FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Thesis for the Master degree in Mathematics
Department of Mathematical Sciences, University of Copenhagen

Speciale for cand.scient graden i matematik
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Labelled String Topology for Classifying Spaces of Compact Lie Groups

A 2-dimensional Homological Field Theory with D-branes

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Revised: 7 February 2012

0.1 Abstract

This master thesis is on the subject of string topology of classifying spaces. Following the article [1] by Chataur and Menichi, which is the main inspiration of the thesis, we will show that for a compact connected Lie group G, the singular homology of strings on BG, when taken with field coefficients, form a homological field theory. Chataur and Menichi treats the case of closed strings, but we will allow for open strings as well to generalize their theory. We also develop the theory to allow for D-branes, or labels on the boundaries of open strings, to further generalize the it. In our case the set of D-branes, or labels, are taken to be oriented connected subgroups of G.

The thesis consists of an introduction followed by three chapters. In the first chapter we give definitions allowing us to state the main result of the thesis, and compare this to the main result of the inspirational article. We also introduce the tools needed to perform the construction found in the following chapter. These tools are those of wrong way maps, determinant lines for free graded modules, and some results on how to decompose certain surfaces as unions of arcs.

The second chapter contains the main body of the thesis. In this we first give an exposition of the work of Chataur and Menichi, but in a slightly more general setting allowing for open strings as well as closed, described in a common setup. Next we introduce necessary terminology to produce similar results for labelled strings. This is done as an extension of the unlabelled case, which is then a special case. Finally the chapter is concluded by a construction of operations which give a homological field theory, and we verify the axioms of this. This construction works equally well in the labelled and unlabelled case, and closely mimics the one by Chataur and Menichi in the closed case. To the best of our knowledge, the construction of an open-closed theory, and that of an open-closed theory with D-branes, are novel contributions to the field.

We finish the thesis in the third chapter by computing some very simple cases of the maps on homology which we obtain by this theory. Among other things, we recover the Pontryagin product on the homology of G. Also we see that the construction does not end up producing a trivial theory, which might have been the case.

Dansk resumé

Dette speciale omhandler strengtopologi for klassificerende rum. Som i artiklen [1] af Chataur og Menichi, der er hovedinspirationen for specialet, vil vi vise at for G en kompakt sammenhængende Lie gruppe, vil den singulære homologi af strengene på BG, taget med legemeskoefficienter, udgøre en homologisk feltteori. Chataur og Menichi behandler tilfældet der omhandler lukkede strenge, mens vi også vil tillade åbne strenge som en generalisering af deres teori. Vi udvider også teorien til at tillade D-branes, eller mærkater på randen af de åbne strenge, som en yderligere generalisering. I vores tilfælde vil mængden af D-branes, eller mærkater, være de orienterede sammenhængende undergrupper af G.

Specialet består af en introduktion efterfulgt af tre kapitler. I det første kapitel vil vi give de nødvendige definitioner for at kunne opstille hovedresultatet

i specialet, og sammenligne det med hovedresultatet i artiklen vi følger. Vi vil også introducere de nødvendige værktøjer for at kunne udføre konstruktionen der følger i det efterfølgende kapitel. Disse værktøjer er umkehr afbildinger, determinantlinier for frie graduerede moduler, og nogle resultater om hvordan vi kan dekomponere bestemte flader som foreninger af stier.

Andet kapitel indeholder hoveddelen af specialet. I dette giver vi først en fremstilling af Chataur og Menichis arbejde i en let generaliseret udgave, der tillader at behandle både lukkede og åbne strenge under ét. Derefter introducerer vi den nødvendige terminologi for at kunne give tilsvarende konstruktioner for mærkede strenge. Dette bliver gjort som en udvidelse af det umærkede tilfælde som dermed bliver et specialtilfælde. Endeligt slutter vi kapitlet ved at konstruere operationer der giver anledning til en homologisk feltteori, og efterviser aksiomerne for denne. Konstruktionen virker lige godt i det umærkede og mærkede tilfælde, og efterligner den konstruktion Chataur og Menichi udfører i det lukkede tilfælde nøje. Efter vores bedste overbevisning er konstruktionen af en åben-lukket teori, og konstruktionen af en åben-lukket teori med D-branes, begge nye resultater inden for emnet.

Vi afslutter specialet i med det tredje kapitel, hvor vi udregner nogle meget simple tilfælde af de afbildinger på homologi som denne teori producerer. Blandt andet genfinder vi Pontryagin produktet på homologien af G. Vi ser også at konstruktionen ikke producerer en triviel teori, hvilket a priori kunne have været tilfældet.

0.2 Changes from original

- Original handed in 14 July 2011
- First revision finished 9 February 2012

First revision Several minor errors corrected. Content-wise the following changes are made (old page numbers)

p. vii The main theorem of the thesis has been reformulated in all three instances

p. 1 Σ^λ now correctly denotes the union of components labelled by λ

p. 10 The symmetry isomorphism now correctly has the sign $(-1)^{w_1 w_2}$ as opposed to the undefined $(-1)^{\lambda\mu}$

p. 27 Notation of final part of proof of proposition 2.2.8 is made more consistent with that of the statement (change obsolete due to next change). Proposition 2.2.8 rewritten, and simpler proof provided. Lemma 2.2.9 removed, as it is no longer needed.

p. 33 The isomorphism on homology induced from excision now goes the right way

p. 36 Rearranged and reformulated part about labelled arc decompositions

p. 46 Definition of $\det \Sigma^{\underline{d}}$ recalled in relation to proposition 2.3.16

Contents

0.3 Introduction

This master thesis is on the subject of string topology of classifying spaces. Following the article [1] by Chataur and Menichi, which is the main inspiration of the thesis, we will show that for a compact connected Lie group G, the singular homology of strings on BG, when taken with field coefficients, form a 2-dimensional homological field theory. Chataur and Menichi treats the case of closed strings, but we will allow for open strings as well to generalize their theory. We also develop the theory to allow for D-branes, or labels on the boundaries of open strings, to further generalize the theory. In our case the set of D-branes, or labels, are taken to be oriented connected subgroups of G.

We give a full definition of the terms that appear in the above paragraph, in section 1.1. For now we will just say that a homological field theory, is a functor from a certain variation of the 2-cobordism category, which assigns a vector space to each 1-manifold, and a linear map to each cobordism of such 1-manifolds. That our theory is *homological*, means that the assigned vector spaces and maps are graded, and that it is actually the homology of surfaces (or even more precisely, the homology of the classifying space of their mapping class groups) representing cobordisms which gives the maps. To say that such a theory has *D-branes*, means that we may decorate the boundary components of the 1-manifolds with elements of a labelling set, or D-branes, and we then require the cobordisms to respect this labelling.

By strings on BG, we mean the mapping space $\mathrm{Map}(I, BG)$ of maps from the interval into BG. Such maps may have the same value on both ends of the interval, in which case they define a map from the circle to BG, and we call such maps for closed strings. We can also label the ends of open strings by elements of some labelling set (the D-branes). This may be interpreted as to give certain conditions on the maps restricted to the ends of the intervals.

Circles and intervals can be taken as representatives for the objects in the open-closed cobordism category, which in this case is the symmetric monoidal category with objects generated by a circle and an interval, and with morphisms the diffeomorphism classes (orientation preserving diffeomorphisms respecting the boundary) of 2-manifolds with boundary, where some boundary is marked as incoming and identified with the source, and some is marked as outgoing and identified with the target. These classes are called cobordisms. There is a labelled version of this also, where the labelled strings represents objects and certain labelled cobordisms are the morphisms. From either of these categories (labelled or not) we create one enriched in graded vector spaces, also symmetric monoidal with the same objects, and morphisms spaces from a to b given as

$$\bigoplus_{[\Sigma_{a,b}]} H_*(B\,\mathrm{Diff}^+(\Sigma_{a,b}, \partial))$$

where the sum ranges over classes of (labelled) cobordisms from a to b. Then a homological field theory (with or without D-branes) is a functor from this category to that of graded vector spaces. This is described in more detail in the section 1.1.

We will show three statements about strings on BG, in increasing generality. The closed strings on BG form a closed homological field theory. This is shown by David Chataur and Luc Menichi in [1]. The open and closed strings on BG

form an open-closed homological field theory extending the closed theory. This is not previously described in literature, to the best of our knowledge. Finally the labelled open and closed strings on BG form an open-closed homological field theory with D-branes, extending the open-closed theory by choosing vacuous boundary conditions for the D-branes. Also this is a novel contribution as far as we know. The most general form of our main result is then the following, which we explain in section 1.2.

Theorem 1.2.3. *Let G be a connected compact Lie group, and $\mathbb{F}$ a field. The singular homology of the labelled strings on BG with coefficients in $\mathbb{F}$, defines a labelled open-closed HFT, i.e. a symmetric monoidal functor*

$$H_*(\mathcal{M}(-);\mathbb{F})\colon \mathcal{H}_G^d \longrightarrow gd - Vect_{\mathbb{F}}$$

In the literature there are results related to this. First we must mention the main article of inspiration to this thesis namely the already referenced [1] by Chataur and Menichi. As mentioned above our results here are extensions of their result on the closed strings, and they have several refinements and corollaries to it, as well as developing a theory which can also handle the case of finite discrete groups.

In a somewhat different setting there are at least the following contributions (we do not claim this to be a full list). Classically string topology is concerned with maps of strings into a manifold M, and was initiated by Moira Chas and Dennis Sullivan, who showed that the homology of free loops (or closed strings, in our language) on a manifold is a Batalin-Vilkovisky algebra [2]. Following this Véronique Godin has shown [3] that there is an open-closed homological field theory whose values are the homology of the strings on M. Andrew J. Blumberg, Ralph L. Cohen, and Constantin Teleman has an article [4], surveying developments in the field, and announcing recent results. They conjecture that the theory of Godin can be extended to one with D-branes, where the set of D-branes is the set of submanifolds of M. One result they announce, is that they have constructed a certain differential graded string category, whose objects are the submanifolds of M. They again conjecture that this can be used to build a open-closed chain-level field theory, which in a sense is a refinement of a homological theory, and that this recovers the homological field theory with D-branes, which they conjecture from the work of Godin.

In [5] Kevin Costello constructs an open-closed chain-level field theory associated to a specific type of category, whose objects becomes the D-branes for the theory. Blumberg, Cohen and Teleman computes that if the differential graded string category is of the required type, the theory obtained in this way, does indeed give the expected value on closed strings, which is evidence for their conjecture.

We have taken some inspiration in the definitions found in [5] by Costello. In this we find a concise way of dealing with labelling, and the general categorical setup, some of which is credited to Graeme Segal [6] by Costello.

Kate Gruher and Craig Westerland [7] show some results about the closed strings in BG for a compact connected Lie group G, i.e. in the same setup as our work here. Their work relates mostly to the parts of the inspirational article [1] which we do not consider in this thesis.

The structure of the thesis is the following. After this introduction the thesis consists of three chapters. In the first chapter we give definitions allowing us to state the main result of the thesis, and compare this to the main result of the inspirational article. We also introduce the tools needed to perform the construction found in the following chapter. These tools are those of wrong way maps, determinant lines for graded vector spaces, and some results on how to decompose certain surfaces as unions of arcs.

The second chapter contains the main body of the thesis. In this we first give an exposition of the work of Chataur and Menichi, but in a slightly more general setting allowing for open strings as well as closed, described in a common setup. Next we introduce necessary terminology to produce similar results for labelled strings. This is done as an extension of the unlabelled case, which is then a special case. Finally the chapter is concluded by a construction of operations which give a homological field theory, and we verify the axioms of this. This construction works equally well in the labelled and unlabelled case, and closely mimics the one by Chataur and Menichi in the closed case. To the best of our knowledge, the construction of an open-closed theory, and that of an open-closed theory with D-branes, are novel contributions to the field.

We finish the thesis by computing some very simple cases of the maps on homology which we obtain by this theory. Among other things, we recover the Pontryagin product on the homology of G as a very special case. This gives hopes that other interesting structures may be recovered. Also we see that the construction does not end up producing a trivial theory, which might have been the case.

This thesis is meant to be read by mathematicians well-versed in algebraic topology, along with the usual tools encountered in this field of study. The thesis is mostly self-contained from a fairly basic level, with the exception of some tools introduced in section 1.3, and a proposition on automorphisms of free groups 2.2.5.

The original ideas in the thesis are mostly due to R. Hepworth (and the people he would like to share this credit with), who has had the foresight to suggest which paths to pursue and which to abandon, and with whom the content has been discussed extensively. The author would like to thank Hepworth for his helpful advice on both content and form of the thesis, his friendly encouragement during the writing, and his patience with the constant challenging of said ideas. This has put the author in a great position to make everything work.

Chapter 1

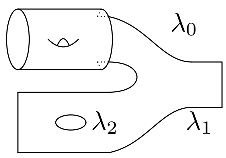
Preliminaries

This first chapter is devoted to set up our definitions, compare results, and introduce the tools needed for later. We will allow ourselves to make some claims in this chapter which will remain claims throughout the thesis, but we will note when this is the case.

1.1 Definitions

In this section we will define what we mean by a homological field theory. We have chosen to present the most general version we will be using at once, and then restrict to simpler cases as needed. Readers familiar with the concept should not skip this section, as we will make small modifications on the definition presented by Costello [5]. In the following, *manifold* will always mean a smooth orientable compact 2-manifold with boundary, unless otherwise specified.

Let Λ be a set of D-branes, or labels. This can be any set, of which we simply call the elements D-branes, or labels. A manifold with labelled open-closed boundary is a manifold with parametrized boundary, where some of the boundary components are marked as closed incoming or outgoing (C_+ and C_- respectively), and with some intervals embedded in the remaining part of the boundary, and marked as open incoming or outgoing boundary (O_+ and O_- respectively). The part of the boundary not marked as incoming or outgoing, is called free boundary, and each component of this is labelled by a D-brane. Further each end of each component of the open boundary (incoming or outgoing), is labelled according to the adjacent free boundary component. If we include the end points as part of the free boundary also, this is itself a labelled 1-manifold with boundary, with some of the boundary marked as incoming and some as outgoing. If Σ is a manifold with labelled open-closed boundary, then we denote by Σ^λ the union of all components of the free boundary labelled by $\lambda \in \Lambda$. Here is an example of a manifold with labelled open-closed boundary.



The circle and interval to the left are marked as incoming boundary, and the interval to the right is marked as outgoing. The remaining components of the boundary is labelled by respectively λ_0, λ_1 and λ_2.

Now fix a set of D-branes Λ, and a vector of integers $\underline{d}$ consisting of d and d_λ, one for each $\lambda \in \Lambda$. We can think of these numbers as an ambient degree d, and a brane degree d_λ for each brane. The main category of interest in this thesis is a symmetric monoidal category $\mathcal{H}_\Lambda^{\underline{d}}$ which we now define, through a series of other categories which we define in turn.

First define a category $\mathcal{C}_\Lambda$ with objects being tuples (O, C, s, t) of finite sets O of intervals, and C of circles, and maps $s, t \colon O \to \Lambda$. The morphisms are diffeomorphism classes of manifolds with labelled open-closed boundaries, where the diffeomorphisms are required to fix the entire boundary pointwise and preserve orientation. A morphism Σ with source (O, C, s, t) and target (O', C', s', t'), is such a class with open and closed incoming boundaries identified with O and C, outgoing open and closed boundaries identified with O' and C', and free boundary labelled by D-branes matching the assignment of both s, t and s', t', in the sense that D-branes assigned by these maps must agree with D-branes at the ends of each component of the open boundary, given by above identification. The set of morphisms from a to b in $\mathcal{C}_\Lambda$ is then

$$\operatorname{mor}_{\mathcal{C}_\Lambda}(a, b) = \coprod_{[\Sigma_{a,b}]} \Sigma_{a,b}$$

where the union ranges over all diffeomorphism classes of manifolds with labelled open-closed boundary, of which the part marked as incoming is identified with a and the part marked as outgoing is identified with b, and $\Sigma_{a,b}$ is a single representative of the class $[\Sigma_{a,b}]$.

We call such a morphism for a labelled open-closed cobordism. Accordingly, we will say open-closed cobordism in case there are not labels present, and closed cobordism for the classical case where all objects only consist of circles and all boundary of the morphisms are marked as either incoming or outgoing. Also note that with a similar definition the free boundary will become labelled 1-cobordisms between the boundaries of the incoming boundaries and those of the outgoing boundaries.

An object will determine a labelled 1-manifold with parametrized boundary, which is a disjoint union of $|C|$ circles and $|O|$ intervals with the boundary labelled according to the assignments of s and t. Thus we will use the terminology of 1-manifolds to describe objects in $\mathcal{C}_\Lambda$, by which we refer to the incoming boundaries of manifolds representing a morphisms from these objects, in obvious analogue with the properties of the objects themselves. E.g. we call an object (O, C, s, t) closed if $O = \emptyset$, and connected if $|O + C| \leq 1$. If we perform any topological constructions with an object, we mean the 1-manifold determined by it as above.

Composition in this category is given by gluing manifolds, such that $\Sigma_2 \circ \Sigma_1$ is the class obtained by gluing the incoming boundary of a representative of Σ_2 along the outgoing boundary of a representative of Σ_1. Clearly different choices of representatives will lead to diffeomorphic gluings. Two morphisms are composable if and only if the entire incoming boundary can be glued to the entire outgoing boundary, such that D-branes are matched at the ends of all the components of the open boundaries. Disjoint union of manifolds, and consequently on classes of manifolds, makes $\mathcal{C}_\Lambda$ into a symmetric monoidal category.

Next we define the category $\mathcal{E}_\Lambda$ which is similar to $\mathcal{C}_\Lambda$, but enriched in the homotopy category of topological spaces, with

$$\mathrm{mor}_{\mathcal{E}_\Lambda}(a,b) = \coprod_{[\Sigma_{a,b}]} B\,\mathrm{Diff}^+(\Sigma_{a,b},\partial)$$

where $\Sigma_{a,b}$ is a cobordism representing a morphism from a to b in $\mathcal{C}_\Lambda$ and the disjoint union ranges over the classes of all such as before. The space $B\,\mathrm{Diff}^+(\Sigma_{a,b},\partial)$ is the classifying space for the (topological) group of diffeomorphisms of $\Sigma_{a,b}$ which fix the entire boundary. For a pair of composable morphisms in $\mathcal{C}_\Lambda$, represented by Σ_1 and Σ_2, there is an obvious map

$$\mathrm{Diff}^+(\Sigma_2,\partial)\times \mathrm{Diff}^+(\Sigma_1,\partial) \xrightarrow{gl} \mathrm{Diff}^+(\Sigma_2\circ\Sigma_1,\partial)$$

which induces the composition map $B(gl)$ in $\mathcal{E}_\Lambda$. This is again a symmetric monoidal category by disjoint union.

We can now consider the category $\mathcal{H}_\Lambda$, still with the same objects as $\mathcal{C}_\Lambda$, and for the morphisms we apply singular homology to each of the morphism spaces of $\mathcal{E}_\Lambda$, i.e.

$$\mathrm{mor}_{\mathcal{H}_\Lambda}(a,b) = H_*(\mathrm{mor}_{\mathcal{E}_\Lambda}(a,b)) = \bigoplus_{[\Sigma_{a,b}]} H_*(B\,\mathrm{Diff}^+(\Sigma_{a,b},\partial);\mathbb{F}).$$

Note that when we apply homology or cohomology, it will always be singular homology with coefficients in a fixed field $\mathbb{F}$ unless explicitly stated otherwise in the notation. This $\mathcal{H}_\Lambda$ is then a category enriched in graded $\mathbb{F}$-vector spaces, with the composition map being the map induced on homology by $B(gl)$, and it is a symmetric monoidal category by tensor product of graded vector spaces.

Now we can finally define $\mathcal{H}_\Lambda^{\underline{d}}$. Again this has the same objects, but now with the homology defining the graded vector spaces of morphism twisted by a certain local system which we will denote by $\det\Sigma^{\underline{d}}$. Thus the morphism spaces are

$$\mathrm{mor}_{\mathcal{H}_\Lambda^{\underline{d}}}(a,b) = \bigoplus_{[\Sigma_{a,b}]} H_*(B\,\mathrm{Diff}^+(\Sigma_{a,b},\partial);\det\Sigma^{\underline{d}}).$$

This local system $\det\Sigma^{\underline{d}}$ is defined as follows

$$\det\Sigma^{\underline{d}} := \det H^*(\Sigma,\partial_{in})^{\otimes d}\otimes\bigotimes_{\lambda\in\Lambda}\det H^*(\Sigma^\lambda,\partial_{in}^\lambda)^{\otimes d_\lambda - d}.$$

For a graded vector space V^*, the determinant $\det V^*$ is the weighted line given as the (signed) tensor product of the top exterior powers of V^n with $n\in\mathbb{Z}$. This is of weight the alternating sum of dimensions of V^n. This is introduced in more detail in section 1.3.2. The big tensor product in the expression ranges over all possible labels, seeing as only those actually on Σ contributes to the tensor product, as we by ∂_{in}^λ mean the incoming boundary of Σ^λ, both of which are potentially empty. General properties about determinant lines ensures that $\mathcal{H}_\Lambda^{\underline{d}}$ is indeed a symmetric monoidal category. We will explain how in section 1.3.2 below. The necessary checks to do for this to hold are done in for example [8]. We shall see later why this expression comes into play.

The symmetric monoidal subcategory generated by closed objects and closed morphism (i.e. coming from closed cobordisms), is denoted by $c\mathcal{H}_\Lambda^d$. In this case the expression $\det \Sigma^{\underline{d}}$ reduces to

$$\det H^*(\Sigma, \partial_{in})^{\otimes d}.$$

because there is no free boundary.

When we talk about monoidal functors in the following, they will always be split. We say that $F\colon (\mathfrak{C}, \otimes) \to (\mathfrak{D}, \otimes)$ is split, if the morphisms

$$F(a) \otimes F(b) \to F(a \otimes b)$$

are all isomorphisms. We are now ready to define.

Definition 1.1.1. *An open-closed homological field theory (ocHFT) with D-branes Λ and degree vector $\underline{d}$, is a symmetric monoidal functor from $\mathcal{H}_\Lambda^{\underline{d}}$ to the category of graded vector spaces.*

Restricting the above to $c\mathcal{H}_\Lambda^d$, is then a definition of a closed homological field theory (cHFT). It is worth noting that with $\underline{d}$ (or just d in the closed case) equal to zero we obtain a field theory with a constant local system equal to the field we are working with.

In this thesis we will only consider cobordisms with at least one incoming and one outgoing boundary component in each connected component, and we will say that such cobordisms have positive boundary. This terminology is adopted from [3] where it means that there is both incoming and non-incoming boundary in every connnected component of a cobordism, so there is a notable difference here, as our requirement is stronger. This restriction is in line with Chataur and Menichi [1].

However in [1] there is a different wording of the definition of homological field theories, and we shall briefly translate that to the one here present. The defintion there is the following

Definition 1.1.2. *A graded vector space V is a closed homological field theory if it is an algebra over the graded linear prop*

$$\bigoplus_{F_{p+q}} H_*(B\,\mathrm{Diff}^+(F_{p+q}, \partial))$$

where the direct sum is taken over representatives of diffeomorphism classes of manifolds with p incoming and q outgoing boundary components, and over all p, q.

We may write $p = \sum_{i\in I} p_i$ and $q = \sum_{i\in I} q_i$, where I indexes the path components of F_{p+q}, and p_i, q_i denotes the number of incoming and outgoing boundary components respectively in the i'th path component. If we require $p_i > 0$ or $q_i > 0$ for all $i \in I$, the theory is called non-unital or non-counital respectively. In particular the work in this thesis will be on a non-unital non-counital theory as both $p_i > 0$ and $q_i > 0$ with our meaning of positive boundary.

That V is an algebra over this prop amounts to the existence of a morphism of graded linear props

$$\bigoplus_{F_{p+q}} H_*(B\,\mathrm{Diff}^+(F_{p+q}, \partial)) \to \mathcal{E}nd_V$$

where $\mathcal{E}nd_V$ is the endomorphism prop of V. Such a morphism is a symmetric monoidal functor (properties listed in [1] p. 9-10) to the symmetric monoidal category $\mathcal{E}nd_V$. To obtain this definition from definition 1.1.1, we should restrict to $c\mathcal{H}_\Lambda^d$ and choose a functor to graded vector spaces, with value V on the closed connected object corresponding to S^1. Then we note that this factors through the symmetric monoidal functor which includes $\mathcal{E}nd_V$ in $gd - Vect_{\mathbb{F}}$.

By this definition, Chataur and Menichi does not quite produce a cHFT because the construction reveals that they too need a local system of coefficients for their homology. In the appendix of [1] they then correct the statement to say that $H_*(\mathcal{L}X)$ is an algebra over the prop

$$\bigoplus_{F_{p+q}} H_*(B\,\mathrm{Diff}^+(F_{p+q}, \partial)); \det H_1(F_{p+q}, \partial_{in})^{\otimes d})$$

where d is the top degree for the homology $H_*(\Omega X)$. The space X is a simply connected space such that $H_*(\Omega X)$ is finite dimensional, and in our setup we have $X = BG$.

1.2 Statement of the main result

The main result by Chataur and Menichi [1] is the following

Theorem 1.2.1. *Let G be a finite group or, let G be a connected topological group such that its singular homology $H_*(G;\mathbb{F})$ with coefficients in a field is finite dimensional, and let d be the top degree of the homology. Then the singular homology of $\mathcal{L}BG$ taken with coefficients in a field, $H_*(\mathcal{L}BG;\mathbb{F})$ is a non-unital non-counital closed homological field theory of degree d.*

This is shown by constructing operations

$$\mu(F_{g,p+q})\colon H_*(B\,\mathrm{Diff}^+(F_{g,p+q}, \partial); \det H_1(F_{g,p+q}, \partial_{in})^{\otimes d}) \otimes H_*(\mathcal{L}BG)^{\otimes p} \to H_{*+d}(\mathcal{L}BG)^{\otimes q}$$

associated to manifolds F with p incoming boundary components, q outgoing boundary components $p_i, q_i > 0$ for all i, and genus g. They then verify that these operations are propic. Note that specifying the genus and the number of incoming and outgoing boundary components, in this closed case, is the same as specifying a diffeomorphism class.

Definition 1.2.2. *Let G be a compact connected Lie group. Define the set of D-branes G to be the set of oriented connected subgroups of the Lie group G. In this case we set $\underline{d}$ to be the tuple consisting of the numbers $d = -\dim G$ and $d_H = -\dim H$ for each oriented connected subgroup $H \leq G$.*

Note that the connected subgroups of G are all trivially orientable, but the emphasis here is on the fact that orientations are part of the data, such that there are two copies of each connected subgroup.

Now equivalent to constructing the above operations, we can say that Chataur and Menichi construct a symmetric monoidal functor

$$c\mathcal{H}_G^d \to gd - Vect_{\mathbb{F}}.$$

In this thesis, we will construct a functor on the category $\mathcal{H}_G^d$ by considering singular homology of labelled strings in BG, taken with field coefficients. Here labelled strings in BG are spaces $\mathcal{M}(\Sigma)$ as in definition 2.1.2, with Σ representing an object in $\mathcal{H}_G^d$. The main theorem of the thesis is

Theorem 1.2.3. *Let G be a connected compact Lie group, and $\mathbb{F}$ a field. The singular homology of the labelled strings on BG with coefficients in $\mathbb{F}$, defines a labelled open-closed HFT, i.e. a symmetric monoidal functor*

$$H_*(\mathcal{M}(-);\mathbb{F})\colon \mathcal{H}_G^d \longrightarrow gd - Vect_{\mathbb{F}}$$

We note that $H_*(\mathcal{M}(S^1)) = H_*(\mathrm{Map}(S^1, BG))$, and thus we recover and extend a special case of the result of [1], namely when G happens to be a compact Lie group. In the article [1] there are various refinements and corollaries of the theorem corresponding to the one above, but we will not go into these in this thesis.

1.3 Tools for construction

In this section we will review a couple of concepts which we will need as tools for our constructions in chapter 2. The first two subsections 1.3.1 and 1.3.2 are purely expository, whereas the last 1.3.3 is a detailed treatment of how to decompose surfaces representing various cobordisms, which we will need for later.

1.3.1 Wrong way maps

As described in [1], there are several versions of wrong way maps, or umkehr maps. We will only be using the following type of those. Let $\mathbb{F}$ be a field, and let G be a space such that the homology $H_*(G;\mathbb{F})$ is concentrated in degree less than n and $H_n(G;\mathbb{F})$ has rank 1. We say that such a space G has *rank 1 top homology*, if there is some such n.

A fibration $G \longrightarrow E \overset{p}{\twoheadrightarrow} B$ over a path connected base B, such that the action of $\pi_1(B)$ on $H_n(G;\mathbb{F})$ is trivial, we call an orientable fibration, and we say that it is oriented if we have made a choice of a $\mathbb{F}$-orientation class $\omega \in H_n(G;\mathbb{F})$. Following [1], an $\mathbb{F}$-orientation class is simply a generator of $H_n(G;\mathbb{F})$.

Now consider the associated homological Serre spectral sequence for such an oriented fibration. On the E^∞ page, we get a filtration of $H_{l+n}(E;\mathbb{F})$ of the form

$$0 = F^0 = \cdots = F^{l-1} \subset F^l \subset \cdots \subset F^{l+n} = H_{l+n}(E;\mathbb{F}).$$

The orientation class ω gives an isomorphism $\mathbb{F} \simeq H_n(F;\mathbb{F})$, and thus induces the first map in the composition

$$H_l(B;\mathbb{F}) \xrightarrow{\omega_*} H_l(B;H_n(F;\mathbb{F})) \simeq E^2_{l,n} \twoheadrightarrow E^\infty_{l,n} \hookrightarrow H_{l+n}(E;\mathbb{F}) .$$

We get a map from E^2 to E^∞ in this case, because there can be no differentials leaving $E^i_{l,n}$ for any value of i, as all entries above is zero by assumption. The last map is an inclusion due to the fact that all the quotients in the filtration

F is zero in degree less than l, so that $E^\infty_{l,n} = F^l$. This composite map is denoted $p_!$, and is called a wrong way map, as it is reverse to the direction of the ordinary induced map on homology $p_*\colon H_*(E;\mathbb{F}) \to H_*(B;\mathbb{F})$. Note that on the top homology of the base space, this map is given by tensoring with the orientation class.

We will now discuss some of the properties satisfied by these wrong way maps. These properties are all listed in [1], and we will elaborate a bit on them here. First, wrong way maps satisfy the following naturality property. Suppose we have a commutative diagram with a fibration p_1 and an oriented fibration p_2

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "h"'] & B_2 \end{tikzcd}$$

such that the induced map between fibres $f\colon F_1 \to F_2$ is an isomorphism on homology. Then with a choice of orientation class $\omega_2 \in H_n(F_2)$ we get an orientation class $\omega_1 = f_*^{-1}(\omega_2) \in H_n(F_1)$, and by comparing spectral sequences, a commutative diagram

$$\begin{tikzcd} H_{*+n}(E_1) \arrow[r, "g_*"] & H_{*+n}(E_2) \\ H_*(B_1) \arrow[r, "h_*"'] \arrow[u, "(p_1)_!"'] & H_*(B_2) \arrow[u, "(p_2)_!"] \end{tikzcd}$$

In particular this is the case if both h and g are homotopy equivalences, or if the square is a pullback. Note that p_1 will be an oriented fibration, since $H_n(F_2)$ is a $\pi_1(B_1)$-module by the map $\pi_1(h)$, and as such isomorphic to $H_n(F_1)$. Now $\pi_1(B_2)$ acts trivially on $H_n(F_2)$, and hence $\pi_1(B_1)$ does so, on both this and on $H_n(F_1)$.

Secondly, suppose we have two oriented fibrations

$$F \longrightarrow E \xrightarrow{p} B,$$

$$F' \longrightarrow E' \xrightarrow{p'} B'$$

with respective orientation classes $\omega \in H_n(F)$ and $\omega' \in H_{n'}(F')$. Then we can consider the product $F \times F' \longrightarrow E \times E' \xrightarrow{p \times p'} B \times B'$, which is also an oriented fibration, and has orientation class $\omega \times \omega' \in H_{n+n'}(F \times F')$. For $b \in H_*(B)$ and $b' \in H_*(B')$ the following relation holds

$$(p \times p')_!(b \times b') = (-1)^{|b|n'} p_!(b) \otimes p'_!(b'). \tag{1.1}$$

By the Künneth isomorphism we may replace the products by tensor products on the left of this. This rule can be shown by comparing the spectral sequence associated to the product $p \times p'$ with the tensor product of the spectral sequences associated to p and p' respectively.

Finally, we have

Proposition 1.3.1. *Let $F \longrightarrow E \longrightarrow B$ be an orientable fibration of spaces such that B and F has rank 1 top homology, and let m, n be the respective top degrees. Then there is a natural isomorphism*

$$H_n(B) \otimes H_m(F) \simeq H_{m+n}(E).$$

If the fibration happens to be a product fibration such that $E = B \times F$, then this isomorphism is the cross product map.

The first part is shown by a simple spectral sequence argument, using that $E^2_{n,m}$ is the top right non-zero entry on the E^2 page, and thus isomorphic to $E^\infty_{n,m}$. This gives us an isomorphism

$$H_n(B) \otimes H_m(F) \simeq E^2_{n,m} \xrightarrow{\simeq} E^\infty_{n,m} \simeq H_{n+m}(E).$$

For the second part, note that there are two associated fibrations $* \longrightarrow B \longrightarrow B$ and $F \longrightarrow F \longrightarrow *$, and all three has associated Serre spectral sequences, respectively $E^*_{i,j}$, $B^*_{i,j}$ and $F^*_{i,j}$. There is a cross product map

$$B^r_{i,j} \otimes F^r_{i',j'} \to E^r_{i+i',j+j'}$$

which is an isomorphism on the E^2-pages. We can then show the claim by inspecting the diagram

$$\begin{tikzcd}
H_n(B) \otimes H_0(*) \arrow[r, "\simeq"] & B^2_{n,0} \arrow[r, "\simeq"] & B^\infty_{n,0} \arrow[r, "\simeq"] & H_n(B) \\
\otimes & \otimes & \otimes & \otimes \\
H_0(*) \otimes H_m(F) \arrow[r, "\simeq"] \arrow[d, "\times\otimes\times"] & F^2_{0,m} \arrow[r, "\simeq"] \arrow[d, "\times"] & F^\infty_{0,m} \arrow[r, "\simeq"] \arrow[d, "\times"] & H_m(F) \arrow[d, "\times"] \\
H_n(B) \otimes H_m(F) \arrow[r, "\simeq"] & E^2_{n,m} \arrow[r, "\simeq"] & E^\infty_{n,m} \arrow[r, "\simeq"] & H_{n+m}(E)
\end{tikzcd}$$

since the vertical maps are the cross product maps.

These observations can be used to show that wrong way maps can be composed in the sense that if f and g are oriented fibrations with orientation classes $\omega_f \in H_m(F_f)$ and $\omega_g \in H_n(F_g)$, then the fibres form an oriented fibration sequence which is pulled back from the oriented fibration f

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, equal] & F_{g\circ f} \arrow[r, two heads, "f'"] \arrow[d] & F_g \arrow[d] \\
F_f \arrow[r] & X \arrow[r, two heads, "f"] \arrow[d, two heads, "g\circ f"'] & Y \arrow[d, two heads, "g"'] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd} \tag{1.2}$$

By the proposition above the top homology of $F_{g\circ f}$ is of rank 1. Further if the composition $g \circ f$ happens to have trivial $\pi_1(Z)$-action on $H_*(F_{g\circ f})$, this is also an oriented fibration, and the orientation class is determined as follows. The pullback of f along the inclusion of the fibre of g is an oriented fibration, denoted by $f'\colon F_{g\circ f} \to F_g$, and the orientation class of f' is again ω_f. The orientation class of $g \circ f$ is then $\omega_{g\circ f} = f'_!(\omega_g) = \omega_g \otimes \omega_f$, and the diagram

$$\begin{tikzcd}
H_{*+m+n}(X;\mathbb{F}) & H_{*+n}(Y;H_m(F_f)\otimes\mathbb{F}) \arrow[l] \\
H_*(Z;H_{m+n}(F_{g\circ f})) \arrow[u] & H_*(Z;H_m(F_f)\otimes H_n(F_g)) \arrow[l, "\simeq"'] \arrow[u] \\
H_*(Z;\mathbb{F}) \arrow[u, "(\omega_f\otimes\omega_g)_*"] \arrow[r, "(\omega_f)_*"'] & H_*(Z;H_m(F_f;\mathbb{F})) \arrow[u, "(\omega_g)_*"']
\end{tikzcd}$$

commutes. The left column is $(g \circ f)_!$, and the right is $g_!$ with coefficients, which is defined just as the one above, but with the Serre spectral sequence associated to g, with coefficients in $H_m(F_f)$. The top map together with the spare orientation choice from the bottom, is $f_!$, and the commutativity expresses that $(g \circ f)_! = f_! \circ g_!$.

We will also say that $p_!\colon H_l(B;H_n(F;\mathbb{F})) \to H_{l+n}(E;\mathbb{F})$, or by the Künneth isomorphism $p_!\colon H_l(B) \otimes H_n(F;\mathbb{F}) \to H_{l+n}(E;\mathbb{F})$ is a wrong way map, where we do not make a choice of orientation class. In fact this will be the case most of the time, as developing a theory with no arbitrary choices removes the worry of making these choices consistently when wanting to compose such maps. With this convention, this last claim about composition, which is then that the top square above commutes, can be shown from the commutativity of the diagram

$$\begin{tikzcd}
H_{*+m+n}(X;\mathbb{F}) & H_{*+n}(Y;H_m(F_f)\otimes\mathbb{F}) \arrow[l, "f_!"'] \\
E^\infty \arrow[u] & E^\infty \arrow[l, dotted] \arrow[u] \\
H_*(Z)\otimes H_{m+n}(F_{g\circ f}) \arrow[u] & H_*(Z)\otimes H_m(F_f)\otimes H_n(F_g) \arrow[l, "id\otimes f'_!"] \arrow[u]
\end{tikzcd}$$

where the left column is $(g \circ f)_!$, and the right is $g_!$. This essentially amounts to showing that $f_!$ is in fact a map of spectral sequences [9].

1.3.2 Determinants

Here we will recall the concept of determinants for free R-modules. The section is primarily based on [8], where the material is treated in a more general setup. For now, we let R be a field, or $\mathbb{Z}$, and set the convention that in this section, cohomology is with coefficients in R.

Definition 1.3.2. *A weighted R-line is a pair (L, w) of a free R-module L of rank 1, and an integer w.*

We may regard weighted lines as a special case of graded modules, and so an isomorphism of weighted lines is an isomorphism of rank 1 R-modules of the same weight. Define the tensor product of two weighted lines (L_1, w_1) and

(L_2, w_2) to be the weighted line $(L_1, w_1)\hat{\otimes}(L_2, w_2) := (L_1 \otimes L_2, w_1 + w_2)$. There is an isomorphism

$$(L_1, w_1)\hat{\otimes}(L_2, w_2) \xrightarrow{\simeq} (L_2, w_2)\hat{\otimes}(L_1, w_1)$$

given by $l \otimes m = (-1)^{w_1 w_2} m \otimes l$.

The inverse of a weighted line (L, w) is defined to be the linear dual, in the negative weight $(L^*, -w)$. We see that

$$(L, w)\hat{\otimes}(L^*, -w) = (L \otimes L^*, 0) \simeq (\mathbb{F}, 0)$$

by a canonical isomorphism. Let K be free R-module of rank 1. We may consider K as a line of weight 0, or any other weight $z \in \mathbb{Z}$, in which case we will write $K[z]$.

Definition 1.3.3. *Let V be a free R-module of finite rank d, and let $W_* = \bigoplus_{n\in\mathbb{Z}} W_n$ be a finitely generated free $\mathbb{Z}$-graded R-module.*

(i) The determinant line of V is the weighted line

$$\det(V) = (\Lambda^d V, d)$$

i.e. the top exterior power of V with weight d.

(ii) The determinant line of W_* is the weighted line

$$\det(W_*) = \hat{\bigotimes}_{n\in\mathbb{Z}} \det(W_{-n})^{(-1)^n}.$$

Often these are written without the weight, just as we will just write $\otimes$ for $\hat{\otimes}$ from now. Note that det is functorial as the top exterior power is, and for a linear map $A\colon V \to V$, the determinant $\det A$ is just multiplication with the usual determinant of the matrix associated to A.

Determinants have the following property

Proposition 1.3.4. *Let A, B, C be free R-modules. There is an isomorphism*

$$\det A \otimes \det C \simeq \det(A \oplus C).$$

Any short exact sequence

$$0 \to A \to B \to C \to 0$$

is split, and there is an isomorphism

$$\det A \otimes \det C \simeq \det B$$

given by a choice of a splitting. This is natural with respect to isomorphisms of short exact sequences.

We will not show this here, but note that the first statement is basic, and the second follows from this. The proof relies on the fact that any two splittings will differ by a map which vanishes when we apply det, and the resulting isomorphism is given as follows. Let $a_1, \ldots, a_n$ be a basis for A, and let $c_1, \ldots, c_m$

be a basis for c. Denote by α_i the image of a_i in B, and by γ_i a lift of c_i to B. Then the isomorphisms maps

$$\alpha_1 \wedge \ldots \wedge \alpha_n \otimes \gamma_1 \wedge \ldots \wedge \gamma_m \mapsto \alpha_1 \wedge \ldots \wedge \alpha_n \wedge \gamma_1 \wedge \ldots \wedge \gamma_m.$$

Further, the determinants satisfies two compatibility properties relating to this isomorphism. First let V, W be free R-modules with basis respectively $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. Then $(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$ is a basis for $V \oplus W$, and we denote by $\tau \colon V \oplus W \to W \oplus V$ the map which for each basis vector transpose the entries, e.g. $(v_i, 0) \mapsto (0, v_i)$. Then the square

$$\begin{tikzcd} \det V \otimes \det W \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \det W \otimes \det V \arrow[d, "\simeq"] \\ \det(V \oplus W) \arrow[r, "\det \tau"] & \det(W \oplus V) \end{tikzcd} \tag{1.3}$$

commutes. The vertical isomorphisms are those of proposition 1.3.4, the top is that described in the beginning of the section.

Secondly, given a diagram of free R-modules, where both vertical and horizontal sequences are short exact

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[d] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

there are isomorphisms

$$\begin{tikzcd} & \det D & \\ \det A \otimes \det E \arrow[ur, "\simeq"] & & \det B \otimes \det F \arrow[ul, "\simeq"'] \\ \det A \otimes (\det C \otimes \det F) \arrow[u, "\simeq"] \arrow[rr, dash, "\simeq"] & & (\det A \otimes \det C) \otimes \det F \arrow[u, "\simeq"'] \end{tikzcd} \tag{1.4}$$

such that this diagram commutes. All but the bottom horizontal map are instances of proposition 1.3.4, and the bottom one itself is just given by rearranging brackets.

Proposition 1.3.4 can be generalized to free graded R-modules, if we add the assumption that these are *bounded graded*, meaning that for a free graded R-module A_* there is a bound $n > 0$, such that $A_i = 0$ for $i < -n$ and for $i > n$.

Proposition 1.3.5. *Let A_*, B_* and C_* be free bounded graded R-modules. For a long exact sequence*

$$\cdots \to A_n \to B_n \to C_n \to A_{n-1} \to \cdots$$

There is an isomorphism

$$\det A_* \otimes \det C_* \simeq \det B_*$$

This isomorphism satisfies the same compatibility properties (1.3) and (1.4) as just discussed for the isomorphism from proposition 1.3.4. Again we refer to [8] for a more thorough treatment of these claims.

In section 1.1 we claimed that general properties about determinants ensures that $\mathcal{H}_\Lambda^d$ is indeed a symmetric monoidal category. To substantiate our claim, suppose that we have a pair of composable open-closed closed cobordisms Σ_1 and Σ_2 with incoming boundary ∂_0 and ∂_1 respectively. Letting Σ_{12} denote the composition, there is a long exact sequence in cohomology associated to the triple $(\Sigma_{12}, \Sigma_1, \partial_0)$.

The free boundary of Σ_1 is itself a 1-cobordism between the boundary of ∂_0 and the boundary of ∂_1, and similar for Σ_2. These 1-cobordisms are themselves composable and from the triple $(\partial_{free}\Sigma_{12}, \partial_{free}\Sigma_1, \partial\partial_0)$ we get a long exact sequence as above. Note that for these long exact sequences we have $H^*(\Sigma_{12}, \Sigma_1) \simeq H^*(\Sigma_2, \partial_1)$, and $H^*(\partial_{free}\Sigma_{12}, \partial_{free}\Sigma_1) \simeq H^*(\partial_{free}\Sigma_2, \partial_{free}\partial_1)$ both by excision. Now by applying proposition 1.3.5 to the two long exact sequences with the above isomorphic replacements, we get an isomorphism

$$\det \Sigma_2^d \otimes \det \Sigma_1^d \simeq \det \Sigma_{12}^d$$

such that the composition map $H_*(B(gl))$ in $\mathcal{H}_\Lambda$ does indeed give us a composition map with local coefficients $H_*(B(gl); \det \Sigma^{\underline{d}})$ in $\mathcal{H}_\Lambda^d$, as we can pull back by this isomorphism. By the second compatibility property described by the pentagon (1.4) above we can verify that this is still associative, and the first compatibility property (1.3) ensures that it respects the symmetric monoidal structure of $\mathcal{H}_\Lambda^d$.

Finally we should note that the identity morphism for an object ∂ in $\mathcal{C}_\Lambda$ is represented by the cylinder $\partial \times I$ on the object. Since this deformation retracts onto the incoming boundary, there is no relative cohomology for this cobordism. Thus if we make sure there is a chosen isomorphism $\det(0) \simeq R$, this verifies that objects also have identity morphisms in $\mathcal{H}_\Lambda^d$.

1.3.3 In-surfaces and arcs

In this section we will (almost literally) dissect the manifolds representing the cobordisms of which we take homology. We show results which at first glance may seem too strong for our purpose, but the intricate role that orientations play will require us to take more care, than one would think. To begin we first note that open-closed cobordisms with positive boundary fall under a larger class of manifolds, as they are all what we will call *in-surfaces*, for lack of a better name. All homology in this section is with coefficients in $\mathbb{Z}$.

Definition 1.3.6. *Let Σ be a smooth compact oriented 2-manifold with boundary. We say that Σ is an* in-surface *if the boundary is not empty, and in each component of Σ, the boundary is partitioned into two non-empty parts,* incoming boundary *which is a submanifold of $\partial\Sigma$, possibly with boundary, and* non-incoming boundary.

We denote the incoming boundary of an in-surface Σ by $\partial_{in}\Sigma$, or simply ∂_{in} when Σ is clear from the context. The main theorem of this section is the following, and most of the section is spent on showing this.

Theorem 1.3.7. *Let Σ be an in-surface. There is a sequence of arcs $\alpha_1, \ldots, \alpha_N$, embedded disjointly in Σ, such that α_i sends the end points and only the end points to $\partial_{in} \cup \alpha_1 \cup \ldots \cup \alpha_{i-1}$, and α_i meets ∂_{in} transversely, and such that $\partial_{in} \cup \alpha_1 \cup \ldots \cup \alpha_N$ is a strong deformation retract of Σ.*

The relative homology $H_*(\Sigma, \partial_{in})$ is concentrated in degree 1, by the condition that both incoming and non-incoming boundary of each component of Σ is non-empty. This also holds the other way around, so that if $H_*(\Sigma, \partial_{in})$ is concentrated in degree 1, then both incoming and non-incoming boundary must be non-empty in every component.

Lemma 1.3.8. *Let Σ be an in-surface. Then $\chi(\Sigma, \partial_{in}) \leq 0$, and if $\chi(\Sigma, \partial_{in}) = 0$ then ∂_{in} is a strong deformation retract of Σ.*

Proof. As noted above $H_*(\Sigma, \partial_{in})$ is concentrated in degree 1, so

$$\chi(\Sigma, \partial_{in}) = -\dim H_1(\Sigma, \partial_{in}) \leq 0.$$

Next, suppose $\chi(\Sigma, \partial_{in}) = 0$ and that Σ is connected. By the long exact sequence associated to the triple $(\Sigma, \partial\Sigma, \partial_{in}\Sigma)$, the relative Euler characteristic is additive in the sense that

$$\chi(\Sigma, \partial_{in}) = \chi(\Sigma, \partial) + \chi(\partial, \partial_{in}).$$

Let n be the number of components of ∂, let q be the number of closed components of ∂_{in}, each a circle, and let p be the number of open components of ∂_{in}, each an interval. Letting g denote the genus of Σ, we then have that $\chi(\Sigma, \partial) = 2-2g-n$, and we have that $\chi(\partial, \partial_{in}) = -p$ by the assumptions about the boundary. Thus

$$\chi(\Sigma, \partial_{in}) = 2 - 2g - n - p = 0$$

where $n > 0$, $p, q \geq 0$, $p+q > 0$, and $n-q > 0$, all by the boundary assumptions. By this, $g = 0$ as otherwise $p < 0$, and so the only options for this to hold are then the two

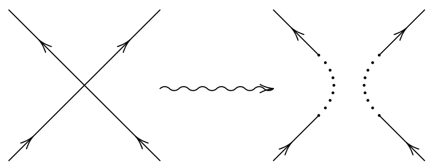
$$g = 0,\ n = 2,\ p = 0,\ q = 1$$
$$g = 0,\ n = 1,\ p = 1,\ q = 0.$$

The first of these is a cylinder with one end labelled as incoming and the other end non-incoming, and the second is a disk, with one interval embedded in the boundary, labelled as incoming. In both cases these in-surfaces deformation retracts strongly onto the incoming boundary.

In case Σ is not connected the argument works for each connected component, since we require that each such has incoming and non-incoming boundary, and the conclusion still holds. □

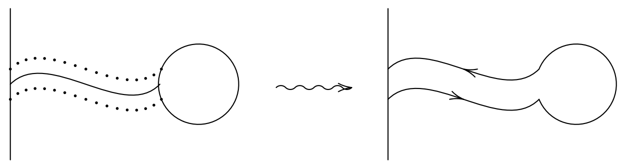
Lemma 1.3.9. *Let Σ be an in-surface with $\chi(\Sigma, \partial_{in}) < 0$. Then there is an arc α embedded in Σ such that α maps the end points and only those to ∂_{in}, and α meets ∂_{in} transversely, and such that $[\alpha] \neq 0$ in $H_1(\Sigma, \partial_{in})$.*

Proof. Since $\chi(\Sigma, \partial_{in}) < 0$ and Σ is an in-surface, whereby the homology is concentrated in degree 1, there is a non-zero class $x \in H_1(\Sigma, \partial_{in})$. There is a compact oriented 1-manifold A and a continuous map $f\colon (A, \partial A) \to (\Sigma, \partial_{in})$ such that $f_*[A, \partial A] = x$. We can assume that A is connected, since otherwise each component will give rise to a summand in the class x, of which at least one is non-zero. Now we perturb f slightly to an immersion g which is homotopic to f, and for which all self-intersections and the intersection with ∂_{in} happen transversely, and with only double points. Further since A is compact, there are only finitely many self-intersections. At each point of intersection we now perform the following simple operation. In a neighborhood of an intersection point, small enough to only contain the two intersecting arcs of $g(A)$, we remove the image $g(A)$ and connect the resulting loose ends in the only other way that respects the orientation of A. This reconnecting of loose ends can be done smoothly.



Also we perform the corresponding operations in the pre-image to get a new domain A', and the resulting map is now a smooth embedding which we denote by $g'\colon (A', \partial A') \to (\Sigma, \partial_{in})$. This represents the same homology class as x, and as before we can assume that A' is connected. Now if A' is an interval, we set $\alpha = g'(A', \partial A')$ and we are done.

If A' is a circle, we choose an arc from a point on $g'(A')$ to the incoming boundary. We can do this as the component of Σ in which g' embeds A' has an incoming boundary since Σ is an in-surface. We then fatten up this arc slightly, and note that traveling along one edge of the fattened arc, followed by $g'(A')$ in positive direction, and then back along the other edge of the fattened arc, defines a based path $g''(I, \partial I)$ in (Σ, ∂_{in}), homologous to $g'(A')$. We then set $\alpha = g''(I, \partial I)$, and we are done (we can make g'' smooth by modifying it slightly where the edges meets the circle).

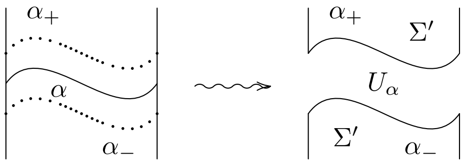


□

Let Σ be an insurface, and let α be an arc as in lemma 1.3.9 above. We can cut Σ along α to obtain a new surface Σ', where this is to be understood as choosing a small open neighborhood U_α of α by thickening α slightly and removing this. The two resulting arcs on the boundary of Σ', previously bounding U_α, are denoted α^+ and α^-. The boundary and incoming boundary of Σ' are then

$$\partial\Sigma' = (\partial\Sigma - (\partial\Sigma \cap U_\alpha)) \cup \alpha^+ \cup \alpha^-$$
$$\partial_{in}\Sigma' = (\partial_{in}\Sigma - (\partial_{in}\Sigma \cap U_\alpha)) \cup \alpha^+ \cup \alpha^-$$

We attempt to illustrate this here below.



Lemma 1.3.10. *Let Σ be an in-surface, and α as in lemma 1.3.9. The surface Σ' obtained from Σ by cutting along α, is an in-surface. Further*

$$\chi(\Sigma', \partial_{in}\Sigma') = \chi(\Sigma, \partial_{in}\Sigma) + 1$$

Proof. For Σ' to be an in-surface it is enough to show that $H_*(\Sigma', \partial_{in}\Sigma')$ is concentrated in degree 1. By excision of U_α we have the isomorphism

$$H_*(\Sigma, \partial_{in}\Sigma \cup U_\alpha) \simeq H_*(\Sigma', \partial_{in}\Sigma - (\partial_{in}\Sigma \cap U_\alpha))$$

so also

$$H_*(\Sigma, \partial_{in}\Sigma \cup \alpha) \simeq H_*(\Sigma', \partial_{in}\Sigma').$$

The long exact sequence associated to the triple $(\Sigma, \partial_{in}\Sigma \cup \alpha, \partial_{in}\Sigma)$ ends in

$$0 = H_0(\Sigma, \partial_{in}\Sigma) \longrightarrow H_0(\Sigma, \partial_{in}\Sigma \cup \alpha) \longrightarrow 0$$

since Σ is an in-surface, so $H_0(\Sigma, \partial_{in}\Sigma \cup \alpha) = 0$. From the same sequence we also get

$$0 = H_2(\Sigma, \partial_{in}\Sigma) \longrightarrow H_2(\Sigma, \partial_{in}\Sigma \cup \alpha) \longrightarrow H_1(\partial_{in}\Sigma \cup \alpha, \partial_{in}\Sigma) \longrightarrow H_1(\Sigma, \partial_{in}\Sigma)$$

where first term is zero, again by noting that Σ is an in-surface. By excision of $\partial_{in}\Sigma - \partial\alpha$, the term second from right is isomorphic to $H_1(\alpha, \partial\alpha)$. Now $H_1(\Sigma, \partial_{in}\Sigma)$ is a free $\mathbb{Z}$-module, and the non-zero class $[\alpha]$ generates all of $H_1(\alpha, \partial\alpha)$. This generator is mapped to a non-zero element $[\alpha] \in H_1(\Sigma, \partial_{in}\Sigma)$ by assumption. From this we conclude that the rightmost map above is injective, and hence $H_2(\Sigma, \partial_{in}\Sigma \cup \alpha) = 0$.

Again noting that $[\alpha] \neq 0$ we also get that

$$\begin{aligned}
\chi(\Sigma', \partial_{in}\Sigma') &= \chi(\Sigma, \partial_{in}\Sigma \cup \alpha) \\
&= \chi(\Sigma, \partial_{in}\Sigma) - \chi(\partial_{in}\Sigma \cup \alpha, \partial_{in}\Sigma) \\
&= \chi(\Sigma, \partial_{in}\Sigma) + 1
\end{aligned}$$

as claimed. □

proof of theorem 1.3.7. We prove the theorem by induction on $-\chi(\Sigma, \partial_{in}\Sigma)$. The initial case $-\chi(\Sigma, \partial_{in}\Sigma) = 0$ is proved by lemma 1.3.8. Now suppose the theorem is true for any Σ with $-\chi(\Sigma, \partial_{in}\Sigma) = n - 1$, and that we are given a Σ with $-\chi(\Sigma, \partial_{in}\Sigma) = n$. Now choose an arc α as in lemma 1.3.9, and construct Σ' by cutting along this α. By lemma 1.3.10 this Σ' is an in-surface with $-\chi(\Sigma, \partial_{in}\Sigma) = n - 1$, so by the induction hypothesis we can choose arcs $\beta_2, \ldots, \beta_N$ in Σ' such that Σ' deformation retracts strongly onto

$$\partial_{in}\Sigma' \cup \beta_2 \cup \ldots \cup \beta_N.$$

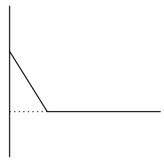
Now set $\alpha_1 = \alpha$, and note that Σ is obtained from Σ' by identifying α^+ and α^-. Under this identification each of the arcs β_i in Σ' for $i = 2, \ldots, N$ becomes an arc α_i in Σ mapping the end points to $\partial_{in}\Sigma \cup \alpha_1 \cup \ldots \cup \alpha_{i-1}$. The strong deformation retract of Σ' onto $\partial_{in}\Sigma' \cup \beta_2 \cup \ldots \cup \beta_N$ induce a strong deformation retract of Σ onto $\partial_{in}\Sigma \cup \alpha_1 \cup \ldots \cup \alpha_{i-1}$ under this identification. □

For later we will also need the following proposition.

Proposition 1.3.11. *Let Σ be an in-surface. The arcs of theorem 1.3.7 can be chosen such that the end points are all in ∂_{in}.*

Lemma 1.3.12. *Let Σ be an in-surface with $\chi(\Sigma, \partial_{in}) < 0$, and let $U \subset V \subset \partial_{in}$ be subsets such that every component of ∂_{in} intersects non-trivially with V, and every component of Σ intersects non-trivially with U. Then there is an arc as in lemma 1.3.9, with both ends in V and at least one end in U.*

Proof. Let α be an arc as produced by lemma 1.3.9. The end points of α can be chosen freely within the boundary components they are in by choosing a path from the current end point to a new within the boundary, and then pushing the concatenated arc slightly off the boundary.



Thus we can assume that V consist of unions of components of ∂_{in}, and in particular that $V = \partial_{in}$. If Σ is connected we can also assume that U is a single such component, as we are only concerned with a single end point in relation to U. If Σ is not connected, the following argument will hold for each component.

If $U = \partial_{in}$, then we are done. If not, then we consider the following part of the long exact sequence associated to the pair (Σ, ∂_{in}).

$$H_1(\Sigma, \partial_{in}) \xrightarrow{\partial_*} H_0(\partial_{in}) \longrightarrow H_0(\Sigma) \simeq \mathbb{Z} .$$

From this we see that the image of the boundary map contains elements for which the coefficient for U is non-zero. E.g. if T is a different boundary component than U, then $U - T$ is in the kernel of $H_0(\partial_{in}) \to H_0(\Sigma)$ since Σ is connected. Recall that $H_0(\partial_{in})$ is free abelian on the boundary components. We can now retrace the steps of the proof of lemma 1.3.9, and check that not only can we produce an embedded arc α for which $[\alpha] \neq 0$ in $H_1(\Sigma, \partial_{in})$, but also such that $\partial_*([\alpha])$ has non-zero U-coefficient.

For this we choose the class x such that $\partial_* x$ has non-zero U-coefficient. The map $f\colon (A, \partial A) \to (\Sigma, \partial_{in})$ representing x will then send ∂A to V, and at least one point in ∂A to U. Then also g' has this feature, and not only can we assume that A' is connected, but also that at least one point of $\partial A'$ goes to U, as $g'(A'\partial A')$ represents a class for which the image of ∂_* has non-zero U-coefficient, and $\partial_*(\beta) = C - D$ when β is an embedded arc starting in C and ending in D. □

proof of proposition 1.3.11. Let Σ be an in-surface. We now show the following statement by induction on $-\chi(\Sigma, \partial_{in})$ as in the proof of theorem 1.3.7. If Y is a subset of $\partial_{in}\Sigma$ which intersects every component of $\partial_{in}\Sigma$ non-trivially, then there is a set of arcs as in theorem 1.3.7, for which all end points are in Y.

The initial case is the same for the theorem 1.3.7. Now if $\chi(\Sigma, \partial_{in}) < 0$ then we apply lemma 1.3.12 with $U = V = Y \subset \partial_{in}$, to produce an arc α which we cut along. As before the resulting in-surface Σ' has

$$\chi(\Sigma', \partial_{in}\Sigma') = \chi(\Sigma, \partial_{in}) + 1$$

and there is a map $r\colon \Sigma' \to \Sigma$, which identifies α_- and α_+. We now set $Y' = r^{-1}Y \subset \partial_{in}\Sigma'$, and by the induction hypothesis Σ', Y' is a pair for which the stronger statement holds. Hence by the map r, it holds for Σ, Y. This shows the proposition as well. □

The proofs about arc decomposition can be tweaked slightly similar to what we just did above to give variations of the conclusions, but we will postpone that to where we have to use it. This concludes the preliminary chapter.

Chapter 2

Construction of operations

The plan of this chapter is to first give a quick overview of the construction we intend to perform in section 2.1. This will give us an idea of which properties we will have to show about our spaces and maps.

We then set out to show the properties referred to above, in a section 2.2 on the simplest possible case, namely when there are no labels to consider. Upon completing this unlabelled special case, we may proceed directly to the actual construction of operations in section 2.4, allowing us to define an ocHFT without labels. The tools developed in section 2.2 on the unlabelled case, will prove valuable for the following section 2.3, where we consider the general case. Here we will develop terminology and tools for showing the same properties in more generality, and relate back to the unlabelled case whenever possible.

Finally we conclude the chapter with section 2.4 by performing the construction of operations needed to define an ocHFT with or without labels, and verify the axioms needed for the construction to actually be a symmetric monoidal functor.

In the following let G be a fixed compact connected Lie group, and let H denote an oriented connected subgroup. We fix EG, and let $BH = EG/H$ for any subgroup $H \leq G$.

2.1 Overview of construction

First we will have to introduce some terminology in order to explain how the construction works in general. Similar to the labelled manifolds from section 1.1, we will need labelled spaces which are not necessarily manifolds. Thus we define

Definition 2.1.1. *Let X be a space, L a set of labels, and $\{X_\alpha\}_{\alpha\in A}$ a set of subspaces indexed by A, each subspace including cofibrantly in X, such that either $X_\alpha \cap X_{\alpha'} = \emptyset$ or $X_\alpha = X_{\alpha'}$ for all $\alpha, \alpha' \in A$. To each subspace we associate a label from L by a map $\lambda\colon A \to L$. We write X^l for a subspace labelled by $l \in L$, and we say that the triple $(X, \{X_\alpha\}_{\alpha\in A}, \lambda)$ is a labelled space, with labels in L.*

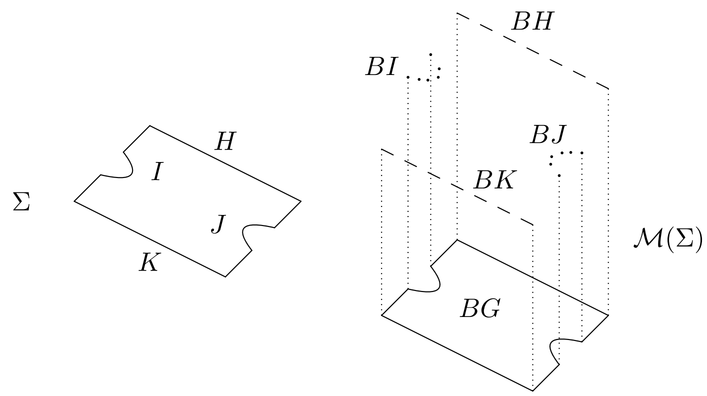
When the labelling is clear from the context, or not a specific one, we may just say that X is a labelled space.

Definition 2.1.2. *Let Σ be labelled space with labels the oriented connected subgroups of G. Define $\mathcal{M}(\Sigma)$ to be the pullback of the diagram*

$$\prod_H \mathrm{Map}(\Sigma^H, BH) \xrightarrow{p} \prod_H \mathrm{Map}(\Sigma^H, BG) \xleftarrow{j} \mathrm{Map}(\Sigma, BG)$$

where p is induced by the fibration $BH \to BG$, j is induced by the inclusions $\Sigma^H \to \Sigma$, and the products ranges over all labels H on Σ.

Thus $\mathcal{M}$ takes the ordinary mapping space $\mathrm{Map}(\Sigma, BG)$, and adds the data of lifts from the subspaces of Σ labelled by H to BH for every connected subgroup $H \leq G$. As notation suggests, we will primarily be working with the case where Σ is an open-closed cobordism, with each component of the free boundary labelled by a connected subgroup of G. Unless otherwise stated, this is assumed to be the case.



Here we have tried to illustrate how a labelled open-closed cobordism Σ, from the two intervals labelled by respectively H, I and I, K, all oriented connected subgroups of G, to the two intervals labelled by respectively H, J and J, K, maps to BG, and how the restrictions to parts labelled by H, I, J and K have lifts to respectively BH, BI, BJ and BK.

Note that $\mathcal{M}(\Sigma)$ is a homotopy pullback, since p and j are fibrations. It is actually enough that either p or j is a fibration for this to hold.

With the above definition in place the operations are constructed as follows. Let ∂_{in} and ∂_{out} denote the incoming and outgoing boundary of Σ respectively, and consider the following diagram

$$\begin{tikzcd} & \mathcal{M}(\Sigma) \arrow[dl, "in"'] \arrow[dr, "out"] & \\ \mathcal{M}(\partial_{in}) & & \mathcal{M}(\partial_{out}) \end{tikzcd}$$

where in and out are the maps induced by the inclusion of the incoming and outgoing boundary respectively. We show that the map in is an orientable fibration, and that in our case, we get an orientable fibration when we apply the Borel construction to the diagram to get

$$\begin{tikzcd}
& \mathcal{M}(\Sigma)_{h\,\mathrm{Diff}^+\,\Sigma} \arrow[dl, "in_{h\,\mathrm{Diff}^+\,\Sigma}"'] \arrow[dr, "out_{h\,\mathrm{Diff}^+\,\Sigma}"] & \\
B\,\mathrm{Diff}^+\,\Sigma \times \mathcal{M}(\partial_{in}) & & B\,\mathrm{Diff}^+\,\Sigma \times \mathcal{M}(\partial_{out})
\end{tikzcd}$$

Note that the diffeomorphism groups are relative to the entire boundary of Σ, but this is suppressed in the notation here and later. Passing to homology we can define a wrong way map, defined using the Serre spectral sequence (section 1.3), associated to this orientable fibration, and get the diagram

$$\begin{tikzcd}
& H_*(\mathcal{M}(\Sigma)_{h\,\mathrm{Diff}^+\,\Sigma}) \arrow[dl, "(in_{h\,\mathrm{Diff}^+\,\Sigma})_*"', shift right] \arrow[dr, "(out_{h\,\mathrm{Diff}^+\,\Sigma})_*"] & \\
H_*(B\,\mathrm{Diff}^+\,\Sigma \times \mathcal{M}(\partial_{in})) \arrow[ur, dotted, "(in_{h\,\mathrm{Diff}^+\,\Sigma})_!"', shift right] & & H_*(B\,\mathrm{Diff}^+\,\Sigma \times \mathcal{M}(\partial_{out}))
\end{tikzcd} \tag{2.1}$$

We may then define the operation associated to Σ to be the composite

$$\mu(\Sigma) := (\pi_2)_* \circ (out_{h\,\mathrm{Diff}^+\,\Sigma})_* \circ (in_{h\,\mathrm{Diff}^+\,\Sigma})_!$$

and by the Künneth isomorphism, we can consider this as a map

$$\mu(\Sigma)\colon H_*(B\,\mathrm{Diff}^+\,\Sigma) \otimes H_*(\mathcal{M}(\partial_{in})) \to H_{*+d}(\mathcal{M}(\partial_{out}))$$

where d is the top dimension of the fibre of in. We will go about showing this in a series of steps. First we will show that in is indeed a fibration. Then we will analyze the fibre, to show that it has rank 1 top homology, and that the action of $\mathrm{Diff}^+\,\Sigma$ is trivial. All this to allow us to form the wrong way map $(in_{h\,\mathrm{Diff}^+\,\Sigma})_!$.

Since the goal is to construct operations, which we will eventually want to be able to compose, it is not enough however, to make sure that we can orient single surfaces and fibres in the above. We must have a way to do this coherently for pairs of composable operations and the resulting composed operation. To have this, we record the orientations in the coefficients of the homology, so that in more detail

$$\mu(\Sigma)\colon H_*(B\,\mathrm{Diff}^+\,\Sigma; \det \Sigma^{\underline{d}}) \otimes H_*(\mathcal{M}(\partial_{in}); \mathbb{F}) \to H_{*+d}(\mathcal{M}(\partial_{out}); \mathbb{F}).$$

Recall that $\det \Sigma^{\underline{d}}$ is short for the expression

$$\det H^*(\Sigma, \partial_{in})^{\otimes -\dim G} \otimes \bigotimes_{H<G} \det H^*(\Sigma^H, \partial_{in}^H)^{\otimes(\dim G - \dim H)}.$$

We will discuss how this expression comes about in sections 2.2 and 2.3, and now proceed to do the actual construction in detail.

The plan is to first establish the claimed properties of the fibre in the unlabelled case. Then we will establish appropriate compatibility properties to ensure that the operations we produce as described above, indeed does give rise to a symmetric monoidal functor. Having done so we may proceed straight to section 2.4 and construct operations in the unlabelled case. However, we will first show that the claims of about the fibre also hold in the labelled case, and that similar compatibility properties also hold in the labelled case. This will allow us to do a common treatment of the two cases in the final section of the chapter, where we first construct the operations, and verify the axioms for this to give rise to a symmetric monoidal functor.

2.2 Unlabelled case

Before we undertake the construction of the operations in the general case, we first establish some theory for the unlabelled case, i.e. the case where all labels are taken to be G itself. We call this unlabelled, as there is no extra data in this labelling. When X is a labelled space with all labels being G, the space $\mathcal{M}(X)$ is just the ordinary mapping space $\mathrm{Map}(X, BG)$.

We will be working with an open-closed cobordism Σ with positive boundary, and the goal of this section is to establish certain properties for the space $\mathrm{Map}_*(\Sigma/\partial_{in}\Sigma, BG)$, occurring as the fibre of the fibration in given by restricting maps $\Sigma \to BG$ to the incoming boundary of Σ. We will show that it has the property we called *rank 1 top homology* in section 1.3.1, i.e. that the homology is concentrated below some degree s and that $H_s(\mathrm{Map}_*(\Sigma/\partial_{in}\Sigma, BG))$ has rank 1.

Next we show that the fibration in, is an orientable fibration, and further that this is still the case when we apply the Borel construction $(-)_{h\,\mathrm{Diff}^+\Sigma}$ to it. This last point follows by showing that $\mathrm{Diff}^+\Sigma$ acts trivially on the fibre, in the process of which we see how the expression $\det \Sigma^{\underline{d}}$ comes into play.

Restricting attention to closed cobordisms, it is the intention that this section can be applied directly to verify various claims in the inspirational reference [1].

2.2.1 Properties of the fibre

The claim that $\mathrm{Map}_*(\Sigma/\partial_{in}, BG)$ has rank 1 top homology follows a corollary of proposition 1.3.11, as we show below. We also determine the degree which bounds the homology.

Corollary 2.2.1. *Let Σ be an in-surface. The space $\mathrm{Map}_*(\Sigma/\partial_{in}, BG)$ has rank 1 top homology, and the top degree is $s = \operatorname{rank} H^1(\Sigma, \partial_{in}; \mathbb{Z}) \cdot \dim G$.*

Proof. The first claim follows directly from noting that G is a Lie group and the homotopy equivalences

$$\begin{aligned}
\mathrm{Map}_*(\Sigma/\partial_{in}, BG) &\simeq \mathrm{Map}_*((\partial_{in} \cup \alpha_1 \cup \ldots \cup \alpha_N)/\partial_{in}, BG) \\
&\simeq \mathrm{Map}_*(\bigvee_N S^1, BG) \\
&\simeq \Omega BG^N \simeq G^N
\end{aligned}$$

where the first is due to proposition 1.3.11. It is clear from the proof of that proposition, that $N = \operatorname{rank} H^1(\Sigma, \partial_{in}; \mathbb{Z})$. □

Note that slightly shorter than $\dim H^1(\Sigma, \partial_{in})$, by the boundary conditions, we may instead write $N = -\chi(\Sigma, \partial_{in})$. This is just $m - \chi(\Sigma)$, where m is the number of open components of ∂_{in}.

Next we verify that the restriction to incoming boundary, i.e. the fibration *in*, is an orientable fibration. In light of the above, it only remains to show that $\pi_1(\mathrm{Map}(\partial_{in}, BG))$ acts trivially on the top homology of the fibre. Following the idea of [1] we note that for arcs as in 1.3.11, the following diagram is a pushout

$$\begin{tikzcd} \partial\alpha_1 \cup \ldots \cup \partial\alpha_N \arrow[r] \arrow[d] & \alpha_1 \cup \ldots \alpha_N \arrow[d] \\ \partial_{in} \arrow[r] & \partial_{in} \cup \alpha_1 \cup \ldots \alpha_N \end{tikzcd}$$

From this we get that the right square below is a pullback

$$\begin{tikzcd} \mathrm{Map}(\Sigma, BG) \arrow[d] & \mathrm{Map}(\partial_{in} \cup \alpha_1 \cup \ldots \alpha_N, BG) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \mathrm{Map}(\alpha_1 \cup \ldots \alpha_N, BG) \arrow[d] \\ \mathrm{Map}(\partial_{in}, BG) \arrow[r, equal] & \mathrm{Map}(\partial_{in}, BG) \arrow[r] & \mathrm{Map}(\partial\alpha_1 \cup \ldots \cup \partial\alpha_N, BG) \end{tikzcd} \tag{2.2}$$

by which we see that the action of $\pi_1(\mathrm{Map}(\partial_{in}, BG))$ on the top homology of the fibre of *in*, is pulled back from the action of

$$\pi_1(\mathrm{Map}(\partial\alpha_1 \cup \ldots \cup \partial\alpha_N, BG)) \simeq \pi_1(BG^{2N}) \simeq \prod_{2N} \pi_0(G) \simeq 1\,.$$

Thus the action is trivial and we have shown

Proposition 2.2.2. *The fibration induced by inclusion of the incoming boundary map*

$$\mathrm{Map}_*(\Sigma/\partial_{in}, BG) \to \mathrm{Map}(\Sigma, BG) \to \mathrm{Map}(\partial_{in}, BG)$$

is an orientable fibration.

Next we proceed to show that the action of $\mathrm{Diff}^+(\Sigma, \partial_{in})$ on the top homology of the fibre of *in* is trivial. To do so we will first discuss the following

Proposition 2.2.3. *There is a well-defined isomorphism of weighted lines*

$$\Phi\colon H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG); \mathbb{Z}) \xrightarrow{\simeq} \det(H^1(\Sigma, \partial_{in}); \mathbb{Z})^{\otimes \dim G}$$

We know that $\Sigma/\partial_{in} \simeq \bigvee_{-\chi(\Sigma,\partial_{in})} S^1$, so $\pi_1(\Sigma/\partial_{in})$ is a free group on $-\chi(\Sigma, \partial_{in})$ generators. Choose a basis $\alpha = \{\alpha_1, \ldots, \alpha_{-\chi(\Sigma,\partial_{in})}\}$, with each element α_i represented by a map $a_i\colon S_1 \to \Sigma/\partial_{in}$.

On one hand this induces the first isomorphism of weighted lines in

$$\begin{aligned} H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG); \mathbb{Z}) &\simeq H_s(\Omega BG^{-\chi(\Sigma,\partial_{in})}; \mathbb{Z}) && (2.3)\\ &\simeq H_s(G^{-\chi(\Sigma,\partial_{in})}; \mathbb{Z}) && (2.4)\\ &\simeq H_{\dim G}(G; \mathbb{Z})^{\otimes -\chi(\Sigma,\partial_{in})} && (2.5)\\ &\simeq \mathbb{Z}[\dim G]^{\otimes -\chi(\Sigma,\partial_{in})} \\ &\simeq \mathbb{Z}[-\chi(\Sigma, \partial_{in}) \dim G] \end{aligned}$$

where (2.5) is the Künneth map which is an isomorphism since $H_{\dim G}(G;\mathbb{Z})$ is torsion free when G is a closed connected Lie group by [10] corollary 3.28. Note that these isomorphisms do not depend on the representatives a_i when we take homology.

On the other hand this chosen basis provides a set of generators of $H^1(\Sigma, \partial_{in}; \mathbb{Z})$ and thus the first isomorphism of

$$\begin{aligned}\det(H^1(\Sigma, \partial_{in}); \mathbb{Z})^{\otimes \dim G} &\simeq \mathbb{Z}[-\chi(\Sigma, \partial_{in})]^{\otimes \dim G} \\ &\simeq \mathbb{Z}[(-\chi(\Sigma, \partial_{in}) \dim G]\end{aligned}$$

Combining these we get an isomorphism

$$\Phi_\alpha \colon H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG); \mathbb{Z}) \xrightarrow{\simeq} \det(H^1(\Sigma, \partial_{in}); \mathbb{Z})^{\otimes \dim G} \tag{2.6}$$

Proposition 2.2.4. *The isomorphism Φ_α does not depend on the choice of basis α.*

Any other choice of basis determines automorphism ϕ of the free group $\pi_1(\Sigma/\partial_{in})$, given by mapping the first basis to the second. For automorphisms of free groups we have the following proposition.

Proposition 2.2.5 ([11] proposition 4.1 chapter 1)**.** *Let $(x_1, \dots, x_n)$ be generators of a free group F_n. Then $\operatorname{Aut} F_n$ is generated by automorphisms of the following two types*

a) Composition by a generator x_j, i.e. composing x_i and x_j on the i'th coordinate, and fixing everything else.

b) Inverting a generator x_i, i.e. inverting x_i on the i'th coordinate, and fixing everything else.

By this proposition, we may assume that ϕ is an automorphisms of one of these two types. The following lemma motivates us to examine what happens locally.

Lemma 2.2.6. *Let $g \colon M \to M$ be a diffeomorphism of a smooth connected closed orientable n-manifold M, and let $x \in M$ be a fixed point. The induced map $g_* \colon H_n(M) \to H_n(M)$ is multiplication by the sign of $\det D_x g$.*

Proof. The diffeomorphism g induces a map on $M - \{x\}$ and consequently on both $H_n(M;\mathbb{Z}) \simeq \mathbb{Z}$ and on $H_n(M, M - \{x\}; \mathbb{Z})$ such that the diagram

$$\begin{array}{ccc} H_n(M;\mathbb{Z}) & \longrightarrow & H_n(M, M-\{x\};\mathbb{Z}) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} \\ H_n(M;\mathbb{Z}) & \longrightarrow & H_n(M, M-\{x\};\mathbb{Z}) \end{array}$$

commutes. By [10] theorem 3.26 the horizontal maps are isomorphisms, and by [12] proposition IV-7.1 the map g'_* is multiplication by the sign of $\det D_x g$. $\square$

proof of proposition 2.2.4. Let $\alpha_1, \dots \alpha_{-\chi(\Sigma,\partial_{in})}$ be a basis for $\pi_1(\Sigma/\partial_{in})$, and let $\beta_1, \dots \beta_{-\chi(\Sigma,\partial_{in})}$ be another basis. We denote by ϕ the automorphism taking α_i to β_i. By proposition 2.2.5 we can assume that ϕ is either of the form a) or of the form b). In either case it induces a diffeomorphism $\bar{\phi}$ on the Lie group $G^{-\chi(\Sigma,\partial_{in})}$ by the following diagram

$$\begin{tikzcd}
\mathrm{Map}_*(\Sigma/\partial_{in}, BG) \arrow[r, "="] \arrow[d, "\alpha"] & \mathrm{Map}_*(\Sigma/\partial_{in}, BG) \arrow[d, "\beta"] \\
\Omega BG^{-\chi(\Sigma,\partial_{in})} \arrow[r] \arrow[d] & \Omega BG^{-\chi(\Sigma,\partial_{in})} \arrow[d] \\
G^{-\chi(\Sigma,\partial_{in})} \arrow[r, "\bar{\phi}"] & G^{-\chi(\Sigma,\partial_{in})}
\end{tikzcd}$$

Now suppose ϕ is of the form a), then $\bar{\phi}$ is given by composing (in G) the i'th factor by the j'th. On the tangent space at the identity e, we see that $D_e\bar{\phi}$ is represented by the following block matrix

$$\begin{pmatrix}
1 & & & & & & & 0 \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & 0 & \cdots & 1_{ij} & 0 \\
& & & & 1 & & & \\
& & & & & \ddots & & \\
& & & & & & & \\
0 & & & & & & & 1
\end{pmatrix}$$

where each entry is a $\dim G$ square matrix. Clearly this has determinant $+1$, so $\bar{\phi}_*$ is multiplication by $+1$ by lemma 2.2.6.

Similarly, if ϕ is of type b), then $\bar{\phi}$ is given by inverting the i'th factor, for which $D_e\bar{\phi}$ is given by the block matrix

$$\begin{pmatrix}
1 & & & & & & 0 \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & -1_{ii} & & & \\
& & & & 1 & & \\
& & & & & \ddots & \\
0 & & & & & & 1
\end{pmatrix}$$

This has determinant $(-1)^{\dim G}$, and by lemma 2.2.6, $\bar{\phi}_*$ is multiplication by $(-1)^{\dim G}$.

We now examine the effect of the two types of automorphisms on the other side of the isomorphism (2.6). The basis of the abelian group $H^1(\Sigma, \partial_{in}; \mathbb{Z})$ is dual to that of the free group $\pi_1(\Sigma/\partial_{in})$, so by choice $\det H^1(\Sigma, \partial_{in}; \mathbb{Z})$ is generated by $\alpha_1 \wedge \cdots \wedge \alpha_{-\chi(\Sigma,\partial_{in})}$, where we by α_i now mean the dual, but we omit this from the notation. If ϕ is of type a), i.e. composition by a generator, then it is given by addition of cohomology classes on $H^1(\Sigma, \partial_{in}; \mathbb{Z})$, and consequently it induces the map as given below on $\det H^1(\Sigma, \partial_{in}; \mathbb{Z})$

$$\begin{aligned}\alpha_1 \wedge \cdots \wedge \alpha_{-\chi(\Sigma,\partial_{in})} \mapsto\, & \alpha_1 \wedge \cdots \wedge \alpha_i + \alpha_j \wedge \cdots \alpha_{-\chi(\Sigma,\partial_{in})} \\ & = \alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_j \wedge \cdots \wedge \alpha_{-\chi(\Sigma,\partial_{in})} \\ & \quad + \alpha_1 \wedge \cdots \wedge \alpha_j \wedge \cdots \wedge \alpha_j \wedge \cdots \wedge \alpha_{-\chi(\Sigma,\partial_{in})} \\ & = \alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_j \wedge \cdots \wedge \alpha_{-\chi(\Sigma,\partial_{in})}\end{aligned}$$

as $\alpha_j \wedge \alpha_j = 0$, and $j \neq i$. Thus it is the identity on the determinant, and consequently on the tensor power of those.

Similarly, if ϕ is of type b), i.e. inverting a generator, then it is given by negating a cohomology class on $H^1(\Sigma, \partial_{in}; \mathbb{Z})$, and thus by multiplication by -1 on the determinant. Hence it is multiplication by $(-1)^{\dim G}$ on the tensor power of $\det H^1(\Sigma, \partial_{in}; \mathbb{Z})$. In particular we see that ϕ induces the same map on either side of the isomorphism (2.6), and so a different choice of basis does not affect the isomorphism. □

From here we will refer to the isomorphism (2.6), just as Φ since it does not depend on α, and this is the isomorphism claimed to exist by proposition 2.2.3. It now remains to show that we get an orientable fibration once we apply the Borel construction $(-)_{h\,\mathrm{Diff}^+\Sigma}$ to the fibration in. For that it is enough to show that $\mathrm{Diff}^+(\Sigma, \partial)$ acts trivially on the top homology of the fibre, since then $\pi_0(\mathrm{Diff}^+ \Sigma)$ acts trivially, and thus $\pi_1(B\,\mathrm{Diff}^+ \Sigma)$ does so.

Now let $f \in \mathrm{Diff}^+(\Sigma, \partial)$ be a diffeomorphism acting on either side of Φ as follows. The diffeomorphism f induces a pointed homeomorphism

$$\bar{f}\colon \Sigma/(\partial_{in}) \to \Sigma/(\partial_{in})$$

acting on $H_s(\mathrm{Map}_*(\Sigma/(\partial_{in}), BG); \mathbb{Z})$ by $H_s(\mathrm{Map}_*(\bar{f}, BG); \mathbb{Z})$. The action on $\det(H^1(\Sigma, \partial_{in}; \mathbb{Z}))^{\otimes \dim G}$ is given by $\det f^{*\otimes \dim G}$.

Proposition 2.2.7. *The isomorphism Φ is $\mathrm{Diff}^+(\Sigma, \partial)$-equivariant, and the action is trivial on both the top homology $H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG))$, and on the line $\det(H^1(\Sigma, \partial_{in}; \mathbb{Z}))^{\otimes \dim G}$.*

Proof. First choose a basis $\alpha_1, \ldots, \alpha_{-\chi(\Sigma,\partial_{in})}$ to define Φ_α. In both cases the action takes the representatives $a_i\colon S^1 \to \Sigma/(\partial_{in})$ with $i = 1, \ldots, -\chi(\Sigma, \partial_{in})$, to $b_i := f \circ a_i$ representing a basis $\beta = \{\beta_1, \ldots, \beta_{-\chi(\Sigma,\partial_{in})}\}$ and we get a diagram

$$\begin{tikzcd} H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG); \mathbb{Z}) \arrow[r, "H_s(\mathrm{Map}_*(\bar{f}, BG))", "\simeq"'] \arrow[d, "\Phi_\alpha"', "\simeq"] & H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG); \mathbb{Z}) \arrow[d, "\Phi_\beta", "\simeq"'] \\ \det(H^1(\Sigma, \partial_{in}; \mathbb{Z}))^{\otimes \dim G} \arrow[r, "\simeq", "\det f^{*\otimes \dim G}"'] & \det(H^1(\Sigma, \partial_{in}; \mathbb{Z}))^{\otimes \dim G} \end{tikzcd}$$

which commutes by inspection, as proposition 2.2.4 tells us that $\Phi_\alpha = \Phi_\beta = \Phi$. Finally by proposition 2.2.8 below, the bottom map of this diagram is multiplication by $+1$ when f preserves orientation. Then also the top map is the identity, and thus $\mathrm{Diff}^+ \Sigma$ acts trivially on the top homology $H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG))$. □

Proposition 2.2.8. *Let Σ be an open-closed cobordism, and $f\colon \Sigma \to \Sigma$ a homeomorphism which fixes the boundary pointwise, preserves orientation and sends every component of Σ to itself. Then the induced isomorphism on cohomology $f^*\colon H^1(\Sigma, \partial_{in}; \mathbb{Z}) \to H^1(\Sigma, \partial_{in}; \mathbb{Z})$ has determinant* 1.

Proof. Since f fixes the entire boundary pointwise, it induces a map not only on $H^1(\Sigma, \partial_{in})$ but also a map on $H^1(\Sigma, \partial\Sigma)$ which then extends to a map on $H^1(\Sigma \sqcup \coprod D^2, \coprod D^2)$ where we have glued disks to all boundary components of Σ such that $\Sigma \sqcup \coprod D^2$ is a closed surface. There are maps

$$H^1(\Sigma \sqcup \coprod D^2, \coprod D^2) \longrightarrow H^1(\Sigma, \partial\Sigma) \longrightarrow H^1(\Sigma, \partial_{in})$$

induced by the obvious maps of pairs, giving a commutative diagram

$$\begin{tikzcd}
H^1(\Sigma, \partial_{in}) \arrow[r, "f^*"] & H^1(\Sigma, \partial_{in}) \\
H^1(\Sigma, \partial\Sigma) \arrow[r, "f^*"] \arrow[u] & H^1(\Sigma, \partial\Sigma) \arrow[u] \\
H^1(\Sigma \sqcup \coprod D^2, \coprod D^2) \arrow[r, "\tilde{f}^*"] \arrow[u] & H^1(\Sigma \sqcup \coprod D^2, \coprod D^2) \arrow[u]
\end{tikzcd}$$

The bottom map is then induced by a map $\tilde{f}$ of closed surfaces. It preserves the symplectic bilinear form $\langle a, b\rangle := [\Sigma](a \smile b)$ on $H^1(\Sigma; \mathbb{Z})$, since it preseves the cup product and the fundamental class of Σ, as f is orientation preserving. By [13] the induced map $\tilde{f}^*$ has determinant $+1$, and by commutativity of the diagram also f^* has determinant $+1$. $\square$

Now that we have established that the action of $\mathrm{Diff}^+ \Sigma$ is trivial on both sides of the isomorphism Φ, we can tensor both sides of the isomorphism with a field $\mathbb{F}$. From this, there are canonical isomorphisms to the expressions with $\mathbb{F}$ coefficients on either side and we get the same result with these. In particular we get

Proposition 2.2.9. *The fibration induced by the inclusion of the incoming boundary*

$$\mathrm{Map}_*(\Sigma/\partial_{in}, BG) \to \mathrm{Map}(\Sigma, BG)_{h\,\mathrm{Diff}^+ \Sigma} \to B\,\mathrm{Diff}^+ \Sigma \times \mathrm{Map}(\partial_{in}, BG)$$

is an orientable fibration.

2.2.2 Composition and monoidality

Our next task will be to show that Φ is compatible with gluing open-closed cobordisms, in the following sense.

Proposition 2.2.10. *Let Σ_1, Σ_2 be a pair of composable open-closed cobordisms with positive boundary, and denote by Σ_{12} the composition. Let ∂_0 denote the common incoming boundary of Σ_1 and Σ_{12}, and let ∂_1 denote the incoming boundary of Σ_2. Denote the isomorphism Φ for each cobordism, by respectively Φ_1, Φ_2 and Φ_{12}. Then* *i) there is a natural isomorphism*

$$H_{s_2}(\mathrm{Map}_*(\Sigma_2/\partial_1, BG)) \otimes H_{s_1}(\mathrm{Map}_*(\Sigma_1/\partial_0, BG)) \simeq H_{s_2+s_1}(\mathrm{Map}_*(\Sigma_{12}/\partial_0, BG))$$

ii) there is a natural isomorphism

$$\det H^1(\Sigma_2, \partial_1)^{\otimes \dim G} \otimes \det H^1(\Sigma_1, \partial_0)^{\otimes \dim G} \simeq \det H^1(\Sigma_{12}, \partial_0)^{\otimes \dim G}$$

iii) the isomorphism of i) and ii) above are such that the following diagram commutes

$$\begin{tikzcd}
H_{s_2}(\mathrm{Map}_*(\Sigma_2/\partial_1, BG)) \otimes H_{s_1}(\mathrm{Map}_*(\Sigma_1/\partial_0, BG)) \arrow[r, "\simeq"] \arrow[d, "\Phi_2\otimes\Phi_1"'] & H_{s_2+s_1}(\mathrm{Map}_*(\Sigma_{12}/\partial_0, BG)) \arrow[d, "\Phi_{12}"] \\
\det H^1(\Sigma_2, \partial_1)^{\otimes \dim G} \otimes \det H^1(\Sigma_1, \partial_0)^{\otimes \dim G} \arrow[r, "\simeq"] & \det H^1(\Sigma_{12}, \partial_0)^{\otimes \dim G}
\end{tikzcd}$$

Before we begin the proof, we will need some modifications of the arc decomposition results from section 1.3.3

Lemma 2.2.11. *Let Σ be an in-surface. Given a strong deformation retract*

$$d\colon \Sigma \to \partial_{in}\Sigma \cup \alpha_1 \cup \ldots \cup \alpha_N$$

as produced by theorem 1.3.11, and any finite subset $p_1, \ldots, p_n$ of the non-incoming boundary of Σ, we can find smoothly embedded arcs $\gamma_1, \ldots, \gamma_n$, disjoint from each other, each γ_i going from p_i to a point in the incoming boundary. Further we may modify d to factor as a strong deformation retract

$$d'\colon \Sigma \to \partial_{in}\Sigma \cup \alpha_1 \cup \ldots \cup \alpha_N \cup \gamma_1 \cup \ldots \cup \gamma_n$$

followed by a strong deformation retract contracting each γ_i to the incoming boundary.

Proof. In the case $\chi(\Sigma, \partial_{in}) = 0$ this is follows obviously from lemma 1.3.8. For $\chi(\Sigma, \partial_{in}\Sigma) < 0$, we note that the arc α produced by lemma 1.3.12 can be modified slightly to a homotopic smooth embedding such that the image is completely disjoint from the non-incoming boundary. This allow us to choose a path from any point of the non-incoming boundary to the incoming boundary which does not intersect α. Therefore this is also a such a path on the surface Σ' obtained by cutting along α, and as we have seen $\chi(\Sigma'\partial_{in}\Sigma') = \chi(\Sigma\partial_{in}\Sigma) + 1$, so by induction on $\chi(\Sigma\partial_{in}\Sigma)$ such a path can be chosen for any in-surface. This argument works for any finite number of points and paths. □

proof of proposition 2.2.10 i). This follows from proposition 1.3.1, since by 2.2.1 all the spaces have rank 1 top homology, and

$$\mathrm{Map}_*(\Sigma_2/\partial_1, BG) \to \mathrm{Map}_*(\Sigma_{12}/\partial_0, BG) \to \mathrm{Map}_*(\Sigma_1/\partial_0, BG)$$

is an orientable fibration as per diagram (1.2). □

proof of proposition 2.2.10 ii). The long exact sequence in cohomology for the triple $(\Sigma_{12}, \Sigma_1, \partial_0)$ collapses to the following by the boundary assumption

$$0 \longrightarrow H^1(\Sigma_{12}, \Sigma_1) \longrightarrow H^1(\Sigma_{12}, \partial_0) \longrightarrow H^1(\Sigma_1, \partial_0) \longrightarrow 0$$

By excision of $\Sigma_1 - \partial_1$ in first term, $H^1(\Sigma_{12}, \Sigma_1) \simeq H^1(\Sigma_2, \partial_1)$ and by proposition 1.3.4 this gives an isomorphism

$$\det H^1(\Sigma_2, \partial_1) \otimes \det H^1(\Sigma_1, \partial_0) \simeq \det H^1(\Sigma_{12}, \partial_0)$$

and hence the same on the $\dim G$'th tensor power. □

proof of proposition 2.2.10 iii). We first show that there is a homotopy equivalence

$$\mathrm{Map}_*(\Sigma_{12}/\partial_0, BG) \simeq \mathrm{Map}_*(\Sigma_1/\partial_0, BG) \times \mathrm{Map}_*(\Sigma_2/\partial_1, BG)$$

such that by proposition 1.2, the isomorphism of $i)$ is given by the cross product map.

As Σ_1, Σ_2 are in-surfaces, there are strong deformation retracts

$$\begin{aligned} d_1 &: \Sigma_1 \xrightarrow{\simeq} \partial_0 \cup \alpha_1 \cup \ldots \cup \alpha_N \\ d_2 &: \Sigma_2 \xrightarrow{\simeq} \partial_1 \cup \beta_1 \cup \ldots \cup \beta_M \end{aligned}$$

by proposition 1.3.11, and these can be chosen such that all arcs α_i and β_j have both end points on the respective incoming boundaries. Denote by β_i^+ the initial point of the arc β_i, and by β_i^- the final point of the arc β_i. These points are in ∂_1. By lemma 2.2.11, there are smoothly embedded arcs $\gamma_i^\pm$ in Σ_1, from each of these $\beta_i^\pm$ to ∂_0. Denote by $\bar{\beta}_i$ the concatenation $\gamma_i^+ \beta_i \gamma_i^-$ and note that we can assume each $\bar{\beta}_i$ to be a smooth embedding. Now there is a strong deformation retract

$$d_{12} : \Sigma_{12} \xrightarrow{\simeq} \partial_0 \cup \alpha_1 \cup \ldots \cup \alpha_N \cup \bar{\beta}_1 \cup \ldots \cup \bar{\beta}_M$$

given by applying first d_2 to the part in Σ_2, and then d_1' to the resulting space, where d_1' is the first map in the factorization as in lemma 2.2.11. By this we get a diagram as follows

$$\begin{tikzcd}
\mathrm{Map}_*(\Sigma_{12}/\Sigma_1, BG) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_*(\Sigma_{12}/\partial_0, BG) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_*(\Sigma_1/\partial_0, BG) \arrow[d, "\simeq"] \\
\Omega BG^M \arrow[r, "i_1"'] & \Omega BG^M \times \Omega BG^N \arrow[r, "\pi_2"'] & \Omega BG^N
\end{tikzcd}$$

where i_1 is the inclusion of the first factor, and π_2 is the projection to the second. The middle vertical map is restriction to $\partial_0 \cup \alpha_1 \cup \ldots \cup \alpha_N \cup \bar{\beta}_1 \cup \ldots \cup \bar{\beta}_M$. Here the first M factors of ΩBG corresponds to the $\bar{\beta}_i$ with $i = 1, \ldots, M$, and the last N factors corresponds to the α_j with $j = 1, \ldots, N$. The top row is the fibration sequence we are interested in as $\mathrm{Map}_*(\Sigma_{12}/\Sigma_1, BG)$ is homeomorphic to $\mathrm{Map}_*(\Sigma_2/\partial_1, BG)$, and the bottom row is a product fibration. We claim that this diagram commutes up to homotopy.

The right square commutes strictly, as restriction to Σ_1, followed by restriction to the arcs $\alpha_1 \cup \ldots \cup \alpha_N$, is the same as restriction first to the arcs $\alpha_1 \cup \ldots \cup \alpha_N \cup \bar{\beta}_1 \cup \ldots \bar{\beta}_M$ and then further to the arcs $\alpha_1 \cup \ldots \cup \alpha_N$.

The left square commutes up to homotopy, as the inclusion of the fibre $\mathrm{Map}_*(\Sigma_{12}/\Sigma_1, BG)$ followed by restriction, maps all the α_i and the $\gamma_j^{\pm}$, to the base point of BG, leaving us with only the $\bar{\beta}_j$ arcs, where the $\gamma_j^{\pm}$ part is constant. On the other hand restricting to $\bar{\beta}_1 \cup \ldots \cup \bar{\beta}_M$ with all the $\gamma_i^{\pm}$'s constant, and including this, provides loops in BG homotopic to $\beta_1 \cup \ldots \cup \beta_M$.

Thus the top fibration is a product fibration up to homotopy. Now recall that the left vertical isomorphism of the square in the proposition, is defined by choosing a set of generators $\alpha_1, \ldots, \alpha_{-\chi(\Sigma_1,\partial_0)}$ for $\pi_1(\Sigma_1/\partial_0)$ and a set of generators $\beta_1, \ldots, \beta_{-\chi(\Sigma_2,\partial_1)}$ for $\pi_1(\Sigma_2/\partial_1)$. Then we know that $\det H^1(\Sigma_2, \partial_1) \otimes \det H^1(\Sigma_1, \partial_0)$ is generated by

$$\beta_1 \wedge \ldots \wedge \beta_{-\chi(\Sigma_2,\partial_1)} \otimes \alpha_1 \wedge \ldots \wedge \alpha_{-\chi(\Sigma_1,\partial_0)}$$

where each generator is now to be understood as the cohomology class dual to the homology class of a representing arc. The lower isomorphism sends this to

$$\tilde{\beta}_1 \wedge \ldots \wedge \tilde{\beta}_{-\chi(\Sigma_2,\partial_1)} \wedge \alpha_1 \wedge \ldots \wedge \alpha_{-\chi(\Sigma_1,\partial_0)}$$

where $\tilde{\beta}_i$ is the image of β_i under the composition

$$H^1(\Sigma_2, \partial_1) \xleftarrow{\simeq} H^1(\Sigma_{12}, \Sigma_1) \longrightarrow H^1(\Sigma_{12}, \partial_0)$$

where the first map is the isomorphism given by excision, and the second map is given by some splitting as noted in proposition 1.3.4. Recall that the choice of splitting does not matter.

We already saw that the top isomorphism is given by the cross product, and so up to homotopy, a pair of maps from the (based) arcs $\alpha_1, \ldots, \alpha_{-\chi(\Sigma_1,\partial_0)}$ and $\beta_1, \ldots, \beta_{-\chi(\Sigma_2,\partial_1)}$ to BG, is sent to the product of maps from those, which is the same a map from the wedge of the arcs. The homotopy is given by the choice of $\gamma_1^{\pm}, \ldots, \gamma_M^{\pm}$ defining a lift of each β_i to $(\Sigma_{12}, \partial_0)$. Thus we get a new set of $-\chi(\Sigma_1, \partial_0) - \chi(\Sigma_2, \partial_1) = -\chi(\Sigma_{12}, \partial_0)$ arcs

$$\alpha_1, \ldots, \alpha_{-\chi(\Sigma_1,\partial_0)}, \bar{\beta}_1, \ldots, \bar{\beta}_{-\chi(\Sigma_2,\partial_1)}$$

which together defines the isomorphism Φ_{12}, by which $\det H^1(\Sigma_{12}, \partial_0)$ is generated by

$$\bar{\beta}_1 \wedge \ldots \wedge \bar{\beta}_{-\chi(\Sigma_2,\partial_1)} \wedge \alpha_1 \wedge \ldots \wedge \alpha_{-\chi(\Sigma_1,\partial_0)}$$

Now we use this particular choice of lifts of the β_i to define a map

$$H^1(\Sigma_2, \partial_1) \oplus H^1(\Sigma_1, \partial_0) \to H^1(\Sigma_{12}, \partial_0)$$

which includes the generators of the second summand, and uses the lift for a splitting on the first. Since the choice of lift did not matter for the lower isomorphism we set $\tilde{\beta}_i := \bar{\beta}_i$, which ensures that the diagram commutes. □

Just as there are certain composition properties as just shown, we have some monoidal structure. The next proposition is obvious from basic properties, and just included for completeness and comparison. We will not show it.

Proposition 2.2.12. *Let Σ', Σ'' be a pair of open-closed cobordisms with positive boundary, and denote by Σ disjoint union of these. Let $\partial'_{in}, \partial''_{in}, \partial_{in}$ denote the respective incoming boundaries. Denote the isomorphism Φ for each cobordism, by respectively Φ', Φ'' and just Φ. Then*

i) there is a natural isomorphism

$$H_{s'}(\mathrm{Map}_*(\Sigma'/\partial'_{in}, BG)) \otimes H_{s''}(\mathrm{Map}_*(\Sigma''/\partial''_{in}, BG)) \simeq H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG))$$

ii) there is a natural isomorphism

$$\det H^1(\Sigma', \partial'_{in})^{\otimes \dim G} \otimes \det H^1(\Sigma'', \partial''_{in})^{\otimes \dim G} \simeq \det H^1(\Sigma, \partial_{in})^{\otimes \dim G}$$

iii) the isomorphism of i) and ii) above are such that the following diagram commutes

$$\begin{tikzcd}
H_{s'}(\mathrm{Map}_*(\Sigma'/\partial'_{in}, BG)) \otimes H_{s''}(\mathrm{Map}_*(\Sigma''/\partial''_{in}, BG)) \arrow[r, "\simeq"] \arrow[d, "\Phi'\otimes\Phi''"'] & H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG)) \arrow[d, "\Phi"] \\
\det H^1(\Sigma', \partial'_{in})^{\otimes \dim G} \otimes \det H^1(\Sigma'', \partial''_{in})^{\otimes \dim G} \arrow[r, "\simeq"] & \det H^1(\Sigma, \partial_{in})^{\otimes \dim G}
\end{tikzcd}$$

With these propositions 2.2.9, 2.2.10 and 2.2.12, we can go straight to section 2.4 to build our operations as in [1], defining an ocHFT with no labels. Doing so, we remind the reader that $\mathcal{M}(\Sigma)$ is just the ordinary mapping space $\mathrm{Map}(\Sigma, BG)$ when we are working in the unlabelled case.

2.3 Labelled case

We will begin this section by introducing some terminology. The goal of the section is to develop a machinery which lets us mimic the unlabelled case, and in some cases reduce the statements about labelled spaces to statements about unlabelled spaces already shown.

First we show that $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$ is a fibration. Then we show that in is an orientable fibration, i.e. that it is a fibration such that the fibre has *rank 1 top homology*, the property introduced in section 1.3.1, and that $\pi_1(\mathcal{M}(\partial_{in}))$ acts trivially on the top homology of the fibre. Even the fact that this is a fibration is non-trivial in this labelled case.

As in the unlabelled case we proceed by showing that this is still an orientable fibration when we apply the Borel construction $(-)_{h\,\mathrm{Diff}^+\Sigma}$, and again this amounts to showing that $\mathrm{Diff}^+\Sigma$ acts trivially on the top homology of the fibre. We will also see why expression $\det \Sigma^{\underline{d}}$, is the right thing to consider when keeping track of orientations.

2.3.1 A fibration

In this subsection we introduce terminology with the goal to establish that $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$ is a fibration. We begin by defining

Definition 2.3.1. *A labelled map $f\colon (X, \{X_\alpha\}_{\alpha\in A}, \lambda) \to (Y, \{Y_\beta\}_{\beta\in B}, \lambda')$, between space that have the same set of labels L, is a map $f\colon X \to Y$ such that the restrictions $f|_l\colon X^l \to Y$, all have image $f|_l(X^l) \subset Y^l$.*

Further we need to say what we mean by a *labelled cofibration*. For us, a labelled cofibration shall mean cofibration of spaces $X \to Y$ such that each restriction $X^l \to Y^l$ is a cofibration, and such that the map

$$\begin{tikzcd} X^l \arrow[r] \arrow[d] & Y^l \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & X \coprod_{X^l} Y^l \arrow[dr, dotted] & \\ & & Y \end{tikzcd}$$

from the pushout to Y is a cofibration. Our goal is now to show

Proposition 2.3.2. *Let X, Y be spaces labelled by the connected subgroups of G, and let $j\colon X \to Y$ be a labelled cofibration. The induced map*

$$\mathcal{M}(j)\colon \mathcal{M}(Y) \to \mathcal{M}(X)$$

is a fibration.

To do so, we first recall the following classical result which we will not show ([14] proposition 15, compare with [15], I proposition 5.2).

Proposition 2.3.3. *If $j\colon W \to X$ is a cofibration and $q\colon E \to B$ is a fibration, then the induced map*

$$F_{j,q}\colon \operatorname{Map}(X, E) \to \operatorname{Map}(W, E) \times_{\operatorname{Map}(W,B)} \operatorname{Map}(X, B)$$

is a fibration.

proof of proposition 2.3.2. We have to produce a lift as indicated in the following diagram, with commuting outer square

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\sim"] & \mathcal{M}(Y) \arrow[d, "h"] \\ B \arrow[r] \arrow[ur, dotted] & \mathcal{M}(X) \end{tikzcd}$$

Such a lift corresponds to two lifts in the following diagram with the outer cube commuting, which agree when composed with g and h respectively. This follows directly from $\mathcal{M}(Y)$ and $\mathcal{M}(X)$ being pullbacks.

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dd] \arrow[dr, hook, "\sim"] & & \operatorname{Map}(Y, BG) \arrow[dr] \arrow[dd, "g"] & \\
& B \arrow[rr] \arrow[dd] \arrow[ur, dotted] \arrow[dl, dotted] & & \operatorname{Map}(X, BG) \arrow[dd] \\
\prod_{H \leq G} \operatorname{Map}(Y^H, BH) \arrow[rr, "h"] \arrow[dr] & & \prod_{H \leq G} \operatorname{Map}(Y^H, BG) \arrow[dr, "f"] & \\
& \prod_{H \leq G} \operatorname{Map}(X^H, BH) \arrow[rr] & & \prod_{H \leq G} \operatorname{Map}(X^H, BG)
\end{tikzcd}
$$

In this diagram both maps from B to the bottom right hand corner agrees, and we can first lift either along f, which is a fibration by proposition 2.3.3. By this lift we get a map from B to the pullback of the three spaces in the bottom face of the cube, and thus a unique map from $\prod_H \operatorname{Map}(Y^H, BH)$ to this pullback, which is a fibration by the homotopy lifting-extension property. Lifting the map from B to the pullback along this fibration gives a dotted arrow, making the left-hand triangle in the cube commute.

$$
\begin{tikzcd}
& B \arrow[dl, dotted] \arrow[d] \\
\prod_{H \leq G} \operatorname{Map}(Y^H, BH) \arrow[r] & \prod_{H \leq G} \operatorname{Map}(X^H, BH)
\end{tikzcd}
$$

Similarly, we get a map from B to the pullback of the three spaces in the right-hand face. Again the unique map from $\operatorname{Map}(Y, BG)$ to the pullback is a fibration, but this time by our definition of a labelled cofibration. The map from B to the pullback lifts along this fibration, and this lift gives the dotted arrow in the top face of the cube and makes the following triangle commute.

$$
\begin{tikzcd}
& \operatorname{Map}(Y, BG) \arrow[dr] & \\
B \arrow[ur, dotted] \arrow[rr] & & \operatorname{Map}(X, BG)
\end{tikzcd}
$$

$\square$

In particular the map $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$ induced by the inclusion of the incoming boundary into Σ, is a fibration. Note that we allow for possible empty subspaces in the definition of a labelled space, and in our case this is indeed a possibility when a labelled boundary component has no incoming boundary. Because of this we can not get the fibre of $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$ as the pullback of fibres from the fibrations defining it. We now turn our attention to this fibre, with the goal to establish the properties which eventually allow us to define $(in_{h\,\mathrm{Diff}^+\,\Sigma})_!$.

2.3.2 Properties of the fibre

As in the unlabelled case we first establish a more handy model for our surfaces, which in turn will give us a way to investigate the properties of the fibre. Our first goal in this section is to show the following proposition inspired by theorem 1.3.7, for which we explain the terminology after it is stated.

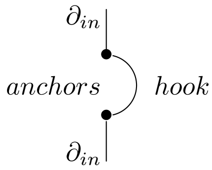
Proposition 2.3.4. *A labelled open-closed cobordism* Σ *with positive boundary, is labelled homotopy equivalent to a labelled arc decomposition* A_Σ*, relative to the incoming boundary* $\partial_{in}\Sigma$

First we to define what we mean by labelled homotopy equivalent.

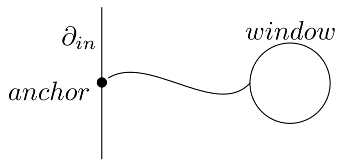
Definition 2.3.5. *Let* X *and* Y *be a pair of labelled space with the same set of labels* L*.* X *and* Y *are labelled homotopy equivalent, if there is a labelled homotopy equivalence* $f\colon X \to Y$*, i.e. if the restrictions* $f|_l\colon X^l \to Y^l$ *are homotopy equivalences for all* $l \in L$*.*

Let $f\colon X \to Y$ be a labelled homotopy equivalence. The functors $\mathrm{Map}(-, BH)$ preserve homotopy equivalences, and $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are homotopy pullbacks, so the induced map here denoted by $\mathcal{M}(f)$, is a homotopy equivalence. Next we define what a labelled arc decomposition is. Where in the unlabelled case we could get away with considering just arcs, the labels complicate matters some.

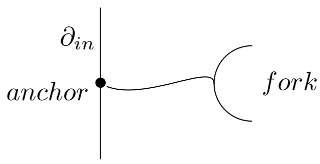
Consider the following four types of spaces, which can all be found as subspaces of a labelled open-closed manifold



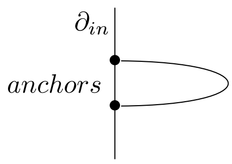
type (i)



type (ii)



type (iii)



type (iv)

The spaces (i) will be called *labelled hooks*, (ii) are *labelled windows*, anchored to the incoming boundary by an unlabelled arc, (iii) are *labelled forks*, anchored to the incoming boundary by an unlabelled arc, and (iv) will not need a special name. Together we call the labelled spaces in (i)-(iv) for *traits* (seeing as they appear as the essential features, or traits, of a labelled open-closed manifold as we will argue), and the points by which they attach to the incoming boundary we will call *anchors*. To be more precise we have

Definition 2.3.6. *A* labelled arc decomposition *is a labelled 1-manifold* B *with boundary, with the boundary components as the set of labelled subspaces, and*

(i) labelled intervals, with each end glued to a distinct boundary point of B *matching the labelling*

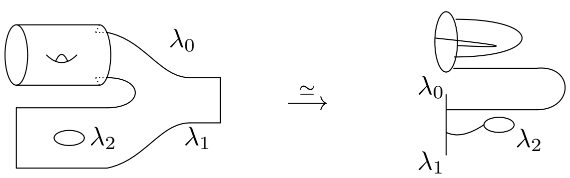
(ii) labelled circles with an unlabelled interval attached by one end, glued to distinct points of $B - \partial B$

(iii) labelled intervals with an unlabelled interval attached by one end, glued to to distinct points of $B - \partial B$

(iv) unlabelled intervals, with ends glued to distinct points of B

with enumeration in obvious connection to the figures above. Naturally, we will say that a labelled arc decomposition A is a labelled arc decomposition *of* a labelled open-closed surface Σ, if A is labelled homotopy equivalent to Σ. We may also write A_Σ, for A in that case, as we note that this is well-defined up to homotopy equivalence.

Example 2.3.7 The surface depicted in section 1.1 has the following labelled arc decomposition,



consisting of the incoming boundary, one anchored window and three arcs of type (iv).

The proposition 2.3.4 will follow as a corollary of the following proposition

Proposition 2.3.8. *Let Σ be a labelled open-closed cobordism with positive boundary, and let X be a subset of the incoming boundary, such that X intersects every component of it non-trivially. There are arcs $\alpha_1, \ldots, \alpha_N$ as in proposition 1.3.11, such that the end points of any single one of these arcs are either both in X, or one in X and the other in a window or a fork. Further each window and each fork has precisely one arc ending on it and*

$$d\colon \Sigma \to \partial_{in} \cup hooks \cup windows \cup forks \cup \alpha_1 \cup \ldots \cup \alpha_N$$

is a strong deformation retract.

Proof. Let Σ be a labelled open-closed cobordism. In particular Σ is an in-surface, with

$$\partial_{in} \cup hooks \cup windows \cup forks$$

marked as incoming. For now we ignore the labelling. Let X be a subset of the incoming boundary, such that X intersects every component of it non-trivially.

The proof is by induction on $-\chi(\Sigma, \partial_{in} \cup hooks \cup windows \cup forks)$. For the base case we note that there are no windows and forks because the boundary must be connected. Thus this follows from lemma 1.3.8, seeing as we can avoid the hooks. If $\chi(\Sigma, \partial_{in} \cup hooks \cup windows \cup forks) < 0$ then we apply lemma 1.3.12 with

$$V = X \cup windows \cup forks, \qquad \text{and} \qquad U = X$$

and call the arc produced for α_1. By the lemma, α_1 has either both ends in X, or one in X and another in either a window or a fork. Now cut along α_1 to obtain an in-surface Σ' for which

$$\begin{aligned}\chi(\Sigma', (\partial_{in} \cup hooks \cup windows \cup forks)') = \\ \chi(\Sigma, \partial_{in} \cup hooks \cup windows \cup forks) + 1\end{aligned}$$

and which comes with a map $r\colon \Sigma' \to \Sigma$ given by identifying α_{1-} and α_{1+}. If we set $X' = r^{-1}X$ then by the induction hypothesis the statement is true for the pair Σ', X', and by regluing along the cut arc α_1, we get the statement for Σ, X as in the proof of proposition 1.3.11. □

With the same notation we continue with

proof of proposition 2.3.4. Set $X = \partial_{in}$. Then by the proof of lemma 2.2.11, the strong deformation retract d can be taken to factor through

$$d'\colon \Sigma \to \partial_{in} \cup hooks \cup windows \cup forks \cup \alpha_1 \cup \ldots \cup \alpha_N \cup \partial_{rest}$$

where ∂_{rest} is the remaining part of the free boundary, which is a set of labelled arcs with one end in the outgoing boundary, and one in ∂_{in}. Now remembering the labels again, the strong deformation retract d is then a labelled homotopy equivalence, as each of these extra labelled arcs retract within themselves, and every other labelled component is fixed. □

Having established a nicer model for the surfaces in question, we can now begin to examine the fibre of $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$. Our goal is to establish

Proposition 2.3.9. *The fibration induced by the inclusion of the incoming boundary*

$$\mathcal{M}(\Sigma)_c \to \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$$

is an orientable fibration.

First consider the following diagram, which is the labelled analogue of the diagram (2.2).

$$\begin{tikzcd}
\mathcal{M}(\Sigma) \arrow[r] \arrow[d, two heads] & \mathcal{M}(A_\Sigma) \arrow[r] \arrow[d, two heads] & \mathcal{M}(traits) \arrow[d, two heads] \\
\mathcal{M}(\partial_{in}) \arrow[r, equal] & \mathcal{M}(\partial_{in}) \arrow[r] & \mathcal{M}(anchors)
\end{tikzcd} \tag{2.7}$$

The right hand fibration is the map induced from the inclusions of anchors in traits. This is a fibration by proposition 2.3.3. We will show that the right square is a pullback, so that the the two fibrations have the same fibre, and by proposition 2.3.4 above the top left map is a homotopy equivalence and the left square commutes. Thus the left and right fibrations have the same fibre.

Lemma 2.3.10. *The right-hand square of diagram* (2.7) *is a pullback.*

Proof. Consider a map $X \to \mathcal{M}(A_\Sigma)$. We will show that the data of such a map is the same as that of a map from X to each of the spaces $\mathcal{M}(traits)$ and $\mathcal{M}(\partial_{in})$, such that these agree on $\mathcal{M}(anchors)$.

By definition a map $X \to \mathcal{M}(A_\Sigma)$ is the same as a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Map}(A_\Sigma, BG) \arrow[d] \\ \prod \mathrm{Map}(A_\Sigma^H, BH) \arrow[r] & \prod \mathrm{Map}(A_\Sigma^H, BG) \end{tikzcd} \tag{2.8}$$

The following square is a pushout for any label H (including G itself, corresponding to no label)

$$\begin{tikzcd} anchors^H \arrow[r] \arrow[d] & traits^H \arrow[d] \\ \partial_{in}^H \arrow[r] & A_\Sigma^H \end{tikzcd}$$

and the $\mathrm{Map}(-, BH)$ functors take pushouts to pullbacks. Thus (2.8) commutes if and only if the four squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Map}(traits, BG) \arrow[d] \\ \prod \mathrm{Map}(traits^H, BH) \arrow[r] & \prod \mathrm{Map}(traits^H, BG) \end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Map}(\partial_{in}, BG) \arrow[d] \\ \prod \mathrm{Map}(\partial_{in}^H, BH) \arrow[r] & \prod \mathrm{Map}(\partial_{in}^H, BG) \end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Map}(traits, BG) \arrow[d] \\ \mathrm{Map}(\partial_{in}, BG) \arrow[r] & \mathrm{Map}(anchors, BG) \end{tikzcd} \tag{2.11}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod \mathrm{Map}(traits^H, BH) \arrow[d] \\ \prod \mathrm{Map}(\partial_{in}^H, BH) \arrow[r] & \prod \mathrm{Map}(anchors^H, BH) \end{tikzcd} \tag{2.12}$$

all commute. The top map of (2.8) corresponds to the top maps of (2.9) and (2.10), together with commutativity of (2.11), and the left map of (2.8) corresponds to the left maps of (2.9) and (2.10), together with commutativity of (2.12).

Also, the first two squares correspond to respectively maps $X \to \mathcal{M}(traits)$ and $X \to \mathcal{M}(\partial_{in})$, and the last two express the commutativity when arranging these maps in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{M}(traits) \arrow[d] \\ \mathcal{M}(\partial_{in}) \arrow[r] & \mathcal{M}(anchors) \end{tikzcd}$$

where the maps to $\mathcal{M}(anchors)$ are those coming from the induced fibrations when applying $\mathrm{Map}(-, BH)$ to the inclusions of anchors in traits and incoming boundary respectively. This commutativity is exactly the condition that $\mathcal{M}(A_\Sigma)$ is the pullback we wanted. □

Now that we have established that the right square of (2.7) is a pullback, we examine the fibre of the right fibration in the diagram. In the unlabelled case this was easy, but there is a little more work to do in the labelled case as we shall see now.

The total space is a pullback of mapping spaces, where the maps are from disjoint unions of spaces. Therefore it is a product with one factor for each component in the disjoint union, and the fibre over the constant map to the base point $\mathcal{M}(traits)_c$ (with slight abuse of notation), is the same product with the free end of each arc now mapping to the base point of BG. Just as there are four types of traits (i)-(iv) 2.3.6, there are four types of factors in this product to study.

Lemma 2.3.11. *For each of the four types of labelled spaces from definition 2.3.6, the fibre of the fibration induced by the inclusion*

$$in\colon \mathcal{M}(traits) \to \mathcal{M}(anchors),$$

has rank 1 top homology.

Factors coming from traits of type

(i) are based loops in BH, when labelled by H

(ii) are homotopy equivalent to paths in BG from the base point, to a point for which we know a lift to BH, when the interval is labelled by H. The homotopy equivalence is given by contracting the interval to the point where it meets the unlabelled interval.

(iii) are treated in detail below

(iv) are treated as factors of type (i) with label G

proof of lemma 2.3.11. A factor of type (i) is a copy of $\Omega BH \simeq H$ which is a connected Lie group. A factor of type (ii) is homotopy equivalent to the homotopy fibre of the standard fibration $BH \to BG$. As such it is homotopy equivalent to the actual fibre G/H, which is a connected manifold. To verify these properties for a factor of type (iii), we note that an anchored window labelled by H, say W_H, admits a fibration $\mathcal{M}(W_H) \to \mathcal{M}(I_1^H)$ induced by the inclusion of the arc labelled by 1 and H at either end, into the whole space. The base $\mathcal{M}(I_1^H)$ is the homotopy fibre of the standard fibration $BH \to BG$, so it is homotopy equivalent to the actual fibre G/H. The fibre is just based loops in BH. Thus we get a fibration sequence

$$\Omega BH \longrightarrow \mathcal{M}(W_H) \longrightarrow \mathcal{M}(I_1^H)$$

where both base and fibre has rank 1 top homology, and if we show that the action of the fundamental group on the homology of the fibre is trivial, we can use the Serre spectral sequence in homology, to get

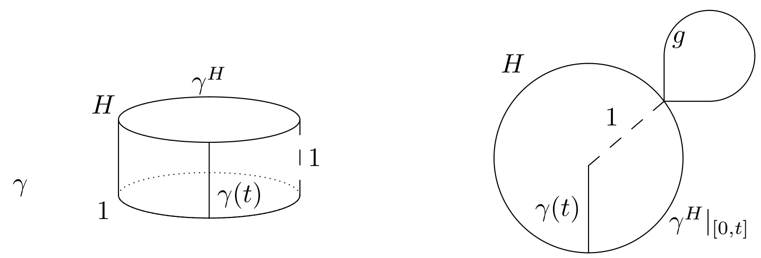
$$E^2_{p,q} = H_p(G/H; H_q(G)) \simeq H_p(G/H) \otimes H_q(G) \simeq \mathbb{F} \otimes \mathbb{F} \simeq \mathbb{F}$$

with $p = \dim G - \dim H$ and $q = \dim G$, as by proposition 1.3.1. Everything above and to the right of this entry is zero, and so $H_{p+q}(\mathcal{M}(M_H)) \simeq \mathbb{F}$ is the top homology group.

The action comes from the assignment of a based path γ in the base, to a homotopy of maps from the fibre to the total space $h_t\colon F_{\gamma(0)} \to E$ such that $h_t(F_{\gamma(0)}) \subset F_{\gamma(t)}$, obtained via the homotopy lifting property. In particular h_1 is a self homotopy equivalence of the fibre $F_{\gamma(0)}$ inside the total space, when γ is a based loop. This corresponds to a map

$$\hat{h}\colon \Omega\mathcal{M}(I_1^H) \times \Omega BH \to \mathcal{M}(W_H)$$

which for a fixed $\gamma \in \Omega\mathcal{M}(I_1^H)$ has the properties above. The map $\hat{h}_t(\gamma, g)$ is then a loop in BH with $\gamma(t)$ attached at the start of the loop. Note the H end of I_1^H trails a based loop γ^H in H under the map γ, and so we set $\hat{h}_t(\gamma, g)$ to be $\gamma^H|_{[0,t]} \centerdot g \centerdot \overline{\gamma^H|_{[0,t]}}$ on the circle. Here $\centerdot$ denotes concatenation, and overline is the path trailed backwards. Then $\hat{h}_t$ has the right properties, and h_1 is just conjugation of g by γ^H in ΩBH. Thus the action of $\pi_1(\Omega\mathcal{M}(I_1^H))$ on $H_*(\Omega BH)$ is trivial, as H is connected. Below we have tried to illustrate how the action arises, by first giving a picture of γ, and next collapsing the bottom part labelled by 1, and attaching the loop g.



□

From this proposition 2.3.9 follows easily.

proof of proposition 2.3.9. As noted above the fibre of *in* is a product. By lemma 2.3.11 each factor in the fibre has rank 1 top homology, by which we conclude that the fibre itself has rank 1 top homology. By diagram (2.7), this fibre is the same as $\mathcal{M}(\Sigma)_c$ (the fibre over the constant map) which then has rank 1 top homology. Finally the action of the fundamental group is trivial since it is pulled back from the action of $\pi_1(\mathcal{M}(anchors))$, which is classes in

$$\Omega\mathcal{M}(anchors) \simeq \Omega\left(\prod_{H \text{ label}} BH \times \prod_{\text{unlabelled}} BG\right) \simeq *$$

since all H and G are connected. $\square$

Next we will show that the action of $\mathrm{Diff}^+\Sigma$ on the top homology of the fibre of in is trivial, which implies that $in_{h\,\mathrm{Diff}^+\Sigma}$ is an oriented fibration. First we define this action in analogue with that in the unlabelled case.

A diffeomorphism $\psi \in \mathrm{Diff}^+(\Sigma, \partial)$ acts on $\mathrm{Map}(\Sigma, BG)$ by $\mathrm{Map}(\psi, BG)$, and acts trivially on $\mathrm{Map}(\Sigma^H, BH)$ and $\mathrm{Map}(\Sigma^H, BG)$ for any label H. This defines a homeomorphism on the pullback $\mathcal{M}(\Sigma)$, which equals $\mathrm{Map}(\psi, BG)$ everywhere but on the boundary, where it is the identity map.

In order to examine the action on the top homology of the fibre, we will establish a diagram as follows

$$\begin{tikzcd}
 & \mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG) \arrow[d] & \\
\Pi \arrow[r] & \mathcal{M}'(\Sigma)_c \arrow[r, two heads, "f"] \arrow[d, "g"] & \mathcal{M}(\Sigma)_c \\
 & \Omega B\Pi_{cl} &
\end{tikzcd} \tag{2.13}$$

for which we now define the spaces and maps.

The space $\mathcal{M}'(\Sigma)$ is obtained from $\mathcal{M}(\Sigma)$ in the following way. In each component of the free boundary of Σ which is not path connected to ∂_{in}, we choose a point. Evaluating at these points defines a map from $\mathcal{M}(\Sigma) \to B\Pi$, where Π is the product of the labels on the boundaries on which we have chosen points. The standard fibration $E\Pi \to B\Pi$ then pulls back to a fibration $f\colon \mathcal{M}'(\Sigma) \to \mathcal{M}(\Sigma)$, with fibre Π. Composing with the fibration $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$, we have two fibrations over $\mathcal{M}(\partial_{in})$, and we see that f is fibre-preserving with respect to the fibration $in\colon \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$. Letting c denote the constant map to the base point of BG, and taking this as a base point for $\mathcal{M}(\partial_{in})$, the pullback of f along the inclusion of the fibre $\mathcal{M}(\Sigma)_c \to \mathcal{M}(\Sigma_{in})$, is a fibration $f_c\colon \mathcal{M}'(\Sigma)_c \to \mathcal{M}(\Sigma)_c$ with the same fibre is as f, and we will just denote this by f later.

$$\begin{tikzcd}
\Pi \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}'(\Sigma)_c \arrow[r, "f_c"] \arrow[d] & \mathcal{M}(\Sigma)_c \arrow[d] \\
\Pi \arrow[r] & \mathcal{M}'(\Sigma) \arrow[r, "f"] \arrow[dr] & \mathcal{M}(\Sigma) \arrow[d, "in"] \\
 & & \mathcal{M}(\partial_{in})
\end{tikzcd}$$

This defines both $\mathcal{M}'(\Sigma)_c$ and the horizontal fibration in the diagram (2.13).

Next we define the vertical fibration in the diagram (2.13). The inclusion of the free boundary induces a fibration $g\colon \mathcal{M}'(\Sigma) \to \mathcal{M}'(\partial_{free})$, where $\mathcal{M}'(\partial_{free})$ denotes $\mathcal{M}(\partial_{free})$ with the preferred points. As with f, this gives a fibration on the fibre $\mathcal{M}'(\Sigma)_c$, in which the free boundary of Σ occurs as components of four types. By the previously introduced terminology we can easily distinguish these.

A closed component of the free boundary of Σ is a *window*. A window labelled by H, is a based loop in BH when seen in $\mathcal{M}'(\Sigma)_c$. An open component with both boundary points in ∂_{in} is a *hook*. A hook labelled by H is also a based loop in BH when seen in $\mathcal{M}'(\Sigma)_c$, since we are in the fibre over the constant map to the basepoint. An open component with one boundary point in ∂_{in} and one in ∂_{out} labelled by H, is a based interval in BH, since ∂_{in} is sent to the base point. Components of this type will not play a role in the following. Finally an open component with both boundary points in ∂_{out}, is a *fork*. A fork labelled by H, is a based interval in BH when seen in $\mathcal{M}'(\Sigma)_c$. In the last two cases, based intervals contract to the base point, so they are homotopy equivalent to points. Note that with this terminology Π is the product of labels on windows and forks.

We can now see g as a fibration to $\Omega B\Pi_{cl} \simeq \mathcal{M}'(\partial_{free})$, where Π_{cl} is the product of labels, labelling windows and hooks (the last two types were contractible). The fibre is the space of maps from Σ to BG, with the restriction that all of the free boundary and incoming boundary is mapped to the base point. This is homotopy equivalent to $\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)$, and with these definitions of f and g we get the diagram (2.13).

Lemma 2.3.12. *The action of* $\mathrm{Diff}^+(\Sigma, \partial)$ *on the top homology of* $\mathcal{M}(\Sigma)_c$ *is trivial if and only if it is trivial on the top homology of* $\mathcal{M}'(\Sigma)_c$.

Proof. The fibration f is a pullback of the standard fibration $E\Pi \to B\Pi$, and $\pi_1(B\Pi) = \pi_0(\Pi) = 0$ since Π is a product of connected subgroups of G. Therefore the action of $\pi_1(\mathcal{M}(\Sigma)_c)$ on $H_*(\Pi)$ is trivial.

Further f is $\mathrm{Diff}^+(\Sigma, \partial)$-equivariant since total space and base only differs on maps on the boundary of Σ which is fixed. Consider the Serre spectral sequence in homology associated to f. We know that both base and fibre has rank 1 top homology, and since the action of $\pi_1(\mathcal{M}(\Sigma)_c)$ is trivial we get

$$H_{p+q}(\mathcal{M}'(\Sigma)_c) \simeq E^{\infty}_{p,q} \simeq E^2_{p,q} = H_p(\mathcal{M}(\Sigma)_c; H_q(\Pi)) \simeq H_p(\mathcal{M}(\Sigma)_c) \otimes H_q(\Pi)$$

as by proposition 1.3.1. Here p and q are the respective top degrees of homology, in particular $q = \dim \Pi$.

Acting by an element $f \in \mathrm{Diff}^+ \Sigma$ induces a isomorphism on all of these homology groups, and naturality of the Serre spectral sequence and the Künneth isomorphism ensures that these actions are coherent. Therefore we see that the action of $\mathrm{Diff}^+ \Sigma$ is trivial on $H_{p+q}(\mathcal{M}'(\Sigma)_c)$ if and only if it is on both $H_p(\mathcal{M}(\Sigma)_c)$ and $H_q(\Pi)$, of which we know it is trivial on the last. $\square$

To establish that the action is trivial on $H_{p+q}(\mathcal{M}'(\Sigma)_c)$, we now turn our attention to the vertical fibration in the cross (2.13) above.

Noting that Σ is an in-surface when we disregard any labelling, and set the incoming boundary to be $\partial_{in} \cup \partial_{free}$, we get from proposition 2.2.1 that $\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)$ has rank 1 top homology, and that the top degree is $s := -\chi(\Sigma, \partial_{in} \cup \partial_{free}) \cdot \dim G$. Also, the base of g is homotopy equivalent to Π_{cl} which is a Lie group, so this has rank 1 top homology bounded by $r := \dim \Pi_{cl}$. As above we want to apply proposition 1.3.1, but for this we must have trivial action of $\pi_1(\Omega B\Pi_{cl})$ on $H_*(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$.

Lemma 2.3.13. *The action of* $\pi_1(\Omega B\Pi_{cl})$ *on* $H_*(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$ *is trivial.*

Proof. As described earlier this action comes from the assignment of a based path γ in the base, to a homotopy of maps from the fibre to the total space $h_t\colon F_{\gamma(0)} \to E$ such that $h_t(F_{\gamma(0)}) \subset F_{\gamma(t)}$. This corresponds to a map

$$\hat{h}_t\colon \Omega(\Omega B\Pi_{cl}) \times \mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG) \to \mathcal{M}'(\Sigma)_c$$

which for any point in $\Omega(\Omega B\Pi_{cl})$ has the above properties. We will describe the map $\hat{h}_t$ here. To a factor H in Π_{cl} corresponds either a window, or a hook labelled by H. These labelled boundary components are disjoint, and the factors of Π_{cl} each gives rise to an action on one component ∂^H in the following way.

There is a small closed neighborhood U of any closed boundary component with label H, diffeomorphic to a cylinder $\psi\colon U \xrightarrow{\cong} S^1 \times [0,1]$ with the boundary mapped to $S^1 \times \{1\}$. For $t \in (0,1]$, define $d_t\colon \Sigma - \psi^{-1}(S^1 \times (1-t,1]) \to \Sigma$ to be the obvious diffeomorphism stretching to all of Σ, and let d_0 be the identity map on Σ.

The boundary of Σ is parametrized, so we identify the closed component labelled by H with S^1. For $\gamma \in \Omega(\Omega BH)$ and $f \in \mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)$ we then define

$$\hat{h}_t(\gamma, f)(x) = \begin{cases} (\gamma|_{[0,t]} \circ \psi)(x) & x \in \psi^{-1}(S^1 \times [1-t,1]) \\ (f \circ d_t)(x) & x \in \Sigma - \psi^{-1}(S^1 \times (1-t,1]) \end{cases}$$

This is well-defined since γ is a based loop, f is a based map, and BG and BH share the same base point. Clearly this is a homotopy in the total space with $\hat{h}_t(\gamma, (\mathcal{M}'(\Sigma)_c)_{\gamma(0)}) \subset (\mathcal{M}'(\Sigma)_c)_{\gamma(t)}$. The claim is that this action is trivial.

By theorem 2.3.4, the in-surface Σ is labelled homotopy equivalent to a labelled arc decomposition relative to the incoming boundary of Σ. Disregarding labelling, and treating the free closed boundary component previously labelled by H as incoming, the collar U on which γ is defined is collapsed by such a homotopy equivalence u. We obtain this in the following way.

The closure of $\Sigma - U$ is an in-surface, when we set the incoming part of the boundary to be $\partial_{in}\Sigma$ and the new boundary at U. By theorem 2.3.4 we can find a homotopy equivalence u' from this in-surface to an arc decomposition A of it. Note that for a component of the incoming boundary with multiple end points of one or more arcs α_i, we can replace these α_i by homologous arcs α_i', such that all end points of α_i' meet that component in exactly one point of our choice. Also an obvious consequence of the theorem 1.3.7 is that any incoming boundary component must have at least one arc ending on it.

Now set u to be the composition of u' and a deformation retract e of U to the boundary such that the point with the arcs attached a, trails the path which is sent to the base point by $\gamma \circ \psi$, when looking at $t \mapsto e_t(a)$. Denote this trailed path by ε, and denote by v' an inclusion of the arc decomposition A in the closure of $\Sigma - U$ which is a homotopy inverse of u'. Now set v to be the inclusion of an arc decomposition C in Σ which is v' everywhere but on the arc and circle mapped to the boundary at U. By v' there is set of arcs ending at a, and as paths in Σ we concatenate these by ε and map the circle to the boundary by U. This v is a homotopy inverse to u, and we can translate the action given by $\hat{h}_1$ to an action on C through u and v, in the sense that for any choice of γ, we have a diagram commuting up to homotopy

$$\begin{tikzcd}
\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG) \arrow[r, "\hat{h}_1(\gamma,-)"] & \mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG) \arrow[d, "v^*"] \\
\mathrm{Map}_*(C/(\partial_{in} \cup \partial_{free}), BG) \arrow[u, "u^*"] \arrow[r, "t(\gamma)"'] & \mathrm{Map}_*(C/(\partial_{in} \cup \partial_{free}), BG)
\end{tikzcd}$$

We see that $(f \circ v)$ is homotopic to $\hat{h}_1(\gamma, f) \circ v$ since they are both constant at the base point on the circle, and on the arc affected, $(f \circ v)$ is a restriction of f, and $\hat{h}_1(\gamma, f) \circ v$ a path homotopic to a restriction of f, concatenated by a constant path, and so is homotopic to a restriction of f. So the translated action $t(\gamma)$ on the arc decomposition is trivial up to homotopy, and thus the action defined by $\hat{h}_1$ of $\pi_1(\Omega BH)$ on $H_*(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$ is trivial. □

Now we get that

Lemma 2.3.14. *The action of $\pi_1(\Omega B\Pi_{cl})$ on the top homology of the fibre $\mathcal{M}'(\Sigma)_c$ is trivial if and only if the action is trivial on the top homology of $\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)$.*

Proof. As for f, the fibration g is $\mathrm{Diff}^+(\Sigma, \partial)$-equivariant

$$\begin{aligned} H_{r+s}(\mathcal{M}'(\Sigma)_c) \simeq E^\infty_{r,s} = E^2_{r,s} &= H_r(\Omega B\Pi_{cl}; H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))) \\ &\simeq H_r(\Omega B\Pi_{cl}) \otimes H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)) \end{aligned}$$

Again we use naturality of the spectral sequence and the Künneth isomorphism to conclude that $\mathrm{Diff}^+ \Sigma$ acts trivially on $H_{r+s}(\mathcal{M}'(\Sigma)_c)$ if and only if it acts trivially on both $H_r(\Omega B\Pi_{cl})$ and $H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$. It is obviously so for $H_r(\Omega B\Pi_{cl})$, which shows the lemma. □

Now we finally have the labelled version of proposition 2.2.9.

Proposition 2.3.15. *The fibration $in_{hDiff^+\Sigma}$ induced by the inclusion of the incoming boundary*

$$\mathcal{M}(\Sigma)_c \longrightarrow \mathcal{M}(\Sigma)_{h\,\mathrm{Diff}^+ \Sigma} \longrightarrow B\,\mathrm{Diff}^+ \Sigma \times \mathcal{M}(\partial_{in})$$

is an oriented fibration.

Proof. By proposition 2.2.7, $\mathrm{Diff}^+ \Sigma$ acts on $H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$ trivially, and by lemmas 2.3.13 and 2.3.14 above the action on the top homology of $\mathcal{M}(\Sigma)_c$ is trivial. Now

$$\pi_1(BDiff^+\Sigma \times \mathcal{M}(\partial_{in})) \simeq Diff^+\Sigma \times \pi_1(\mathcal{M}(\partial_{in}))$$

and both factors act trivially on the top homology of the fibre, which is of rank 1 by proposition 2.3.9. □

2.3.3 Composition and monoidality

Our next task is to establish propositions as 2.2.10 and 2.2.12 in the unlabelled case. First off we need an isomorphism as in proposition 2.2.3. First recall that we defined $\det \Sigma^{\underline{d}}$, in section 1.1, to be the weighted line

$$\det H^*(\Sigma, \partial_{in})^{\otimes -\dim G} \otimes \bigotimes_{H \leq G} \det H^*(\Sigma^H, \partial_{in}^H)^{\otimes(\dim G - \dim H)}$$

where we tensor over all labelled components (the labelling set still being the appropriate subgroups of G).

Proposition 2.3.16. *There is a well-defined isomorphism*

$$\Psi \colon H_p(\mathcal{M}(\Sigma)_c) \xrightarrow{\simeq} \det \Sigma^{\underline{d}}$$

By examining the diagram (2.13), we have established the isomorphism

$$H_p(\mathcal{M}(\Sigma)_c) \otimes H_q(\Pi) \simeq H_r(\Pi_{cl}) \otimes H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$$

This is even an isomorphism of weighted lines when we take the weights to be the degree of the homology. We will now examine this a bit more carefully to produce the desired isomorphism.

First recall that $\Pi = \Pi_w \times \Pi_f$, where Π_w is the product of factors (labels) in Π coming from closed boundary components (windows), and Π_f is the product of factors in Π coming from open boundary components (forks). Similar we have $\Pi_{cl} = \Pi_w \times \Pi_h$, where Π_w is as above, and Π_h is the product of factors coming from open boundary with both ends in the incoming boundary (hooks).

Now we have

$$\begin{aligned} H_p(\mathcal{M}(\Sigma)_c) \otimes & H_{q_w}(\Pi_w) \otimes H_{q_f}(\Pi_f) \\ & \simeq H_{r_w}(\Pi_w) \otimes H_{r_h}(\Pi_h) \otimes H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG)) \end{aligned}$$

where all homology is still the top degree. Noting that $q_w = r_w$, we can reduce this by tensoring with the inverse of the weighted line $H_{q_w}(\Pi_w) \otimes H_{q_f}(\Pi_f)$, to get by canonical isomorphisms

$$H_p(\mathcal{M}(\Sigma)_c) \simeq H_{r_h}(\Pi_h) \otimes H_{q_f}(\Pi_f)^* \otimes H_s(\mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG))$$

From this we are not quite able to produce an isomorphism as we want yet, but note that we can control what happens when we relabel Σ by the following double cross

$$\begin{tikzcd}
 & \Omega B\tilde{\Pi}_{cl} & \\
\tilde{\Pi} \arrow[r] & \mathcal{M}'(\tilde{\Sigma})_c \arrow[u, two heads] \arrow[r, two heads] & \mathcal{M}(\tilde{\Sigma})_c \\
 & \mathrm{Map}_*(\Sigma/(\partial_{in} \cup \partial_{free}), BG) \arrow[u] \arrow[d] & \\
\Pi \arrow[r] & \mathcal{M}'(\Sigma)_c \arrow[r, two heads] \arrow[d, two heads] & \mathcal{M}(\Sigma)_c \\
 & \Omega B\Pi_{cl} &
\end{tikzcd}$$

Here $\tilde{\Sigma}$ is the same open-closed cobordism as Σ but with a different labelling, giving different products of labels $\tilde{\Pi}$ and $\tilde{\Pi}_{cl}$ with each factor in these products corresponding to exactly one of the same type in Π and Π_{cl}. In particular we are interested in the case where $\tilde{\Sigma}$ is the unlabelled version of Σ. I.e. $\tilde{\Pi}$ and $\tilde{\Pi}_{cl}$ are both just products of G's, and $\mathcal{M}(\tilde{\Sigma})_c = \mathrm{Map}_*(\Sigma/\partial_{in}, BG)$. For the respective top degree homology groups, which are all of rank 1, we then get

$$\psi\colon H_p(\mathcal{M}(\Sigma)_c) \xrightarrow{\simeq} H_s(\mathrm{Map}_*(\Sigma/\partial_{in}, BG)) \otimes H_{r_f}(\Pi_f)^* \otimes \bigotimes_{forks} H_{\dim G}(G)$$
$$\otimes H_{q_h}(\Pi_h) \otimes \bigotimes_{hooks} H_{\dim G}(G)^*$$

by the same type of simple arithmetic of weighted lines as we did above. This leaves us with the task to reduce the right hand side to some expression of determinants.

From the proposition 2.2.3 in the unlabelled case we know the first factor. The factor $H_{r_h}(\Pi_h)$ is isomorphic to the tensor product with one factor $H_{\dim H}H$ for each hook labelled by an oriented copy of $H \leq G$. Fixing such a hook Σ^H, the line $H_{\dim H}(H)$ of weight $\dim H$ is isomorphic to $\mathbb{F}[\dim H]$ with the isomorphism given by the (predetermined) choice of orientation. The graded vector space $H^*(\Sigma^H, \partial_{in}^H)$ is concentrated in degree 1, and has a single tautological generator in this degree. Thus $\det H^*(\Sigma^H, \partial_{in}^H)^{\otimes -\dim H}$ is canonically isomorphic to $\mathbb{F}[\dim H]$.

Similarly $H_{q_f}(\Pi_f)^*$ is isomorphic to the tensor product with one factor $H_{\dim F}(F)^*$ for each fork labelled by an oriented copy of $F \leq G$. Fixing such a fork Σ^F, the line $H_{\dim F}(F)^*$ of weight $-\dim F$ is isomorphic to $\mathbb{F}[-\dim F]$ with the isomorphism given by the choice of orientation. The graded vector space $H^*(\Sigma^F, \partial_{in}^F)$ is concentrated in degree 0, and has a single tautological generator in this degree. Thus $\det H^*(\Sigma^F, \partial_{in}^F)^{\otimes -\dim F}$ is canonically isomorphic to $\mathbb{F}[-\dim F]$.

By setting $\Psi := \Phi \otimes \bigotimes_H c_H \circ \psi$, where c_H either the canonical isomorphism $H_{\dim H}(H) \to \det H^*(\Sigma^H, \partial_{in}^H)$ or $H_{\dim H}(H)^* \to \det H^*(\Sigma^H, \partial_{in}^H)$ just established above, depending on whether H is a hook or fork, we have now shown proposition 2.3.16, and have a isomorphism

$$\Psi\colon H_p(\mathcal{M}(\Sigma)_c) \xrightarrow{\simeq} \det H^*(\Sigma, \partial_{in})^{\otimes -\dim G} \otimes \bigotimes_{H \leq G} \det H^*(\Sigma^H, \partial_{in}^H)^{\otimes(\dim G - \dim H)}$$

Recall that windows cancel out. Note also that the line on the right hand side of this isomorphism is $\det \Sigma^{\underline{d}}$.

The action of $\mathrm{Diff}^+ \Sigma$ is trivial on the left hand side, and also on the right hand side, since this was shown for the first factor in the unlabelled case by proposition 2.2.7 and for the other factors since the diffeomorphisms are required to fix the boundary. This gives the labelled version of 2.2.7

Proposition 2.3.17. *The isomorphism Ψ is $\mathrm{Diff}^+(\Sigma, \partial)$-equivariant, and the action is trivial on both the top homology $H_p(\mathcal{M}(\Sigma)_c)$, and on the line $\det \Sigma^{\underline{d}}$.*

Determining p the top degree of the homology of the fibre, is now just the same as determining the weight of $\det \Sigma^{\underline{d}}$. This weight can be expressed as

$$-\chi(\Sigma, \partial_{in}) \cdot \dim G + \sum_{H \leq G} \chi(\Sigma^H, \partial_{in}^H) \cdot (\dim G - \dim H) \tag{2.14}$$

which is arguably more readily available for computations, and captures the additivity property of the fibre dimension since each summand is additive with respect to gluing of surfaces.

We can now state the labelled analogues to proposition 2.2.10, and further down below proposition 2.2.12.

Proposition 2.3.18. *Let Σ_1, Σ_2 be a pair of composable labelled open-closed cobordisms with positive boundary, and denote by Σ_{12} the composition. Denote the isomorphism Ψ for each cobordism, by respectively Ψ_1, Ψ_2 and Ψ_{12}. Then*

i) there is a natural isomorphism

$$H_{p_2}(\mathcal{M}(\Sigma_2)_c) \otimes H_{p_1}(\mathcal{M}(\Sigma_1)_c) \simeq H_{p_2+p_1}(\mathcal{M}(\Sigma_{12})_c)$$

ii) there is a natural isomorphism

$$\det \Sigma_2^{\underline{d}} \otimes \det \Sigma_1^{\underline{d}} \simeq \det \Sigma_{12}^{\underline{d}}$$

iii) the isomorphism of i) and ii) above are such that the following diagram commutes

$$\begin{tikzcd}
H_{p_2}(\mathcal{M}(\Sigma_2)_c) \otimes H_{p_1}(\mathcal{M}(\Sigma_1)_c) \arrow[r, "\simeq"] \arrow[d, "\Psi_2\otimes\Psi_1"'] & H_{p_2+p_1}(\mathcal{M}(\Sigma_{12})_c) \arrow[d, "\Psi_{12}"] \\
\det \Sigma_2^{\underline{d}} \otimes \det \Sigma_1^{\underline{d}} \arrow[r, "\simeq"] & \det \Sigma_{12}^{\underline{d}}
\end{tikzcd}$$

The proofs of $i)$ and $ii)$ are the same as for proposition 2.2.10, if we replace $H_s(\mathrm{Map}_*(\Sigma/\partial, BG))$ by $H_p(\mathcal{M}(\Sigma)_c)$, and $\det H^1(\Sigma, \partial_{in})^{\otimes \dim G}$ by $\det \Sigma^{\underline{d}}$.

proof of proposition 2.3.18 i). This follows from proposition 1.3.1, since by 2.3.9 all the spaces have rank 1 top homology, and

$$\mathcal{M}(\Sigma_2)_c \to \mathcal{M}(\Sigma_{12})_c \to \mathcal{M}(\Sigma_1)_c$$

is an orientable fibration as per diagram (1.2). □

proof of proposition 2.3.18 ii). Consider the long exact sequences in cohomology for the triples $(\Sigma_{12}, \Sigma_1, \partial_0)$, and $(\Sigma_{12}^H, \Sigma_1^H, \partial_0^H)$. As in the unlabelled case we use excision of $\Sigma_1 - \partial_1$, and $\Sigma_1^H - \partial_1^H$ to get $H^*(\Sigma_{12}, \Sigma_1) \simeq H^*(\Sigma_2, \partial_1)$, and $H^*(\Sigma_{12}^H, \Sigma_1^H) \simeq H^*(\Sigma_2^H, \partial_1^H)$. By 1.3.5 we get isomorphisms

$$\det H^*(\Sigma_2, \partial_1) \otimes \det H^*(\Sigma_1, \partial_0) \simeq \det H^*(\Sigma_{12}, \partial_0),$$
$$\det H^*(\Sigma_2^H, \partial_1^H) \otimes \det H^*(\Sigma_1^H, \partial_0^H) \simeq \det H^*(\Sigma_{12}^H, \partial_0^H)$$

and thus also on the tensor product

$$\det \Sigma_2^{\underline{d}} \otimes \det \Sigma_1^{\underline{d}} \simeq \det \Sigma_{12}^{\underline{d}}$$

□

proof of proposition 2.3.18 iii). Seeing as Ψ is defined to factor as ψ followed by a product of the isomorphisms Φ and a factor of c_H for each label H, it is enough to show that there is a commutative square for ψ as in the proposition, and one for each factor in the product. The square for ψ is loosely speaking

$$\begin{tikzcd}
H_{p_2}(\mathcal{M}(\Sigma_2)_c) \otimes H_{p_1}(\mathcal{M}(\Sigma_1)_c) \arrow[r, "\simeq"] \arrow[d, "\Psi_2\otimes\Psi_1"'] & H_{p_2+p_1}(\mathcal{M}(\Sigma_{12})_c) \arrow[d, "\Psi_{12}"] \\
\bigotimes homology_1 \otimes \bigotimes homology_2 \arrow[r, "\simeq"] & \bigotimes homology_{12}
\end{tikzcd}$$

with reference to the homology groups on the right hand side of ψ.

First, ψ is obtained from the diagram (2.13) of which there are one for each of the three cobordisms Σ_1, Σ_2 and Σ_{12}. These are easily seen to fit in a commutative diagram as follows, where we have shortened notation in the obvious way.

$$\begin{tikzcd}
 & & & \mathrm{Map}_*(\Sigma_2) \arrow[dl] \arrow[dd] & \\
\Pi_2 \arrow[rr] \arrow[dd] & & \mathcal{M}'(\Sigma_2)_c \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{M}(\Sigma_2)_c \arrow[dd] \\
 & \Pi_{cl,2} \arrow[dd] & & \mathrm{Map}_*(\Sigma_{12}) \arrow[dl] \arrow[dd] & \\
\Pi_{12} \arrow[rr] \arrow[dd] & & \mathcal{M}'(\Sigma_{12})_c \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{M}(\Sigma_{12})_c \arrow[dd] \\
 & \Pi_{cl,12} \arrow[dd] & & \mathrm{Map}_*(\Sigma_1) \arrow[dl] & \\
\Pi_1 \arrow[rr] & & \mathcal{M}'(\Sigma_1)_c \arrow[rr] \arrow[dl] & & \mathcal{M}(\Sigma_1)_c \\
 & \Pi_{cl,1} & & &
\end{tikzcd}$$

Each of the vertical maps is part of a fibration sequence of orientable fibrations, which by proposition 1.3.1 gives natural isomorphisms, and hence the relevant square for ψ commutes.

Next, by proposition 2.2.12 the relevant square for Φ commutes, so it is enough to keep track of the canonical isomorphisms c_H, defined for each labelled component of the three surfaces in question. Implicitly c_H is also defined for a window labelled by H, but in this case both sides are canonical isomorphic to $\mathbb{F}$. We proceed by considering components case by case.

For H associated to a hook in Σ_1 or a fork in Σ_2, the square trivially commutes. This account for all hooks and forks of Σ_{12}. Similar for H associated to a window in either Σ_1 or Σ_2 the square is trivial. The last case is if H is associated to a window in Σ_{12} coming from a fork in Σ_1 glued to a hook in Σ_2. In this case the square is the following

$$\begin{tikzcd}
(H_{\dim H}(H)\otimes H_{\dim G}(G)^*)\otimes(H_{\dim H}(H)^*\otimes H_{\dim G}(G)) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & H_{\dim H}(H)\otimes H_{\dim G}(G)^*\otimes H_{\dim H}(H)^*\otimes H_{\dim G}(G) \arrow[d, "\simeq"] \\
\det H^*(\Sigma_2^H,\partial_1^H)^{\dim G-\dim H}\otimes\det H^*(\Sigma_1^H,\partial_0^H)^{\dim G-\dim H} \arrow[r, "\simeq"] & \det H^*(\Sigma_{12}^H,\partial_0^H)^{\dim G-\dim H}
\end{tikzcd}$$

where both entries in the right column are canonically isomorphic to $\mathbb{F}$, as noted above. This square commutes by inspection, and tensoring all these commutative squares for the c_H and the one for Φ, we then get a single commutative square as claimed. □

As in the unlabelled case we will skip the proof of the following theorem. Again, it follows from basic properties.

Proposition 2.3.19. *Let Σ', Σ'' be a pair of labelled open-closed cobordisms with positive boundary, and denote by Σ disjoint union of these. Let $\partial'_{in}, \partial''_{in}, \partial_{in}$ denote the respective incoming boundaries. Denote the isomorphism Ψ for each cobordism, by respectively Ψ', Ψ'' and just Ψ. Then*

i) there is a natural isomorphism

$$H_{s'}(\mathcal{M}(\Sigma')_c) \otimes H_{s''}(\mathcal{M}(\Sigma'')_c) \simeq H_s(\mathcal{M}(\Sigma)_c)$$

ii) there is a natural isomorphism

$$\det \Sigma'^{\underline{d}} \otimes \det \Sigma''^{\underline{d}} \simeq \det \Sigma^{\underline{d}}$$

iii) the isomorphism of i) and ii) above are such that the following diagram commutes

$$\begin{tikzcd} H_{s'}(\mathcal{M}(\Sigma')_c) \otimes H_{s''}(\mathcal{M}(\Sigma'')_c) \arrow[r, "\simeq"] \arrow[d, "\Psi'\otimes\Psi''"'] & H_s(\mathcal{M}(\Sigma)_c) \arrow[d, "\Psi"] \\ \det \Sigma'^{\underline{d}} \otimes \det \Sigma''^{\underline{d}} \arrow[r, "\simeq"] & \det \Sigma^{\underline{d}} \end{tikzcd}$$

This concludes this last section before we proceed to do the actual construction of our ocHFT.

2.4 Operations

The plan for this section is to define operations similar to those in [1], and show that they give rise to an ocHFT with or without labels. For the reader primarily interested in the unlabelled case, we once again remind, that in this case $\mathcal{M}(X)$ is simply the ordinary mapping space $\operatorname{Map}(X, BG)$.

Let Σ be a labelled open-closed cobordism with positive boundary. We have the commutative diagram

$$\begin{tikzcd} & E\operatorname{Diff}^+\Sigma \times_{\operatorname{Diff}^+\Sigma} \mathcal{M}(\Sigma) \arrow[dl, "in_{h\operatorname{Diff}^+\Sigma}"'] \arrow[dd, "p_c"] \arrow[dr, "out_{h\operatorname{Diff}^+\Sigma}"] & \\ B\operatorname{Diff}^+\Sigma \times \mathcal{M}(\partial_{in}) \arrow[dr, "p_l"'] & & B\operatorname{Diff}^+\Sigma \times \mathcal{M}(\partial_{out}) \arrow[dl, "p_r"] \\ & B\operatorname{Diff}^+\Sigma & \end{tikzcd}$$

with p_l and p_r projections. In the following we will be explicit about coefficients of the homology, as a crucial point in the construction is to keep track of these. Define the operation $\mu(\Sigma)$ by the following sequence of maps and isomorphisms. First

$$\begin{aligned}&H_*(B\operatorname{Diff}^+\Sigma; \det\Sigma^{\underline{d}}) \otimes H_*(\mathcal{M}(\partial_{in}); \mathbb{F})\\ \simeq &H_*(B\operatorname{Diff}^+\Sigma \times \mathcal{M}(\partial_{in}); p_l^* \det\Sigma^{\underline{d}} \otimes \pi_2^*\mathbb{F})\end{aligned}$$

by the Künneth map and the fact that p_l is the projection on first coordinate. By proposition 2.3.15 (or just proposition 2.2.9 in the unlabelled case) the map $in_{h\operatorname{Diff}^+\Sigma}$ is an orientable fibration, and we can form the wrong way map $(in_{h\operatorname{Diff}^+\Sigma})_!$, as described in section 1.3.1. Further by proposition 2.3.16 (or proposition 2.2.3 in the unlabelled case) we can replace the top homology of the fibre with $\det\Sigma^{\underline{d}}$ when defining the wrong way map. This gives us the map (2.15) below

$$\begin{aligned}&H_*(B\operatorname{Diff}^+\Sigma \times \mathcal{M}(\partial_{in}); p_l^* \det\Sigma^{\underline{d}} \otimes \pi_2^*\mathbb{F})\\ \xrightarrow{(in_{h\operatorname{Diff}^+\Sigma})_!} &H_{*+p}(E\operatorname{Diff}^+\Sigma \times_{\operatorname{Diff}^+\Sigma} \mathcal{M}(\Sigma); (in_{h\operatorname{Diff}^+\Sigma})^* p_l^*\mathbb{F} \otimes \pi_2^*\mathbb{F}) \qquad (2.15)\\ \simeq &H_{*+p}(E\operatorname{Diff}^+\Sigma \times_{\operatorname{Diff}^+\Sigma} \mathcal{M}(\Sigma); p_c\mathbb{F} \otimes \pi_2^*\mathbb{F})\\ \to &H_{*+p}(B\operatorname{Diff}^+\Sigma \times \mathcal{M}(\partial_{out}); p_r^*\mathbb{F} \otimes \pi_2^*\mathbb{F}) \qquad (2.16)\\ \xrightarrow{(\pi_2)_*} &H_{*+p}(\mathcal{M}(\partial_{out}); \mathbb{F})\end{aligned}$$

where (2.16) is the map $(out_{h\operatorname{Diff}^+\Sigma})_*$. We now claim that these operations give us a functor

$$H_*(\mathcal{M}(-))\colon \mathcal{H}_G^{\underline{d}} \to gd - Vect_{\mathbb{F}}$$

Namely, for an object ∂ in $\mathcal{H}_G^{\underline{d}}$ we associate the graded vector space $H_*(\mathcal{M}(\partial))$, and for a morphism $\sigma \in H_*(B\operatorname{Diff}^+\Sigma; \det\Sigma^{\underline{d}})$ we assign the graded linear map

$$\mu(\Sigma)(\sigma \otimes -)\colon H_*(\mathcal{M}(\partial_{in}\Sigma)) \longrightarrow H_{*+p}(\mathcal{M}(\partial_{out}\Sigma))$$

of degree p, the top degree of the homology of the fibre $\mathcal{M}(\Sigma)_c$. We now verify that these assignments gives us a symmetric monoidal functor, i.e. that this preserves identity, composition, symmetry and monoidality.

2.4.1 Verification of functorial properties

The arguments that this construction preserves identities and symmetry isomorphisms, carries over almost directly from [1]. In the following we will make the abbreviation $BD\Sigma$ for $B\operatorname{Diff}^+\Sigma$, whenever the diagrams get too big.

Lemma 2.4.1. *The operation*

$$\mu(\Sigma)\colon H_*(B\operatorname{Diff}^+\Sigma; \det\Sigma^{\underline{d}}) \otimes H_*(\mathcal{M}(\partial_{in}\Sigma)) \to H_{*+p}(\mathcal{M}(\partial_{out}\Sigma))$$

restricted to $H_0(B\operatorname{Diff}^+\Sigma; \det\Sigma^{\underline{d}})$*, coincides with the operation*

$$H_*(\mathcal{M}(\partial_{in}); \det\Sigma^{\underline{d}}) \xrightarrow{in_!} H_{*+p}(\mathcal{M}(\Sigma)) \xrightarrow{out_*} H_{*+p}(\mathcal{M}(\partial_{out}))$$

Proof. Note first that all squares in the following diagram are pullbacks

$$\begin{tikzcd}
BD\Sigma \times \mathcal{M}(\partial_{in}) & \mathcal{M}(\Sigma)_{hD\Sigma} \arrow[l, "in_{hD\Sigma}"'] \arrow[r, "out_{hD\Sigma}"] & BD\Sigma \times \mathcal{M}(\partial_{out}) \\
ED\Sigma \times \mathcal{M}(\partial_{in}) \arrow[u] \arrow[d, "\pi_2"'] & ED\Sigma \times \mathcal{M}(\Sigma) \arrow[l, "id\times in"'] \arrow[r, "id\times out"] \arrow[u] \arrow[d, "\pi_2"'] & ED\Sigma \times \mathcal{M}(\partial_{out}) \arrow[u] \arrow[d, "\pi_2"] \\
\mathcal{M}(\partial_{in}) & \mathcal{M}(\Sigma) \arrow[l, "in"] \arrow[r, "out"'] & \mathcal{M}(\partial_{out})
\end{tikzcd}$$

Since wrong way maps are natural with respect to pullbacks, the degree shifts match and following diagram then commutes

$$\begin{tikzcd}
H_0(BD\Sigma;\det\Sigma^{\underline{d}})\otimes H_*(\mathcal{M}(\partial_{in})) \arrow[r, "(in_{hD\Sigma})_!"] & H_{*+p}(\mathcal{M}(\Sigma)_{h\,\mathrm{Diff}^+\Sigma}) \arrow[r, "(out_{hD\Sigma})_*"] & H_0(BD\Sigma)\otimes H_{*+p}(\mathcal{M}(\partial_{out})) \\
H_0(ED\Sigma;\det\Sigma^{\underline{d}})\otimes H_*(\mathcal{M}(\partial_{in})) \arrow[u] \arrow[d] \arrow[r, "(id\times in)_!"] & H_0(ED\Sigma)\otimes H_{*+p}(\mathcal{M}(\Sigma)) \arrow[u] \arrow[d] \arrow[r, "(id\times out)_*"] & H_0(ED\Sigma)\otimes H_{*+p}(\mathcal{M}(\partial_{out})) \arrow[u] \arrow[d] \\
H_*(\mathcal{M}(\partial_{in});\det\Sigma^{\underline{d}}) \arrow[r, "in_!"'] & H_{*+p}(\mathcal{M}(\Sigma)) \arrow[r, "out_*"'] & H_{*+p}(\mathcal{M}(\partial_{out}))
\end{tikzcd}$$

Now note that there is a canonical 0-cycle of $ED\Sigma \simeq *$ giving canonical isomorphisms in the first, and last column. ☐

Denote by $i_0, i_1 \colon \partial \to \partial \times I$ the standard inclusion of ∂ into either end of the cylinder $\partial \times I$ labelled according to the labels on ∂.

Lemma 2.4.2. *Let ∂ be an object of $\mathcal{H}^{\underline{d}}_G$, and let φ be a labelled diffeomorphism of the corresponding labelled 1-manifold. Consider the labelled open-closed cobordism $\partial_\varphi \times I$, which is the cylinder $\partial \times I$ with incoming boundary $\partial \overset{i_0\circ\varphi}{\hookrightarrow} \partial \times I$ and outgoing boundary $\partial \overset{i_1}{\hookrightarrow} \partial \times I$. Denote by in_φ and out the induced maps on $\mathcal{M}(-)$. The operation*

$$H_*(\mathcal{M}(\partial); \det(\partial \times I)^{\underline{d}}) \xrightarrow{(in_\varphi)_!} H_*(\mathcal{M}(\partial_\varphi \times I)) \xrightarrow{out_*} H_*(\mathcal{M}(\partial))$$

is an isomorphism, and is inverse to the map $\mathcal{M}(\varphi)_*$ induced by φ on $H_*(\mathcal{M}(\partial))$.

Recall that a labelled homotopy equivalence (in particular a diffeomorphism) induces a homotopy equivalence on $\mathcal{M}(-)$, and thus an isomorphism on homology of this. Also note that $\det\Sigma^{\underline{d}} \simeq \mathbb{F}$ by a canonical isomorphism when $H^1(\Sigma, \partial_{in}) = H^1(\Sigma^H, \partial^H_{in}) = 0$ for all H.

Proof. Wrong way maps and induced maps on homology are natural with respect to homotopy equivalences, and both in_{id} and out are homotopy equivalences, so on homology we have the diagram

$$
\begin{tikzcd}
H_*(\mathcal{M}(\partial);\det\Sigma^{\underline{d}}) \arrow[r, "(in_\varphi)_!"] \arrow[d, "\mathcal{M}(\varphi)_!=\mathcal{M}(\varphi)_*^{-1}"'] & H_{*+0}(\mathcal{M}(\partial_\varphi\times I)) \arrow[r, "out_*"] \arrow[d, "\simeq"] & H_{*+0}(\mathcal{M}(\partial)) \arrow[d, "id_*"] \\
H_{*+0}(\mathcal{M}(\partial);\det\Sigma^{\underline{d}}) \arrow[r, "(in_{id})_!"] \arrow[dr, "id_!"'] & H_{*+0}(\mathcal{M}(\partial\times I)) \arrow[r, "out_*"] \arrow[d, "\simeq"] & H_{*+0}(\mathcal{M}(\partial)) \\
 & H_{*+0}(\mathcal{M}(\partial)) \arrow[ur, "id_*"'] &
\end{tikzcd}
$$

where the bottom part commutes, which shows the case $\varphi = id$. For general φ we note that $\mathcal{M}(\varphi)_! = \mathcal{M}(\varphi)_*^{-1}$ and $(in_\varphi)_! = (in_{id})_! \circ \mathcal{M}(\varphi)_!$, by the composition property of wrong way maps. Thus the squares also commute, and we are done. $\square$

By the two lemmas 2.4.1 and 2.4.2 above, a morphism in $\mathcal{H}_G^{\underline{d}}$ which is in $H_0(B\,\mathrm{Diff}^+(\partial_\varphi \times I);\det(\partial_\varphi\times I)^{\underline{d}})$ for some diffeomorphism φ, gives rise to the isomorphism $H_*(\mathcal{M}(\partial)) \overset{\mathcal{M}(\varphi)_*^{-1}}{\longrightarrow} H_*(\mathcal{M}(\partial))$. In particular this is the case for the identity and symmetry isomorphisms of the category $\mathcal{H}_G^{\underline{d}}$, as we explain now.

The identity morphism on an object ∂ in $\mathcal{H}_G^{\underline{d}}$ is a 0-cycle id_∂ associated to a single point in $H_0(B\,\mathrm{Diff}^+(\partial\times I);\det(\partial\times I)^{\underline{d}})$, where $\partial \times I$ is the cylinder on ∂ labelled according to the labels on ∂.

When ∂ is an object consisting of more than one connected component, say $\partial = \partial_1 \coprod \partial_2$, any labelled diffeomorphism $\tau_{1,2}$, which interchanges the components will also be a 0-cycle $t_{1,2} \in H_0(B\,\mathrm{Diff}^+(\partial_\tau\times I);\det(\partial_\tau\times I)^{\underline{d}})$, associated to a single point . These are the symmetry isomorphisms of the category.

Therefore we get that

$$\mu(\partial\times I)(id_\partial \otimes -)\colon H_*(\mathcal{M}(\partial)) \to H_*(\mathcal{M}(\partial))$$

is the identity, and that

$$\mu(\partial_\tau\times I)(t_{1,2} \otimes -)\colon H_*(\mathcal{M}(\partial)) \to H_*(\mathcal{M}(\partial))$$

is given as $\mathcal{M}(\tau_{1,2})_*^{-1}$.

Now when ∂ has more than one connected component $\mathcal{M}(\partial)$ is a product with one factor for each component, and so the homology is a tensor product with one factor for each component. The induced map $\mathcal{M}(\tau_{1,2})_*^{-1}$ then interchanges the factors from ∂_1 with those from ∂_2 at the cost of a sign coming from the graded commutativity of graded vector spaces. This is the symmetry isomorphism in graded vector spaces. Next we check that the monoidal structure is respected.

Suppose we have two morphisms $\sigma'\colon \partial'_{in} \to \partial'_{out}$ and $\sigma''\colon \partial''_{in} \to \partial''_{out}$ in respectively $H_*(B\,\mathrm{Diff}^+\Sigma';\det\Sigma'^{\underline{d}})$ and $H_*(B\,\mathrm{Diff}^+\Sigma'';\det\Sigma''^{\underline{d}})$, giving rise to graded linear maps, $\mu(\Sigma')(\sigma'\otimes -)$ of degree p' and $\mu(\Sigma'')(\sigma''\otimes -)$ of degree p''. Then consider the morphism $\sigma'\otimes\sigma''\colon \partial'_{in}\coprod\partial''_{in} \to \partial'_{out}\coprod\partial''_{out}$ in $H_*(B\,\mathrm{Diff}^+\Sigma'\coprod\Sigma'';\det\Sigma'^{\underline{d}}\otimes\det\Sigma''^{\underline{d}})$. Since all surfaces are assumed to have positive boundary we have an isomorphism of groups

$$\mathrm{Diff}^+(\Sigma'\coprod\Sigma'') \simeq \mathrm{Diff}^+\Sigma' \times \mathrm{Diff}^+\Sigma''$$

As mentioned before there is a homeomorphism $\mathcal{M}(\Sigma' \coprod \Sigma'') \simeq \mathcal{M}(\Sigma') \times \mathcal{M}(\Sigma'')$, which is natural in Σ' and Σ'', since $\mathcal{M}(\Sigma' \coprod \Sigma'')$ is a pullback of spaces of maps from disjoint unions. For groups D, D', a D-space X and a D'-space X' there is a natural (in X and X') homotopy equivalence $(X \times X')_{hD \times D'} \simeq X_{hD} \times X_{hD'}$, coming from the homotopy equivalence $ED \times ED' \simeq E(D \times D')$. Now set $\Sigma := \Sigma' \coprod \Sigma''$, $\partial_- := \partial'_- \coprod \partial''_-$, $p := p' + p''$ and let D, D', D'' be the respective diffeomorphism groups. There is then a commutative diagram

$$\begin{tikzcd}
BD' \times \mathcal{M}(\partial'_{in}) \times BD'' \times \mathcal{M}(\partial''_{in}) \arrow[r, "\simeq"] & BD \times \mathcal{M}(\Sigma) \\
\mathcal{M}(\Sigma')_{hD'} \times \mathcal{M}(\Sigma'')_{hD''} \arrow[r, "\simeq"] \arrow[u, "in'_{hD'} \times in''_{hD''}"'] \arrow[d, "out'_{hD'} \times out''_{hD''}"'] & \mathcal{M}(\Sigma)_{hD} \arrow[u, "in_{hD}"'] \arrow[d, "out_{hD}"] \\
BD' \times \mathcal{M}(\partial'_{out}) \times BD'' \times \mathcal{M}(\partial''_{out}) \arrow[r, "\simeq"'] & BD \times \mathcal{M}(\partial_{out})
\end{tikzcd}$$

and if we apply homology to this, we get the diagram

$$\begin{tikzcd}
H_*(BD' \times \mathcal{M}(\partial'_{in}) \times BD'' \times \mathcal{M}(\partial''_{in}); \det \Sigma'^{\underline{d}} \otimes \det \Sigma''^{\underline{d}}) \arrow[r, "\simeq"] \arrow[d, "(in'_{hD'} \times in''_{hD''})_!"'] & H_*(BD \times \mathcal{M}(\partial_{in}); \det \Sigma^{\underline{d}}) \arrow[d, "(in_{hD})_!"] \\
H_{*+p'+p''}(\mathcal{M}(\Sigma')_{hD'} \times \mathcal{M}(\Sigma'')_{hD''}; \mathbb{F}) \arrow[r, "\simeq"] \arrow[d, "(\pi_2 \circ out'_{hD'} \times \pi_2 \circ out''_{hD''})_*"'] & H_{*+p}(\mathcal{M}(\Sigma)_{hD}; \mathbb{F}) \arrow[d, "(\pi_2 \circ out_{hD})_*"] \\
H_{*+p'+p''}(\mathcal{M}(\partial'_{out}) \times \mathcal{M}(\partial''_{out}); \mathbb{F}) \arrow[r, "\simeq"] & H_{*+p}(\mathcal{M}(\partial_{out}); \mathbb{F})
\end{tikzcd}$$

The right column of this is $\mu(\Sigma)$, and by propositions 2.2.12 and 2.3.19 the isomorphism of the coefficients respects the monoidal structure of disjoint union. The top horizontal isomorphism is obtained by interchanging the factors BD'' and $\mathcal{M}(\partial'_{in})$ and applying the isomorphism of diffeomorphism groups, and the bottom map is induced by the obvious isomorphism of spaces. We check commutativity of the outer square by evaluating on elements $v \in H_*\mathcal{M}(\partial'_{in})$ and $w \in H_*\mathcal{M}(\partial''_{in})$. From top left across the top of the diagram to bottom right we get

$$(\sigma' \otimes v \otimes \sigma'' \otimes w) \mapsto (-1)^{|\sigma''||v|}\mu(\Sigma)((\sigma' \otimes \sigma'') \otimes (v \otimes w))$$

Across the bottom of the diagram we get

$$\begin{aligned}(\sigma' \otimes v \otimes \sigma'' \otimes w) \mapsto & (\mu(\Sigma') \times \mu(\Sigma''))((\sigma' \otimes v) \times (\sigma'' \otimes w)) \\ \mapsto & \mu(\Sigma)(\sigma' \otimes \sigma'') \otimes (v \otimes w)\end{aligned}$$

with some abuse of notation. The point being that the outer square commutes up to the sign $(-1)^{|\sigma''||v|}$. By the Künneth isomorphism we can also write the diagram

$$\begin{tikzcd}
H_*(BD'\times\mathcal{M}(\partial'_{in}))\otimes H_*(BD''\times\mathcal{M}(\partial''_{in})) \arrow[d, "(in'_{hD'})_!\otimes(in''_{hD''})_!"'] & H_*(BD'\times\mathcal{M}(\partial'_{in})\times BD''\times\mathcal{M}(\partial''_{in})) \arrow[l, "\simeq"'] \arrow[d, "(in'_{hD'}\times in''_{hD''})_!"'] \\
H_{*+p'}(\mathcal{M}(\Sigma')_{hD'})\otimes H_{*+p''}(\mathcal{M}(\Sigma'')_{hD''}) \arrow[d, "(\pi_2\circ out'_{hD'})\otimes(\pi_2\circ out''_{hD''})_*"'] & H_{*+p}(\mathcal{M}(\Sigma')_{hD'}\times\mathcal{M}(\Sigma'')_{hD''}) \arrow[l, "\simeq"'] \arrow[d, "(\pi_2\circ out'_{hD'}\times\pi_2\circ out''_{hD''})_*"'] \\
H_{*+p'}(\mathcal{M}(\partial'_{out}))\otimes H_{*+p''}(\mathcal{M}(\partial''_{out})) & H_{*+p}(\mathcal{M}(\partial'_{out})\times\mathcal{M}(\partial''_{out})) \arrow[l, "\simeq"']
\end{tikzcd}$$

where the coefficients are $\det\Sigma'^{\underline{d}}\otimes\det\Sigma''^{\underline{d}}$ in the top row, and $\mathbb{F}$ elsewhere. Now the left column is $\mu(\Sigma')\otimes\mu(\Sigma'')$, and again we evaluate on $v\in H_*\mathcal{M}(\partial'_{in})$ and $w\in H_*\mathcal{M}(\partial''_{in})$. Across the bottom we get

$$\begin{aligned}(\sigma'\otimes v\otimes\sigma''\otimes w)\mapsto&(\mu(\Sigma')\times\mu(\Sigma''))((\sigma'\otimes v)\times(\sigma''\otimes w))\\ \mapsto&\mu(\Sigma')(\sigma'\otimes v)\otimes\mu(\Sigma'')(\sigma''\otimes w)\end{aligned}$$

and across the top we get

$$\begin{aligned}(\sigma'\otimes v\otimes\sigma''\otimes w)&\mapsto(\sigma'\otimes v)\otimes(\sigma''\otimes w)\\ &\mapsto(-1)^{p''(|\sigma'|+|v|)}\mu(\Sigma')(\sigma'\otimes v)\otimes\mu(\Sigma'')(\sigma''\otimes w)\end{aligned}$$

by the properties of wrong way maps. So the square commutes up to the sign $(-1)^{p''(|\sigma'|+|v|)}$, and thus we get that

$$\mu(\Sigma)((\sigma'\otimes\sigma'')\otimes(v\otimes w))=(-1)^{|\sigma''||v|+p''(|\sigma'|+|v|)}\mu(\Sigma')(\sigma'\otimes v)\otimes\mu(\Sigma'')(\sigma''\otimes w) \tag{2.17}$$

The reader may notice that this sign is not the usual one for the monoidal isomorphism of graded vector spaces. However, what we have not discussed is that we ought to use *graded coefficients* for the homology. This would have no impact on the thesis in any other way than the fact that with an appropriate notion of such graded coefficients, the sign $(-1)^{p''(|\sigma'|+|v|)}$ would disappear. This is because the square

$$\begin{tikzcd}
H_*(BD'\times\mathcal{M}(\partial'_{in}))\otimes H_*(BD''\times\mathcal{M}(\partial''_{in})) \arrow[d] & H_*(BD'\times\mathcal{M}(\partial'_{in})\times BD''\times\mathcal{M}(\partial''_{in})) \arrow[l, "\simeq"'] \arrow[d] \\
H_{*+p'}(\mathcal{M}(\partial'_{out}))\otimes H_{*+p''}(\mathcal{M}(\partial''_{out})) & H_{*+p}(\mathcal{M}(\partial'_{out})\times\mathcal{M}(\partial''_{out})) \arrow[l, "\simeq"']
\end{tikzcd}$$

commutes strictly in such a setup (coming from (1.1) which would have no sign). The resulting sign in (2.17) above would then just be $(-1)^{|\sigma''||v|}$ which is indeed the sign expected for the usual monoidal structure on graded vector spaces.

Finally we should show that compositions are taken to compositions. We do this by calculating both the map coming from the composition of two morphisms, and composing the maps coming from either of them. Let ∂_i, $i=0,1,2$ be objects, and let $\sigma_i\in H_*(B\operatorname{Diff}^+\Sigma_i;\det\Sigma_i^{\underline{d}})$, $i=1,2$ be a pair of composable morphisms, with σ_i going from ∂_{i-1} to ∂_i. Denote the composition $\sigma_2\circ\sigma_1$ by $\sigma_{12}\in H_*(B\operatorname{Diff}^+\Sigma_{12};\det\Sigma_{12}^{\underline{d}})$, and the respective diffeomorphism groups D_1, D_2 and D_{12}. Thus we want to show that

$$\mu(\Sigma_{12})(\sigma_{12}\otimes -)=\mu(\Sigma_2)(\sigma_2\otimes -)\circ\mu(\Sigma_1)(\sigma_1\otimes -).$$

There is an obvious homomorphism $D_2 \times D_1 \to D_{12}$, which induces a map $gl\colon BD_2 \times BD_1 \to BD_{12}$, and consequently a map

$$glue\colon \mathcal{M}(\Sigma_{12})_{h(D_2 \times D_1)} \to \mathcal{M}(\Sigma_{12})_{hD_{12}}$$

Note that $gl_*(\sigma_2, \sigma_1) = \sigma_{12}$. Further the inclusion of $\partial_0 \to \Sigma_{12}$ induces both the usual $(in_0)_{hD_{12}}$, and a map

$$(in_0)_{h(D_2 \times D_1)}\colon \mathcal{M}(\Sigma_{12})_{h(D_2 \times D_1)} \to BD_2 \times BD_1 \times \mathcal{M}(\partial_0)$$

There is a commutative diagram

$$\begin{tikzcd}
\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)} \arrow[r, "glue"] \arrow[d, "(in_0)_{h(D_2\times D_1)}"'] & \mathcal{M}(\Sigma_{12})_{hD_{12}} \arrow[d, "(in_0)_{hD_{12}}"] \\
BD_2 \times BD_1 \times \mathcal{M}(\partial_0) \arrow[r, "gl\times id"] \arrow[d, "\pi_1"'] & BD_{12} \times \mathcal{M}(\partial_0) \arrow[d, "\pi_1"] \\
BD_2 \times BD_1 \arrow[r, "gl"] & BD_{12}
\end{tikzcd}$$

where both vertical compositions are bundles with fibre $\mathcal{M}(\Sigma_{12})$, so the outer square is a pullback. The lower square is obviously a pullback, so also the top square is a pullback. From this we get the diagram in homology

$$\begin{tikzcd}
H_{*+p_{12}}(\mathcal{M}(\partial_2);\mathbb{F}) \arrow[r, "id_*"] & H_{*+p_{12}}(\mathcal{M}(\partial_2);\mathbb{F}) \\
H_{*+p_{12}}(\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)};\mathbb{F}) \arrow[r, "glue_*"] \arrow[u, "(\pi_2\circ out_{h(D_2\times D_1)})_*"] & H_{*+p_{12}}(\mathcal{M}(\Sigma_{12})_{hD_{12}};\mathbb{F}) \arrow[u, "(\pi_2\circ out_{h(D_{12})})_*"'] \\
H_*(BD_2 \times BD_1 \times \mathcal{M}(\partial_0); \det \Sigma_{12}^d) \arrow[r, "(gl\times id)_*"] \arrow[u, "((in_0)_{h(D_2\times D_1)})_!"] & H_*(BD_{12} \times \mathcal{M}(\partial_0); \det \Sigma_{12}^d) \arrow[u, "((in_0)_{hD_{12}})_!"']
\end{tikzcd} \tag{2.18}$$

where the right column is $\mu(\Sigma_{12})$. The diagram commutes since wrong way maps are natural with respect to pullbacks. Now we turn our attention to the composition of a pair of operations, coming from a pair of composable morphisms. First note that we have a pullback

$$\begin{tikzcd}
\mathcal{M}(\Sigma_{12}) \arrow[r] \arrow[d] & \mathcal{M}(\Sigma_1) \arrow[d] \\
\mathcal{M}(\Sigma_2) \arrow[r] & \mathcal{M}(\partial_1)
\end{tikzcd}$$

giving us that the outer square of the following diagram is a pullback

$$\begin{tikzcd}
\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)} \arrow[r] \arrow[d] & \mathcal{M}(\Sigma_2)_{hD_2} \arrow[d, "in_{hD_2}"] \\
BD_2 \times \mathcal{M}(\Sigma_1)_{hD_1} \arrow[r, "id\times\pi_2\circ out_{hD_2}"] \arrow[d, "\pi_2"'] & BD_2 \times \mathcal{M}(\partial_1) \arrow[d, "\pi_2"] \\
\mathcal{M}(\Sigma_1)_{hD_1} \arrow[r] & \mathcal{M}(\partial_1)
\end{tikzcd}$$

The bottom square is obviously a pullback, so also the top square is. Now consider the following diagram

$$\begin{tikzcd}
 & & & \mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)} \arrow[dl, "(in_{\Sigma_1})_{hD_2\times D_1}"'] \arrow[dr] \arrow[ddll, bend right, "(in_0)_{hD_2\times D_1}"'] \arrow[ddrr, bend left, "\pi_2\circ out_{hD_2\times D_1}"] & & & \\
 & & BD_2 \times \mathcal{M}(\Sigma_1)_{hD_1} \arrow[dl, "id\times(in_0)_{hD_1}"'] \arrow[dr, "\pi_2\circ out_{hD_1}"] & & \mathcal{M}(\Sigma_2)_{hD_2} \arrow[dl, "(in_1)_{hD_2}"'] \arrow[dr, "\pi_2\circ out_{hD_2}"] & & \\
 & BD_2 \times BD_1 \times \mathcal{M}(\partial_0) & & BD_2 \times \mathcal{M}(\partial_1) & & \mathcal{M}(\partial_2) &
\end{tikzcd} \tag{2.19}$$

We have just seen that the middle square is a pullback. Next note that the pullback of $(in_1)_{hD_2}$ along $\pi_2 \circ out_{hD_1}$ is induced by the inclusion of Σ_1 into Σ_{12}. In obvious analogue to the rest of the notation we denote this by $(in_{\Sigma_1})_{hD_2\times D_1}$. Then

$$(in_{\Sigma_1})_{hD_2\times D_1} \circ (id \times (in_0)_{hD_1}) = (in_0)_{hD_2\times D_1}$$

and for the associated wrong way maps we have the diagram

$$\begin{tikzcd}
H_{*+p_1+p_2}(\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)};\mathbb{F}) \arrow[r, equal] & H_{*+p_{12}}(\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)};\mathbb{F}) \\
H_{*+p_1}(BD_2 \times \mathcal{M}(\Sigma_1)_{hD_1};\mathbb{F}\otimes\det\Sigma_2^d) \arrow[u, "((in_{\Sigma_1})_{hD_2\times D_1})_!"] & \\
H_*(BD_2 \times BD_1 \times \mathcal{M}(\partial_0);\det\Sigma_1^d\otimes\det\Sigma_2^d) \arrow[u, "(id\times(in_0)_{hD_1})_!"] \arrow[r, "\simeq"] & H_*(BD_2 \times BD_1 \times \mathcal{M}(\partial_0);\det\Sigma_{12}^d) \arrow[uu, "((in_0)_{hD_2\times D_1})_!"']
\end{tikzcd} \tag{2.20}$$

which commutes by propositions 2.2.10 and 2.3.18. Similar the right hand composition in the diagram (2.19) is $\pi_2 \circ out_{hD_1\times D_2}$, induced by the inclusion of the outgoing boundary. Because wrong way maps respect pullbacks, the left column of diagram (2.20) also fits in a commutative diagram

$$
\begin{tikzcd}
H_{*+p_1+p_2}(\mathcal{M}(\partial_2);\mathbb{F}) & H_{*+p_1+p_2}(\mathcal{M}(\Sigma_{12})_{h(D_2\times D_1)};\mathbb{F}) \arrow[l, "\pi_2\circ out_{hD_2\times D_1}"'] \\
H_{*+p_1+p_2}(\mathcal{M}(\Sigma_2)_{hD_2};\mathbb{F}) \arrow[u, "\pi_2\circ out_{hD_2}"] & \\
H_{*+p_1}(BD_2\times\mathcal{M}(\partial_1);\mathbb{F}\otimes\det\Sigma_2^d) \arrow[u, "((in_1)_{hD_2})_!"] & H_{*+p_1}(BD_2\times\mathcal{M}(\Sigma_1)_{hD_1};\mathbb{F}\otimes\det\Sigma_2^d) \arrow[l, "\pi_2\circ out_{hD_1}"] \arrow[uu, "((in_{\Sigma_1})_{hD_2\times D_1})_!"'] \\
& H_*(BD_2\times BD_1\times\mathcal{M}(\partial_0);\det\Sigma_1^d\otimes\det\Sigma_2^d) \arrow[u, "(id\times(in_0)_{hD_1})_!"']
\end{tikzcd}
\tag{2.21}
$$

where the composition from bottom right to top left along the left-hand side of the diagram, is $\mu(\Sigma_2)\circ(id_*\times\mu(\Sigma_1))$. We now combine these diagrams (2.20) and (2.21) with the diagram (2.18) to finally get the commutative diagram

$$
\begin{tikzcd}
H_*(BD_2\times BD_1\times\mathcal{M}(\partial_0);\det\Sigma_{12}^d) \arrow[r, "\mu(\Sigma_1)\circ(id_*\times\mu(\Sigma_2))"] \arrow[d, "(gl\times id)_*"'] & H_{*+p_{12}}(\mathcal{M}(\partial_2);\mathbb{F}) \\
H_*(BD_{12}\times\mathcal{M}(\partial_0);\det\Sigma_{12}^d) \arrow[ur, "\mu(\Sigma_{12})"'] &
\end{tikzcd}
$$

and so we conclude that

$$\mu(\Sigma_{12})(\sigma_{12}\otimes -) = \mu(\Sigma_2)(\sigma_2\otimes -)\circ\mu(\Sigma_1)(\sigma_1\otimes -)$$

This the verifies that the construction $H_*(\mathcal{M}(-))$ respects composition of morphisms. So after verifying all the properties, we conclude that this is a symmetric monoidal functor. Thus we have finally proven our main theorem

Theorem 1.2.3. *Let G be a connected compact Lie group, and $\mathbb{F}$ a field. The singular homology of the labelled strings on BG with coefficients in $\mathbb{F}$, defines a labelled open-closed HFT, i.e. a symmetric monoidal functor*

$$H_*(\mathcal{M}(-);\mathbb{F})\colon \mathcal{H}_G^d \longrightarrow gd-Vect_{\mathbb{F}}$$

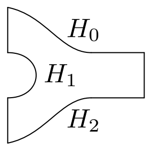
Chapter 3

Examples

Here we give some examples of the operations we have constructed, to see that our work has not been for nothing. For all we knew, these operations might all have been trivial. Fortunately this is not the case, and as we shall see there are in fact not only non-trivial, but also interesting examples.

As in previous chapter, we fix a compact connected Lie group G together with a model for EG and set $BH = EG/H$ for any subgroup $H \leq G$. We will also be using different models for BH in this chapter, but only when noted, and they are all based on EG. We take the set of oriented connected subgroups as the set of labels.

Throughout the chapter we will work with the following labelled open-closed cobordism Σ, as a representative for an element $\sigma \in H_0(B\,\mathrm{Diff}^+\,\Sigma; \det \Sigma^{\underline{d}})$.



where the left vertical boundary components are incoming, and the right vertical boundary component is outgoing. For arbitrary labels H_0, H_1 and H_2 we can compute the fibrations induced by inclusions of the incoming and outgoing boundaries, after which we choose more specific labellings.

Example 3.1 The maps induced by the inclusion of incoming respectively outgoing boundary of Σ are given as

$$\begin{tikzcd}
\mathcal{M}(\partial_{in}) \arrow[d, "\simeq"] & \mathcal{M}(\Sigma) \arrow[l] \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}(\partial_{out}) \arrow[d, "\simeq"] \\
H_0\backslash\backslash G//H_1 \times H_1\backslash\backslash G//H_2 & H_0\backslash\backslash G//H_1\backslash\backslash G//H_2 \arrow[l] \arrow[r] & H_0\backslash\backslash G//H_2
\end{tikzcd}$$

$$\begin{tikzcd}
([g_1],[g_2]) & {[g_1, g_2]} \arrow[l, mapsto] \arrow[r, mapsto] & {[g_1 g_2]}
\end{tikzcd}$$

This is somewhat vague at this point, but we will explain it in the following.

Let us first identify the spaces

$$\mathcal{M}(\partial_{in}) = \mathcal{M}(I_{H_2}^{H_1} \sqcup I_{H_1}^{H_0})$$
$$\mathcal{M}(\partial_{out}) = \mathcal{M}(I_{H_2}^{H_0}).$$

We note that spaces of the form $\mathcal{M}(I_{H'}^{H})$ are homotopy pullbacks

$$\begin{tikzcd} \mathcal{M}(I_{H'}^{H}) \arrow[r] \arrow[d] & BH \arrow[d] \\ BH' \arrow[r] & BG \end{tikzcd}$$

This is obvious from writing out the definitions, and noting that pullbacks of fibrations are also homotopy pullbacks. Hence we can identify these spaces up to homotopy by finding another homotopy pullback of the same three spaces. We define the two-sided homotopy quotient

$$H\backslash\backslash G//H' := EG \times_H G \times_{H'} EG.$$

This is an actual pullback

$$\begin{tikzcd} H\backslash\backslash G//H' \arrow[r] \arrow[d] & EG \times EG/H \arrow[d] \\ EG \times EG/H' \arrow[r] & EG \times EG/G \end{tikzcd}$$

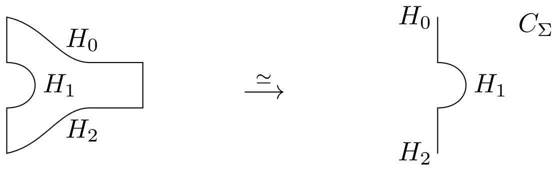
and thus a homotopy pullback

$$\begin{tikzcd} H\backslash\backslash G//H' \arrow[r] \arrow[d] & BH \arrow[d] \\ BH' \arrow[r] & BG \end{tikzcd}$$

This gives homotopy equivalences

$$\mathcal{M}(\partial_{out}) \simeq H_0\backslash\backslash G//H_2$$
$$\mathcal{M}(\partial_{in}) \simeq H_0\backslash\backslash G//H_1 \times H_1\backslash\backslash G//H_2.$$

Next we must identify the total space $\mathcal{M}(\Sigma)$. For this we note that Σ is labelled homotopy equivalent to its incoming boundary union the hook labelled by H_1, and call this space C_Σ.



Then consider the following diagram

$$\begin{tikzcd}
\mathcal{M}(C_\Sigma) \arrow[rr, dotted, "\simeq"'] \arrow[dr] \arrow[dd] & & H_0\backslash\backslash G//H_1\backslash\backslash G//H_2 \arrow[dl] \arrow[dd] \\
& BH_1 \arrow[dd, "\Delta"] & \\
\mathcal{M}(\partial_{in}) \arrow[rr, "\simeq"] \arrow[dr, "ev_t \times ev_b"'] & & H_0\backslash\backslash G//H_1 \times H_1\backslash\backslash G//H_2 \arrow[dl, "\pi_3\times\pi_4"] \\
& BH_1 \times BH_1 &
\end{tikzcd} \tag{3.1}$$

which we explain shortly hereafter. We claim that both the left and right front squares are homotopy pullbacks, and since the lower horizontal map is a homotopy equivalence we get a homotopy equivalence in the top as well, such that the back square commutes up to homotopy. We first investigate the right square, and in particular we define the top right entry.

$$\begin{tikzcd}
H_0\backslash\backslash G//H_1\backslash\backslash G//H_2 & := & \frac{EG\times_{H_0} G\times EG\times EG\times G\times_{H_2} EG}{\Delta H_1} \arrow[r] \arrow[d] & \frac{EG\times EG}{\Delta H_1} \arrow[d, "\Delta"] \\
H_0\backslash\backslash G//H_1 \times H_1\backslash\backslash G//H_2 & := & \frac{EG\times_{H_0} G\times EG\times EG\times G\times_{H_2} EG}{H_1\times H_1} \arrow[r, "\pi_3\times\pi_4"'] & \frac{EG\times EG}{H_1\times H_1}
\end{tikzcd}$$

By ΔH_1 we mean a single copy of H_1 acting diagonally on the two factors $(G \times EG) \times (EG \times G)$ in the top middle entry, and diagonally on the left and right of $EG \times EG$ in the top right entry. This is an actual pullback with these models for BH_1 and $BH_1 \times BH_1$ in the right column, and since the map $\pi_3 \times \pi_4$ induced by projection to third and fourth factor is a fibration, the square is a homotopy pullback.

Next we show that the left square of diagram (3.1) is also a homotopy pullback. With the same models for BH_1 and $BH_1 \times BH_1$ as for the right square, we have

$$\begin{tikzcd}
\mathcal{M}(C_\Sigma) \arrow[r] \arrow[d] & \frac{EG\times EG}{\Delta H_1} \arrow[d, "\Delta"] \\
\mathcal{M}(\partial_{in}) \arrow[r, "ev_t\times ev_b"'] & \frac{EG\times EG}{H_1\times H_1}
\end{tikzcd}$$

where the bottom map is given by evaluating at the two points labelled by H_1, i.e. ev_t for the evaluation map at the top point, and ev_b for the same at the bottom point. The actual pullback identifies these two points because of the diagonal map, whereas a homotopy pullback identifies these two points up to a path between them in BH_1, and we see that this is exactly $\mathcal{M}(C_\Sigma)$. Hence we can identify, up to homotopy

$$\mathcal{M}(\Sigma) \simeq \mathcal{M}(C_\Sigma) \simeq H_0\backslash\backslash G//H_1\backslash\backslash G//H_2$$

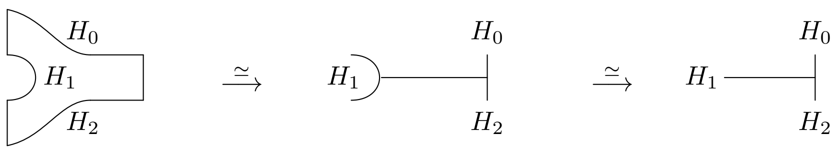
as we wanted.

Knowing the spaces however, is only the first step. We want to know the maps between them as well. Now consider the diagram

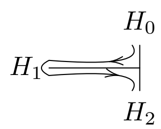
$$\begin{tikzcd}
\mathcal{M}(\partial_{in}) \arrow[d, "\simeq"] & \mathcal{M}(\Sigma) \arrow[l, "in"'] \arrow[r, "out"] \arrow[d, "\simeq"] & \mathcal{M}(\partial_{out}) \arrow[d, "\simeq"] \\
H_0\backslash\backslash G//H_1 \times H_1\backslash\backslash G//H_2 & H_0\backslash\backslash G//H_1\backslash\backslash G//H_2 \arrow[l, "\pi_{123}\times\pi_{456}"] \arrow[r, "\mu"'] & H_0\backslash\backslash G//H_2
\end{tikzcd}$$

We explain what we mean by the names of the lower horizontal maps. From the middle to the left, $\pi_{123} \times \pi_{456}$ is given by taking a class $[g_1, g_2]$ and mapping it to the pair of classes $([g_1], [g_2])$. From the middle to the right, μ is the map induced by the product in G, and maps a class $[g_1, g_2]$ to the class $[g_1g_2]$. It is now our claim that this diagram commutes up to homotopy.

The left square does so by definition, since this is the back square of the diagram (3.1). For the right square we note that relative to the outgoing boundary, Σ is labelled homotopy equivalent to the space drawn to the right below



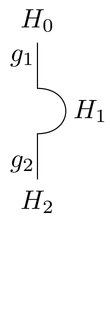
The first labelled homotopy equivalence is due to proposition 2.3.4 (or the proof of it), where we regard the outgoing boundary of Σ as incoming now, and the second is just contracting the labelled hook within itself. Now Σ was also labelled homotopy equivalent to C_Σ, which we can regard as a path in BG together with a lift of some of the path. Through the labelled homotopy equivalence above this is mapped to the a path trailing the picture to the right from the top point which has a lift to BH_0, down to the middle and out to the point with a lift to BH_1, back along the same path to the middle and down to the bottom point which has a lift to BH_2. This must be the case, as we took a *labelled* homotopy equivalence.



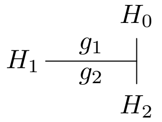
This is homotopic in BG to the path in BG given by the outgoing boundary with a lift of either end point to respectively H_0 and H_2. The homotopy equivalence

$$\mathcal{M}(\Sigma) \to H_0\backslash\backslash G//H_1\backslash\backslash G//H_2$$

factors through the homotopy equivalence $\mathcal{M}(\Sigma) \to \mathcal{M}(C_\Sigma)$, so there is a homotopy inverse which does this too. By such a homotopy inverse a class $([g_1, g_2])$ is mapped to $\mathcal{M}(C_\Sigma)$ where the image is a path in BG with lifts as follows



The paths between the labelled parts are determined by the elements $g_1, g_2 \in G$. As discussed, this is labelled homotopic to a path given as



Restricting to the incoming boundary we then get a path in BG with lifts of the ends to respectively BH_0 and BH_2, which is homotopic in BG to this path, but where we forget the lift of the point to H_1. Thus we get a path which is homotopic to the concatenation of g_1 and g_2. Fixing a path from the point where we concatenate to the base point, then makes the concatenation itself homotopic to the map induced by the composition in G under $G \simeq \Omega BG$. The choice of path to fix does not matter as $\pi_1(BG^I) \simeq \pi_0(BG) = 1$. Thus the map is induced by composition, and forgets the action of H_1 as we claimed.

Example 3.2 Now we can specialize a bit further by setting $H_0 = H_1 = H_2 = 1$, and denote the operation associated to the homology of this particular labelled open-closed cobordism by M. From the above, the restriction maps are homotopic to

$$G \times G \xleftarrow{id} G \times G \xrightarrow{\mu} G$$

where now μ is the actual composition in G. The wrong way map for the identity is just the identity on homology, so we get

$$M\colon H_*(G) \otimes H_*(G) \xrightarrow{\mu_*} H_*(G),$$

the usual Pontryagin product on the homology of G taken with field coefficients.

Before we consider the next example, we will briefly review a fact about wrong way maps for fibre bundles.

Let $F \longrightarrow E \xrightarrow{p} B$ be an orientable fibration which is also a fibre bundle, with $H_n(F) \simeq \mathbb{F}$ being the top homology group of the fibre, and with B connected and satisfying Poincaré duality. In particular this means that B has rank 1 top homology, the term introduced in section 1.3.1, with homology degree bounded by, say t.

As noted in section 1.3.1 a generator y of $H_t(B)$ (an orientation class) is sent to a generator $p_!(y)$ of $H_{t+n}(E)$ by the wrong way map associated to p, with $p_!(y)$ determined by a choice of orientation of the fibre.

The homology $H_*(B)$ is a module over $H^*(B)$ with the structure given by the cap product. By pulling back along p^*, and then using the cap product, the homology $H_*(E)$ is also a $H^*(B)$-module. The wrong way map $p_!$ itself, is then a map of $H^*(B)$ modules. We will not show this here, but leave it as a claim.

By Poincaré duality, we can generate any class in $H_*(B)$ by capping with y, and so the wrong way map is determined by the value on y. Since $p_!(y)$ is determined by the choice of orientation of the fibre, these two choices completely determines $p_!$.

Example 3.3 We use the same setup as in the above example 3.2, but this time we interchange incoming and outgoing boundaries, and denote the resulting operation by N. The identification of the restriction maps still applies, but now we read the diagram backwards, and get

$$G \xleftarrow{\mu} G \times G \xrightarrow{id} G \times G \, .$$

Recall that μ is pulled back from a G-bundle, so by the above discussion the wrong way map for μ is a $H^*(G)$-module map given by mapping the fundamental class $[G]$ to $[G]\otimes[G]$ (note that we may say *the* fundamental class, as orientations are fixed). Thus the operation

$$N\colon H_*(G) \longrightarrow H_*(G) \otimes H_*(G),$$

is also given by this.

The two examples above are in a sense extremes for the choice of subgroups, and we will now consider the other extreme.

Example 3.4 Set $H_0 = H_1 = H_2 = G$, and denote the associated operation by K. Again from our identification of the restriction maps, we get

$$BG \times BG \xleftarrow{\Delta} BG \xrightarrow{id} BG \, .$$

The diagonal map $\Delta\colon EG \times EG/\Delta H \to EG \times EG/H \times H$ is actually a H-bundle with structure group $H \times H$ acting on the fibre H by $(x, y).h = xhy^{-1}$, which is elementary to check. So in our case Δ is a G-bundle, and in principle this allow us to compute K if we know enough about the group G, and in particular the Serre spectral sequence associated to the principal G-bundle $G \to EG \to BG$. We will not pursue this here.

We immediately get another example from the above simply by interchanging incoming and outgoing boundary of the underlying cobordism as before.

Example 3.5 Set $H_0 = H_1 = H_2 = G$, and denote the operation associated to Σ with incoming and outgoing boundaries interchanged, by L. We get

$$BG \xleftarrow{id} BG \xrightarrow{\Delta} BG \times BG \, ,$$

and the operation L is then given by the map induced by the diagonal map Δ on homology. This map on homology is then in a sense dual to the cup product map on the cohomology of BG.

$$L\colon H_*(BG) \xrightarrow{\Delta_*} H_*(BG \times BG) \simeq H_*(BG) \otimes H_*(BG)$$

Finally we will compute an example for a concrete Lie group.

Example 3.6 Set $G = SU(2)$, and consider the operation associated to the homology of the labelled open-closed cobordism we have considered all along in this chapter, now with $H_0 = H_2 = 1$ and with $H_1 = T$ where T denotes the maximal torus subgroup. I.e. T is the subgroup consisting of all the elements of the form $\left(\begin{smallmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{smallmatrix}\right)$.

With this setup we get

$$\begin{tikzcd}
SU(2)/T \times T\backslash SU(2) \arrow[d, "\simeq"] & SU(2)/T\backslash SU(2) \arrow[l] \arrow[r] \arrow[d, "\simeq"] & SU(2) \arrow[d, "\simeq"] \\
S^2 \times S^2 & S^3 \times S^2 \arrow[l] \arrow[r] & S^3
\end{tikzcd}$$

with the lower maps defined by the top ones and the homotopy equivalences. Note that we may use the ordinary quotient instead of a homotopy quotient, as the action of T is free in all cases above. Take the middle homotopy equivalence to be given by mapping a class $[u,v]$ to $(uv,[v])$. This is well-defined since T acts on (u,v) by $t.(u,v) = (ut^{-1},tv)$ in the middle top entry. This map has a homotopy inverse given by mapping $(x,[y])$ to $[xy^{-1},y]$. Then the left lower map is given by mapping $(x,[y])$ to $([xy^{-1}],[y])$, and we see that for fixed y this is a non-trivial S^1-bundle over S^2, i.e. a Hopf bundle.

The wrong way map associated to this Hopf bundle takes the fundamental class of S^2 to the fundamental class of S^3 (again notice that we may say *the* fundamental class, as orientations are chosen). In particular if we restrict the operation produced to $H_0(S^2)$ on the second factor of the incoming boundary, we get a map

$$H_*(S^2)\otimes H_0(S^2) \to H_{*+1}(S^3)\otimes H_0(S^2) \to H_{*+1}(S^3).$$

which is defined by mapping $[S^2]\otimes 1 \mapsto [S^3]$.

There are certainly more interesting examples, but we will stop here for now.

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