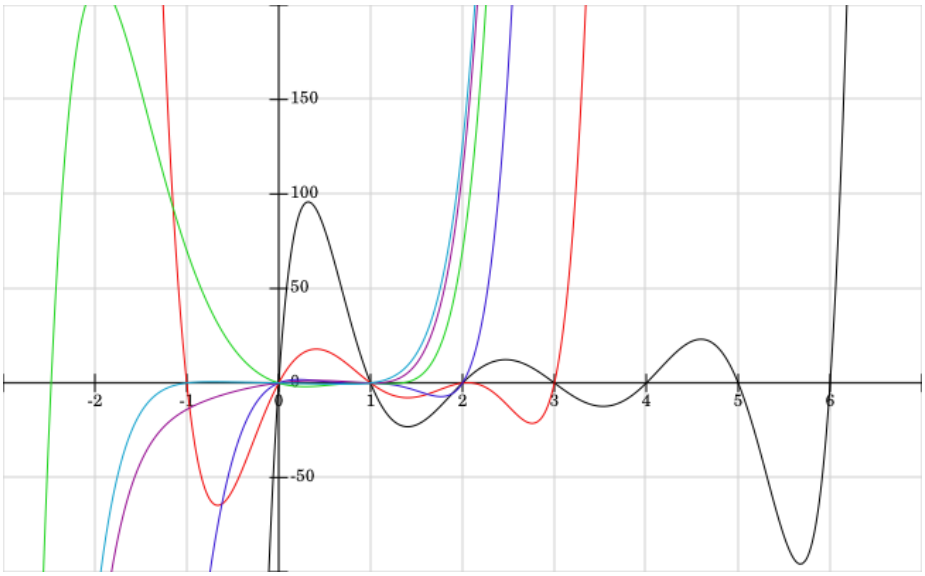
The s-Chromatic Polynomial

Gesche Nord



MSc Thesis

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Cover illustration: The six s-chromatic polynomials of the 6-simplex Δ^6.

Abstract

In this thesis we will define and prove the existence of the *s-chromatic polynomial* for finite abstract simplicial complexes. First we will give an overview on some of the most important results on the chromatic polynomial for graphs and introduce vertex colorings for finite abstract simplicial complexes. Then we will prove the existence of chromatic polynomials for finite abstract simplicial complexes based on this vertex colorings and show that they can be calculated as a sum of graph chromatic polynomials. Furthermore we will generalize some of the properties of the graph chromatic polynomial to s-chromatic polynomials of abstract simplicial complexes.

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Introduction

The chromatic polynomial is an invariant for graphs that was introduced in 1912 by George David Birkhoff. As a function of the number of colors it counts all possible distinct vertex colorings of a given graph. Through edge deletion and contraction the chromatic polynomial can be expressed as a sum of smaller graphs with less vertices or less edges. Therefore it can be calculated recursively and many properties can be proven inductively. Birkhoff originally introduced the chromatic polynomial in an attempt to prove the *Four Color Problem*, which states that any map can be colored with four colors such that no two adjacent regions have the same color. The link to Graph Theory is that any map can be expressed as a planar graph. Birkhoff wanted to prove that 4 is not a chromatic root for planar graphs. Especially the roots of the chromatic polynomial and its coefficients are still subjects of research in Graph Theory today.
Abstract simplicial complexes are mathematical objects that are mainly studied in Topology but in principle they can be viewed as a higher dimensional generalization of graphs. Therefore it seems natural to try to generalize some ideas of Graph Theory to simplicial complexes in order to possibly get new results on their underlying topological structures. In the preprint, *Vertex Colorings of Simplicial Complexes* [5], Dobrinskaya, Møller and Notbohm started to develop some theoretical ideas based on special vertex colorings for abstract simplicial complexes. This vertex colorings raise the question if a similar invariant as the chromatic polynomial can be found for simplicial complexes.
In the following we will show that there exists a *s-chromatic polynomial* for simplicial complexes and generalize some properties of the graph chromatic polynomial to the s-chromatic polynomials.
Chapter 1 gives an overview of some of the most important results for graph colorings and the chromatic polynomial for graphs.
The second chapter is an introduction to the vertex colorings of simplicial complexes as defined by by Dobrinskaya/Møller/Notbohm and is mainly based on [5]. We will also introduce a generalization of uniquely colorability of graphs to the colorings of simplicial complexes.
The third chapter is based on personal investigations. We will prove the existence of the s-chromatic polynomials for finite abstract simplicial complexes, based on the vertex colorings introduced in Chapter 2. We will also show that the s-chromatic polynomials can be expressed as a sum of graph chromatic polynomials and prove some generalizations of the results for graph chromatic polynomials.

Chapter 1

Vertex Colorings and the Chromatic Polynomial for Graphs

The structure of a graph is defined by its *vertices* and their pairwise relations indicated by *edges*. A *vertex coloring* assigns colors to the vertices of a graph and thereby defines a partition of its vertex set. Therefore, in order to learn more about the possible structures that graphs can have, it is often useful to study vertex colorings.

Definition 1.1. Let $G = (V, E)$ be a finite graph, with vertex set V and edge set E. A *vertex coloring* (or simply *coloring*) of G is a map $f : V \to P$ from the vertex set V of G to a palette P of colors. The map f is called a *proper coloring* if $|f^{-1}(c) \cap e| \leq 1$, for all $c \in P$ and $e \in E$.
A coloring f using at most r colors is called a *(proper) r-coloring*. A graph that admits a proper r-coloring is *r-colorable*.

If we want to learn more about the underlying structure of a graph only proper vertex colorings will be of greater interest, therefore all colorings in the following are assumed to be proper colorings and we will often drop the term "proper". It is easy to see, that a graph is 1-colorable if and only if its edge set is empty and 2-colorable if and only if it is bipartite. Since only loopless graphs admit proper colorings and a loopless graph is r-colorable if and only if its underlying *simple graph*[1] is r-colorable, all graphs we consider in the following are assumed to be simple graphs.
An *independent set* (or *stable set*) of a graph G is a subset V' of its vertex set, such that non of the vertices in V' are adjacent. Thus a vertex coloring is in fact a partition of the vertex set into independent subsets. The blocks of a partition induced by a proper r-coloring f are also called the *color classes* of f.

Example 1.2. The graph G, defined by the edges $[1, 2]$, $[1, 3]$, $[1, 4]$, $[2, 3]$, $[3, 4]$, $[3, 5]$, $[4, 5]$ is 4-colorable. Consider for instance the two 4-colorings $f_1, f_2 : V \longrightarrow \{c_1, c_2, c_3, c_4\}$, given

[1] A *simple graph* is a graph without loops and parallel edges

by

$$
\begin{aligned}
f_1(1) = f_1(5) = f_2(2) = f_2(5) = c_1,\\
f_1(2) = f_2(1) = c_2,\\
f_1(3) = f_2(3) = c_3,\\
f_1(4) = f_2(4) = c_4.
\end{aligned}
$$

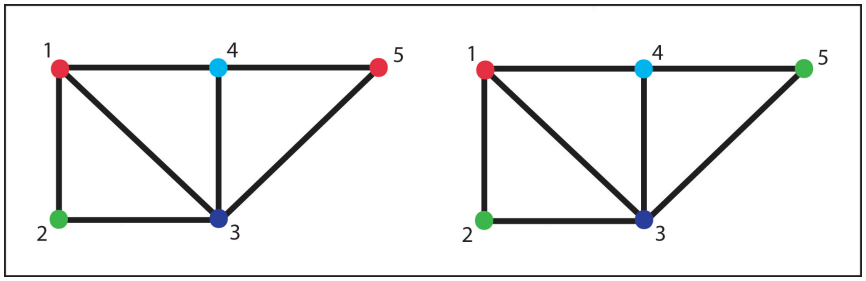

Figure 1.1: Two 4-colorings and of the graph G of Example 1.2.

The map $f_3 : V \longrightarrow \{c_1, c_2, c_3\}$, given by

$$
\begin{aligned}
f_3(1) = f_3(5) = c_1,\\
f_3(2) = f_3(4) = c_2,\\
f_3(3) = c_3
\end{aligned}
$$

is a proper 3-coloring of G.

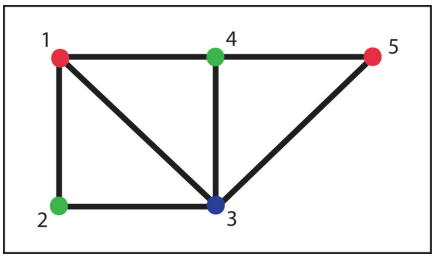

Figure 1.2: A 3-coloring of the graph G of Example 1.2.

Clearly, every simple graph is r-colorable, for $r \geq |V|$. More difficult is the question how many colors are minimally needed for a given graph G in order to be r-colorable. Such a minimal $r \in \mathbb{N}$ is called the *chromatic number* of a graph.

Definition 1.3. Let $G = (V, E)$ be a finite graph. The *chromatic number* $\chi(G)$ of G is the smallest natural number r such that G is r-colorable. If $\chi(G) = r$ we also say that G is *r-chromatic*.

The graph of Example 1.2 is 3-chromatic. The chromatic number is a *graph invariant*. That means that isomorphic graphs always have the same chromatic number. The opposite is obviously not true - there are many examples of graphs that are not isomorphic, having the same chromatic number. For instance every bipartite graph is 2-chromatic. In fact, by definition a graph is 2-chromatic if and only if it is bipartite.
A lot of research has been done in order to find good bounds on the chromatic number of a graph. For example, if $\gamma(G)$ is the cardinality of a maximal stable subset of V and $|V| = n$, then clearly

$$\chi(G) \geq \frac{n}{\gamma(G)},$$

for every graph $G = (V, E)$. Furthermore, a graph G that has a subgraph isomorphic to a complete graph K_l obviously has chromatic number $\chi(G) \geq l$. A subset $C \subseteq V$ that induces a complete subgraph of a graph is called a *clique* of G. Therefore we find that

$$\chi(G) \geq \omega(G),$$

where $\omega(G) = \max_{\text{clique } C \subseteq V} |C|$.
Since every $\chi(G)$-coloring induces a partition of the vertex set such that there is at least one edge between every two blocks, it is not difficult to see that $|E| \geq \frac{1}{2}\chi(G)(\chi(G) - 1)$. Thus

$$\chi(G) \leq \frac{1}{2} + \sqrt{2|E| + \frac{1}{4}},$$

for every graph $G = (V, E)$.
Another upper bound for the chromatic number is

$$\chi(G) \leq \Delta(G) + 1,$$

where $\Delta(G) := \max_{v \in V} d(v) := \max_{v \in V} |\{u \in V \mid [u, v] \in E\}|$ is the *maximum degree* of G and $d(v) := |\{u \in V \mid [u, v] \in E\}|$ is the *degree* of $v \in V$. To see that $\chi(G) \leq \Delta(G) + 1$ consider the following algorithm, called *greedy algorithm*:
We start by enumerating the vertices of G in some way, so $V = \{v_1, v_2, ..., v_n\}$. Then we consider the vertices one by one in this order and assign to each vertex the"smallest" color of some linearly sorted color pallet (or the smallest positive integer) that is still available, so the smallest color that is not used yet on any of the neighbors of v_i among $v_1, v_2, ..., v_{i-1}$. In this way, we never use more then $\Delta(G) + 1$ colors [4].
From this upper bound of $\chi(G)$ it easily follows, that every r-chromatic graph has a vertex of degree at least $r - 1$. In fact every r-chromatic graph has at least r vertices of degree at least $r - 1$:

Lemma 1.4. *If $G = (V, E)$ is a r-chromatic graph, then G has at least r vertices of degree at least $r - 1$.*

Proof. Let $f : V \longrightarrow P := \{c_1, c_2, ..., c_r\}$ be a proper r-coloring of G. If $V_i := f^{-1}(c_i)$ is a color class of f, then clearly V_i must contain a vertex v that is adjacent to at least one vertex in every color class $V_j \neq V_i$ of f, since if such a vertex would not exist, we could easily give every vertex in V_i one of the other $r-1$ colors and thus $\chi(G) \neq r$. Therefore there is at least one vertex in every of the r color classes that has degree at least $r-1$. □

It is obvious, that the bound $\chi(G) \leq \Delta(G) + 1$ is quite generous. For example, a complete bipartite graph K_{n_1,n_2} has maximum degree $\Delta(G) = \max\{n_1, n_2\}$. Odd cycles, on the other hand, have chromatic number 3 and maximum degree 2. Also every complete graph has chromatic number $\Delta(K_n) + 1$. For all other graphs, however, the upper bound $\chi(G) \leq \Delta(G) + 1$ can be slightly improved.

Theorem 1.5 (Brooks, 1941)**.** *Let G be a connected graph. If G is neither an odd cycle nor a complete graph, then*

$$\chi(G) \leq \Delta(G).$$

For a prove of this theorem see for example *Graph Theory*, by Bondy and Murty [3].

Uniquely Colorability

Let $G = (V, E)$ be a graph, $\mathbf{S}_i$ the symmetric group on i elements and

$$\mathrm{Aut}(G) = \{\rho \in \mathbf{S}_{|V|} \mid [u, v] \in E \Leftrightarrow [\rho(u), \rho(v)] \in E, \forall\, u, v \in V\}$$

the automorphism group of G.

Definition 1.6. Two r-colorings f, f' of G are said to be *equivalent* (write $f \sim f'$) if there exist $\sigma \in \mathbf{S}_r$ and $\rho \in \mathrm{Aut}(G)$, such that $f \circ \rho = \sigma \circ f'$.

Definition 1.7. A graph G is called *equivalently r-colorable* if all proper surjective r-colorings of G are equivalent.

It is obvious, that every graph G is equivalently $|V|$-colorable and that every connected bipartite graph is equivalently 2-colorable. Here are some more examples.

Example 1.8. The graph G, defined by the edges $[1,2]$, $[1,3]$, $[1,4]$, $[1,5]$, $[2,3]$, $[2,4]$, $[2,5]$ is 3-chromatic and equivalently i-colorable for $i \in \{3,4,5\}$ (Figure 1.3).

Example 1.9. The graph G, defined by the edge-set $E := \{[1,2],[2,3],[4,5],[5,6]\}$ (Figure 1.4) is an example of a bipartite graph, that is not equivalently 2-colorable: Consider the two surjective 2-colorings $f_1, f_2 : V \to \{c_1, c_2\}$, given by

$$f_1(1) = f_1(3) = f_1(5) = f_2(2) = f_2(5) = c_1,$$
$$f_1(2) = f_1(4) = f_1(6) = f_2(1) = f_2(3) = f_2(4) = f_2(6) = c_2.$$

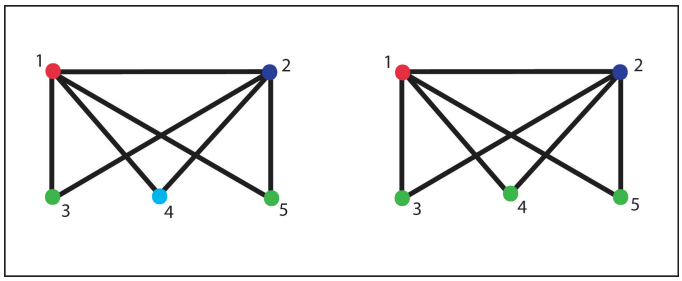

Figure 1.3: A 3-coloring and a 4-coloring of the graph G of Example 1.8.

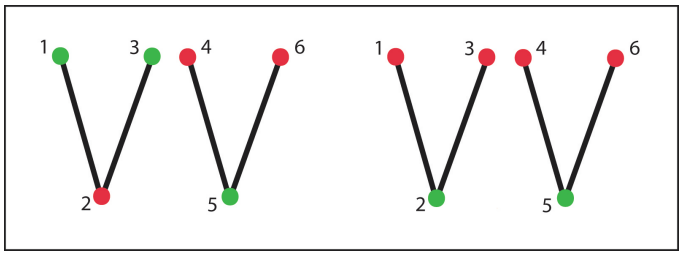

Figure 1.4: Two non-equivalent 2-colorings of the graph G of Example 1.9.

Example 1.10. The graph G of Example 1.2 is equivalently i-colorable, for $i = 3$ and $i = 5$ but not for $i = 4$: Consider for instance the two surjective 4-colorings f_1, f_2 as defined in Example 1.2.

If f is a r-coloring of G, then clearly $\sigma \circ f$ is also a r-coloring, for any $\sigma \in \mathbf{S}_r$. In fact f and $\sigma \circ f$ are two r-colorings, that induce the same partition of V into $i \leq r$ independent subsets. Write $[f] = \{\sigma \circ f \mid \sigma \in \mathbf{S}_r\}$ for the *coloring class* of f. For every $r \in \mathbb{N}$, $\mathrm{Aut}(G)$ acts on the set $\mathcal{F}^r(G) := \{[f] \mid f \text{ is a } r\text{-coloring of } G\}$ and on the subset $\mathcal{F}_S^r(G) := \{[f] \mid f \text{ is a surjective } r\text{-coloring of } G\}$ by composition. Thus equivalently r-colorability means nothing else but that the action of $\mathrm{Aut}(G)$ on $\mathcal{F}_S^r(G)$ is transitive and it follows that G is equivalently r-colorable if and only if

$$|\mathcal{F}_S^r(G)| = |\mathrm{Aut}(G)f| = \frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(G)_f|},$$

for $f \in \mathcal{F}_S^r(G)$, where $\mathrm{Aut}(G)f$ is the orbit of f and $\mathrm{Aut}(G)_f$ its stabilizer.

Definition 1.11. Let $G = (V, E)$. For any partition $P = \{V_1, V_2, ..., V_k\}$ of V, we call the graph $G_P := (P, E_P)$, with vertex set P and edge set $E_P := \{[V_i, V_j] \mid \exists v \in V_i, \exists u \in V_j \text{ such that } [v, u] \in E\}$ the *partition-graph* of G induced by P.

Theorem 1.12. *Let G be an equivalently r-colorable graph, $f, f' : V \longrightarrow \{c_1, ..., c_r\}$ two surjective r-colorings and $\sigma \in \mathbf{S}_r$ and $\rho \in \mathrm{Aut}(G)$, such that $\sigma \circ f = f' \circ \rho$. Then $|f^{-1}(c_i)| = |f'^{-1} \circ \sigma(c_i)|$, for all $1 \leq i \leq r$. Furthermore, if P respectively P' are the corresponding partitions of the vertex set V, then ρ defines a graph isomorphism between G_P and $G_{P'}$.*

Proof. Let $c_i, c_j \in \{c_1, ..., c_r\}$, such that $\sigma(c_i) = c_j$ and define $V_i := f^{-1}(c_i)$ and $V'_j := f'^{-1}(c_j)$. Since ρ is a graph automorphism it clearly follows, that $|f^{-1}(c_i)| = |V_i| = |\rho(V_i)|$ and since

$$f' \circ \rho(V_i) = \sigma \circ f(V_i) = \sigma(c_i) = c_j,$$

we find that $\rho(V_i) \subseteq V'_j$. For the same reasons we have that $|f'^{-1}(c_j)| = |V'_j| = |\rho^{-1}(V'_j)|$ and that

$$f \circ \rho^{-1}(V'_j) = \sigma^{-1} \circ f'(V'_j) = \sigma^{-1}(c_j) = c_i.$$

Thus $\rho^{-1}(V'_j) \subseteq V_i$. It follows therefore that

$$|f^{-1}(c_i)| = |V_i| = |V'_j| = |f'^{-1} \circ \sigma(c_i)|.$$

Now, for every $1 \leq i \leq r$, $\rho(V_i) = \rho \circ f^{-1}(c_i) = f'^{-1} \circ \sigma(c_i) = V'_j$, for some $1 \leq j \leq r$ and since ρ is an automorphism of G, ρ defines a bijection between G_P and $G_{P'}$. Furthermore,

$$\begin{aligned}[V_i, V_j] \in E_P &\Longleftrightarrow \exists v_i \in V_i, v_j \in V_j, \text{ s.th. } [v_i, v_j] \in E \\ &\Longleftrightarrow [\rho(v_i), \rho(v_j)] \in E \\ &\Longrightarrow [\rho(V_i), \rho(V_j)] \in E_{P'}.\end{aligned}$$

The other direction follows trivially, since ρ is an automorphism. Therefore G_P and $G_{P'}$ are isomorphic. □

If there is just one partition of V into r independent sets, then $|\mathcal{F}_S^r(G)| = 1$ and G is equivalently r-colorable. Clearly $|\mathcal{F}_S^{|V|}(G)| = 1$ for every graph G. The next lemma shows that if $|\mathcal{F}_S^r(G)| = 1$, such that $r \neq |V|$, then $r = \chi(G)$.

Lemma 1.13. *Let $G = (V, E)$ be a graph and $\chi(G) \leq r < |V|$. If there exists a permutation $\sigma \in \mathbf{S}_r$, for every two surjective r-colorings f, f', such that $f' = \sigma \circ f$, then $r = \chi(G)$.*

Proof. Assume that $|V| - 1 \geq r \neq \chi(G)$, then $\chi(G) \leq |V| - 2$ and so there exists either an independent set $V' \subseteq V$, such that $|V'| \geq 3$ or there are (at least) two independent sets $V', V'' \subseteq V$, such that $V' \cap V'' = \emptyset$ and $|V'| = |V''| = 2$.
In the first case let $v_1, v_2, v_3 \in V'$ be three distinct vertices. Note that there must exist a surjective $(r+1)$-coloring $h : V \longrightarrow P := \{c_1, ..., c_{r+1}\}$, such that $h(v_1) \neq h(v_2) \neq h(v_3)$, since $r + 1 \geq \chi(X) + 2$ (simply "change" a $\chi(G)$-coloring into a surjective $(r+1)$-coloring, such that v_1, v_2, v_3 are colored distinctively).
Now consider the two maps $f, f' : V \longrightarrow P := \{c_1, ..., c_r\}$, given by

$$f(v) = \begin{cases} h(v_1) & \text{for } v \in \{v_1, v_2\}, \\ h(v) & \text{for } v \in V \backslash \{v_1, v_2\} \end{cases}$$

and

$$f'(v) = \begin{cases} h(v_1) & \text{for } v \in \{v_1, v_3\}, \\ h(v) & \text{for } v \in V \backslash \{v_1, v_3\}. \end{cases}$$

Clearly f, f' are surjective r-colorings, but there is no $\sigma \in \mathbf{S}_r$, such that $f = \sigma \circ f'$.

In the second case let $V' = \{v_1, v_2\}$ and $V'' = \{v_3, v_4\}$. Again there must exist a surjective $(r+1)$-coloring h, such that $h(v_1) \neq h(v_2)$ and $h(v_3) \neq h(v_4)$, since $r \neq \chi(G)$. Therefore the two maps $f, f' : V \longrightarrow P := \{c_1, ..., c_r\}$, given by

$$f(v) = \begin{cases} h(v_1) & \text{for } v \in \{v_1, v_2\}, \\ h(v) & \text{for } v \in V \backslash \{v_1, v_2\} \end{cases}$$

and

$$f'(v) = \begin{cases} h(v_3) & \text{for } v \in \{v_3, v_4\}, \\ h(v) & \text{for } v \in V \backslash \{v_3, v_4\} \end{cases}$$

are surjective r-colorings. But then again there is no $\sigma \in \mathbf{S}_r$, such that $f = \sigma \circ f'$.

$$\Rightarrow r = \chi(G).$$

□

Definition 1.14. A graph G is called *uniquely colorable* if G has only one (proper) $\chi(G)$-coloring up to permutation of the colors. In that case a $\chi(G)$-coloring f is called a *unique coloring* of G.

The graphs in Example 1.8 and 1.10 are both uniquely colorable. Every connected bipartite graph is uniquely colorable.

Example 1.15. The bipartite graph G, defined by the edge-set $E := \{[1,2], [3,4], [4,5]\}$ (Figure 1.5) is equivalently 2-colorable, but not uniquely colorable. To see this, consider the two 2-colorings $f_1, f_2 : V \longrightarrow \{a, b\}$, defined by

$$f_1(1) = f_1(4) = f_2(1) = f_2(3) = f_2(5) = a$$
$$f_1(2) = f_1(3) = f_1(5) = f_2(2) = f_2(4) = b.$$

If G is uniquely colorable then there exists just one possible partition of V into $r = \chi(G)$ independent sets. Therefore, if f is a r-coloring, every vertex in any color class V_i of f must have at least one edge to any other color class. Thus

$$|E| \geq \frac{(r-1)|V|}{2}.$$

In fact

$$|E| \geq (r-1)|V| - \frac{r(r-1)}{2},$$

as shown by Shaoji Xu in 1990.

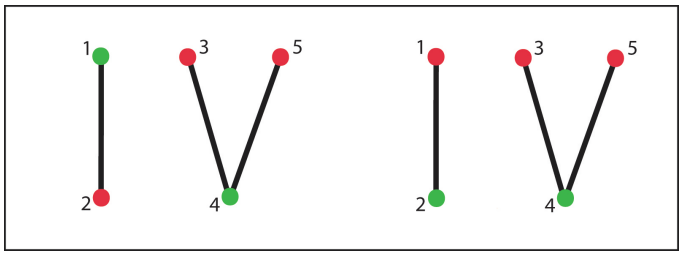

Figure 1.5: The graph G of Example 1.15 is equivalently 2-colorable but not uniquely colorable.

Theorem 1.16 (Xu, 1990 [18]). *Let $G = (V, E)$ be a finite graph and $r = \chi(G)$. If G is uniquely colorable then*

$$|E| \geq (r-1)|V| - \frac{r(r-1)}{2}.$$

Proof. Let $P := \{V_1, V_2, ..., V_r\}$ be the only partition of V into r independent subsets and note that every subgraph $G(V_i \cup V_j)$ induced by any two color classes V_i and V_j must be connected:
Suppose, by contradiction, that there are $V_i, V_j \in P$, such that $G(V_i \cup V_j)$ is not connected and let C_1, C_2 be two connected components of $G(V_i \cup V_j)$. Define the independent sets $V_i' := (C_1 \cap V_i) \cup (C_2 \cap V_j)$ and $V_j' = (C_1 \cap V_j) \cup (C_2 \cap V_i)$. Then $P' := (P \backslash \{V_i, V_j\}) \cup \{V_i', V_j'\}$ is a second partition of V into r independent subsets, in contradiction to the assumption that G is uniquely colorable. Therefore $G(V_i \cup V_j)$ is connected, for all $1 \leq i < j \leq r$.

Now, if we write $E_{i,j} := E(G(V_i \cup V_j))$ for the edge set of $G(V_i \cup V_j)$, then it follows from the previous, that

$$|E_{i,j}| \geq |V_i \cup V_j| - 1,$$

and so

$$\begin{aligned} |E| &= \sum_{1 \leq i < j \leq r} |E_{i,j}| \\ &\geq \sum_{1 \leq i < j \leq r} |V_i \cup V_j| - \frac{r(r-1)}{2} \\ &= (r-1) \sum_{i=1}^{r} |V_i| - \frac{r(r-1)}{2} \\ &= (r-1)|V| - \frac{r(r-1)}{2}. \end{aligned}$$

□

The Chromatic Polynomial for Graphs

Definition 1.17. Let G be a graph. We write $\mathrm{C}_r(G)$ to denote the number of different r-colorings of G.

It is not difficult to see, that for any $r, n \in \mathbb{N}$ the number of distinct r-colorings of the empty graph $\overline{K_n}$ on n vertices equals $\mathrm{C}_r(\overline{K_n}) = r^n$ and the number of distinct r-colorings of the complete graph K_n equals $\mathrm{C}_r(K_n) = [r]_n$, where $[r]_n$ denotes the product $r(r-1)(r-1)\cdots(r-(n-1))$. Moreover for every graph G we easily see that $\mathrm{C}_r(G) = [r]_{|V|}$, for $r \geq |V|$, and $\mathrm{C}_r(G) = 0$, for $\chi(G) > r$. Furthermore, if G is uniquely colorable, then clearly $\mathrm{C}_{\chi(G)}(G) = [\chi(G)]_{\chi(G)}$. As the chromatic number, the number $\mathrm{C}_r(G)$ is a graph invariant for every $r \in \mathbb{N}$. For graphs with large numbers of vertices and edges it is a difficult task to find the number $\mathrm{C}_r(G)$. However, since $\mathrm{C}_r(G)$ satisfies a property called the *deletion-contraction property*, it is possible to compute the number of r-colorings of G recursively. Before we can state the deletion-contraction property we first need to define the two graphs $G\backslash e$ and G/e associated to a given graph $G = (V, E)$:
For any edge $e = [u, v] \in E$, $G\backslash e$ is simply the subgraph $G\backslash e := (V, E\backslash\{e\})$ and G/e is the graph we obtain by contracting the edge e in G. So, if P_e is the partition of V defined as $P_e := \{\{u, v\}\} \cup \{\{v'\} \mid v' \in V\backslash\{u, v\}\}$, then $G/e \cong G_{P_e}$.

Lemma 1.18 (Deletion-Contraction Property)**.** *Let $G = (V, E)$ be a finite simple graph and let $\mathrm{C}_r(G)$ denote the number of possible r-colorings of G. Then*

$$\mathrm{C}_r(G) = \mathrm{C}_r(G\backslash e) - \mathrm{C}_r(G/e),$$

for every $e \in E$ and $r \in \mathbb{N}$.

Proof. Choose $e = [u, v] \in E$ and $r \in \mathbb{N}$. Clearly every proper r-coloring of G is a proper r-coloring of $G\backslash e$.
On the other hand, every r-coloring f of $G\backslash e$ is a r-coloring of G if and only if $f(u) \neq f(v)$. Since every coloring f of $G\backslash e$, with $f(u) = f(v)$ corresponds uniquely with a coloring f' of G/e, such that $f'(\{v'\}) = f(v')$, for all $v' \in V\backslash\{u, v\}$ and $f'(\{u, v\}) = f(u) = f(v)$, the lemma follows. □

The formula in Lemma 1.18 can be rewritten as

$$\mathrm{C}_r(G) = \mathrm{C}_r(G + e) + \mathrm{C}_r((G + e)/e).$$

By applying one of the two formulas recursively, we can now calculate $\mathrm{C}_r(G)$ for any finite graph $G = (V, E)$; either we start with $\overline{K_{|V|}}$ and add edges or we start with $K_{|V|}$ and delete edges. Even more interesting is the fact that, since $\mathrm{C}_r(\overline{K_n}) = r^n$ and $\mathrm{C}_r(K_n) = [r]_n$ are polynomials in r, the deletion-contraction formula leads us to the following conclusion:

Theorem/Definition 1.19. *For every finite simple graph $G = (V, E)$ there exists a polynomial $\mathcal{P}(G, x)$ in x, such that $\mathcal{P}(G, r) = \mathrm{C}_r(G)$, for all $r \in \mathbb{Z}_{\geq 0}$.*
Furthermore

$$\mathcal{P}(G, x) = \mathcal{P}(G\backslash e, x) - \mathcal{P}(G/e, x),$$

for all $x \in \mathbb{R}$ and $e \in E$. $\mathcal{P}(G, x)$ is called the chromatic polynomial of G.

Proof. Let $|V| = n$ and $|E| = m$. We prove by induction on m. Clearly, if $E = \emptyset$, then $\mathrm{C}_r(G) = r^n$ and thus $\mathcal{P}(G, x) := x^n$ is a polynomial satisfying the conditions.
Assume now, that the theorem holds for all graphs with less then m edges and let G be a graph with m edges. For every edge $e \in E$, the two graphs $G\backslash e$ and G/e have $m-1$ edges and so it follows by induction, that there exist polynomials $\mathcal{P}(G\backslash e, x)$ and $\mathcal{P}(G/e, x)$, such that $\mathcal{P}(G\backslash e, r) = \mathrm{C}_r(G\backslash e)$ and $\mathcal{P}(G/e, r) = \mathrm{C}_r(G/e)$, for all $r \in \mathbb{N}$. Furthermore, from Lemma 1.18 it follows that

$$\mathrm{C}_r(G) = \mathrm{C}_r(G\backslash e) - \mathrm{C}_r(G/e) = \mathcal{P}(G\backslash e, r) - \mathcal{P}(G/e, r),$$

for all $e \in E$ and $r \in \mathbb{N}$. Therefore the polynomial $\mathcal{P}(G, x) := \mathcal{P}(G\backslash e, x) - \mathcal{P}(G/e, x)$ satisfies the conditions and the theorem follows. □

Again we can rewrite this to get a second formula:

$$\mathcal{P}(G, x) = \mathcal{P}(G + e, x) + \mathcal{P}((G + e)/e, x).$$

Theorem 1.19 shows that for every finite graph G there exists a single invariant, that combines the invariants $\chi(G)$ and $\mathrm{C}_r(G)$, we had already associated to colarability. Furthermore this invariant is a polynomial and we even know how to calculate it without specifically looking at any coloring.

Example 1.20. The graph G from Example 1.2 is isomorphic to the graph

$$K_5\backslash\{[1,5],[2,4],[2,5]\}.$$

Since $\mathcal{P}(K_n, x) = [x]_n$, for all $n \in \mathbb{N}$, the chromatic polynomial of G is equal to

$$\begin{aligned}
\mathcal{P}(G, x) &= \mathcal{P}((K_5\backslash\{[1,5],[2,4]\}), x) + \mathcal{P}((K_5\backslash\{[1,5],[2,4]\})/[2,5], x) \\
&= \mathcal{P}(K_5\backslash[1,5], x) + \mathcal{P}((K_5\backslash[1,5])/[2,4], x) + \mathcal{P}((K_5\backslash\{[1,5],[2,4]\})/[2,5], x) \\
&= \mathcal{P}(K_5, x) + \mathcal{P}(K_5/[1,5], x) + \mathcal{P}((K_5\backslash[1,5])/[2,4], x) \\
&\quad + \mathcal{P}((K_5\backslash\{[1,5],[2,4]\})/[2,5], x) \\
&= \mathcal{P}(K_5, x) + \mathcal{P}(K_4, x) + (\mathcal{P}(K_4, x) + \mathcal{P}(K_3, x)) + \mathcal{P}(K_4, x) \\
&= [x]_5 + 3[x]_4 + [x]_3 \\
&= x^5 - 7x^4 + 18x^3 - 20x^2 + 8x.
\end{aligned}$$

Example 1.21. The graph G with six vertices and edges $[1,2]$, $[2,3]$ is isomorphic to the graph $\overline{K_6} + \{[1,2],[2,3]\}$. Since $\mathcal{P}(\overline{K_n}, x) = x^n$, for all $n \in \mathbb{N}$, the chromatic polynomial of G is equal to

$$\begin{aligned}
\mathcal{P}(G, x) &= \mathcal{P}(G\backslash[1,2]), x) - \mathcal{P}(G/[1,2], x) \\
&= \mathcal{P}(\overline{K_6}, x) - \mathcal{P}((G\backslash[1,2])/[2,3], x) - \mathcal{P}(G/[1,2], x) \\
&= \mathcal{P}(\overline{K_6}, x) - \mathcal{P}(\overline{K_5}, x) - \mathcal{P}(\overline{K_5} + [2,3], x)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}(\overline{K_6}, x) - \mathcal{P}(\overline{K_5}, x) - (\mathcal{P}(\overline{K_5}, x) - \mathcal{P}((\overline{K_5} + [2,3])/[2,3], x)) \\
&= \mathcal{P}(\overline{K_6}, x) - 2 \cdot \mathcal{P}(\overline{K_5}, x) + \mathcal{P}(\overline{K_4}, x) \\
&= x^6 - 2x^5 + x^4 \\
&= [x]_5 + 13[x]_5 + 46[x]_4 + 46[x]_3 + 8[x]_2.
\end{aligned}
$$

Even though, for any finite simple graph, this two versions of the deletion-contraction formula will eventually lead to a chromatic polynomial it is also clear that the computation time will be very high for big graphs. However, since the deletion-contraction formula is a recursive formula it has the advantage, that it can easily be used for induction. It can, for instance, be shown inductively that every tree T with n vertices has the same chromatic polynomial. In order to prove this note first that if G is the disjoint union of k connected components $G_1, G_2, ..., G_k$ we can color each component independently. Therefore the number of r-colorings is $\mathrm{C}_r(G) = \mathrm{C}_r(G_1) \cdots \mathrm{C}_r(G_k)$, for every $r \in \mathbb{N}$ and thus

$$
\mathcal{P}(G_1 \sqcup G_2 \sqcup ... \sqcup G_k, x) = \mathcal{P}(G_1, x) \cdots \mathcal{P}(G_k, x).
$$

Lemma 1.22. *Let T be a tree with n vertices, then*

$$
\mathcal{P}(T, x) = x(x-1)^{n-1}.
$$

Proof. We prove by induction on the number $|V|$ of vertices. Clearly if $|V| = 1$ then the claim is true. Assume now that the lemma holds for $|V| < n$ and let T be a tree with n vertices. Let e be an edge of T such that one of its vertices has degree 1, then T/e is a tree with $n-1$ vertices and so it follows by induction that

$$
\mathcal{P}(T/e, x) = x(x-1)^{n-2}.
$$

Furthermore, we find that $T \backslash e = G_1 \sqcup G_2$ is the disjoint union of two connected subgraphs, where G_1 is a single point and G_2 is a tree with $n-1$ vertices. Now induction and the previous remark give that

$$
\mathcal{P}(T \backslash e, x) = \mathcal{P}(G_1, x) \cdot \mathcal{P}(G_2, x) = x^2(x-1)^{n-2}.
$$

Therefore, by Theorem 1.19

$$
\begin{aligned}
\mathcal{P}(T, x) &= \mathcal{P}(T \backslash e, x) - \mathcal{P}(T/e, x) \\
&= x^2(x-1)^{n-2} - x(x-1)^{n-2} \\
&= x(x(x-1)^{n-2} - (x-1)^{n-2}) \\
&= x(x-1)^{n-1}.
\end{aligned}
$$

This proves the claim. □

Clearly all isomorphic graphs have the same chromatic polynomial. On the other hand we have just proven, that all trees with the same number of vertices have the same chromatic polynomial also when they are not isomorphic. Therefore all trees with the same number of vertices are *chromatic equivalents*:

Definition 1.23. Two graphs G, H are called *chromatically equivalent* if

$$\mathcal{P}(G, x) = \mathcal{P}(H, x).$$

It is not hard to see, that each of the three graphs defined by the edge-sets

$$E_1 := \{[1,2],[1,3],[2,3],[1,4],[1,5]\},$$
$$E_2 := \{[1,2],[1,3],[2,3],[1,4],[4,5]\}$$

and

$$E_3 := \{[1,2],[1,3],[2,3],[1,4],[2,5]\}$$

of Figure 1.5 have chromatic polynomial equal to $x(x-1)^3(x-2)$. Thus they are chromatically equivalent.

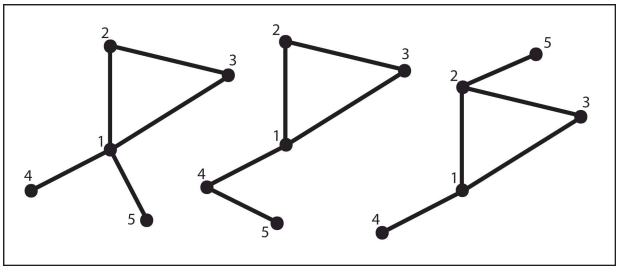

Figure 1.6: Three chromatically equivalent graphs.

The graph of Example 1.8 has chromatic polynomial

$$x(x-1)(x-2)^3 = x^5 - 7x^4 + 18x^3 - 20x^2 + 8x$$

and is therefore chromatically equivalent to the graph of Example 1.2, as follows from 1.20. The next two lemmas show that we can easily construct chromatically equivalent graphs.

Theorem 1.24. *Let $G = (V, E)$, $H = (V', E')$ be two graphs, such that $|V \cap V'| = 1$, then*

$$\mathcal{P}(G \cup H, x) = \frac{\mathcal{P}(G, x) \cdot \mathcal{P}(H, x)}{x}.$$

Proof. Let $p_i \in \{p_1, ..., p_r\}$ be a fixed color. Note that for every vertex v in a graph G there are exactly $\frac{\mathrm{C}_r(G)}{r}$ r-colorings, that color v with the color p_i. Therefore, if G and H share one vertex v, any given r-coloring of H leaves $\frac{\mathrm{C}_r(G)}{r}$ ways to properly color the vertices $V \backslash \{v\}$ in $G \cup H$. Thus

$$\mathrm{C}_r(G \cup H) = \mathrm{C}_r(H) \cdot \frac{\mathrm{C}_r(G)}{r}$$
$$\Longrightarrow \mathcal{P}(G \cup H, x) = \frac{\mathcal{P}(G, x) \cdot \mathcal{P}(H, x)}{x}.$$

□

Theorem 1.25. *Let $G = (V, E)$, $H = (V', E')$ be two graphs, such that $G \cap H = K_n$, for some $n \leq \min\{|V|, |V'|\}$ then*

$$\mathcal{P}(G \cup H, x) = \frac{\mathcal{P}(G, x) \cdot \mathcal{P}(H, x)}{[x]_n}.$$

Proof. Since $G \cap H = K_n$ is a subgraph of G every r-coloring of G must color K_n with n different colors. Therefore there are exactly $\frac{\mathrm{C}_r(G)}{\mathrm{C}_r(K_n)} = \frac{\mathrm{C}_r(G)}{[r]_n}$ r-colorings of G, for every fixed coloring of K_n. Now every r-coloring of H leaves $\frac{\mathrm{C}_r(G)}{[r]_n}$ ways to properly color the vertices $V \backslash \{V(K_n)\}$ in $G \cup H$. Thus

$$\mathrm{C}_r(G \cup H) = \mathrm{C}_r(H) \cdot \frac{\mathrm{C}_r(G)}{[r]_n}$$

$$\Longrightarrow \mathcal{P}(G \cup H, x) = \frac{\mathcal{P}(G, x) \cdot \mathcal{P}(H, x)}{[x]_n}.$$

□

It is in general not true that $\mathcal{P}(G \cup H, x) = \frac{\mathcal{P}(G,x) \cdot \mathcal{P}(H,x)}{\mathcal{P}(G \cap H,x)}$. Take as a counter example a cycle C_5 of length five, induced by the edges $[1, 2]$, $[2, 3]$, $[3, 4]$, $[4, 5]$, $[5, 1]$ and let H be the subgraph with the edges $[1, 2]$, $[2, 3]$ and G the subgraph consisting of the edges $[3, 4]$, $[4, 5]$, $[5, 1]$. Then

$$\mathcal{P}(G \cup H, x) = \mathcal{P}(C_5, x) = (x-1)^5 - (x-1),$$

as we will see in Lemma 1.27, while

$$\frac{\mathcal{P}(G, x) \cdot \mathcal{P}(H, x)}{\mathcal{P}(G \cap H, x)} = \frac{x(x-1)^2 \cdot x(x-1)^3}{x^2} = (x-1)^5,$$

according to Lemma 1.22.

Remark. To prove Lemma 1.22 we could have also applied Lemma 1.24 repeatedly: It is obvious that a path P_n with n vertices has chromatic polynomial $\mathcal{P}(P_n, x) = x(x-1)^{n-1}$. Therefore, since every tree is constructed by repeatedly "gluing" edges or paths together in one vertex, Lemma 1.22 is simply a result of 1.24.

Since the chromatic polynomial of a graph G with n vertices is obtained recursively from either $\mathcal{P}(K_n, x) = [x]_n$ or $\mathcal{P}(\overline{K_n}, x) = x^n$ it is not hard to see, that the coefficient of x^k in $\mathcal{P}(G, x)$ is zero for every $k > n$ and the coefficient of x^n is equal to one. Therefore we can improve Lemma 1.22 to the following result:

Lemma 1.26. *A graph G, with n vertices is a tree if and only if*

$$\mathcal{P}(G, x) = x(x-1)^{n-1}.$$

Proof. We already proved in Lemma 1.22 that $\mathcal{P}(T,x) = x(x-1)^{n-1}$, for every tree T with n vertices. Assume therefore that G is a graph with $\mathcal{P}(G,x) = x(x-1)^{n-1}$ and let T be a spanning tree of G, then $T = G\backslash\{e_1, e_2, ..., e_k\}$ for some $e_1, e_2, ..., e_k \in E$. The deletion-contraction property of the chromatic polynomial gives, that

$$\begin{aligned}\mathcal{P}(T,x) &= \mathcal{P}(G\backslash\{e_1,e_2,...,e_k\}+e_k,x) + \mathcal{P}((T+e_k)/e_k,x)\\ &= \mathcal{P}(G\backslash\{e_1,e_2,...,e_{k-1}\},x) + \mathcal{P}((T+e_k)/e_k,x)\\ &= (\mathcal{P}(G\backslash\{e_1,e_2,...,e_{k-2}\},x) + \mathcal{P}(G\backslash\{e_1,e_2,...,e_{k-2}\}/e_{k-1},x)) + \mathcal{P}((T+e_k)/e_k,x)\\ &= ...\\ &= \mathcal{P}(G,x) + \mathcal{P}(G/e_1,x) + \sum_{i=2}^{k} \mathcal{P}(G\backslash\{e_1,e_2,...,e_{i-1}\}/e_i,x).\end{aligned}$$

Since T is a tree with n vertices we know that $\mathcal{P}(T,x) = x(x-1)^{n-1}$. Therefore we find, that

$$\mathcal{P}(G/e_1,x) = -\sum_{i=2}^{k} \mathcal{P}(G\backslash\{e_1,e_2,...,e_{i-1}\}/e_i,x).$$

Because G/e_1 is a graph with $n-1$ vertices the highest power of $\mathcal{P}(G/e_1,x)$ is $n-1$ and its coefficient is 1. The same is true for every $G\backslash\{e_1,e_2,...,e_{i-1}\}/e_i$, for all $2 \le i \le k$. Therefore $\sum_{i=2}^{k} -\mathcal{P}(G\backslash\{e_1,e_2,...,e_{i-1}\}/e_i,x) = -(k-1)x^{n-1} \pm ... = x^{n-1} \pm ... = \mathcal{P}(G/e_1,x)$. It follows that $k=0$ and so $G=T$ is a tree. □

Here is another example how the deletion-contraction property of the chromatic polynomial can be used for finding the chromatic polynomial for a certain group of graphs.

Lemma 1.27. *Let C_n be the cycle of length n, then*

$$\mathcal{P}(C_n,x) = (x-1)^n + (-1)^n(x-1).$$

Proof. We prove by induction on the number n of vertices. If $n=3$, $C_3 = K_3$ and so

$$\begin{aligned}\mathcal{P}(C_3,x) &= x(x-1)(x-2)\\ &= (x-1)((x-1)^2-1)\\ &= (x-1)^3 + (-1)^3(x-1).\end{aligned}$$

Now assume the lemma is true for cycles of length less than n. Note that for any edge e, $C_n\backslash e$ is a tree with n vertices and so Lemma 1.22 gives that $\mathcal{P}(C_n\backslash e,x) = x(x-1)^{n-1}$. Furthermore, since $C_n/e \cong C_{n-1}$, induction and the deletion-contraction property of the chromatic polynomial give:

$$\begin{aligned}\mathcal{P}(C_n,x) &= \mathcal{P}(C_n\backslash e,x) - \mathcal{P}(C_n/e,x)\\ &= x(x-1)^{n-1} - (x-1)^{n-1} - (-1)^{n-1}(x-1)\\ &= (x-1)(x-1)^{n-1} + (-1)^n(x-1)\\ &= (x-1)^n + (-1)^n(x-1).\end{aligned}$$

□

There do exist graphs that are uniquely determined by their chromatic polynomials. The most simple examples are K_n and its complement the empty graph $\overline{K_n}$, therefore K_n and $\overline{K_n}$ are *chromatically unique*:

Definition 1.28. A graph G is called *chromatically unique* if

$$\mathcal{P}(H,x) = \mathcal{P}(G,x) \Longleftrightarrow G \cong H,$$

for every graph H.

We saw that all trees with n vertices are chromatically equivalent. In particular no tree with more then three vertices is chromatically unique. In the next section we will see that cycles of length n are chromatically unique. In order to prove this we need to learn a bit more about the coefficients of the chromatic polynomial first.

The Coefficients of the Chromatic Polynomial

For every finite graph $G = (V, E)$ we can write its chromatic polynomial in two ways:

$$\mathcal{P}(G,x) = \sum_{i=0}^{k} a_i x^i = \sum_{j=0}^{k} b_j [x]_j.$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the Stirling numbers of the first and second kind, respectively. Recall that the *Stirling numbers of the first kind* are equal to $\mathcal{S}_1(n,k) = (-1)^{n-k} c(n,k)$, where $c(n,k)$ is the number of permutations of n elements with k disjoint cycles. Furthermore, the *Stirling numbers of the second kind* $\mathcal{S}_2(n,k)$ count the number of distinct partitions of n elements into k non-empty blocks. It is a well-known fact that $x^k = \sum_{j=0}^{k} \mathcal{S}_2(k,j)[x]_j$ and $[x]_k = \sum_{i=0}^{k} \mathcal{S}_1(k,i)x^i$ (see for instance [1]). We know therefore, that

$$a_i = \sum_{j=i}^{k} b_j \mathcal{S}_1(j,i) \quad \text{and} \quad b_j = \sum_{i=j}^{k} a_i \mathcal{S}_2(i,j).$$

Clearly $a_0 = b_0 = 0$, as $\mathrm{C}_0(G) = 0$ for any G. Furthermore, we already noted that $a_k = b_k = 0$, for all $k > n$ and that $a_n = b_n = 1$, since the chromatic polynomial of a graph with n vertices is obtained recursively from either $\mathcal{P}(K_n,x) = [x]_n$ or $\mathcal{P}(\overline{K_n},x) = x^n$.
Before we will try to give an interpretation of the coefficients a_i and b_i of $\mathcal{P}(G,x)$ we first prove a couple of facts that follow directly from the deletion-contraction property of the chromatic polynomial.

Lemma 1.29. *Let G be a graph with n vertices, m edges and chromatic polynomial $\mathcal{P}(G,x) = \sum_{i=1}^{n} a_i x^i$. Then $a_{n-1} = -m$.*

Proof. We prove by induction on the number of edges m. Clearly the claim is true for $G = \overline{K_n}$. Assume therefore the lemma is true for all graphs with n vertices and less then m edges. For any $e \in E$ the associated graph $G\backslash e$ is a graph with n vertices and $m-1$ edges and so $\mathcal{P}(G\backslash e, x) = x^n - (m-1)x^{n-1} \pm ...$, by induction. Furthermore G/e is a graph with $n-1$ vertices, and so $\mathcal{P}(G/e, x) = x^{n-1} \pm ....$ Therefore the deletion-contraction formula gives, that $a_{n-1} = -m$. □

Lemma 1.30. *Let G be a graph with n vertices, that is the disjoint union of k connected components and let $\mathcal{P}(G, x) = \sum_{i=1}^{n} a_i x^i$ be its chromatic polynomial. Then $a_k \neq 0$ and $a_i = 0$ for all $i < k$.*

Proof. We already saw that the chromatic polynomial of a graph is the product of the chromatic polynomials of its connected components. Since the constant in every of those k polynomials is zero it follows that $\mathcal{P}(G, x)$ is divisible by x^k and thus $a_i = 0$, for all $i < k$.

In order to prove that $a_k \neq 0$ assume first that $k = 1$ and G is connected. We will prove by induction on the number m of edges that $a_1 = (-1)^{n-1} l$, for some $l \in \mathbb{N}_{>0}$. For $m = 1$ the claim is clearly true. Since G is connected it is either a tree, and $a_1 = (-1)^{n-1}$, or we can choose an edge $e \in E$ such that $G\backslash e$ and G/e are both connected and have less than m edges. Therefore the claim follows by induction and the deletion-contraction formula.
For $k > 1$ the result follows directly from the fact that the chromatic polynomial of a graph is the product of the chromatic polynomials of its connected components. □

Lemma 1.31. *Let G be a graph with n vertices and let $\mathcal{P}(G, x) = \sum_{i=1}^{n} a_i x^i$. Then, for all $i \geq k$, where k is the number of disjoint connected components of G, we have that $a_i > 0$, for $i \equiv n (\mathbf{mod}\ 2)$ and $a_i < 0$ otherwise.*

Proof. For any graph with just one edge the statement is obviously true. Now assume that it is true for all graphs with less then m edges. Let G be a graph with m edges, then the coefficients a'_i of $\mathcal{P}(G\backslash e, x)$ alternate in sign, with $a'_i > 0$, for $i \equiv n (\mathbf{mod}\ 2)$ and $a'_i < 0$ otherwise and the coefficients a''_i of $\mathcal{P}(G/e, x)$ alternate in sign, with $a''_i > 0$, for $i \equiv n-1 (\mathbf{mod}\ 2)$ and $a''_i < 0$ otherwise, for all $n-1 \geq i \geq k$. Therefore the lemma follows from the deletion-contraction formula.
Note that if $G\backslash e$ has $k+1$ connected components, then $a'_k = 0$, but since $a''_k \neq 0$, by the previous lemma this does not change the result. □

We can prove now that all cycles are chromatically unique.

Lemma 1.32. *Every cycle C_n of length n is chromatically unique, with polynomial*

$$\mathcal{P}(C_n, x) = (x-1)^n + (-1)^n (x-1).$$

Proof. Lemma 1.27 showed, that $\mathcal{P}(C_n, x) = (x-1)^n + (-1)^n(x-1)$. Now let G be a graph with $n \geq 3$ vertices and

$$\begin{aligned}\mathcal{P}(G,x) &= (x-1)^n + (-1)^n(x-1) \\ &= x^n - nx^{n-1} + ... + (-1)^{n-1}\left(\binom{n}{n-1} - 1\right)x.\end{aligned}$$

From Lemma 1.29 and 1.30 it follows, that $|E| = n = |V|$ and that G is connected. Therefore we conclude that G has exactly one cycle C and so every spanning tree T is equal to $G\backslash e$, for some $e \in C \subseteq E$.
Now we prove by induction on n. If $n = 3$ then G must be a cycle, because of the above observations. Assume that the claim holds for all graphs with less then n vertices and let $e \in E$, such that $T = G\backslash e$ is a spanning tree (thus e is part of the only cycle C of G). Then G/e is a graph with $n-1$ vertices and the deletion-contraction formula, together with Lemma 1.22, gives that

$$\begin{aligned}\mathcal{P}(G/e) &= \mathcal{P}(T,x) - \mathcal{P}(G,x) \\ &= x(x-1)^{n-1} - (x-1)^n - (-1)^n(x-1) \\ &= (x-1)^{n-1}(x-(x-1)) + (-1)^{n-1}(x-1) \\ &= (x-1)^{n-1} + (-1)^{n-1}(x-1).\end{aligned}$$

By induction we find that G/e must be a cycle and since e was an edge from the cycle C, it follows that G also must be a cycle. □

A sequence $k_1, k_2, ..., k_n$ of integers is called *unimodal* if there exists an index $i \in \{1, ..., n\}$, such that

$$\leq k_1 \leq k_2 \leq ... \leq k_{i_1} \leq k_i \geq k_{i+1} \geq ... \geq k_n.$$

In 1968 Read observed that for any graph G the absolute values of the coefficients a_i of the chromatic polynomial appeared to form a unimodal sequence. This became known as the *Unimodal Conjecture* [12]. Although this conjecture has been proven for many classes of graphs, for a long time the best possible result for general G was the following lemma:

Lemma 1.33. *Let G be a connected graph, with n vertices. If n is odd, then*

$$1 < |a_n| < |a_{n-1}| < ... < |a_{\frac{n+1}{2}}|,$$

and if n is even, then

$$1 < |a_n| < |a_{n-1}| < ... < |a_{\frac{n}{2}+1}| \leq |a_{\frac{n}{2}}|,$$

where $a_{\frac{n}{2}+1} = a_{\frac{n}{2}}$ if and only if G is a tree.

A proof of Lemma 1.33 can be found in [6].
In 1974 Hoggar proposed the so-called *(Strong) Logarithmic Concavity Conjecture*. A sequence $k_1, k_2, ..., k_n$ of integers is called *logarithmically concave* (or *log-concave*), if

$$k_i^2 \geq k_{i-1}k_{i+1}$$

holds for all $2 \leq i \leq n-1$, and *strongly logarithmically concave* (or *strongly log-concave*), if

$$k_i^2 > k_{i-1}k_{i+1},$$

for all $2 \leq i \leq n-1$. Hoggar conjectured that the coefficients a_i of the chromatic polynomial of any graph G form a strongly log-concave sequence. Clearly log-concavity implies unimodality. The research has been concentrated on the Log-Concavity Conjecture rather than on the Unimodal Conjecture ever since and with success. After several results for different classes of graphs in the 1970's and 1980's a report of Lundow and Markström in 2002 showed that the Strong Logarithmic Concavity Conjecture holds for all graphs with $V(G) \leq 11$ and for all graphs with $V(G) = 12$ and either $E(G) < 20$ or $E(G) > 45$ [11]. Recently, in 2010, June Huh proved the Log-Concavity Conjecture and thus the Unimodal Conjecture in the paper *Milnor Numbers of Projective Hypersurfaces and the Chromatic Polynomial of Graphs*, that has been published in an improved version in the beginning of 2012 [9].

Interpretation of the Coefficients of $\mathcal{P}(G, x)$

Every proper r-coloring of a graph G corresponds with a partition of the vertex set V of G into $j \leq r$ independent subsets. So if $\beta_j(G)$ is the number of all distinct proper coloring partitions of V into j blocks, then permutations of the r colors give that there are exactly $\beta_j(G)[r]_j$ proper r-colorings that use j of the r colors. Since $\beta_j(G) = 0$, for $j < \chi(G)$ and $[r]_j = 0$, for $j > r$ this simply means, that

$$\mathrm{C}_r(G) = \sum_{j=\chi(G)}^{n} \beta_j(G)[r]_j$$

and so

$$\mathcal{P}(G, x) = \sum_{j=\chi(G)}^{n} \beta_j(G)[x]_j.$$

Thus the coefficient b_j in the factorial form of the chromatic polynomial of a graph G is exactly the number of partitions of V into j independent subsets.

Theorem 1.34. *For every finite graph G the chromatic polynomial is equal to*

$$\mathcal{P}(G, x) = \sum_{j=\chi(G)}^{n} \beta_j(G)[x]_j,$$

where $\beta_j(G)$ is the number of partitions of V into j independent subsets. □

An easy consequence of this is now:

Corollary 1.35. *Let $G = (V, E)$ be a graph with $\mathcal{P}(G, x) = \sum_{j=\chi(G)}^{n} b_j [x]_j$. Then*

$$G \text{ is uniquely colorable } \iff b_{\chi(G)} = 1.$$

Proof. Uniquely colorability means that there is exactly one partition of V into $\chi(G)$ independent subset, thus the claim follows from the interpretation of the coefficients b_j of the factorial form of $\mathcal{P}(G)$. □

To give an interpretation of the coefficients a_i of the normal form of $\mathcal{P}(G, x)$ is a bit less obvious and we will need to understand what happens in the process of deletion/contraction.
Let $G = (V, E)$ be a graph with n points and m edges, and let $e_1, e_2, ..., e_m$ be a fixed order of the edge set E. Then the deletion-contraction property gives, that

$$\begin{aligned}
\mathcal{P}(G, x) &= \mathcal{P}(G \backslash e_1, x) - \mathcal{P}(G/e_1, x) \\
&= \mathcal{P}(G \backslash \{e_1, e_2\}, x) - \mathcal{P}((G \backslash e_1)/e_2, x) - \mathcal{P}(G/e_1, x) \\
&= ... \\
&= \mathcal{P}(G \backslash \{e_1, e_2, ..., e_m\}, x) \\
&\quad - \Big(\mathcal{P}(G/e_1, x) + \sum_{i=2}^{m} \mathcal{P}((G \backslash \{e_1, e_2, ..., e_{i-1}\})/e_i, x) \Big) \\
&= \mathcal{P}(\overline{K_n}, x) - \Big(\mathcal{P}(G/e_1, x) + \sum_{i=2}^{m} \mathcal{P}((G \backslash \{e_1, e_2, ..., e_{i-1}\})/e_i, x) \Big) \\
&= x^n - \Big(\mathcal{P}(G/e_1, x) + \sum_{i=2}^{m} \mathcal{P}((G \backslash \{e_1, e_2, ..., e_{i-1}\})/e_i, x) \Big).
\end{aligned}$$

If an edge $e_i \in E$ is contained in a triangle $\{e_i, e_{j_1}, e_{j_2}\}$ in $G \backslash \{e_1, e_2, ..., e_{i-1}\}$, then contracting $e_i = [a, b]$ means that we 'melt' $e_{j_1} = [a, x]$ and $e_{j_2} = [b, x]$ together and obtain the edge $[e_i, x] \in E((G \backslash \{e_1, e_2, ..., e_{i-1}\})/e_i)$. Therefore, if we identify $[e_i, x]$ with the edge $e_k \in E$, where $k = \max\{j_1, j_2\}$, then $E((G \backslash \{e_1, e_2, ..., e_{i-1}\})/e_i)$ corresponds with a subset E' of E. We can now again use the deletion-contraction formula on the polynomial $\mathcal{P}((G \backslash \{e_1, ..., e_{i-1}\})/e_i, x)$ and find that

$$\begin{aligned}
\mathcal{P}((G \backslash \{e_1, ..., e_{i-1}\})/e_i, x) = \mathcal{P}((G \backslash \{e_1, ..., e_{i-1}, e_{i+1}, ..., e_m\})/e_i, x) \\
- \Big(\mathcal{P}(((G \backslash \{e_1, ..., e_{i-1}\})/e_i)/e_{i+1}, x) \\
+ \sum_{i+2 \le j \le m} \mathcal{P}((G \backslash \{e_1, ..., e_{i-1}, e_{i+1}, ..., e_{j-1}\}/e_i)/e_j, x) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}(\overline{K_{n-1}}, x) - \Big(\mathcal{P}(((G\backslash\{e_1, ..., e_{i-1}\})/e_i)/e_{i+1}, x) \\
&\qquad + \sum_{i+2\leq j\leq m} \mathcal{P}((G\backslash\{e_1, ..., e_{i-1}, e_{i+1}, ..., e_{j-1}\}/e_i)/e_j, x)\Big) \\
&= x^{n-1} - \Big(\mathcal{P}(((G\backslash\{e_1, ..., e_{i-1}\})/e_i)/e_{i+1}, x) \\
&\qquad + \sum_{i+2\leq j\leq m} \mathcal{P}((G\backslash\{e_1, ..., e_{i-1}, e_{i+1}, ..., e_{j-1}\}/e_i)/e_j, x)\Big).
\end{aligned}
$$

We assume here, that the sum

$$
\mathcal{P}(((G\backslash\{e_1, ..., e_{i-1}\})/e_i)/e_{i+1}, x) + \sum_{i+2\leq j\leq m} \mathcal{P}((G\backslash\{e_1, ..., e_{i-1}, e_{i+1}, ..., e_{j-1}\}/e_i)/e_j, x)
$$

is taken over all edges in E'. Repeatedly deleting and contracting edges with the formula will always lead to a graph

$$
\begin{aligned}
&G(\{i_1,i_2, ..., i_k\}) \\
&\qquad := (...((G\backslash\{e_1, ..., e_{i_1-1}, e_{i_1+1}, ..., e_{i_2-1}, e_{i_2+1}, ..., e_{i_k-1}, e_{i_k+1}, ..., e_m\}/e_{i_1})/e_{i_2})/...)/e_{i_k},
\end{aligned}
$$

for certain $1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq m$. Note that

$$
G(\{i_1, i_2, ..., i_k\}) \cong \overline{K_{n-k}}
$$

is the empty graph on $n-k$ vertices and that every vertex in $G(\{i_1, i_2, ..., i_k\})$ is either a vertex of G or obtained by contracting edges. Therefore $G(\{i_1, i_2, ..., i_k\})$ corresponds uniquely with a spanning subgraph of G, having $n-k$ connected components and k edges. Observe further, that if we contract edges of a cycle $C = \{e_{i_1}, e_{i_2}, ..., e_{i_l}\}$ in G, then $(...((C/e_{i_1})/e_{i_2})/...)/e_{i_{l-3}}$ is a triangle and thus $(...((C/e_{i_1})/e_{i_2})/...)/e_{i_{l-2}}$ is equal to $e_{i_l} \sim e_{i_{l-1}}$ and will be denoted as e_{i_l}, by the earlier made convention. Therefore, if we repeatedly delete and contract edges from G we will never contract all but the last edge of a cycle (the edge with the highest index). Thus, for any graph

$$
((G\backslash\{e_1, ..., e_{i_1-1}, e_{i_1+1}, ..., e_{i_2-1}, e_{i_2+1}, ..., e_{i_k-1}, e_{i_k+1}, ..., e_m\}/e_{i_1})/e_{i_2})/...)/e_{i_k}
$$

that we obtain by deletion/contraction the set $\{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$ is a so called *broken cycle-free* subset of E.

Definition 1.36. Let $C = \{e_1, e_2, ..., e_k\}$ be a cycle and $e_1, e_2, ..., e_k$ be some fixed order of its edges. The subset $\{e_1, e_2, ..., e_{k-1}\}$, that misses the edge with the highest index is called a *broken cycle*. If $G = (V, E)$ is a finite graph with a fixed linear ordering of the edge set, then a subset $E' \subseteq E$, that contains no broken cycles (as subsets) is called a *broken cycle-free* subset of E.

Now we see that any graph $G(\{i_1, i_2, ..., i_k\})$ that we obtain by deletion/contraction corresponds with a broken cycle-free spanning subgraph of G, having $n-k$ connected components and k edges. Let $\alpha_i(G)$ denote the number of all spanning subgraphs of G with $n-i$ connected components (or equivalently i edges), that contain no broken cycles, then recursion gives, that

$$\begin{aligned}
\mathcal{P}(G,x) &= \mathcal{P}(\overline{K_n},x) - \Big(\mathcal{P}(G/e_1,x) + \sum_{i=2}^{m} \mathcal{P}((G\backslash\{e_1,e_2,...,e_{i-1}\})/e_i,x)\Big) \\
&= x^n - \Big(\mathcal{P}(G/e_1,x) + \sum_{i=2}^{m} \mathcal{P}((G\backslash\{e_1,e_2,...,e_{i-1}\})/e_i,x)\Big) \\
&= ... \\
&= \alpha_0(G)x^n - \alpha_1(G)x^{n-1} + \alpha_2(G)x^{n-2} - ... \\
&= \sum_{i=0}^{n-1} (-1)^i \alpha_i(G) x^{n-i}.
\end{aligned}$$

Note that a spanning subgraph is uniquely defined by its edge set, thus $\alpha_i(G)$ is also equal to the number of all broken cycle-free subsets of $E(G)$ of size i. It follows a formal proof of the observations we made above.

Theorem 1.37 (Whitney, 1932 [17])**.** *For every finite graph G the chromatic polynomial is equal to*

$$\mathcal{P}(G,x) = \sum_{i=0}^{n-1} (-1)^i \alpha_i(G) x^{n-i},$$

where $\alpha_i(G)$ denotes the number of all broken cycle-free subsets of $E(G)$ of size i.

Proof. We proof by induction on the number of edges $|E| = m$. For $m = 0$ the claim is certainly true. Assume therefore that it is true for all graphs with less than m edges, and let G be a graph with m edges. Fix an order $e_1, e_2, ..., e_m$ of the edges of G. The graph $G\backslash e_1$ has the edge set $E' = \{e_2, ..., e_m\}$, which is a subset of E. The edge set of G/e_1 is not a subset of E, but we can identify the edge set of the graph G/e_1 with a subset E'' of E as follows:
If e_1 is contained in a triangle $\{e_1, e_i, e_j\}$, then contracting e_1 means 'melting' e_i and e_j together. Thus $e_i \sim e_j$ in G/e_1. Now, if $e_1 = [v_1, v_2]$ and $e_i = [v_1, x], e_j = [v_2, x] \in E$, we identify the edge $[e_1, x]$ in $E(G/e_1)$ with e_k in E, where $k = \max\{i, j\}$.
Now induction and the deletion-contraction property of the chromatic polynomial gives, that

$$\begin{aligned}
\mathcal{P}(G,x) &= \mathcal{P}(G\backslash e_1, x) - \mathcal{P}(G/e_1) \\
&= \sum_{i=0}^{n-1} (-1)^i \alpha_i(G\backslash e_1) x^{n-i} - \sum_{i=0}^{n-2} (-1)^i \alpha_i(G/e_1) x^{n-1-i}
\end{aligned}$$

$$= \sum_{i=0}^{n-1}((-1)^i\alpha_i(G\backslash e_1) - (-1)^{i-1}\alpha_{i-1}(G/e_1))x^{n-i}$$
$$= \sum_{i=0}^{n-1}(-1)^i(\alpha_i(G\backslash e_1) + \alpha_{i-1}(G/e_1))x^{n-i},$$

and so, if a_{n-i} is the $(n-i)$'th coefficient of the chromatic polynomial of G, then

$$a_{n-i} = (-1)^i(\alpha_i(G\backslash e_1) + \alpha_{i-1}(G/e_1)),$$

for every $0 \leq i \leq n-1$.
It remains to proof that $\alpha_i(G) = \alpha_i(G\backslash e_1) + \alpha_{i-1}(G/e_1)$. Consider first any subset $F \subseteq E$ of size i that does not contain e_1 and observe that F contains broken cycles in G if and only if F contains broken cycles in $G\backslash e_1$. Therefore, the number $\alpha_i(G\backslash e_1)$ is equal to all broken cycle-free subsets of size i of E that do not contain e_1. Now suppose, that $e_1 \in F \subseteq E$, then $F\backslash\{e_1\}$ is a subset of E'' if and only if there are no broken triangles in F that contain e_1. Thus, a subset $F \subseteq E$ of size i, that does contain e_1 is broken cycle-free if and only if $F\backslash\{e_1\} \subset E''$ and $F\backslash\{e_1\}$ is broken cycle-free in G/e_1. Therefore the number $\alpha_{i-1}(G/e_1)$ is equal to all broken cycle-free subsets of size i of E that contain e_1. And so the theorem follows. □

The Roots of the Chromatic Polynomial

Much research has been done concerning the roots of chromatic polynomials. In fact when George David Birkhoff originally defined the chromatic polynomial he hoped to be able to prove that 4 can never be a root of the chromatic polynomial of any planar graph.[2]

Conjecture 1.38 (Birkhoff-Lewis)**.** *If G is a planar graph, then $\mathcal{P}(G,x)$ has no real roots in $[4,\infty)$.*

Birkhoff and Lewis were able to prove that planar graphs have no chromatic roots in $[5,\infty)$, but were unsuccessful with the interval $[4,5)$. The ultimate goal of formulating this conjecture was to establish the *Four Color Conjecture*,[3] but instead Apple and Haken proved the Four Color Conjecture with the help of computers and through this established that no planar graph has a chromatic root at $x = 4$. The Birkhoff-Lewis Conjecture, however remains unsolved for the interval $(4,5)$.

In general, for any graph G, it is obvious that every non-negative integer $k < \chi(G)$ is a root of $\mathcal{P}(G,x)$ and that there are no integer roots $k \geq \chi(G)$. From the falling factorial form of the chromatic polynomial it follows that $\mathcal{P}(G,x)$ has no real roots greater than $n-1 = |V|-1$, since every term $b_i[x]_i$ is clearly greater than 0, for all $x > n-1 \geq i-1$.

[2] A *planar graph* is a graph that can be embedded in the plane.

[3] The *Four Color Conjecture* states that, given any map, which is simply a separation of a plane into contiguous regions, no more than four colors are needed to color the regions, such that no two adjacent regions have the same color. It was proven in 1989 by Apple and Haken and is since then known as the *Four Color Theorem*.

Lemma 1.39. *The chromatic polynomial of a graph has no negative real roots.*

Proof. From Lemma 1.31 we know that the coefficient a_i in the normal form of the chromatic polynomial is greater than 0 whenever i is equal to $n(\mathbf{mod}\ 2)$ and smaller than 0 otherwise. Therefore $(-1)^n a_i x^i > 0$ for any $x < 0$ and so

$$(-1)^n \mathcal{P}(G,x) = \sum_{i=1}^{n} (-1)^n a_i x^i > 0,$$

for any negative $x \in \mathbb{R}$. It follows that $\mathcal{P}(G,x)$ has no negative real roots. □

Thus all real roots of $\mathcal{P}(G,x)$ lie in the interval $[0, |V|)$.

Lemma 1.40. *There exists no chromatic polynomial that has real roots between* 0 *and* 1*.*

Proof. Since the chromatic polynomial of any disconnected graph is the product of the polynomials of the connected components it is enough to proof the claim only for connected graphs.
We proof by induction on the number of edges. Clearly the claim holds for any empty graph. Assume the claim is true for all graphs with less than m edges. Let G be a graph with m edges. If G is a tree the lemma follows from 1.26. If G is not a tree then $n := |V| > 2$ and we can choose an edge $e \in E$, such that $G\backslash e$ and G/e are both connected. Assume that G has a real root $x' \in (0,1)$, then it follows from the deletion-contraction property, that

$$\mathcal{P}(G\backslash e, x') = \mathcal{P}(G/e, x').$$

Since $G\backslash e$ is a connected graph with $n \geq 3$ and G/e is a connected graph with $n-1 \geq 2$ vertices, both graphs have a root at 0 and 1. From the previous lemma it follows, that the derivatives $\mathcal{P}'(G\backslash e, x)$ and $\mathcal{P}'(G/e, x)$ have opposite sign at $x = 0$. Therefore induction gives, that they cannot meet in the interval $(0,1)$ and so G has no real root in $(0,1)$. □

From Lemma 1.30 it follows, that the root $x = 0$ of $\mathcal{P}(G,x)$ has multiplicity equal to the number of connected components. The chromatic polynomial of any graph has a root at $x = 1$ with multiplicity bigger or equal to the number of *blocks*[4] of G: If $G = G_1 \cup G_2 \cup ... \cup G_k$, where $G_1, ..., G_k$ are the blocks of G, then repeatedly applying Lemma 1.24 gives, that

$$\mathcal{P}(G,x) = \frac{\mathcal{P}(G_1,x) \cdot \mathcal{P}(G_2,x) \cdots \mathcal{P}(G_k,x)}{x^{k-1}}.$$

Since every G_i is nonempty it follows that each $\mathcal{P}(G_i,x)$ has at least one root at 1. Therefore $\mathcal{P}(G,x)$ has a root at 1 with multiplicity at least k.
In fact the multiplicity of the root 1 of $\mathcal{P}(G,x)$ is equal to the number of blocks but we will not prove this here.

[4] A *block* of a graph is a maximal biconnected subgraph. Any connected graph decomposes uniquely into the sum of its blocks.

In 1993 Bill Jackson showed that there exists no graph with real chromatic roots in the interval $(1, \frac{32}{27}]$ [10]. Which leads us to the following conclusion:

Lemma 1.41. *For any graph G the chromatic polynomial $\mathcal{P}(G, x)$ has no real roots in $(-\infty, \frac{32}{27}]$ except for 0 and 1.*

Thomassen proved in 1997, that the real roots of all chromatic polynomials are dense in the interval $[\frac{32}{27}, \infty)$ [15], which shows that $(-\infty, 0]$, $(0, 1)$ and $(1, \frac{32}{27}]$ are the only intervals that are free from any real chromatic roots. However, for certain classes these intervals can be extended. Thomassen showed, for instance, that graphs with a Hamiltonian path[5] have no real chromatic roots in the interval $(1, \frac{1}{3}(2 + \sqrt[3]{26 + 6\sqrt{33}} + \sqrt[3]{26 - 6\sqrt{33}})) \approx (1, 1.29559...)$ [16] and of course we already saw that planar graphs have no real chromatic roots in $x = 4$ and $[5, \infty)$.
For complex chromatic roots a result of Sokal [14] showed that the roots of all chromatic polynomials are dense in $\mathbb{C}$. Sokal showed further, that:

Theorem 1.42 (Sokal, 2001 [13])**.** *For any $r \in \mathbb{N}$ there exists a universal constant $C(r) < \infty$, such that, for all simple graphs, with maximum degree $\Delta(G) \leq r$ all chromatic roots lie in the disc $|x| < C(r)$. For all graphs of second-largest degree $\leq r$ the chromatic roots lie in the disc $|x| < C(r) + 1$. Furthermore, $C(r) \leq 7.963907 \cdot r$.*

Fernández and Procacci improved Sokals result to $C(r) \leq 6.91 \cdot r$ in 2007 [8] and Dong and Koh showed in their paper *Bounds for the Real Zeros of Chromatic Polynomials* [7] that all real zeros of $\mathcal{P}(G, x)$ lie in the interval $[0\ ,\ 5.664 \cdot \Delta(G))$. Furthermore, as a special case, they showed that for $\Delta(G) = 3$ all real roots of $\mathcal{P}(G, x)$ lie in $[0\ ,\ 4.765 \cdot \Delta(G))$.

[5] A *Hamiltonian path* is a path that visits every vertex in a graph exactly once.

Chapter 2

Vertex Colorings for Simplicial Complexes

In the previous chapter we explained how vertex colorings are used in Graph Theory to study the structure of graphs. Similarly as in Graph Theory we can define colorings of the vertex set of finite abstract simplicial complexes (ASC). Since the 1-skeleton X^1 of a finite ASC X has the structure of a graph it is sometimes called the *underlying graph* of the complex X. Therefore the colorings as defined in Definition 1.1 can certainly describe some of the structure of a finite ASC. However, in order to take the higher dimensional structures of simplicial complexes into account we obviously need to make some other restrictions to the vertex colorings as in Chapter 1. Dobrinskaya/Møller/Notbohm defined vertex colorings for simplicial complexes in *Vertex Colorings of Simplicial Complexes* [5] as follows:

Definition 2.1. Let X be a finite abstract simplicial complex and let $s \in \mathbb{N}$. A (proper) (r,s)-*coloring* of X is a map $f : V \to P$ from the vertex set $V \cong X^0$ of X to a palette P of r colors, such that $|f^{-1}(c) \cap \sigma| \leq s$, for every $c \in P$ and every $\sigma \in X$. A simplicial complex that admits a (r,s)-coloring is (r,s)-*colorable*.

Clearly, a $(r,1)$-coloring is just a normal coloring of the underlying graph $G(X) := X^1$ of X as defined in the previous chapter.[1] It is also obvious, that all (r,s)-colorings, with $s > \dim(X)$ are proper. Here are some examples for $1 < s \leq \dim(X)$:

Example 2.2. Consider the 3-dimensional simplicial complex X defined by the two 3-simplices, [1,2,3,4] and [2,3,4,5]. The map $f : V \longrightarrow \{c_1, c_2\}$, defined by

$$f(1) = f(2) = f(5) = c_1 \quad \text{and} \quad f(3) = f(4) = c_2$$

is a proper $(2,2)$-coloring of X and the map $h : V \longrightarrow \{c_1, c_2\}$, defined by

$$h(1) = h(5) = c_1 \quad \text{and} \quad h(2) = h(3) = h(4) = c_2$$

is a proper $(2,3)$-coloring of X (Figure 2.1).

[1] Unless it leads to confusion we will simply write G instead of $G(X)$ or X^1.

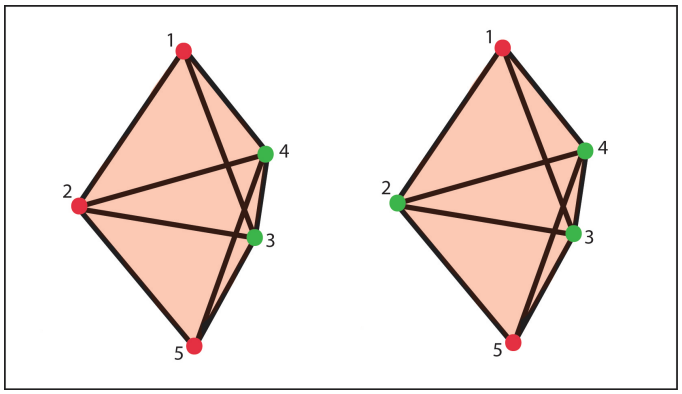

Figure 2.1: A $(2,2)$-coloring and a $(2,3)$-coloring of the simplicial complex X of Example 2.2.

Example 2.3. The triangulation $T2$ of the torus, defined by the fourteen 2-simplices $[1,2,4]$, $[1,2,6]$, $[1,3,4]$, $[1,3,7]$, $[1,5,6]$, $[1,5,7]$, $[2,3,5]$, $[2,3,7]$, $[2,4,5]$, $[2,6,7]$, $[3,4,6]$, $[3,5,6]$, $[4,5,7]$ and $[4,6,7]$ is $(3,2)$-colorable. Consider for instance the $(3,2)$-coloring $f : V \longrightarrow \{c_1, c_2, c_3\}$, defined by

$$f(1) = f(2) = f(3) = c_1, \ f(4) = f(5) = f(6) = c_2 \quad \text{and} \quad f(7) = c_3.$$

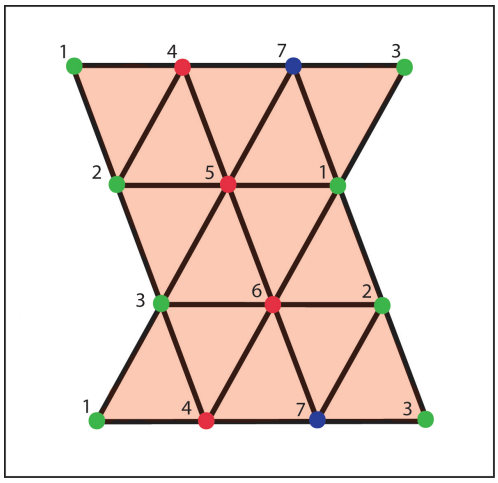

Figure 2.2: A $(3,2)$-coloring of the simplicial complex $T2$ of Example 2.3.

Example 2.4. The 3-dimensional complex M_{3-6-1}, defined by the eight 3-simplices

$$[1,2,3,4], \ [1,2,3,5], \ [1,2,4,6], \ [1,2,5,6], \ [1,3,4,5], \ [1,4,5,6], \ [2,3,4,5] \text{ and } [2,4,5,6]$$

is $(2,3)$-colorable. Take as an example the $(2,3)$-coloring $f : V \longrightarrow \{c_1, c_2\}$, defined by

$$f(1) = f(2) = f(3) = f(6) = c_1 \quad \text{and} \quad f(4) = f(5) = c_2.$$

Unlike graph vertex-colorings (r,s)-colorings do not necessarily induce partitions of V into independent sets. A map $f : V \longrightarrow P$ is a $(|P|, s)$-coloring of X if and only if $|f(\sigma)| > 1$ for every s-dimensional simplex $\sigma \in X$. Thus every (r,s)-coloring induces a partition of the vertex set into sets that do not contain any s-simplices, we will call these sets *s-simplex-independent sets*, or simply *s-independent sets*.
It is not very hard to see, that every (r,s)-coloring of a simplex X only depends on its s-skeleton X^s:

Lemma 2.5 (Dobrinskaya/Møller/Notbohm, [5])**.** *Let X be a finite ASC, then f is a (r,s)-coloring of X if and only if f is a (r,s)-coloring of the s-skeleton X^s.*

Proof. If f is a (r,s)-coloring of X it is obvious that it is also a (r,s)-coloring of X^s. On the other hand, if f is a proper (r,s)-coloring of X^s and $\sigma := [v_1, v_2, ..., v_{n+1}] \in X$ is any n-simplex, with $n > s$, then all s-faces of σ are in X^s and thus are properly colored. Therefore σ must also be properly colored, otherwise, if there would exist vertices $v_{i_1}, v_{i_2}, ..., v_{i_{s+1}} \in \sigma$ colored with the same color, then the s-simplex $[v_{i_1}, v_{i_2}, ..., v_{i_{s+1}}]$ would not be properly colored by f. Thus the lemma follows. □

Similarly as for graphs it can be interesting to ask how many colors are minimally necessary in order to color a finite ASC with a s-coloring. Therefore we define the *s-chromatic number*:

Definition 2.6. Let X be an ASC. The *s-chromatic number* $\chi^s(X)$ of X is the smallest natural number r such that X is (r,s)-colorable. If $\chi^s(X) = r$ we also say that X is *(r,s)-chromatic*.

Example 2.7. The 3-dimensional complex from Example 2.2 is obviously $(2,2)$ and $(2,3)$-chromatic.

Example 2.8. The complex $T2$ and M_{3-6-1} from Example 2.3 and 2.4 are both $(3,2)$-chromatic, furthermore M_{3-6-1} is $(2,3)$-chromatic.

For every $s \in \mathbb{N}$ the s-chromatic number is an invariant for finite ASC's. Clearly, every finite ASC X is $(r, \lceil \frac{|V|}{r} \rceil)$-colorable. Furthermore, since every 1-skeleton of a (complete) n-simplex is isomorphic to a complete graph K_{n+1} it clearly follows, that

$$m(X) := \max\{|\sigma| \mid \sigma \in X\} \leq \chi^1(X) = \chi(X^1) \leq |X^0| = |V|,$$

and so

$$1 = \chi^{m(X)}(X) \leq ... \leq \chi^2(X) \leq \chi^1(X) \leq |V|,$$

since every (r,s)-coloring is also a $(r, s+1)$-coloring of X.
We can improve this bounds a little bit, depending on s. Note first, that for every complete

n-simplex $\chi^s(\Delta^n) = \lceil \frac{n+1}{s} \rceil$, and that $\chi^s(X') \leq \chi^s(X)$, for every subcomplex X' of X. In particular we find that

$$\left\lceil \frac{|\sigma|}{s} \right\rceil = \chi^s(\Delta^{|\sigma|-1}) = \chi^s(\sigma) \leq \chi^s(X) \leq \chi^s(\Delta^{|V|-1}) = \left\lceil \frac{|V|}{s} \right\rceil,$$

for every $\sigma \in X$, so that

$$\left\lceil \frac{\dim(X)+1}{s} \right\rceil = \left\lceil \frac{\max\{|\sigma| \mid \sigma \in X\}}{s} \right\rceil \leq \chi^s(X) \leq \left\lceil \frac{|V|}{s} \right\rceil,$$

for every finite ASC X [5]. Therefore any finite ASC with vertex set V is $(\lceil \frac{|V|}{s} \rceil, s)$-colorable. These bounds are the best we can get, since for every (complete) simplex equality holds.

Uniquely s-Colorability

Let X be a finite ASC and

$$\mathrm{Aut}(X) = \{\rho \in \mathbf{S}_{|V|} \mid [v_1, v_2, ..., v_k] \in X \Leftrightarrow [\rho(v_1), \rho(v_2), ..., \rho(v_k)] \in X, \forall\, v_1, v_2, ..., v_k \in V\}$$

the automorphism group of X.

Definition 2.9. Two (r, s)-colorings f, f' of X are called *equivalent* (write $f \sim f'$) if there exist $\sigma \in \mathbf{S}_r$ and $\rho \in \mathrm{Aut}(X)$, such that $f \circ \rho = \sigma \circ f'$.

Definition 2.10. A finite ASC X is called *equivalently* (r, s)*-colorable* if all proper surjective (r, s)-colorings of X are equivalent.

Every finite ASC X, with vertex set V is equivalently $(|V|, 1)$-colorable and equivalently $(1, |V|)$-colorable. Complete n-simplices Δ^n are equivalently $(n, 2)$-colorable. Here are some more examples.

Example 2.11. Let X be the 2-dimensional simplicial complex, defined by the 2-simplices $[1,2,3]$, $[1,2,4]$, $[1,3,4]$, $[2,3,4]$, $[2,3,5]$, $[2,4,5]$ and $[3,4,5]$ then there are only three possible partitions of the vertex set V into two 2-independent sets:

$$P_1 := \{\{1,2,5\},\{3,4\}\},\ P_2 := \{\{1,3,5\},\{2,4\}\} \text{ and } P_3 := \{\{1,4,5\},\{2,3\}\}.$$

Clearly the map $\rho_{i,j} := (i\ j) \in \mathbf{S}_5$ is an automorphism of X, for $i, j \in \{2,3,4\}$. Furthermore, $\rho_{2,3}(P_1) = P_2$, $\rho_{2,4}(P_1) = P_3$ and $\rho_{3,4}(P_2) = P_3$. Thus, X is equivalently $(2,2)$-colorable (Figure 2.3).

Example 2.12. The subcomplex X of the 3-skeleton $(\Delta^6)^3$ of the complete 6-simplex, missing the two 3-simplices $[1,2,3,4]$ and $[1,5,6,7]$ is equivalently $(2,3)$-colorable: the only possible partitions of the vertex set into two 3-independent sets are

$$P_1 := \{\{1,2,3,4\},\{5,6,7\}\} \text{ and } P_2 := \{\{1,5,6,7\},\{2,3,4\}\}.$$

The map $\rho := (2\ 5\ 3\ 6\ 4\ 7) \in \mathbf{S}_7$ is an automorphism of X. Furthermore, $\rho(P_1) = P_2$, therefore X is equivalently $(2,3)$-colorable.

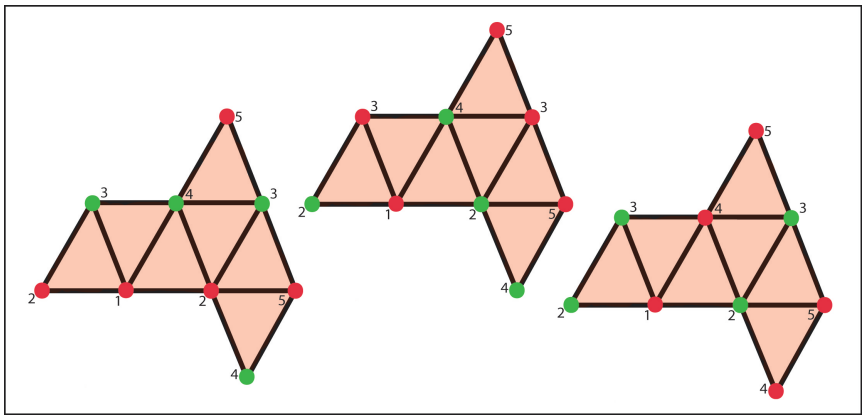

Figure 2.3: Three equivalent $(2,2)$-colorings of the simplicial complex X of Example 2.11.

If f is a (r,s)-coloring of a finite ASC X, then, similarly as for graphs, f and $\sigma \circ f$ are two (r,s)-colorings that induce the same partition of V into $i \leq r$ s-independent subsets, for every $\sigma \in \mathbf{S}_r$. So if we write $[f] := \{\sigma \circ f \mid \sigma \in \mathbf{S}_r\}$ for the *(r,s)-coloring class of f*, then, as with graph-colorings, equivalently (r,s)-colorability means that the action of $\mathrm{Aut}(X)$ on $\mathcal{F}_S^{(r,s)}(X) := \{[f] \mid f \text{ is a surjective } (r,s)\text{-coloring of } X\}$ is transitive. Thus X is equivalently (r,s)-colorable if and only if

$$|\mathcal{F}_S^{(r,s)}(X)| = |\mathrm{Aut}(X)f| = \frac{|\mathrm{Aut}(X)|}{|\mathrm{Aut}(X)_f|},$$

for $f \in \mathcal{F}_S^{(r,s)}(X)$, where $\mathrm{Aut}(X)f$ is the orbit of f and $\mathrm{Aut}(X)_f$ its stabilizer.

It is not hard to see, that Theorem 1.12 translates directly into the simplicial case. If $P := \{V_1, ..., V_r\}$ is a partition of the vertex set of a finite ASC X, define

$$X_P := \{[V_{i_1}, V_{i_2}, ..., V_{i_k}] \mid \exists v_{i_j} \in V_{i_j}, \text{ such that } [v_{i_1}, v_{i_2}, ..., v_{i_k}] \in X, 1 \leq k \leq r\}.$$

Now we find:

Theorem 2.13. *Let X be an equivalently (r,s)-colorable finite abstract simplicial complex, $f, f' : V \longrightarrow \{c_1, ..., c_r\}$ two surjective (r,s)-colorings and $\sigma \in \mathbf{S}_r$ and $\rho \in \mathrm{Aut}(X)$, such that $\sigma \circ f = f' \circ \rho$. Then $|f^{-1}(c_i)| = |f'^{-1} \circ \sigma(c_i)|$, for all $1 \leq i \leq r$. Furthermore, if P respectively P' are the corresponding partitions of the vertex set V, then ρ defines a simplicial isomorphism between X_P and $X_{P'}$.*

The proof of this theorem is nearly identical to the proof of 1.12 and will therefore be left out.

If there is just one partition of V into r s-independent sets, then $|\mathcal{F}_S^{(r,s)}(X)| = 1$ and X is equivalently (r,s)-colorable. Clearly $|\mathcal{F}_S^{(|V|,s)}(X)| = 1$ for every finite ASC X. As for graphs we find, that if $|\mathcal{F}_S^{(r,s)}(X)| = 1$ then $r = \chi^s(X)$ or $r = |V|$.

Lemma 2.14. *Let X be a finite ASC and $\chi^s(X) \leq r < |V|$. If there exists a permutation $\sigma \in \mathbf{S}_r$ for every two surjective (r,s)-colorings f, f', such that $f' = \sigma \circ f$, then $r = \chi^s(X)$.*

The proof of this lemma is very similar to the proof of Lemma 1.13.

Definition 2.15. A finite ASC X is called *uniquely s-colorable* if X has only one (proper) $\chi^s(X)$-coloring up to permutation of the colors. In that case a $\chi^s(X)$-coloring f is called a *unique s-coloring* of X.

Example 2.16. Let X be the 2-dimensional simplicial complex, defined by the 2-simplices $[1,2,3]$, $[1,2,4]$, $[1,3,4]$, $[1,3,5]$, $[1,4,5]$, $[2,3,4]$, $[2,3,5]$, $[2,4,5]$ and $[3,4,5]$, then X is uniquely 2-colorable, since the only possible partition of the vertex set V into two 2-independent sets is $\{\{1,2,5\},\{3,4\}\}$.

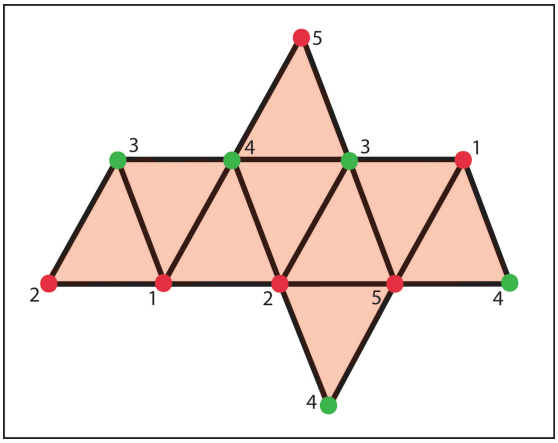

Figure 2.4: The uniquely 2-colorable simplicial complex X of Example 2.16.

Example 2.17. Consider the 3-skeleton $(\Delta^7)^3$ of the complete 7-simplex and let X be the subcomplex missing the two 3-simplices $[1,2,3,4]$ and $[5,6,7,8]$. X is $(2,3)$-chromatic and uniquely 3-colorable: the only possible partition of the vertex set into two 3-independent sets is clearly $\{\{1,2,3,4\},\{5,6,7,8\}\}$.
The subcomplex Y of the 2-skeleton $(\Delta^7)^2$ of the complete 7-simplex, missing the two 2-simplices $[1,2,3]$ and $[4,5,6]$ is $(3,2)$-chromatic and uniquely 2-colorable: the only possible partition of the vertex set into three 2-independent sets is $\{\{1,2,3\},\{4,5,6\},\{7,8\}\}$.

Let X be an uniquely s-colorable finite ASC and f a $\chi^s(X)$-coloring. If V_i and V_j are two distinct color classes of f, then every vertex $v \in V_i$ must be part of at least one s-simplex $[v, u_1, ..., u_s]$, such that $u_1, u_2, ..., u_s \in V_j$ and every vertex $u \in V_j$ must be part of at least one s-simplex $[u, v_1, ..., v_s]$, such that $v_1, v_2, ..., v_s \in V_i$. Thus, if $X(s)$ is the subset of all s-simplices, then:

$$|X(s)| \geq (\chi^s(X) - 1)|V|,$$

for $s > 1$.

Example 2.16 and 2.17 show, that the bound given above is very generous. For instance, complex X in 2.17 has 68 3-simplices, that is quite a lot more than $(\chi^3(X) - 1) \cdot 8 = 8$. Certainly that there is a lot more to say about uniquely s-colorability, but to find out if this is the best possible bound or to find better ones needs some further investigations.

Chapter 3

The s-Chromatic Polynomial

In Chapter 1 we proved the existence of a chromatic polynomial for vertex colorings of graphs and we showed that this polynomial can be calculated recursively for every finite graph. In this chapter we want to find similar polynomials associated to the s-colorings of finite abstract simplicial complexes. Equivalently as the chromatic polynomial for graphs, a "s-chromatic" polynomial for an ASC should return the number of distinct (r, s)-colorings for all $r \in \mathbb{N}$.

Definition 3.1. If X is a finite ASC, then we write $\mathrm{C}_{(r,s)}(X)$ to denote the number of all possible (r, s)-colorings of X.

Any s-coloring of a given abstract simplicial complex X corresponds with a partition of the vertex set V into s-simplex-independent sets. Define the following sub-class of s-simplex-independent partitions:

Definition 3.2. Let X be a finite ASC, $s \in \mathbb{N}$ and let $P := \{V_1, V_2, ..., V_k\}$ be a partition of the vertex set V of X into s-independent sets, such that each block V_i induces a connected subgraph of the underlying graph G of X. Then the partition P is called a *block-connected s-partition* or simply a *s-partition* of V.

If f is a (r, s)-coloring that induces a s-partition P of the vertex set V of X, then f is in fact an injective graph-vertex coloring of the partition-graph G_P. Now let f' be any (r, s)-coloring of X and P the partition of V induced by f'. Observe, that if P' is the coarsest block-connected refinement of P, then it is a s-partition of V and f' is a graph-vertex coloring of the partition-graph $G_{P'}$, using r colors.

Theorem 3.3. *Let X be a finite abstract simplicial complex, $r, s \in \mathbb{N}$ and $\mathcal{B}_s$ the collection of all block-connected s-partitions of V, then*

$$\mathrm{C}_{(r,s)}(X) = \sum_{D \in \mathcal{B}_s} \mathrm{C}_r(G_D).$$

Proof. We want to show, that every (r, s)-coloring f of X corresponds uniquely with a pair (D, f'), where $D \in \mathcal{B}_s$ is a s-partition of the vertex set V and f' is a graph-vertex coloring

of G_D.
If $f : V \longrightarrow P$ is a (r,s)-coloring of X, then $f^{-1}(P)$ defines a partition of V. Let $V_1 := f^{-1}(c_1), V_2 := f^{-1}(c_2), ..., V_r := f^{-1}(c_r)$ be the corresponding color classes. For each $1 \leq i \leq r$ let further $V_{i_1}, V_{i_2}, .., V_{i_{n(i)}}$ be the maximal subsets of V_i, such that each V_{i_j} induces a connected subgraph of G. Then $D_f := \{V_{1_1}, V_{1_2}, .., V_{1_{n(1)}}, V_{2_1}, V_{2_2}, ..., V_{r_{n(r)}}\}$ is the coarsest block-connected refinement of $f^{-1}(P)$ and therefore D_f is a s-partition of V. The map $f' : D_f \longrightarrow P$, defined as $f'(V_{i_j}) := c_i$ for all $1 \leq j \leq n(i)$, is a graph-vertex coloring of G_{D_f}. From this construction it follows, that the map

$$\varphi : \{(r,s)\text{-colorings of } X\} \longrightarrow \{(D, f') \mid D \in \mathcal{B}_s, f' \text{ is a } r\text{-coloring of } G_D\} :$$

$$f \mapsto (D_f, f')$$

is injective: if $h : V \longrightarrow P$ is another (r,s)-coloring of X, $h \neq f$, then there is a color $c_i \in P$, such that $V_i' := h^{-1}(c_i) \neq f^{-1}(c_i) =: V_i$. Therefore, if $D_h = D_f$ it must follow that $h' \neq f'$.
For the surjectivity of the map φ let $D = \{V_1, ..., V_k\} \in \mathcal{B}_s$ and let $g : D \longrightarrow P$ be a r-coloring of G_D. Now define the map $f : V \longrightarrow P$ as $f(x) := g(V_i)$, for all $x \in V_i$, $1 \leq i \leq r$, then f is a (r,s)-coloring of X. Since for all $V_i, V_j \in D$, $V_i \neq V_j$, with $g(V_i) = g(V_j)$ clearly $[V_i, V_j]$ can not be an edge in G_D and every V_i induces a connected subgraph of G it follows that $D_f = D$ and $f' = g$, therefore $\varphi(f) = (D, g)$.
It follows that φ is a bijection and so $\mathrm{C}_{(r,s)}(X) = \sum_{D \in \mathcal{B}_s} \mathrm{C}_r(G_D)$. □

Corollary/Definition 3.4. *For every finite abstract simplicial complex X and every $s \in \mathbb{N}$ the polynomial*

$$\mathcal{P}^s(X, x) := \sum_{D \in \mathcal{B}_s} \mathcal{P}(G_D, x)$$

is equal to $\mathrm{C}_{(r,s)}(X)$ for every $r \in \mathbb{Z}_{\geq 0}$. $\mathcal{P}^s(X, x)$ is therefore called the s-chromatic polynomial of X.

Proof. From 1.19 it follows that $\mathcal{P}(G_D, r) = \mathrm{C}_r(G_D)$, for every s-partition D of V and every $r \in \mathbb{Z}_{\geq 0}$. Therefore Theorem 3.3 gives that

$$\mathcal{P}^s(X, r) = \sum_{D \in \mathcal{B}_s} \mathrm{C}_r(G_D) = \mathrm{C}_{(r,s)}(X).$$

□

Corollary/Definition 3.4 not only establishes the existence of a s-chromatic polynomial, it also shows how to calculate it. However, since the computation of the chromatic polynomial for graphs is not particularly easy, it is clear that the s-chromatic polynomials of big simplicial complexes must have very high computation times.
Before we will try to find out more about the s-chromatic polynomial we look at some examples.

Example 3.5. The block-connected 2-partitions $\mathcal{B}_2$ of the 2-simplex Δ^2 are equal to

$$D_1 = \{\{1\},\{2\},\{3\}\},\ D_2 = \{\{1,2\},\{3\}\},\ D_3 = \{\{1,3\},\{2\}\},\ D_4 = \{\{2,3\},\{1\}\}.$$

Thus, if $G := G(\Delta^2)$, then $G_{D_1} = K_3$ and $G_{D_i} = K_2$, for $i = 2,3,4$. We find that

$$\begin{aligned}\mathcal{P}^2(\Delta^2, x) &= \sum_{D\in\mathcal{B}_2} \mathcal{P}(G_D, x)\\ &= \mathcal{P}(K_3, x) + 3\,\mathcal{P}(K_2, x)\\ &= [x]_3 + 3[x]_2\\ &= x^3 - x.\end{aligned}$$

Example 3.6. The underlying graph G of the 3-dimensional simplicial complex X from Example 2.2, defined by the two 3-simplices, $[1,2,3,4]$ and $[2,3,4,5]$ is isomorphic to the complete graph on five vertices missing the edge $[1,5]$, $K_5\backslash[1,5]$. The 2-partitions $\mathcal{B}_2$ of X are equal to

$D_1 = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$,
$D_{2_i} = \{\{1\},\{5\},\{i \notin \{1,5\}\},\{j,k \notin \{1,5,i\}\}\}$ $(i = 2,3,4)$,
$D_{3_{1i}} = \{\{1,i\},\{5\},\{j\},\{k\}\}$ $(i = 2,3,4)$,
$D_{3_{5i}} = \{\{5,i\},\{1\},\{j\},\{k\}\}$ $(i = 2,3,4)$,
$D_{4_i} = \{\{1,i,5\},\{j\},\{k\}\}$ $(i = 2,3,4)$,
$D_{5_{in}} = \{\{i\},\{j,k\ ([j,k] \in X)\},\{l,m\ ([l,m] \in X)\}\}$ (i=1,..,5; n=1,2,(3)),
$D_{6_i} = \{\{1,i,5\},\{j,k\}\}$ $(i = 2,3,4)$

and so

$G_{D_1} = K_5\backslash[1,5]$
$G_{D_{2_i}} = K_4\backslash[1,5]$ $(i = 2,3,4)$,
$G_{D_{3_{1i}}} = G_{D_{3_{5i}}} = K_4$ $(i = 2,3,4)$,
$G_{D_{4_i}} = K_3$ $(i = 2,3,4)$,
$G_{D_{5_{in}}} = K_3$ $(i = 1,..,5; n = 1,2,(3))$,
$G_{D_{6_i}} = K_2$ $(i = 2,3,4)$.

We find, that

$$\begin{aligned}\mathcal{P}^2(X, x) &= \sum_{D\in\mathcal{B}_2} \mathcal{P}(G_D, x)\\ &= \mathcal{P}(K_5\backslash[1,5], x) + 6\,\mathcal{P}(K_4, x) + 3\,\mathcal{P}(K_4\backslash[1,5], x) + 15\,\mathcal{P}(K_3, x) + 3\,\mathcal{P}(K_2, x)\\ &= (x-3)[x]_4 + 6[x]_4 + 3(x-2)[x]_3 + 15[x]_3 + 3[x]_2\\ &= [x]_5 + 10[x]_4 + 18[x]_3 + 3[x]_2\\ &= x^5 - 7x^3 + 9x^2 - 3x.\end{aligned}$$

To write down the 3-partitions in an understandable way is quite complicated, nevertheless, the 3-chromatic polynomial of X is given by:

$$\begin{aligned}\mathcal{P}^3(X) &= \sum_{D\in\mathcal{B}_2} \mathcal{P}(G_D, x)\\ &= [x]_5 + 10[x]_4 + 25[x]_3 + 13[x]_2\\ &= x^5 - 2x^2 + x.\end{aligned}$$

Since the s-chromatic polynomial for simplicial complexes is a sum of chromatic polynomials some properties follow directly from the properties of the chromatic polynomial for graphs. We easily see for instance, that the constant term of $\mathcal{P}^s(X,x)$ must be zero for all ASC X, and since the sum of the coefficients of the chromatic polynomial equals zero for every graph with at least one edge, this is certainly also true for the s-chromatic polynomial of any simplicial complex X, whenever $s \leq \dim(X)$. Furthermore, the highest power of x in the s-chromatic polynomial of any complex X equals the number of vertices $n = |V|$, since $X^0 \in \mathcal{B}_s$, and, as X^0 is clearly the only s-partition with n blocks, it follows that the coefficient of x^n equals 1. Another easy observation is, that the s-chromatic polynomial of the disjoint union of k connected ASC's $X_1 \sqcup X_2 \sqcup ... \sqcup X_k$ is equal to

$$\mathcal{P}^s(X_1 \sqcup X_2 \sqcup ... \sqcup X_k, x) = \mathcal{P}^s(X_1, x) \cdot \mathcal{P}^s(X_2, x) \cdots \mathcal{P}^s(X_k, x).$$

If $X := X_1 \sqcup X_2 \sqcup ... \sqcup X_k$, then also every partition-graph of the underlying graph, induced by a s-partition, is a disjoint union of k components. Thus, if we write $\mathcal{P}^s(X,x) = \sum_{i=1}^{n} a_i x^i$, it follows from Lemma 1.30, that $a_i = 0$, for all $i < k$. However, we can clearly not conclude that $a_k \neq 0$, since the sign of the coefficient of x^k in each polynomial $\mathcal{P}(G_D, x)$ depends on the number of vertices of the partition-graph G_D. In fact we will come across some examples of connected simplicial complexes, where $a_1 = 0$ (see Example 3.9 or 3.11) .

Lemma 3.7. *Let X, Y be two finite abstract simplicial complexes, such that $|X^0 \cap Y^0| = 1$, then*

$$\mathcal{P}^s(X \cup Y, x) = \frac{\mathcal{P}^s(X,x) \cdot \mathcal{P}^s(Y,x)}{x},$$

for all $s \in \mathbb{N}$

Proof. Let $s \in \mathbb{N}$, $X^0 \cap Y^0 = \{v\}$ and $D \in \mathcal{B}_s(X \cup Y)$. If $C \in D$ is the block that contains v, then, since D is a block-connected s-partition and v is the only connection between X and Y, this is the only block of D that contains vertices of X and vertices of Y. Let $C_{|X} := C \cap X^0$ and $C_{|Y} := C \cap Y^0$ and define $D_X := \{C_i \in D \mid C_i \subset X^0\} \cup \{C_{|X}\}$ and $D_Y := \{C_i \in D \mid C_i \subset Y^0\} \cup \{C_{|Y}\}$. It is not difficult to see that $D_X \in \mathcal{B}_s(X)$ and

$D_Y \in \mathcal{B}_s(Y)$. Since $G(X \cup Y)_D \cong G(X)_{D_X} \cup G(Y)_{D_Y}$ and $V(G(X)_{D_X}) \cap V(G(Y)_{D_Y}) = C$, we know from Chapter 1 that

$$\mathcal{P}(G(X \cup Y)_D, x) = \frac{\mathcal{P}(G(X)_{D_X}, x) \cdot \mathcal{P}(G(Y)_{D_Y}, x)}{x},$$

for every $D \in \mathcal{B}_s$.
On the other hand, if $D'_X \in \mathcal{B}_s(X)$, $D'_Y \in \mathcal{B}_s(Y)$ and $C_X \in D'_X$, $C_Y \in D'_Y$ are the blocks that contain v then there are no vertices in $C := C_X \cup C_Y$ that span a s-simplex (for $s > 1$) and C induces a connected subgraph of $G(X \cup Y)$. Therefore every two s-partitions $D'_X \in \mathcal{B}_s(X)$, $D'_Y \in \mathcal{B}_s(Y)$ define a unique s-partition $D := D'_X \backslash \{C_X\} \cup D'_Y \backslash \{C_Y\} \cup \{C\} \in \mathcal{B}_s(X \cup Y)$ and so

$$\begin{aligned}
\mathcal{P}^s(X \cup Y, x) &= \sum_{D \in \mathcal{B}_s(X \cup Y)} \mathcal{P}(G(X \cup Y)_D, x) \\
&= \sum_{D_X \in \mathcal{B}_s(X)} \sum_{D_Y \in \mathcal{B}_s(Y)} \frac{\mathcal{P}(G(X)_{D_X}, x) \cdot \mathcal{P}(G(Y)_{D_Y}, x)}{x} \\
&= \frac{\mathcal{P}^s(X, x) \cdot \mathcal{P}^s(Y, x)}{x}.
\end{aligned}$$

□

Definition 3.8. Two finite abstract simplicial complexes, X and Y are called *s-chromatically equivalent* if

$$\mathcal{P}^s(X, x) = \mathcal{P}^s(Y, x).$$

Example 3.9. In Example 3.5 and 3.6 we calculated the s-chromatic polynomials of the 2-simplex Δ^2 and the simplicial complex X defined by the two 3-simplices $[1, 2, 3, 4], [2, 3, 4, 5]$. Let X' be the simplicial complex defined by the simplices $[1, 2, 3, 4], [2, 3, 4, 5], [5, 6, 7]$ and X'' the simplicial complex defined by the simplices $[1, 2, 3, 4], [2, 3, 4, 5], [4, 6, 7]$ (Figure 3.1). Then Lemma 3.7 gives that X' and X'' are s-chromatically equivalent, for all $s \in \mathbb{N}$:

$$\begin{aligned}
\mathcal{P}^2(X', x) = \mathcal{P}^2(X'', x) &= \frac{\mathcal{P}^2(X, x) \cdot \mathcal{P}^2(\Delta^2, x)}{x} \\
&= \frac{(x^5 - 7x^3 + 9x^2 - 3x)(x^3 - x)}{x} \\
&= x^7 - 8x^5 + 9x^4 + 4x^3 - 9x^2 + 3x \\
&= [x]_7 + 21[x]_6 + 132[x]_5 + 279[x]_4 + 159[x]_3 + 9[x]_2,
\end{aligned}$$

$$\begin{aligned}\mathcal{P}^3(X',x) = \mathcal{P}^3(X'',x) &= \frac{\mathcal{P}^3(X,x)\cdot\mathcal{P}^3(\Delta^2,x)}{x}\\ &= \frac{(x^5-2x^2+x)x^3}{x}\\ &= x^7-2x^4+x^2\\ &= [x]_7+21[x]_6+140[x]_5+348[x]_4+289[x]_3+50[x]_2.\end{aligned}$$

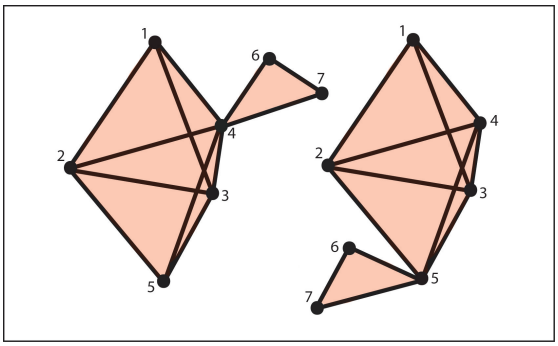

Figure 3.1: The chromatically equivalent simplicial complexes X' and X'' of Example 3.9.

Here are a couple of more examples of chromatic polynomials of some of the complexes of Chapter 2:

Example 3.10. The 2-chromatic polynomial of the simplicial complex $T2$, of Example 2.3 is given by:

$$\begin{aligned}\mathcal{P}^2(T2,x) &= x^7-14x^5+21x^4+7x^3-21x^2+6x\\ &= [x]_7+21[x]_6+126[x]_5+231[x]_4+84[x]_3.\end{aligned}$$

Example 3.11. The 3-dimensional complex M_{3-6-1} from Example 2.4 has the following chromatic polynomials:

$$\begin{aligned}\mathcal{P}(M_{3-6-1},x) &= x^6-14x^5+75x^4-190x^3+224x^2-96x\\ &= [x]_6+[x]_5,\\ \mathcal{P}^2(M_{3-6-1},x) &= x^6-16x^4+33x^3-18x^2\\ &= [x]_6+15[x]_5+49[x]_4+27[x]_3,\\ \mathcal{P}^3(M_{3-6-1},x) &= x^6-8x^3+10x^2-3x\\ &= [x]_6+15[x]_5+65[x]_4+82[x]_3+17[x]_2.\end{aligned}$$

The s-Chromatic Polynomial of Δ^n

Every finite abstract simplicial complex with n vertices can be obtained by deleting certain simplices of the n-simplex Δ^n and every finite ASC is uniquely defined by its maximal simplices. Therefore we want to find out more about the s-chromatic polynomials of Δ^n. Since the underlying graph G of Δ^n is equal to the complete graph K_{n+1} on $n+1$ vertices, every partition-graph G_D is a complete graph on $|D|$ vertices. It follows therefore that $\mathcal{P}^s(\Delta^n, x) = \sum_{D\in\mathcal{B}_s} \mathcal{P}(K_{|D|}, x) = \sum_{j=1}^{n+1} b_j[x]_j$, where $b_j = |\{D \in \mathcal{B}_s \mid |D| = j\}|$.

Theorem 3.12. *Let $\mathcal{S}'(n+1, j, s)$ denote the number of partitions of $n+1$ elements into j blocks, such that no block contains more than s elements, then*

$$\mathcal{P}^s(\Delta^n, x) = \sum_{j=\lceil \frac{n+1}{s} \rceil}^{n+1} \mathcal{S}'(n+1, j, s)[x]_j.$$

Proof. Note that if V' is a subset of the vertex set V such that $|V'| = s+1$, then the vertices of V' span a s-simplex in Δ^n. Furthermore, every subset of the vertex set induces a connected subgraph of the underlying graph $G(\Delta^n)$. Thus, for every $1 \leq j \leq n+1$, the number of s-partitions, with j blocks is exactly $\mathcal{S}'(n+1, j, s)$. Since $\mathcal{S}'(n+1, j, s) = 0$, for $\frac{n+1}{s} > j$ the theorem follows. □

If $j - 1 \geq n + 1 - s$ then $\mathcal{S}'(n+1, j, s)$ is equal to the Stirling number of the second kind $\mathcal{S}_2(n+1, j)$ and so it follows, that

$$\mathcal{P}^s(\Delta^n, x) = \sum_{j=n+2-s}^{n+1} \mathcal{S}_2(n+1, j)[x]_j + \sum_{j=\lceil \frac{n+1}{s} \rceil}^{n+1-s} \mathcal{S}'(n+1, j, s)[x]_j,$$

for $n + 1 - s \geq \lceil \frac{n+1}{s} \rceil$ and

$$\mathcal{P}^s(\Delta^n, x) = \sum_{j=n+2-s}^{n+1} \mathcal{S}_2(n+1, j)[x]_j,$$

for $n+1-s < \lceil \frac{n+1}{s} \rceil$. Note, that for $n+1-s < \lceil \frac{n+1}{s} \rceil$ it follows, that $n+1 < \frac{s^2}{s-1} = s+1+rest$ and thus, that $s \geq n = \dim \Delta^n$.

Lemma 3.13. *Let $\mathcal{I}_{(n,r,s)} := \{\hat{i} = (i_1, i_2, ..., i_r) \in \mathbb{N}^r \mid 0 < i_1 \leq i_2 ... \leq i_r \leq s \ ; \ \sum_{j=1}^r i_j = n\}$ and let $\eta_j : \mathbb{N}^r \to \mathbb{N}$ be the map, defined as $\eta_j(\hat{i}) := |\{i \in \{i_1, i_2, ..., i_r\} \mid i = j\}|$, for $\hat{i} = (i_1, i_2, .., i_r)$ and $j, r \in \mathbb{N}$, then*

$$\mathcal{S}'(n, r, s) = \frac{1}{r!} \sum_{\hat{i}\in\mathcal{I}_{(n,r,s)}} \binom{r}{\eta_1(\hat{i}), \eta_2(\hat{i}), ..., \eta_s(\hat{i})} \binom{n}{i_1, i_2, ..., i_r}.$$

Proof. Clearly, if $P := \{P_1, P_2, .., P_r\}$ is a partition of n elements into r blocks, such that no block contains more than s elements, and we sort the the blocks of P such that $|P_{i_1}| \leq |P_{i_2}| \leq ... \leq |P_{i_r}|$, then $(|P_{i_1}|, |P_{i_2}|, ..., |P_{i_r}|) \in \mathcal{I}_{(n,r,s)}$. Thus every such partition corresponds with a vector $\hat{i} \in \mathcal{I}_{(n,r,s)}$.
Given a vector $\hat{i} \in \mathcal{I}_{(n,r,s)}$ we want to count how many different partitions of a set V of n elements exist that correspond with $\hat{i}$. Assume first, that $\hat{i} = (i_1, i_2, ..., i_r)$, such that $0 < i_1 < i_2 < ... < i_r \leq s$. Then, starting with the first block, there are $\binom{n}{i_1}$ possibilities to choose i_1 elements from V. For the next block there are $\binom{n-i_1}{i_2}$ possibilities left and so on. Thus there are a total of

$$\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-\sum_{k=1}^{r-1} i_k}{i_r} = \frac{n!}{i_1! \cdot i_2! \cdots i_r!} = \binom{n}{i_1, i_2, ..., i_r}$$

partitions of V that correspond with $\hat{i}$.
The multinomial coefficient $\binom{n}{i_1, i_2, ..., i_r}$ gives the number of ways how to distribute n elements into r distinct 'boxes', such that in the j^{th} box there are i_j elements. Therefore, if $\hat{i} \in \mathcal{I}(n, r, s)$, such that there are $1 \leq j_1 \leq j_2 \leq ... \leq j_k \leq r$, with $i_{j_1} = i_{j_2} = ... = i_{j_k}$, then the multinomial coefficient $\binom{n}{i_1, i_2, ..., i_r}$ will count every partition corresponding with $\hat{i}$, $k!$ times. Thus, for every $\hat{i} \in \mathcal{I}(n, r, s)$ there are

$$\frac{1}{\eta_1(\hat{i})! \cdot \eta_2(\hat{i})! \cdots \eta_s(\hat{i})!}\binom{n}{i_1, i_2, ..., i_r} = \frac{1}{r!}\binom{r}{\eta_1(\hat{i}), \eta_2(\hat{i}), ..., \eta_s(\hat{i})}\binom{n}{i_1, i_2, ..., i_r}$$

corresponding partitions of V.
It follows, that

$$\mathcal{S}'(n, r, s) = \frac{1}{r!}\sum_{\hat{i} \in \mathcal{I}_{(n,r,s)}} \binom{r}{\eta_1(\hat{i}), \eta_2(\hat{i}), ..., \eta_s(\hat{i})}\binom{n}{i_1, i_2, ..., i_r}.$$

□

Example 3.14. We want to calculate the 4-chromatic polynomial of Δ^6. Therefore we need to find the numbers $\mathcal{S}'(7, 2, 4)$ and $\mathcal{S}'(7, 3, 4)$. Since $\mathcal{I}_{(7,3,4)} = \{(3, 2, 2), (3, 3, 1), (4, 2, 1)\}$ and $\mathcal{I}_{(7,2,4)} = \{(4, 3)\}$, it follows that

$$\begin{aligned}
\mathcal{S}'(7, 2, 4) &= \frac{1}{2!}\binom{2}{1, 1}\binom{7}{4, 3} = 35 \\
\mathcal{S}'(7, 3, 4) &= \frac{1}{3!}\left(\binom{3}{2, 1}\binom{7}{3, 2, 2} + \binom{3}{1, 2}\binom{7}{3, 3, 1} + \binom{3}{1, 1, 1}\binom{7}{4, 2, 1}\right) \\
&= \frac{7!}{2!3!2!2!} + \frac{7!}{2!3!3!} + \frac{7!}{4!2!} \\
&= 280.
\end{aligned}$$

Thus the 4-chromatic polynomial of Δ^6 is equal to:

$$\begin{aligned}\mathcal{P}^4(\Delta^6, x) &= \sum_{i=4}^{7} \mathcal{S}_2(7,i)[x]_i + \sum_{i=2}^{3} \mathcal{S}'(7,i,4)[x]_i \\ &= [x]_7 + 21[x]_6 + 140[x]_5 + 350[x]_4 + 280[x]_3 + 35[x]_2 \\ &= x^7 - 21x^3 + 35x^2 - 15x.\end{aligned}$$

In Appendix A is a list of the s-chromatic polynomials of Δ^n, for $3 \leq n \leq 9$.

The Coefficients of the s-Chromatic Polynomial

We already saw earlier that some properties of the coefficients of the s-chromatic polynomial for higher s are equivalent to those of $s = 1$. For instance that the constant of every s-chromatic polynomial must be zero and that the highest power of the s-chromatic polynomial is always equal to the number of vertices of the simplicial complex. On the other hand we can see in Example 3.9 that Lemma 1.31 is not true for higher s and that the coefficients of the s-chromatic polynomial in general do not form a log-concave or unimodal sequence. Furthermore, the 3-chromatic polynomial of the complexes in Example 3.9 and the 2-chromatic polynomial of the complex M_{3-6-1} in Example 3.11 show that also Lemma 1.30 is not (entirely) true for $s > 1$. Nevertheless we want to try to find out more about the coefficients of the s-chromatic polynomials for simplicial complexes.
As for the chromatic polynomial for graphs we can write every s-chromatic polynomial in two ways:

$$\mathcal{P}^s(X, x) = \sum_{i=1}^{n} a_i x^i = \sum_{j=1}^{n} b_j [x]_j.$$

where

$$a_i = \sum_{j=i}^{n} b_j \mathcal{S}_1(j,i) \quad \text{and} \quad b_j = \sum_{i=j}^{n} a_i \mathcal{S}_2(i,j).$$

In Chapter 1 we saw that

$$\mathcal{P}(G, x) = \sum_{j=\chi(G)}^{n} \beta_j(G)[x]_j,$$

where n is the number of vertices of G and $\beta_j(G)$ is the number of ways of coloring G in exactly j colors with color indifference. That means that $\beta_j(G)$ is equal to the number of proper coloring partitions of the vertex set V of G, namely the partitions of V into j blocks, such that each block is an independent vertex-set.

Definition 3.15. If X is a finite abstract simplicial complex, let $\mathcal{S}_\Delta(X, r, s)$ denote the number of partitions of the vertex set V of X into r blocks, such that no block contains vertices that span a s-simplex in X.

With this definition $\beta_j(G)$ equals $\mathcal{S}_\Delta(G, j, 1)$ and so we see that if X has n vertices

$$\mathcal{P}^s(X, x) = \sum_{D \in \mathcal{B}_s} \sum_{j=1}^{n} \mathcal{S}_\Delta(G(X)_D, j, 1)[x]_j.$$

If $D \in \mathcal{B}_s$ then no block of D contains vertices that span a s-simplex in X and so it follows, that

$$\mathcal{P}^s(X, x) = \sum_{j=1}^{n} \mathcal{S}_\Delta(X, j, s)[x]_j.$$

Since all non-negative integer roots of $\mathcal{P}^s(X, x)$ must lay in the interval $(0, \chi^s(X))$ it follows that

Theorem 3.16. *Let X be a finite abstract simplicial complex with $n \in \mathbb{N}$ vertices, then*

$$\mathcal{P}^s(X, x) = \sum_{j=\chi^s(X)}^{n} \mathcal{S}_\Delta(X, j, s)[x]_j.$$

□

We see now that, for every $n \in \mathbb{N}$, $\mathcal{S}_\Delta(\Delta^n, r, s) = \mathcal{S}'(n+1, r, s)$, for all $r, s \in \mathbb{N}$ and that Theorem 3.12 is just a special case of Theorem 3.16. In fact, for every simplicial complex X with n vertices we find that

$$\mathcal{S}'(n, r, s) \leq \mathcal{S}_\Delta(X, r, s),$$

and whenever $r - 1 < n - s$ then $\mathcal{S}'(n, r, s) = \mathcal{S}_\Delta(X, r, s)$ if and only if every subset of V consisting of $s + 1$ vertices spans a s-simplex in X. It is also easy to see, that

$$\mathcal{S}_\Delta(X, r, s) \leq \mathcal{S}_2(n, r),$$

and that equality holds if $r - 1 \geq n - s$. So, as before we find that

$$\mathcal{P}^s(X, x) = \sum_{j=n+1-s}^{n} \mathcal{S}_2(n, j)[x]_j + \sum_{j=\chi^s(X)}^{n-s} \mathcal{S}_\Delta(X, j, s)[x]_j.$$

Example 3.17. Consider the triangulation MB of the Möbius band, defined by the five 2-simplices $[1,2,4],[1,3,5],[1,4,5],[2,3,4]$ and $[2,3,5]$ (Figure 3.3). $\mathcal{S}_\Delta(MB,4,2)$ is equal to $\mathcal{S}_2(5,4)=10$. If D is a partition of the vertex set V into three blocks, then D contains either one 3-element-block and two 1-element-blocks or two 2-element-blocks and one 1-element-block. Since there are $\binom{5}{3}-5=5$ subsets of three elements of V, that do not span a 2-simplex in MB there are clearly five distinct partitions of the first kind. Furthermore there are $\frac{1}{2!}\binom{5}{2}\binom{3}{2}=15$ distinct partitions of the second kind. Therefore $\mathcal{S}_\Delta(MB,3,2)=20$. There are two different kind of partitions of V into two blocks: either a partition contains one 4-element-block and one 1-element-block or one 3-element-block and one 2-element-block. Since every subset of four elements of V spans a 2-simplex in M there are no partitions of the first kind, but there are five distinct partitions of the second kind, therefore $\mathcal{S}_\Delta(MB,2,2)=5$. It follows now that

$$\begin{aligned}\mathcal{P}^2(MB,x) &= [x]_5+10[x]_4+20[x]_3+5[x]_2\\ &= x^5-5x^3+5x^2-x.\end{aligned}$$

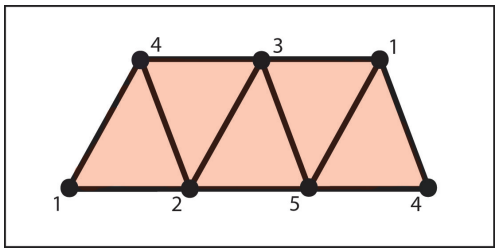

Figure 3.2: The triangulation MB of the Möbius band of Example 3.17.

Example 3.18. The simplicial complex X of Example 2.16 has four 2-simplices more than the triangulation MB of the Möbius band, therefore $\mathcal{S}_\Delta(X,3,2)=16$ and $\mathcal{S}_\Delta(X,2,2)=1$. We have:

$$\begin{aligned}\mathcal{P}^2(X,x) &= [x]_5+10[x]_4+16[x]_3+[x]_2\\ &= x^5-9x^3+13x^2-5x.\end{aligned}$$

We see that $\mathcal{S}_\Delta(X,2,2)=1$ for the uniquely 2-colorable simplicial complex of Example 3.18. Of course this follows, as for the chromatic polynomial, directly from the interpretation of the coefficients of the s-chromatic polynomial.

Corollary 3.19. *Let X be a finite abstract simplicial complex with s-chromatic polynomial $\mathcal{P}^s(X,x)=\sum_{j=\chi^s(G)}^{n}\mathcal{S}_\Delta(X,j,s)[x]_j$. Then*

$$X \text{ is uniquely } s\text{-colorable} \iff \mathcal{S}_\Delta(X,j,s)=1.$$

□

Another consequence of Theorem 3.16 is the following:

Lemma 3.20. *If X is a finite abstract simplicial complex, then*

$$\mathcal{P}^s(X,x) = \mathcal{P}^s(\Delta^n,x), \text{ for some } s \leq n \iff X^s \cong (\Delta^n)^s.$$

Proof. Clearly $\mathcal{P}^s(X,x) = \mathcal{P}^s(\Delta^n,x)$, whenever $X^s = (\Delta^n)^s$.
If X is a finite simplicial complex, such that $\mathcal{P}^s(X,x) = \mathcal{P}^s(\Delta^n,x)$ for some $s \leq n$, then $|V(X)| = n+1$ and we know from Theorem 3.12 and Theorem 3.16, that

$$\begin{aligned}\mathcal{P}^s(X,x) &= \sum_{j=\chi^s(X)}^{n+1} \mathcal{S}_\Delta(X,j,s)[x]_j \\ &= \sum_{j=\chi^s(\Delta^n)}^{n+1} \mathcal{S}'(n+1,j,s)[x]_j \\ &= \mathcal{P}^s(\Delta^n,x).\end{aligned}$$

Therefore $\mathcal{S}_\Delta(X,j,s) = \mathcal{S}'(n+1,j,s)$, for all $\chi^s(X) = \chi^s(\Delta^n) \leq j \leq n+1$. As remarked earlier this means that every subset of V that contains $s+1$ vertices must span a s-simplex in X and so $X^s = (\Delta^n)^s$. □

In correspondence with Definition 1.28 we make the following definition:

Definition 3.21. A finite abstract simplicial complex X is called *s-chromatically unique* if

$$\mathcal{P}^s(X,x) = \mathcal{P}^s(Y,x) \iff X^s \cong Y^s,$$

for every simplicial complex Y.

In the preceding examples it sticks out that, like for the chromatic polynomial of graphs, the coefficient of the second highest power in the normal form of $\mathcal{P}^s$ always equals the number of s-simplices of the simplicial complex. This is not a coincidence as the following theorem shows.

Theorem 3.22. *Let X be a finite abstract simplicial complex with n vertices and let m be the number of s-simplices. If $\mathcal{P}^s(X,x) = \sum_{i=1}^n a_i x^i$, $a_i \in \mathbb{Z}$, is the s-chromatic polynomial of X, then $a_i = 0$, for $n > i > n-s$ and $a_{n-s} = -m$.*

Proof. Theorem 3.16 gives that

$$\mathcal{P}^s(X,x) = \sum_{j=\chi^s(X)}^{n} \mathcal{S}_\Delta(X,j,s)[x]_j = \sum_{j=n+1-s}^{n} \mathcal{S}_2(n,j)[x]_j + \sum_{j=\chi^s(X)}^{n-s} \mathcal{S}_\Delta(X,j,s)[x]_j,$$

and so

$$a_i = \sum_{j=i}^{n} S_\Delta(X,j,s)S_1(j,i) = \sum_{j=n+1-s}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,i) + \sum_{j=i}^{n-s} \mathcal{S}_\Delta(X,j,s)\mathcal{S}_1(j,i),$$

for $1 \leq i \leq n$.

Claim 1 :
For every $i < n$

$$\sum_{j=i}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,i) = 0$$

proof of Claim 1: Since $x^k = \sum_{j=0}^{k} \mathcal{S}_2(k,j)[x]_j$ and $[x]_k = \sum_{i=0}^{k} \mathcal{S}_1(k,i)x^i$ for any $k \leq n$, the matrices $A = [\mathcal{S}_1(i,j)]_{n\times n}$ and $B = [\mathcal{S}_2(i,j)]_{n\times n}$ are each others inverse. Therefore it follows, that

$$\sum_{j=i}^{k} \mathcal{S}_2(k,j)\mathcal{S}_1(j,i) = \delta_{ik}$$

and thus

$$\sum_{j=i}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,i) = 0,$$

for every $i < n$. □

Claim 2 :

$$\mathcal{S}_\Delta(X, n-s, s) = \mathcal{S}_2(n, n-s) - m$$

proof of Claim 2: If D is a partition of the n vertices of X into $n-s$ blocks, then no block of D contains more then $s+1$ vertices and if $C \in D$ is a block that contains $s+1$ vertices then all other blocks of D contain exactly one vertex. Therefore there are exactly $\binom{n}{s+1}$ partitions of this form. Since X has m s-simplices the claim follows. □

From Claim 1 and Claim 2 it follows now that

$$a_i = \sum_{j=i}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,i) = 0,$$

for all $n-s < i < n$ and

$$\begin{aligned} a_{n-s} &= \sum_{j=n+1-s}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,n-s) + \mathcal{S}_\Delta(X,n-s,s)\mathcal{S}_1(n-s,n-s) \\ &= \sum_{j=n+1-s}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,n-s) + (\mathcal{S}_2(n,n-s) - m)\mathcal{S}_1(n-s,n-s) \end{aligned}$$

$$= \sum_{j=n-s}^{n} \mathcal{S}_2(n,j)\mathcal{S}_1(j,n-s) - m$$
$$= -m.$$

□

This proof also provides an alternative to the proof of Lemma 1.29, purely based on combinatorial observations.
Unfortunately this is about all, what can be said about the coefficients of the normal form of $\mathcal{P}^s$, for $s > 1$ at this point. All attempts to give a good interpretation of the coefficients a_i of the s-chromatic polynomial failed until now and will need more research. However, since the Sterling numbers of the first and second kind form log-concave sequences (see for example [1]) and considering the close connection between the numbers $\mathcal{S}_\Delta$ and $\mathcal{S}_2$, it is reasonable to believe that the coefficients of the falling factorial form of the s-chromatic polynomial are log-concave:

Conjecture 3.23. *For every abstract simplicial complex X the coefficients $\mathcal{S}_\Delta(X,j,s)$ of the s-chromatic polynomial $\mathcal{P}^s(X,x) = \sum_{j=\chi^s(X)}^{n} \mathcal{S}_\Delta(X,j,s)[x]_j$ form a log-concave sequence.*

The Roots of the s-Chromatic Polynomial

The roots of the s-chromatic polynomial, as the coefficients of the normal form will need more research. What can be said is that the interval $(0,1)$ is not always a zero-free interval. The 2-chromatic polynomial $\mathcal{P}^2(\Delta^4,x)$ of the 4-simplex for instance, has a root at $x \to 0.791288$. Also negative roots can obviously occur in s-chromatic polynomials, for $s > 1$, since $\mathcal{P}^n(\Delta^n,x) = x^{n+1} - x$ and so $x = -1$ is always a root for even n. Both is not very surprising, since the s-chromatic polynomial is the positive sum of chromatic polynomials with both, odd and even number of vertices. Therefore it seems quite unlikely that there exist intervals in $\mathbb{R}$ that are free of s-chromatic roots for some $s \in \mathbb{N}$. However, it might be possible and it is surly possible that there are classes of simplicial complexes that have no s-chromatic roots in certain intervals or points of $\mathbb{R}$. Furthermore, it would be interesting to find similar bounds on the real and complex roots of the s-chromatic polynomial, for $s > 1$, as Sokal, Fernández/Procacci and Dong/Koh have found for $s = 1$.

Appendix A: The s-Chromatic Polynomials of Δ^n, for $3 \leq n \leq 9$

$$
\begin{aligned}
\mathcal{P}(\Delta^3, x) &= [x]_4 \\
&= x^4 - 6x^3 + 11x^2 - 6x \\
\mathcal{P}^2(\Delta^3, x) &= [x]_4 + 6[x]_3 + 3[x]_2 \\
&= x^4 - 4x^2 + 3x \\
\mathcal{P}^3(\Delta^3, x) &= [x]_4 + 6[x]_3 + 7[x]_2 \\
&= x^4 - x.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(\Delta^4, x) &= [x]_5 \\
&= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x \\
\mathcal{P}^2(\Delta^4, x) &= [x]_5 + 10[x]_4 + 15[x]_3 \\
&= x^5 - 10x^3 + 15x^2 - 6x \\
\mathcal{P}^3(\Delta^4, x) &= [x]_5 + 10[x]_4 + 25[x]_3 + 10[x]_2 \\
&= x^5 - 5x^2 + 4x \\
\mathcal{P}^4(\Delta^4, x) &= [x]_5 + 10[x]_4 + 25[x]_3 + 15[x]_2 \\
&= x^5 - x.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(\Delta^5, x) &= [x]_6 \\
&= x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x \\
\mathcal{P}^2(\Delta^5, x) &= [x]_6 + 15[x]_5 + 45[x]_4 + 15[x]_3 \\
&= x^6 - 20x^4 + 45x^3 - 26x^2 \\
\mathcal{P}^3(\Delta^5, x) &= [x]_6 + 15[x]_5 + 65[x]_4 + 75[x]_3 + 10[x]_2 \\
&= x^6 - 15x^3 + 24x^2 - 10x \\
\mathcal{P}^4(\Delta^5, x) &= [x]_6 + 15[x]_5 + 65[x]_4 + 90[x]_3 + 25[x]_2 \\
&= x^6 - 6x^2 + 5x \\
\mathcal{P}^5(\Delta^5, x) &= [x]_6 + 15[x]_5 + 65[x]_4 + 90[x]_3 + 31[x]_2 \\
&= x^6 - x.
\end{aligned}
$$

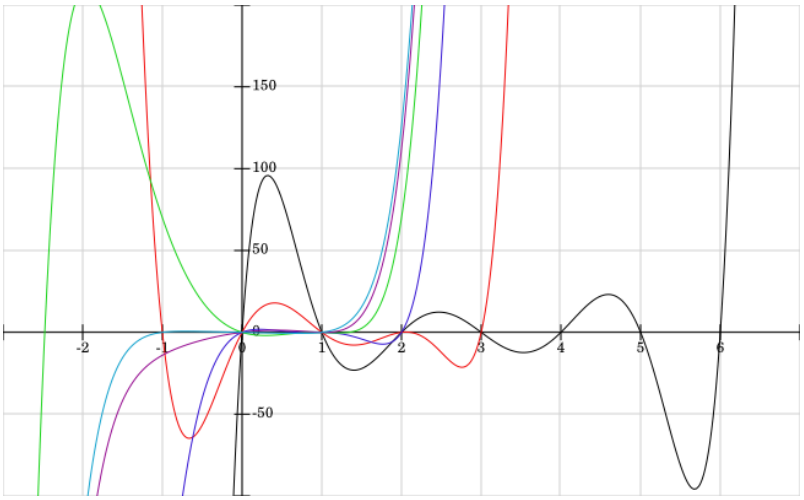

Figure 3.3: The six s-chromatic polynomials of the 6-simplex Δ^6. $s = 1$ (black), $s = 2$ (red), $s = 4$ (blue), $s = 5$ (green), $s = 6$ (purple).

$$
\begin{aligned}
\mathcal{P}(\Delta^6, x) &= [x]_7 \\
&= x^7 - 21x^6 + 175x^5 - 735x^4 + 1624x^3 - 1764x^2 + 720x \\
\mathcal{P}^2(\Delta^6, x) &= [x]_7 + 21[x]_6 + 105[x]_5 + 105[x]_4 \\
&= x^7 - 35x^5 + 105x^4 - 56x^3 - 105x^2 + 90x \\
\mathcal{P}^3(\Delta^6, x) &= [x]_7 + 21[x]_6 + 140[x]_5 + 315[x]_4 + 175[x]_3 \\
&= x^7 - 35x^4 + 84x^3 - 70x^2 + 20x \\
\mathcal{P}^4(\Delta^6, x) &= [x]_7 + 21[x]_6 + 140[x]_5 + 350[x]_4 + 280[x]_3 + 35[x]_2 \\
&= x^7 - 21x^3 + 35x^2 - 15x \\
\mathcal{P}^5(\Delta^6, x) &= [x]_7 + 21[x]_6 + 140[x]_5 + 350[x]_4 + 301[x]_3 + 56[x]_2 \\
&= x^7 - 7x^2 + 6x \\
\mathcal{P}^6(\Delta^6, x) &= [x]_7 + 21[x]_6 + 140[x]_5 + 350[x]_4 + 301[x]_3 + 63[x]_2 \\
&= x^7 - x.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(\Delta^7, x) &= [x]_8 \\
&= x^8 - 28x^7 + 322x^6 - 1960x^5 + 6769x^4 - 13132x^3 + 13068x^2 - 5040x \\
\mathcal{P}^2(\Delta^7, x) &= [x]_8 + 28[x]_7 + 210[x]_6 + 420[x]_5 + 105[x]_4 \\
&= x^8 - 56x^6 + 210x^5 - 56x^4 - 840x^3 + 1371x^2 - 630x \\
\mathcal{P}^3(\Delta^7, x) &= [x]_8 + 28[x]_7 + 266[x]_6 + 980[x]_5 + 1225[x]_4 + 280[x]_3 \\
&= x^8 - 70x^5 + 224x^4 - 280x^3 + 195x^2 - 70x
\end{aligned}
$$

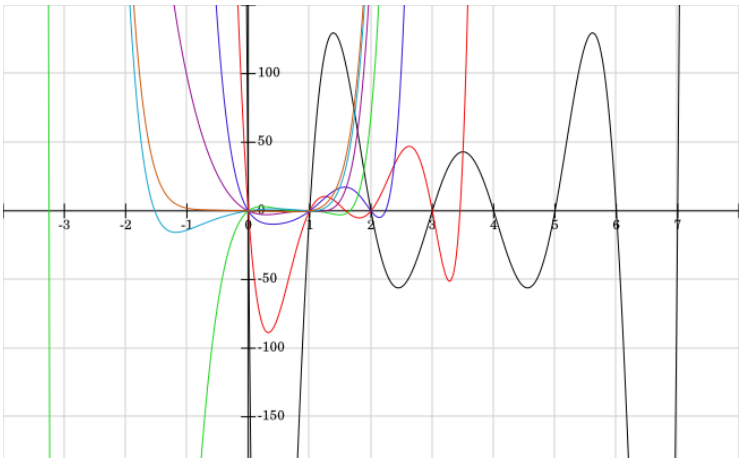

Figure 3.4: The seven s-chromatic polynomials of the 7-simplex Δ^7. $s = 1$ (black), $s = 2$ (red), $s = 4$ (blue), $s = 5$ (green), $s = 6$ (purple), $s = 7$ (turquoise).

$$
\begin{aligned}
\mathcal{P}^4(\Delta^7, x) &= [x]_8 + 28[x]_7 + 266[x]_6 + 1050[x]_5 + 1645[x]_4 + 770[x]_3 + 35[x]_2 \\
&= x^8 - 56x^4 + 140x^3 - 120x^2 + 35x \\
\mathcal{P}^5(\Delta^7, x) &= [x]_8 + 28[x]_7 + 266[x]_6 + 1050[x]_5 + 1701[x]_4 + 938[x]_3 + 91[x]_2 \\
&= x^8 - 28x^3 + 48x^2 - 21x \\
\mathcal{P}^6(\Delta^7, x) &= [x]_8 + 28[x]_7 + 266[x]_6 + 1050[x]_5 + 1701[x]_4 + 966[x]_3 + 119[x]_2 \\
&= x^8 - 8x^2 + 7x \\
\mathcal{P}^7(\Delta^7, x) &= [x]_8 + 28[x]_7 + 266[x]_6 + 1050[x]_5 + 1701[x]_4 + 966[x]_3 + 127[x]_2 \\
&= x^8 - x.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(\Delta^8, x) &= [x]_9 \\
&= x^9 - 36x^8 + 546x^7 - 4536x^6 + 22449x^5 - 67284x^4 + 118124x^3 - 109584x^2 \\
&\quad + 40320x \\
\mathcal{P}^2(\Delta^8, x) &= [x]_9 + 36[x]_8 + 378[x]_7 + 1260[x]_6 + 945[x]_5 \\
&= x^9 - 84x^7 + 378x^6 + 84x^5 - 3780x^4 + 8819x^3 - 7938x^2 + 2520x \\
\mathcal{P}^3(\Delta^8, x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2520[x]_6 + 5565[x]_5 + 3780[x]_4 + 280[x]_3 \\
&= x^9 - 126x^6 + 504x^5 - 840x^4 + 1035x^3 - 1134x^2 + 560x \\
\mathcal{P}^4(\Delta^8, x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2646[x]_6 + 6825[x]_5 + 6930[x]_4 + 1855[x]_3 \\
&= x^9 - 126x^5 + 420x^4 - 540x^3 + 315x^2 - 70x
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}^5(\Delta^8,x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2646[x]_6 + 6951[x]_5 + 7686[x]_4 + 2737[x]_3 + 126[x]_2 \\
&= x^9 - 84x^4 + 216x^3 - 189x^2 + 56x \\
\mathcal{P}^6(\Delta^8,x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2646[x]_6 + 6951[x]_5 + 7770[x]_4 + 2989[x]_3 + 210[x]_2 \\
&= x^9 - 36x^3 + 63x^2 - 28x \\
\mathcal{P}^7(\Delta^8,x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2646[x]_6 + 6951[x]_5 + 7770[x]_4 + 3025[x]_3 + 246[x]_2 \\
&= x^9 - 9x^2 + 8x \\
\mathcal{P}^8(\Delta^8,x) &= [x]_9 + 36[x]_8 + 462[x]_7 + 2646[x]_6 + 6951[x]_5 + 7770[x]_4 + 3025[x]_3 + 255[x]_2 \\
&= x^9 - x.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(\Delta^9,x) &= [x]_{10} \\
&= x^{10} - 45x^9 + 870x^8 - 9450x^7 + 63273x^6 - 269325x^5 + 723680x^4 \\
&\quad - 1172700x^3 + 1026576x^2 - 362880x \\
\mathcal{P}^2(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 630[x]_8 + 3150[x]_7 + 4725[x]_6 + 945[x]_5 \\
&= x^{10} - 120x^8 + 630x^7 + 588x^6 - 12600x^5 + 37295x^4 - 44730x^3 + 18936x^2 \\
\mathcal{P}^3(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5670[x]_7 + 19425[x]_6 + 26145[x]_5 + 9100[x]_4 \\
&= x^{10} - 210x^7 + 1008x^6 - 2100x^5 + 3975x^4 - 8190x^3 + 9716x^2 - 4200x \\
\mathcal{P}^4(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22575[x]_6 + 39795[x]_5 + 25750[x]_4 \\
&\quad + 3675[x]_3 \\
&= x^{10} - 252x^6 + 1050x^5 - 2475x^4 + 5625x^3 - 7999x^2 + 4050x \\
\mathcal{P}^5(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22827[x]_6 + 42315[x]_5 + 32050[x]_4 \\
&\quad + 7455[x]_3 + 126[x]_2 \\
&= x^{10} - 210x^5 + 45x^4 + 3105x^3 - 6865x^2 + 3924x \\
\mathcal{P}^6(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22827[x]_6 + 42525[x]_5 + 33310[x]_4 \\
&\quad + 8925[x]_3 + 336[x]_2 \\
&= x^{10} - 795x^4 + 4365x^3 - 7705x^2 + 4134x \\
\mathcal{P}^7(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22827[x]_6 + 42525[x]_5 + 34105[x]_4 \\
&\quad + 9285[x]_3 + 456[x]_2 \\
&= x^{10} - 45x^3 + 80x^2 - 36x \\
\mathcal{P}^8(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22827[x]_6 + 42525[x]_5 + 34105[x]_4 \\
&\quad + 9330[x]_3 + 501[x]_2 \\
&= x^{10} - 10x^2 + 9x \\
\mathcal{P}^9(\Delta^9,x) &= [x]_{10} + 45[x]_9 + 750[x]_8 + 5880[x]_7 + 22827[x]_6 + 42525[x]_5 + 34105[x]_4 \\
&\quad + 9330[x]_3 + 511[x]_2 \\
&= x^{10} - x.
\end{aligned}
$$

Populaire Samenvatting

Een graaf is een wiskundig object dat bestaat uit een verzameling punten en een verzameling lijnen, die sommige punten met elkaar verbinden enop die manier relaties tussen twee punten aangeven. In de Grafentheorie worden de verschillende structuren die grafen kunnen hebben bestudeerd. Een manier om dit te doen is door de punten van een graaf in te kleuren, waarbij elk tweetal punten die een lijn delen verschillende kleuren moeten hebben. Het aantal kleuren die nodig zijn om een graaf in te kleuren en het aantal van kleuringen met een bepaald aantal kleuren zegt iets over de structuur van een graaf. George David Birkhoff heeft in 1912 het *chromatisch polynoom* ingevoerd. Het chromatisch polynoom telt het aantal verschillende kleuringen van een gegeven graaf als functie van het aantal kleuren. Dat wil zeggen, dat als men het chromatisch polynoom van een graaf weet men het aantal verschillende kleuringen met een bepaald aantal kleuren gewoon af kan lezen. Het chromatisch polynoom wordt tot vandaag bestudeerd en heeft vele interessante eigenschappen.
Simpliciale complexen zijn wiskundige objecten die vornamelijk bestudeerd worden in de Topologie. Het zijn in principe hoger dimensionale grafen. Zoals grafen bestaan simpliciale complexen uit punten en lijnen, maar daarnaast hebben ze ook vlakken en hypervlakken (hoger dimensionale vlakken). Net als een lijn wordt opgespannen door twee punten, wordt een vlak in een simpliciaal complex opgespannen door drie lijnen en één 3-dimensionaal hypervlak door vier vlakken. Ook zijn simpliciale complexen iets ingewikkelder, zo hebben ze toch veel gemeen met grafen en het is daarom interessant sommige ideeën uit de Grafentheorie te gebruiken om een soortgelijke theorie voor simpliciale complexen te ontwikkelen. Men kan bij voorbeeld een kleuring voor de punten verzameling van simpliciale complexen defineren, die de hoger dimensionale structuren van een complex mee betrekken.
In dit scriptie wordt een overzicht gegeven over belangrijke resultaten uit de Grafentheorie met betrekking tot kleuringen en het chromatisch polynoom voor grafen. Daarnaast wordt een chromatisch polynoom voor simpliciale complexen gedefinieerd, het *s-chromatisch polynoom*. Bovendien worden enkele eigenschappen van het chromatisch polynoom voor grafen gegeneraliseerd tot het s-chromatisch polynoom voor simpliciale complexen.

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