isometry the rotation with respect to U. Its determinant is clearly $(-1)^2 = 1$. In general we will refer to such isometries as codimension 2 rotations.

Corollary 25. *Let $n = \dim V$. Every element of $\mathrm{SO}(V, Q)$ is the composition of at most n codimension 2 rotations when $n \geq 3$.*

Proof. Let $g \in \mathrm{SO}(V, Q)$. From Theorem 24 we know that $g = s_1 \cdots s_k$ where $s_1, \ldots, s_k \in \mathrm{O}(V, Q)$ are hyperplane reflections and $k \leq n$. Let $v_1^\perp, \ldots, v_k^\perp$ be the associated hyperplanes. We may assume that v_i and v_{i+1} are linearly independent over F for $i = 1, \ldots, k - 1$. Set $U = \mathrm{span}(v_1, v_2)$. From the discussion above we have that $V = U \oplus U^\perp$, and we can find a nonisotropic element $v_{1,2} \in U^\perp$, since (V, Q) is nonsingular and $\dim V \geq 3$. Let $s_{1,2}$ be the reflection in the hyperplane $s_{1,2}^\perp$. Then since $s_1 \perp s_{1,2}$, the composition $s_1 s_{1,2}$ is the rotation with respect to $\mathrm{span}(v_1, v_{1,2})$. In the same manner, $s_{1,2} s_2$ is a codimension 2 rotation. Since k must be even (g has determinant $(-1)^k$), we can continue in this fashion and get

$$g = (s_1 s_{1,2})(s_{1,2} s_2)(s_3 s_{3,4})(s_{3,4} s_4) \cdots (s_{k-1} s_{k-1,k})(s_{k-1,k} s_k)$$

□

7 Spin groups

Throughout this section F will be a field of prime characteristic $\neq 2$, V a finite vector space over F, and $Q\colon V \to F$ a nonsingular quadratic form on V. We shall see how to construct the spin group $\mathrm{Spin}(V, Q)$; the fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ is defined over a Sylow-2-subgroups of a certain spin group.

For each $i \in \mathbb{N}$, the nth tensor power of V is

$$T^i(V) = \underbrace{V \otimes \cdots \otimes V}_{i}$$

i.e. the tensor product over F of i copies of V. We set $T^0(V) = F$. The tensor algebra is the direct sum of all tensor powers of V:

$$T(V) = \bigoplus_{i \in \mathbb{N}_0} T^i(V)$$

$T(V)$ has an obvious product structure given by the tensor product $T^i(V) \otimes T^j(V) \to T^{i+j}(V)$, so $T(V)$ is a graded algebra over F. We may view $F = T^0(V)$ as a subalgebra of $T(V)$, and $V = T^1(V)$ as a an F-submodule.
The Clifford algebra is defined as the quotient $C(V, Q) = T(V)/\langle v \otimes v - Q(v) \mid v \in V \rangle$. We can still view F as a subring of $C(V, Q)$ and V as an F-submodule; the relation $v \otimes v = Q(v)$ makes no identifications in F or in V. To ease the notation, we will write $v_1 \cdots v_i$ to indicate the class of $v_1 \otimes \cdots \otimes v_i \in T^i(V)$.
V has an orthogonal basis $\{v_1, \ldots, v_n\}$ of nonisotropic elements by Proposition 23. The set $\{1\} \cup \{v_{i_1} \cdots v_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$ then constitutes an F-basis for $C(V, Q)$. To see this, we note that given $v, w \in V$, we have $v \perp w \Leftrightarrow wv = -vw$:

$$vw + wv = (v + w)^2 - v^2 - w^2 = Q(v + w) - Q(v) - Q(w) = 2B_Q(w, v)$$

Every element of $C(V,Q)$ is a sum of products of elements of V. Writing each element as an F-linear combination of the elements $v_1, \ldots, v_n$ and rearranging using the relations $v_j v_i = -v_i v_j$ for $j \neq i$, and $v_i^2 = Q(v_i)$, the claim follows.
The grading of $T(V)$ is only preserved modulo 2 when passing to the quotient $C(V,Q)$, so we can write $C(V,Q) = C_0 \oplus C_1$, where C_0 (or C_1) is the subalgebra consisting of all elements of even (or odd) degrees. In other words

$$C_0 = \bigoplus_{i \in \mathbb{N}_0} T^{2i} / \langle v \otimes v - Q(V) \mid v \in V \rangle, \quad \text{and}$$
$$C_0 = \bigoplus_{i \in \mathbb{N}_0} T^{2i+1} / \langle v \otimes v - Q(V) \mid v \in V \rangle$$

The Clifford group is defined as $G(V,Q) = \{u \in C(V,Q)^* \mid uVu^{-1} = V\}$ where, as usual, $C(V,Q)^*$ denotes the multiplicative group of invertible elements of $C(V,Q)$. One can show that if $u \in G(V,Q)$ then either $u \in C_0$ or $u \in C_1$. This leads to the definition of a homomorphism $\pi\colon G(V,Q) \to O(V,Q)$ defined by letting $\pi(u)$ be the isometry of V given by $v \mapsto (-1)^i uvu^{-1}$, with $u \in C_i$. It is easy to check that $\pi(u)$ is an isometry:

$$Q(\pm uvu^{-1}) = (uvu^{-1})^2 = uQ(v)u^{-1} = Q(v)$$

Every nonisotropic element $w \in V$ is an element of $G(V,Q)$: Its inverse is clearly $Q(w)^{-1}w$, and if we extend $\{w\}$ to an orthogonal basis $\{w_1 = w, w_2, \ldots, w_n\}$ of V we see that $\pi(w)$ is the reflection in hyperplane $w^\perp$.
By Theorem 24, every isometry of V is a composition of hyperplane reflections. This shows that π is surjective. Clearly $F^* \subseteq \ker \pi$, and we claim that equality holds. To see this let $u \in G(V,Q)$ have even degree and assume $\pi(u)$ is the identity $V \to V$. We can write $u = u_0 + v_1 u_1$ such that $u_0 \in C_0$, $u_1 \in C_1$, and neither u_0 nor u_1 contain a factor v_1 when written in the basis $\{1\} \cup \{v_{i_1} \cdots v_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$ of $C(V,Q)$. Now $uv_1u^{-1} = \pi(u)(v_1) = v_1$, and so

$$(u_0 + v_1 u_1)v_1 = v_1(u_0 + v_1 u_1) = (u_0 - v_1 u_1)v_1,$$

hence $u_0 + v_1 u_1 = u_0 - v_1 u_1$, i.e. $u = u_0$. This means that when u can be written as a linear combination the basis vectors 1 and $v_{i_1} \cdots v_{i_k}$ with $1 < i_1 < \ldots < i_k \leq n$. A similar argument for each of the other basis vectors $v_2, \ldots, v_n$ of V, shows that $u_0 \in F^*$. Likewise, if $u' \in G(V,Q)$ has odd degree and $\pi(u')$ is the identity, then $u' \in F^*$, which is a contradiction. This implies that $G(V,Q)$ is exactly the set of elements of $C(V,Q)$ of the form $\lambda w_1 \cdots w_k$ with $\lambda \in F^*$, and $w_1, \ldots, w_k \in V$ nonisotropic.
$C(V,Q)$ has a canonical anti-automorphism t given by $t(w_1 \cdots w_i) = w_i \cdots w_1$ for all $w_1, \ldots, w_i \in V$ and then extended linearly to all of $C(V,Q)$. We obtain a map $\widetilde{\theta}_{V,Q}\colon G(V,Q) \to F^*$ given by $\widetilde{\theta}_{V,Q}(u) = u \cdot t(u)$. It maps into F^* (and is thus a homomorphism) since every $u \in G(V,Q)$ has the form $\lambda w_1 \cdots w_k$ as described above and

$$\widetilde{\theta}_{V,Q}(\lambda w_1 \cdots w_k) = \lambda w_1 \cdots w_k \lambda w_k \cdots w_1 = \lambda^2 Q(w_1) \cdots Q(w_k)$$

$\widetilde{\theta}_{V,Q}$ maps $\ker \pi = F^*$ to F^{*2}, so we obtain an induced homomorphism $\theta_{V,Q}\colon \mathrm{O}(V,Q) \to F^*/F^{*2}$, called the spinor norm. We will drop the subscripts and write $\widetilde{\theta}$ and θ when the space (V,Q) is clear from the context.
Given $u = \lambda w_1 \cdots w_k \in G(V,Q)$, $\pi(u)$ is the composition of k hyperplane reflections, so $\pi(u)$ has determinant 1 if and only if k is even. The preimage of $\mathrm{SO}(V,Q)$ under π therefore equals $G(V,Q) \cap C_0 =: G^+(V,Q)$. We now define two subgroups

- $\mathrm{Spin}(V,Q) = \ker(\widetilde{\theta}_{V,Q}|_{G^+(V,Q)})$, called the spin group.
- $\Omega(V,Q) = \ker(\theta_{V,Q}|_{\mathrm{SO}(V,Q)})$.

The groups and homomorphisms introduced in the above paragraphs can be summarized in a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & F^* & \xrightarrow[\cdot^2]{} & F^{*2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Spin}(V,Q) & \longrightarrow & G^+(V,Q) & \xrightarrow[\widetilde{\theta}_{V,Q}]{} & F^* & & \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
1 & \longrightarrow & \Omega(V,Q) & \longrightarrow & \mathrm{SO}(V,Q) & \xrightarrow[\theta_{V,Q}]{} & F^*/F^{*2} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{13}$$

where all rows and columns are exact. (In general, θ and $\widetilde{\theta}$ need not be surjective. For the spaces we study later they will be, but we don't need that fact). Every nonsingular quadratic space gives rise to such a diagram. We need to see how such diagrams are compatible when the base field F is extended.

If the field E is an extension of F we obtain a vector space V_E by replacing the field of scalars of V by E. Formally, we set $V_E = E \otimes_F V$, and then $\{1 \otimes v_1, \dots, 1 \otimes v_n\}$ is an E-basis of V_E. Next we define a quadratic form Q_E on V_E by setting $Q_E = (\lambda \mapsto \lambda^2) \otimes Q$, where $\lambda \mapsto \lambda^2 \colon E \to E$ is the square map. Then $Q_E|_V = Q$ and we may view $C(V,Q)$ as a subring of $C(V_E, Q_E)$. However, this formalism is quite awkward, especially when one needs to consider the compatibility of the Clifford groups, spin groups, etc. Ultimately we just need to understand what happens when F is replaced by its algebraic closure $\overline{F}$. Informally, we let the vector space $\overline{V}$ consist of all (formal) $\overline{F}$-linear combinations of the basis elements $v_1, \dots, v_n$ of V. (In this sense, one can think of V as some sort of lattice of $\overline{V}$). We extend the bilinear form B_Q of V by linearity to a bilinear form $\overline{B}_Q$ of $\overline{V}$, and thus obtain a quadratic form $\overline{Q}$ of $\overline{V}$ which extends Q. $G^+(V,Q)$ is clearly a subgroup of $G^+(\overline{V},\overline{Q})$, and the image of $u \in G^+(V,Q)$ under $\overline{\pi}\colon G(\overline{V},\overline{Q}) \to \mathrm{SO}(\overline{V},\overline{Q})$ has the same matrix as $\pi(u) \in \mathrm{SO}(V,Q)$. All in all, we can combine the diagrams (13) of the spaces $C(V,Q)$ and $C(\overline{V},\overline{Q})$ by mapping the groups in the diagram of $C(V,Q)$ to their counterparts in the diagram of $C(\overline{V},\overline{Q})$ via inclusions.
The Glaois group $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{V}$ by $\psi.(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \psi(a_1)v_1 + \ldots \psi(a_n)v_n$, thus a vector is fixed if and only if all its coordinates are. This action induces an action on each of the groups in the diagram of $C(\overline{V},\overline{Q})$. Taking the subgroups fixed by the action on each of the groups derived from $C(\overline{V},\overline{Q})$ recovers their counterparts with respect to $C(V,Q)$, since $\overline{F}/F$ is a Galois extension ([Mi, §7]). In particular $\mathrm{Spin}(V,Q)$ is the subgroup of all elements fixed by each $\psi \in \mathrm{Gal}(\overline{F}/F)$. For any intermediate field $F \subseteq E \subseteq \overline{F}$, $\mathrm{Spin}(V_E, Q_E)$ is just the subgroup of $\mathrm{Spin}(\overline{V},\overline{Q})$ fixed by all $\psi \in \mathrm{Gal}(\overline{F}/E)$.

Before we turn our attention to a specific quadratic space, we state two results of general character. These will be used extensively in section 9.

Lemma 26. *Let $E \leqslant \mathrm{O}(V,Q)$ be an elementary abelian 2-subgroup. To each homomorphism $\chi\colon E \to \{\pm 1\}$, associate the subspace*

$$V_\chi \stackrel{def}{=} \{v \in V \mid \forall g \in E, v \in V \colon g(v) = \chi(g) \cdot v\}$$

Then $Q|_{V_\chi}$ is nonsingular for each χ, and V is the direct, orthogonal sum the all the V_χ.

We shall refer to the subspaces V_χ as the eigenspaces of E. It is worth noting that V_χ is contained in an eigenspace (in the usual sense) of each $g \in E$, with eigenvalue is $\chi(g)$. The lemma is stated without proof in [LO].

Proof. Let $\chi_1, \chi_2 \in \mathrm{Hom}(E, \{\pm 1\})$ be given such that $\chi_1 \neq \chi_2$. There is $g \in E$ such that $\chi_1(g) \neq \chi_2(g)$; without loss of generality, assume $\chi_1(g) = 1$, $\chi_2(g) = -1$. Then given $v_1 \in V_{\chi_1}$ and $v_2 \in V_{\chi_2}$ one has

$$B_Q(v_1, v_2) = B_Q(g(v_1), g(v_2)) = B_Q(v_1, -v_2) = -B_Q(v_1, v_2)$$

hence $B_Q(v_1, v_2) = 0$, i.e. $v_1 \perp v_2$. This shows that the eigenspaces are orthogonal to one another. In addition, given $v \in V_{\chi_1} \cap V_{\chi_2}$, we have

$$v = \chi_1(g) \cdot v = g(v) = \chi_2(g) \cdot v = -v$$

so $v = 0$, and each pair of eigenspaces has trivial intersection. The sum of the eigenspaces is therefore a direct, orthogonal sum. We next show that the sum is all of V. This is somewhat technical, first we introduce some notation: E has the form $(\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$ (if E is the trivial group, there is nothing to show). Let g_i, $i = 1, \ldots, k$, be the generators of the k factors, that is $E = \langle g_1 \rangle \times \ldots \times \langle g_k \rangle$. Each homomorphism $\chi \colon E \to \{\pm 1\}$ is determined by its values on the g_i, and these can be chosen however one pleases, i.e. $|\mathrm{Hom}(E, \{\pm 1\})| = 2^k$. Given $\chi \colon E \to \{\pm 1\}$, it is clear that

$$\forall v \in V \colon v \in V_\chi \Leftrightarrow \forall i \in \{1, \ldots, k\} \colon g_i(v) = \chi(g_i) \cdot v$$

For each subset $S \subseteq \{1, \ldots, k\}$, let g_S denote the element $\prod_{s \in S} g_s$, and let $g_\emptyset$ be the identity element of E, i.e. $g_\emptyset = \mathrm{id}_V$. Every element of E has the form g_S for some $S \subseteq \{1, \ldots, k\}$. To each $v \in V$ and $\chi \in \mathrm{Hom}(E, \{\pm 1\})$, set

$$v_\chi = \sum_{S \subseteq \{1, \ldots, k\}} \chi(g_S) g_S(v)$$

We claim that $v_\chi \in V_\chi$. It is enough to show that $g_i(v_\chi) = \chi(v_i) \cdot v$ for all $i \in \{1, \ldots, k\}$.

$$\begin{aligned}
g_i(v_\chi) &= g_i\left(\sum_{S \subseteq \{1, \ldots, k\}} \chi(g_S) g_S(v)\right) \\
&= \sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \notin S}} \chi(g_S)(g_S g_i)(v) + \sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \in S}} \chi(g_S)(g_S g_i)(v) \\
&= \chi(g_i)\left(\sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \notin S}} \chi(g_S g_i)(g_S g_i)(v)\right) + \chi(g_i)\left(\sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \in S}} \chi(g_S g_i)(g_S g_i)(v)\right) \\
&= \sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \in S}} \chi(g_S) g_S(v) + \sum_{\substack{S \subseteq \{1, \ldots, k\} \\ i \notin S}} \chi(g_S) g_S(v) \\
&= \chi(g_i) v_\chi
\end{aligned}$$

We need two more results before we can prove that the sum of all the eigenspaces V_χ equals V. Let $S \subseteq \{1, \ldots, k\}$ be given. If $S = \emptyset$, then clearly $\sum_\chi \chi(g_S) = 2^k$, where the sum is taken over all $\chi \in \mathrm{Hom}(E, \{\pm 1\})$. If $S \neq \emptyset$, we claim that $\sum_\chi \chi(g_S) = 0$: Let $i \in S$ and define $\chi_- \colon E \to \{\pm 1\}$ by

$$\chi_-(g_j) = \begin{cases} 1 & \text{when } j \neq i \\ -1 & \text{when } j = i \end{cases}$$

In particular, $\chi_-(g_S) = -1$. Let H_+ denote the subgroup of $\mathrm{Hom}(E, \{\pm 1\})$ consisting of those homomorphisms that map g_i to 1. ($\mathrm{Hom}(E, \{\pm 1\})$ has the group structure given by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$). Then $\mathrm{Hom}(E, \{\pm 1\}) = H_+ \cup \chi_- H_+$ is the coset decomposition with respect to the subgroup H_+, and

$$\sum_{\chi} \chi(g_S) = \sum_{\chi \in H_+} \chi(g_S) + \sum_{\chi \in H_+} (\chi_- \chi)(g_S) = 0$$

Let $v \in V$ be given. We can now show that $v \in \bigoplus_\chi V_\chi$:

$$\begin{aligned} \frac{1}{2^k} \sum_\chi v_\chi &= \frac{1}{2^k} \sum_\chi \sum_S \chi(g_S) g_S(v) \\ &= \frac{1}{2^k} \sum_S g_S(v) \sum_\chi \chi(g_S) \\ &= \frac{1}{2^k} g_\emptyset(v) 2^k = v \end{aligned}$$

Finally we show that $Q|_{V_\chi}$ is nonsingular for each $\chi \in \mathrm{Hom}(E, \{\pm 1\})$. Choose a basis of each eigenspace; their union is a basis of V with the property that basis vectors belonging to different eigenspaces are orthogonal. The matrix associated to B_Q with respect to this basis is therefore a block diagonal matrix and its determinant is nonzero since Q is nonsingular. The determinant is also equal to the product of the determinants of the blocks of the diagonal. But these blocks are exactly the matrices associated to the restrictions of Q to each of the eigenspaces V_χ. These matrices must all have nonzero determinant, i.e. $Q|_{V_\chi}$ is nonsingular for all $\chi \in \mathrm{Hom}(E, \{\pm 1\})$. □

Lemma 27. *[LO, A.4]. Let x be an involution of $\mathrm{SO}(V, Q)$, and let $V_- \oplus V_+$ be its eigenspace decomposition as given by Lemma 26. (V_+ corresponds to the constant homomorphism $\langle x \rangle \to \{\pm 1\}$ and V_- corresponds to the other one, i.e. the one given by $x \mapsto -1$).*
x satisfies the following three properties

1. *$x \in \Omega(V, Q)$ if and only if the discriminant of $Q|_{V_-}$ is a square in F^*.*
2. *If $x \in \Omega(V, Q)$, then a lifting of x in $\mathrm{Spin}(V, Q)$ has order 2 if and only if $4 \mid \dim(V_-)$.*
3. *If $x \in \Omega(V, Q)$ and $\alpha \in C_{\Omega(V,Q)}(x)$, there is an $\alpha_- \in \mathrm{O}(V_-, Q|_{V_-})$ and an $\alpha_+ \in \mathrm{O}(V_+, Q|_{V_+})$ such that α decomposes as $\alpha = \alpha_- \oplus \alpha_+$. Futhermore, if $\widetilde{x}, \widetilde{\alpha} \in \mathrm{Spin}(V, Q)$ are liftings of x and α respectively, then $\widetilde{x}\widetilde{\alpha} = \widetilde{\alpha}\widetilde{x}$ if and only if $\alpha_- \in \mathrm{SO}(V_-, Q|_{V_-})$.*

Proof. ad 1. $Q|_{V_-}$ is nonsingular, by Lemma 26. We can therefore choose an orthogonal basis $\{v_1, \ldots, v_k\}$ of V_- of nonisotropic elements, and then $x(v_i) = -v_i$ for $i = 1, \ldots, k$. Since $x|_{V_+}$ is the identity, the determinant of x equals the determinant of $x|_{V_-}$, and this determinant is $(-1)^k$. As $x \in \mathrm{SO}(V, Q)$, k must be even.
For all $w \in V_+$ we have $x(w) = w$. x is therefore the composition of all the hyperplane reflections $\pi(v_i)$, $i = 1, \ldots, k$, i.e. $x = \pi(v_1 \cdots v_k)$. This mean that

$$\theta_{V,Q}(x) = \widetilde{\theta}_{V,Q}(v_1 \cdots v_k) \mod F^{*2}$$

But $\widetilde{\theta}(v_1 \cdots v_k) = Q(v_1) \cdots Q(v_k)$ is the discriminant of $Q|_{V_-}$, hence $\theta_{V,Q}(x) = 1$ if and only if $Q|_{V_-} \in F^{*2}$.

ad 2. Assume $x \in \Omega(V, Q)$. Since $Q(v_1) \cdots Q(v_k)$ is a square, we may assume that $Q(v_1) \cdots Q(v_k) = 1$ by replacing v_1 with λv_1 for a suitable $\lambda \in F^*$, and $\pi(v_1 \cdots v_k) = \pi(\lambda v_1 \cdots v_k)$, so x is unchanged under such a substitution. As k is even, $\widetilde{x} = v_1 \cdots v_k \in G^+$ (with inverse $v_k \cdots v_1$) is a lifting of x, and since $\widetilde{\theta}_{V,Q}(\widetilde{x}) = Q(v_1)\cdots Q(v_k) = 1$, we get that $\widetilde{x} \in \mathrm{Spin}(V,Q)$. From the relations $v \perp w \Leftrightarrow wv = -vw$ and $v^2 = Q(v)$ of the Clifford algebra we see that

$$\widetilde{x}^2 = v_1 \cdots v_k v_1 \cdots v_k = (-1)^{\frac{1}{2}k(k-1)} {v_1}^2 \cdots {v_k}^2 = (-1)^{\frac{1}{2}k(k-1)}$$

I.e. $\widetilde{x}^2 = 1$ if and only if $4 \mid k = \dim(V_-)$. The order of $\widetilde{x}$ cannot be 1, since x is not the identity.

ad 3. Assume that $x \in \Omega(V,Q)$ and let $\alpha \in C_{\Omega(V,Q)}(x)$ be given. Set $\alpha_- = \alpha|_{V_-}$, $\alpha_+ = \alpha|_{V_+}$. For all $v_- \in V_-$ and $v_+ \in V_+$

$$x(\alpha_-(v_-)) = \alpha(x(v_-)) = -\alpha(v_-) = -\alpha_-(v_-) \quad \text{and} \quad x(\alpha_+(v_+)) = \alpha(x(v_+)) = \alpha(v_+) = \alpha_+(v_+)$$

Hence $\alpha_-(V_-) = V_-$ and $\alpha_+(V_+) = V_+$. So $\alpha_\pm$ is an isometry of $(V, Q|_{V_\pm})$. This proves the first part.

Now let $\widetilde{\alpha}, \widetilde{x} \in \mathrm{Spin}(V,Q)$ be lifitngs of α and x, respectively. Both $\widetilde{\alpha}\widetilde{x}$ and $\widetilde{x}\widetilde{\alpha}$ are liftings of $x\alpha = \alpha x$, but then $\widetilde{\alpha}\widetilde{x} = \pm\widetilde{x}\widetilde{\alpha}$ since there are only these two liftings of $x\alpha$. We wish to determine exactly when $\widetilde{\alpha}\widetilde{x} = \widetilde{x}\widetilde{\alpha}$ holds. Let $\widetilde{\alpha}_- \in G(V_-, Q|_{V_-})$ and $\widetilde{\alpha}_+ \in G(V_+, Q|_{V_+})$ be liftings under π of α_- and α_+, respectively. We know that $\widetilde{\alpha}_- = \lambda_- u_1 \cdots u_s$ for some nonisotropic elements $u_1, \ldots, u_s \in V_-$ and a $\lambda_- \in F^*$. Likewise $\widetilde{\alpha}_+ = \lambda_+ w_1 \cdots w_t$ for some nonisotropic elements $w_1, \ldots, w_t \in V_+$ and a $\lambda_+ \in F^*$. $\widetilde{\alpha}_-$ and $\widetilde{\alpha}_+$ are also elements of $C(V,Q)^*$; we wish to see that they belong to $G(V,Q)$. Given $v_+ \in V_+$ we have that

$$\widetilde{\alpha}_- v_+ (\widetilde{\alpha}_-)^{-1} = \lambda_- w_1 \cdots w_t v_+ w_t \cdots w_1 \frac{1}{\lambda_- Q(w_1)\cdots Q(w_t)} = (-1)^t v_+ \tag{14}$$

Thus conjugation by $\widetilde{\alpha}_-$ maps V_+, and thereby also V, to itself, that is $\widetilde{\alpha}_- \in G(V,Q)$. From (14) and the definition of $\widetilde{\alpha}_-$ we see that $\pi(\widetilde{\alpha}_-) = \alpha_- \oplus \mathrm{id}_{V^+}$. A similar argument shows that $\widetilde{\alpha}_+ \in G(V,Q)$ and that $\pi(\widetilde{\alpha}_+) = \mathrm{id}_{V_-} \oplus \alpha_+$. Thus the product $\widetilde{\alpha}_-\widetilde{\alpha}_+$ in $G(V,Q)$ is a lifting of α, i.e. $\lambda\widetilde{\alpha}_-\widetilde{\alpha}_+ = \widetilde{\alpha}$ for some $\lambda \in F^*$. In particular, $\widetilde{\alpha}_-\widetilde{\alpha}_+ \in G^+(V,Q)$, and then $\widetilde{\alpha}_-\widetilde{\alpha}_+ = (-1)^{st}\widetilde{\alpha}_+\widetilde{\alpha}_- = \widetilde{\alpha}_+\widetilde{\alpha}_-$, since st must be even. Let $\widetilde{x} = v_1 \cdots v_k$, $v_1, \ldots, v_k \in V_-$, just like above. Since k is even and $V_+ \perp V_-$, it is clear that $\widetilde{x}$ and $\widetilde{\alpha}_+$ commute. Now

$$\widetilde{x}^{-1}\widetilde{\alpha}^{-1}\widetilde{x}\widetilde{\alpha} = \widetilde{x}^{-1}\lambda^{-1}\widetilde{\alpha}_-^{-1}\widetilde{\alpha}_+^{-1}\widetilde{x}\lambda\widetilde{\alpha}_+\widetilde{\alpha}_- = \widetilde{x}^{-1}\widetilde{\alpha}_-^{-1}\widetilde{x}\widetilde{\alpha}_-$$

Which proves the last part. □

8 The group $Spin_7(q)$

We now set $F = \mathbb{F}_{q^n}$ where q is a given odd prime power, and consider a 7-dimensional vector space V over $\mathbb{F}_{q^n}$. [As, (21.4) p. 87] describes the nonsingular quadratic forms over V; they are all similar under appropriate scalar transformations and all have square discriminants. In particular they all have the same group of isometries. As mentioned in the section on spin groups, we will need to consider extensions of $\mathbb{F}_q$.

It turns out to be advantegeous to work with the 7-dimensional vector spaces $V_n := M_2(\mathbb{F}_{q^n}) \oplus M_2^0(\mathbb{F}_{q^n})$, where $M_2(\mathbb{F}_{q^n})$ denotes the vector space of 2×2-matrices over $\mathbb{F}_{q^n}$ and $M_2^0(\mathbb{F}_{q^n})$ its subspace consisting of the matrices of trace 0. To be precise, V_n is the external direct sum of the two vector spaces, i.e., as a set, it is the Cartesian product. If we set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then the set $\{(e_1,0),(e_2,0),(e_3,0),(e_4,0),(0,e_2),(0,e_3),(0,e_4)\}$ is an $\mathbb{F}_{q^n}$-basis of V_n. With $\mathbb{F}_{q^\infty} = \overline{\mathbb{F}}_q$ we also allow $n = \infty$. It is a routine exercise to check that $(M_2(\mathbb{F}_{q^n}), \det)$ and

$(M_2^0(\mathbb{F}_{q^n}), \det)$ are nonsingular quadratic spaces with $\{e_1, e_2, e_3, e_4\}$ and $\{e_2, e_3, e_4\}$ as orthogonal bases of nonisotropic elements and that both spaces have square discriminants. We define a nonsingular quadratic form on V_n by $Q(A_1, A_2) = \det A_1 + \det A_2$, for all $A_1 \in M_2(\mathbb{F}_{q^n})$, $A_2 \in M_2^0(\mathbb{F}_{q^n})$. The basis already mentioned is orthogonal and the basis elements are nonisotropic. We see that the discriminant of Q is a square.
We write $\mathrm{Spin}_7(q^n) = \mathrm{Spin}(V_n, Q)$, $\mathrm{Spin}_7(q^\infty) = \mathrm{Spin}(V_\infty, Q)$. Likewise $\Omega_7(q^n) = \Omega(V_n, Q)$. The groups are nested inside one another in the same way the fields $\mathbb{F}_{q^n}$ are.

Proposition 28. *$Z(\mathrm{Spin}_7(q^n)) = \{\pm 1\}$ for $n = 1, 2 \ldots, \infty$.*

Proof. We set $G^+ = G^+(V_n, Q)$ and show that $C_{G^+}(\mathrm{Spin}_7(q^n)) = \mathbb{F}_{q^n}^*$. Let $\{v_1, \ldots, v_7\}$ be the orthogonal basis of nonisotropic elements of V_n described above. Let S denote the set of subsequences of $(1, \ldots, 7)$, and for each $s = (i_1, \ldots, i_k) \in S$ define $v_s = v_{i_1} \cdots v_{i_k}$, and $v_s = 1 \in F$ when s is the empty sequence. Hence $\{v_s \mid s \in S\}$ is a basis of $C(V_n, Q)$. Let $S_0 \subseteq S$ denote the sequences of even length. Clearly $v_s \in G^+$ when $s \in S_0$, and every element $u \in \mathrm{Spin}_7(q^n)$ has the form $u = \sum_{s \in S_0} \lambda_s v_s$ for some $\lambda_s \in \mathbb{F}_{q^n}$ not all zero. If $u \in \mathbb{F}_{q^n}^*$, then $u \in C_{G^+}(\mathrm{Spin}_7(q^n))$. Assume $\lambda_{s'} \neq 0$ for some $s' \in S_0$ of positive length. Given $t \in S_0$ we see that

$$\left(\sum_{s \in S_0} \lambda_s v_s \right) v_t = v_t \left(\sum_{s \in S_0} \pm \lambda_s v_s \right)$$

where the signs depends on s and t. We see that u and v_t commute exactly if $v_s v_t = v_t v_s$ for all $s \in S_0$ for which $\lambda_s \neq 0$. Since s' has positive and even length, we can find $1 \leq i < j \leq 7$ such that $i \in s'$ but $j \notin s'$ (or vice versa). Then

$$v_{s'}(v_i v_j) = -(v_i v_j) v_{s'}$$

hence u and $v_i v_j \in G^+$ do not commute, and so $C_{G^+}(\mathrm{Spin}_7(q^n)) \subseteq \mathbb{F}_{q^n}^*$. The reverse inclusion has already been shown to hold. Now

$$Z(\mathrm{Spin}_7(q^n)) = \mathrm{Spin}_7(q^n) \cap C_{G^+}(\mathrm{Spin}_7(q^n)) = \{\pm 1\} \qquad \square$$

We write $\mathrm{Spin}_4(q^n) = \mathrm{Spin}(M_2(\mathbb{F}_{q^n}), \det)$ and $\mathrm{Spin}_3(q^n) = \mathrm{Spin}(M_2^0(\mathbb{F}_{q^n}), \det)$. We will use the same notation with Ω in place of Spin. To avoid confusing matrix multiplication and multiplication in the Clifford algebra, we will use '*' to denote the latter.
The embeddings of $M_2(\mathbb{F}_{q^n})$ and $M_2^0(\mathbb{F}_{q^n})$ into V_n induce embeddings

$$\iota_4 \colon \mathrm{Spin}_4(q^n) \hookrightarrow \mathrm{Spin}_7(q^n), \quad \text{and} \quad \iota_3 \colon \mathrm{Spin}_3(q^n) \hookrightarrow \mathrm{Spin}_7(q^n)$$

where $\mathrm{Spin}_7(q^n) = \mathrm{Spin}_7(M_2(\mathbb{F}_{q^n}) \oplus M_2^0(\mathbb{F}_{q^n}), Q)$. It is clear that $\mathrm{Spin}_i(q^n)$ embeds in $C_0(V_n, Q)$, and to see that the images of the embeddings are in $\mathrm{Spin}_7(q^n)$, we just need to note that $M_2(\mathbb{F}_{q^n}) \perp M_2^0(\mathbb{F}_{q^n})$ as subspaces of V_n, and that $\widetilde{\theta}_{V_n,Q}$ equals $\widetilde{\theta}_{M_2(\mathbb{F}_{q^n}),\det}$ and $\widetilde{\theta}_{M_2^0(\mathbb{F}_{q^n}),\det}$ when restricted to $G(M_2(\mathbb{F}_{q^n}), \det)$ and $G(M_2^0(\mathbb{F}_{q^n}), \det)$ respectively. The fact that $M_2(\mathbb{F}_{q^n}) \perp M_2^0(\mathbb{F}_{q^n})$ and that elements of $\mathrm{Spin}_i(q^n)$ have even degree implies that $\mathrm{Spin}_3(q^n)$ and $\mathrm{Spin}_4(q^n)$ centralize each other as subgroups of $\mathrm{Spin}_7(q^n)$. Thus we obtain a homomorphism

$$\iota_{4,3} \colon \mathrm{Spin}_4(q^n) \times \mathrm{Spin}_3(q^n) \to \mathrm{Spin}_7(q^n), \quad \iota_{4,3}(x, y) = x * y$$

The idea is that $\mathrm{Spin}_3(q^n)$ and $\mathrm{Spin}_3(q^n)$ can be related to well-known groups.

Proposition 29. *[LO, A.5]. For $n = 1, 2, \ldots, \infty$ there are homomorphisms*

$$\bar{\rho}_{4,n} \colon \{(A, B) \in \mathrm{GL}_2(q^n) \times \mathrm{GL}_2(q^n) \mid \det A = \det B\} \to \mathrm{SO}_4(q^n),$$
$$\bar{\rho}_{3,n} \colon \mathrm{GL}_2(q^n) \to \mathrm{SO}_3(q^n)$$

given by

$$\bar\rho_{4,n}(A,B)(X) = AXB^{-1}, \quad \textit{for all } A,B \in \mathrm{GL}_2(q^n) \textit{ with } \det A = \det B,\ X \in M_2(\mathbb{F}_{q^n}),$$
$$\bar\rho_{3,n}(A)(X) = AXA^{-1}, \quad \textit{for all } A \in \mathrm{GL}_2(q^n),\ X \in M_2^0(\mathbb{F}_{q^n}).$$

$\bar\rho_{4,n}$ *and* $\bar\rho_{3,n}$ *restrict to sujective homomorphisms*

$$\rho_{4,n} \colon \mathrm{SL}_2(q^n) \times \mathrm{SL}_2(q^n) \to \Omega_4(q^n),$$
$$\rho_{3,n} \colon \mathrm{SL}_2(q^n) \to \Omega_3(q^n)$$

$\rho_{4,n}$ *and* $\rho_{3,n}$ *lift to isomorphisms*

$$\widetilde\rho_4 \colon \mathrm{SL}_2(q^n) \times \mathrm{SL}_2(q^n) \to \mathrm{Spin}_4(q^n),$$
$$\widetilde\rho_3 \colon \mathrm{SL}_2(q^n) \to \mathrm{Spin}_3(q^n)$$

i.e. $\pi\widetilde\rho_{i,n} = \rho_{i,n}$, $i = 4,3$. *Additionally, these liftings are unique.*

Proof. It is elementary to check that $\bar\rho_{i,n}$, $i = 4,3$, as defined are homorphisms with images in $\mathrm{O}_i(q^n)$. Basic (but lengthy) calculations show that $\det(\bar\rho_{3,n}(A)) = \det(\bar\rho_{4,n}(A,B)) = 1$ for all $A, B \in \mathrm{GL}_2(\mathbb{F}_{q^n})$ with $\det A = \det B$. $\mathrm{SO}_i(q^n)/\Omega_i(q^n)$ is abelian, so the commutator subgroup of $\mathrm{SO}_i(q^n)$ is contained in $\Omega_i(q^n)$. Assume $q^n \neq 3$ (this case must be treated separately). Since $\mathrm{SL}_2(q^n)$ is perfect, the image of $\rho_{i,n}$ is contained in $\Omega_i(q^n)$. We wish to see that $\rho_{4,n}$ and $\rho_{3,n}$ are surjective when $n < \infty$. To do so we describe how values of ρ_i and π can be related.

Let $g \in \Omega_3(q^n)$ be given. From Corollary 25, we know $g = r_{U_1} \cdots r_{U_k}$ where r_{U_i} is the rotation with respect to the 2-dimensional subspace U_i. Furthermore $\theta(g) = 1 \in \mathbb{F}_{q^n}^*/(\mathbb{F}_{q^n}^*)^2$. For $i = 1, \dots, k$, fix an orthogonal basis $\{B_i, C_i\}$ of U_i with $\det(B_i), \det(C_i) \neq 0$, and choose $A_i \in U_i^\perp$, $A_i \neq 0$. Then $\det(A_i) \neq 0$ and $\{A_i, B_i, C_i\}$ is an orthogonal basis of $M_2^0(\mathbb{F}_{q^n})$. We see that

$$\pi(B_i * C_i)(A_i) = A_i, \quad \pi(B_i * C_i)(B_i) = -B_i, \quad \pi(B_i * C_i)(C_i) = -C_i,$$

which means that $\pi(B_i * C_i) = r_{U_i}$. Now

$$g = \pi(B_1 * C_1 * \cdots * B_k * C_k)$$

and in particular $\det(B_1)\det(C_1)\cdots\det(B_k)\det(C_k)$ is a sqaure in $\mathbb{F}_{q^n}^*$. We know that $\det(A_i)\det(B_i)\det(C_i)$ is a square as well since $\{A_i, B_i, C_i\}$ is an orthogonal basis of $M_2^0(\mathbb{F}_{q^n})$. Thus $\det(A_1 \cdots A_k)$ is also a square, i.e. $\det(A_1 \cdots A_k) = \lambda^2$ for some $\lambda \in \mathbb{F}_{q^n}^*$. Set $A_1 := \lambda^{-1}A_1$. Then $\det(A_1 \cdots A_k) = 1$. For each $i \in \{1, \dots, k\}$, since A_i has trace zero, $A_i^2 = \mathrm{diag}\,(-\det A_i, -\det A_i)$. The kernel of $\bar\rho_{3,n}$ is clearly the matrices $\{\mathrm{diag}\,(\lambda,\lambda) \mid \lambda \in \mathbb{F}_{q^n}^*\}$, none of which have trace zero. Therefore $\bar\rho_{3,n}(A_i)$ has order 2. Its restriction to $\mathrm{span}(A_i)$ is clearly the identity. Its restriction to $A_i^\perp = \mathrm{span}(B_i, C_i) = U_i$ must therefore be an isometry of order 2 and determinant 1. There is only one such isometry; multiplication by -1. Hence $\rho_{3,n}(A_i) = r_{U_i}$. Now

$$g = \bar\rho_{3,n}(A_1)\cdots\bar\rho_{3,n}(A_k) = \bar\rho_{3,n}(A_1 \cdots A_k) = \rho_{3,n}(A_1 \cdots A_k)$$

Hence $\rho_{3,n}$ is surjective.

Showing that $\rho_{4,n}$ is surjective can be done in a similar way. With $g \in \Omega_4(q^n)$ we have the exact same setup until the choice of the $A_i \in U_i^\perp$. This time $U_i^\perp$ is 2-dimensional and for each $i \in \{1, \dots, k\}$ we choose an orthogonal basis $\{Y_i, Z_i\}$ of $U_i^\perp$ with $\det(Y_i), \det(Z_i) \neq 0$. In the same way as above, we can conlude that $\det(Y_1Z_1 \cdots Y_kZ_k)$ is a square. But then

$$\det(Z_1Y_1^{-1} \cdots Z_kY_k^{-1}) = \det(Y_1^{-1}Z_1 \cdots Y_k^{-1}Z_k)$$

is also a square, and by replacing Y_1 with a scalar multiple, we may assume the determinants equal 1. I.e. $(Z_1Y_1^{-1}\cdots Z_kY_k^{-1}, Y^{-1}Z_1\cdots Y_k^{-1}Z_k) \in \mathrm{SL}_2(q^n)\times \mathrm{SL}_2(q^n)$. We still have the identity

$$g = \pi(B_1 * C_1 * \cdots * B_k * C_k)$$

We claim that $\pi(B_i * C_i) = \bar{\rho}_{4,n}(Z_iY_i^{-1}, Y_i^{-1}Z_i)$ for all $i \in \{1,\dots,k\}$, which will prove that $\rho_{4,n}$ is surjective.
As $Z_i \perp Y_i$ we get that

$$0 = \det(Y_i + Z_i) - \det(Y_i) - \det(Z_i) = \det(Y_i)(\det(Y_i^{-1}Z_i + I) - \det(I) - \det(Y_i^{-1}Z_i))$$

which shows that $Y_i^{-1}Z_i \perp I$. Similarly $Z_iY_i^{-1} \perp I$. A 2×2-matrix is orthogonal to I exactly if its trace is zero. The square of a 2×2-matrix X of trace zero equals $\mathrm{diag}\,(-\det X, -\det X)$. Thus $\rho_{4,n}(Z_iY_i^{-1}, Y_i^{-1}Z_i)$ has order 2. In addition, it fixes both Y_i and Z_i, and thereby U_i. By the same argument as above, we conclude that $r_{U_i} = \bar{\rho}_{4,n}(Z_iY_i^{-1}, Y_i^{-1}Z_i)$.

We now know that $\Omega_3(q^n) \cong \mathrm{SL}_2(\mathbb{F}_{q^n}/\{\pm I\} = \mathrm{PSL}_2(\mathbb{F}_{q^n})$. It is well known that $\mathrm{SL}_2(q^n)$ is the universal central extension of $\mathrm{PSL}_2(q^n)$ (we have assumed $q^n \neq 3$). Since $\mathrm{Spin}_3(q^n)$ is another central extension of $\Omega_3(q^n)$ there is a unique homomorphism $\widetilde{\rho}_{3,n}\colon \mathrm{SL}_2(q^n) \to \mathrm{Spin}_3(q^n)$ such that $\pi\widetilde{\rho}_{3,n} = \rho_{3,n}$. We show that $\widetilde{\rho}_{3,n}$ is an isomorphism by showing that it is surjective. $\Omega_3(q^n)$ is perfect, since $\mathrm{PSL}_2(q^n)$ is. Thus $\mathrm{Spin}_3(q^n)$ decomposes as $\mathrm{Spin}_3(q^n) = \{\pm 1\}[\mathrm{Spin}_3(q^n), \mathrm{Spin}_3(q^n)]$. If we let $\{A_1, A_2, A_3\}$ be an orthogonal basis of $M_2^0(\mathbb{F}_{q^n})$ with $\det A_1, \det A_2, \det A_3 \neq 0$, then at least two of the elements $\pm A_1 * A_2, \pm A_1 * A_3, \pm A_2 * A_3 \in G^+(M_2^0(\mathbb{F}_{q^n}), det)$ map to squares in $\mathbb{F}_{q^n}^*$ under θ. Without loss of generality we may assume that $\theta(A_1 * A_2)$ and $\theta(A_1 * A_3)$ are squares. After multiplying A_1 by some scalar, we may even assume $\theta(A_1 * A_2) = \theta(A_1 * A_3) = 1$, i.e. $A_1 * A_2, A_1 * A_3 \in \mathrm{Spin}_3(q^n)$. Now

$$\begin{aligned}[A_1 * A_2, A_1 * A_3] &= A_2 * A_1 * A_3 * A_1 * A_1 * A_2 * A_1 * A_3 \\ &= -A_1 * A_2 * A_2 * A_1 * A_1 * A_3 * A_3 * A_1 = -1\end{aligned}$$

Hence $\mathrm{Spin}_3(q^n)$ is perfect. By [As, (33.6), p. 168] $\widetilde{\rho}_{3,n}$ is surjective. Whenever $m \mid n$, the inclusion of fields $\mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^n}$ gives us inclusions of groups $\mathrm{SL}_2(q^m) \leqslant \mathrm{SL}_2(q^n)$, and $\mathrm{Spin}_3(q^m) \leqslant \mathrm{Spin}_3(q^n)$. The isomorphism $\widetilde{\rho}_{3,n}\colon \mathrm{SL}_2(q^n) \to \mathrm{Spin}_3(q^n)$ restricts to an isomorphism $\mathrm{SL}_2(q^m) \to \mathrm{Spin}_3(q^m)$ which is also a lift of π. By uniqueness, this ismorphism is $\widetilde{\rho}_{3,m}$. We can now obtain an isomorphism $\rho_{3,\infty}\colon \mathrm{SL}_2(q^\infty) \to \mathrm{Spin}_3(q^\infty)$ as the direct limit of the $\widetilde{\rho}_{3,n}$.
A similar argument shows that there are unique isomorphisms $\widetilde{\rho}_{4,n}\colon \mathrm{SL}_2(q^n)\times \mathrm{SL}_2(q^n) \to \mathrm{Spin}_4(q^n)$ for all $n < \infty$. $\rho_{4,\infty}$ is constructed in the same way as $\rho_{3,\infty}$.
In case $q = 3$, the isomorphisms $\rho_{i,1}$, $i = 4, 3$, can be obtained by restriction. Uniqueness can be shown by considering central extensions of $\mathrm{PSL}_2(\mathbb{F}_3)$ by $\{\pm 1\}$. □

9 The fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$

In this section we will construct the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^n)$, $n \in \mathbb{N}$, as fusion system over certain Sylow-2-subgroups of $\mathrm{Spin}_7(q^n)$. (The construction is as in [LO], modified by its corrections [LOc]).
The isomorphisms

$$\widetilde{\rho}_4\colon \mathrm{SL}_2(\mathbb{F}_{q^n})\times \mathrm{SL}_2(\mathbb{F}_{q^n}) \to \mathrm{Spin}_4(q^n), \quad \widetilde{\rho}_3\colon \mathrm{SL}_2(\mathbb{F}_{q^n}) \to \mathrm{Spin}_3(q^n)$$

combine with the homomorphism

$$\iota_{4,3}\colon \mathrm{Spin}_4(q^n)\times \mathrm{Spin}_3(q^n) \to \mathrm{Spin}_7(q^n)$$

resulting in a homomorphism

$$\omega\colon \mathrm{SL}_2(q^n)^3 \to \mathrm{Spin}_7(q^n)$$

The kernel of $\iota_{4,3}$ is $\{\pm(1,1)\}$ because $\mathrm{Spin}_4(q^n) \cap \mathrm{Spin}_3(q^n) = \{\pm 1\}$ (we view $\mathrm{Spin}_4(q^n)$ and $\mathrm{Spin}_3(q^n)$ as subgroups of $\mathrm{Spin}_7(q^n)$). Therefore the kernel of ω is $\{\pm(I,I,I)\}$. By Proposition 28, $\mathrm{Spin}_i(q^n)$ has center $\{\pm 1\}$ for $i = 4,3,7$. I.e. the centers are the kernels of $\pi\colon \mathrm{Spin}_i(q^n) \to \Omega_i(q^n)$. Let z denote the generator of $Z(\mathrm{Spin}_7(q))$. Given $A, B, C \in \mathrm{SL}_2(q^n)$ we see that

$$\begin{aligned} &\omega(A,B,C) = \widetilde{\rho}_4(A,B) * \widetilde{\rho}_3(C) \in Z(\mathrm{Spin}_7(q^n)) && \Leftrightarrow \\ &\pi(\widetilde{\rho}_4(A,B)) \oplus \pi(\widetilde{\rho}_3(C)) = \mathrm{id}_V && \Leftrightarrow \\ &(A,B) = \pm(I,I) \quad \text{and} \quad C = \pm I \end{aligned}$$

The elements of $\mathrm{SL}_2(q^n)^3$ that map to z under ω are therefore precisely $(-I,-I,I)$ and $(I,I,-I)$. Set $z_1 = \omega(-I,I,I)$; it is some other element of $\mathrm{Spin}_7(q^n)$ of order 2. Define $U = \langle z, z_1\rangle$. We introduce the notation $[A_1,A_2,A_3] = \omega(A_1,A_2,A_3)$. With this notation $U = [\pm I, \pm I, \pm I]$, $z = [-I,-I,I] = [I,I,-I]$, and $z_1 = [-I,I,I]$. U is isomorphic to the Klein four-group; $[A_1,A_2,A_3] = [B_1,B_2,B_3]$ exactly if $(A_1,A_2,A_3) = \pm(B_1,B_2,B_3)$, and multiplication of elements is performed coordinatewise.
A lot of quantities are needed to define $\mathcal{F}_{\mathrm{Sol}}(q^n)$. The next lemma introduces another one.

Lemma 30. *[LO, 2.3]. $N_{\mathrm{Spin}_7(q)}(U)$ contains an element τ of order 2 which satisfies*

$$\tau[A_1,A_2,A_3]\tau = [A_2,A_1,A_3], \quad \textit{for all } A_1,A_2,A_3 \in \mathrm{SL}_2(q^\infty).$$

Proof. Let $\theta\colon M_2(\overline{\mathbb{F}}_q) \to M_2(\overline{\mathbb{F}}_q)$ be the linear map given by

$$\theta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and note that $\theta(X) = X^{-1}$ if $\det(X) = -1$. Let $\widetilde{\tau}$ be the endomorphism of $M_2(\overline{\mathbb{F}}_q) \oplus M_2^0(\overline{\mathbb{F}}_q) = V_\infty$ given by

$$\widetilde{\tau}(X,Y) = (-\theta(X), -Y) \quad \text{for all } (X,Y) \in M_2(\overline{\mathbb{F}}_q) \oplus M_2^0(\overline{\mathbb{F}}_q).$$

Note that $\widetilde{\tau}$ has order 2. We claim that $\widetilde{\tau} \in \mathrm{SO}_7(q^\infty)$. Given $(X,Y) \in M_2(\overline{\mathbb{F}}_q) \oplus M_2^0(\overline{\mathbb{F}}_q)$,

$$Q(\widetilde{\tau}(X,Y)) = \det(-\theta(X)) + \det(-Y) = \det(X) + \det(Y) = Q(X,Y)$$

The matrix of $\widetilde{\tau}$ with respect to our orthogonal basis of V_∞ is $\mathrm{diag}\,(-1,1,1,1,-1,-1,-1)$, hence $\widetilde{\tau}$ is an isometry of (V_∞, Q) of determinant 1. As all entries of the matrix of $\widetilde{\tau}$ are in $\mathbb{F}_q$, we may also view $\widetilde{\tau}$ as an isometry of V_1. Its (-1)-eigenspace is clearly the 4-dimensional subspace V_- with orthogonal basis $\{(e_1,0),(0,e_2),(0,e_3),(0,e_4)\}$, and the discriminant of $Q|_{V_-}$ is $(-1)^4 = 1$ in this basis. By property 1. of Lemma 27, $\widetilde{\tau} \in \Omega_7(q)$. Choose a lifting $\tau \in \mathrm{Spin}_7(q^\infty)$ of $\widetilde{\tau}$. By property 2. of Lemma 27, τ has order 2.
The elements of $H(q^\infty)$ act on V_∞ by

$$[A_1,A_2,A_3].(X,Y) = \pi([A_1,A_2,A_3])(X,Y) = (\rho_4(A_1,A_2), \rho_3(A_3))(X,Y) = (A_1XA_2^{-1}, A_3XA_3^{-1})$$

for all $A_1,A_2,A_3 \in SL_2(q^\infty)$ and all $(X,Y) \in M_2(\overline{\mathbb{F}}_q) \oplus M_2^0(\overline{\mathbb{F}}_q)$. The action is well-defined, since the only other choice of (A_1,A_2,A_3) that produces the element $[A_1,A_2,A_3]$ is $(-A_1,-A_2,-A_3)$. Matrix computations show that $\theta(XY) = \theta(Y)\theta(X)$ for all $X,Y \in M_2(\overline{\mathbb{F}}_q)$. Given $A_1,A_2,A_3 \in SL_2(q^\infty)$ and $(X,Y) \in M_2(\overline{\mathbb{F}}_q) \oplus M_2^0(\overline{\mathbb{F}}_q)$ we see that

$$\begin{aligned} (\tau[A_1,A_2,A_3]\tau).(X,Y) &= (\widetilde{\tau} \circ (\rho_4(A_1,A_2), \rho_3(A_3)) \circ \widetilde{\tau})(X,Y) \\ &= \widetilde{\tau}(-A_1\theta(X)A_2^{-1}, -A_3YA_3^{-1}) \\ &= (\theta(A_2^{-1})\theta(\theta(X))\theta(A_1), A_3YA_3^{-1}) \\ &= (A_2XA_1^{-1}, A_3YA_3^{-1}) = [A_2,A_1,A_3].(X,Y) \end{aligned}$$

This means that

$$\tau[A_1, A_2, A_3]\tau \equiv [A_2, A_1, A_3] \pmod{\langle z\rangle} \tag{15}$$

The assignments $[A_1, A_2, A_3] \mapsto \tau[A_1, A_2, A_3]\tau$ and $[A_1, A_2, A_3] \mapsto [A_2, A_1, A_3]$ both define automorphisms of $H(q^\infty)$. (The first one is conjugation by τ; it maps into $H(q^\infty)$ by (15) since $z \in H(q^\infty)$). Both automorphisms map z to z, so they induce automorphisms of $H(q^\infty)/\langle z\rangle$, and these automorphisms are equal by (15). It is well known that $\mathrm{SL}_2(q^\infty)$ is perfect, and then so is $H(q^\infty)$ being a quotient of $\mathrm{SL}_2(q^\infty)^3$. Hence any lifting of an automorphism of $H(q^\infty)/\langle z\rangle$ to an automorphism of $H(q^\infty)$ is unique, since any two liftings must agree on all commutators of $H(q^\infty)$. We conclude that $\tau[A_1, A_2, A_3]\tau = [A_2, A_1, A_3]$ for all $A_1, A_2, A_3 \in \mathrm{SL}_2(q^\infty)$. Finally, it is clear that $\tau \in N_{\mathrm{Spin}_7(q)}(U)$, since $U = \{[\pm I, \pm I, \pm I]\}$ with all eight combinations of signs. □

We introduce a few more subgroups of $\mathrm{Spin}_7(q^n)$:

$$H(q^\infty) = \omega(\mathrm{SL}_2(q^\infty)^3)$$

And for $n < \infty$:

$$H(q^n) = H(q^\infty) \cap \mathrm{Spin}_7(q^n)$$
$$H_0(q^n) = \omega(\mathrm{SL}_2(q^n)^3)$$

Note that $H_0(q^n) \leqslant H(q^n)$

Lemma 31. *[LO, 2.5]. Let $n \in \mathbb{N}$ or let $n = \infty$. Then*

1. *$C_{\mathrm{Spin}_7(q^n)}(U) = H(q^n)$.*
2. *$N_{\mathrm{Spin}_7(q^n)}(U) = H(q^n) \cdot \langle\tau\rangle$.*

When $n < \infty$, $H(q^n) \cdot \langle\tau\rangle$ contains a Sylow-2-subgroup of $\mathrm{Spin}_7(q^n)$.

Proof. ad 1. We first show the equality in case $n = \infty$. As $U = \langle z, z_1\rangle$, clearly $C_{\mathrm{Spin}_7(q^\infty)}(U) = C_{\mathrm{Spin}_7(q^\infty)}(z_1)$. Recall that $z_1 = \omega(-I, I, I)$. We see that $\rho_4(-I, I) \oplus \rho_3(I) = \pi(z_1) \in \Omega_7(q^\infty) = \mathrm{SO}_7(q^\infty)$ has order 2, and with the notation of Lemma 27, $V_- = M_2(\mathbb{F}_{q^\infty})$ and $V_+ = M_2^0(\mathbb{F}_{q^\infty})$. Given $\widetilde{\alpha} \in \mathrm{Spin}_7(q^\infty)$, the lemma tells us that there are $\alpha_\pm \in \mathrm{O}(V_\pm, \det)$ such that $\pi(\widetilde{\alpha}) = \alpha_+ \oplus \alpha_-$ and that $\widetilde{\alpha} \in C_{\mathrm{Spin}_7(q^\infty)}(z_1)$ if and only if $\alpha_- \in \mathrm{SO}(V_-, \det)$. Since $\alpha_+ \oplus \alpha_- \in \mathrm{SO}_7(q^\infty)$ we see that $\alpha_- \in \mathrm{SO}(V_-, \det)$ if and only if $\alpha_+ \in \mathrm{SO}(V_+, \det)$. Hence $\widetilde{\alpha} \in C_{\mathrm{Spin}_7(q^\infty)}(z_1)$ if and only if

$$\pi(\widetilde{\alpha}) \in (\mathrm{SO}(V_-, \det) \oplus \{\mathrm{id}_{V+}\}) \circ (\{\mathrm{id}_{V-}\} \oplus \mathrm{SO}(V_+, \det)) \tag{16}$$

But since $\pi \colon \mathrm{Spin}_7(q^\infty) \to \mathrm{SO}_7(q^\infty)$ is surjective, (16) holds exactly when

$$\widetilde{\alpha} \in (\mathrm{Spin}_4(q^\infty) \times \{\pm 1\}) (\{\pm 1\} \times \mathrm{Spin}_3(q^\infty))$$

i.e.

$$C_{\mathrm{Spin}_7(q^\infty)}(U) = C_{\mathrm{Spin}_7(q^\infty)}(z_1) = \iota_{4,3}(\mathrm{Spin}_4(q^\infty) \times \mathrm{Spin}_4(q^\infty)) = H(q^\infty)$$

And then

$$C_{\mathrm{Spin}_7(q^n)}(U) = C_{\mathrm{Spin}_7(q^\infty)}(z_1) \cap \mathrm{Spin}_7(q^n) = H(q^n)$$

ad 2. $U = \langle z, z_1\rangle$ and z is central in $\mathrm{Spin}_7(q^\infty)$. It follows that $\mathrm{Aut}_{\mathrm{Spin}(q^\infty)}(U)$ has order at most 2. The potential non-identity automorphism is the one given by $z_1 \mapsto zz_1$. But that one is in fact a $\mathrm{Spin}_7(q^\infty)$-automorphism of U: With τ as defined in Lemma 30 we see that

$$\tau z_1 \tau = \tau[-I, I, I]\tau = [I, -I, I] = [-I, -I, I][-I, I, I] = zz_1$$

Thus $\tau \notin C_{\mathrm{Spin}_7(q^\infty)}(U)$ and

$$N_{\mathrm{Spin}_7(q^n)}(U) = C_{\mathrm{Spin}_7(q^\infty)}(U)\langle\tau\rangle = H(q^\infty)\langle\tau\rangle$$

Consequently

$$N_{\mathrm{Spin}_7(q^n)}(U) = N_{\mathrm{Spin}_7(q^\infty)}(z_1) \cap \mathrm{Spin}_7(q^n) = H(q^n)\langle\tau\rangle$$

because $\tau \in N_{\mathrm{Spin}_7(q)}(U)$ by definition.
ad 3. This can be shown by comparing the orders of $\mathrm{Spin}_7(q^n)$ and $H(q^n)\langle\tau\rangle$. See [LO, p. 932]. □

It is well known that every element of $\mathbb{F}_q$ can be written as a sum of two squares (the squares make up more than half the elements of $\mathbb{F}_{q^n}$). Let $\alpha, \beta \in \mathbb{F}_q$ satisfy $\alpha^2 + \beta^2 = -1$, and define

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then $A, B \in \mathrm{SL}_2(q)$ and $\langle A, B\rangle \cong Q_8$, the quaternion group. The characteristic polynomial of A is $x^2 + 1$, so A has two distinct eigenvalues $\pm\lambda \in \mathrm{SL}_2(q^2)$ and is conjugate in $\mathrm{SL}_2(q^2)$ to the diagonal matrix $\mathrm{diag}\,(\lambda, -\lambda)$. One can explicitly determine an $L \in \mathrm{SL}_2(q^2)$ such that $L^{-1}AL = \mathrm{diag}\,(\lambda, -\lambda)$. Then $C_{\mathrm{SL}_2(q^\infty)}(A) = LDL^{-1}$, where D denotes the subgroup of diagonal matrices of $\mathrm{SL}_2(q^\infty)$. Direct computations show that every element $C \in C_{\mathrm{SL}_2(q^\infty)}(A)$ satisfies $CBC = -B^{-1} = B$.
Define $C(q^\infty) = \{X \in C_{\mathrm{SL}_2(q^\infty)}(A) \mid \exists k \in \mathbb{N}\colon A^{2^k} = I\}$. We see that $C(q^\infty) \cap \mathrm{SL}_2(q^n)$ is conjugate to the subgroup of all diagonal matrices of $\mathrm{SL}_2(q^n)$ whose orders are powers of 2. Since the group diagonal matrices of $\mathrm{SL}_2(q^n)$ is isomorphic to $\mathbb{F}^*_{q^n}$, $C(q^\infty) \cap \mathrm{SL}_2(q^n)$ is a cyclic 2-group, and so $C(q^\infty)$ is a union of cyclic 2-groups. We say that $C(q^\infty)$ is the infinite cyclic 2-group. It is isomorphic to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.
We define the following groups:

$$\begin{aligned} Q(q^\infty) &= \langle C(Q^\infty), B\rangle \leqslant \mathrm{SL}_2(q^\infty), \\ C(q^n) &= C(q^\infty) \cap \mathrm{SL}_2(q^n), \\ Q(q^n) &= Q(q^\infty) \cap \mathrm{SL}_2(q^n), \end{aligned}$$

and

$$\begin{aligned} A(q^\infty) &= \omega(C(q^\infty)^3), & A(q^n) &= A(q^\infty) \cap \mathrm{Spin}_7(q^n), \\ S_0(q^\infty) &= \omega(Q(q^\infty)^3) \leqslant H(q^\infty), & S_0(q^n) &= S_0(q^\infty) \cap \mathrm{Spin}_7(q^n), \\ S(q^\infty) &= S_0(q^\infty)\langle\tau\rangle \leqslant H(q^\infty)\langle\tau\rangle, & S(q^n) &= S(q^\infty) \cap \mathrm{Spin}_7(q^n). \end{aligned}$$

We also set $\hat{A} = [A, A, A] \in A(q) \leqslant S_0(q)$, and $\hat{B} = [B, B, B] \in S_0(q)$. Note that $\hat{A}$ and $\hat{B}$ both have order 2, and that $\hat{A} \neq \hat{B}$.

Lemma 32. *[LO, Lemma 2.7]. For all $n < \infty$, $S(q^n)$ is a Sylow-2-subgroup of $H(q^n)\langle\tau\rangle$, and thereby also of* $\mathrm{Spin}_7(q^n)$. □

In general, given a finite p-group S_0 and a subgroup $\Gamma \leqslant \mathrm{Aut}(S_0)$ we will let $\mathcal{F}_{S_0}(\Gamma)$ denote the fusion system over S_0 whose morphisms are all possible restrictions of elements of Γ to homomorphisms $P \to Q$, $P, Q \leqslant S_0$.
The fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^n)$ will be constructed as the smallest fusion system containing the group fusion system $\mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$ and the fusion system $\mathcal{F}_{S_0(q^n)}(\Gamma_n)$ for some specific $\Gamma_n \leqslant \mathrm{Aut}(S_0(q^n))$. We will use the notation $\langle \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n)), \mathcal{F}_{S_0(q^n)}(\Gamma_n)\rangle$ for this fusion system. In the following, we describe how to choose Γ_n.

Given $X \in C(q^\infty)$ of order 2^k, we have $X^{u+2^{k'}} = X^u$ for all $k' \geq k$. If we let $\mathbb{Z}_2$ denote the 2-adic integers, it therefore makes sense to write X^u for $u \in \mathbb{Z}_2$. For each $u \in (\mathbb{Z}_2)^*$, define $\delta_u \in \mathrm{Aut}(A(q^\infty))$ by $\delta_u([X_1, X_2, X_3]) = [X_1, X_2, X_3^u]$ (it is clearly a homomorphism with $\delta_{u^{-1}}$ as inverse). Define $\gamma \in \mathrm{Aut}(A(q^\infty))$ by $\gamma([X_1, X_2, X_3]) = [X_3, X_1, X_2]$, i.e. γ permutes the coordinates cyclically. Finally set $\gamma_u = \delta_u \gamma \delta_u^{-1}$.

Lemma 33. *[LOc, 1.5]. There is an element $u' \in (\mathbb{Z}_2)^*$ with $u' \equiv 1 \pmod 4$, such that*

$$\langle \mathrm{Aut}_{\mathrm{Spin}_7(q^\infty)}(A(q^\infty)), \gamma_{u'}\rangle \cong C_2 \times \mathrm{GL}_3(\mathbb{F}_2) \qquad \square$$

If $X \in Q(q^\infty) = \langle C(q^\infty), B\rangle$, then since $CBC = B$ for all $C \in C(q^\infty)$, X can be given the form $X = A'B^j$ for some $A' \in C(q^\infty)$. As B has order 4, $X^{u'} = (A')^{u'}B^j$. We now define extensions of $\delta_{u'}$, γ, and $\gamma_{u'}$ to automorphisms of $S_0(q^n)$ as follows:

$$\widetilde{\delta}_{u'}([X_1, X_2, A'B^j]) = [X_1, X_2, (A')^{u'}B^j]$$
$$\widetilde{\gamma}([X_1, X_2, X_3]) = [X_3, X_1, X_2]$$
$$\widetilde{\gamma}_{u'} = \widetilde{\delta}_{u'}\widetilde{\gamma}\widetilde{\delta}_{u'}^{-1}$$

Note that conjugation by τ also defines an automorphism of $S_0(q^n)$ since it just permutes the first two coordinates; let $c_\tau \in S_0(q^n)$ denote this automorphism. For all $n \in \mathbb{N}$, we choose $\Gamma_n \leqslant \mathrm{Aut}(S_0(q^n))$ as follows:

$$\Gamma_n = \langle \mathrm{Inn}(S_0(q^n)), c_\tau, \widetilde{\gamma}_{u'}\rangle$$

And we define

$$\mathcal{F}_n = \mathcal{F}_{\mathrm{Sol}}(q^n) = \langle \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n)), \mathcal{F}_{S_0(q^n)}(\Gamma_n)\rangle$$

Theorem 34. *[LO, 2.11]. The fusion system $\mathcal{F}_n$ over $S(q^n)$ defined above is saturated. In addition it satisfies*

1. *$C_{\mathcal{F}_n}(z) = \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$.*
2. *All elements of order 2 of $S(q^n)$ are $\mathcal{F}_n$-conjugate.* $\square$

$\mathcal{F}_n$ is also exotic, see [LO, 3.4].

10 The Euler characteristic of $\mathcal{F}_{\mathrm{Sol}}(q)$

We wish to compute the Euler characteristic of $\mathcal{F}_{\mathrm{Sol}}(q)$. This requires an in-depth study of the structure and relations of the elementary abelian subgroups of $S(q)$. We refer to the appendix of [LO].
Let $\mathcal{E}$ be the set of all elementary abelian 2-subgroups of $\mathrm{Spin}_7(q)$ that contain z, and for each $n \in \mathbb{N}$, let $\mathcal{E}_n \subseteq \mathcal{E}$ be those of rank n. In general given $E \in \mathcal{E}$, we will let $\overline{E}$ denote the image of E in $\mathrm{SO}_7(q)$, i.e. $\overline{E} = \pi(E)$. Similarly, $\overline{g} = \pi(g)$ when $g \in \mathrm{Spin}_7(q)$. If $E \in \mathcal{E}$ and $h \in \mathrm{Spin}_7(q)$, then $\overline{hEh^{-1}} = \overline{h} \cdot \overline{E} \cdot \overline{h}^{-1}$, and $\chi \mapsto \chi \circ c_{\overline{h}^{-1}}$ defines a bijection of characters

$\mathrm{Hom}(\overline{E}, \{\pm 1\}) \to \mathrm{Hom}(\overline{h} \cdot \overline{E} \cdot \overline{h}^{-1}, \{\pm 1\})$. If we let $V = \bigoplus_\chi V_\chi$ be the eigenspace decomposition of $\overline{E}$ as given in Lemma 26, then $\bigoplus_\chi \overline{h} V_\chi$ is the eigenspace decomposition of $\overline{h} \cdot \overline{E} \cdot \overline{h}^{-1}$. This implies that if E and E' are conjugate elementary abelian 2-subgroups of $\mathrm{Spin}_7(q)$, then $\overline{E}$ and $\overline{E}'$ act similarly on $V = M_2(\mathbb{F}_q) \oplus M_2^0(\mathbb{F}_q)$, and it is enough to understand one of them.
Let $E \in \mathcal{E}$ be of rank $n \geq 2$, and let V_1 denote the eigenspace of the trivial character $\overline{E} \to \{\pm 1\}$. If $n = 2$, then $\overline{E} \cong C_2$; let $1 \neq \chi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$ be the nontrivial character. V_χ is then the (-1)-eigenspace which is not trivial. Since $\dim V = 7$, we get that $\dim V_\chi = 4$ by applying Lemma 27, and so $\dim V_1 = 3$.
Now assume that $n > 2$. We claim that $\dim V_1$ only depends on n. We prove this inductively: Assume the eigenspace of the trivial character of $\overline{E}'$ has dimension r for all $E' \in \mathcal{E}_{n-1}$. Let $1 \neq \chi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$. Then $\ker \chi$ has index 2 in $\overline{E}$, so there is an $E' \in \mathcal{E}_{n-1}$ such that $\overline{E}' = \ker \chi$. By construction, the trivial character of $\ker \chi$ has eigenspace $V_1 \oplus V_\chi$. Thus $\dim V_\chi = r - \dim V_1$. This shows that all eigenspaces except V_1 in the eigenspace decomposition $V = \bigoplus_\chi V_\chi$ have the same dimension. There are $|E| - 1 = 2^{n-1} - 1$ nontrivial characters of $\overline{E}$, so we get the formula $\dim V_1 + \dim V_\chi \cdot (2^{n-1} - 1) = 7$. The two formulas show that $\dim V_1$ only depends on r and n, which proves our claim. We can now compute the dimensions of the eigenspaces of all characters of $\overline{E}$ for all $E \in \mathcal{E}$ recursively; we have already dealt with the rank 2 case. If $n = 3$, then $\dim V_1 = 1$ and $\dim V_\chi = 2$ for each of the three nontrivial characters of $\overline{E}$. If $n = 4$, then $\dim V_1 = 0$ and $\dim V_\chi = 1$ for each of the seven nontrivial characters of $\overline{E}$.
The formula $\dim V_1 + \dim V_\chi \cdot (2^{n-1} - 1) = 7$ shows that if $n \geq 5$, then $V_\chi = 0$ for all nontrivial characters. This means that each element of $\overline{E}$ is the identity of V, but $\overline{E}$ has rank at least 4, which is a contradiction. Hence $\mathcal{E}$ contains no subgroups of rank 5 or greater that contain z. In particular $\mathcal{E}_n = \emptyset$ for all $n \geq 5$, and every elementary abelian 2-subgroup of $\mathrm{Spin}_7(q)$ of rank 4 contains z.
In the continued study of $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$, it turns out to be useful to introduce the following notion: Given $E \in \mathcal{E}$ we say that E and $\overline{E}$ are of type I if the eigenspaces of $\overline{E}$ all have square discriminants. Otherwise we say that E and $\overline{E}$ are of type II. We write $\mathcal{E}_n^I$ for the set of elements of $\mathcal{E}_n$ of type I, and $\mathcal{E}_n^{II}$ for the set of elements of $\mathcal{E}_n$ of type II; $\mathcal{E}_n$ equals the disjoint union of $\mathcal{E}_n^I$ and $\mathcal{E}_n^{II}$. If E and E' are conjugate in $\mathrm{Spin}_7(q)$ they have the same type. We first show that $\mathcal{E}_2^{II} = \emptyset$. If $E \in \mathcal{E}_2$, then $E = \langle z, \alpha \rangle$ for some element $z \neq \alpha \in \mathrm{Spin}_7(q)$ of order 2. Now $\overline{E} = \langle \overline{\alpha} \rangle$, and by Lemma 27 the eigenspace of the character $\overline{E} \to \{\pm 1\}$ given by $\overline{\alpha} \mapsto -1$ has square discriminant. The trivial character is the only other character, and its eigenspace must therefore also have square discriminant since V does.
We can summarize many of the results in the following table:

	$\mathcal{E}_2^I$	$\mathcal{E}_3^I$	$\mathcal{E}_3^{II}$	$\mathcal{E}_4^I$	$\mathcal{E}_4^{II}$
$\dim V_1$	3	1	1	0	0
$\dim V_\chi$	4	2	2	1	1
discV_1	sq.	sq.	nonsq.	-	-
discV_1	sq.	sq.	nonsq.	sq.	sq./nonsq.

Here $\dim V_1$ is the dimension of the eigenspace of the trivial character, and $\dim V_\chi$ is the dimension of the eigenspaces of each of the other characters. Whether these subspaces (with the restriction of the quadratic form Q) have square or nonsquare discriminants is listed in the last two rows. We have yet to prove that these discriminants are as in the table in the rank 3 and 4 cases:
If $E \in \mathcal{E}_3$, then $\overline{E} = \{\mathrm{id}, \overline{g}_1, \overline{g}_2, \overline{g}_3 := \overline{g}_1 \overline{g}_2\}$. We can define the three nontrivial characters by χ_i, $i = 1, 2, 3$ by $\chi_i(g_i) = 1$ and $\chi_i(g_j) = -1$ when $j \neq i$. We see that $V_{\chi_1} \oplus V_{\chi_2}$ is the eigenspace of $\overline{g}_3$ of eigenvalue -1. But $V_{\chi_1} \oplus V_{\chi_2}$ is the eigenspace of the nontrivial character of $\overline{\langle z, g_3 \rangle}$ which we know has square discriminant. Thus V_{χ_1} and V_{χ_2} have the same discriminant modulo squares.

Similarly, V_{χ_2} and V_{χ_3} have the same determinant modulo squares. So the discriminants of the eigenspaces of $\overline{E}$ are as presented in the table.
If $E \in \mathcal{E}_4^{II}$, then there are seven 1-dimensional eigenspaces of $\overline{E}$. As V has square discriminant, not all of them can have nonsquare discriminants.

Next we examine how many conjugacy classes of subgroups of $\mathrm{Spin}_7(q)$ each of the sets $\mathcal{E}_2^I$, $\mathcal{E}_3^I$, $\mathcal{E}_3^{II}$, $\mathcal{E}_4^I$, and $\mathcal{E}_4^{II}$ contain. At the same time we determine $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ when $E \notin \mathcal{E}_4^{II}$. z is of course fixed under conjugation by every element of $\mathrm{Spin}_7(q)$; define $\mathrm{Aut}(E, z)$ as the subgroup of automorphisms of E that map z to itself.

Lemma 35. *Let $E \in \mathcal{E}$. Then*

1. *If $E \in \mathcal{E}_4$, then $C_{\mathrm{Spin}_7(q)}(E) = E$.*
2. *If $E \in \mathcal{E}^{II}$, then there is a unique element $\overline{x}(E) \in \overline{E}$ such that for every nontrivial character χ of $\overline{E}$, $\mathrm{disc}(V_\chi)$ is a square if and only if $\chi(\overline{x}(E)) = 1$.*
3. *If $E \notin \mathcal{E}_4^{II}$, then $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) = \mathrm{Aut}(E, z)$.*
4. *If $E \in \mathcal{E}_4^{II}$ and if $X \leqslant E$ denotes the preimage of $\langle \overline{x}(E) \rangle$, then*

$$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) \geqslant \{\alpha \in \mathrm{Aut}(E) \mid \alpha|_X = \mathrm{id}_X, \alpha \equiv \mathrm{id} \mod \langle z \rangle\}$$

Proof. ad 1. Let $E \in \mathcal{E}_4$, let $a \in C_{\mathrm{Spin}_7(q)}(E)$, and let $1 \neq \chi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$. Applying Lemma 27.3 with $\alpha = \overline{a}$ to each $\overline{x}$ of E, we see that $\overline{a}(V_\chi) = V_\chi$. Since V_χ is 1-dimensional $\overline{a}$ is plus or minus the identity on V_χ, hence $\overline{a}$ has order 2. Let V_- be the sum of the V_χ on which $\overline{a}$ acts as $-\mathrm{id}$ and let V_+ be the sum of the other ones, i.e. the ones on which $\overline{a}$ acts as id. Because $a \in \mathrm{Spin}_7(q)$, we have $\overline{a} \in \Omega_7(q)$ and in particular $\det(\overline{a}) = 1$, so $\dim V_-$ is even. By applying Lemma 27 to $\overline{a}$ we also get that V_- has square discriminant. If $\dim V_- = 2$, then there are $\chi_1, \chi_2 \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$, $\chi_1 \neq \chi_2$, such that $V_- = V_{\chi_1} \oplus V_{\chi_2}$. Choose $g \in E$ such that $\chi_1(\overline{g}) \neq \chi_2(\overline{g})$. Then $\det(\overline{g}|_{V_-}) = -1$, and point 3. of Lemma 27 tells us that g and a do not commute, which contradicts that $a \in C_{\mathrm{Spin}_7(q)}(E)$. If $\dim V_- = 4$, then by Lemma 27.2, a has order 2. But then $a \in E$ since otherwise $\langle E, a \rangle$ has rank 5. If $\dim V_- = 6$ one obtains a contradiction like in the case $\dim V_- = 2$. We conclude that $C_{\mathrm{Spin}_7(q)}(E) = E$.

ad 2. Let $E \in \mathcal{E}_4^{II}$, and define $\overline{x}(E)$ as the isometry of V which is $-\mathrm{id}$ on eigenspaces V_χ of nonsquare discriminant, and id on eigenspaces V_χ of square discriminant. As there is an even number of eigenspaces of nonsquare dimension, $\overline{x}(E) \in \Omega_7(q)$ by Lemma 27.1. Let $x \in \mathrm{Spin}_7(q)$ be a lifting of $\overline{x}(E)$. To show that $x \in E$, let $g \in E$ be given. As $\overline{g} \in \Omega_7(q)$, Lemma 27.1 says that the (-1)-eigenspace of g has square discriminant, and so it must contain an even number of the eigenspaces of $\overline{E}$ that have nonsqaure discriminant. By Lemma 27.3, g and x commute. Thus $x \in C_{\mathrm{Spin}_7(q)}(E) = E$.
It is clear that there is at most one element with the stated property, since there are no two distinct elements of $\overline{E}$ on which all the nontrivial characters agree.

ad 3.+4. Assume $E \notin \mathcal{E}_4^{II}$ and let $E' \in \mathcal{E}$ have the same rank and type as E. Given $\alpha \in \mathrm{Iso}(\overline{E}, \overline{E}')$ and $\chi \in \mathrm{Hom}(\overline{E}', \{\pm 1\})$, $\chi \circ \alpha$ defines a character of $\overline{E}$, which is the trivial character if and only if χ is. Therefore V_χ and $V_{\chi \circ \alpha}$ have the same dimension and their discriminants are equal modulo squares. We can therefore define an isometry $V_\chi \to V_{\chi \circ \alpha}$ for each $\chi \in \mathrm{Hom}(\overline{E}', \{\pm 1\})$, and this defines an isometry $g \in \mathrm{O}_7(q)$ which satisfies $c_g = \alpha$ as isomorphisms $\overline{E} \to \overline{E}'$. Since $c_{-g} = c_g$ we may assume $g \in \mathrm{SO}_7(q)$. If $E, E' \in \mathcal{E}_4^{II}$ and we also require $\alpha(\overline{x}(E)) = \overline{x}(E')$, the same line of arguments work. Thus we have shown that if $E, E' \in \mathcal{E}$ have the same rank and type, then $\overline{E}$ and $\overline{E}'$ are conjugate by an element of $\mathrm{SO}_7(q)$, and furthermore that

$$\begin{aligned} \mathrm{Aut}(\overline{E}) &= \mathrm{Aut}_{\mathrm{SO}_7(q)}(\overline{E}), \text{ if } E \notin \mathcal{E}_4^{II}, \text{ and} \\ \{\alpha \in \mathrm{Aut}(\overline{E}) \mid \alpha(\overline{x}(E)) = \overline{x}(E)\} &= \mathrm{Aut}_{\mathrm{SO}_7(q)}(\overline{E}), \text{ if } E \in \mathcal{E}_4^{II} \end{aligned}$$

We claim that

$$C_{\mathrm{SO}_7(q)}(\overline{E}) \leqslant \Omega_7(q) \Leftrightarrow E \in \mathcal{E}_4^I \tag{17}$$

Assume $E \notin \mathcal{E}_4^I$. If $E \in \mathcal{E}_2^I$ or if $E \in \mathcal{E}_3^I$ choose $w \in V_1$ such that $Q(w)$ is a square. Let χ be a nontrivial character of $\overline{E}$; then $\dim V_\chi \geq 2$. Choose an orthogonal basis $\{v_i\}$ of nonisotropic elements of V_χ. If $Q(v_i)$ is a square for all i, then there is a linear combination w' of two of them such that $Q(w')$ is not a square. If $E \in \mathcal{E}_3^{II}$ or if $E \in \mathcal{E}_4^{II}$ choose an orthogonal basis of each eigenspace of E. Their union is an orthogonal basis $\{v_1, \ldots, v_7\}$ of V, but $Q(v_i)$ cannot be a square for all i, since E has eigenspaces of nonsquare discriminants. In this case we can therefore also choose w and w' from different eigenspaces, such that $Q(w)$ is a square while $Q(w')$ is not. Let W and W' be the 1-dimensional subspaces spanned by w and w', respectively. Then $W \perp W'$ are 1-dimensional subspaces of distinct eigenspaces of $\overline{E}$, W has square discriminant, and W' has nonsquare discriminant. Define $\gamma \in \mathrm{SO}_7(q)$ as the involution which is $-\mathrm{id}$ on $W \oplus W'$ and id on $(W \oplus W')^\perp$. Then $W \oplus W'$ is the (-1)-eigenspace of γ, and it has nonsquare discriminant by construction. Hence $\gamma \notin \Omega_7(q)$ by Lemma 27.1. But $\gamma(V_\chi) = V_\chi$ for each $\chi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$, so γ commutes with every element of $\overline{E}$. This proves '$\Rightarrow$' of (17). Assume conversely that $E \in \mathcal{E}_4^I$, and let $g \in C_{\mathrm{SO}_7(q)}(\overline{E})$ be given. Then $g(V_\chi) = V_\chi$ for each of the 1-dimensional eigenspaces of $\overline{E}$, and g must be multiplication by $+1$ or -1 on each of them. As $\det(g) = 1$, V_- must have even dimension, hence $g \in \Omega_7(q)$ by Lemma 27.1.
Next we show that $N_{\mathrm{SO}_7(q)}(\overline{E}) \leqslant \Omega_7(q)$ when $E \in \mathcal{E}_4^I$. We know that

$$N_{\mathrm{SO}_7(q)}(\overline{E})/C_{\Omega_7(q)}(\overline{E}) = N_{\mathrm{SO}_7(q)}(\overline{E})/C_{\mathrm{SO}_7(q)}(\overline{E}) \cong \mathrm{Aut}_{\mathrm{SO}_7(q)}(\overline{E}) = \mathrm{Aut}(\overline{E}) \cong \mathrm{GL}_3(\mathbb{F}_2) = \mathrm{SL}_3(\mathbb{F}_2)$$

and that $\mathrm{SL}_3(\mathbb{F}_2)$ is simple. As $\Omega_7(q)$ has index 2 in $\mathrm{SO}_7(q)$, $N_{\Omega_7(q)}(\overline{E})/C_{\Omega_7(q)}(\overline{E})$ has index 1 or 2 in $N_{\mathrm{SO}_7(q)}(\overline{E})/C_{\Omega_7(q)}(\overline{E})$. The index can't be 2, hence $N_{\mathrm{SO}_7(q)}(\overline{E}) = N_{\Omega_7(q)}(\overline{E})$.
We can now determine the number of conjugacy classes among subgroups of the same rank and type. Obviously $E, E' \in \mathcal{E}$ are conjugate in $\mathrm{Spin}_7(q)$ if and only if $\overline{E}$ and $\overline{E}'$ are $\Omega_7(q)$-conjugate. If E and E' have the same rank and type, then $\overline{E} = g\overline{E}'g^{-1}$ for some $g \in \mathrm{SO}_7(q)$. Assume $g \notin \Omega_7(q)$.

- If $E, E' \notin \mathcal{E}_4^I$, then by (17) there is a $\gamma \in \mathrm{SO}_7(q) \setminus \Omega_7(q)$ such that $\gamma \in C_{\mathrm{SO}_7(q)}(\overline{E})$. But then $c_g|_{\overline{E},\overline{E}'} = c_{g\gamma}|_{\overline{E},\overline{E}'}$, and $g\gamma \in \Omega_7(q)$. This shows that E is conjugate to every other subgroup of the same rank and type, and that $\mathrm{Aut}_{\mathrm{SO}_7(q)}(\overline{E}) = \mathrm{Aut}_{\Omega_7(q)}(\overline{E})$.
- If $E \in \mathcal{E}_4^I$, then $\overline{E}$ and $g\overline{E}g^{-1}$ represent different $\Omega_7(q)$-conjugacy classes, since otherwise $g \in \Omega_7(q) \cdot N_{\mathrm{SO}_7(q)}(\overline{E}) = \Omega_7(q)$. If $h \in \mathrm{SO}_7(q) \setminus \Omega_7(q)$, then $h\overline{E}h^{-1}$ and $g\overline{E}g^{-1}$ represent the same $\Omega_7(q)$-conjugacy class, because $h^{-1}g \in \Omega_7(q)$. We also note that $\mathrm{Aut}_{\mathrm{SO}_7(q)}(\overline{E}) = \mathrm{Aut}_{\Omega_7(q)}(\overline{E})$ since $N_{\mathrm{SO}_7(q)}(\overline{E}) \leqslant \Omega_7(q)$.

We have now shown that there are six conjugacy classes of elementary abelian subgroups of $\mathrm{Spin}_7(q)$: The sets $\mathcal{E}_2^I$, $\mathcal{E}_3^I$, $\mathcal{E}_3^{II}$, and $\mathcal{E}_4^{II}$ are four of them, and the union of the last two equals $\mathcal{E}_4^I$.
We now turn to the question of determining $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ for $E \notin \mathcal{E}_4^{II}$. Every element of $\mathrm{Aut}(E, z)$ induces an automorphism of $\overline{E}$ by composition with π. This defines a homomorphism $\Psi\colon \mathrm{Aut}(E, z) \to \mathrm{Aut}(\overline{E})$. We have shown that $\mathrm{Aut}(\overline{E}) = \mathrm{Aut}_{\Omega_7(q)}(\overline{E})$, and so Ψ is an extension

of the surjective homomorphism $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) \to \mathrm{Aut}_{\Omega_7(q)}(\overline{E})$ induced by $\pi\colon \mathrm{Spin}_7(q) \to \Omega_7(q)$. The kernel of Ψ is

$$\ker(\Psi) = \{\alpha \in \mathrm{Aut}(E, z) \mid \forall x \in E\colon \alpha(x) \equiv x \pmod{\langle z \rangle}\}$$

If we can show that $\ker(\Psi) \leqslant \mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ it will follow that $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) = \mathrm{Aut}(E, z)$. Let $\alpha \in \ker(\Psi)$ be given. Define $\chi\colon \overline{E} \to \{\pm 1\}$ by $\chi(\overline{x}) = 1$ if $\alpha(x) = x$ and $\chi(\overline{x}) = -1$ if $\alpha(x) = zx$. χ is well-defined and a homomorphism since α is. We may assume that χ is not the trivial character, since otherwise $\alpha = \mathrm{id}$. Choose $\psi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$ such that $V_\psi \neq 0$ and $V_{\psi\chi} \neq 0$, and note that $V_\psi \neq V_{\psi\chi}$. By an argument similar to one used to prove (17), there are 1-dimensional, non-isotropic 1-dimensional subspaces $W \subseteq V_\psi$ and $W' \subseteq V_{\psi\chi}$ such that W and W' have the *same* discriminant modulo squares. Define $\overline{g} \in \mathrm{O}_7(q)$ as the involution with (-1)-eigenspace $W \oplus W'$. By Lemma 27.1, $g \in \Omega_7(q)$; let $g \in \mathrm{Spin}_7(q)$ be a lifting of $\overline{g}$. We claim that $c_g = \alpha$. Given $x \in \mathrm{Spin}_7(q)$, $\overline{g}$ and $\overline{x}$ commute because $\overline{g}$ maps each eigenspace of $\overline{E}$ to itself. Thus $c_g(x) = gxg^{-1} \in \{zx, x\}$. By use of Lemma 27.3,

$$c_g(x) = \begin{cases} x & \text{when } \det(\overline{x}|_{W \oplus W'}) = 1 \\ zx & \text{when } \det(\overline{x}|_{W \oplus W'}) \neq 1 \end{cases}$$

$\overline{x}|_{W \oplus W'}$ is multiplication by $\psi(\overline{x})$ on W and multiplication by $\psi\chi(\overline{x})$ on W'. This shows that $\det(\overline{x}|_{W \oplus W'}) = \chi(\overline{x})$. We conclude that $c_g(x) = \alpha(x)$.

By a similar argument, one can show that

$$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) \geqslant \{\alpha \in \mathrm{Aut}(E) \mid \alpha|_X = \mathrm{id}_X, \alpha \equiv \mathrm{id} \pmod{\langle z \rangle}\}$$

The extra requirement of α (i.e. that $\alpha|_X = \mathrm{id}_X$) implies $\chi(\overline{x}(E)) = \overline{x}(E)$, which ensures that $V_{\psi\chi}$ and V_ψ have the same discriminant. We choose $\psi \neq \chi$, since otherwise $V_{\chi\psi} = V_1 = 0$. Then set $W = V_\psi$, $W' = V_{\psi\chi}$. The rest of the argument is the same. □

Determining $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ when $E \in \mathcal{E}_4^{II}$ is slightly more technical. Let f_n denote the Frobenius automorphism of $\mathbb{F}_{q^n}$ for each $n \in \mathbb{N}$. Then $f_n^{n/m} = f_m$ whenever $m \mid n$; let f denote the automorphism of $\overline{\mathbb{F}}_q$ they define. Then f acts on $\mathrm{Spin}_7(q^\infty)$ as described earlier, and the action defines an automorphism of $\mathrm{Spin}_7(q^\infty)$ which we denote ψ. $\mathrm{Spin}_7(q)$ is exactly the subgroup fixed by ψ^q.

Lemma 36. *Let $\mathcal{C}$ and $\mathcal{C}'$ denote the two conjugacy of subgroups of* $\mathrm{Spin}_7(q)$ *whose union is $\mathcal{E}_4^I$, and let $E \in \mathcal{E}_4$. Then*

1. *There is an $a \in \mathrm{Spin}_7(q^\infty)$ such that $aEa^{-1} \in \mathcal{C}$. For such an a, the element $a^{-1}\psi^q(a) \in \mathrm{Spin}_7(q^\infty)$ depends only on E; we denote it by $x_{\mathcal{C}}(E)$.*
2. *$x_{\mathcal{C}}(E) \in E$.*
3. *$E \in \mathcal{C}$ if and only $x_{\mathcal{C}}(E) = 1$, and $E \in \mathcal{C}'$ if and only $x_{\mathcal{C}}(E) = z$.*
4. *If E has type II (such that $x_{\mathcal{C}}(E) \notin \langle z \rangle$ by 3.), then*

$$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) = \{\alpha \in \mathrm{Aut}(E, z) \mid \alpha(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(E)\}$$

Proof. ad 1. The results proved so far for $\mathrm{Spin}_7(q)$ obviously hold with q^n in place of q, since q^n is just another odd prime power. Viewing E as a subgroup of $\mathrm{Spin}_7(q^2)$, E has type I, since $\mathbb{F}_q$ is precisely the subfield of squares of $\mathbb{F}_{q^2}$. So if $E' \in \mathcal{C}$, then $E, E' \leqslant \mathrm{Spin}_7(q^2)$ have the same rank and type, and there is an $\overline{a} \in \mathrm{SO}_7(q^2)$ such that $\overline{a}\overline{E}\overline{a}^{-1} = \overline{E}'$. As $\mathbb{F}_{q^2}$ is the subfield

of squares in $\mathbb{F}_{q^4}$, the image under θ of $\mathrm{SO}_7(q^2)$ equals 1 in $\mathbb{F}_{q^4}^*/(\mathbb{F}_{q^4}^*)^2$, i.e. $\mathrm{SO}_7(q^2) \leqslant \Omega_7(q^4)$. Thus $\overline{a}$ lifts to an element $a \in \mathrm{Spin}_7(q^4) \leqslant \mathrm{Spin}_7(q^\infty)$, and $aEa^{-1} = E'$.
Now assume $b \in \mathrm{Spin}_7(q^\infty)$ is another element which satisfies $bEb^{-1} \in \mathcal{C}$. Then aEa^{-1} and bEb^{-1} are conjugate in $\mathrm{Spin}_7(q)$; choose $g \in \mathrm{Spin}_7(q)$ such that $gbE(gb)^{-1} = aEa^{-1}$. Then $gba^{-1} \in N_{\mathrm{Spin}_7(q^\infty)}(E')$. Let $x \in N_{\mathrm{Spin}_7(q^\infty)}(E')$ be given, then $x \in N_{\mathrm{Spin}_7(q^k)}(E')$ for some sufficiently large k. We know that $C_{\mathrm{Spin}_7(q^k)}(E') = E'$ (Lemma 35.1 with $q = q^k$), and that $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E') = \mathrm{Aut}(E', z)$. As $c_x \in \mathrm{Aut}(E', z)$ we conclude that $x \in N_{\mathrm{Spin}_7(q)}(E')$. This argument shows that $N_{\mathrm{Spin}_7(q^\infty)}(E') = N_{\mathrm{Spin}_7(q)}(E')$. Hence gba^{-1} and thereby also ba^{-1} are elements of $\mathrm{Spin}_7(q)$. In particular $\psi^q(ba^{-1}) = ba^{-1}$, and

$$b^{-1}\psi^q(b) = b^{-1}\psi^q(ba^{-1}a) = a^{-1}\psi^q(a)$$

ad 2. Let $g \in E$ be given, and let $a \in \mathrm{Spin}_7(q^\infty)$ be such that $aEa^{-1} \in \mathcal{C}$. Then $aga^{-1} \in \mathrm{Spin}_7(q)$, so

$$aga^{-1} = \psi^q(aga^{-1}) = \psi^q(a)g\psi^q(a)^{-1},$$

and then g conjugated by $x_{\mathcal{C}}(E)$ is

$$x_{\mathcal{C}}(E)gx_{\mathcal{C}}(E)^{-1} = a^{-1}\psi^q(a)g\psi^q(a)^{-1}a = g$$

Hence $x_{\mathcal{C}}(E) \in C_{\mathrm{Spin}_7(q^\infty)}(E)$. But then $x_{\mathcal{C}}(E) \in C_{\mathrm{Spin}_7(q^k)}(E)$ for some $k \in \mathbb{N}$, and $C_{\mathrm{Spin}_7(q^k)}(E) = E$.

ad 3. If $E \in \mathcal{C}$, choose $a = 1$. Then clearly $x_{\mathcal{C}}(E) = 1$. If $E \in \mathcal{C}'$ and $E' \in \mathcal{C}$, then E and E' have the same rank and type, and we can find $\overline{a} \in \mathrm{SO}_7 \setminus \Omega_7(q)$ such that $\overline{a}\overline{E}\overline{a}^{-1} = \overline{E}'$. As in the proof of 1., $\overline{a} \in \Omega_7(q^2)$, so $\overline{a}$ has a lifting $a \in \mathrm{Spin}_7(q^2)$. Now $aEa^{-1} \in \mathcal{C}$, but $a \notin \mathrm{Spin}_7(q)$ since $E \notin \mathcal{C}$. Hence $x_{\mathcal{C}}(E) \neq 1$. But $\overline{a} \in \mathrm{SO}_7(q)$, so $\psi^q(\overline{a}) = \overline{a}$. Thus $\overline{x_{\mathcal{C}}(E)} = \mathrm{id}$, and this forces $x_{\mathcal{C}}(E) = z$.
To complete this part of the proof we need to see that $x_{\mathcal{C}}(E) \in \langle z \rangle$ implies that E has type I. Let $a \in \mathrm{Spin}_7(q^\infty)$ be such that $aEa^{-1} \in \mathcal{C}$, and assume that $x_{\mathcal{C}}(E) \in \langle z \rangle$. Then either $\psi^q(a) = a$ or $\psi^q(a) = za$. In both cases we have $\psi^q(\overline{a}) = \overline{a}$, i.e. $\overline{a} \in \mathrm{SO}_7(q)$. We have already seen that if $\bigoplus_\chi V_\chi$ is the eigenspace decomposition of $\overline{E}$, then $\bigoplus_\chi \overline{a}V_\chi$ is the eigenspace decomposition of $\overline{a}\overline{E}\overline{a}^{-1}$. As $\overline{a}$ is an isometry, V_χ and $\overline{a}V_\chi$ have the same discriminants modulo squares. As $a^{-1}Ea^{-1}$ has type I, so does E.

ad 4. Let $E \in \mathcal{E}_4^{II}$. Then by 3., $x_{\mathcal{C}}(E) \notin \langle z \rangle$. If $g \in N_{\mathrm{Spin}_7(q)}(E)$, then

$$gx_{\mathcal{C}}(E)g^{-1} = (ag^{-1})^{-1}\psi^q(ag^{-1}) = x_{\mathcal{C}}(E)$$

where $a \in \mathrm{Spin}_7(q^\infty)$ is such that $aEa^{-1} \in \mathcal{C}$. This provides one inclusion. In the proof of Lemma 35 we saw that

$$\mathrm{Aut}_{\Omega_7(q)}(E) = \mathrm{Aut}_{\mathrm{SO}_7(q)}(E) = \{\alpha \in \mathrm{Aut}(\overline{E}) \mid \alpha(\overline{x}(E)) = \overline{x}(E)\}$$

We also have a surjective homomorphism $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) \to \mathrm{Aut}_{\Omega_7(q)}(\overline{E})$ induced by $\pi\colon \mathrm{Spin}_7(q) \to \Omega_7(q)$. Since every $\alpha \in \mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ maps $x_{\mathcal{C}}(E)$ to itself and $\overline{x_{\mathcal{C}}(E)} \neq \mathrm{id}$, we conclude that $\overline{x_{\mathcal{C}}(E)} = \overline{x}(E)$. This implies that every element of

$$\mathrm{Aut}(E, z, x_{\mathcal{C}}(E)) \overset{def}{=} \{\alpha \in \mathrm{Aut}(E, z) \mid \alpha(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(E)\}$$

induces an automorphism of $\overline{E}$ which maps $\overline{x}(E)$ to itself, and clearly $\mathrm{Aut}(E, z, x_{\mathcal{C}}(E)) \geqslant \mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$. I.e. we have a surjective homomorphism $\Psi\colon \mathrm{Aut}(E, z, x_{\mathcal{C}}(E)) \to \mathrm{Aut}_{\Omega_7(q)}(E)$. Lemma 35.4 states that

$$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) \geqslant \{\alpha \in \mathrm{Aut}(E) \mid \alpha|_X = \mathrm{id}_X;\ \alpha \equiv \mathrm{id} \pmod{\langle z \rangle}\}$$

where X was defined as the preimage in E of $\langle \overline{x}(E)\rangle \leqslant \overline{E}$. But the right hand side is precisely the kernel of Ψ. This finishes the proof. □

The results presented so far tell us what we need to know about every elementary abelian subgroups E of $S(q) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$: We know what the $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$-automorphisms of E are, and how the elementary abelian subgroups of $S(q)$ are $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$-conjugate. Adding the morphisms of Γ_1 to $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$ to construct $\mathcal{F}_{\mathrm{Sol}}(q)$ may cause some of the $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$-conjugacy classes to be $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate, and the automorphism groups may become larger. In the following we investigate what the effects are.

Lemma 37. *[LOc, 1.8]. Let X be a generator of the cyclic group $C(q)$, and let $Y \in C(q^2)$ be such that $Y^2 = X$. Define $E_{000} = \langle z, z_1, \hat{A}, \hat{B}\rangle \leqslant S_0(q)$. Then E_{000} is elementary abelian of rank 4 and type I. Furthermore the following holds:*

1. *The subgroups*

$$E_{001} \overset{def}{=} \langle z, z_1, \hat{A}, [B, B, XB]\rangle, \quad and \quad E_{100} \overset{def}{=} \langle z, z_1, \hat{A}, [XB, B, B]\rangle$$

   are both elements of $\mathcal{E}_4$.

2. *E_{001} represents the other $\mathrm{Spin}_7(q)$ conjugacy class of subgroups of type I, and E_{100} represents the class of subgroups of type II.*

3. *$\phi(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(\phi(E))$, for all $\phi \in \Gamma_1$, and $E = E_{000}, E_{001}, E_{100}$.*

Proof. One checks that AB has trace 0, and that

$$\{(I, 0), (A, 0), (B, 0), (AB, 0), (0, A), (0, B), (0, AB)\}$$

is a basis corresponding to the eigenspace decomposition

$$V_1 = M_2(\mathbb{F}_q) \oplus M_2^0(\mathbb{F}_q) = \bigoplus_{\chi} V_\chi$$

of E_{000}. Since $\det A = \det B = \det AB = 1$, all the eigenspaces have discriminant 1 in this basis, hence E_{000} has type I.
In greater generality, define $E_{ijk} = \langle z, z_1, \hat{A}, [X^iB, X^jB, X^kB]\rangle$. To see that E_{ijk} is elementary abelian, we just need to check that $(X^mB)^2 = -I$. However, we know that $XBX = B$, i.e. $BX = X^{-1}B$, and then clearly $X^mBX^mB = B^2 = -I$. It is clear that E_{ijk} has rank 4.
Next we compute $x_{\mathcal{C}}(E_{ijk})$. Note that

$$Y^{-m}(X^iB)Y^m = Y^{-m}X^iY^{-m}B = Y^{-2m}X^mB = B$$

where we have used that Y and X commute (they are both elements of $C(q^2)$ which is abelian) and that $Y^2 = X$. Y^{-m} clearly centralizes z, z_1 and $\hat{A}$. Thus

$$[Y^i, Y^j, Y^k]^{-1}E_{ijk}[Y^i, Y^j, Y^k] = E_{000}$$

As $Y \in \mathrm{SL}_2(q^2)$ we have that $\psi^q(Y^m) = (-Y)^m$. Now

$$x_{\mathcal{C}}(E_{ijk}) = [Y^i, Y^j, Y^k] * \psi^q([Y^i, Y^j, Y^k]^{-1}) = [(-I)^i, (-I)^j, (-I)^k]$$

In particular, $x_{\mathcal{C}}(E_{001}) = [I, I, -I] = z$, and $x_{\mathcal{C}}(E_{100}) = [-I, I, I] = z_1$. This proves that E_{001} has type I, but that $E_{001} \notin \mathcal{C}$, and that E_{100} has type II, by use of Lemma 36.

When $E \in \mathcal{E}_4$, we already know that $gx_{\mathcal{C}}(E)g^{-1} = x_{\mathcal{C}}(gEg^{-1})$ for all $g \in \mathrm{Spin}_7(q)$. As $\Gamma_1 = \langle \mathrm{Inn}(S_0(q^n)), c_\tau, \widetilde{\gamma}_{u'}\rangle$, we just need to check that $\widetilde{\gamma}_{u'}(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(\widetilde{\gamma}_{u'}(E))$. As $u' \equiv 1$

(mod 4) and A has order 4, we see that $\widetilde{\gamma}_{u'}(\hat{A}) = \hat{A}$. Furthermore, it is clear that $\widetilde{\gamma}_{u'}$ maps z and z_1 to themselves. By definition of $\widetilde{\gamma}_{u'}$ we get that

$$\widetilde{\gamma}_{u'}([X^iB, X^jB, X^kB]) = [X^{-ku'}B, X^iB, X^{ju'}B], \quad \text{and}$$
$$\widetilde{\gamma}_{u'}(x_{\mathcal{F}}(E_{ijk})) = \widetilde{\gamma}_{u'}([(-I)^i, (-I)^j, (-I)^k]) = [(-I)^k, (-I)^i, (-I)^{-j}].$$

And then

$$x_{\mathcal{C}}(\widetilde{\gamma}_{u'}(E_{ijk})) = [(-I)^{-ku'}, (-I)^i, (-I)^{ju'}] = [(-I)^{-k}, (-I)^i, (-I)^j]$$

which proves that $\widetilde{\gamma}_{u'}(x_{\mathcal{C}}(E_{ijk})) = x_{\mathcal{C}}(\widetilde{\gamma}_{u'}(E_{ijk}))$ as desired. □

We can now determine the $\mathcal{F}_{\mathrm{Sol}}(q)$-automorphisms of E_{000}, E_{001} and E_{100} and whether or not they are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate. (The results presented below are more or less what Lemma 3.1 of [LO] states. However, the corrections [LOc] to [LO] invalidate the proof, and the proof did not even treat elementary abelian subgroups that do not contain z).

As noted in the proof of Lemma 37, if c_g, $g \in \mathrm{Spin}_7(q)$, is an $\mathcal{F}_{\mathrm{Sol}}(q)$-isomorphism with source $E \in \mathcal{E}_4$, then $c_g(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(c_g(E))$. The $\mathcal{F}_{\mathrm{Sol}}(q)$-homomorphism are all possible compositions of restrictions of elements of Γ_1 and of $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$-homomorphisms. By repeated use of part 3. of the above lemma and the identity $c_g(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(c_g(E))$, we see that if $E \in \mathcal{C}$, then $\phi(E) \in \mathcal{C}$ for all $\phi \in \mathrm{Iso}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E, \phi(E))$, since $x_{\mathcal{C}}(E) = 1$. Therefore E_{000} is not $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate to E_{001} or E_{100}. On the other hand

$$\widetilde{\gamma}_{u'}(x_{\mathcal{C}}(E_{001})) = \widetilde{\gamma}_{u'}(z) = \widetilde{\gamma}(z) = z_1$$

and we get that $\widetilde{\gamma}_{u'}(E_{001})$ has type II, i.e. is in the $\mathrm{Spin}_7(q)$-conjugacy class of E_{100}. Thus E_{001} and E_{100} are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate.

If $\alpha \in \mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{001})$, then α has the form $\alpha = \phi \circ \phi'$, where $\phi = c_g$ for some $g \in \mathrm{Spin}_7(q)$ or $\phi \in \Gamma_1$, and ϕ' is a composition of elements of Γ_1 and $\mathcal{F}_{S(q)}(\mathrm{Spin}_7(q))$-homomorphisms. If $\phi = c_g$, then $x_{\mathcal{C}}(\alpha(E_{001})) = \phi(x_{\mathcal{C}}(\phi'(E_{001})))$. If $\phi \in \Gamma_1$, then

$$x_{\mathcal{C}}(\alpha(E_{001})) = \phi\phi^{-1}(x_{\mathcal{C}}(E_{001})) = \phi(x_{\mathcal{C}}(\phi^{-1}(E_{001}))) = \phi(x_{\mathcal{C}}(\phi'(E_{001})))$$

In either case, we get inductively that $\alpha(x_{\mathcal{C}}(E_{001})) = x_{\mathcal{C}}(\alpha(E_{001})) = x_{\mathcal{C}}(E_{001})$. As $x_{\mathcal{C}}(E_{001}) = z$, this shows that every $\mathcal{F}_{\mathrm{Sol}}(q)$-automophism of E_{001} maps z to itself. But then $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{001}) = \mathrm{Aut}(E_{001}, z)$, since $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E_{001}) = \mathrm{Aut}(E, z)$.

We also know that $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E_{000}) = \mathrm{Aut}(E, z)$, and we see that $\widetilde{\gamma}_{u'}$ and c_τ map E_{000} to itself; they both fix $\hat{A}$ and $\hat{B}$, and they define automorphisms of $\langle z, z_1\rangle$. There are six automorphisms of $\langle z, z_1\rangle$, and these can all be realized by $c_\tau^r \circ \widetilde{\gamma}_{u'}^s$, $r = 0, 1$, $s = 0, 1, 2$. This shows that $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{000}) = \mathrm{Aut}(E_{001})$.

We now turn to the elementary abelian subgroups E of $S(q)$ of rank 3. We claim that they are all $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate. Note that if $z \notin E$, then $\langle E, z\rangle$ has rank 4 and is $\mathrm{Spin}_7(q)$-conjugate to either E_{000}, E_{001}, or E_{100}. This implies that E is $\mathrm{Spin}_7(q)$ conjugate to $\langle z_1, \hat{A}, \hat{B}\rangle$, $\langle z_1, \hat{A}, [B, B, XB]\rangle$, or $\langle z_1, \hat{A}, [B, B, XB]\rangle$. But $\widetilde{\gamma}_{u'}^2(z_1) = z$, which shows that each of these groups are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate to an elementary abelian subgroup of rank 3 that contains z. It is therefore enough to show that the two $\mathrm{Spin}_7(q)$-conjugacy classes of elementary abelian subgroups of $S(q)$ of rank 3 that contain z are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate.

A subgroups E of E_{001} of rank 3 that contains z has type I, since every eigenspace of E_{001} is contained in an eigenspace of E. A subgroup E of E_{100} of rank 3 that contains z may have type I or type II, but it is possible to choose an E of type II. Now let $\phi \in \mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{001}, E_{100})$, then $\phi(z) = z_1$. Choose $\alpha \in \mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{100}) = \mathrm{Aut}(E_{100}, z)$ such that $z_1 \in \alpha(E)$. Then $z \in \phi^{-1}(\alpha(E))$, and $\phi^{-1}(\alpha(E)) \leqslant E_{001}$ has type I. This finishes the proof of the claim.

If $E \leqslant E_{000}$ has rank 3, then $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E) = \mathrm{Aut}(E)$. All in all we have shown that all elementary abelian subgroups $E \leqslant S(q)$ of rank 3 are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate, and that $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E) = \mathrm{Aut}(E)$.

In particular all elementary abelian subgroups of rank 1 or 2 are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate, and $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E) = \mathrm{Aut}(E)$ for any such subgroup.

We can now determine the Euler characteristic of $\mathcal{F}_{\mathrm{Sol}}(q)^*$. For $r = 1, 2, 3$, choose $E_r \leqslant S(q)$ such that E_r is elementary abelian of rank r. The number of $\mathcal{F}_{\mathrm{Sol}}(q)$-automorphisms of E_r is

$$\left|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_r)\right| = |\mathrm{Aut}(E_r)| = |\mathrm{GL}_r(\mathbb{F}_2)| = \prod_{i=0}^{r}(2^r - 2^i)$$

Thus

$$\begin{aligned}
|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_1)| &= 1, \\
|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_2)| &= 6, \\
|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_3)| &= 168, \\
|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{000})| &= 20160
\end{aligned}$$

To obtain the number of $\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}$-automorphisms of E_{001}, we just divide $|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{000})|$ by $2^4 - 1 = 15$. I.e.

$$|\mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q)}(E_{001})| = 1344$$

The values of the Möbius function are

$$\mu(E_1) = -1, \quad \mu(E_2) = 2, \quad \mu(E_3) = -8, \quad \mu(E_{000}) = \mu(E_{001}) = 64$$

And the Euler characteristic is

$$\chi(\mathcal{F}_{\mathrm{Sol}}(q)) = \frac{1}{1} + \frac{-2}{6} + \frac{8}{168} + \frac{-64}{20160} + \frac{-64}{1244} = \frac{209}{315}$$

Note that $\chi(\mathcal{F}_{\mathrm{Sol}}(q))$ does not depend on q at all; it depends on neither the characteristic of $\mathbb{F}_q$ nor its order.

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