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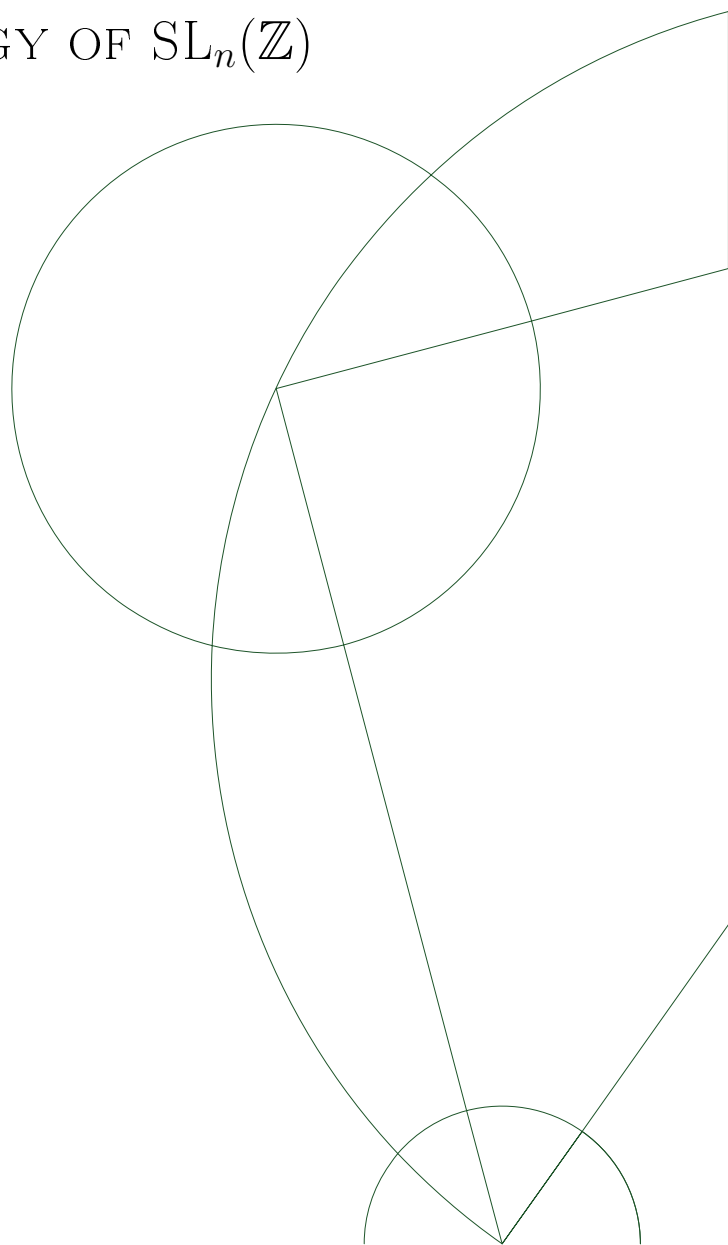
Stable Real Cohomology of $SL_n(\mathbb{Z})$

Master's Thesis in Mathematics

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Abstract

The aim of this thesis is to calculate the real cohomology of the special linear group $\mathrm{SL}_n(\mathbb{Z})$ in low degrees. This is a special case of Borel's article *Stable Real Cohomology of Arithmetic Groups* from 1974 and Borel and Serre's article *Corners and Arithmetic Groups* from 1973. In fact, the ambition of this project is to provide a stepping stone towards understanding these articles by looking at the details of the special case while avoiding use of the general theory.

To calculate the real cohomology of $\mathrm{SL}_n(\mathbb{Z})$, we exploit the geometric setting: We cover Siegel reduction theory, the Borel-Serre compactification, logarithmic differential forms and Matsushima's Vanishing Theorem.

Resumé

Formålet med dette speciale er at udregne den reelle kohomologi af den specielle lineære gruppe $\mathrm{SL}_n(\mathbb{Z})$ i lave grader. Dette er et specialtilfælde af Borels artikel *Stable Real Cohomology of Arithmetic Groups* fra 1974 og Borel og Serres artikel *Corners and Arithmetic Groups* fra 1973. Målet og ønsket er i virkeligheden, at dette projekt kan være en hjælp til at forstå disse artikler ved at give en gennemgang af detaljerne i specialtilfældet uden brug af den generelle teori.

Vi udnytter et geometrisk perspektiv til at udregne den reelle kohomologi af $\mathrm{SL}_n(\mathbb{Z})$: Vi gennemgår Siegels reduktionsteori, Borel-Serre-kompactificeringen, logaritmiske differentialformer og en sætning af Matsushima.

CONTENTS

Introduction and Notation

Introduction

The aim of this project is to cover the content of the article *Stable Real Cohomology of Arithmetic Groups* by Armand Borel ([6]) in the special case of the arithmetic group $\mathrm{SL}_n(\mathbb{Z})$. We also go through the construction of the article *Corners and Arithmetic Groups* by Armand Borel and Jean-Pierre Serre ([3]) as this is used explicitly in [6]. These articles are written in great generality and our intention is to bring it all down to a more digestible level; we therefore try to avoid the general theory as much as possible, but we do, now and then, make some remarks relating our way of doing it to the more general way.

We wish to compute the real cohomology of the discrete group $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. It sits as a lattice inside the Lie group $G := \mathrm{SL}_n(\mathbb{R})$ and this enables us to move into the world of geometry. We consider the smooth manifold X of positive definite quadratic forms on $\mathbb{R}^n$ inducing the same volume as the standard inner product. There is a natural action of G on X, and with the inherited action, Γ acts properly discontinuously on X. It turns out that

$$H^*(\Gamma; \mathbb{R}) \cong H^*\big(\Omega^*(X)^\Gamma\big),$$

where $\Omega(X)^\Gamma$ denotes the complex of Γ-invariant differential forms on X. Hence, the computation of $H^*(\Gamma; \mathbb{R})$ boils down to understanding the manifold X and the action of Γ on X. With the above isomorphism, it is natural to consider the inclusion

$$\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma.$$

The chain complex $\Omega^*(X)^G$ can be calculated using Lie algebra cohomology, so if we could use this complex to calculate $H^*(\Gamma, \mathbb{R})$, we would be in a much more favourable situation. The aim of the rest of the thesis is then to prove that this inclusion induces an isomorphism on cohomology in low degrees, more specifically in degrees $* \leq \frac{n+1}{4}$ for $n \neq 3$, and in degree zero for $n = 3$.

The first inconvenience we encounter is that the quotient X/Γ is not compact. To solve this, we find a nice compact replacement; this is the Borel-Serre compactification and the content of [3]. The construction proceeds as follows: We add some boundary to the smooth manifold X yielding a smooth manifold with corners $\overline{X}$, which contains X as its interior — one could say that we construct a partial compactification of X. We do this in a way that enables us to extend the action of Γ to $\overline{X}$. This action is also properly discontinuous and the quotient $\overline{X}/\Gamma$ is compact and contains X/Γ as its interior.

It turns out that it suffices to work with a normal torsion free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of finite index, so from now on we let Γ denote such a subgroup instead. The Borel-Serre compactification remains the same. As Γ is torsion free and acts properly discontinuously on X and $\overline{X}$, it acts freely on X and $\overline{X}$. Then X/Γ is a smooth manifold and $\overline{X}/\Gamma$ is a compact smooth manifold with corners.

The second inconvenience is that the de Rham complex of X/Γ is too big. The problem is that we cannot control the growth of an arbitrary differential form on X/Γ near the boundary of $\overline{X}/\Gamma$. To solve this, we define some growth conditions and consider the subcomplex of differential forms satisfying these conditions; crudely put, the differential forms have to grow logarithmically as

they approach the boundary. Now, this subcomplex is particularly nice: The inclusion into the de Rham complex is a quasi isomorphism, it contains the image of $\Omega^*(X)^G$ under the isomorphism $\Omega^*(X)^\Gamma \cong \Omega^*(X/\Gamma)$, and in low degrees the forms with logarithmic growth are square integrable. The final ingredient needed is the fact that in low degrees, harmonic forms on X/Γ are pulled back to G-invariant forms on X via the projection $X \to X/\Gamma$. This is a version of the Matsushima Vanishing Theorem. With this fact, the existence of the above mentioned subcomplex and some classical results on harmonic and square integrable differential forms, we are able to prove that $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma$ does indeed induce an isomorphism on cohomology in low degrees as desired. To finish off, we use that we can now express $H^*(\Gamma; \mathbb{R})$, for $*$ sufficiently small, in terms of Lie algebra cohomology to calculate these cohomology groups. We use a clever little trick, allowing us to consider a compact Lie group and a well known calculation. The range is in fact rather small for a given n, but the bound tends to infinity as n does. Our calculations show that the real cohomology of $\mathrm{SL}_n(\mathbb{Z})$ stabilises as $n \to \infty$ and we are also able to calculate the real cohomology of $\mathrm{SL}_\infty(\mathbb{Z})$.

The thesis is structured as follows:

- In Chapter 1, we go through some preliminary theory that we will need throughout: First, we review several matrix decompositions, then we revise the definition of the Haar measure and of unimodularity and finally we go over the details of induced inner products on exterior algebras and Riemannian manifolds.

- In Chapter 2, we set the scene: We define the smooth manifold X, the action of G on it and make some immediate observations, and we take a closer look at the case $n = 2$. We then show that we have the above mentioned relationship between the real cohomology of Γ and the homology of the complex of Γ-invariant forms on X.

- In Chapter 3, we look at Siegel reduction theory, the aim of which is simply to find a nice subset of X that intersects all Γ-orbits. Again, we take a closer look at the case $n = 2$ and we finish off by proving some technical results.

- Chapters 4,5 and 6 is where the real work is done. The three chapters differ greatly in method: One could say that Chapter 4 is geometric in nature, Chapter 5 analytic and Chapter 6 algebraic.

- In Chapter 4, we go through the Borel-Serre compactification. We define the geodesic action on X, which is the key ingredient, and then directly construct $\overline{X}$. The construction is very technical, so to illustrate the geometry behind it, we include a section on the cases $n = 2$ and $n = 3$.

- In Chapter 5, we define the subcomplex of logarithmic forms, that is, the differential forms on X/Γ satisfying some suitable growth conditions near the boundary of $\overline{X}/\Gamma$. We show that this subcomplex satisfies the three properties mentioned in the above.

- In Chapter 6, we first prove a version of the Matsushima Vanishing Theorem: That a harmonic form on X/Γ of sufficiently low degree is pulled back to a G-invariant form on X via the projection onto X/Γ. Then, finally, we are able to prove that the inclusion $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma$ induces an isomorphism on cohomology in small degrees. We finish off with an actual calculation of $H^*(\Gamma; \mathbb{R})$ in low degrees and we consider the issue of stability.

NOTATION AND CONVENTIONS

We try, as far as possible, to stick to standard notation.

Let M be a manifold (all manifolds are smooth) of dimension n. We opt for the definition of a differential k-form on M as a smooth section

$$\omega \colon M \to \Lambda^k(M) = \bigcup_{p \in M} \Lambda^k(T_p M^*),$$

where $\Lambda^k(M)$ is equipped with the natural smooth structure for which the projection onto M is smooth — the charts are of the form

$$U \times \mathbb{R}^{\binom{n}{k}} \to \Lambda^k(M), \quad (x, v) \mapsto \Lambda^k((D_p \theta^{-1})^*) \circ \varphi(v) \in \Lambda^k(T_p M^*),$$

for a chart $\theta \colon U \to M$ and an isomorphism $\varphi \colon \mathbb{R}^{\binom{n}{k}} \to \Lambda^k((\mathbb{R}^n)^*)$.

Given a chart $\theta \colon \mathbb{R}^n \to U \subseteq M$, the maps $x_i := \mathrm{pr}_i \circ \theta^{-1} \colon U \to \mathbb{R}$ are local coordinates on V, where $\mathrm{pr}_i \colon \mathbb{R}^n \to \mathbb{R}$ is projection onto the i'th coordinate.

Consider the differentials $dx_i \colon U \times \mathbb{R}^n \to \mathbb{R}$, which we shall regard as differential 1-forms on U by evaluating in the first coordinate, $dx_i \colon U \to (\mathbb{R}^n)^* = \Lambda^1((\mathbb{R}^n)^*)$. For $p = \theta(x)$, $\{D_x \theta(e_i)\}_{i=1}^n$ form a basis of $T_p M$ with dual basis $\{dx_i(p)\}_{i=1}^n$ of $T_p M^*$, where (e_i) is the standard basis of $\mathbb{R}^n$. Then the set $\{dx_\sigma = dx_{\sigma(1)}(p) \wedge \cdots \wedge dx_{\sigma(k)}(p)\}_{\sigma \in \Sigma_{k,n-k}}$ forms a basis of $\Omega^k(T_p M^*)$, where $\Sigma_{k,m} \subseteq \Sigma_{k+m}$ denotes the set of permutations $\sigma \in \Sigma_{k+m}$ satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+m)$. For $1 \leq i \leq n$, we can define a vector field $X_i \colon M \to TM$ corresponding to the coordinate x_i: This is simply given by $X_i(p) = D_x \theta(e_i)$ for $p = \theta(x)$, $x \in \mathbb{R}^n$.

As usual, a hat denotes that an element is omitted, for example

$$(x_1, \ldots, \hat{x}_i, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

1 | PRELIMINARIES AND TOOLS

1.1 MATRIX DECOMPOSITIONS

We make use of various matrix decompositions in this project: Iwasawa, Cartan, Langlands, Cholesky and Bruhat. We collect them all here.

Let $G = \mathrm{SL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$.

THEOREM 1.1.1 (Iwasawa Decomposition). Let $A \leq G$ be the subgroup of diagonal matrices with positive entries, $N \leq G$ the subgroup of upper triangular matrices with 1's on the diagonal, and $K = \mathrm{SO}(n) \leq G$ the subgroup of orthogonal matrices. Then the multiplication map

$$K \times A \times N \to G, \qquad (k, a, u) \mapsto kau,$$

is a diffeomorphism.

This is a well-known decomposition and in this special case a standard exercise — we refer to [17] for a proof of the general case, i.e. the decomposition for any connected semisimple real Lie group.
We will also need the following useful Lie algebra decomposition of $\mathfrak{g}$, which is also standard and easy to prove:

PROPOSITION 1.1.2 (Cartan Decomposition). Let $\mathfrak{k} = \mathfrak{so}(n)$ be the set of skew-symmetric matrices with trace zero, and $\mathfrak{p}$ the set of symmetric matrices with trace zero. Then $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

There is also an Iwasawa decomposition of the Lie algebra:

PROPOSITION 1.1.3. Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the subspace of diagonal matrices with trace 0, $\mathfrak{n} \subseteq \mathfrak{g}$ the subspace of strictly upper triangular matrices, and $\mathfrak{k} = \mathfrak{so}(n)$ the subspace of skew-symmetric matrices. Then $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We refer also to [17] for a proof of the general case.

Remark 1.1.4. Note that with the above definitions, $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{a}$ is the Lie algebra of A, and $\mathfrak{n}$ is the Lie algebra of N.

Let $P \leq G$ be a subgroup of block upper triangular matrices (a BUT) defined by a partition κ_P of n: κ_P is given by an increasing sequence of natural numbers $0 = l_0 < l_1 < \cdots < l_k = n$, or equivalently by a tuple of natural numbers $(m_1, \ldots, m_k)$ satisfying $\sum_{i=1}^{k} m_i = n$ (here $m_i = l_i - l_{i-1}$ and $l_j = \sum_{i=1}^{j} m_i$). Then P is the subgroup of matrices $u = (u_{ij}) \in G$ satisfying $u_{ij} = 0$ for $j \leq l_r < i$, $r = 1, \ldots, k$, i.e. the elements of P are block upper triangular matrices such that the i'th diagonal block is an $m_i \times m_i$-matrix, $i = 1, \ldots, k$. Define

$$A_P = \{(a_i) \text{ diagonal} \mid a_{l_i+j} = a_{l_{i+1}} > 0 \text{ for } j = 1, \ldots, m_{i+1}, \ i = 0, \ldots, k-1\},$$
$$N_P = \{(u_{ij}) \text{ upper triangular} \mid u_{ii} > 0 \text{ for } i = 1, \ldots, n$$
$$\text{and } \Pi_{j=1}^{m_{i+1}} u_{l_i+j, l_i+j} = 1 \text{ for } i = 0, \ldots, k-1\}.$$

In other words, A_P is the subgroup of diagonal matrices with positive entries and determinant 1 such that the entries of the i'th block as defined by the partition κ_P are all equal; and N_P is the subgroup of upper triangular matrices with positive diagonal entries and determinant 1 such that the i'th block as defined by κ_P has determinant 1. We let $A_P, N_P \leq G$ inherit the smooth structure of G; thus the maps

$$N_P \longrightarrow \mathbb{R}_{>0}^{n-k} \times \mathbb{R}^{\frac{n(n-1)}{2}}, \quad (u_{ij}) \mapsto ((u_{ii})_{i \neq l_j}, (u_{ij})_{i<j}), \quad A_P \longrightarrow \mathbb{R}_{>0}^{k-1}, \quad a = (a_i) \mapsto (a_{l_i})_{i=1}^{k-1}$$

are diffeomorphisms. In fact, this last map is a Lie group isomorphism into the multiplicative group $\mathbb{R}_{>0}^{k-1}$. If $k = 1$, i.e. $P = G$, then $A_P = \{\mathrm{id}\}$, and we interpret $\mathbb{R}_{>0}^{k-1}$ as a point.

PROPOSITION 1.1.5 (Langlands Decomposition). The multiplication map

$$(K \cap P) \times A_P \times N_P \longrightarrow P, \qquad (k, a, u) \mapsto kau$$

is a diffeomorphism.

This is a consequence of the Iwasawa decomposition (see [17] for the general case, that is, the Langlands decomposition for parabolic subgroups of reductive Lie groups).

Remark 1.1.6. P is a so called *standard parabolic subgroup* of G, and there is a Langlands decomposition of all parabolic subgroups. The group B of upper triangular matrices is a *standard Borel subgroup*, and we see that $A_B = A$ and $N_B = N$, so the Langlands decomposition of B coincides with the restriction of the Iwasawa decomposition to $(K \cap B) \times A \times N$.

DEFINITION 1.1.7. A symmetric matrix $s \in G$ is *positive definite* if $x^t s x > 0$ for all $x \in \mathbb{R}^n - \{0\}$, and it is *positive semi-definite* if $x^t s x \geq 0$ for all $x \in \mathbb{R}^n - \{0\}$.

Note that any positive definite matrix is invertible. In addition we have the following useful result:

PROPOSITION 1.1.8 (Cholesky Decomposition). Any positive definite matrix s can be written uniquely as a product $s = b^t b$, where b is an upper triangular matrix with positive diagonal values.

Proof. We prove the claim by induction on the dimension of s. If $s = (s_{11})$ is a 1×1 positive definite matrix, then $s = b^t b$ for $b = (\sqrt{s_{11}})$, where we use that s being positive definite implies that $s_{11} > 0$. Now, assume that the claim holds for positive definite $(n \times n)$-matrices, and let $s = (s_{ij})$ be an $(n+1) \times (n+1)$-matrix which is positive definite. Write

$$s = \begin{pmatrix} s_{11} & s_{12,1n} \\ s_{12,1n}^t & s' \end{pmatrix}, \quad \text{where } s_{12,1n} = (s_{12}, \ldots, s_{1n}) \text{ and } s' = (s_{ij})_{i,j=2}^n.$$

Consider the matrix $r := s' - \frac{1}{s_{11}} s_{12,1n}^t s_{12,1n}$. Clearly, r is symmetric. We claim that it is also positive definite: Indeed, for any n-dimensional vector $x \neq 0$, let $y = \begin{pmatrix} -\frac{1}{s_{11}}(s_{12,1n}x) \\ x \end{pmatrix}$, and note that

$$0 < y^t s y = \begin{pmatrix} -\frac{1}{s_{11}}(s_{12,1n}x) & x^t \end{pmatrix} \begin{pmatrix} s_{11} & s_{12,1n} \\ s_{12,1n}^t & s' \end{pmatrix} \begin{pmatrix} -\frac{1}{s_{11}}(s_{12,1n}x) \\ x \end{pmatrix}$$

$$= x^t s' x - \frac{1}{s_{11}} x^t s_{12,1n}^t s_{12,1n} x = x^t r x.$$

Our induction hypothesis implies that $r = b_0^t b_0$ for an upper triangular matrix, b_0, with positive diagonal elements. Noting finally that $s_{11} = e_1^t s e_1 > 0$, where e_1 is the first standard basis vector of $\mathbb{R}^{n+1}$, we can define

$$b := \begin{pmatrix} \sqrt{s_{11}} & \frac{1}{\sqrt{s_{11}}} s_{12,1n} \\ 0 & b_0 \end{pmatrix} \quad \text{and we see that} \quad b^t b = \begin{pmatrix} s_{11} & s_{12,1n} \\ s_{12,1n}^t & \frac{1}{s_{11}} s_{12,1n}^t s_{12,1n} + b_0^t b_0 \end{pmatrix} = s.$$

Thus we have proved existence. If $b = (b_{ij})$ is an upper triangular matrix with positive diagonal entries satisfying $b^t b = s = (s_{ij})$, then one can compute the entries b_{ij} recursively:

$$b_{ll} = \sqrt{s_{ll} - \sum_{k=1}^{l-1} b_{kl}^2}, \qquad b_{lj} = \frac{1}{b_{ll}} \left(s_{lj} - \sum_{k=1}^{l-1} b_{kl} b_{kj} \right), \quad l = 1, \ldots, n, \ j > l.$$

From this uniqueness is immediate. $\qquad\qquad\square$

COROLLARY 1.1.9. A matrix g is positive definite, if and only if it admits a Cholesky decomposition, i.e. $g = b^t b$ for an upper triangular matrix, b, with positive diagonal entries.

We will make use of one final matrix decomposition, the Bruhat decomposition.

DEFINITION 1.1.10. For every permutation $\sigma \in \Sigma_n$, define a matrix $w_\sigma \in \mathrm{SL}_n(\mathbb{R})$ such that

$$(w_\sigma)_{1,\sigma(1)} = \mathrm{sign}\,\sigma, \quad (w_\sigma)_{i,\sigma(i)} = 1 \quad \text{for } i \geq 2, \quad \text{and} \quad (w_\sigma)_{i,j} = 0 \quad \text{for } j \neq \sigma(i).$$

The *Weyl group* is the group whose underlying set is $W := \{w_\sigma \mid \sigma \in \Sigma_n\}$, and with composition given by $w_\sigma w_\tau = w_{\sigma \circ \tau}$.

THEOREM 1.1.11 (Bruhat Decomposition). For B the subgroup of upper triangular matrices and N the subgroup of upper triangular matrices with 1's on the diagonal, the sets NwB, $w \in W$, form a partition of G. In particular, any $g \in G$ can be written uniquely as $g = uwav$ for $u, v \in N$, $a \in A$, and $w \in W$.

We refer to [5, Theorem 3.3] for the proof.

1.2 HAAR MEASURE AND UNIMODULARITY

In this section, we consider the Haar measure and the notion of unimodularity. We cover the basics, take a look at the specific matrix groups appearing in this project, and go on to prove an immensely useful result, namely the Iwasawa decomposition of the Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

Let G be an arbitrary real Lie group of dimension n.

We shall make a very brief recap of some basic definitions and results from measure theory, which we need in the following — we refer the reader to [14] for details. Recall that a *Borel measure* on G is a measure on the measurable space $(G, \mathscr{B}(G))$, where $\mathscr{B}(G)$ is the Borel σ-algebra on G, that is the σ-algebra generated by the open subsets of G. If $\varphi \colon G \to H$ is a Borel-measurable map between Lie groups G and H, and μ a Borel measure on G, then we denote by $\varphi_* \mu$ the *image measure* on H, i.e. the measure given by $\varphi_* \mu(U) = \mu(\varphi^{-1}(U))$ for $U \in \mathscr{B}(H)$. We have an abstract *change of variable*-formula:

$$\int_H f(h) \, d\varphi_* \mu(h) = \int_G f \circ \varphi(g) \, d\mu(g), \qquad f \colon H \to \mathbb{R} \text{ integrable}.$$

For a positive Borel-measurable function $f\colon G \to [0, \infty)$, we define a measure ν with *density* f with respect to μ, denoted by $\nu = f.\mu$ and given by $\nu(U) = \int_U f(g)\, d\mu(g)$ for $U \in \mathscr{B}(G)$. Note that if $\varphi\colon G \to H$ and $f\colon G \to [0, \infty)$ are as above, and in addition φ is invertible with φ^{-1} Borel-measurable, then $\varphi_*(f.\mu) = (f \circ \varphi^{-1}).(\varphi_*\mu)$. If we have Borel measures μ and ν on Lie groups G respectively H, then we define the *product measure* $\mu \otimes \nu$ on the product $G \times H$ by setting $(\mu \otimes \nu)(U \times V) = \mu(U)\nu(V)$ for all $U \in \mathscr{B}(G)$, $V \in \mathscr{B}(H)$; as the sets $U \times V$, $U \in \mathscr{B}(G)$, $V \in \mathscr{B}(H)$, generate $\mathscr{B}(G \times H)$, this defines a Borel measure on $G \times H$. If μ and ν are σ-finite, then *Tonelli's Theorem* states that for any integrable function $f\colon G \times H \to \mathbb{R}$.

$$\int_{G \times H} f(g, h)\, d(\mu \otimes \nu)(g, h) = \int_G \left(\int_H f(h, l)\, d\nu(h) \right) d\mu(g).$$

We will in this case write $\int_{G \times H} f(g, h)\, d\mu(g)d\nu(h)$. If φ and ψ are Borel-measurable maps from G respectively H into some other Lie groups, then $(\varphi \times \psi)_*(\mu \otimes \nu) = \varphi_*\mu \otimes \psi_*\nu$.

DEFINITION 1.2.1. A *left Haar measure* on G is a non-zero Borel measure μ on G which is left invariant, i.e. $(L_g)_*\mu = \mu$ for all $g \in G$, where $L_g\colon G \to G$ denotes left translation by g, and which satisfies $\mu(K) < \infty$ for all compact $K \subseteq G$.
Analogously, a *right Haar measure* on G is a non-zero Borel measure μ on G which is right invariant and satisfies $\mu(K) < \infty$ for all compact $K \subseteq G$.
We say that a measure μ on G is *biinvariant*, if it is both left and right invariant.

Remark 1.2.2. As G is locally compact and second countable, and a Haar measure is finite on compact subsets, it is immediate that the Haar measure is σ-finite. It follows from [11, Proposition 7.2.3] that any Haar measure μ is regular, i.e.

$$\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ compact}\} \quad \text{for all } U \subseteq G \text{ open,}$$
$$\mu(F) = \inf\{\mu(U) \mid F \subseteq U \text{ open}\} \qquad \text{for all } F \in \mathscr{B}(G).$$

We refer to [17] for a proof of the following important theorem.

THEOREM 1.2.3. There exists a left (right) Haar measure on G and it is unique up to multiplication by a positive constant.

The existence is simply a consequence of the existence of a left (right) invariant volume form ω on G (see the proof of Lemma 1.2.6 below) and Riesz Representation Theorem: $f \mapsto \int f\omega$, $f \in C_c(G)$, is a linear functional and as such defines a unique regular Borel measure μ on G satisfying $\int_G f\, d\mu = \int f\omega$, $f \in C_c(G)$ (here $C_c(G)$ denotes the compactly supported continuous functions $G \to \mathbb{R}$). Proportionality of any two left (right) Haar measures is proved using the Radon-Nikodym Theorem and Fubini's Theorem.

DEFINITION 1.2.4. Let μ be a left Haar measure on G. Define the *modular function* of G, $\Delta_G\colon G \to \mathbb{R}_{>0}$, such that $(R_g)_*\mu = \Delta_G(g)\mu$ for all $g \in G$, where $R_g\colon G \to G$ denotes right multiplication by g. We write $\Delta = \Delta_G$, when no confusion can occur.

Remark 1.2.5. The above definition makes sense, as $(R_g)_*\mu$ is also left invariant, so Theorem 1.2.3 implies that it is equal to $\lambda\mu$ for some $\lambda \in \mathbb{R}_{>0}$. The same theorem also implies that the definition is independent of the choice of μ.

LEMMA 1.2.6. The modular function $\Delta\colon G \to \mathbb{R}_{>0}$ is given by $\Delta(g) = |\det \operatorname{Ad}(g)|$ for all $g \in G$.

Proof. Let $\{e_i\}_{i=1}^n$ be a basis of the Lie algebra $\mathfrak{g}$ of G with dual basis $\{\epsilon_i\}_{i=1}^n$. Set $\epsilon := \epsilon_1 \wedge \cdots \wedge \epsilon_n$, and define $\omega \in \Omega^n(G)$ by $\omega_g(v_1, \ldots, v_n) = \epsilon(D_g L_{g^{-1}}(v_1), \ldots, D_g L_{g^{-1}}(v_n))$ for all $g \in G$. This is a left-invariant volume form on G, and as such defines a Haar measure μ on G satisfying $\int_G f \, d\mu = \int f \omega$ for all $f \in C_c(G)$. Recall that for linear maps $\omega_i : \mathfrak{g} \to \mathbb{R}$, $v_i \in \mathfrak{g}$, we have

$$(\omega_1 \wedge \cdots \wedge \omega_n)(v_1, \ldots, v_n) = \det\left(\omega_i(v_j)\right)_{i,j}.$$

From this it follows that if $T : T_h G \to T_e G = \mathfrak{g}$ is a linear transformation for some $h \in G$ and $\{f_i\}_{i=1}^n$ is any basis of $T_h G$, then $\epsilon(T(f_1), \ldots, T(f_n)) = \det(T)$. Now, let $g, h \in G$ and let $\{f_i\}_{i=1}^n$ be some basis of $T_h G$. Then

$$
\begin{aligned}
((R_g)^* \omega)_h (f_1, \ldots, f_n) &= \omega_{hg}(D_h R_g(f_1), \ldots, D_h R_g(f_n)) \\
&= \epsilon(D_h(L_{(hg)^{-1}} \circ R_g)(f_1), \ldots, D_h(L_{(hg)^{-1}} \circ R_g)(f_n)) \\
&= \epsilon(\mathrm{Ad}(g^{-1}) \circ D_h L_{h^{-1}}(f_1), \ldots, \mathrm{Ad}(g^{-1}) \circ D_h L_{h^{-1}}(f_n)) \\
&= \det(\mathrm{Ad}(g^{-1}))\epsilon(D_h L_{h^{-1}}(f_1), \ldots, D_h L_{h^{-1}}(f_n)) \\
&= \det(\mathrm{Ad}(g^{-1}))\omega_h(f_1, \ldots, f_n).
\end{aligned}
$$

A differential top form is uniquely determined by its value on an ordered basis of every tangent space, so we conclude that $(R_g)^* \omega = \det(\mathrm{Ad}(g^{-1}))\omega$ for all $g \in G$. Recall the following transformation formula for integrating differential forms (see [17, Proposition 8.19]): If $\Phi : M \to N$ is a diffeomorphism of n-dimensional manifolds, and $\omega \in \Omega^n(N)$, then for all $f \in C_c(N)$

$$\int_N f \omega = \delta \int_M (f \circ \Phi) \Phi^* \omega,$$

where $\delta = 1$ if Φ is orientation-preserving, and $\delta = -1$ if it is orientation-reversing. Then

$$
\begin{aligned}
\int_G f \, d(R_g)_* \mu &= \int_G (f \circ R_g) \, d\mu = \int (f \circ R_g) \omega = \delta \int f (R_{g^{-1}})^* \omega \\
&= \delta \det(\mathrm{Ad}(g)) \int f \omega = \delta \det(\mathrm{Ad}(g)) \int_G f \, d\mu, \qquad \text{for all } f \in C_c(G).
\end{aligned}
$$

Now, $R_{g^{-1}}$ is orientation-preserving if and only if ω and $(R_{g^{-1}})^* \omega = \det(\mathrm{Ad}(g))\omega$ determine the same orientation, which clearly happens if and only if $\det(\mathrm{Ad}(g)) > 0$. Thus, we conclude that

$$\int_G f \, d(R_g)_* \mu = |\det(\mathrm{Ad}(g))| \int_G f \, d\mu, \quad \text{for all } f \in C_c(\mathbb{R}),$$

and the Riesz Representation Theorem implies that $\Delta(g)\mu = (R_g)_* \mu = |\det(\mathrm{Ad}(g))|\mu$ for all $g \in G$. $\qquad \square$

It is easy to see that $\Delta(gh) = \Delta(g)\Delta(h)$ for all $g, h \in G$ and the above lemma in particular shows that Δ is smooth. Thus:

COROLLARY 1.2.7. $\Delta : G \to \mathbb{R}_{>0}$ is a smooth homomorphism into the multiplicative group $\mathbb{R}_{>0}$.

PROPOSITION 1.2.8. Let μ be a left Haar measure on G, and let $\iota : G \to G$ denote the inversion. Then $\iota_* \mu$ and $\Delta.\mu$ are right Haar measures and are equal.

Proof. We see directly that $(R_g)_*\iota_*\mu = (R_g \circ \iota)_*\mu = (\iota \circ L_{g^{-1}})_*\mu = \iota_*(L_{g^{-1}})_*\mu = \iota_*\mu$ for all $g \in G$. Clearly, $\iota_*\mu(K) < \infty$ for all compacts $K \subseteq G$, so $\iota_*\mu$ is a right Haar measure.

As Δ is smooth, it makes sense to define $\Delta.\mu$. It is clear that $\Delta.\mu(K) < \infty$ for all compacts $K \subseteq G$. To see that it is right invariant, let $g \in G$; then

$$(R_g)_*(\Delta.\mu)(U) = \Delta.\mu(Ug^{-1}) = \int \mathbb{1}_{Ug^{-1}}(h)\Delta(h)\,d\mu(h) = \int \mathbb{1}_U(hg)\Delta(h)\,d\mu(h)$$

$$= \int \mathbb{1}_U(h)\Delta(hg^{-1})d(R_g)_*\mu(h) = \int \mathbb{1}_U(h)\Delta(h)\Delta(g)^{-1}\Delta(g)\,d\mu(h) = \Delta.\mu(U)$$

for all $U \in \mathscr{B}(G)$. It follows that $\iota_*\mu = \lambda\Delta.\mu$ for some $\lambda \in \mathbb{R}_{>0}$. Then

$$\mu = \iota_*\iota_*\mu = \lambda\iota_*(\Delta.\mu) = \lambda(\Delta \circ \iota).(\iota_*\mu) = \lambda^2((\Delta \circ \iota) \cdot (\Delta)).\mu = \lambda^2\mu.$$

Thus, we must have $\lambda = 1$, and as desired $\iota_*\mu = \Delta.\mu$. $\qquad\square$

DEFINITION 1.2.9. We say that G is *unimodular*, if the modular function is identically 1.

PROPOSITION 1.2.10. *G is unimodular if and only if any left or right Haar measure on G is biinvariant.*

Proof. This follows from Proposition 1.2.8. For the left to right implication, let μ be a left Haar measure on G. Then $\mu = \Delta.\mu$ is a right Haar measure, so μ is biinvariant. It follows that any right Haar measure is proportional to μ and as such also left invariant. For the converse implication, note that $\lambda\mu = \Delta.\mu$ for some $\lambda > 0$, as both μ and $\Delta.\mu$ are right Haar measures. Hence, Δ is constantly equal to λ, and as $\Delta(e) = 1$, we must have $\lambda = 1$. $\qquad\square$

PROPOSITION 1.2.11. *The following types of Lie groups are unimodular:*

1. Compact Lie groups.

2. Abelian Lie groups.

3. Nilpotent Lie groups.

Proof. The first two are easy: On an abelian Lie group, any Haar measure is biinvariant; the image of a compact Lie group under the modular function is a compact subgroup of $\mathbb{R}_{>0}$ and therefore equal to $\{1\}$. For the third case, suppose G is a nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then for any $x \in \mathfrak{g}$, $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ is a nilpotent linear transformation, so it has all eigenvalues equal to zero and

$$\det\mathrm{Ad}(\exp(x)) = \det e^{\mathrm{ad}(x)} = e^{\mathrm{tr}(\mathrm{ad}(x))} = e^0 = 1.$$

As G is connected by assumption, $\exp(\mathfrak{g})$ generates G, and we conclude that

$$\Delta_G(g) = |\det\mathrm{Ad}(g)| = 1 \quad \text{for all} \quad g \in G.$$

$$\square$$

The following proposition gives a useful decomposition of the Haar measure, when the Lie group itself can be decomposed as a product.

PROPOSITION 1.2.12. *Let $H, L \leq G$ be closed subgroups such that the multiplication map $\Phi\colon H \times L \to G$ is a homeomorphism. Then the left Haar measures on G, H and L, denoted by μ_G, μ_H, μ_L, respectively, can be scaled such that*

$$(\Phi^{-1})_*\mu_G = (\tfrac{\Delta_L}{\Delta_G} \circ \pi_L).(\mu_H \otimes \mu_L),$$

where $\pi_L\colon H \times L \to L$ denotes the projection onto L

Proof. Note first that $H \times L$ acts on G from the left by $(h,l).g = hgl^{-1}$, and it acts naturally on itself by left multiplication. Then the map $\varphi \colon H \times L \to G$, $(h,l) \mapsto hl^{-1}$ is equivariant and a homeomorphism.

Set $\mu := (\varphi^{-1})_* \mu_G$ and note that

$$(R_{l^{-1}} \circ L_h)_* \mu_G = (R_{l^{-1}})_* \mu_G = \Delta_G(l)^{-1} \mu_G \quad \text{for any } (h,l) \in H \times L.$$

Then, as φ^{-1} is equivariant, we have

$$(L_{(h,l)})_* \mu = (\varphi^{-1})_* (R_{l^{-1}})_* (L_h)_* \mu_G = \Delta_G(l)^{-1} \mu \quad \text{for any } (h,l) \in H \times L.$$

Set $\nu := (\Delta_G|_L \circ \iota \circ \pi_L).\mu$, where ι is the inversion on L. For all $(h,l) \in H \times L$, we have $\Delta_G \circ \iota \circ \pi_L \circ L_{(h^{-1},l^{-1})} = \Delta_G(l)(\Delta_G \circ \iota \circ \pi_L)$ and therefore

$$(L_{(h,l)})_* \nu = (\Delta_G|_L \circ \iota \circ \pi_L \circ L_{(h^{-1},l^{-1})}).(L_{(h,l)})_* \mu = \Delta_G(l)^{-1} \Delta_G(l)(\Delta_G \circ \iota \circ \pi_L).\mu = \nu.$$

The measure ν is finite on compact sets as μ is; thus, ν is a left Haar measure on $H \times L$ and therefore after appropriately scaling μ_H and μ_L, we have $\nu = \mu_H \otimes \mu_L$. This in turn implies that $\mu = (\Delta_G \circ \pi_L).(\mu_H \otimes \mu_L)$.

From this we see that for any Borel-measurable function $f \colon H \times L \to [0, \infty)$, we have

$$
\begin{aligned}
\int_{H \times L} f(h,l) \, d(\Phi^{-1})_* \mu_G(h,l) &= \int_G f \circ \Phi^{-1}(g) \, d\mu_G(g) \\
&= \int_G f \circ \Phi^{-1} \circ \varphi(h,l) \, d(\varphi^{-1})_* \mu_G(h,l) \\
&= \int_{H \times L} f(h,l^{-1}) \, d\mu(h,l) \\
&= \int_{H \times L} f(h,l^{-1}) \Delta_G(l) \, d\mu_H(h) d\mu_L(l) \\
&= \int_{H \times L} f(h,l) \Delta_G(l)^{-1} \, d\mu_H(h) d\iota_* \mu_L(l) \\
&= \int_{H \times L} f(h,l) \frac{\Delta_L(l)}{\Delta_G(l)} \, d\mu_H(h) d\mu_L(l),
\end{aligned}
$$

implying $(\Phi^{-1})_* \mu_G = (\frac{\Delta_L}{\Delta_G} \circ \pi_L).(\mu_H \otimes \mu_L)$, as desired. □

Remark 1.2.13. The above proposition can be generalised considerably: The intersection $H \cap L$ need only be compact and HL need not be equal to G, but the difference should be a null set (see [17]). We will, however, only need the above version.

IWASAWA DECOMPOSITION OF THE HAAR MEASURE

Let $G = \mathrm{SL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$, and let $\Psi \colon K \times A \times N \to G$, $(k,a,u) \mapsto kau$ denote the Iwasawa diffeomorphism of Theorem 1.1.1, where $K = \mathrm{SO}(n)$, $A \leq G$ is the subgroup of diagonal matrices with positive entries, and $N \leq G$ the subgroup of upper triangular matrices with 1's on the diagonal. Recall from Proposition 1.1.3 that we also have a decomposition of the Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{a}$ is the set of diagonal matrices with positive entries and trace 0, and $\mathfrak{n}$ is the set of strictly upper triangular matrices.

PROPOSITION 1.2.14. The Lie groups K, A and N are all unimodular.

Proof. The Lie algebra of N is the set $\mathfrak{n}$ of strictly upper triangular matrices; it is clear that this is a nilpotent Lie algebra, so N is a nilpotent Lie group. A is abelian and K is compact. Proposition 1.2.11 then implies that K, A and N are unimodular. $\qquad\square$

PROPOSITION 1.2.15. *The Lie group G is unimodular.*

Proof. To see this, we show directly that $\Delta = \Delta_G \colon G \to \mathbb{R}_{>0}$ is identically 1. As Δ is multiplicative, it suffices to consider the elements of K, A and N separately. Note first that $\Delta(K) = \{1\}$ as it is a compact subgroup of $\mathbb{R}_{>0}$. For A and N, we will use Lemma 1.2.6. As $\mathrm{Ad}(g)$ is linear for all $g \in G$, we can evaluate $\mathrm{Ad}(g) \colon \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ separately for any $g \in G$. Note that $\{E_{ij} - E_{ji}\}_{i<j}$ is a basis of $\mathfrak{k}$, and that $\{E_{ij}\}_{i<j}$ is a basis of $\mathfrak{n}$, where E_{ij} is the matrix with (i,j)'th entry equal to 1 and all other entries equal to zero.

Let $a = (a_i) \in A$. For any $\hat{a} \in \mathfrak{a}$, $\mathrm{Ad}(a)(\hat{a}) = a\hat{a}a^{-1} = \hat{a}$, so $\mathrm{Ad}(a)|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$. For any $1 \leq i < j \leq n$,

$$\mathrm{Ad}(a)(E_{ij}) = aE_{ij}a^{-1} = \frac{a_i}{a_j}E_{ij},$$

and

$$\mathrm{Ad}(a)(E_{ij} - E_{ji}) = a(E_{ij} - E_{ji})a^{-1} = \frac{a_i}{a_j}E_{ij} - \frac{a_j}{a_i}E_{ji} = \frac{a_j}{a_i}(E_{ij} - E_{ji}) + \frac{a_i^2 - a_j^2}{a_i a_j}E_{ij}.$$

If we take any basis $\{\hat{a}_i\}_i$ of $\mathfrak{a}$, then with respect to the basis

$$\{E_{ij} - E_{ji}\}_{i<j} \cup \{\hat{a}_i\}_i \cup \{E_{ij}\}_{i<j} \quad \text{of} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

$\mathrm{Ad}(a)$ is given by the matrix

$$\begin{pmatrix} \left(\frac{a_j}{a_i}\right)_{i<j} & 0 & 0 \\ 0 & \mathrm{id}_{\mathfrak{a}} & 0 \\ \left(\frac{a_i^2 - a_j^2}{a_i a_j}\right)_{i<j} & 0 & \left(\frac{a_i}{a_j}\right)_{i<j} \end{pmatrix},$$

where $(x_{i,j})_{i<j}$ is the diagonal matrix with entries $x_{i,j}$, and thus

$$\det \mathrm{Ad}(a) = \prod_{i<j} \frac{a_j}{a_i} \cdot \prod_{i<j} \frac{a_i}{a_j} = 1,$$

which gives us $\Delta(a) = 1$ for all $a \in A$. Let $\hat{u} = (\hat{u}_{ij}) \in \mathfrak{n}$. Then

$$\det \mathrm{Ad}(\exp(\hat{u})) = \det e^{\mathrm{ad}(\hat{u})} = e^{\mathrm{tr}(\mathrm{ad}(\hat{u}))}.$$

We will show that $\mathrm{ad}(\hat{u})$ has trace zero. Note first that as $\mathrm{ad}(\hat{u})$ is nilpotent on $\mathfrak{a} \oplus \mathfrak{n}$, so $\mathrm{ad}(\hat{u})|_{\mathfrak{a} \oplus \mathfrak{n}}$ has trace 0, having all eigenvalues equal to zero. For any basis $\{\hat{b}_i\}_i$ of $\mathfrak{a} \oplus \mathfrak{n}$, we can write the matrix of $\mathrm{ad}(\hat{u})$ with respect to the basis $\{E_{ij} - E_{ji}\}_{i<j} \cup \{\hat{b}_i\}_i$ of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as

$$\begin{pmatrix} \widetilde{u} & 0 \\ * & \mathrm{ad}(\hat{u})|_{\mathfrak{a} \oplus \mathfrak{n}} \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \widetilde{u} \\ * \end{pmatrix} \text{ is the matrix representing } \mathrm{ad}(\hat{u})|_{\mathfrak{k}} \colon \mathfrak{k} \to \mathfrak{g}.$$

Calculations show that for any $1 \leq i < j \leq n$

$$
\begin{aligned}
\mathrm{ad}(\hat{u})(E_{ij} - E_{ji}) &= \hat{u}(E_{ij} - E_{ji}) - (E_{ij} - E_{ji})\hat{u} \\
&= \sum_{i<l} \hat{u}_{il} E_{jl} - \sum_{l<j} \hat{u}_{lj} E_{li} + \sum_{l<i} \hat{u}_{li} E_{lj} - \sum_{j<l} \hat{u}_{jl} E_{il} \\
&= \sum_{i<l<j} \left(\hat{u}_{lj}(E_{il} - E_{li}) - \hat{u}_{il}(E_{lj} - E_{jl}) \right) + \hat{u}_{ij}(E_{jj} - E_{ii}) \\
&\quad + \sum_{l<i}(\hat{u}_{li} E_{lj} - \hat{u}_{lj} E_{li}) + \sum_{j<l}(\hat{u}_{il} E_{jl} - \hat{u}_{lj} E_{il}) + \sum_{i<l<j}(\hat{u}_{il} E_{lj} - \hat{u}_{lj} E_{il}).
\end{aligned}
$$

Here the first sum is an element of $\mathfrak{k}$, the element $\hat{u}_{ij}(E_{jj} - E_{ii})$ is an element of $\mathfrak{a}$, and the last three sums are elements of $\mathfrak{n}$. We see that $E_{ij} - E_{ji}$ does not appear in the $\mathfrak{k}$-element, and thus conclude that all diagonal entries of $\widetilde{u}$ are zero. Hence, $\mathrm{tr}(\mathrm{ad}(\hat{u})) = 0$. As $\exp(\mathfrak{n})$ generates N, we conclude that $\Delta(u) = |\det \mathrm{Ad}(u)| = 1$ for all $u \in N$. $\qquad\square$

Remark 1.2.16. In fact, all semisimple Lie groups are unimodular (see for example [17]).

Remark 1.2.17. Some of the computations in the above proof will come in handy further on.

PROPOSITION 1.2.18. Let μ_G denote the Haar measure on G. There is a decomposition of μ_G corresponding to the Iwasawa decomposition of G:

$$
(\Psi^{-1})_* \mu_G = (\rho \circ \pi_A).(\mu_K \otimes \mu_A \otimes \mu_N)
$$

for appropriate scalings of the Haar measures on K, A and N, and where $\rho \colon A \to \mathbb{R}_{>0}$, $\rho(a) = \prod_{i<j} \frac{a_i}{a_j}$ for $a = (a_i) \in A$, and $\pi_A \colon K \times A \times N \to A$ is the projection.

Proof. Let $\varphi \colon K \times AN \to G$, denote the diffeomorphism $(k, b) \mapsto kb$. Then by Proposition 1.2.12

$$
(\varphi^{-1})_* \mu_G = (\tfrac{\Delta_{AN}}{\Delta_G} \circ \pi_{AN}).(\mu_K \otimes \mu_{AN}) = (\Delta_{AN} \circ \pi_{AN}).(\mu_K \otimes \mu_{AN}).
$$

Letting $\psi \colon A \times N \to AN$ denote the diffeomorphism $(a, u) \mapsto au$, and applying Proposition 1.2.12 again, we have that

$$
(\psi^{-1})_* \mu_{AN} = (\tfrac{\Delta_N}{\Delta_{AN}} \circ \pi_N).(\mu_K \otimes \mu_{AN}) = (\Delta_{AN} \circ \iota \circ \pi_N).(\mu_A \otimes \mu_N).
$$

Now, we determine the modular function Δ_{AN}; recall that its Lie algebra is $\mathfrak{a} \oplus \mathfrak{n}$, the set of upper triangular matrices with trace zero. From the proof of Proposition 1.2.15, we see that $\Delta_{AN}(u) = |\det(\mathrm{Ad}_{\mathfrak{a}\oplus\mathfrak{n}}(u))| = 1$ for all $u \in N$ as $\mathrm{ad}_{\mathfrak{a}\oplus\mathfrak{n}}(\hat{u})$ is a nilpotent linear transformation on $\mathfrak{a} \oplus \mathfrak{n}$ for all $\hat{u} \in \mathfrak{n}$. The proof of Proposition 1.2.15 also shows that for any $a = (a_i) \in A$

$$
\det \mathrm{Ad}_{\mathfrak{a}\oplus\mathfrak{n}}(a) = \prod_{i<j} \frac{a_i}{a_j}.
$$

We conclude that $\Delta_{AN}(au) = \Delta_{AN}(a) = \prod_{i<j} \frac{a_i}{a_j}$ for all $a = (a_i) \in A$, $u \in N$.

Thus we see that $(\psi^{-1})_* \mu_{AN} = \mu_A \otimes \mu_N$ and as $\Psi = \varphi \circ (\mathrm{id} \times \psi)$, we have

$$
\begin{aligned}
(\Psi^{-1})_* \mu_G &= (\mathrm{id} \times \psi^{-1})_*(\varphi^{-1})_* \mu_G = (\mathrm{id} \times \psi^{-1})_*((\Delta_{AN} \circ \pi_{AN}).(\mu_K \otimes \mu_{AN})) \\
&= (\Delta_{AN} \circ \pi_{AN} \circ (\mathrm{id} \times \psi)).(\mu_K \otimes (\psi^{-1})_* \mu_{AN}) = (\rho \circ \pi_A).(\mu_K \otimes \mu_A \otimes \mu_N).
\end{aligned}
$$

$\qquad\square$

Remark 1.2.19. The above decomposition holds for all reductive Lie groups as they all admit an Iwasawa decomposition.

1.3 Inner products

There is a natural choice of inner product on the exterior algebra of an oriented inner product space. In this section, we go through the details involved as it will be useful to have a good understanding of the steps in this construction. To finish off, we apply it to an oriented Riemannian manifold and define the notion of a square integrable differential form.

Exterior Algebra

Let V be an n-dimensional real vector space equipped with an inner product $\langle -, - \rangle$. Let $e_1, \ldots, e_n$ be a basis of V and denote by $e^1, \ldots, e^n$ the dual basis of V^*, i.e. $e^i(e_j) = \delta_{ij}$. The inner product gives rise to an isomorphism

$$V \longrightarrow V^*, \qquad v \mapsto \hat{v} := \langle v, - \rangle.$$

In terms of our chosen bases, this is given by the matrix

$$g = (g_{ij}) \quad \text{with entries} \quad g_{ij} = \langle e_i, e_j \rangle.$$

This dualises to an isomorphism

$$(V^*)^* \longrightarrow V^*, \qquad f \mapsto (\check{f} \colon v \mapsto f(\hat{v})),$$

which is also given by the matrix g as g is symmetric. So we have an isomorphism $V^* \to (V^*)^*$ given by the matrix $g^{-1} = (g^{ij})$, and thus an inner product

$$\langle -, - \rangle^* \colon V^* \times V^* \to \mathbb{R}, \qquad \langle v, w \rangle^* = g^{-1}(v)(w).$$

There is an induced inner product on the k'th exterior power of V, $\langle -, - \rangle_k \colon \Lambda^k V \times \Lambda^k V \to \mathbb{R}$ defined on elementary wedges by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle_k = \det \left(\langle v_i, w_j \rangle \right),$$

i.e. the determinant of the matrix with entries $\langle v_i, w_j \rangle$, and extended bilinearly.
The elements

$$\epsilon_\sigma := e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}, \qquad \sigma \in \Sigma_{k, n-k},$$

form a basis of $\Lambda^k V$ with respect to $\langle -, - \rangle_k$ and

$$\langle \epsilon_\sigma, \epsilon_\tau \rangle_k = \det \left(g_{\sigma(i)\tau(j)} \right).$$

If $e_1, \ldots, e_n$ is an orthonormal basis of V, then $\{\epsilon_\sigma\}_{\sigma \in \Sigma_{k, n-k}}$ is an orthonormal basis of $\Lambda^k V$. The inner product gives rise to an orientation form on V, the choice of which is canonical up to sign, and if V is already oriented, then there is a canonical choice respecting this orientation. Indeed, we have an inner product $\langle -, - \rangle_n^*$ on the one-dimensional vector space $\Lambda^n V^*$, which then has exactly two unit vectors. If V is oriented, then the unit vector respecting this orientation is the canonical choice of orientation form induced by $\langle -, - \rangle$. If (e_i) is a positively oriented basis, then the orientation form induced by $\langle -, - \rangle$ is

$$\sqrt{\det g} \; e^1 \wedge \cdots \wedge e^n$$

as $\|e^1 \wedge \cdots \wedge e^n\|^2 = \det(g^{-1})$.

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Let M be a connected oriented Riemannian manifold of dimension n with metric tensor g. We alter between interpreting g as a smoothly varying family of inner products on the tangent spaces and a $C^\infty(M)$-bilinear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$. Then g induces a volume form $\omega_0 \in \Omega^n(M)$ as it induces a canonical orientation form on T_xM for every x by the above. On a positively oriented coordinate system $U = (x_1, \ldots, x_n)$ on M, ω_0 is given by

$$\omega_0|_U = \sqrt{\det g^U}\, dx_1 \wedge \cdots \wedge dx_n,$$

where g^U is the matrix representation of g in the local coordinates, i.e. $g_{ij}^U = g(X_i, X_j)$ with X_i the vector field corresponding to the coordinate x_i as defined in the preliminaries. Denote by μ_M the corresponding Borel measure.

Let $x \in M$. As in the above section the inner product g_x on T_xM induces an inner product on the k'th exterior power of the cotangent space, $\Lambda^k T_xM^*$, which we denote by $\langle -, - \rangle_g^x$ (k will be implicit). As $x \mapsto g_x$ is smooth, so is $x \mapsto \langle -, - \rangle_g^x$.

DEFINITION 1.3.1. A differential form $\omega \in \Omega^k(M)$ is *square integrable*, if

$$\|\omega\|_M^2 := \int \langle \omega_x, \omega_x \rangle_g^x\, d\mu_M. < \infty.$$

We denote by $\Omega_{(2)}^k(M)$ the set of all square integrable k-forms on M.

Remark 1.3.2. On $\Omega_{(2)}^k(M)$, there is an inner product, $(-, -)_M$, given by

$$(-, -)_M \colon \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathbb{R}, \qquad (\alpha, \beta)_M = \int \langle \alpha_x, \beta_x \rangle_g^x\, d\mu_M.$$

We denote by $\| - \|_M$ the corresponding norm.

Suppose N is another connected oriented Riemannian manifold with metric tensor h, and that $f \colon N \to M$ is an immersion. Suppose in addition that $h = f^*g$, that is, for $p \in N$, $v, w \in T_pN$, $h_p(v, w) = g_{f(p)}(D_pf(v), D_pf(w))$. Then $\omega_h = f^*\omega_g$ and $\mu_g = f_*\mu_h$. Moreover,

$$\langle \omega_{f(p)}, \omega'_{f(p)} \rangle_g^{f(p)} = \langle (f^*\omega)_p, (f^*\omega')_p \rangle_h^p \qquad \text{for all } p \in N,\ \omega, \omega' \in \Omega^k(M).$$

2 | SETTING THE SCENE

To calculate the real cohomology of $\mathrm{SL}_n(\mathbb{Z})$, we exploit the action of $\mathrm{SL}_n(\mathbb{Z})$ on a specific manifold X. In this chapter, we construct the manifold X, define an action of $\mathrm{SL}_n(\mathbb{R})$, and hence of $\mathrm{SL}_n(\mathbb{Z})$, on it and finally show how this setting can be exploited to express the group cohomology of $\mathrm{SL}_n(\mathbb{Z})$ in terms of a certain subcomplex of the de Rham complex of X. This places us in the realm of geometry with an immense selection of tools at hand; the rest of the project then exploits this. We take a closer look at the construction in the case $n = 2$, which will be our running example.

2.1 QUADRATIC FORMS

In this section, we construct the space X of quadratic forms inducing the same volume as the standard inner product and equip it with a natural smooth structure. There is a natural action of $\mathrm{SL}_n(\mathbb{R})$ on X and this action is proper. We also show that the four manifolds which we will be considering in this project are orientable.

CONSTRUCTION

Let $n \geq 2$.

DEFINITION 2.1.1. An n-ary *quadratic form* over a field k is a homogeneous polynomial of degree two in n variables.

A quadratic form q must be of the form $q(x_1, \ldots, x_n) = \sum_{1 \leq i,j \leq n} s_{ij} x_i x_j$, for $s_{ij} \in k$. If we write $x = (x_1, \ldots, x_n)$, this reads

$q(x) = x^t s x$, where $s = (s_{ij})$ is the $n \times n$-matrix with entries the coefficients above.

As the matrix $(s + s^t)/2$ is symmetric and gives rise to the same quadratic form q, we may and do always assume that s is symmetric. With this assumption, s is uniquely given, and

$$q(x) = x^t s x = \sum_{i=1}^n s_{ii} x_i^2 + \sum_{1 \leq i < j \leq n} 2\, s_{ij} x_i x_j.$$

From now on, we consider only real quadratic forms, $k = \mathbb{R}$, and we will additionally assume that the variables are real, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Under these assumptions, we say that q is a quadratic form on $\mathbb{R}^n$.

DEFINITION 2.1.2. A quadratic form q is *positive definite*, if $q(x) > 0$ for all $x \neq 0$, and it is *positive semi-definite*, if $q(x) \geq 0$ for all $x \neq 0$.

Remark 2.1.3. A quadratic form is positive definite if and only if the symmetric matrix defining it is (cf. Definition 1.1.7).

Remark 2.1.4. A positive definite quadratic form q on $\mathbb{R}^n$ induces an inner product on $\mathbb{R}^n$: $\langle x, y \rangle^q = x^t s y$, $x, y \in \mathbb{R}^n$, where s is the symmetric matrix defining q.

Let (e_i) be the standard basis of $\mathbb{R}^n$, e^i the dual basis of $\mathbb{R}_n^*$. Applying the machinery of Section 1.3, q induces an orientation form on $\mathbb{R}^n$ given by

$$\epsilon_q := \sqrt{\det(s)}\, e^1 \wedge \cdots e^n.$$

DEFINITION 2.1.5. We say that ϵ_q as defined above is the *volume* induced by q on $\mathbb{R}^n$.

PROPOSITION 2.1.6. Two quadratic forms q and q' given by the matrices s respectively s' induce the same volume on $\mathbb{R}^n$, if and only if $\det s = \det s'$. In particular, q induces the same volume as the standard inner product, i.e. as the quadratic form $\iota\colon x \mapsto x^t x$, if and only if $\det s = 1$.

Construction 2.1.7. We are interested in these latter quadratic forms: Let X denote the set of positive definite quadratic forms on $\mathbb{R}^n$ inducing the same volume as the standard inner product. By the above observations, this is in bijection with the set of real positive definite symmetric matrices of determinant 1, which we will denote by $\mathscr{S}(\mathbb{R})$. Cholesky decomposition (cf. Proposition 1.1.8) implies that this is in bijection with the set of upper triangular matrices with positive diagonal entries and determinant 1, AN; here A denotes the set of diagonal matrices with positive entries and determinant 1, and N denotes the set of upper triangular matrices with 1's on the diagonal. Denote these bijections by

$$\Phi\colon X \longrightarrow \mathscr{S}(\mathbb{R}), \qquad (q\colon x \mapsto x^t s x) \;\mapsto\; s$$
$$\Psi\colon \mathscr{S}(\mathbb{R}) \longrightarrow AN, \qquad s = b^t b \;\mapsto\; b.$$

Now, AN inherits the smooth structure of $\mathrm{SL}_n(\mathbb{R})$, being a subgroup, and we let X and $\mathscr{S}(\mathbb{R})$ inherit the smooth structure of AN under the bijections Φ and Ψ.

PROPOSITION 2.1.8. The inclusion $\mathscr{S}(\mathbb{R}) \hookrightarrow \mathrm{SL}_n(\mathbb{R})$ is smooth.

Proof. The inclusion is equal to the following composition of smooth maps

$$\mathscr{S}(\mathbb{R}) \xrightarrow{\cong} AN \xrightarrow{\mathrm{incl}} \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \xrightarrow{t \times \mathrm{id}} \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \xrightarrow{m} \mathrm{SL}_n(\mathbb{R}),$$

where $\mathrm{incl}(b) = (b,b)$, t is transposition, and m is the multiplication map. $\qquad\square$

So in fact, $\mathscr{S}(\mathbb{R})$ is a submanifold of $\mathrm{SL}_n(\mathbb{R})$.

PROPOSITION 2.1.9. X is diffeomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$. In particular, it is contractible.

This is immediate from the construction.

Remark 2.1.10. For illustrative purposes it is sometimes helpful to picture a positive definite quadratic form on $\mathbb{R}^n$ as an ellipsoid in $\mathbb{R}^n$, namely the unit ball with respect to the norm induced by the given quadratic form. This will be particularly helpful when considering group actions on X, which we look at in the following section. For example, the quadratic form $\iota\colon x \mapsto x^t x$ gives rise to the unit ball in $\mathbb{R}^n$, as $\langle -, - \rangle^\iota$ is the standard inner product — when we need a basepoint, this will be our choice. If $q \in X$ is a quadratic form represented by a diagonal matrix $d = (d_i)$, then it gives rise to the ellipsoid given by the equation $\sum_{i=1}^n d_i x_i^2 = 1$; thus its semi-principal axes follow the standard axes of $\mathbb{R}^n$ and the i'th semi-principal axis is of length $\frac{1}{\sqrt{d_i}}$. The non-diagonal positive definite symmetric matrices give rise to ellipsoids with axes not following the standard axes. The condition that the quadratic forms induce the same volume as ι simply implies that the ellipsoids all have the same Euclidean volume.

GROUP ACTION

There is an obvious action of $G = \mathrm{SL}_n(\mathbb{R})$ on X, namely precomposition:

$$(q.g)(x) = q(gx) \quad \text{for } x \in \mathbb{R}^n,\ q \in X,\ g \in G.$$

This is a right action and it is immediately seen that if q is represented by the symmetric matrix s, then $q.g$ is represented by $g^t s g$.

PROPOSITION 2.1.11. The action $\alpha\colon X \times G \to X$, $(q, g) \mapsto q.g$ is smooth.

Proof. It is obvious that the map $\alpha'\colon G \times G \to G$, $(s, g) \mapsto g^t s g$ is smooth. Consider the inclusion $i\colon \mathscr{S}(\mathbb{R}) \hookrightarrow G$ and the diffeomorphism $\Phi\colon X \to \mathscr{S}(\mathbb{R})$ given in Construction 2.1.7. We have the following commutative diagram

$$
\begin{array}{ccccc}
X \times G & \xrightarrow{\ \Phi \times \mathrm{id}\ } & \mathscr{S}(\mathbb{R}) \times G & \xhookrightarrow{\ i \times \mathrm{id}\ } & G \times G \\
{\scriptstyle \alpha}\big\downarrow & & & & \big\downarrow {\scriptstyle \alpha'} \\
X & \xrightarrow[\ \Phi\ ]{} & \mathscr{S}(\mathbb{R}) & \xhookrightarrow[\ i\ ]{} & G
\end{array}
$$

As the image of $\alpha' \circ (i \times \mathrm{id})$ is in the image of i, this diagram shows that α is smooth. $\qquad\square$

Cholesky decomposition (cf. Proposition 1.1.8) implies that the subgroup B of upper triangular matrices with determinant 1 acts transitively on X; hence any BUT P acts transitively on X as it contains B. The stabiliser of $X \curvearrowleft G$ at ι is $K = \mathrm{SO}(n)$; more generally, for a BUT P, the stabiliser at ι of the restriction of this action, $X \curvearrowleft P$, is $K \cap P$, which is the subgroup of block diagonal matrices where each block is an orthogonal matrix. Hence, in view of Proposition A.2.3, we have:

PROPOSITION 2.1.12. X is diffeomorphic to the homogeneous spaces $(K \cap P)\backslash P$ via the map $[g] \mapsto \iota.g$ for any BUT P.

Remark 2.1.13. The diffeomorphism

$$(K \cap P)\backslash P \longrightarrow \mathscr{S}(\mathbb{R})$$

is given by $[g] \mapsto g^t g$.

Let P be a BUT, and recall the Langlands decomposition $(K \cap P) \times A_P \times N_P \to P$ (cf. Proposition 1.1.5). This combined with the above proposition immediately yields the following result:

COROLLARY 2.1.14. The map $A_P \times N_P \to X$, $(a, u) \mapsto \iota.(au)$ is a diffeomorphism.

Recall that an action, $Y \curvearrowleft H$, is proper if the map $Y \times H \to Y \times Y$, $(y, h) \mapsto (y.h, y)$ is proper (see also Appendix A.1). As a corollary of the results in the appendix, we have the following proposition.

PROPOSITION 2.1.15. The action $X \curvearrowleft G$ is proper, and the action $X \curvearrowleft \mathrm{SL}_n(\mathbb{R})$ is properly discontinuous.

Proof. As X is diffeomorphic to the homogeneous space $K\backslash G$, with the action of G on X corresponding to right multiplication on the coset space, the first claim is a direct consequence of Corollary A.1.5. The second is a consequence of the first and Corollary A.1.7 since $\mathrm{SL}_n(\mathbb{R})$ is discrete and closed in G. $\qquad\square$

In the following, we identify X with $K\backslash G$. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We alter between interpreting $\mathfrak{g}$ as $T_e G$ and as the set of right-invariant vector fields on G — it should not cause any confusion. Write $\Gamma = \mathrm{SL}_n(\mathbb{R})$.

LEMMA 2.1.16. *Let $\omega \in \Omega^k(G)$. Then $\omega = \rho^*\eta$ for some $\eta \in \Omega^k(X)$, if and only if $\iota_x \omega = \mathscr{L}_x \omega = 0$ for all $x \in \mathfrak{k}$, where ι_x is the interior product on differential forms and $\mathscr{L}_x$ is the Lie derivative.*

Proof. Consider the diagram below, where the downwards directed vertical maps are the projections: we have to show that η exists if and only if $\iota_x \omega = \mathscr{L}_x \omega = 0$ for all $x \in \mathfrak{k}$.

$$
\begin{array}{ccc}
(\Lambda^k TG)^* & \xleftarrow{\;(\Lambda^k D\rho)^*\;} & (\Lambda^k T(K\backslash G))^* \\
\omega \Big\uparrow \Big\downarrow & & \Big\downarrow \Big\uparrow \eta \\
G & \xrightarrow{\quad\rho\quad} & K\backslash G
\end{array}
$$

To prove the left to right implication, let $\eta \in \Omega^k(K\backslash G)$, $x \in \mathfrak{k}$, and note that for all $g \in G$, $v_1, \ldots, v_{k-1} \in T_g G$,

$$
\begin{aligned}
(\iota_x \rho^*\eta)_g(v_1, \ldots, v_{k-1}) &= (\rho^*\eta)_g(D_e R_g(x), v_1, \ldots, v_{k-1}) \\
&= \eta_{\rho(g)}(D_g\rho \circ D_e R_g(x), D_g\rho(v_1), \ldots, D_g\rho(v_{k-1})) = 0,
\end{aligned}
$$

as $D_g\rho \circ D_e R_g(x) = D_{\rho(g)} g \circ D_e \rho(x) = 0$, where as usual $g \colon X \to X$ denotes the action of g on X given by right multiplication on $K\backslash G$.

Recall that $\mathbb{R} \times G \to G$, $(t, g) \mapsto L_{\exp(tx)}(g)$, is the flow of the right invariant vector field x. For any $g \in G$ we therefore have

$$
\mathscr{L}_x(\rho^*\eta)_g = \frac{d}{dt}\Big|_{t=0}(L^*_{\exp(tx)}\rho^*\eta)_g = \frac{d}{dt}\Big|_{t=0}((\rho \circ L_{\exp(tx)})^*\eta)_g = \frac{d}{dt}\Big|_{t=0}(\rho^*\eta)_g = 0.
$$

For the converse implication, assume that $\iota_x \omega = \mathscr{L}_x \omega = 0$ for all $x \in \mathfrak{k}$ and define a map $\widetilde{\omega} \colon G \to (\Lambda^k T(K\backslash G))^*$ as follows: For $g \in G$, $v_1, \ldots, v_k \in T_{\rho(g)}(K\backslash G)$, set

$$
\widetilde{\omega}_g(v_1, \ldots, v_k) := \omega_g(w_1, \ldots, w_k) \qquad \text{for } w_i \in T_g G \text{ such that } D_g\rho(w_i) = v_i.
$$

If $w \in T_g G$ is such that $D_g\rho(w) = v_i$, then $w_i - w \in \ker D_g\rho$ and $y = D_g R_{g^{-1}}(w_i - w) \in \mathfrak{k}$, so

$$
\omega_g(w_1, \ldots, w_i - w, \ldots, w_k) = (-1)^{i-1}(\iota_y \omega)_g(w_1, \ldots, \hat{w}_i, \ldots, w_k) = 0,
$$

Hence, $\widetilde{\omega}$ is well-defined and it is smooth as ω is smooth.

Let $x \in \mathfrak{k}$, $g \in G$, and consider the map $t \mapsto \widetilde{\omega}_{\exp(tx)g}$. Given $v_1, \ldots, v_k \in T_{\rho(g)}(K\backslash G)$ and $t_0 \in \mathbb{R}$, let $w_1, \ldots, w_k \in T_{\exp(t_0 x)g} G$ such that $D_{\exp(t_0 x)g}\rho(w_i) = v_i$. Then for any $t \in \mathbb{R}$,

$$
\begin{aligned}
D_{\exp(tx)g}\rho\big(D_g L_{\exp(tx)} \circ D_{\exp(t_0 x)g} L_{\exp(-t_0 x)}(w_i)\big) &= D_{\exp(t_0 x)g}(\rho \circ L_{\exp((t-t_0)x)})(w_i) \\
&= D_{\exp(t_0 x)g}(\rho)(w_i) = v_i,
\end{aligned}
$$

so

$$
\begin{aligned}
\widetilde{\omega}_{\exp(tx)g}(v_1, \ldots, v_k) &= \omega_{\exp(tx)g}(D_g L_{\exp(tx)} \circ D_{\exp(t_0 x)g} L_{\exp(-t_0 x)}(w_1), \ldots) \\
&= (L^*_{\exp(tx)}\omega)_g(D_{\exp(t_0 x)g} L_{\exp(-t_0 x)}(w_1), \ldots, D_{\exp(t_0 x)g} L_{\exp(-t_0 x)}(w_k))
\end{aligned}
$$

and

$$\frac{d}{dt}\bigg|_{t=t_0} \widetilde{\omega}_{\exp(tx)g}(v_1,\dots,v_k) = \frac{d}{dt}\bigg|_{t=t_0} (L^*_{\exp(tx)}\omega)_g(D_{\exp(t_0x)g}L_{\exp(-t_0x)}(w_1),\dots)$$

$$= \frac{d}{dt}\bigg|_{t=0} (L^*_{\exp((t+t_0)x)}\omega)_g(D_{\exp(t_0x)g}L_{\exp(-t_0x)}(w_1),\dots)$$

$$= \frac{d}{dt}\bigg|_{t=0} \omega_{\exp(tx)\exp(t_0x)g}(D_g(L_{\exp(tx)} \circ L_{\exp(t_0x)}) \circ D_{\exp(t_0x)g}L_{\exp(-t_0x)}(w_1),\dots)$$

$$= \frac{d}{dt}\bigg|_{t=0} (L^*_{\exp(tx)}\omega)_{\exp(t_0x)g}(w_1,\dots,w_k) = (\mathscr{L}_x\omega)_{\exp(t_0x)g}(w_1,\dots,w_k) = 0.$$

In other words, the map $t \mapsto \widetilde{\omega}_{\exp(tx)g}$ is constant. As $K = \langle\exp\mathfrak{k}\rangle$, K being connected, we conclude that $\widetilde{\omega}$ is constant on the fibres, Kg, of ρ. Therefore it factors through ρ, i.e. there exists an η as in the diagram. $\qquad\square$

Suppose $\Gamma' \leq \Gamma$ is a torsion-free subgroup. Then the action of Γ' on X is free: Indeed, the action is properly discontinuous, so the stabiliser subgroups are finite and as Γ' is torsion-free, they must be trivial.

PROPOSITION 2.1.17. If $\Gamma' \leq \Gamma$ is a torsion-free subgroup, then the manifolds G, X, G/Γ' and X/Γ' are all orientable.

Proof. Being a Lie group, G is orientable. Moreover, we may take a right and left invariant volume form $\omega \in \Omega^{n^2-1}(G)$ on G as it is unimodular (cf. Proposition 1.2.15). X is diffeomorphic to Euclidean space and as such is orientable, but we would like to give a specific volume form. More specifically, we will show that ω descends to a volume form on the quotients $K\backslash G$, G/Γ' and $K\backslash G/\Gamma'$.
Let $\rho\colon G \to X$, $\pi\colon X \to X/\Gamma'$ and $\widetilde{\pi}\colon G \to G/\Gamma'$ denote the projections.
We have an isomorphism of chain complexes $\widetilde{\pi}^*\colon \Omega^*(G/\Gamma') \to \Omega^*(G)^{\Gamma'}$. As ω is right invariant under G, it is in particular right invariant under Γ', so it descends to a form $\zeta \in \Omega^{n^2-1}(G/\Gamma')$ satisfying $\widetilde{\pi}^*\zeta = \omega$.
Let $x_1,\dots,x_k$ be a basis of $\mathfrak{k}$, $k = \frac{n(n-1)}{2}$. Consider the form $\omega' := i_{x_1}\cdots i_{x_k}\omega \in \Omega^m(G)$, $m = n - k = \frac{n(n+1)}{2} - 1$. Clearly, $i_x\omega' = 0$ for all $x \in \mathfrak{k}$. In addition, $\mathscr{L}_x\omega' = 0$ for all $x \in \mathfrak{k}$, but this requires a little more work to prove.
Note first that as ω is left invariant,

$$\mathscr{L}_x\omega = \frac{d}{dt}\bigg|_{t=0} (L^*_{\exp(tx)}\omega)_g) = \frac{d}{dt}\bigg|_{t=0} \omega_g = 0 \qquad \text{for all } x \in \mathfrak{g}.$$

Using succesively the identity $\mathscr{L}_y i_x - i_x\mathscr{L}_y = i_{[y,x]}$, $x, y \in \mathfrak{g}$, and the above, we get

$$\mathscr{L}_y\omega' = \sum_{i=1}^{k} i_{x_1}\cdots i_{[y,x_i]}\cdots i_{x_k}\omega \qquad \text{for all } y \in \mathfrak{g}.$$

It follows that $\mathscr{L}_{y+y'} = \mathscr{L}_y + \mathscr{L}_{y'}$, so it suffices to check the identity $\mathscr{L}_x\omega' = 0$ on basis vectors $x \in \mathfrak{k}$. Choosing as our basis $x_1,\dots,x_k$, the basis $\{E_{ij} - E_{ji}\}_{i<j}$, straightforward calculations show that

$$[x_i,x_j] = \sum_{l=1}^{k} c^l_{i,j}x_l \qquad \text{with} \quad c^i_{i,j} = c^j_{i,j} = 0.$$

Therefore

$$\mathscr{L}_{x_i}\omega' = \sum_{j=1}^{k} i_{x_1} \cdots i_{[x_i,x_j]} \cdots i_{x_k}\omega = \sum_{j=1}^{k} \sum_{l \neq j} c_{i,j}^l i_{x_1} \cdots i_{x_l} \cdots i_{x_k}\omega = 0$$

for all $i = 1, \ldots, k$, and we conclude that $\mathscr{L}_x\omega' = 0$ for all $x \in \mathfrak{k}$.

By Lemma 2.1.16, ω' descends to X, i.e. there is a form $\eta \in \Omega^m(X)$ such that $\rho^*\eta = \omega'$. Note now that ω' is right invariant: Indeed, for any right invariant form $\alpha \in \Omega^{q+1}(G)$, $x \in \mathfrak{g}$, the form $i_x\alpha$ is also right invariant by the simple calculation

$$(R_g^* i_x\alpha)_h(v_1, \ldots, v_q) = \alpha_{hg}(D_e R_{hg}(x), D_h R_g(v_1), \ldots, D_h R_g(v_k))$$
$$= (R_g^*\alpha)_h(D_e R_h(x), v_1, \ldots, v_q) = (i_x\alpha)_h(v_1, \ldots, v_q)$$

for any $g, h \in G$, $v_1, \ldots, v_q \in T_h G$.

It follows that η is G-invariant; in particular, $\eta \in \Omega^m(X)^{\Gamma'}$, so it descends to X/Γ'. Let $\xi \in \Omega^m(X/\Gamma')$ such that $\pi^*\xi = \eta$.

As ω is non-zero everywhere, so is ω', and it follows immediately that the forms ζ, η and ξ are non-zero everywhere, and as such are volume forms on the respective manifolds. $\square$

2.2   THE CASE $n = 2$ PART I

In order to understand the setting better, we take a closer look at the case $n = 2$. Let X denote the set of positive definite binary forms on $\mathbb{R}^2$ inducing the same volume as the standard inner product equipped with the smooth structure defined in the previous section, set $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $\mathscr{H} \subseteq \mathbb{C}$ denote the upper half-plane. We claim that X is diffeomorphic to $\mathscr{H}$, and we will use this relationship to get a better understanding of the space X. We then investigate the action of G and Γ on X by defining an appropriate action of G on $\mathscr{H}$.

PROPOSITION 2.2.1. X is diffeomorphic to $\mathscr{H}$.

Proof. Given $q \in X$, let $s = (s_{ij})$ be the positive definite symmetric matrix defining q, and define a complex number $z_q = \frac{s_{12}}{s_{11}} + \frac{1}{s_{11}}i \in \mathscr{H}$. Conversely, given $z \in \mathscr{H}$, define a matrix $s_z = (s_{ij})$ by

$$s_{11} = \frac{1}{\mathrm{Im}(z)}, \qquad s_{12} = s_{21} = \frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}, \qquad s_{22} = \frac{1 + s_{12}^2}{s_{11}}.$$

Clearly, s_z is symmetric and $\det(s_z) = 1$. As s_{11} is positive, s_z is positive definite: Indeed, for $a > 0$, $c \in \mathbb{R}$,

$$\begin{pmatrix} a & c \\ c & \frac{1+c^2}{a} \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 \\ \frac{c}{\sqrt{a}} & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} \sqrt{a} & \frac{c}{\sqrt{a}} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix},$$

is a Cholesky decomposition. Let q_z be the quadratic form $x \mapsto x^t s_z x$. The maps

$$\Phi \colon X \to \mathscr{H}, \quad q \mapsto z_q, \qquad \Psi \colon \mathscr{H} \to X, \quad z \mapsto q_z$$

are smooth and each other's inverses, so we have $X \cong \mathscr{H}$, as claimed. $\square$

Remark 2.2.2. Recall that we can picture an element $q \in X$ as an ellipse in $\mathbb{R}^2$ by plotting the unit ball with respect to the norm induced by q. Our basepoint is $\iota \colon x \mapsto x^t x$, which can be identified with the unit disk in $\mathbb{R}^2$; in $\mathscr{H}$, this basepoint is $\Phi(\iota) = i$.
For $\lambda > 1$, consider the matrices

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{\lambda}{2} + \frac{1}{2\lambda} & \frac{\lambda}{2} - \frac{1}{2\lambda} \\ \frac{\lambda}{2} - \frac{1}{2\lambda} & \frac{\lambda}{2} + \frac{1}{2\lambda} \end{pmatrix}, \quad g_3 = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}, \quad g_4 = \begin{pmatrix} \frac{\lambda}{2} + \frac{1}{2\lambda} & -\frac{\lambda}{2} + \frac{1}{2\lambda} \\ -\frac{\lambda}{2} + \frac{1}{2\lambda} & \frac{\lambda}{2} + \frac{1}{2\lambda} \end{pmatrix}.$$

These are all positive definite with determinant 1. Let $q_i \in X$ denote the binary form $x \mapsto x^t g_i x$ and let E_i denote the ellipse arising from q_i, $i = 1, \ldots, 4$. The ellipses, E_i, all have major radius $\sqrt{\lambda}$ and minor radius $\frac{1}{\sqrt{\lambda}}$, but their orientations differ. Let e_1, e_2 denote the standard basis of $\mathbb{R}^2$.

- E_1 has major axis in the direction of e_2 and minor axis in the direction of e_1;

- E_2 has major axis in the direction of $e_2 - e_1$ and minor axis in the direction of $e_1 + e_2$;

- E_3 has major axis in the direction of e_1 and minor axis in the direction of e_2;

- E_4 has major axis in the direction of $e_1 + e_2$ and minor axis in the direction of $e_2 - e_1$.

In $\mathscr{H}$, the four binary forms q_1, q_2, q_3 and q_4 are

$$\Phi(q_1) = 1/\lambda i, \quad \Phi(q_2) = \frac{\lambda - 1/\lambda}{\lambda + 1/\lambda} + \frac{2}{\lambda + 1/\lambda}i, \quad \Phi(q_3) = \lambda i, \quad \Phi(q_4) = -\frac{\lambda - 1/\lambda}{\lambda + 1/\lambda} + \frac{2}{\lambda + 1/\lambda}i.$$

Note that $|\Phi(q_2)| = |\Phi(q_4)| = 1$. The four points are plotted in Figure 2.1 for the values $\lambda = 2, 4$ including sketches of the corresponding ellipses; the basepoint is also plotted. Looking at this figure, the reader may think of the hyperbolic plane; after defining an appropriate action of G on $\mathscr{H}$, we show that the hyperbolic metric is G-invariant.

Construction 2.2.3. Define an action $\mathscr{H} \curvearrowleft G$ by

$$z.g = \frac{dz + b}{cz + a} \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

It is clear that $z.\mathrm{id} = z$, and a direct calculation shows that $(z.g).h = z.(gh)$ for any $g, h \in G$. Note that for any $z \in \mathscr{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have

$$\mathrm{Im}(z.g) = \frac{\mathrm{Im}(z)}{|cz + a|^2},$$
$$\mathrm{Re}(z.g) = \frac{cd|z|^2 + (bc + ad)\mathrm{Re}(z) + ab}{|cz + a|^2}. \tag{2.1}$$

We also have $z.g = z.(-g)$ for all $z \in \mathscr{H}$, $g \in G$.

PROPOSITION 2.2.4. The maps Φ and Ψ defined in the proof of Proposition 2.2.1 above are equivariant.

Proof. Let $q \in X$ be given by the matrix $s = (s_{ij})$, and let $g = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in G$. Then $q.g$ is given by the matrix

$$g^t s g = \begin{pmatrix} a^2 s_{11} + 2ac\, s_{12} + c^2 s_{22} & ab\, s_{11} + (bc + ad)\, s_{12} + cd\, s_{22} \\ ab\, s_{11} + (bc + ad)\, s_{12} + cd\, s_{22} & b^2 s_{11} + 2bd\, s_{12} + d^2 s_{22} \end{pmatrix},$$
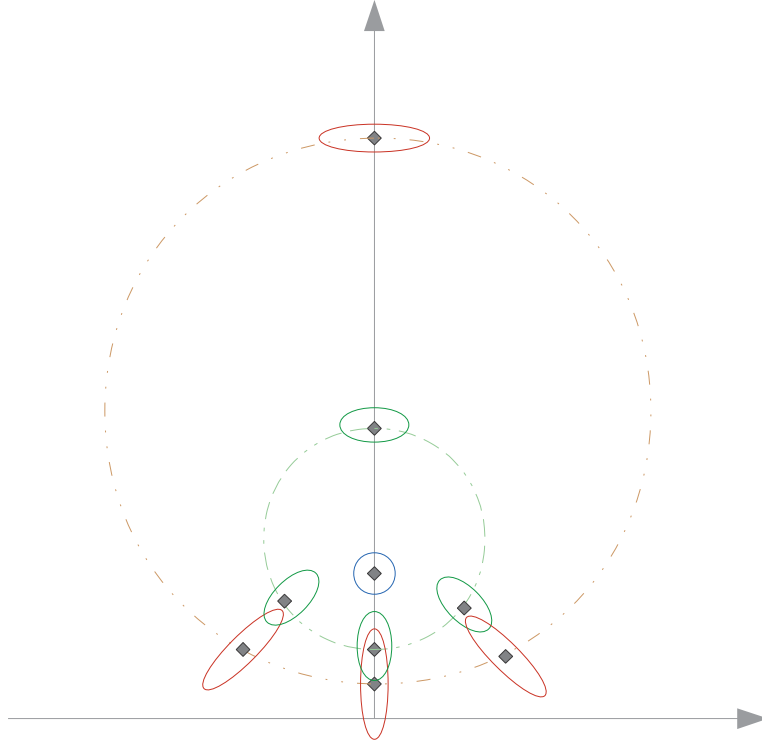
Figure 2.1: The upper half-plane with ellipses. The blue circle is the basepoint ι; the dashed green circle marks the set of points which give rise to ellipses with major radius $\sqrt{2}$ and minor radius $1/\sqrt{2}$; the dashed orange circle marks the set of points which give rise to ellipses with major radius 2 and minor radius $1/2$. The four points plotted on the two circles denote the four points in $\mathscr{H}$ corresponding to $q_1, q_2, q_3, q_4 \in X$ for $\lambda = 2$, respectively, $\lambda = 4$. The numbering runs anticlockwise with E_1 at the bottom.

and by Equation (2.1), we have

$$
\begin{aligned}
\mathrm{Im}(z_q.g) = \frac{\mathrm{Im}(z_q)}{|cz_q + a|^2} &= \left(s_{11}(c^2 \tfrac{s_{12}^2+1}{s_{11}^2} + 2ac \tfrac{s_{12}}{s_{11}} + a^2) \right)^{-1} \\
&= \left(c^2 s_{22} + 2ac\, s_{12} + a^2\, s_{11} \right)^{-1} = (g^t s g)_{11}^{-1}, \\
\mathrm{Re}(z_q.g) &= \frac{cd\,|z_q|^2 + (bc + ad)\,\mathrm{Re}(z_q) + ab}{|cz_q + a|^2} \\
&= \frac{cd\,\tfrac{s_{12}^2+1}{s_{11}^2} + (bc + ad)\,\tfrac{s_{12}}{s_{11}} + ab}{c^2 \tfrac{s_{12}^2+1}{s_{11}^2} + 2ac \tfrac{s_{12}}{s_{11}} + a^2} \\
&= \frac{cd\, s_{22} + (bc + ad)\, s_{12} + ab\, s_{11}}{c^2 s_{22} + 2ac\, s_{12} + a^2\, s_{11}} = \frac{(g^t s g)_{12}}{(g^t s g)_{11}}.
\end{aligned}
$$

We see that $z_{(q.g)} = z_q.g$, implying that Φ and Ψ are indeed equivariant. $\square$

Thus X and $\mathscr{H}$ are isomorphic as smooth G-spaces.

PROPOSITION 2.2.5. The hyperbolic metric on $\mathcal{H}$, $d\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}_{\geq 0}$, given by

$$d(z_1, z_2) = 2\tanh^{-1}\left(\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|}\right)$$

is G-invariant.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $z_1, z_2 \in \mathcal{H}$. Then

$$
\begin{aligned}
\frac{z_1.g - z_2.g}{z_1.g - \overline{z_2.g}} &= \left(\frac{dz_1 + b}{cz_1 + a} - \frac{dz_2 + b}{cz_2 + a}\right)\left(\frac{dz_1 + b}{cz_1 + a} - \frac{d\overline{z_2} + b}{c\overline{z_2} + a}\right)^{-1} \\
&= \left(\frac{c\overline{z_2} + a}{cz_2 + a}\right)\left(\frac{(dz_1 + b)(cz_2 + a) - (dz_2 + b)(cz_1 + a)}{(dz_1 + b)(c\overline{z_2} + a) - (d\overline{z_2} + b)(cz_1 + a)}\right) \\
&= \left(\frac{c\overline{z_2} + a}{cz_2 + a}\right)\left(\frac{z_1 - z_2}{z_1 - \overline{z_2}}\right),
\end{aligned}
$$

implying $d(z_1, z_2) = d(z_1.g, z_2.g)$ as the first factor has norm 1. □

Remark 2.2.6. We take a closer look at the geometry of the action of G on $\mathcal{H}$. Note that for any $\lambda \in \mathbb{R}$, $z \in \mathcal{H}$

$$z.\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) = z + \lambda, \qquad z.\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) = \frac{z}{\lambda z + 1}.$$

Thus we see that the orbits of N, the upper triangular matrices of the form $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $\lambda \in \mathbb{R}$, are horizontal lines in $\mathcal{H}$. In terms of ellipses, they are stretched out, tending towards a horizontal line.

The orbits of lower triangular matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$, $\lambda \in \mathbb{R}$, are circles centered at a point on the imaginary axis and with the real axis as a tangent line: Indeed, if $z = \mathrm{Im}(z)i$ is purely imaginary, then for any $\lambda \in \mathbb{R}$

$$
\begin{aligned}
\left|z.\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) - \frac{\mathrm{Im}(z)}{2}i\right|^2 &= \left|\frac{2\mathrm{Im}(z)i - \mathrm{Im}(z)i(\lambda\mathrm{Im}(z)i + 1)}{2(\lambda\mathrm{Im}(z)i + 1)}\right|^2 \\
&= \left|\frac{\mathrm{Im}(z)i + \lambda\mathrm{Im}(z)^2}{2(\lambda\mathrm{Im}(z)i + 1)}\right|^2 = \frac{\mathrm{Im}(z)^2 + \lambda^2\mathrm{Im}(z)^4}{4(\lambda\mathrm{Im}(z))^2 + 4} = \frac{\mathrm{Im}(z)^2}{4}.
\end{aligned}
$$

In other words, the orbit of z is a circle of radius $\frac{1}{2}\mathrm{Im}(z)$ centered at $\frac{1}{2}\mathrm{Im}(z)i$ (here we are using the standard Euclidean norm). Now, for arbitrary $z \in \mathcal{H}$, set $\lambda = -\frac{\mathrm{Re}(z)}{|z|^2}$; then

$$z.\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) = \frac{z}{\lambda z + 1} = \frac{\lambda|z|^2 + z}{|\lambda z + 1|^2} = \frac{1}{|\lambda z + 1|^2}(\lambda|z|^2 + \mathrm{Re}(z) + \mathrm{Im}(z)i) = \frac{\mathrm{Im}(z)}{|\lambda z + 1|^2}i,$$

so by the above, the orbit of z under $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$, $\lambda \in \mathbb{R}$, is the circle of radius $\frac{\mathrm{Im}(z)}{2|\lambda z + 1|^2}$ centered at $\frac{\mathrm{Im}(z)}{2|\lambda z + 1|^2}i$. In terms of ellipses, they are stretched out, tending towards a vertical line.

Finally, note that $z.\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = -\frac{1}{z} = -\frac{\overline{z}}{|z|^2}$. In terms of ellipses, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ simply rotates the ellipse such that major and minor axes are swapped around.

2.3 Real Cohomology of $\mathrm{SL}_n(\mathbb{Z})$

In this section, we relate the group cohomology of $\mathrm{SL}_n(\mathbb{Z})$ with real coefficients to the de Rham cohomology of X. This relationship naturally leads us to consider a specific chain map, and the focus of the rest of the project is to show that this map induces an isomorphism on cohomology in low degrees.

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and X as constructed in Section 2.1. For any $g \in G$, we denote by $g\colon X \to X$ the action map $x \mapsto x.g$. Recall that there is an induced action of G on the de Rham complex $\Omega^*(X)$ via the pullbacks of these maps: $\omega.g = g^*\omega$, $\omega \in \Omega^*(X)$.

PROPOSITION 2.3.1. Γ contains a normal torsion free subgroup of finite index.

Proof. Let $p \geq 3$ be some prime; we claim that the principal congruence subgroup

$$\Gamma' := \{\gamma \in \Gamma \mid \gamma \equiv \mathrm{id} \;(\mathrm{mod}\; p)\}$$

is normal, torsion free and of finite index. That it is normal is immediate, and clearly the quotient Γ/Γ' is isomorphic to $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ and thus finite. Assume for contradiction that Γ' is not torsion free: Then it contains an element of prime order, i.e. there is an element $\gamma = \mathrm{id} + p^k\alpha \in \Gamma'$, where $\alpha \in \Gamma$ such that $\alpha \not\equiv 0 \;(\mathrm{mod}\; p)$, $k \geq 1$, and a prime q such that $\gamma^q = \mathrm{id}$. But then

$$\mathrm{id} = \gamma^q = \sum_{l=0}^{q} \binom{q}{l} p^{k(q-l)}\alpha^{q-l}, \quad \text{implying that} \quad p^k q\alpha = -p^{2k}\sum_{l=0}^{q-2} \binom{q}{l} p^{k(q-l-2)}\alpha^{q-l}.$$

Then we must have $p = q$ and $k = 1$, and the equality yields

$$\alpha = -\sum_{l=0}^{p-2} \binom{p}{l} p^{k(p-l-2)}\alpha^{p-l}.$$

But $p \mid \binom{p}{l}$ for all $0 < l < p$, and $p \mid p^{k(p-2)}$ as $p \geq 3$, so p divides the right-hand side, hence α, a contradiction. $\qquad\square$

Let $\Gamma' \leq \Gamma$ be a normal, torsion-free subgroup of finite index and let Γ' inherit the action of Γ on X. Then Γ' acts freely on X. As X is contractible, it follows immediately that X/Γ' is a classifying space of Γ'.

PROPOSITION 2.3.2. The group cohomology of Γ' with real coefficients is isomorphic to the homology of the the chain complex of Γ'-invariant differential forms on X: $H^*(\Gamma') \cong H_*(\Omega(X)^{\Gamma'})$.

Proof. As Γ' acts properly and freely on X, X/Γ' is a smooth manifold (cf. [18, Theorem 9.16]) and by the remark above, it is a classifying space of Γ'. Hence we have $H^*(\Gamma') \cong H^*(X/\Gamma'; \mathbb{R})$, and by De Rham's Theorem (cf. [18, Theorem 11.34]), this latter group is isomorphic to the de Rham cohomology of X/Γ', $H_*(\Omega^*(X/\Gamma'))$.

Let $\pi\colon X \to X/\Gamma'$ denote the projection and consider the chain map $\pi^*\colon \Omega^*(X/\Gamma') \to \Omega^*(X)$. Note that for any $\gamma \in \Gamma'$, $\omega \in \Omega^*(X/\Gamma')$, we have $\gamma^*\pi^*\omega = (\pi \circ \gamma)^*\omega = \pi^*\omega$, so $\mathrm{im}\,\pi^* \subseteq \Omega^*(X)^{\Gamma'}$. Now, given $\omega \in \Omega^k(X)^{\Gamma'}$, define $\widetilde{\omega} \in \Omega^k(X/\Gamma')$ as follows: for $x \in X/\Gamma'$, $w_1, \ldots w_k \in T_x(X/\Gamma')$, set

$$\widetilde{\omega}_x(w_1, \ldots, w_k) = \omega_q(v_1, \ldots, v_k)$$

for $q \in X$, $v_i \in T_q X$ such that $\pi(q) = x$ and $D_q\pi(v_i) = w_i$. This is well-defined since if $q' \in X$, $v_i' \in T_{q'} X$ satisfy $\pi(q') = x$ and $D_{q'}\pi(v_i') = w_i$, then $q' = q.\gamma$ and $v_i' = D_q\gamma(v_i)$ for some $\gamma \in \Gamma'$, and thus

$$\omega_{q'}(v_1', \ldots, v_k') = \omega_{q.\gamma}(D_q\gamma(v_1), \ldots, D_q\gamma(v_k)) = (\gamma^*\omega)_q(v_1, \ldots, v_k) = \omega_q(v_1, \ldots, v_k)$$

as ω is Γ' invariant. The form $\widetilde{\omega}$ is smooth as π is a local diffeomorphism. Finally, the map $\Omega^*(X)^{\Gamma'} \longrightarrow \Omega^*(X/\Gamma')$, $\omega \mapsto \widetilde{\omega}$, is a chain map and inverse to π^*, so π^* is a chain isomorphism and we conclude that

$$H^*(\Gamma') \cong H_*(\Omega^*(X/\Gamma')) \cong H_*(\Omega^*(X)^{\Gamma'}).$$

$\square$

If C is a chain complex of H-modules for some group H, then there is an obvious action on $H_*(C)$: $h.[c] = [h.c]$, $h \in H$, $[c] \in H_*(C)$.

LEMMA 2.3.3. For any finite group H and chain complex of H-modules C, the inclusion of the chain complex of invariants $C^H \hookrightarrow C$ induces an isomorphism $H_*(C^H) \to H_*(C)^H$, which is natural in C.

Proof. It is obvious that the image of the map $H_*(C^H) \to H_*(C)$ is a subset of the H-invariants of $H_*(C)$. Consider the map $C \to C^H$, $c \mapsto {}^1\!/|H| \sum_{h \in H} hc$, and the induced map on homology restricted to the submodule of H-invariants $H_*(C)^H \to H_*(C^H)$. The composition $C^H \hookrightarrow C \to C^H$ is the identity, and thus so is the induced map on homology. For the other composition, note that for $[c] \in H_*(C)^H$, we have

$$\left[{}^1\!/|H| \sum_{h \in H} hc \right] = {}^1\!/|H| \sum_{h \in H} h[c] = {}^1\!/|H| \sum_{h \in H} [c] = [c].$$

Hence, the composition $C \to C^H \to C$ induces the identity on $H_*(C)^H$.
Naturality follows directly from naturality of $H^*(-)$, requiring of course the chain map in question to be equivariant. $\square$

We need the following important result:

PROPOSITION 2.3.4. Let H be a group, H' a normal subgroup of finite index in H, and M an H-module. If multiplication by $|H : H'|$ is an isomorphism of M, then the restriction map induced by the inclusion $H' \hookrightarrow H$ yields an isomorphism $H^*(H, M) \to H^*(H', M)^{H/H'}$.

Proof. See [9, Proposition 10.4]. $\square$

PROPOSITION 2.3.5. The group cohomology of Γ with real coefficients is isomorphic to the homology of the chain complex of Γ-invariant differential forms on X: $H^*(\Gamma) \cong H_*(\Omega^*(X)^\Gamma)$

Proof. For any Γ-module, M, the submodule $M^{\Gamma'}$ has a natural structure of a Γ/Γ'-module and $M^\Gamma = (M^{\Gamma'})^{\Gamma/\Gamma'}$. Hence, using Proposition 2.3.2, Lemma 2.3.3 and Proposition 2.3.4, we have an isomorphism

$$H^*(\Gamma) \xrightarrow{\cong} H^*(\Gamma')^{\Gamma/\Gamma'} \xrightarrow{\cong} H_*(\Omega^*(X)^{\Gamma'})^{\Gamma/\Gamma'} \xrightarrow{\cong} H_*((\Omega^*(X)^{\Gamma'})^{\Gamma/\Gamma'}) =\!\!= H_*(\Omega^*(X)^\Gamma)$$

$\square$

The aim of this project is to calculate $H^*(\Gamma)$ in low degrees and the way we do this is by exploiting the following theorem. The proof of this theorem is where all the hard work of this project lies and this will be the content of the next four chapters.

THEOREM 2.3.6. The inclusion $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma$ induces an isomorphism on homology in degrees $* \leq \frac{n+1}{4}$ for $n \neq 3$ and in the zero'th degree for $n = 3$.

Evaluation at the identity yields an isomorphism $\Omega^*(X)^G \cong C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$, where the latter denotes the Chevalley-Eilenberg chain complex of the relative Lie algebra cohomology (we do a very brief recap of Lie algebra cohomology in Section 6.1). Hence, the above theorem and Proposition 2.3.5 enable us, in low degrees, to express the group cohomology of Γ with real coefficients in terms of Lie algebra cohomology. We exploit this in Section 6.3.
It suffices to prove the theorem for a torsion-free normal subgroup of finite index:

THEOREM 2.3.7. Let $\Gamma' \subseteq \Gamma$ be a normal torsion-free subgroup of finite index. If the inclusion $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^{\Gamma'}$ induces an isomorphism on homology in degree k, then so does $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma$.

Proof. The claim is seen immediately from the following commutative diagram on homology, where the vertical maps are induced by the inclusions, and the second square commutes by naturality of the isomorphism in Lemma 2.3.3:

$$
\begin{array}{ccccc}
H_k(\Omega^*(X)^G) & = & H_k((\Omega^*(X)^G)^{\Gamma/\Gamma'}) & \xrightarrow{\cong} & H_k(\Omega^*(X)^G)^{\Gamma/\Gamma'} \\
\downarrow & & \downarrow & & \downarrow \\
H_k(\Omega^*(X)^\Gamma) & = & H_k((\Omega^*(X)^{\Gamma'})^{\Gamma/\Gamma'}) & \xrightarrow{\cong} & H_k(\Omega^*(X)^{\Gamma'})^{\Gamma/\Gamma'}
\end{array}
$$

$\square$

3 | Siegel Reduction Theory

In this chapter we try to understand better the manifold X as defined in the previous chapter and the action of $\mathrm{SL}_n(\mathbb{Z})$ on it. The definitions and results of this chapter will be essential in what is to come. More specifically, we define a "nice" type of subset of $\mathrm{SL}_n(\mathbb{R})$ and of X, namely Siegel sets, and we show that sufficiently large Siegel sets intersect all Γ-orbits in X. This eases the study of the quotient space X/Γ. Again, we take a closer look at the case $n = 2$ and finish off with some technical results for use later on. The chapter is based on [5].

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and let X be as in Section 2.1.

3.1 Siegel Sets

Using the Iwasawa decomposition of G, we define Siegel sets and show that sufficiently large Siegel sets in G intersect all Γ-orbits in G, with Γ acting by right multiplication. This is done by showing that a certain function has a minimum and that this minimum is attained in a point belonging to a certain Siegel set. We then use the quotient map $G \to X$ to translate these definitions and results onto X.

Recall the Iwasawa decomposition of G (Theorem 1.1.1): The multiplication map $K \times A \times N \to G$ is a diffeomorphism, with $A \leq G$ the subgroup of diagonal matrices with positive entries, $N \leq G$ the subgroup of upper triangular matrices with 1's on the diagonal, and $K = \mathrm{SO}(n)$. In the following, $1 \leq i, j \leq n$.

Definition 3.1.1. For $\lambda, \delta > 0$, set

$$A_\lambda := \{a = (a_i) \in A \mid a_i \leq \lambda a_{i+1} \text{ for all } i\}$$
$$N_\delta := \{u = (u_{ij}) \in N \mid |u_{ij}| \leq \delta \text{ for all } i < j\}.$$

A *Siegel set* in G is a subset of the form $\mathfrak{S}_{\lambda,\delta} := K A_\lambda N_\delta \subseteq G$ for some $\lambda, \delta > 0$.

We will prove that the Siegel set $\mathfrak{S}_{2/\sqrt{3}, 1/2}$ intersects all Γ-orbits in G, and hence satisfies $G = \mathfrak{S}_{2/\sqrt{3}, 1/2}.\Gamma$ when Γ acts on G by right translation. It will be a consequence of Theorem 3.1.6 below.

Lemma 3.1.2. $N = N_{1/2} N_\mathbb{Z}$, where $N_\mathbb{Z} = N \cap \Gamma$.

Proof. Let $u = (u_{ij}) \in N$. Note that for any $z = (z_{ij}) \in N_\mathbb{Z}$, the product uz has entries $(uz)_{ii} = 1$ for all i,

$$(uz)_{ij} = \sum_{k=i}^{j} u_{ik} z_{kj} = z_{ij} + u_{i\,i+1} z_{i+1\,j} + \ldots + u_{ij} \qquad \text{for } i < j, \tag{3.1}$$

and zero elsewhere. We construct $z = (z_{ij}) \in N_\mathbb{Z}$ such that $|(uz)_{ij}| \leq \frac{1}{2}$ for all $i < j$ by defining the entries recursively: First, set $z_{ii} = 1$ for all i, and let $z_{n-1\,n} \in \mathbb{Z}$ such that

$|z_{n-1\,n} + u_{n-1\,n}| \leq \frac{1}{2}$; by 3.1, $|(uz)_{n-1\,n}| \leq \frac{1}{2}$. Let $1 \leq l < n - 1$, and suppose that z_{ij} have been defined for all $l < i < j$ such that $|(uz)_{ij}| \leq \frac{1}{2}$. For $j > l$, let $z_{lj} \in \mathbb{Z}$ such that

$$\left| z_{lj} + \sum_{k=l+1}^{j} u_{lk} z_{kj} \right| \leq \frac{1}{2}.$$

With $z = (z_{ij})$ as above, we have $u = (uz)z^{-1}$ with $uz \in N_{1/2}$, $z^{-1} \in N_{\mathbb{Z}}$ (this last claim can be seen by recursively calculating the entries of z^{-1} using Equation (3.1)). $\qquad \square$

Construction 3.1.3. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. Define $\Phi \colon G \to \mathbb{R}_{>0}$ by $\Phi(g) = \|g(e_1)\|$, where $\|\cdot\|$ is the standard norm on $\mathbb{R}^n$. Note that Φ is continuous, and that if $g = kau$ is the Iwasawa decomposition with $a = (a_i)$, then

$$\Phi(g) = \|kau(e_1)\| = \|ka(e_1)\| = \|a(e_1)\| = a_1 = \Phi(a),$$

as $u(e_1) = e_1$, and k is orthogonal.

LEMMA 3.1.4. For any $g \in G$, the map $\varphi_g \colon \Gamma \to \mathbb{R}_{>0}$, $\gamma \mapsto \Phi(g\gamma)$, has a minimum.

Proof. Let $g \in G$. As $\Gamma e_1 \subseteq \mathbb{Z}^n - \{0\}$ is a closed, discrete subset of $\mathbb{R}^n$, and $g \colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, $g\Gamma e_1$ is a closed, discrete subset of $\mathbb{R}^n$. It follows that the norm function $\|\cdot\| \colon \mathbb{R}^n - \{0\} \to \mathbb{R}_{>0}$ restricted to $g\Gamma e_1$ has a minimum. $\qquad \square$

LEMMA 3.1.5. Let $g \in G$ and let $g = k_g a_g u_g$ be its Iwasawa decomposition with $a_g = (a_i)$. If $\Phi(g) \leq \Phi(g\gamma)$ for all $\gamma \in \Gamma$, then $a_1 \leq \frac{2}{\sqrt{3}} a_2$.

Proof. Note first that for any $z \in N_{\mathbb{Z}}$, we have $\Phi(gz) = \Phi(g)$ as $z(e_1) = e_1$, and moreover, $k_g a_g(u_g z) = gz = k_{gz} a_{gz} u_{gz}$, so we must have $a_g = a_{gz}$ by uniqueness of the Iwasawa decomposition. Hence, in view of Lemma 3.1.2, it suffices to consider the case where $g = kau$ for $u \in N_{1/2}$. In particular, $|u_{12}| \leq \frac{1}{2}$. Consider the element $\gamma = (\gamma_{ij}) \in \Gamma$ with $\gamma_{12} = -1$, $\gamma_{21} = 1$, $\gamma_{ii} = 1$ for all $i \geq 3$ and zero elsewhere. Then $\gamma \in \Gamma$ and

$$g\gamma(e_1) = g(e_2) = kau(e_2) = ka(e_2 + u_{12}e_1) = k(a_1 u_{12} e_1 + a_2 e_2),$$

and therefore

$$a_1^2 = \Phi(g)^2 \leq \Phi(g\gamma)^2 = a_1^2 u_{12}^2 + a_2^2 \leq \tfrac{1}{4} a_1^2 + a_2^2,$$

from which the desired inequality follows. $\qquad \square$

THEOREM 3.1.6. For any $g \in G$, the minimum of Φ on $g.\Gamma$ is attained in a point belonging to $g.\Gamma \cap \mathfrak{S}_{\lambda,\delta}$ for $\lambda = 2/\sqrt{3}$, $\delta = 1/2$. In particular, $g.\Gamma \cap \mathfrak{S}_{2/\sqrt{3},1/2} \neq \emptyset$.

Proof. Write $\mathfrak{S}_0 := \mathfrak{S}_{2/\sqrt{3},1/2}$. We prove the claim by induction on the dimension n.
For $n = 1$, we have $G = \mathfrak{S}_0 = \{(1)\}$. Now, let $n > 1$ and assume that the claim holds for $n - 1$. Let $g \in G$, and take, in view of Lemma 3.1.4, an $h \in g.\Gamma$ such that $\Phi(h) \leq \Phi(g\gamma)$ for all $\gamma \in \Gamma$. Write $h = k_h a_h u_h$ as its Iwasawa decomposition. Again, we can assume that $u_h \in N_{1/2}$ as $\Phi(hz) = \Phi(z)$ for all $z \in N_{\mathbb{Z}}$. Then

$$k_h^{-1} h = \begin{pmatrix} a_1 & a_1 v \\ 0 & g' \end{pmatrix} \quad \text{for some } g' \in \mathrm{GL}_{n-1}(\mathbb{R}) \text{ with } \det g' = \frac{1}{a_1},$$

where a_1 is the first entry of a_h and $v = (u_{12} \cdots u_{1n})$. Then $\sqrt[n-1]{a_1}\, g' \in \mathrm{SL}_{n-1}(\mathbb{R})$ and our induction hypothesis implies that $g'.\mathrm{SL}_{n-1}(\mathbb{Z}) \cap \mathfrak{S}_0^{(n-1)} \neq \emptyset$, where $\mathfrak{S}_0^{(n-1)} = \mathfrak{S}_{2/\sqrt{3},1/2}^{(n-1)}$ denotes

the Siegel set in $\mathrm{SL}_{n-1}(\mathbb{R})$, i.e. there is a $\gamma' \in \mathrm{SL}_{n-1}(\mathbb{Z})$ such that $\sqrt[n-1]{a_1}\, g'\gamma' \in \mathfrak{S}_0^{(n-1)}$. Set $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \gamma' \end{pmatrix}$ and write $\sqrt[n-1]{a_1}\, g'\gamma' = k'a'u'$, $h\gamma = k''a''u''$ as their Iwasawa decompositions. As $\sqrt[n-1]{a_1}\, g'\gamma' \in \mathfrak{S}_0^{(n-1)}$, we have $a' \in A_{2/\sqrt{3}}^{(n-1)}$ and $u' \in N_{1/2}^{(n-1)}$. Note that

$$k_h^{-1} k'' a'' u'' = k_h^{-1} h\gamma = \begin{pmatrix} a_1 & a_1 v \\ 0 & g'\gamma' \end{pmatrix} = \begin{pmatrix} a_1 & a_1 v \\ 0 & \frac{1}{\sqrt[n-1]{a_1}} k'a'u' \end{pmatrix}.$$

By uniqueness of the Iwasawa decomposition, we must have

$$k'' = k_h \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix}, \quad a'' = \begin{pmatrix} a_1 & 0 \\ 0 & \frac{1}{\sqrt[n-1]{a_1}} a' \end{pmatrix}, \quad u'' = \begin{pmatrix} 1 & v \\ 0 & u' \end{pmatrix}.$$

It is clear that $u'' \in N_{1/2}$. To finish the proof, we must show that $a'' \in A_{2/\sqrt{3}}$: By construction,

$$a_i'' = \frac{1}{\sqrt[n-1]{a_1}} a_i' \le \frac{1}{\sqrt[n-1]{a_1}} \frac{2}{\sqrt{3}} a_{i+1}' = \frac{2}{\sqrt{3}} a_{i+1}'' \quad \text{for all } i \ge 2.$$

Note that $\Phi(h\gamma) = \Phi(h)$ as $\gamma(e_1) = e_1$. Hence, $\Phi(h\gamma) \le \Phi(g\eta)$ for all $\eta \in \Gamma$. In particular, $\Phi(h\gamma) \le \Phi(h\gamma\eta)$ for all $\eta \in \Gamma$, and therefore by Lemma 3.1.5,

$$a_1'' \le \frac{2}{\sqrt{3}} a_2''.$$

We conclude that $a'' \in A_{2/\sqrt{3}}$. Thus $h\gamma \in g.\Gamma \cap \mathfrak{S}_0$ and $h\gamma$ is a minimum point of $\Phi|_{g.\Gamma}$. $\square$

We have proved that:

THEOREM 3.1.7. *The Siegel set $\mathfrak{S}_{2/\sqrt{3},1/2}$ intersects all Γ-orbits in G; that is, $G = \mathfrak{S}_{2/\sqrt{3},1/2}.\Gamma$.*

Remark 3.1.8. Recall that $\mathscr{S}(\mathbb{R})$ denotes the set of positive definite matrices with determinant 1. Cholesky decomposition (Proposition 1.1.8) implies that an element $s \in \mathscr{S}(\mathbb{R})$ can be written uniquely as $s = u^t a u$ for $u \in N$, $a \in A$.

DEFINITION 3.1.9. A *Siegel set* in X is the image of a set of the form

$$\{ u^t a u \mid u \in N_\delta, a \in A_\lambda \} \subseteq \mathscr{S}(\mathbb{R})$$

under the diffeomorphism $\mathscr{S}(\mathbb{R}) \cong X$ described in Construction 2.1.7 for some $\lambda, \delta > 0$; in other words, it is the set of quadratic forms in X represented by matrices of the form $u^t a u$ with $u \in N_\delta$, $a \in A_\lambda$. It is denoted by $\mathfrak{S}'_{\lambda,\delta}$.

Remark 3.1.10. Let $\pi \colon G \to X$ be the projection $g \mapsto \iota.g$, where $\iota \colon x \mapsto x^t x$. It is equivariant, when G acts on itself by right multiplication and $X \curvearrowleft G$ as defined in the previous section. Note that $g = kau$ is the Iwasawa decomposition of g if and only if $g^t g = u^t a^2 u$ is the Cholesky decomposition of $g^t g$. It follows that

$$\pi(\mathfrak{S}_{\lambda,\delta}) = \mathfrak{S}'_{\lambda^2,\delta}, \quad \text{and} \quad \pi^{-1}(\mathfrak{S}'_{\lambda^2,\delta}) = \mathfrak{S}_{\lambda,\delta}.$$

The above remark and Theorem 3.1.7 yields the following result

THEOREM 3.1.11. *The Siegel set $\mathfrak{S}'_{4/3,1/2}$ intersects all Γ-orbits in X, so $X = \mathfrak{S}'_{4/3,1/2}.\Gamma$.*

From now on we denote by $\mathfrak{S}_{\lambda,\delta}$ both the Siegel sets in G and X; it will be immediate from the context, where they belong.

Remark 3.1.12. We will see that at least for $n = 2$, we cannot take λ and δ any smaller (see Section 3.2), so this is the best we can do to find a Siegel set which works for all n.

Construction 3.1.13 (Siegel Normal Coordinates). Consider the map $\tau\colon A \to \mathbb{R}_{>0}^{n-1}$ given by $\mathrm{pr}_i \circ \tau(a) = \frac{a_i}{a_{i+1}}$ for $a = (a_i) \in A$. This is bijective with inverse $\tau^{-1}\colon \mathbb{R}_{>0}^{n-1} \to A$ given by

$$\tau^{-1}(b)_j = \frac{\sqrt[n]{\prod_{i=1}^{n-1} b_i^{n-i}}}{b_1 \dots b_{j-1}} \qquad \text{for } b = (b_i) \in \mathbb{R}_{>0}^{n-1}.$$

Indeed, for all $b = (b_i) \in \mathbb{R}_{>0}^{n-1}$,

$$\mathrm{pr}_j(\tau \circ \tau^{-1}(b)) = \frac{\tau^{-1}(b)_j}{\tau^{-1}(b)_{j+1}} = \frac{\sqrt[n]{\prod_{i=1}^{n-1} b_i^{n-i}}}{b_1 \dots b_{j-1}} \cdot \frac{b_1 \dots b_j}{\sqrt[n]{\prod_{i=1}^{n-1} b_i^{n-i}}} = b_j$$

and for all $a = (a_i) \in A$, where we write $\tau(a)_j := \mathrm{pr}_j(\tau(a))$,

$$a_1^n = \prod_{i=1}^n \frac{a_1}{a_i} = \prod_{i=2}^n \tau(a)_1 \dots \tau(a)_{i-1} = \prod_{i=1}^{n-1} \tau(a)_i^{n-i}$$

and thus

$$\tau^{-1}(\tau(a))_j = (\tau(a)_1 \dots \tau(a)_{j-1})^{-1} \sqrt[n]{\prod_{i=1}^{n-1} \tau(a)_i^{n-i}} = \frac{a_j}{a_1} a_1 = a_j.$$

Both τ and τ^{-1} are smooth, so τ is a diffeomorphism, and it is easily seen that it is in fact a group isomorphism into the multiplicative group $\mathbb{R}_{>0}^{n-1}$.

DEFINITION 3.1.14. The maps $t_i\colon A \to \mathbb{R}_{>0}$, $a = (a_i) \mapsto \frac{a_i}{a_{i+1}}$, are called the *Siegel normal coordinates* on A, and the map $\tau\colon A \to \mathbb{R}_{>0}^{n-1}$ given by $\mathrm{pr}_i \circ \tau = t_i$ is the *Siegel normal coordinatisation map*.

3.2   THE CASE $n = 2$ PART II

We return to the case $n = 2$. Set $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, let X be the manifold constructed in Section 2.1 and let $\mathscr{H} \subseteq \mathbb{C}$ denote the upper half-plane. We know that X and $\mathscr{H}$ are diffeomorphic, and we have compatible actions of G on X and $\mathscr{H}$ (Section 2.2). In this section, we determine a fundamental domain of the action $\mathscr{H} \curvearrowleft \Gamma$ and compare it with $\mathfrak{S}_0 := \mathfrak{S}_{4/3,1/2} \subseteq X$.

Recall that the action of G on $\mathscr{H}$ is given by

$$z.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{dz + b}{cz + a} \qquad \text{for } z \in \mathscr{H}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

PROPOSITION 3.2.1. The set $D = \{z \in \mathscr{H} \mid |z| > 1, |\mathrm{Re}(z)| < \frac{1}{2}\}$ satisfies $\mathscr{H} = \overline{D}.\Gamma$.

Proof. To see that $\mathscr{H} = \overline{D}.\Gamma$, let $z \in \mathscr{H}$. For any $M > 0$, there are only finitely many integers $a, b \in \mathbb{Z}$ such that $|az + b| \leq M$. Hence, as $\mathrm{Im}(z.\gamma) = \frac{\mathrm{Im}(z)}{|\gamma_{21}z + \gamma_{11}|^2}$ for any $\gamma = (\gamma_{ij}) \in \Gamma$, there exists $\gamma_0 \in \Gamma$ such that $\mathrm{Im}(z.\gamma_0) \geq \mathrm{Im}(z.\gamma)$ for all $\gamma \in \Gamma$. With γ_0 as above, set $z' := z.\gamma_0$. Let

$m \in \mathbb{Z}$ such that $|\mathrm{Re}(z') + m| \leq \frac{1}{2}$ and define $\eta_0 := \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$; then $z'.\eta_0 = z' + m$. We claim that $z'' := z'.\eta_0 = z.(\gamma_0\eta_0)$ is an element of $\overline{D}$. By construction $|\mathrm{Re}(z'')| \leq \frac{1}{2}$; to see that $|z''| \geq 1$, note that

$$\frac{\mathrm{Im}(z'')}{|z''|^2} = \mathrm{Im}(z''.\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)) = \mathrm{Im}(z.(\gamma_0\eta_0\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))) \leq \mathrm{Im}(z.\gamma_0) = \mathrm{Im}(z.(\gamma_0\eta_0)) = \mathrm{Im}(z'').$$

We conclude that $z'' \in \overline{D} = \{z \in \mathscr{H} \mid |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}\}$. $\square$

Claim 1 of the following proposition shows that we cannot take D smaller; Claims 2-4 show what happens on the boundary.

PROPOSITION 3.2.2. Let $z \in \mathscr{H}$, $\gamma \in \Gamma$. Then the following hold:

1. If $z, z.\gamma \in D$, then $\gamma = \pm\mathrm{id}$.

2. If $z, z.\gamma \in \overline{D} - \{z \in \mathscr{H} \mid |z| = 1\}$ and $\gamma \neq \pm\mathrm{id}$, then $\mathrm{Re}(z) = \pm\frac{1}{2}$ and $\gamma = \pm\left(\begin{smallmatrix} 1 & \mp 1 \\ 0 & 1 \end{smallmatrix}\right)$.

3. If $z, z.\gamma \in \overline{D} - \{\pm\frac{1}{2} + \frac{\sqrt{3}}{2}i\}$ and $\gamma \notin \left\{ \pm\mathrm{id}, \pm\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right) \right\}$, then $|z| = 1$ and $\gamma = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

4. If $z, z.\gamma \in \overline{D}$ and $\gamma \notin \left\{ \pm\mathrm{id}, \pm\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\}$, then

$$z, z.\gamma \in \{\pm\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\} \quad \text{and} \quad \gamma \in \left\{ \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & \pm 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 0 \\ \pm 1 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} \pm 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\}.$$

Proof. For all four claims, we can without loss of generality assume that $\mathrm{Im}(z.\gamma) \geq \mathrm{Im}(z)$ — if the opposite is the case, we simply consider $z.\gamma$ and $(z.\gamma).\gamma^{-1}$ instead. Then, as $\mathrm{Im}(z.\gamma) = \frac{\mathrm{Im}(z)}{|\gamma_{21}z + \gamma_{11}|^2}$, we must have $|\gamma_{21}z + \gamma_{11}| \leq 1$. First note that

$$1 \geq |\gamma_{21}z + \gamma_{11}|^2 = \gamma_{21}^2|z|^2 + 2\gamma_{11}\gamma_{21}\mathrm{Re}(z) + \gamma_{11}^2 \geq \gamma_{21}^2|z|^2 - \gamma_{11}\gamma_{21} + \gamma_{11}^2$$
$$= \gamma_{21}^2|z|^2 - \tfrac{1}{4}\gamma_{21}^2 + (\tfrac{1}{2}\gamma_{21} - \gamma_{11})^2 \geq \gamma_{21}^2(|z|^2 - \tfrac{1}{4}) \geq \tfrac{3}{4}\gamma_{21}^2. \qquad (3.2)$$

It follows that $|\gamma_{21}| \leq \frac{2}{\sqrt{3}} < 2$, i.e. $\gamma_{21} \in \{0, \pm 1\}$.
For the first two claims we have $|z| > 1$. Hence, if $\gamma_{21} = \pm 1$, then

$$1 \geq |\pm z + \gamma_{11}|^2 = |z|^2 \pm 2\gamma_{11}\mathrm{Re}(z) + \gamma_{11}^2 \geq |z|^2 \mp \gamma_{11} + \gamma_{11}^2 \geq |z|^2 + \gamma_{11}^2 - |\gamma_{11}| \geq |z|^2 > 1,$$

so we conclude that $\gamma_{21} = 0$, and thus

$$\gamma = \pm\begin{pmatrix} 1 & \gamma_{12} \\ 0 & 1 \end{pmatrix} \quad \text{is an upper triangular matrix and} \quad z.\gamma = z + \gamma_{12}.$$

In Claim 1, $z, z.\gamma \in D$, and therefore $|\gamma_{21}| = |\mathrm{Re}(z.\gamma) - \mathrm{Re}(z)| < 1$, so $\gamma_{21} = 0$, and as desired, we have $\gamma = \pm\mathrm{id}$ and $z.\gamma = z$.
In Claim 2, $z, z.\gamma \in \overline{D}$, and therefore $|\gamma_{21}| = |\mathrm{Re}(z.\gamma) - \mathrm{Re}(z)| \leq 1$, and as $\gamma \neq \pm\mathrm{id}$, we must have $\gamma_{12} = \pm 1$. It follows that $\mathrm{Re}(z) = \mp\frac{1}{2}$ and $z.\gamma = z \pm 1$.
For the third and fourth claim, note first that if $\gamma_{21} = 0$, then $\mathrm{Re}(z.\gamma) = \mathrm{Re}(z) + \gamma_{12}$ would imply $\gamma_{12} \in \{0, \pm 1\}$; as we have assumed $\gamma \neq \pm\mathrm{id}, \pm\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$, we conclude that $\gamma_{21} = \pm 1$. The inequality

$$1 \geq |\pm z + \gamma_{11}|^2 \geq |z|^2 + \gamma_{11}^2 - |\gamma_{11}|$$

implies that $|z| = 1$ and $\gamma_{11} \in \{0, \pm 1\}$. Then also $|\pm z + \gamma_{11}| = 1$, and therefore $\mathrm{Im}(z) = \mathrm{Im}(z.\gamma)$. If $\gamma_{11} = \pm 1$, then

$$1 = |z \pm 1| = |z|^2 \pm 2\mathrm{Re}(z) + 1 = 2(1 \pm \mathrm{Re}(z)),$$

implying $\mathrm{Re}(z) = \mp\frac{1}{2}$ and thus $z = \mp\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Therefore, in Claim 3, we must have $\gamma_{11} = 0$, so $\gamma = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & \gamma_{22} \end{smallmatrix} \right)$. If $\gamma_{22} \neq 0$, then

$$z.\gamma = \gamma_{22} - \overline{z} \quad \text{and} \quad |\mathrm{Re}(z.\gamma)| = |\gamma_{22} - \mathrm{Re}(z)| \geq \frac{1}{2}.$$

But then $z = \pm\frac{1}{2} + \frac{\sqrt{3}}{2}i$, contradicting our assumption. Hence, as desired $\gamma = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and $z.\gamma = \frac{1}{z} = -\overline{z}$.

Now, for the fourth and final claim: If $\gamma_{11} = 0$, then $\gamma = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & \gamma_{22} \end{smallmatrix} \right)$ for $\gamma_{22} \neq 0$ and the above arguments show that $\gamma_{22} = \pm 1$ and $z = z.\gamma = \pm\frac{1}{2} + \frac{\sqrt{3}}{2}$.

If $\gamma_{11} = \pm 1$, then $z = \mp\frac{1}{2} + \frac{\sqrt{3}}{2}$ by the above and $\gamma = \pm \left(\begin{smallmatrix} \pm 1 & \gamma_{22} - 1 \\ 1 & \pm \gamma_{22} \end{smallmatrix} \right)$. Then, as $|z| = 1$ and $\mathrm{Im}(z.\gamma) = \mathrm{Im}(z) = \frac{\sqrt{3}}{2}$, we must have $|\mathrm{Re}(z.\gamma)| = \frac{1}{2}$. It follows that

$$1 = |z.\gamma|^2 = \frac{|\pm \gamma_{22}z + \gamma_{22} - 1|^2}{|z \pm 1|^2} = \gamma_{22}^2|z|^2 \pm 2\gamma_{22}(\gamma_{22} - 1)\mathrm{Re}(z) + (\gamma_{22} - 1)^2$$
$$\geq \gamma_{22}^2 \mp \gamma_{22}(\gamma_{22} - 1) + (\gamma_{22} - 1)^2 = (2 \mp 1)\gamma_{22}(\gamma_{22} - 1) + 1$$

and thus $\gamma_{22} \in \{0, 1\}$, so $\gamma \in \left\{ \pm \left(\begin{smallmatrix} 1 & 0 \\ \pm 1 & 1 \end{smallmatrix} \right), \pm \left(\begin{smallmatrix} \pm 1 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right\}$. $\square$

DEFINITION 3.2.3. A *fundamental domain* for a group action on a set Y is a subset of Y which intersects each orbit exactly once.

COROLLARY 3.2.4. The following set is a fundamental domain for the action $\mathscr{H} \curvearrowleft \Gamma$

$$D \cup \{z \in \overline{D} \mid |z| > 1, \mathrm{Re}(z) = \tfrac{1}{2}\} \cup \{z \in \overline{D} \mid |z| = 1, \mathrm{Re}(z) \leq 0\}.$$

Remark 3.2.5. The image of $\mathfrak{S}_{\lambda,\delta} \subseteq X$ under the diffeomorphism Φ is the set

$$\Phi(\mathfrak{S}_{\lambda,\delta}) = \left\{ z \in \mathscr{H} \;\middle|\; \mathrm{Im}(z) \geq \frac{1}{\sqrt{\lambda}}, \; |\mathrm{Re}(z)| \leq \delta \right\}$$

This can be seen from the Cholesky decomposition of $s = (s_{ij})$ defining $q \in X$:

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{s_{11}} & 0 \\ \frac{s_{12}}{\sqrt{s_{11}}} & \frac{1}{\sqrt{s_{11}}} \end{pmatrix} \begin{pmatrix} \sqrt{s_{11}} & \frac{s_{12}}{\sqrt{s_{11}}} \\ 0 & \frac{1}{\sqrt{s_{11}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{s_{12}}{s_{11}} & 1 \end{pmatrix} \begin{pmatrix} s_{11} & 0 \\ 0 & \frac{1}{s_{11}} \end{pmatrix} \begin{pmatrix} 1 & \frac{s_{12}}{s_{11}} \\ 0 & 1 \end{pmatrix}.$$

By definition of $\mathfrak{S}_{\lambda,\delta}$, if $q \in \mathfrak{S}_{\lambda,\delta}$, then $s_{11} \leq \lambda\frac{1}{s_{11}}$, implying $s_{11} \leq \sqrt{\lambda}$ and $|\frac{s_{12}}{s_{11}}| \leq \delta$. Hence, $\Phi(q) = z_q = \frac{s_{12}}{s_{11}} + \frac{1}{s_{11}}i$ is contained in the set above. This decomposition also shows that $\Psi(z) = q_z \in \mathfrak{S}_{\lambda,\delta}$ for any z satisfying $\mathrm{Im}(z) \geq \frac{1}{\sqrt{\lambda}}$ and $|\mathrm{Re}(z)| \leq \delta$.

We see that $\Phi(\mathfrak{S}_0)$ is only a little bit bigger than the fundamental domain of $X \curvearrowleft \Gamma$, and that it is the smallest Siegel set $\mathfrak{S}_{\lambda,\delta}$ in X such that $D \subseteq \Phi(\mathfrak{S}_{\lambda,\delta})$. The sets D and $\Phi(\mathfrak{S}_0)$ are pictured in Figure 3.1.
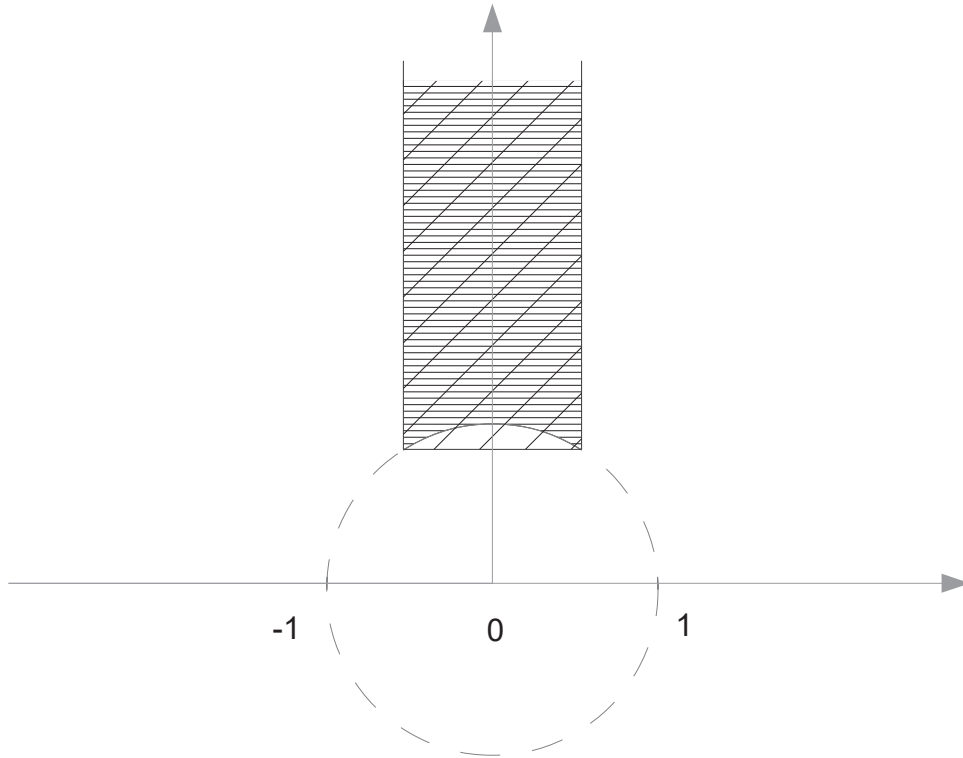
Figure 3.1: The subsets D (horizontal shading) and $\Phi(\mathfrak{S}_0)$ (diagonal shading) of the upper half-plane $\mathscr{H}$.

3.3 Technical Results

As the title suggests, this section contains some technical results about Siegel sets. Though we will not need it, we prove that a Siegel set has finite measure with respect to a Haar measure on G. There are two reasons for doing this: It is in itself an interesting result, but more importantly the idea of the proof reappears later on, albeit in a much more complicated version (see Proposition 5.2.9). We also prove that there are certain limitations to how an element of $\mathrm{SL}_n(\mathbb{R})$ may act on a Siegel set; this we will need later on.

Lemma 3.3.1. For any compact subset $C \subseteq N$, $\lambda > 0$, the set $\bigcup_{a \in A_\lambda} aCa^{-1} \subseteq N$ is relatively compact.

Proof. Note first that that the exponential map $\exp \colon \mathfrak{n} \to N$ is a diffeomorphism and that

$$\exp \circ \mathrm{Ad}(a)|_{\mathfrak{n}} = c_a|_N \circ \exp, \quad \text{where } c_a \text{ is conjugation by } a \in A.$$

Hence, it suffices to show that for any compact $C \subseteq \mathfrak{n}$, $\lambda > 0$, the set $\bigcup_{a \in A_\lambda} \mathrm{Ad}(a)C$ is relatively compact in $\mathfrak{n}$. We know from the proof of Proposition 1.2.15 that for $a = (a_i) \in A$, $\mathrm{Ad}(a)|_{\mathfrak{n}}$ is given by the diagonal matrix $((\frac{a_i}{a_j})_{i<j})$. Then $\{\mathrm{Ad}(a)|_{\mathfrak{n}}\}_{a \in A_\lambda}$ is a bounded family of operators on $\mathfrak{n}$ as $\frac{a_i}{a_j} < \lambda^{j-i}$ for all $i < j$, $a = (a_i) \in A_\lambda$. It follows that $\bigcup_{a \in A_\lambda} \mathrm{Ad}(a)C$ is bounded and thus relatively compact for any $C \subseteq \mathfrak{n}$ compact. $\qquad\square$

Proposition 3.3.2. Let $(x_j)_{j \in \mathbb{N}} \subseteq G$ and $g \in G$. Let $0 = l_0 < \ldots < l_k = n$ be a partition of n defining a BUT P. If $\{x_j\}, \{x_j g\} \subseteq \mathfrak{S}_{\lambda, \delta}$ and $\frac{(a_j)_{l_i}}{(a_j)_{l_i + 1}} \xrightarrow{j \to \infty} 0$ for all $i = 1, \ldots, k - 1$, where $a_j \in A_\lambda$ is the diagonal matrix of the Iwasawa decomposition of x_j, then $g \in P$.

Proof. Assume $x_j, x_j g \in \mathfrak{S}_{\lambda, \delta}$ for all $j \in \mathbb{N}$ and write $x_j = k_j a_j x_j$, $x_j g = k_j' a_j' u_j'$ for $k_j, k_j' \in K$, $a_j, a_j' \in A_\lambda$, $u_j, u_j' \in N_\delta$. Assume that $\frac{(a_j)_{l_i}}{(a_j)_{l_i + 1}} \xrightarrow{j \to \infty} 0$ for all $i = 1, \ldots, k - 1$. Let $g = uwzv$ be the Bruhat decomposition of g, i.e. $u, v \in N$, $w \in W$ and $z \in A$. Then

$$k_j' a_j' u_j' = x_j g = k_j a_j u_j uwzv = k_j (a_j u_j u a_j^{-1}) a_j wzv = k_j w (w^{-1} a_j u_j u a_j^{-1} w)(w^{-1} a_j w) zv.$$

Setting $d_j := w^{-1} a_j u_j u a_j^{-1} w = k_{d_j} a_{d_j} u_{d_j}$, $k_{d_j} \in K$, $a_{d_j} \in A$, $u_{d_j} \in N$, we have

$$k_j' a_j' u_j' = k_j w (k_{d_j} a_{d_j} u_{d_j})(w^{-1} a_j w) zv = (k_j w k_{d_j})(a_{d_j} w^{-1} a_j wz)((w^{-1} a_j wz)^{-1} u_{d_j} (w^{-1} a_j wz) v),$$

where

$$k_j w k_{d_j} \in K, \quad a_{d_j} w^{-1} a_j wz \in A, \quad (w^{-1} a_j wz)^{-1} u_{d_j} (w^{-1} a_j wz) v \in N.$$

In particular, we must have $a_j' = a_{d_j} w^{-1} a_j wz$. We use this identity to prove that $w \in P$ from which it directly follows that $g = uwzv \in P$. Let $\sigma^{-1} \in \Sigma_n$ be the permutation defining w, i.e. $w_{i\sigma^{-1}(i)} = \pm 1$ and $w_{ij} = 0$ for $j \neq \sigma^{-1}(i)$ (we take the inverse permutation to simplify notation a little). Then $w^{-1} a_j w$ is the diagonal matrix with i'th entry equal to $(a_j)_{\sigma(i)}$. We see that $d_j \in w^{-1}(\bigcup_{j \in \mathbb{N}} a_j N_\delta u a_j^{-1}) w$; this set is bounded by Lemma 3.3.1, so the set $\{d_j\}_{j \in \mathbb{N}}$ is bounded in G and hence the set $\{a_{d_j}\}$ is bounded in $A \cong \mathbb{R}_{>0}^{n-1}$. Let $C \in \mathbb{R}$ such that $\frac{(a_{d_j})_i}{(a_{d_j})_{i+1}} \geq C$ for all $i = 1, \ldots, n - 1$ and set $Z := \min_i \frac{z_i}{z_{i+1}}$. Then

$$\lambda > \frac{(a_j')_i}{(a_j')_{i+1}} = \frac{z_i}{z_{i+1}} \cdot \frac{(a_{d_j})_i}{(a_{d_j})_{i+1}} \cdot \frac{(a_j)_{\sigma(i)}}{(a_j)_{\sigma(i+1)}} \geq ZC \frac{(a_j)_{\sigma(i)}}{(a_j)_{\sigma(i+1)}} \quad \text{for all } i = 1, \ldots n - 1.$$

Now, we claim that σ preserves the partition $0 = l_0 < \cdots < l_k = n$, i.e.

$$\sigma(\{l_s + 1, \ldots, l_{s+1}\}) = \{l_s + 1, \ldots, l_{s+1}\} \quad \text{for all } s = 1, \ldots, k - 1.$$

Then so does σ^{-1} and hence $w \in P$. Assume for contradiction that σ does not preserve the partition and let s be minimal such that $\sigma(\{l_{s-1}+1, \ldots, l_s\}) \neq \{l_{s-1}+1, \ldots, l_s\}$. By minimality of s, we must have some $i \leq l_s$ such that $\sigma(i) > l_s$ and some $i' > l_s$ such that $\sigma(i') \leq l_s$. But then

$$\prod_{m=\sigma(i')}^{\sigma(i)-1} \frac{(a_j)_{m+1}}{(a_j)_m} = \frac{(a_j)_{\sigma(i)}}{(a_j)_{\sigma(i')}} = \prod_{m=i}^{i'-1} \frac{(a_j)_{\sigma(m)}}{(a_j)_{\sigma(m+1)}} \leq \left(\frac{\lambda}{ZC} \right)^{i'-i}.$$

Now, $\frac{(a_j)_{m+1}}{(a_j)_m} > \lambda^{-1}$ for all $m = 1, \ldots, n - 1$, $j \in \mathbb{N}$, and $\frac{(a_j)_{l_s+1}}{(a_j)_{l_s}} \to \infty$ as $j \to \infty$. Then as $l_s \in \{\sigma(i'), \ldots, \sigma(i) - 1\}$, the product on the very left tends to infinity, a contradiction. $\square$

Theorem 3.3.3. The measure of a Siegel set $\mathfrak{S}_{\lambda, \delta} \subseteq G$ with respect to a Haar measure on G is finite.

Proof. Let μ_G be a Haar measure on G. Then by Proposition 1.2.18, we have

$$\mu_G(\mathfrak{S}_{\lambda,\delta}) = \int_{\mathfrak{S}_{\lambda,\delta}} d\mu_G(g) = \int_{K \times A_\lambda \times N_\delta} \rho(a) \, d\mu_K(k) d\mu_A(a) d\mu_N(n)$$

$$= \left(\int_K d\mu_K(k) \right) \left(\int_{A_\lambda} \rho(a) \, d\mu_A(a) \right) \left(\int_{N_\delta} d\mu_N(n) \right).$$

As K and N_δ are compact, we have

$$\mu_G(\mathfrak{S}_{\lambda,\delta}) = c_1 \int_{A_\lambda} \rho(a) \, d\mu_A(a)$$

for some constant $c_1 > 0$. Let $\tau \colon A \to \mathbb{R}_{>0}^{n-1}$, $a = (a_i) \mapsto b = (b_i)$ with $b_i = \frac{a_i}{a_{i+1}}$ denote the Siegel normal coordinatisation map (Definition 3.1.14). Then $\tau_*\mu_G$ is a Haar measure on $\mathbb{R}_{>0}^{n-1}$. Note that

$$\tau(A_\lambda) = (0, \lambda]^{n-1} \quad \text{and} \quad \rho \circ \tau^{-1}(b) = \prod_{i=1}^{n-1} b_i^{i(n-i)} \quad \text{for all } b = (b_i) \in \mathbb{R}_{>0}^{n-1}.$$

The map $\xi \colon \mathbb{R}^{n-1} \to \mathbb{R}_{>0}^{n-1}$, $y = (y_i) \mapsto (\exp(y_i))$ is a group isomorphism mapping the Lebesgue measure ν on $\mathbb{R}^{n-1}$ to a Haar measure on $\mathbb{R}_{>0}^{n-1}$, so $\xi_*\nu = c_2\tau_*\mu_G$ for some $c_2 > 0$. Setting $C = \frac{c_1}{c_2}$, we then have

$$\mu_G(\mathfrak{S}_{\lambda,\delta}) = c_1 \int_{A_\lambda} \rho(a) \, d\mu_A(a) = c_1 \int_{(0,\lambda]^{n-1}} \rho \circ \tau^{-1}(b) \, d\tau_*\mu_A(b)$$

$$= C \int_{(-\infty,\log(\lambda)]^{n-1}} \rho \circ \tau^{-1} \circ \xi(y) \, d\nu(y).$$

Now, $\rho \circ \tau^{-1} \circ \xi(y) = \prod_{i=1}^{n-1} \exp(i(n-i)y_i)$ for all $y = (y_i) \in \mathbb{R}^{n-1}$. Hence, as desired

$$\mu_G(\mathfrak{S}_{\lambda,\delta}) = C \prod_{i=1}^{n-1} \left(\int_{-\infty}^{\log(\lambda)} \exp(i(n-i)y_i) \, dy_i \right) < \infty.$$

$\square$

4 | Borel-Serre Compactification

We still consider $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and X as in Section 2.1 and we are interested in the quotient space X/Γ. The main inconvenience is that X/Γ is not compact. In this chapter, we construct a compact replacement, that is a compactification of X/Γ. We partially compactify X to get a manifold with corners $\overline{X}$, to which we can extend the action of Γ, and such that the quotient $\overline{X}/\Gamma$ is compact. Again, we look at the case $n = 2$ in detail to get a better understanding of the geometry of this construction. We also take a look at the case $n = 3$, as the case $n = 2$ is in some ways too simple to really illustrate the construction.

The construction is due to Borel and Serre and is done for general arithmetic groups in their paper [3]. In our case, as Borel and Serre also point out in their article, it is an application of Siegel reduction theory of quadratic forms, which was dealt with in the previous chapter.

We will not prove here that X/Γ is not compact — it is a consequence of the Godement compactness criterion (see [7, Proposition III.2.15]). Note that in the case $n = 2$, it is seen immediately that X/Γ is not compact in Section 3.2 as we determined explicitly the fundamental domain of the action $X \curvearrowleft \Gamma$.

We refer to [15, Appendix C] for some background on manifolds with corners. We will in the following use the terms smooth, submanifold etc. without specifying explicitly that it is of course meant in the sense of manifolds with corners.

Let $G = \mathrm{SL}_n(\mathbb{R})$ and let $A \leq G$ denote the subgroup of diagonal matrices with positive entries, $N \leq G$ the subgroup of upper triangular matrices with 1's on the diagonal and set $K = \mathrm{SO}(n)$. For $g \in G$, we write $q_g := \iota.g$ for the quadratic form given by the matrix $g^t g$.

4.1 Geodesic Action

We begin by defining a left action on X by A. We will consider the orbits of this action and of certain restrictions of it when we construct the partial compactification.

Let $P \leq G$ be a subgroup of block upper triangular matrices (a BUT) given by a partition κ_P, with κ_P given by $0 = l_0 < l_1 < \cdots < l_k = n$, or equivalently $(m_1, \ldots, m_k)$ such that $\sum_{i=1}^k m_i = n$. Recall the Langlands decomposition (cf. 1.1.5): $(K \cap P) \times A_P \times N_P \xRightarrow{\cong} P$, where

$$A_P = \{(a_i) \text{ diagonal} \mid a_{l_i+j} = a_{l_{i+1}} > 0 \text{ for } j = 1, \ldots, m_{i+1}, \ i = 0, \ldots, k-1\},$$
$$N_P = \{(u_{ij}) \text{ upper triangular} \mid u_{ii} > 0 \text{ for } i = 1, \ldots, n$$
$$\text{and } \Pi_{j=1}^{m_{i+1}} u_{l_i+j, l_i+j} = 1 \text{ for } i = 0, \ldots, k-1\}.$$

Recall the Siegel normal coordinates on A (Definition 3.1.14), $t_i \colon A \to \mathbb{R}_{>0}$, $t_i(a) = \frac{a_i}{a_{i+1}}$, and consider the map

$$\tau_P \colon A_P \longrightarrow \mathbb{R}_{>0}^{k-1}, \quad \text{given by} \quad \mathrm{pr}_i \circ \tau = t_{l_i}. \tag{4.1}$$

This is an isomorphism of Lie groups (by the same arguments as in Construction 3.1.13).

Recall also that we have a diffeomorphism $A_P \times N_P \to X$, $(a, u) \mapsto \iota.(au) = q_{au}$ (Corollary 2.1.14).

Construction 4.1.1 (Geodesic Action).
Consider the action of A on X given by left multiplication of A on $(K \cap B)\backslash B \cong X$. This is well-defined because A commutes with $K \cap B$, the subgroup of diagonal matrices with ± 1 on the diagonal. In terms of quadratic forms and symmetric matrices, the action is given as follows: if $q \in X$ is represented by $s = b^t b$ for $b \in B$, then for $a \in A$, $a.q$ is represented by $(ab)^t ab$.
For a BUT P, let $A_P \le A$ inherit the above left action on X. This is easily seen to be equivalent to defining the action as left multiplication of A_P on $(K \cap P)\backslash P \cong X$, exploiting that A_P commutes with $K \cap P$: If $q \in X$ is represented by $s = b^t b$ for $b \in P$, then for $a \in A_P$, $a.q$ is represented by $(ab)^t ab$.
This action is called the *geodesic action* of A_P on X.

PROPOSITION 4.1.2. *The action of A, and hence of any A_P, on X is smooth.*

Proof. The multiplication map $m\colon A \times AN \to AN$ is smooth as it is the restriction of multiplication in G and the inclusions $AN \hookrightarrow G$ and $A \hookrightarrow G$ are smooth. Then $A \times X \to X$, $(a, q) \mapsto a.q$, is smooth, being equal to the composite

$$A \times X \xrightarrow{\text{id} \times \varphi^{-1}} A \times AN \xrightarrow{m} AN \xrightarrow{\varphi} X,$$

where $\varphi\colon AN \to X$ is the diffeomorphism $b \mapsto (q\colon x \mapsto x b^t b x)$. $\square$

PROPOSITION 4.1.3. *The action of A, and hence of any A_P, on X is free.*

Proof. Let $b \in B$ and assume that $a \in A$ satisfies $(ab)^t(ab) = b^t b$. Then $a^t a = \text{id}$, so a is orthogonal. Being a diagonal matrix with positive entries, we must have $a = \text{id}$. $\square$

Remark 4.1.4. It follows from the above propositions that the orbits of A_P in X are diffeomorphic to A_P, and therefore via τ_P to $\mathbb{R}_{>0}^{k-1}$, so we get a partition of X into copies of $\mathbb{R}_{>0}^{k-1}$.
If we let A_P act on $A_P \times N_P$ by left multiplication on the first term, then the diffeomorphism $A_P \times N_P \to X$, $(a, u) \mapsto q_{au}$, is equivariant, so the partition of X into A_P-orbits is given by this product.

4.2 CONSTRUCTION

In this section, we get our hands dirty: The aim is to construct a manifold with corners $\overline{X}$ with the desired properties. For every subgroup of block upper triangular matrices P, we add the boundary to each A_P-orbit and, in addition, to each γ-translate of an A_P-orbit, $\gamma \in \Gamma$. In this way, we partially compactify X, obtaining a space $\overline{X}$. We can equip this with the structure of a manifold with corners, extend the action of Γ to it and using Siegel reduction theory, we see that $\overline{X}/\Gamma$ is compact.

Construction 4.2.1 (Corner associated to a Subgroup of Block Upper Triangular Matrices I).
Let P be a BUT given by a partition κ_P, with κ_P given by $0 = l_0 < l_1 < \cdots < l_k = n$, or equivalently $(m_1, \ldots, m_k)$ such that $\sum_{i=1}^{k} m_i = n$. Set $\overline{A}_P := \mathbb{R}_{\geq 0}^{k-1}$ and interpret A_P as a subspace of $\overline{A}_P$ using the diffeomorphism $\tau_P\colon A_P \to \mathbb{R}_{>0}^{k-1} \subseteq \mathbb{R}_{\geq 0}^{k-1} = \overline{A}_P$. Let A_P act on $\overline{A}_P$ by coordinatewise multiplication and note that this simply extends the action of A_P on itself. In the case $P = G$, $A_P = \overline{A}_P = *$.
Given $\gamma \in \Gamma$, let A_P act on $X.\gamma$ from the left by

$$A_P \times X.\gamma \xrightarrow{\text{id} \times \gamma^{-1}} A_P \times X \longrightarrow X \to X.\gamma,$$

where the middle map is the action map of the geodesic action of A_P on X, so the composition maps $(a, q.\gamma) \in A_P \times X.\gamma$ to $(a.q).\gamma \in X.\gamma$. The orbits of this action are simply the γ-translates of the A_P-orbits in X.

For any $[\gamma]_P \in (\Gamma \cap P)\backslash\Gamma$, set

$$X(P)_{[\gamma]_P} := \overline{A}_P \times_{A_P} X.\gamma = \overline{A}_P \times X/\sim_{\gamma,P},$$

where the equivalence relation $\sim_{\gamma,P}$ is defined as $(\overline{a}.a, q.\gamma) \sim_{\gamma,P} (\overline{a}, (a.q).\gamma)$ for all $\overline{a} \in \overline{A}$, $p = q.\gamma \in X$, $a \in A_P$.

We must show that $X(P)_{[\gamma]_P}$ is well-defined: Suppose $\gamma = \beta\eta$ for some $\eta \in \Gamma$, $\beta \in \Gamma \cap P$, and let $q \in X$, $\overline{a} \in \overline{A}_P$ and $a \in A_P$. Let $b \in P$ such that q is given by $x \mapsto x^t(b^t b)x$. Then the quadratic form $(a.q).\beta$ is represented by the matrix $(ab\beta)^t(ab\beta)$. The quadratic form $q.\beta$ is represented by $(b\beta)^t(b\beta)$, and as $b\beta \in P$, $a.(q.\beta)$ is represented by $(ab\beta)^t(ab\beta)$, so $(a.q).\beta = a.(q.\beta)$. Therefore

$$(\overline{a}.a, q.\gamma) = (\overline{a}.a, (q.\beta).\eta) \sim_{\eta,P} (\overline{a}, (a.(q.\beta)).\eta) = (\overline{a}, ((a.q).\beta).\eta) = (\overline{a}, (a.q).\gamma).$$

We conclude that $\sim_{\gamma,P}$ and $\sim_{\eta,P}$ are equal, and thus that $X(P)_{[\gamma]_P}$ is well-defined.

Equipping $X(P)_{[\gamma]_P}$ with the quotient topology, $\iota_{[\gamma]_P} \colon X \hookrightarrow X(P)_{[\gamma]_P}$, $q \mapsto [(\mathrm{id}, q)]_{[\gamma]_P}$ is an open inclusion. We identify X with $\iota_{[\gamma]_P}(X)$ for all $[\gamma]_P$.

PROPOSITION 4.2.2. The map $\overline{A}_P \times N_P \to X(P)_{[\gamma]_P}$, $(\overline{a}, u) \mapsto [\overline{a}, q_u.\gamma]_{[\gamma]_P}$, is a homeomorphism.

Proof. Consider the smooth quotient $\overline{A}_P \times A_P \times N_P \to \overline{A}_P \times N_P$, $(\overline{a}, a, u) \mapsto (\overline{a}a, u)$ and let $\varphi \colon A_P \times N_P \to X$ denote the diffeomorphism $(a, u) \mapsto q_{au}$. Then the diffeomorphism $\mathrm{id} \times (\gamma \circ \varphi) \colon \overline{A}_P \times A_P \times N_P \to \overline{A}_P \times X$ induces a homeomorphism on the quotients, as illustrated in the following diagram, given by $(\overline{a}, u) \mapsto [\overline{a}, q_u.\gamma]_{[\gamma]_P}$:

$$
\begin{array}{ccc}
\overline{A}_P \times A_P \times N_P & \xrightarrow{\mathrm{id} \times (\gamma \circ \varphi)} & \overline{A}_P \times X \\
\downarrow & & \downarrow \\
\overline{A}_P \times N_P & \longrightarrow & X(P)_{[\gamma]_P}
\end{array}
$$

$\square$

Construction 4.2.3 (Corner associated to a Subgroup of Block Upper Triangular Matrices II). We equip the space $X(P)_{[\gamma]_P}$ with the structure of a manifold with corners inherited from $\overline{A}_P \times N_P \cong \mathbb{R}^{k-1}_{\geq 0} \times \mathbb{R}^{\frac{n(n+1)}{2}-k}$ under the above homeomorphism. With this structure, the quotient map $A_P \times X \twoheadrightarrow X(P)_{[\gamma]_P}$ is smooth. $X(P)_{[\gamma]_P}$ is the *corner* associated to P and γ. Let $0_P \in \overline{A}_P$ denote the origin and let $e(P)_{[\gamma]_P} \subseteq X(P)_{[\gamma]_P}$ denote the image of $\{0_P\} \times N_P$ under the diffeomorphism $\overline{A}_P \times N_P \to X(P)_{[\gamma]_P}$. This is the *boundary component* associated to P and γ.

Remark 4.2.4. Note that the boundary component associated to P and γ is not necessarily the same as the boundary of the corner associated to P and γ. This is unfortunate, but defining them in this way eases the construction considerably.

Note also that for $P = G$, we have $X(G)_{[\mathrm{id}]_G} = e(G)_{[\mathrm{id}]_G} = X$.

PROPOSITION 4.2.5. For BUTs $P \leq Q$, $\gamma \in \Gamma$, we have an open embedding of manifolds with corners

$$X(Q)_{[\gamma]_Q} \longrightarrow X(P)_{[\gamma]_P}, \quad [\overline{a}, q]_{[\gamma]_Q} \mapsto [\overline{a}, q]_{[\gamma]_P}.$$

Proof. For R equal to P or Q, denote by $\kappa_R\colon 0 = l_{0,R} < l_{1,R} < \cdots < l_{k_R,R} = n$ the partition defining R (note that κ_P is finer that κ_Q) and denote by τ_R the Lie group isomorphism $\tau_R\colon A_R \to \mathbb{R}_{>0}^{k_R-1}$, $\mathrm{pr}_i(\tau(a)) = \frac{a_{l_{i,R}}}{a_{l_{i+1,R}}}$. The inclusion $A_Q \hookrightarrow A_P$ extends to an inclusion of $\overline{A}_Q \hookrightarrow \overline{A}_P$: Interpreting A_P and A_Q as $\mathbb{R}_{>0}^{k_P-1}$, respectively, $\mathbb{R}_{>0}^{k_Q-1}$ via the maps τ_P and τ_Q, $A_Q \hookrightarrow A_P$ is given by

$$\mathbb{R}_{>0}^{k_Q-1} \hookrightarrow \mathbb{R}_{>0}^{k_P-1}, \qquad (r_i)_{i=1}^{k_Q-1} \mapsto (s_i)_{i=1}^{k_P-1}$$

with

$$s_i = r_j, \quad \text{if } l_{i,P} = l_{j,Q} \quad \text{and} \quad s_i = 1 \quad \text{if } l_{i,P} \neq l_{j,Q} \text{ for all } j = 1,\ldots,k_Q-1.$$

Replacing strictly positive with weakly positive in the above gives us the inclusion $\overline{A}_Q \hookrightarrow \overline{A}_P$. Interpreting $\overline{A}_Q$ as a subset of $\overline{A}_P$, it is easy to see that if $\overline{a} \in \overline{A}_Q$, $a \in A_P$ satisfy that $\overline{a}a \in \overline{A}_Q$ (where the multiplication takes place in $\overline{A}_P$), then we must have $a \in A_Q$:

Indeed, let $(r_i) \in \mathbb{R}_{\geq 0}^{k_Q-1}$ and $(a_i) \in \mathbb{R}_{>0}^{k_P-1}$ and denote by $(s_i) \in \mathbb{R}_{geq0}^{k_P-1}$ the image of (r_i) under the above map. Then $(r_i)(a_i) = (s_i)(a_i) = (s_i a_i)$ and this element belongs to $\overline{A}_Q \subseteq \overline{A}_P$ if and only if $s_i a_i = 1$ for all i such that $l_{i,P} \neq l_{j,Q}$ for all $j = 1,\ldots,k_Q-1$. This in turn implies that $a_i = s_i a_i = 1$ for all such i since (s_i) belongs to $\overline{A}_Q \subseteq \overline{A}_P$.

It follows that $\sim_{\gamma,P}$ restricts to $\sim_{\gamma,Q}$ on $\overline{A}_Q \times X$.

Now, the inclusion $\overline{A}_Q \times X \hookrightarrow \overline{A}_P \times X$ induces a smooth map on the quotients as illustrated below, and it is given by $[\overline{a}, q]_{[\gamma]_Q} \mapsto [\overline{a}, q]_{[\gamma]_P}$. It is injective as $\sim_{\gamma,P}$ restricts to $\sim_{\gamma,Q}$ on $\overline{A}_Q \times X$ by the above:

$$
\begin{array}{ccc}
\overline{A}_Q \times X & \longrightarrow\!\!\!\!\!\rightarrow & X(Q)_{[\gamma]_Q} \\
\big\downarrow & & \big\downarrow \\
\overline{A}_P \times X & \longrightarrow\!\!\!\!\!\rightarrow & X(P)_{[\gamma]_P}
\end{array}
$$

$\square$

Remark 4.2.6. In view of the above observation, we will interpret $X(Q)_{[\gamma]_Q}$ as a subspace of $X(P)_{[\gamma]_P}$ for BUTs $P \leq Q$, $\gamma \in \Gamma$. We can also view $e(Q)_{[\gamma]_Q}$ as a subspace of $X(P)_{[\gamma]_P}$, which leads to the proposition below. Note that for a different choice of representative of $[\gamma]_Q$, we may get a different class in $(P \cap \Gamma)\backslash\Gamma$ and thus an inclusion into a different corner — this will be quite essential in the understanding of the construction of the Borel-Serre compactification (see Remark 4.2.9).

PROPOSITION 4.2.7. For any BUT $P \leq G$, $[\gamma]_P \in (\Gamma \cap P)\backslash P$, we have

$$X(P)_{[\gamma]_P} = \coprod_{P \leq Q \text{ BUT}} e(Q)_{[\gamma]_Q}.$$

Proof. For a given BUT Q containing P, denote by $\kappa_Q\colon 0 = l_{0,Q} < l_{1,Q} < \cdots < l_{k_Q,Q} = n$, the partition defining Q (note again that κ_P is finer than κ_Q). Recall the Lie group isomorphisms $\tau_Q\colon A_Q \to \mathbb{R}_{>0}^{k_Q-1}$, $\mathrm{pr}_i(\tau(a)) = \frac{a_{l_{i,Q}}}{a_{l_{i+1,Q}}}$. Define

$$A_{P,Q} := \{\, a \in A_P \mid a_{l_{j,Q}+1} \cdots a_{l_{j+1,Q}} = 1, \ j = 0,\ldots,k_Q-1 \,\}.$$

In other words, $A_{P,Q}$ consists of the elements in A_P (diagonal matrices with positive entries and determinant 1 such that the entries in each block defined by P are equal) such that each block defined by Q has determinant 1. Then the multiplication map

$$A_{P,Q} \times N_P \longrightarrow N_Q$$

is a diffeomorphism, and we have the following commutative diagram, where the second upper map is the identity on the first factor and multiplication of the second and third factor, and the third upper map is multiplication of the first and second factor and the identity on the third factor:

$$\{0_Q\} \times N_Q \lhook\joinrel\longrightarrow \overline{A}_Q \times N_Q \overset{\cong}{\longleftarrow} \overline{A}_Q \times A_{P,Q} \times N_P \longrightarrow \overline{A}_P \times N_P \overset{\cong}{\longrightarrow} X(P)_{[\gamma]_P}$$
$$\cong \Big\downarrow$$
$$X(Q)_{[\gamma]_Q}$$

Thus, to identify $e(Q)_{[\gamma]_Q}$ as a subset of $X(P)_{[\gamma]_P}$, we simply need to identify the image of the composition of the upper sequence of maps; in other words, we must identify the image of $\{0_Q\} \times A_{P,Q} \times N_P$ in $\overline{A}_P \times N_P$ under multiplication of the first two factors.
Recall that the inclusion $\overline{A}_Q \subseteq \overline{A}_P$ is given by

$$\mathbb{R}_{\geq 0}^{k_Q-1} \hookrightarrow \mathbb{R}_{\geq 0}^{k_P-1}, \qquad (r_i)_{i=1}^{k_Q-1} \mapsto (s_i)_{i=1}^{k_P-1}$$

with

$$s_i = r_j \quad \text{if } l_{i,P} = l_{j,Q} \quad \text{and} \quad s_i = 1, \quad \text{if } l_{i,P} \neq l_{j,Q} \text{ for all } j = 1, \ldots, k_Q - 1.$$

Note that $0_Q = (r_i)_{i=1}^{k_P-1} \in \overline{A}_P$ is given by $r_i = 0$ if $l_{i,P} = l_{j,Q}$ for some $j = 1, \ldots, k_Q - 1$, and $r_i = 1$ for all other i.
The inclusion $A_{P,Q} \hookrightarrow A_P \cong \mathbb{R}_{>0}^{k_P-1}$ is given by $a = (a_i) \mapsto (s_i)_{i=1}^{k_P-1}$ with

$$s_i = \begin{cases} \dfrac{a_{l_{i,P}}}{a_{l_{i+1,P}}} & \text{if } l_{i,P} \neq l_{j,Q} \text{ for all } j = 1, \ldots, k_Q - 1, \\ \varphi_i(a) & \text{if } l_{i,P} = l_{j,Q} \text{ for some } j = 1, \ldots, k_Q - 1. \end{cases}$$

for some functions $\varphi_i \colon \mathbb{R}_{>0}^{k_P-k_Q} \longrightarrow \mathbb{R}$: The conditions on the elements of $A_{P,Q}$ are such that the $l_{j,Q}$'th coordinates are completely determined by the rest, which may take any strictly positive value. Thus the image of the composition

$$\{0_Q\} \times A_{P,Q} \longrightarrow \overline{A}_P \longrightarrow \mathbb{R}_{\geq 0}^{k_P-1},$$

where the first map is multiplication and the second is τ_P, is $\prod_{i=1}^{k_P-1} R_i^Q$ with

$$R_i^Q = \begin{cases} \{0\} & l_{i,P} \neq l_{j,Q} \text{ for all } j = 1, \ldots, k_Q - 1 \\ \mathbb{R}_{>0} & l_{i,P} = l_{j,Q} \text{ for some } j = 1, \ldots, k_Q - 1 \end{cases}.$$

It is easy to see that

$$\mathbb{R}_{\geq 0}^{k_P-1} = \coprod_{P \leq Q} \prod_{i=1}^{k_P-1} R_i^Q,$$

and thus we conclude that

$$\overline{A}_P \times N_P = \coprod_{P \leq Q} \mathrm{Im}(\{0_Q\} \times A_{P,Q} \times N_P),$$

which finally gives the desired

$$X(P)_{[\gamma]_P} = \coprod_{P \leq Q} e(Q)_{[\gamma]_Q}.$$

$\square$

Construction 4.2.8 (Partial Compactification of X).
We can now define our space $\overline{X}$ as the disjoint union of the boundary components $e(P)_{[\gamma]_P}$:

$$\overline{X} := \coprod_{P, [\gamma]_P} e(P)_{[\gamma]_P},$$

where P runs over all BUTs and $[\gamma]_P$ runs through all the elements in $(\Gamma \cap P)\backslash\Gamma$.
In view of Proposition 4.2.7, we identify $X(P)_{[\gamma]_P}$ with $\coprod_{P \leq Q} e(Q)_{[\gamma]_Q} \subseteq \overline{X}$ for all BUTs P and $[\gamma]_P \in (P \cap \Gamma)\backslash\Gamma$.
Now, for any BUTs P, Q and $\gamma, \eta \in \Gamma$, we have

$$X(P)_{[\gamma]_P} \cap X(Q)_{[\eta]_Q} = \coprod_{\substack{P,Q \leq R \\ \gamma\eta^{-1} \in R}} e(R)_{[\gamma]_R} = X(R_0)_{[\gamma]_{R_0}},$$

where R_0 is the smallest BUT such that $\gamma\eta^{-1} \in R_0$ and $P, Q \leq R_0$.
The inclusions $X(Q)_{[\gamma]_Q} \hookrightarrow X(P)_{[\gamma]_P}$ are open embeddings of manifolds with corners, and for a BUT P with k blocks, we have diffeomorphisms

$$\mathbb{R}^{k-1}_{\geq 0} \times \mathbb{R}^{\frac{n(n+1)}{2} - k} \longrightarrow \overline{A}_P \times N_P \longrightarrow X(P)_{[\gamma]_P}.$$

Hence, the inclusions $X(P)_{[\gamma]_P} \hookrightarrow \overline{X}$ form an atlas on $\overline{X}$, defining a structure of a manifold with corners on $\overline{X}$ such that the corners $X(P)_{[\gamma]_P}$ are open submanifolds with corners of $\overline{X}$ (note that we do not here require a manifold with corners to be Hausdorff — we show below that $\overline{X}$ is in fact Hausdorff and thus $\overline{X}$ is a manifold with corners in the usual sense).
Note that the interior of $\overline{X}$ is equal to $e(G)_{[\mathrm{id}]_G} = X$. It follows that the inclusion $X \hookrightarrow \overline{X}$ is a homotopy equivalence as a topological manifold with boundary is homotopy equivalent to its interior (cf. [22, p. 297]).

Remark 4.2.9. What we have done in the above construction is to add the boundary to all A_P-orbits and their Γ-translates in X. Picturing the A_P-orbit or its translate as $\mathbb{R}^{k-1}_{>0}$, we simply add all points with at least one coordinate equal to zero, obtaining $\mathbb{R}^{k-1}_{\geq 0}$.
Note that we can interpret $\overline{X}$ as the union of the corners $X(P)_{[\gamma]_P}$ under the condition that we glue these corners together along the inclusions of subcorners $X(Q)_{[\gamma]_Q} \subseteq X(P)_{[\gamma]_P}, P \leq Q$. This interpretation is nice to have in mind when trying to visualise the construction. It does not unfortunately come into play in our running example $n = 2$ as we have just two BUTs, namely B and G, and here we easily see the enormous difference in complexity between the cases $n = 2$ and $n = 3$. We go through the construction in the case $n = 2$ below and also try to give an idea of what happens in the case $n = 3$ to get a better grasp of this glueing interpretation.

Construction 4.2.10 (Extending the action of Γ).
For a BUT P, set $X(P) := \bigcup_{[\gamma]_P} X(P)_{[\gamma]_P} \subseteq \overline{X}$, where $[\gamma]_P$ runs through all elements in $(P \cap \Gamma) \backslash \Gamma$ and define a right action of Γ on $X(P)$ by

$$[(\overline{a}, q)]_{[\eta]_P}.\gamma = [(\overline{a}, q.\gamma)]_{[\eta\gamma]_P}, \qquad \text{for any } [(\overline{a}, q)]_{[\eta]_P} \in X(P)_{[\eta]_P}, \ \gamma \in \Gamma.$$

This action is well-defined, as

$$(\overline{a}.a, (q.\eta).\gamma) = (\overline{a}.a, q.(\eta\gamma)) \sim_{\eta\gamma, P} (\overline{a}, (a.q).(\eta\gamma)) = (\overline{a}.a, ((a.q).\eta).\gamma),$$

for all $\overline{a} \in \overline{A}_P$, $a \in A_P$, $q \in X$, $\eta, \gamma \in \Gamma$. In addition, it extends the action of Γ on $X \subseteq X(P)$: $[\mathrm{id}, q]_{[\gamma]_P}.\eta = [\mathrm{id}, q.\eta]_{[\gamma\eta]_P}$.
For BUTs $P \leq Q$, the inclusion $X(Q) \hookrightarrow X(P)$ given by $[\overline{a}, q]_{[\gamma]_Q} \mapsto [\overline{a}, q]_{[\gamma]_P}$ for all $\overline{a} \in \overline{A}_Q$, $q \in X$, $\gamma \in \Gamma$, is equivariant. Hence, as the sets $X(P)$ form an open cover of $\overline{X}$, this defines a right action of Γ on $\overline{X}$, which extends the action of Γ on X.

Remark 4.2.11. Γ acts on the boundary components of $\overline{X}$ as follows: $\gamma \in \Gamma$ maps the boundary component $e(P)_{[\eta]_P}$ to the boundary component $e(P)_{[\eta\gamma]_P}$; in particular, if $\gamma \in P \cap \Gamma$, then γ simply translates the elements of a boundary component along the component itself.

PROPOSITION 4.2.12. For $\gamma \in \Gamma$, the map $\gamma \colon \overline{X} \to \overline{X}$, $x \mapsto x.\gamma$, is smooth.

Proof. As $X(P) \subseteq \overline{X}$ is an open submanifold, it suffices to prove that the restriction of γ to $X(P)$ is smooth for all BUTs P. Consider the commutative diagram

$$
\begin{array}{ccc}
X(P) & \xrightarrow{\ \gamma\ } & X(P) \\
\cong\downarrow & & \downarrow\cong \\
\overline{A}_P \times N_P & \xrightarrow{\hat{\gamma}} & \overline{A}_P \times N_P
\end{array}
\qquad\qquad
\begin{array}{ccc}
[\overline{a}, q_u]_{[\eta]_P} & \xmapsto{\ \gamma\ } & [\overline{a}, q_u.\gamma]_{[\eta\gamma]_P} \\
\uparrow & & \uparrow \\
(\overline{a}, u) & \xmapsto{\hat{\gamma}} & (\overline{a}a, v)
\end{array}
$$

where $u\gamma = kav$ is the Langlands decomposition, so $q_u.\gamma$ is given by $(av)^t(av)$. Then $\hat{\gamma}$ is smooth as it is equal to the composition

$$\overline{A}_P \times N_P \xrightarrow{\mathrm{id}\times\gamma} \overline{A}_P \times G \xrightarrow{\mathrm{id}\times\mathrm{Langlands}} \overline{A}_P \times K \times A_P \times N_P \xrightarrow{m\times\mathrm{id}} \overline{A}_P \times N_P$$

of smooth maps, where γ here denotes multiplication by γ restricted to N_P, and m is the map which multiplies the first and third factor while forgetting the second, and we have extended the Langlands decomposition to G. That the multiplication map $K \times A_P \times N_P \to G$ is a diffeomorphism is a consequence of the Iwasawa decomposition. We conclude that γ is smooth. $\qquad\square$

4.3 OBSERVATIONS AND PROPERTIES

In this section, we prove that $\overline{X}$ satisfies the properties that we are after: It is Hausdorff, so an actual manifold with corners, the action of Γ on $\overline{X}$ is properly discontinuous and the quotient $\overline{X}/\Gamma$ is compact.

We will consider sequences $(q_m)_{m\in\mathbb{N}}$ of quadratic forms. This should not be confused with our notation q_g for the quadratic form given by the matrix $g^t g$, $g \in G$ — it should be clear from the subscripts which it is.

LEMMA 4.3.1. Let P be a BUT, $\gamma, \eta \in \Gamma$ and let $(q_m)_{m \in \mathbb{N}}$ be a sequence in X converging to a point $x \in e(P)_{[\gamma]_P}$. If $(q_m.\eta)_{m \in \mathbb{N}}$ converges in $X(B)_{[\gamma]_B}$, then $\gamma\eta\gamma^{-1} \in P$.

Proof. Let P be given by the partition $0 = l_0 < l_1 < \cdots < l_k = n$ and assume that $(q_m.\eta)_{m \in \mathbb{N}}$ converges in $X(B)_{[\gamma]_B}$.
Assume first that $\gamma = \mathrm{id}$. As $\{q_m\}_{m \in \mathbb{N}}, \{q_m.\eta\}_{m \in \mathbb{N}}$ are relatively compact in $X(B)_{[\mathrm{id}]_B}$, we have $\{q_m\}_{m \in \mathbb{N}}, \{q_m.\eta\}_{m \in \mathbb{N}} \subseteq \mathfrak{S}_{\lambda,\delta}$ for some $\lambda, \delta > 0$. For every $m \in \mathbb{N}$, let $a_m \in A_\lambda$, $u_m \in N_\delta$ such that q_m is given by $(a_m u_m)^t(a_m u_m)$.
Now, write $x = [0_P, q_{au}]_{[\mathrm{id}]_P} \in e(P)_{[\mathrm{id}]_P}$ for $u \in N$, $a \in A$, using the Iwasawa decomposition restricted to N_P. Then in $X(B)_{[\mathrm{id}]_B}$ we have

$$x = [0_P, q_{au}]_{[\mathrm{id}]_P} = [0_P, q_{au}]_{[\mathrm{id}]_B} = [0_P a, q_u]_{[\mathrm{id}]_B}.$$

Note that $0_P \in \overline{A}_P \subseteq \overline{A} = \mathbb{R}_{\geq 0}^{n-1}$ is the element with $(0_P)_{l_i} = 0$ for all $i = 1, \ldots, k-1$ and all other coordinates equal to 1. Then, since $(a_m, u_m) \to (0_p a, u)$ in $\overline{A} \times N$ as $m \to \infty$, we have

$$a_m u_m, \ a_m u_m \eta \in K A_\lambda N_\delta, \qquad \text{with} \qquad \frac{(a_m)_{l_i}}{(a_m)_{l_i+1}} \to 0 \quad \text{for all } i = 1, \ldots, k-1,$$

and Proposition 3.3.2 yields $\eta \in P$, as desired.
For $\gamma \neq \mathrm{id}$, consider the sequence $(q_m.\gamma^{-1})_{m \in \mathbb{N}} \subseteq X(B)_{[\mathrm{id}]_B}$ converging to $x.\gamma^{-1} \in X(B)_{[\gamma]_B}$. Then the sequence $((q_m.\gamma^{-1}).(\gamma\eta\gamma^{-1}))_{m \in \mathbb{N}} = ((q_m.\eta).\gamma^{-1})_{m \in \mathbb{N}}$ converges in $X(B)_{[\mathrm{id}]_B}$, and by the above $\gamma\eta\gamma^{-1} \in P$ as desired. $\qquad\square$

PROPOSITION 4.3.2. The space $\overline{X}$ is Hausdorff.

Proof. Let $y, y' \in \overline{X}$ and assume that $V_m, V'_m \subseteq \overline{X}$, $m \in \mathbb{N}$, are open neighbourhoods of y, respectively, y' such that $V_m \cap V'_m \neq \emptyset$ for all $m \in \mathbb{N}$ and the sequences $(V_m)_{m \in \mathbb{N}}, (V'_m)_{m \in \mathbb{N}}$ are strictly decreasing. We will show that $y = y'$.
If $y \in X$, then there is a relatively compact neighbourhood $V \subseteq X$ of y. This is bounded, hence $\overline{V} \cap \partial X = \emptyset$. It follows that $y' \in X$, and hence, $y = y'$ as X is Hausdorff.
We may therefore assume that $y, y' \in \partial X$. We will show that y and y' belong to the same corner. Let P, P' be BUTs and $\gamma, \gamma' \in \Gamma$ such that $y \in e(P)_{[\gamma]_P}$, $y' \in e(P')_{[\gamma']_{P'}}$. As $V_m \cap V'_m$ is non-empty and open for all $m \in \mathbb{N}$ and X is an open dense subspace of $\overline{X}$, we have $V_m \cap V'_m \cap X \neq \emptyset$ for all $m \in \mathbb{N}$. Let for all $m \in \mathbb{N}$, $x_m \in V_m \cap V'_m \cap X$. For $\eta = \gamma'^{-1}\gamma$, we have

$$X(P')_{[\gamma']_{P'}}.\eta = X(P')_{[\gamma]_{P'}} \subseteq X(B)_{[\gamma]_B}.$$

Therefore, as the action of Γ on $\overline{X}$ is continuous, the sequence $(x_m)_{m \in \mathbb{N}} \subseteq X$ satisfies

$$x_m \longrightarrow y \in e(P)_{[\gamma]_P} \subseteq X(B)_{[\gamma]_B}, \qquad x_m.\eta \longrightarrow y'.\eta \in e(P')_{[\gamma]_{P'}} \subseteq X(B)_{[\gamma]_B} \quad \text{as } m \to \infty.$$

Then by Lemma 4.3.1, $\gamma\gamma'^{-1} = \gamma\eta\gamma^{-1} \in P$, implying $[\gamma]_P = [\gamma']_P$, and thus

$$e(P)_{[\gamma]_P} = e(P)_{[\gamma']_P} \subseteq X(B)_{[\gamma']_B}.$$

But then $y, y' \in X(B)_{[\gamma']_B}$, and we have $y = y'$ as $X(B)_{[\gamma']_B} \cong \overline{A} \times N$ is Hausdorff. $\qquad\square$

The following observation is immediate from the composition of diffeomorphisms

$$X(B)_{[\mathrm{id}]_B} \longrightarrow \overline{A} \times N \longrightarrow \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}^{\frac{n(n-1)}{2}},$$

which maps the Siegel set $\mathfrak{S}_{\lambda,\delta}$ to $(0, \lambda]^{n-1} \times [-\delta, \delta]^{\frac{n(n-1)}{2}}$:

LEMMA 4.3.3. For any $\lambda, \delta > 0$, the closure of the Siegel set $\mathfrak{S}_{\lambda,\delta} \subseteq X$ in $\overline{X}$ is equal to

$$\overline{\mathfrak{S}}_{\lambda,\delta} = \{[(\overline{a}, q_u)]_{[\mathrm{id}]_B} \in X(B)_{[\mathrm{id}]_B} \mid \overline{a} \in [0,\lambda]^{n-1} \subseteq \overline{A}, \ u \in N_\delta\}.$$

DEFINITION 4.3.4. The closure in $\overline{X}$ of a Siegel set in X is a *Siegel set* in $\overline{X}$.

LEMMA 4.3.5. Let $\mathfrak{S} = \mathfrak{S}_{\lambda,\delta} = KA_\lambda N_\delta \subseteq G$, $\lambda, \delta > 0$, be a Siegel set in G, and let $M \subseteq M_n(\mathbb{Z})$ be a set of invertible $(n \times n)$-matrices with integer coefficients such that $|\det m| \leq c$ for all $m \in M$ and some constant $c > 0$. Then the set $\{m \in M \mid \mathfrak{S}.m \cap \mathfrak{S} \neq \emptyset\}$ is finite.

Proof. See [5, Theorem 4.6]. $\qquad\square$

PROPOSITION 4.3.6. The action of Γ on $\overline{X}$ is properly discontinuous.

Proof. We have to prove that for any two compacts $K_1, K_2 \subseteq \overline{X}$, the set $\{\gamma \in \Gamma \mid K_1 \cap K_2.\gamma \neq \emptyset\}$ is finite. Since $\overline{X}$ is locally compact, we may for any $K_1, K_2 \subseteq \overline{X}$ take compact neighbourhoods $C_1, C_2 \subseteq \overline{X}$ of K_1 respectively K_2. As X is dense in $\overline{X}$, $C_1 \cap C_2.\gamma \neq \emptyset$ will imply $C_1 \cap C_2.\gamma \cap X \neq \emptyset$. Hence,

$$\{\gamma \in \Gamma \mid K_1 \cap K_2.\gamma \neq \emptyset\} \subseteq \{\gamma \in \Gamma \mid C_1 \cap C_2.\gamma \cap X \neq \emptyset\},$$

so it suffices to show that for any two compacts $K_1, K_2 \subseteq \overline{X}$, the set $\{\gamma \in \Gamma \mid K_1 \cap K_2.\gamma \cap X \neq \emptyset\}$ is finite.

Let $K \subseteq \overline{X}$ be a compact subset. The corners $X(B)_{[\gamma]_B}$, $[\gamma]_B \in (\Gamma \cap B)\backslash\Gamma$, form an open cover of $\overline{X}$, so K is covered by finitely many of them, say the ones associated to $\gamma_1, \ldots, \gamma_m \in \Gamma$. Being bounded, any compact subset of $X(B)_{[\gamma]_B}$ is contained in $\overline{\mathfrak{S}}_{\lambda,\delta}.\gamma$ for some $\lambda, \delta > 0$. It follows that there exist $\lambda, \delta > 0$ such that $K \subseteq \bigcup_{i=1}^m \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma_i$. For a second compact subset $K' \subseteq \overline{X}$, let $\gamma_1', \ldots, \gamma_{m'}' \in \Gamma$ and $\lambda', \delta' > 0$ such that $K \subseteq \bigcup_{i=1}^{m'} \overline{\mathfrak{S}}_{\lambda',\delta'}.\gamma_i'$. We may assume $\lambda = \lambda'$ and $\delta = \delta'$ and we see that

$$\{\gamma \in \Gamma \mid K \cap K'.\gamma \cap X \neq \emptyset\} \subseteq \{\gamma \in \Gamma \mid \bigcup_{i=1}^m \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma_i \cap \bigcup_{i=1}^{m'} \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma_i'\gamma \cap X \neq \emptyset\}$$

$$= \bigcup_{i,j}\{\gamma \in \Gamma \mid \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma_i \cap \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma_j'\gamma \cap X \neq \emptyset\},$$

and for any $\eta, \zeta \in \Gamma$,

$$\{\gamma \in \Gamma \mid \overline{\mathfrak{S}}_{\lambda,\delta}.\eta \cap \overline{\mathfrak{S}}_{\lambda,\delta}.\zeta\gamma \cap X \neq \emptyset\} = \{\gamma \in \Gamma \mid \overline{\mathfrak{S}}_{\lambda,\delta} \cap \overline{\mathfrak{S}}_{\lambda,\delta}.\zeta\gamma\eta^{-1} \cap X \neq \emptyset\}$$

$$= \{\gamma \in \Gamma \mid \overline{\mathfrak{S}}_{\lambda,\delta} \cap \overline{\mathfrak{S}}_{\lambda,\delta}.\gamma \cap X \neq \emptyset\}$$

$$= \{\gamma \in \Gamma \mid \mathfrak{S}_{\lambda,\delta} \cap \mathfrak{S}_{\lambda,\delta}.\gamma \neq \emptyset\}$$

$$= \{\gamma \in \Gamma \mid KA_\lambda N_\delta \cap KA_\lambda N_\delta\gamma \neq \emptyset\}.$$

It is enough to prove that this latter set is finite; this is a consequence of Lemma 4.3.5 above. $\quad\square$

PROPOSITION 4.3.7. $\overline{\mathfrak{S}}_{\lambda,\delta}.\Gamma = \overline{X}$ for any $\lambda \geq {}^4\!/\!{}_3$, $\delta \geq {}^1\!/\!{}_2$.

Proof. As $\overline{\mathfrak{S}} := \overline{\mathfrak{S}}_{\lambda,\delta}$ is compact in $\overline{X}$, the family $\{\overline{\mathfrak{S}}.\gamma\}_{\gamma \in \Gamma}$ is locally finite: Indeed, for any $x \in \overline{X}$, we may take a relatively compact neighbourhood U of x; then $U \cap \overline{\mathfrak{S}}.\gamma \neq \emptyset$ for only finitely many γ as Γ acts properly discontinuously on $\overline{X}$.
Being locally finite, the union $\bigcup_{\gamma \in \Gamma} \overline{\mathfrak{S}}.\gamma = \overline{\mathfrak{S}}.\Gamma$ is closed in $\overline{X}$. From Theorem 3.1.7, we know that $X \subseteq \overline{\mathfrak{S}}.\Gamma$, and as X is dense in $\overline{X}$, we have $\overline{\mathfrak{S}}.\Gamma = \overline{X}$ as desired. $\quad\square$

As an immediate consequence, we see that this construction does indeed satisfy what we set out to fix:

COROLLARY 4.3.8. *The space $\overline{X}/\Gamma$ is compact.*

Remark 4.3.9. Recall that Γ contains a normal subgroup of finite index which is torsion free. For such a subgroup, Γ', there is a finite set $C \subseteq \Gamma$ such that $\Gamma = C\,\Gamma'$. The inherited action of Γ' on $\overline{X}$ is properly discontinuous and free, and so the space $\overline{X}/\Gamma'$ inherits the structure of a manifold with corners from $\overline{X}$. Moreover, $(\overline{\mathfrak{S}}_{\lambda,\delta}.C).\Gamma' = \overline{X}$ for any $\lambda \geq {}^4/_3$, $\delta \geq {}^1/_2$. Hence, $\overline{X}/\Gamma'$ is compact, being the image of a finite union of compact sets. The inclusion $X/\Gamma' \hookrightarrow \overline{X}/\Gamma'$ is a homotopy equivalence (cf. [22, p. 297]).

4.4   THE CASE $n = 2$ PART III AND THE CASE $n = 3$

We will take a closer look at the cases $n = 2$ and $n = 3$ to give ourselves a better understanding of the construction of the Borel-Serre compactification. The formal construction above is very technical, but the geometry behind it is actually not so bad (if we ignore the fact that the dimensions very quickly get out of hand). In the case $n = 2$, we can go through the complete construction and are able to visualise it as X and $\overline{X}$ are two-dimensional. The case $n = 2$ is, however, almost too simple: It all becomes a lot more complicated for $n > 2$ and not just because the dimension of X exceeds our abilities of perception. Essentially, this is because there is only one proper subgroup of block upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$, namely the subgroup B of upper triangular matrices, therefore there is no glueing to be done (cf. Remark 4.2.9). Therefore we also try to give a picture of the case $n = 3$, where we have three proper subgroups of block upper triangular matrices, so we see how this glueing comes into play.

THE CASE $n = 2$

We identify the geodesic action on X on the model $\mathscr{H}$ and show what happens when we add the boundary to the orbits under this action. We will use both the upper half plane model and the Poincaré disk model as both have their advantages in visualising the construction.

Recall from Section 2.2 that we can identify X with the hyperbolic plane, $\mathscr{H}$, under the map

$$X \longrightarrow \mathscr{H}, \qquad q \mapsto z_q = {}^1/_{s_{11}}\,(s_{12} + i),$$

where $q \in X$ is given by the positive definite matrix $s = (s_{ij})$.
Simple calculations show that the geodesic action of A on $\mathscr{H}$ is given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & {}^1/_\lambda \end{pmatrix}.z = \mathrm{Re}(z) + {}^1/_{\lambda^2}\,\mathrm{Im}(z)\,i, \qquad \lambda > 0,\ z \in \mathscr{H}.$$

The following proposition is immediate:

PROPOSITION 4.4.1. *The orbits of A are vertical lines in the upper half plane, $\mathscr{H}$. Under the diffeomorphism $A \to \mathbb{R}_{>0}$, $(a_i) \mapsto \frac{a_1}{a_2} = a_1^2$, the orientation of the orbits is such that 0 sits at infinity.*

Adding zero to each A-orbit then corresponds to "adding a real line at infinity", i.e. the corner $X(B)_{[\mathrm{id}]_B}$ can be interpreted as the set $\mathbb{R} \times (0, \infty]$, and the boundary component $e(B)_{[\mathrm{id}]_B}$ as the line $\{(x, \infty) \in \mathbb{R} \times \mathbb{R} \times (0, \infty]\}$. We then have:

COROLLARY 4.4.2. The corner $X(B)_{[\mathrm{id}]_B}$ is diffeomorphic to a strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) \leq a\}$ for any choice of $a > 0$, and the boundary component $e(B)_{[\mathrm{id}]_B}$ corresponds to the boundary of this corner, i.e. the line $\{z \in \mathbb{C} \mid \mathrm{Im}(z) = a\}$.

Remark 4.4.3. Below, in Figures 4.1 and 4.2, we draw the corner $X(B)_{[\mathrm{id}]_B}$ in both the upper half plane model and the Poincaré disk model, where we interpret the corner as a strip as in the above corollary; we also include some A-orbits.
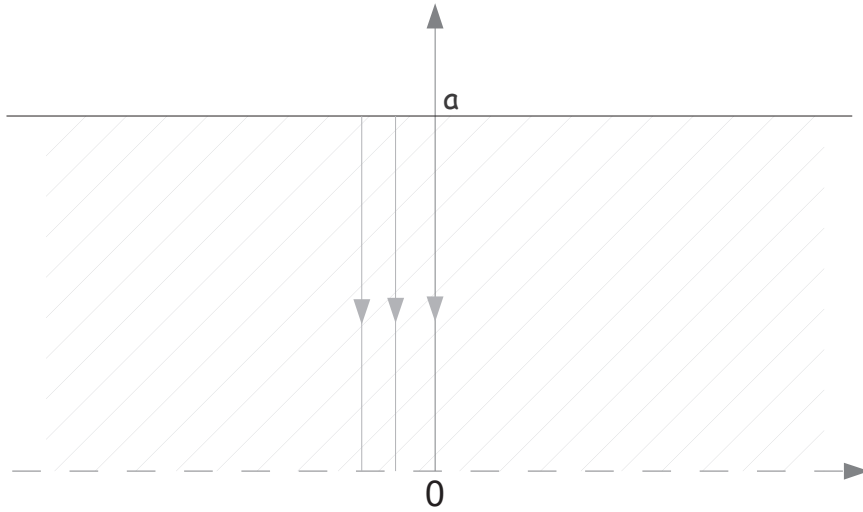

Figure 4.1: The corner $X(B)_{[\mathrm{id}]_B}$ for $n = 2$ in the upper half plane. The hatched area is the corner $X(B)_{[\mathrm{id}]_B}$ interpreted as the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) \leq a\}$; the boundary component $e(B)_{[\mathrm{id}]_B}$ is then the line $\{z \in \mathbb{C} \mid \mathrm{Im}(z) = a\}$. We have drawn some of the A-orbits, the arrows denoting the orientation under the diffeomorphism $A \to \mathbb{R}_{>0}$, $(a_i) \mapsto \frac{a_1}{a_2}$.

PROPOSITION 4.4.4. Let $\gamma \in \Gamma$. The γ-translates of the A-orbits are parallel lines in $\mathscr{H} \subseteq \mathbb{C}$, and the orientation is such that all the orbits have zero located at infinity or at the same rational point on the real axis.

Proof. As the metric is Γ-invariant and the A-orbits are parallel, clearly the γ-translates of the orbits are parallel. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \lambda.z := \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}.z \ \text{ for } \lambda > 0,\ z \in \mathbb{C}.$$

Then for any $z \in \mathbb{C}$, $\lambda > 0$, we have

$$\begin{aligned}(\lambda.z).\gamma &= \frac{d(\lambda.z) + b}{c(\lambda.z) + a} = \frac{(d(\lambda.z) + b)(c(\overline{\lambda.z}) + a)}{(c(\lambda.z) + a)(c(\overline{\lambda.z}) + a)} \\ &= \frac{cd\,\mathrm{Re}(z)^2 + (ad + bc)\,\mathrm{Re}(z) + cd\lambda^{-4}\mathrm{Im}(z)^2 + ab + \lambda^{-2}\mathrm{Im}(z)i}{c^2\,\mathrm{Re}(z)^2 + c^2\lambda^{-4}\mathrm{Im}(z)^2 + 2ac\,\mathrm{Re}(z) + a^2}.\end{aligned}$$

We see that if $c \neq 0$, then

$$\mathrm{Im}((\lambda.z).\gamma) \longrightarrow 0 \quad \text{and} \quad \mathrm{Re}((\lambda.z).\gamma) \longrightarrow d/c \qquad \text{as } \lambda \to 0.$$
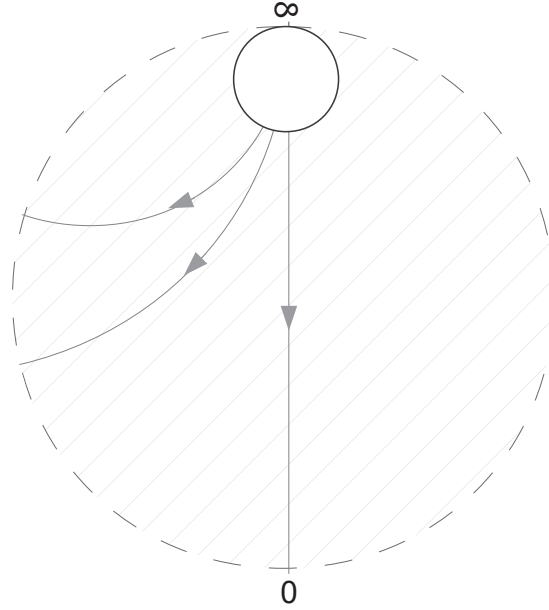
Figure 4.2: The corner $X(B)_{[\mathrm{id}]_B}$ for $n = 2$ in the Poincaré disk. The hatched area is the corner $X(B)_{[\mathrm{id}]_B}$ interpreted as the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) \leq a\} \subseteq \mathscr{H}$ mapped into the disk model; the boundary component $e(B)_{[\mathrm{id}]_B}$ is the circumference of the smaller circle minus the point on the boundary of the disk. The lines are A-orbits, where the arrows denote the orientation under the diffeomorphism $A \to \mathbb{R}_{>0}$, $(a_i) \mapsto \frac{a_1}{a_2}$.

If $c = 0$, we simply get

$$(\lambda.z).\gamma = \frac{1}{a^2}(ad\,\mathrm{Re}(z) + ab + \lambda^{-2}\,\mathrm{Im}(z)i),$$

and thus

$$\mathrm{Im}((\lambda.z).\gamma) \longrightarrow \infty \quad \text{as } \lambda \to 0 \quad \text{and} \quad \mathrm{Re}((\lambda.z).\gamma) = {}^1\!/\!a(d\,\mathrm{Re}(z) + b) \quad \text{for all } \lambda > 0.$$

$\square$

Remark 4.4.5. We view the corner $X(B)_{[\gamma]_B}$ as a strip in $\mathscr{H}.\gamma$ as in Figure 4.1. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the point $\frac{d}{c}$ takes the place of infinity, so this amounts to cutting out an open disk in $\mathscr{H}$ which touches the real line at $\frac{d}{c}$. Doing this for all $[\gamma]_B \in (\Gamma \cap B)\backslash\Gamma$, we get the upper half plane with a disk cut out at every rational point on the real axis and also at infinity. See Figures 4.3 and 4.4 for illustrations of $\overline{X}$ in the upper half plane and Poincaré disk models.

Remark 4.4.6. The element $\gamma \in \Gamma$ acts on $\overline{X}$ as follows:

- On X, γ acts as it did before.

- Suppose $\eta \in \Gamma$ is such that $[\eta]_B = [\gamma]_B$. Viewing $e(B)_{[\eta]_B}$ as a copy of the real line (see Figure 4.3 and Figure 4.4), γ acts by translations.

- Suppose $\eta \in \Gamma$ is such that $[\eta]_B \neq [\gamma]_B$. Then γ maps $e(B)_{[\eta]_B}$ to $e(B)_{[\eta\gamma]_B}$, possibly with some translation as well.

Figure 4.3: The Borel-Serre compactification of X for $n = 2$ in the half plane model. The hatched area is diffeomorphic to $\overline{X}$, which is the upper half plane $\mathscr{H}$ with an open disk removed at every rational point on the real axis and at infinity. The different sizes of the disks are simply for the sake of fitting in more of them.
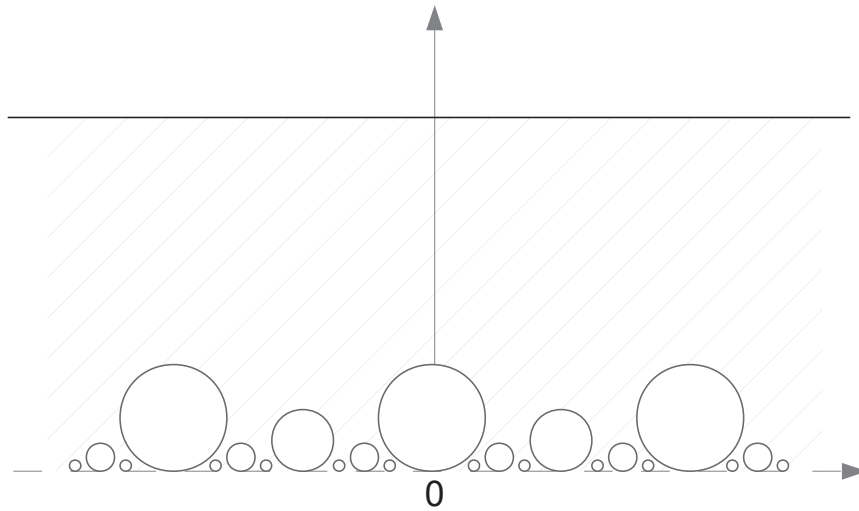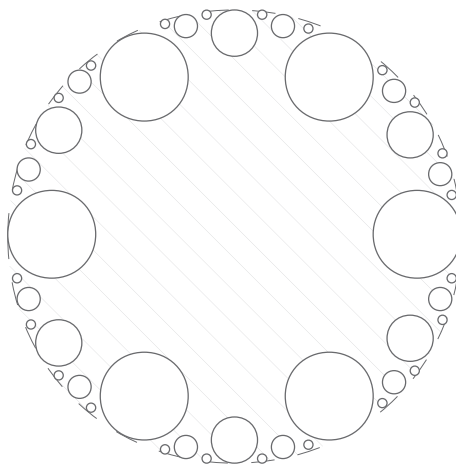

Figure 4.4: The Borel-Serre compactification of X for $n = 2$ in the Poincaré disk model. The hatched area is diffeomorphic to $\overline{X}$, which is the Poincaré disk with a small open disk removed at every rational point on the boundary. The different sizes of the disks are again simply for the sake of fitting in more of them.

THE CASE $n = 3$

We go through the ideas of the Borel-Serre compactification for $n = 3$ to get a better grasp of the glueing involved in the construction.

In $\mathrm{SL}_3(\mathbb{R})$, there are three proper subgroups of block upper triangular matrices

$$B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \qquad P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Since $B = P \cap Q$, we have to glue the different corners associated to B together along the corners associated to P and Q. In $\mathrm{SL}_3(\mathbb{R})$, $A = A_B$ is diffeomorphic to $\mathbb{R}_{>0}^2$, and both A_P and A_Q are diffeomorphic to $\mathbb{R}_{>0}$. For $\gamma \in \Gamma$, consider the corner associated to B and γ:

$$X(B)_{[\gamma]_B} \cong \overline{A} \times N = \mathbb{R}_{\geq 0}^2 \times N.$$

Its boundary looks like the product of the boundary of the upper quadrant in $\mathbb{R}^2$ and N. The boundary component associated to B and γ, $e(B)_{[\gamma]_B}$, is the preimage of $\{0\} \times N$ under the above composition of diffeomorphisms.

For $\gamma \in \Gamma$, consider now the corner associated to P and γ:

$$X(P)_{[\gamma]_P} \cong \overline{A}_P \times N_P = \mathbb{R}_{\geq 0} \times N_P.$$

Under the inclusion $X(P)_{[\gamma]_P} \subseteq X(B)_{[\gamma]_B}$, the corner corresponds to $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times N \subseteq \mathbb{R}_{\geq 0}^2 \times N$ under the above diffeomorphisms. This is better seen from the commutative diagram below, where $u \in N_P$ is decomposed as $u = bv$ for $b \in A$, $v \in N$; note that $N_P = (A \cap N_P)N$, so $b \in A \cap N_P$ is of the form $b = (b, 1/b, 1)$, $b \in \mathbb{R}_{>0}$. Recall that the inclusion $\mathbb{R}_{\geq 0} = \overline{A}_P \hookrightarrow \overline{A} = \mathbb{R}_{\geq 0}^2$ is given by $\overline{a} \mapsto (1, \overline{a})$.

$$
\begin{array}{ccc}
\overline{A}_P \times N_P \xrightarrow{\;\cong\;} X(P)_{[\gamma]_P} & \qquad & (\overline{a}, u) \longmapsto [\overline{a}, q_u.\gamma]_{[\gamma]_P} \\
\Big\downarrow \qquad\qquad \Big\uparrow & & \Big\uparrow \qquad\qquad\qquad \Big\uparrow \\
\overline{A} \times N \xrightarrow[\;\cong\;]{} X(B)_{[\gamma]_B} & & (\overline{a}b, v) \longmapsto [\overline{a}, q_u.\gamma]_{[\gamma]_B} = [\overline{a}b, q_v.\gamma]_{[\gamma]_B}
\end{array}
$$

From this diagram, it is also easy to see that the boundary component associated to P and γ, $e(P)_{[\gamma]_P}$, i.e. the image of $\{0\} \times N_P$ under the upper horizontal diffeomorphism, corresponds to $\mathbb{R}_{>0} \times \{0\} \times N \subseteq \mathbb{R}_{\geq 0}^2 \times N$. Similarly, the corner associated to Q and γ, $X(Q)_{[\gamma]_Q}$, corresponds $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \times N \subseteq \mathbb{R}_{\geq 0}^2 \times N$ in $X(B)_{[\gamma]_B}$, and the boundary component, $e(Q)_{[\gamma]_Q}$, corresponds to $\{0\} \times \mathbb{R}_{>0} \times N$.

For a given $\gamma \in \Gamma$, there exists $\eta \in \Gamma$ such that $[\gamma]_P = [\eta]_P$, but $[\gamma]_B \neq [\eta]_B$. Then we have to identify the corners $X(B)_{[\gamma]_B}$ and $X(B)_{[\eta]_B}$ along the corner $X(P)_{[\gamma]_P}$. Consider the commutative diagram below, where the middle isomorphism is the one defined using γ: $\overline{A}_P \times N_P \to X(P)_{[\gamma]_P}$, $(\overline{a}, u) \mapsto [\overline{a}, q_u.\gamma]_{[\gamma]_P}$. The upper right inclusion is given by the equality $X(P)_{[\gamma]_P} = X(P)_{[\eta]_P}$.

$$
\begin{array}{ccc}
X(B)_{[\gamma]_B} \longleftarrow X(P)_{[\gamma]_P} \longleftarrow X(B)_{[\eta]_B} \\
\cong \Big\downarrow \qquad\qquad \cong \Big\downarrow \qquad\qquad \Big\downarrow \cong \\
\mathbb{R}_{\geq 0}^2 \times N \longleftarrow \mathbb{R}_{\geq 0} \times N_P \longrightarrow \mathbb{R}_{\geq 0}^2 \times N
\end{array}
$$

The lower row induces a map $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times N \longrightarrow \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times N$. It can be shown that this map reverses the orientation of the first factor. So glueing $X(B)_{[\gamma]_B}$ and $X(B)_{[\gamma]_B}$ together along $X(P)_{[\gamma]_P} = X(P)_{[\eta]_P}$ corresponds to glueing two copies of $\mathbb{R}^2_{\geq 0} \times N$ together along $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times N$ with the orientation of the first factor reversed in one of the copies, see Figure 4.5. There is also some translation going on in the other factors, but we will not go into that.

Glueing along a corner associated to Q is similar. Now, we have to glue all the corners together along their "intersections", i.e. along the corners associated to P and Q; this amounts to glueing lots of copies of $\mathbb{R}^2_{\geq 0}$ together as above, yielding a kind of "infinite polygon", see Figure 4.6.
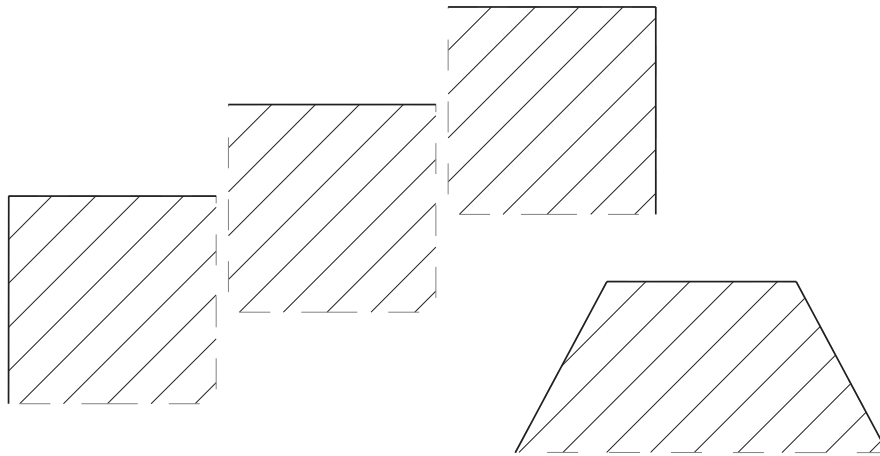

Figure 4.5:   Glueing the corners associated to B together for $n = 3$. A cross section of the corner $X(B)_{[\gamma]_B} \cong \mathbb{R}^2_{\geq 0} \times N$ at some $u \in N$ looks like $\mathbb{R}^2_{\geq 0}$, similarly a cross section of $X(P)_{[\gamma]_P}$ at u looks like $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$. In the cross section (which may be twisted in some way), glueing two corners associated to B together along a corner associated to P looks like identifying the upper and lower square above along the middle square included into the upper, respectively, lower square in the way they are drawn. In the cross section, the resulting space then looks like the shape on the right.
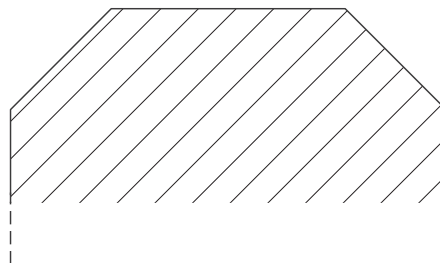
Figure 4.6:   Glueing several corners associated to B together for $n = 3$. Glueing several corners associated to B together along the corners associated to P and Q amounts to, when viewing the cross section, glueing several copies of $\mathbb{R}^2_{\geq 0}$ together along $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ where the orientation of the first, respectively, second factor are reversed in one of the copies. This results in a kind of "infinite polygon".

5 | Logarithmic Forms

We want to exploit the geometric setting, so we are going to work with the de Rham complex of the manifold X/Γ for an appropriate torsion free subgroup $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$. The complex is, however, too wild to consider all at once. What we would like is to be able control the growth of the differential forms as they approach the boundary of $\overline{X}$. To this end, we consider a subcomplex of forms which behave "nicely" near the boundary and it turns out that the inclusion of this into the de Rham complex is a quasi-isomorphism. In addition, this subcomplex satisfies two very convenient properties, which will be essential in the final chapter, where we show that $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma$ induces an isomorphism on cohomology in low degrees. This chapter is a technical nightmare — enjoy!

As usual, $G = \mathrm{SL}_n(\mathbb{R})$, X is the manifold of Section 2.1, A the subgroup of diagonal matrices with positive entries, N the subgroup of upper triangular matrices with 1's on the diagonal, and $q_g \in X$ denotes the quadratic form given by the matrix $g^t g$, $g \in G$. Let $\mathfrak{a}$ denote the Lie algebra of A, that is the set of diagonal matrices with trace zero, and let $\mathfrak{n}$ denote the Lie algebra of N, that is the set of strictly upper triangular matrices.

5.1 Preliminaries

Before we dive into the real content of this chapter, we need to do some foot work: We explore the structure of X in more detail. More specifically, we fix convenient bases of the tangent and cotangent bundles, $T(A \times N)$ and $T^*(A \times N)$, go on to define a G-invariant metric on X and do a lot of calculations.

Let $t_i \colon A \to \mathbb{R}_{>0}$ denote the Siegel normal coordinates (Definition 3.1.14), $t_i(a) = \frac{a_i}{a_{i+1}}$, $a \in A$. Then the maps

$$\log t_i \colon A \to \mathbb{R}, \qquad \text{respectively}, \qquad d(\log t_i) = \frac{dt_i}{t_i} \colon A \to TA^*, \qquad i = 1, \ldots, n-1$$

form a coordinate system on A, respectively a basis of the cotangent bundle, T^*A. Moreover, if $\theta \colon \mathbb{R}^{n-1} \to A$ is the diffeomorphism such that $\mathrm{pr}_i \circ \theta^{-1} = \log t_i$, then for $a = \theta(x)$, $\{D_x\theta(e_i)\}_{i=1}^{n-1}$ is a basis of the tangent space T_aA with dual basis $\{d(\log t_i)(a)\}_{i=1}^{n-1}$ of $(T_aA)^*$.

Consider the basis $\{E_{ij}\}_{i<j}$ of $\mathfrak{n} = T_{\mathrm{id}}N$ and the dual basis $\{\hat{E}_{ij}\}_{i<j}$ of $\mathfrak{n}^*$. For $i < j$, let η_{ij} denote the right-invariant differential 1-form on N which is equal to the dual of E_{ij} at the identity, i.e. $\eta_{ij}(u)(x) = \hat{E}_{ij}(D_u R_{u^{-1}}(x))$ for all $u \in N$, $x \in T_uN$. Then $\{\eta_{ij}\}_{i<j}$ is a right-invariant basis of the cotangent bundle TN^*. For $u \in N$, $\eta_{ij}(u) \in T_uN^*$ is the dual of $D_e R_u E_{ij} \in T_uN$. Fix an enumeration of this basis, $\{\eta_i\}_{i=1}^{1/2\,n(n-1)}$; we will write $l_i < k_i$ for the pair corresponding to i under the chosen enumeration, i.e. $\eta_i = \eta_{l_i k_i}$.

Now, set $\epsilon_i := \pi_A^* d(\log t_i)$ for $i = 1, \ldots, n-1$ and set $\epsilon_i := \pi_N^* \eta_i$ for $i = n, \ldots, m$, where π_A, π_N denote the projections onto A and N, respectively, and $m = 1/2\,n(n+1) - 1$. Then $\{\epsilon_i\}_{i=1}^m$ is a basis of the cotangent bundle $T^*(A \times N)$ and the elements

$$\epsilon_\sigma = \epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(k)}, \qquad \sigma \in \Sigma_{k,m-k},$$

form a basis of $\Omega^k(A \times N)$. Any differential k-form on an open subset $V \subseteq A \times N$ can then be written uniquely as a linear combination of the ϵ_σ. In particular, if $\varrho \colon A \times N \to X$ denotes the

diffeomorphism $(a, u) \mapsto q_{au}$, then for any open subset $U \subseteq X$ and $\omega \in \Omega^k(U)$, we can uniquely write

$$\varrho^*\omega = \sum_\sigma f_\sigma \epsilon_\sigma, \qquad f_\sigma \in C^\infty(\varrho^{-1}(U)).$$

Let $e_1, \ldots, e_{n-1}$ denote the standard basis of $\mathbb{R}^{n-1}$.

PROPOSITION 5.1.1. With $\theta \colon \mathbb{R}^{n-1} \to A$ as above, $x \in \mathbb{R}^{n-1}$, $\theta(x) = a = (a_i)$ and $1 \le j \le n-1$, we have

$$D_x\theta(e_j) = \frac{1}{n}\left((n{-}j)\,a_1, \ldots, (n{-}j)\,a_{j-1}, -j\,a_j, \ldots, -j\,a_{n-1}, (j{-}n)\sum_{i=1}^{j-1} a_i + j\sum_{i=j}^{n-1} a_i\right),$$

where the tuple should be interpreted as a diagonal matrix in $T_a A$.

Proof. Let $\tau \colon A \to \mathbb{R}^{n-1}_{>0}$ denote the Siegel normal coordinatisation map, $\mathrm{pr}_i \circ \tau = t_i$. Then $\theta^{-1} = \log \circ \tau$ and $\theta = \tau^{-1} \circ \exp$, where log and exp denote the maps which apply the logarithm, respectively, the exponential map coordinatewise. Recall that the inverse $\tau^{-1} \colon \mathbb{R}^{n-1}_{>0} \to A$ is given by

$$\tau^{-1}(b)_j = \frac{\sqrt[n]{\prod_{i=1}^{n-1} b_i^{n-i}}}{b_1 \cdots b_{j-1}} \qquad \text{for } b = (b_i) \in \mathbb{R}^{n-1}_{>0}.$$

Composing with the coordinatisation $\kappa \colon A \to \mathbb{R}^{n-1}_{>0}$, $\mathrm{pr}_i \circ \kappa(a) = a_i$ for $a = (a_i) \in A$, we see that

$$D_{\theta^{-1}(a)}(\kappa \circ \theta) = D_{\tau(a)}(\kappa \circ \tau^{-1}) \circ D_{\log \tau(a)} \exp \colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}.$$

We have $\frac{\partial}{\partial x_i}\mathrm{pr}_i \exp(x) = e^{x_i}$, $x = (x_i) \in \mathbb{R}^{n-1}$, and $\frac{\partial}{\partial x_j}\mathrm{pr}_i \exp(x) = 0$ for $i \ne j$, so $D_{\log \tau(a)} \exp$ is a diagonal matrix with i'th diagonal entry $e^{\log \tau(a)} = \frac{a_i}{a_{i+1}}$.
Now we determine the matrix $D_{\tau(a)}(\kappa \circ \tau^{-1})$. For $b = (b_i) \in \mathbb{R}^{n-1}$,

$$\frac{\partial}{\partial x_j}(\mathrm{pr}_k \circ \kappa \circ \tau^{-1})(b) = \frac{\partial}{\partial x_j}\left(\sqrt[n]{\prod_{i=1}^{n-1} x_i^{n-i}} \cdot \frac{1}{x_1 \cdots x_{k-1}}\right)\Bigg|_{x=b} = c_{j,k}\,\frac{\mathrm{pr}_k \circ \kappa \circ \tau^{-1}(b)}{\mathrm{pr}_j(b)},$$

where $c_{j,k} = \frac{(n-j)}{n}$ if $k \le j$, and $c_{j,k} = \frac{-j}{n}$ if $j < k$. Thus, $D_{\tau(a)}(\kappa \circ \tau^{-1})$ has entry (k, j) equal to

$$D_{\tau(a)}(\kappa \circ \tau^{-1})_{kj} = \frac{\partial}{\partial x_j}(\mathrm{pr}_k \circ \kappa \circ \tau^{-1})(\tau(a)) = c_{j,k}\,\frac{\mathrm{pr}_k \circ \kappa \circ \tau^{-1}(\tau(a))}{\mathrm{pr}_j(\tau(a))} = c_{j,k}\,\frac{a_{j+1}}{a_j}\,a_k.$$

Hence,

$$(D_{\theta^{-1}(a)}(\kappa \circ \theta))_{ij} = (D_{\tau(a)}(\kappa \circ \tau^{-1}))_{ij} \circ (D_{\log \tau(a)} \exp)_j = c_{j,i} a_i$$

and

$$D_{\theta^{-1}(a)}(\kappa \circ \theta)(e_j)_i = \begin{cases} \frac{n-j}{n} a_i & i \le j \\ -\frac{j}{n} a_i & j < i \end{cases}.$$

To finish off, note that $D_a\kappa\colon T_aA \to \mathbb{R}^{n-1}$ is given by $\mathrm{pr}_i \circ D_a\kappa(x) = x_i$ for $x = (x_i) \in T_aA$, and thus $(D_a\kappa)^{-1}\colon \mathbb{R}^{n-1} \to T_aA$ is given by

$$(D_a\kappa)^{-1}(b)_i = \begin{cases} b_i & \text{for } i < n, \\ -\sum_{i=1}^{n-1} b_i & \text{for } i = n. \end{cases}$$

With this we get

$$D_{\theta^{-1}(a)}\theta(e_j) = (D_a\kappa)^{-1} \circ D_{\theta^{-1}(a)}(\kappa \circ \theta)(e_j)$$

$$= \frac{1}{n}\bigg((n-j)a_1, \ldots, (n-j)a_j, -ja_{j+1}, \ldots, -ja_{n-1}, (j-n)\sum_{i=1}^{j} a_i + j \sum_{i=j+1}^{n-1} a_i \bigg),$$

as claimed. $\square$

Remark 5.1.2. In particular, the diagonal matrices expressed as tuples

$$D_0\theta(e_j) = (1, \ldots, 1, -\tfrac{j}{n-j}, \ldots, -\tfrac{j}{n-j}), \quad j = 1, \ldots, n-1,$$

where the first j entries of $D_0\theta(e_j)$ are 1 and the last $n-j$ entries are $-\frac{j}{n-j}$, form a basis of $\mathfrak{a} = T_{\mathrm{id}}A$, and the dual basis of $\mathfrak{a}^*$ is $\{d(\log t_i)(\mathrm{id})\}_{i=1}^{n-1}$. Note also that θ is a group isomorphism from the additive group $\mathbb{R}^{n-1}$ into A, so $R_a \circ \theta = \theta \circ +_{\theta^{-1}(a)}$ for all $a \in A$, where R_a is right multiplication by a and $+_b$ is translation by $b \in \mathbb{R}^{n-1}$. Hence, $D_a R_{a^{-1}} \circ D_{\theta^{-1}(a)}\theta = D_0\theta$.

PROPOSITION 5.1.3. *The map $g_0\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $g_0(x,y) = \mathrm{tr}(xy^t)$, $x, y \in \mathfrak{g}$, defines a right invariant Riemannian metric on G.*

Proof. Define a smooth section $g\colon G \longrightarrow T^*G \otimes T^*G$ as follows: For $g \in G$, let

$$g_g\colon T_gG \times T_gG \to \mathbb{R} \quad \text{be given by} \quad g_g(x,y) = g_0(D_gR_{g^{-1}}x, D_gR_{g^{-1}}y), \quad x, y \in T_gG,$$

where we interpret $T_gG^* \otimes T_gG^*$ as the set of bilinear maps $T_gG \times T_gG \to \mathbb{R}$. This is a $(0,2)$-tensor field on G and clearly it is right invariant.

To see that g is a Riemannian metric, we simply need to show that g_g is an inner product on T_gG: It is symmetric as $\mathrm{tr}(xy^t) = \mathrm{tr}((xy^t)^t) = \mathrm{tr}(yx^t)$ for all $x, y \in \mathfrak{g}$, and it is positive definite as a positive semi-definite matrix, xx^t, has trace zero, if and only if $x = 0$. $\square$

Remark 5.1.4. The above defined $g_0\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is proportional to the positive definite form

$$(x,y) \mapsto -B(x, \theta(y)) \qquad \text{for all } x, y \in \mathfrak{g},$$

where B is the Killing form on $\mathfrak{g}$ and θ the Cartan involution $y \mapsto -y^t$.

Let G act on AN from the right such that the diffeomorphism $AN \to X$, $p \mapsto q_p$, is equivariant; explicitly, this action is given by $p.g = au$, $p \in AN$, $g \in G$, where $pg = kau$ is the Iwasawa decomposition of pg. Let $\lambda_g\colon AN \to AN$ denote map $p \mapsto p.g$; it is equal to the composite

$$AN \hookrightarrow G \xrightarrow{R_g} G \xrightarrow{Iwasawa} K \times AN \twoheadrightarrow AN.$$

Note that for $g = au \in AN$, λ_{au} is given by $\lambda_{au}(bv) = (ba)(a^{-1}vau)$ for $b \in A$, $v \in N$. The restriction of g to AN, $g\colon AN \to T^*(AN) \otimes T^*(AN)$, is a G-invariant Riemannian metric on AN (we also denote it by g). It is given by

$$g_p(x,y) = g_0(D_p\lambda_{p^{-1}}x, D_p\lambda_{p^{-1}}y), \qquad x, y \in T_{an}(AN).$$

Let $\nu\colon A\times N\to AN$ denote the multiplication map and set $\hbar:=\nu^*(\mathcal{g})$; this is a metric on $A\times N$ which is invariant under the inherited action of G, namely the one given by $(a,u).g=(b,v)$, where $aug=kbv$ is the Iwasawa decomposition, $a,b\in A$, $u,v\in N$, $g\in G$, $k\in K$ — we denote also by $\lambda_g\colon A\times N\to A\times N$ the map $(a,u)\mapsto(a,u).g$. Note that for $g=au\in AN$, we have $(b,v).g=(ba,a^{-1}vau)$. We wish to determine $\hbar$ a little more explicitly. Recall that $\{D_a\theta(e_i)\}_{i=1}^{n-1}\cup\{E_{ij}u\}_{i<j}$ forms a basis of the tangent space $T_{(a,u)}(A\times N)=T_aA\times T_uN$ at $(a,u)\in A\times N$, where we interpret $\hat{a}\in T_aA$ and $\hat{u}\in T_uN$ as the elements $(\hat{a},0)$, respectively, $(0,\hat{u})$ in $T_aA\times T_uN$.

PROPOSITION 5.1.5. For $(a,u)\in A\times N$, $a=(a_i)$, we have

$$\hbar_{(a,u)}(D_a\theta(e_l),D_eR_uE_{ij})=0 \qquad \text{for any } l \text{ and any } i<j,$$
$$\hbar_{(a,u)}(D_eR_uE_{ij},D_eR_uE_{lk})=0 \qquad \text{for any distinct } i<j \text{ and } l<k,$$
$$\hbar_{(a,u)}(D_eR_uE_{ij},D_eR_uE_{ij})=(\tfrac{a_i}{a_j})^2 \qquad \text{for any } i<j,$$
$$\hbar_{(a,u)}(D_a\theta(e_i),D_a\theta(e_j))={}^{1\!}/{}_{n^2}\, m_{i,j,n} \qquad \text{for any } i,j,$$

for an integer $m_{i,j,n}\in\mathbb{Z}$ depending only on n, i and j.

Proof. As $(au)^{-1}=a^{-1}(au^{-1}a^{-1})$ is the Iwasawa decomposition of $(au)^{-1}$, we see that

$$\lambda_{(au)^{-1}}(bv)=(ba^{-1},ava^{-1}(au^{-1}a^{-1}))=(ba^{-1},avu^{-1}a^{-1}),$$

so $\lambda_{(au)^{-1}}=R_{a^{-1}}\times(c_a\circ R_{u^{-1}})$, where $R_{(-)}$ is right multiplication and $c_{(-)}$ is conjugation. Hence, $D_{au}\lambda_{(au)^{-1}}=D_aR_{a^{-1}}\times\mathrm{Ad}(a)\circ D_uR_{u^{-1}}$. Recalling that $D_{(a,u)}\nu\colon T_aA\times T_uN\to T_{au}(AN)$ is given by $(v,w)\mapsto v+w$, we see that for any $(a,u)\in A\times N$ and $(v,w),(v',w')\in T_aA\times T_uN$,

$$\begin{aligned}
\hbar_{(a,u)}((v,w),(v',w')) &= (\lambda^*_{(au)^{-1}}\hbar)_{(a,u)}((v,w),(v',w'))\\
&= \hbar_{(\mathrm{id},\mathrm{id})}(D_{(a,u)}\lambda_{(au)^{-1}}(v,w),D_{(a,u)}\lambda_{(au)^{-1}}(v',w'))\\
&= (\nu^*\mathcal{g})_{(\mathrm{id},\mathrm{id})}(D_{(a,u)}\lambda_{(au)^{-1}}(v,w),D_{(a,u)}\lambda_{(au)^{-1}}(v',w'))\\
&= \mathcal{g}_0(D_{\mathrm{id}}\nu(D_aR_{a^{-1}}v,\mathrm{Ad}(a)\circ D_uR_{u^{-1}}w),D_{\mathrm{id}}\nu(D_aR_{a^{-1}}v',\mathrm{Ad}(a)\circ D_uR_{u^{-1}}w'))\\
&= \mathcal{g}_0(D_aR_{a^{-1}}v+\mathrm{Ad}(a)\circ D_uR_{u^{-1}}w,D_aR_{a^{-1}}v'+\mathrm{Ad}(a)\circ D_uR_{u^{-1}}w').
\end{aligned}$$

We now apply this expression to our chosen basis of $T_aA\times T_uN$. Recall first that

$$D_aR_{a^{-1}}\circ D_{\theta^{-1}(a)}\theta=D_0\theta \quad\text{and}\quad \mathrm{Ad}(a)E_{ij}=\frac{a_i}{a_j}E_{ij}, \quad i<j,$$

(cf. Remark 5.1.2 and the proof of Proposition 1.2.15).
For any $i<j$, l, we have

$$\begin{aligned}
\hbar_{(a,u)}(D_{\theta^{-1}(a)}\theta(e_l),D_eR_uE_{ij}u) &= \mathcal{g}_0(D_aR_{a^{-1}}\circ D_{\theta^{-1}(a)}\theta(e_l),\mathrm{Ad}(a)\circ D_uR_{u^{-1}}\circ D_eR_uE_{ij})\\
&= \mathcal{g}_0(D_0\theta(e_l),\frac{a_i}{a_j}E_{ij})=\frac{a_i}{a_j}\,\mathrm{tr}(D_0\theta(e_l)E_{ij}^t)=0.
\end{aligned}$$

For $i<j$, $l<k$, we have

$$\begin{aligned}
\hbar_{(a,u)}(D_eR_uE_{ij},D_eR_uE_{lk}) &= \mathcal{g}_0(\mathrm{Ad}(a)E_{ij},\mathrm{Ad}(a)E_{lk})\\
&= \frac{a_i}{a_j}\frac{a_l}{a_k}\,\mathrm{tr}(E_{ij}E_{lk}^t)=\begin{cases}(a_i/a_j)^2 & i=l,\ j=k,\\ 0 & \text{else}\end{cases}.
\end{aligned}$$

Lastly, for some i, j, we may assume that $i \leq j$. Then, using Proposition 5.1.1, we have

$$\hbar_{(a,u)}(D_a\theta(e_i), D_a\theta(e_j)) = g_0(D_0\theta(e_i), D_0\theta(e_j)) = \mathrm{tr}(D_0\theta(e_i)D_0\theta(e_j))$$

$$= i - (j-i)\frac{n-i}{n} + (n-j)\frac{(n-i)(n-j)}{n^2}$$

$$= \frac{1}{n^2}(in^2 - n(j-i)(n-i) + (n-i)(n-j)^2).$$

$\square$

Recall from Section 1.3 that the metric $\hbar$ induces inner products $\langle -, - \rangle_{\hbar}^{(a,u)}$ on $\Lambda^k(T_{(a,u)}(A \times N))^*$ and a measure $\mu_{\hbar}$ on $A \times N$. We have the following corollaries.

COROLLARY 5.1.6. In terms of the basis $\{D_{\theta^{-1}(a)}\theta(e_j)\}_{j=1}^{n-1} \cup \{D_e R_u E_{ij}\}_{i<j}$, the isomorphism $T_aA \times T_uN \to (T_aA \times T_uN)^*$ induced by $\hbar_{(a,u)}$ is given by the matrix

$$\boldsymbol{x}_{(a,u)} = ((\boldsymbol{x}_{(a,u)})_{ij}) = \begin{pmatrix} \frac{1}{n^2}M & 0 \\ 0 & \mathrm{Ad}(a)|_{\mathfrak{n}}^2 \end{pmatrix},$$

where $M \in \mathrm{GL}_{n-1}(\mathbb{Z})$ is independent of the choice of (a,u), and $\mathrm{Ad}(a)|_{\mathfrak{n}}$ is the diagonal matrix with entry $i < j$ equal to $\frac{a_i}{a_j}$.

COROLLARY 5.1.7. The volume form $\omega_{\hbar}$ on $A \times N$ induced by $\hbar$ is given by

$$(\omega_{\hbar})_{(a,u)} = \sqrt{|\det \boldsymbol{x}_{(a,u)}|}\, \epsilon_{\mathrm{id}} = \frac{1}{n^{n-1}}\prod_{i<j}\frac{a_i}{a_j}\, \epsilon_{\mathrm{id}}, \qquad (a,u) \in A \times N.$$

Let μ_A be the measure on A given by the volume form $d(\log t_1) \wedge \ldots \wedge d(\log t_{n-1})$, and let μ_N be the measure on N given by the volume form $\eta_n \wedge \ldots \wedge \eta_m$. The η_i are by definition right invariant and the forms $d(\log t_i)$ are right invariant, as $t_i \circ R_b = t_i(b)\, t_i$. Thus

$$\lambda_b^* d(\log t_i) = R_b^*(\frac{1}{t_i}\, dt_i) = \frac{1}{t_i \circ R_b}d(t_i \circ R_b) = \frac{1}{t_i(b)\, t_i}t_i(b)\, dt_i = d(\log t_i) \text{ for all } i = 1, \ldots, n-1,$$

where $R_b \colon A \to A$ denotes right multiplication by $b \in A$. So the measures μ_A and μ_N are right Haar measures (and therefore also left Haar measures as A and N are unimodular). Finally, let $\mu_{\hbar}$ be the measure on $A \times N$ defined by $\hbar$, i.e. given by the volume form $\omega_{\hbar}$.

COROLLARY 5.1.8. With $\rho \colon A \to \mathbb{R}$ given by $\rho(a) = \prod_{i<j}\frac{a_i}{a_j}$, $a \in A$, and π_A the projection onto A, we have $\mu_{\hbar} = (\frac{1}{n})^{n-1}(\rho \circ \pi_A).(\mu_A \otimes \mu_N)$.

For $(a,u) \in A \times N$, let $\langle -, - \rangle_{\hbar}^{(a,u)}$ denote the inner product on $\Lambda^k(T_aA \times T_uU)^*$ induced by $\hbar$ (the power k will be implicit from the context).

COROLLARY 5.1.9. Let $\sigma, \tau \in \Sigma_{k,m-k}$, $(a,u) \in A \times N$. Then for the elements $\epsilon_\sigma, \epsilon_\tau \in \Omega^k(A \times N)$, we have

$$\langle (\epsilon_\sigma)_{(a,u)}, (\epsilon_\tau)_{(a,u)} \rangle_{\hbar}^{(a,u)} = 0 \qquad \qquad \text{if } \{\sigma(i)\}_{\substack{\sigma(i)\geq n \\ i=1,\ldots k}} \neq \{\tau(i)\}_{\substack{\tau(i)\geq n \\ i=1,\ldots k}},$$

$$\left| \langle (\epsilon_\sigma)_{(a,u)}, (\epsilon_\tau)_{(a,u)} \rangle_{\hbar,}^{(a,u)} \right| \leq c_n \prod_{\sigma(i)\geq n}\left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}}\right)^2 \qquad \text{if } \{\sigma(i)\}_{\substack{\sigma(i)\geq n \\ i=1,\ldots k}} = \{\tau(i)\}_{\substack{\tau(i)\geq n \\ i=1,\ldots k}},$$

for some constant $c_n > 0$ depending only on n.

Proof. The point (a, u) will be implicit: We write $\epsilon_\sigma = (\epsilon_\sigma)_{(a,u)}$, $\langle -, - \rangle_\hbar = \langle -, - \rangle_\hbar^{(a,u)}$, and $x = x_{(a,u)}$. Writing $x^{-1} = (x^{ij})$, we have by definition that

$$\langle \epsilon_\sigma, \epsilon_\tau \rangle_\hbar = \det \left(x^{\sigma(i)\tau(j)} \right).$$

From Corollary 5.1.6 we see that

$$x^{-1} = \begin{pmatrix} n^2 M^{-1} & 0 \\ 0 & \mathrm{Ad}(a^{-1})|_\mathfrak{n}^2 \end{pmatrix}$$

and we know that M is independent of the point (a, u). Write $M^{-1} = (m^{ij})$ and define for any $l = 1, \dots, n-1$, $\alpha, \beta \in \Sigma_{l,n-1-l}$, an $l \times l$-matrix $M_{\alpha,\beta}^{-1} := (m^{\alpha(i)\beta(j)})$. Set

$$c'_n := \max\{ |\det M_{\alpha,\beta}^{-1}| \mid l = 1, \dots, n-1, \ \alpha, \beta \in \Sigma_{l,n-1-l} \}.$$

If

$$\{ \sigma(i) \mid \sigma(i) \geq n, \ i = 1, \dots k \} \neq \{ \tau(i) \mid \tau(i) \geq n, \ i = 1, \dots k \},$$

then the elements η_{ij} appearing in ϵ_σ and ϵ_τ are not the same and therefore the matrix $(x^{\sigma(i)\tau(j)})$ has a zero row or column, so $\langle \epsilon_\sigma, \epsilon_\tau \rangle_\hbar = 0$.
If

$$\{ \sigma(i) \mid \sigma(i) \geq n, \ i = 1, \dots k \} = \{ \tau(i) \mid \tau(i) \geq n, \ i = 1, \dots k \},$$

let $1 \leq p \leq k$, such that $\sigma(i), \tau(i) \leq n-1$ for $i \leq p$ and $\sigma(i), \tau(i) \geq n$ for $i \geq p$. Let $\sigma', \tau' \in \Sigma_{p,n-1-p}$ denote the restriction of σ, respectively, τ to $\{1, \dots, p\}$, extended to permutations on $\{1, \dots, n-1\}$ in whatever way possible. Then

$$\left| \langle \epsilon_\sigma, \epsilon_\tau \rangle_\hbar \right| = \left| \det \left(x^{\sigma(i)\tau(j)} \right) \right| = n^{2p} \left| \det M_{\sigma',\tau'}^{-1} \right| \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \leq n^{2(n-1)} c'_n \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2.$$

$\square$

5.2 Logarithmic Forms

The scene is set and we can now define the notion of a differential form having logarithmic growth near the boundary. This in turn allows us to define the subcomplex of logarithmic forms on X/Γ for an appropriate torsion free subgroup $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$. We prove that the inclusion into the de Rham complex is a quasi-isormorphism, that logarithmic forms of low degrees are square integrable and that the G-invariant forms on X are mapped to logarithmic forms under the chain isomorphism $\Omega^*(X)^\Gamma \cong \Omega^*(X/\Gamma)$. The section relies heavily on the calculations of the previous section and is in itself very heavy on calculations.

Recall the diffeomorphisms

$$\varrho \colon A \times N \xrightarrow{\cong} X, \quad (a, u) \mapsto q_{au}, \qquad \varrho' \colon \overline{A} \times N \xrightarrow{\cong} X(B)_{[\mathrm{id}]_B} \hookrightarrow \overline{X}, \quad (\overline{a}, u) \mapsto [\overline{a}, q_u].$$

In the following, the notion of open Siegel sets will be useful, the definition of which is rather obvious:

DEFINITION 5.2.1. For $\lambda, \delta > 0$, set

$$A_{(\lambda)} := \{a = (a_i) \in A \mid \tfrac{a_i}{a_{i+1}} < \lambda, \; i = 1, \ldots, n-1\},$$
$$\overline{A}_{(\lambda)} := \{a = (a_i) \in \overline{A} \mid a_i < \lambda, i = 1, \ldots, n-1\},$$
$$N_{(\delta)} := \{u = (u_{ij}) \in N \mid |u_{ij}| < \delta, \; i < j\}.$$

The *open Siegel sets* of X, respectively, $\overline{X}$ given by λ and δ are defined to be the sets

$$\mathfrak{S}_{(\lambda,\delta)} := \varrho(A_{(\lambda)} \times N_{(\delta)}), \quad \text{respectively,} \quad \overline{\mathfrak{S}}_{(\lambda,\delta)} := \varrho(\overline{A}_{(\lambda)} \times N_{(\delta)}).$$

Remark 5.2.2. It is immediate that $\mathfrak{S}_{(\lambda,\delta)}$ is open in X and in fact equal to the interior of $\mathfrak{S}_{\lambda,\delta}$; likewise, $\overline{\mathfrak{S}}_{(\lambda,\delta)}$ is open in $\overline{X}$ and equal to the interior of $\overline{\mathfrak{S}}_{\lambda,\delta}$.

Recall that $\mathrm{SL}_n(\mathbb{Z})$ contains a normal torsion free subgroup Γ of finite index. Then X/Γ is a smooth manifold, $\overline{X}/\Gamma$ is a compact smooth manifold with corners and the inclusion $X/\Gamma \hookrightarrow \overline{X}/\Gamma$ is a homotopy equivalence (cf. Remark 4.3.9). Let $\pi \colon \overline{X} \to \overline{X}/\Gamma$ denote the projection and let $C \subseteq \mathrm{SL}_n(\mathbb{Z})$ be a finite subset such that $\mathrm{SL}_n(\mathbb{Z}) = C\Gamma$. Then $X/\Gamma = \pi(\mathfrak{S}_0.C)$ and $\overline{X}/\Gamma = \pi(\overline{\mathfrak{S}}_0.C)$, where $\mathfrak{S}_0 = \mathfrak{S}_{4/3,1/2}$.

DEFINITION 5.2.3.

i) For an open subset $U \subseteq \mathfrak{S}_{(\lambda,\delta)}$, $\lambda, \delta > 0$, and $\omega \in \Omega^k(U)$, write $\varrho^*\omega = \sum f_\sigma \epsilon_\sigma$ for $f_\sigma \in C^\infty(\varrho^{-1}(U))$. We say that ω has *logarithmic growth* if there exists a real polynomial p in $n-1$ variables such that for all $(a, u) \in \varrho^{-1}(U) \subseteq A_{(\lambda)} \times N_{(\delta)}$, $\sigma \in \Sigma_{k,m-k}$,

$$|f_\sigma(a, u)| \leq |p(\log t_1(a), \ldots, \log t_{n-1}(a))|. \tag{5.1}$$

ii) For an open subset $U \subseteq \mathfrak{S}_{(\lambda,\delta)}.C$, $\lambda, \delta > 0$, a differential form $\omega \in \Omega^*(U)$ has *logarithmic growth* if the restriction of $\gamma^*\omega$ to $U.\gamma^{-1} \cap \mathfrak{S}_{(\lambda,\delta)}$ has logarithmic growth for all $\gamma \in C$.

iii) For an open subset $V \subseteq X/\Gamma$, a differential form $\omega \in \Omega^k(V)$ has *logarithmic growth* if $\pi^*\omega$ has logarithmic growth on $\pi^{-1}(V) \cap \mathfrak{S}_{(\lambda,\delta)}.C$ for some $\lambda > 4/3$, $\delta > 1/2$.

DEFINITION 5.2.4. Let $V \subseteq \overline{X}/\Gamma$ be an open subspace and let $\omega \in \Omega^*(V \cap X/\Gamma)$. We say that ω has *logarithmic growth near the boundary of* V if for every point $y \in \partial V$, there exists a neighbourhood U of y such that the restriction of ω to $U \cap X/\Gamma$ has logarithmic growth.
If $V = \overline{X}/\Gamma$, and $\omega \in \Omega^k(X/\Gamma)$ has logarithmic growth near the boundary of $\overline{X}/\Gamma$, we say that ω has *logarithmic growth at infinity*, and we let $\Omega^*_{\log}(X/\Gamma) \subseteq \Omega^*(X/\Gamma)$ denote the subcomplex of forms $\omega \in \Omega^*(X/\Gamma)$ for which both ω and $d\omega$ have logarithmic growth at infinity. We also call these forms *logarithmic*.

LEMMA 5.2.5. A differential form $\omega \in \Omega^*(X/\Gamma)$ has logarithmic growth at infinity, if and only if there exist $\lambda > 4/3$, $\delta > 1/2$ such that $\gamma^*\pi^*\omega$ has logarithmic growth on $\mathfrak{S}_{(\lambda,\delta)}$ for all $\gamma \in C$.

Proof. The right to left implication is clear. For the converse, note that $\partial \overline{X}/\Gamma$ is compact, being the image of $\partial \overline{\mathfrak{S}}_0.C$, and $\partial \overline{\mathfrak{S}}_0 \cong \partial \overline{A}_{4/3} \times N_{1/2}$. Hence, if ω has logarithmic growth at infinity, then there is a finite cover $U_1, \ldots, U_k$ of $\partial \overline{X}/\Gamma$ such that the restriction of ω to each U_i has logarithmic growth. Then there exist $\lambda > 4/3$, $\delta > 1/2$ for which the restriction of $\gamma^*\pi^*\omega$ to $\pi^{-1}(U_i).\gamma^{-1} \cap \mathfrak{S}_{(\lambda,\delta)}$ has logarithmic growth for all $\gamma \in C$, $i = 1, \ldots, k$.
Let $\gamma \in C$, set $V_i := \varrho^{-1}(\pi^{-1}(U_i).\gamma^{-1}) \cap (A_{(\lambda)} \times N_{(\delta)})$ and write $\varrho^*\gamma^*\pi^*\omega|_{V_i} = \sum f_\sigma^i \epsilon_\sigma$ for some $f_\sigma^i \in C^\infty(V_i)$ for all i. Let $p_1, \ldots, p_k$ be real polynomials in $n-1$ variables such that

$$|f_\sigma^i(a, u)| \leq |p_i(\log t_1(a), \ldots, \log t_{n-1}(a))| \quad \text{for all } (a, u) \in V_i, \, \sigma \in \Sigma_{k,m-k}, \; i = 1, \ldots, k.$$

Now, let $\lambda > \lambda' > {}^4\!/_3$, $\delta > \delta' > {}^1\!/_2$ and write

$$\varrho^* \gamma^* \pi^* \omega|_{A_{(\lambda)} \times N_{(\delta)}} = \sum f_\sigma \, \epsilon_\sigma \quad \text{for} \quad f_\sigma \in C^\infty(A_{(\lambda)} \times N_{(\delta)}).$$

Then we must have $f_\sigma|_{V_i} = f_\sigma^i$ for all $i = 1, \ldots, k$. Note that $F := A_{\lambda'} \times N_{\delta'} - \bigcup_{i=1}^k V_i \subseteq A_{\lambda'} \times N_{\delta'}$ is compact, being closed in $\overline{A}_{\lambda'} \times N_{\delta'}$ and not intersecting the boundary. Let p be a real polynomial in $n-1$ variables satisfying

$$|p(\log t_1(a), \ldots, \log t_{n-1}(a))| \geq |p_i(\log t_1(a), \ldots, \log t_{n-1}(a))|$$
$$\text{and} \quad |p(\log t_1(a), \ldots, \log t_{n-1}(a))| \geq \max_{\sigma,\,(b,v) \in F} \{|f_\sigma(b, v)|\}.$$

for all $(a, u) \in A_{\lambda'} \times N_{\delta'}$, $i = 1, \ldots k$. This polynomial satisfies

$$|f_\sigma(a, u)| \leq |p(\log t_1(a), \ldots, \log t_{n-1}(a))|, \quad \text{for all } (a, u) \in A_{(\lambda')} \times N_{(\delta')}, \quad \sigma \in \Sigma_{k, m-k},$$

and we conclude that $\gamma^* \pi^* \omega$ has logarithmic growth on $\mathfrak{S}_{(\lambda', \delta')}$. $\qquad\square$

We go on to prove three important properties of the complex $\Omega^*_{\log}(X/\Gamma)$.

PROPOSITION 5.2.6. Under the chain isomorphism $\Omega^*(X)^\Gamma \cong \Omega^*(X/\Gamma)$, the subcomplex of G-invariant forms on X is mapped into the subcomplex of logarithmic forms.

Proof. Let $\omega \in \Omega^*(X)^G$ and let $\varphi \colon \Omega^*(X)^\Gamma \to \Omega^*(X/\Gamma)$ denote the chain isomorphism (see Proposition 2.3.2). To see that $\varphi(\omega) \in \Omega^*_{\log}(X/\Gamma)$, we have to show that $\varphi(\omega)$ and $d\varphi(\omega)$ have logarithmic growth at infinity. Note first that ω is closed: This is an immediate consequence of the chain isomorphism $\Omega^*(X)^G \cong C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$ as the latter chain complex has trivial differential (see Proposition 6.2.3). Then $d\varphi(\omega) = \varphi(d\omega) = 0$ trivially has logarithmic growth at infinity.
In view of Lemma 5.2.5, $\varphi(\omega)$ has logarithmic growth at infinity if for some $\lambda > {}^4\!/_3$, $\delta > {}^1\!/_2$, $\gamma^* \pi^* \varphi(\omega)$ has logarithmic growth on $\mathfrak{S}_{\lambda, \delta}$ for any $\gamma \in C$. As $\pi^* = \varphi^{-1}$ and ω is G-invariant, we have $\gamma^* \pi^* \varphi(\omega) = \omega$.
Write $\varrho^* \omega = \sum f_\sigma \epsilon_\sigma$, $f_\sigma \in C^\infty(A \times N)$. At this point we are only going to use that ω is invariant under the inherited right action of A. Recall the right action of G on $A \times N$ from the previous section (with this action on $A \times N$, ϱ is equivariant), and let for $b \in A$, $\lambda_b \colon A \times N \to A \times N$ denote this action, i.e. $\lambda_b(a, u) = (ab, b^{-1}ub)$.
Recall that $d(\log t_i)$ is right invariant and note that for $c_{b^{-1}} \colon N \to N$ conjugation by b^{-1}, we have $c_{b^{-1}}^* \eta_{ij} = \frac{b_j}{b_i} \eta_{ij}$ as

$$(c_{b^{-1}}^* \eta_{ij})_{\mathrm{id}}(E_{lk}) = (\eta_{ij})_{\mathrm{id}}(\mathrm{Ad}(b^{-1}) E_{lk}) = \frac{b_k}{b_l} \hat{E}_{ij}(E_{lk}) \quad \text{for all } i < j, \, l < k.$$

Hence, if ι_A, ι_N denote the canonical inclusions of A, respectively, N into $A \times N$, we have

$$\lambda_b^* \epsilon_i = \pi_A^* \iota_A^* \lambda_b^* \pi_A^* d(\log t_i) = \pi_A^* R_b^* d(\log t_i) = \epsilon_i \qquad \text{for all } i = 1, \ldots, n-1,$$
$$\text{and} \quad \lambda_b^* \epsilon_i = \pi_N^* \iota_N^* \lambda_b^* \pi_N^* \eta_i = \pi_N^* c_{b^{-1}}^* \eta_i = \frac{b_{k_i}}{b_{l_i}} \epsilon_i \qquad \text{for all } i = n, \ldots, m,$$

where $l_i < k_i$ correspond to i under the chosen enumeration. With these observations, we see that

$$\sum f_\sigma \, \epsilon_\sigma = \varrho^* \omega = \varrho^* b^* \omega = \lambda_b^* \varrho^* \omega = \sum f_\sigma \circ \lambda_b \left(\prod_{\substack{\sigma(i) \geq n \\ i=1, \ldots, k}} \frac{b_{k_{\sigma(i)}}}{b_{l_{\sigma(i)}}} \right) \epsilon_\sigma,$$

so

$$f_\sigma = \left(\prod_{\substack{\sigma(i) \geq n \\ i=1,\dots,k}} \frac{b_{k_{\sigma(i)}}}{b_{l_{\sigma(i)}}} \right) f_\sigma \circ \lambda_b, \qquad \text{for all } \sigma \in \Sigma_{k,m-k}.$$

In particular, for a given $(a, u) \in A \times N$ we can take $b = a^{-1}$ and then

$$f_\sigma(a, u) = \left(\prod_{\substack{\sigma(i) \geq n \\ i=1,\dots,k}} \frac{a_{l_{\sigma(i)}}}{a_{k_{\sigma(i)}}} \right) f_\sigma(\mathrm{id}, aua^{-1}) = \left(\prod_{i=1}^{n-1} t_i(a)^{n_{i,\sigma}} \right) f_\sigma(\mathrm{id}, aua^{-1})$$

for some $n_{i,\sigma} \in \mathbb{N}_0$.

Let $\lambda > 4/3$, $\delta > 1/2$. We know from Lemma 3.3.1 that $V = \bigcup_{a \in A_\lambda} a N_\delta a^{-1}$ is relatively compact. Thus there exists $c > 0$ such that $|f_\sigma(\mathrm{id}, aua^{-1})| \leq c$ for all $(a, u) \in A_{(\lambda)} \times N_{(\delta)}$, and therefore by the above

$$|f_\sigma(a, u)| \leq c\lambda^{\sum_i n_{i,\sigma}} \qquad \text{for all } (a, u) \in A_{(\lambda)} \times N_{(\delta)}.$$

Being bounded on $A_{(\lambda)} \times N_{(\delta)}$, the f_σ trivially have logarithmic growth, which is what we needed to show. $\square$

LEMMA 5.2.7. If $\lambda \leq 3/2$ and $f \in C^\infty(A_{(\lambda)} \times N_{(\delta)})$ satisfies

$$|f(a, u)| \leq |p(\log t_1(a), \dots, \log t_{n-1}(a))|$$

for some polynomial p and all $(a, u) \in A_{(\lambda)} \times N_{(\delta)}$, then for any $0 < \varepsilon < 1$ there is a constant $c_\epsilon > 0$ such that

$$|f(a, u)| \leq c_\varepsilon \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^\varepsilon \quad \text{for all } (a, u) \in A_{(\lambda)} \times N_{(\delta)}.$$

Proof. Suppose the map $f \in C^\infty(A_{(\lambda)} \times N_{(\delta)})$ and the polynomial p satisfy the inequality above. Note first that for any $0 < \varepsilon < 1$, $|\log(x)| \leq \frac{1}{\varepsilon} x^{-\varepsilon}$ for all $0 < x < \lambda$. Then for a given $0 < \varepsilon < 1$ and any $n \in \mathbb{N}$, replacing ϵ by ϵ/n in the above inequality yields

$$|\log(x)|^n \leq (n/\varepsilon \, x^{-\varepsilon/n})^n \leq (n/\varepsilon)^n x^{-\varepsilon} \quad \text{and} \quad |\log(x)|^0 = 1 < (\lambda)^\varepsilon \, x^{-\varepsilon} \qquad \text{for all } 0 < x < \lambda.$$

Then there is some constant $c_\varepsilon > 0$ for which

$$|p(\log t_1(a), \dots, \log t_{n-1}(a))| \leq c_\varepsilon \prod_{i=1}^{n-1} t_i(a)^{-\varepsilon} = c_\varepsilon \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^\varepsilon \quad \text{for all } a \in A_\lambda.$$

$\square$

Equip X with the G-invariant Riemannian metric $g^X := \varrho^*(\hbar)$. Equip X/Γ with the inherited metric g', i.e. the metric given by $g'_{\pi(p)}(v, w) = g^X_p((D_p\pi)^{-1}(v), (D_p\pi)^{-1}(w))$ for $p \in X$, $v, w \in T_{\pi(p)}(X/\Gamma)$ (this is well-defined as g^X is G-invariant and thus in particular Γ-invariant). Then $g^X = \pi^* g'$. Similarly, we let G/Γ inherit the right invariant metric from G. For future use, we note the following:

PROPOSITION 5.2.8. The manifolds G, X, G/Γ and X/Γ are all complete.

Proof. G and X are complete as they are homogeneous and the metrics are G-invariant, so the isometries act transitively (cf. [1, Lemma 5.2]). As the projections $G \to G/\Gamma$, $X \to X/\Gamma$ are local isometries, it follows that G/Γ and X/Γ are complete. $\qquad\square$

Now X and X/Γ are connected oriented Riemannian manifolds (cf. the above and Proposition 2.1.17), so we can apply the machinery of Section 1.3. Let μ_{g^X}, $\mu_{g'}$ denote the measures induced by g^X, respectively, g' and note that $\mu_{g'} = \pi_* \mu_{g^X}$. Recall that g^X and g' induce inner products $\langle -, - \rangle^p_{g^X}$ on $\Lambda^k(T_p X)^*$, respectively, $\langle -, - \rangle^{\pi(p)}_{g'}$ on $\Lambda^*(T_{\pi(p)}(X/\Gamma))^*$ for $p \in X$ and that

$$\langle \omega_{\pi(p)}, \omega'_{\pi(p)} \rangle^{\pi(p)}_{g'} = \langle (\pi^*\omega)_p, (\pi^*\omega')_p \rangle^p_{g^X}, \qquad \omega, \omega' \in \Omega^k(X/\Gamma),\ p \in X.$$

Recall finally that a differential form $\omega \in \Omega^*(X/\Gamma)$ is square integrable if

$$\|\omega\|^2_{g'} := \int \langle \omega_p, \omega_p \rangle^p_{g'}\, d\mu_{g'}(p) < \infty.$$

PROPOSITION 5.2.9. *For $k < \frac{n-1}{2}$, any $\omega \in \Omega^k_{\log}(X/\Gamma)$ is square integrable.*

Proof. Let $\omega \in \Omega^k_{\log}(X/\Gamma)$ for some $k < \frac{n-1}{2}$. By definition of the metric g' on X/Γ, and using that $X/\Gamma = \pi(\mathfrak{S}_0.C)$, we have by the abstract change of variable formula

$$\|\omega\|^2_{g'} = \int_{X/\Gamma} \langle \omega_{\pi(p)}, \omega_{\pi(p)} \rangle^{\pi(p)}_{g'}\, d\mu_{g'}(\pi(p)) \le \int_{\mathfrak{S}_0.C} \langle (\pi^*\omega)_p, (\pi^*\omega)_p \rangle^p_{g^X}\, d\mu_{g^X}(p).$$

Noting that

$$\int_{\mathfrak{S}_0.\gamma} \langle (\pi^*\omega)_p, (\pi^*\omega)_p \rangle^k_{g^X}\, d\mu_{g^X}(p) = \int_{\mathfrak{S}_0.\gamma} \langle (\pi^*\omega)_p, (\pi^*\omega)_p \rangle^k_{g^X}\, d\gamma_* \mu_{g^X}(p)$$

$$= \int_{\mathfrak{S}_0} \langle (\gamma^*\pi^*\omega)_p, (\gamma^*\pi^*\omega)_p \rangle^k_{g^X}\, d\mu_{g^X}(p),$$

we have

$$\|\omega\|^2_{g'} \le \int_{\mathfrak{S}_0.C} \langle (\pi^*\omega)_p, (\pi^*\omega)_p \rangle^p_{g^X}\, d\mu_{g^X}(p)$$

$$\le \sum_{\gamma \in C} \int_{\mathfrak{S}_0.\gamma} \langle (\pi^*\omega)_p, (\pi^*\omega)_p \rangle^p_{g^X}\, d\mu_{g^X}(p)$$

$$= \sum_{\gamma \in C} \int_{\mathfrak{S}_0} \langle (\gamma^*\pi^*\omega)_p, (\gamma^*\pi^*\omega)_p \rangle^p_{g^X}\, d\mu_{g^X}(p)$$

$$= \sum_{\gamma \in C} \int_{A_{4/3} \times N_{1/2}} \langle (\varrho^*\gamma^*\pi^*\omega)_{(a,u)}, (\varrho^*\gamma^*\pi^*\omega)_{(a,u)} \rangle^{(a,u)}_{\hbar}\, d\mu_{\hbar}(p)$$

$$= \frac{1}{n^{n-1}} \sum_{\gamma \in C} \int_{A_{4/3} \times N_{1/2}} \langle (\varrho^*\gamma^*\pi^*\omega)_{(a,u)}, (\varrho^*\gamma^*\pi^*\omega)_{(a,u)} \rangle^{(a,u)}_{\hbar}\, \rho(a)\, d(\mu_A \otimes \mu_N)(a,u),$$

where we use that $g^X = \varrho^*(\hbar)$ and $\mu_{\hbar} = (1/n)^{n-1}(\rho \circ \pi_A).(\mu_A \otimes \mu_N)$.
For an arbitrary $\gamma \in C$, write $\varrho^*\gamma^*\pi^*\omega = \sum f_\sigma \epsilon_\sigma$ for $f_\sigma \in C^\infty(A \times N)$. Then

$$\langle (\varrho^*\gamma^*\pi^*\omega)_{(a,u)}, (\varrho^*\gamma^*\pi^*\omega)_{(a,u)} \rangle^{(a,u)}_{\hbar} = \sum_{\sigma,\tau} f_\sigma(a,u) f_\tau(a,u) \langle ((\epsilon_\sigma)_{(a,u)}, (\epsilon_\tau)_{(a,u)} \rangle^{(a,u)}_{\hbar}$$

and the only non-zero terms in this sum are the ones for σ, τ satisfying

$$\{\sigma(i) \mid \sigma(i) \geq n,\, i = 1, \ldots, k\} = \{\tau(i) \mid \tau(i) \geq n,\, i = 1, \ldots, k\}$$

(cf. Corollary 5.1.9). If we can show that for all such σ, τ, we have

$$\int_{A_{4/3} \times N_{1/2}} \left| f_\sigma(a, u) f_\tau(a, u) \langle (\epsilon_\sigma)_{(a,u)}, (\epsilon_\tau)_{(a,u)} \rangle_{\hbar}^{(a,u)} \rho(a) \right| d(\mu_A \otimes \mu_N)(a, u) < \infty,$$

then by the above calculations, we must have $\|\omega\|_{\mathscr{G}'}^2 < \infty$, i.e. $\omega \in \Omega_{(2)}^*(X/\Gamma)$.
Using Corollary 5.1.9, we see that

$$\int_{A_{4/3} \times N_{1/2}} \left| f_\sigma(a, u) \right| \left| f_\tau(a, u) \right| \left| \langle (\epsilon_\sigma)_{(a,u)}, (\epsilon_\tau)_{(a,u)} \rangle_{\hbar}^{(a,u)} \right| \left| \rho(a) \right| d(\mu_A \otimes \mu_N)(a, u)$$

$$\leq \int_{A_{4/3} \times N_{1/2}} \left| f_\sigma(a, u) \right| \left| f_\tau(a, u) \right| c_n \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \prod_{i < j} \left(\frac{a_i}{a_j} \right) d(\mu_A \otimes \mu_N)(a, u).$$

Let $0 < \varepsilon < \frac{1}{2}$. We know from Lemma 5.2.5 that the f_σ have logarithmic growth in the sense of Definition 5.2.3 Equation (5.1) on some neighbourhood of $A_{4/3} \times N_{1/2}$. Then, by Lemma 5.2.7, we have

$$\int_{A_{4/3} \times N_{1/2}} \left| f_\sigma(a, u) \right| \left| f_\tau(a, u) \right| c_n \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \prod_{i < j} \left(\frac{a_i}{a_j} \right) d(\mu_A \otimes \mu_N)(a, u)$$

$$\leq c_\varepsilon c_n \int_{A_{4/3} \times N_{1/2}} \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^{2\varepsilon} \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \prod_{i < j} \left(\frac{a_i}{a_j} \right) d(\mu_A \otimes \mu_N)(a, u)$$

$$= \mu_N(N_{1/2}) c_n c_\varepsilon \int_{A_{4/3}} \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^{2\varepsilon} \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \prod_{i < j} \left(\frac{a_i}{a_j} \right) d\mu_A(a)$$

for some constant $c_\varepsilon > 0$, and where $\mu_N(N_{1/2}) < \infty$ as $N_{1/2}$ is compact. Note that

$$\prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^{2\varepsilon} \prod_{\sigma(i) \geq n} \left(\frac{a_{k_{\sigma(i)}}}{a_{l_{\sigma(i)}}} \right)^2 \prod_{i < j} \left(\frac{a_i}{a_j} \right) = \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^{2\varepsilon} \prod_{i=1}^{n-1} \left(\frac{a_{i+1}}{a_i} \right)^{2n_{\sigma,i}} \prod_{i=1}^{n-1} \left(\frac{a_i}{a_{i+1}} \right)^{i(n-i)}$$

$$= \prod_{i=1}^{n-1} \left(\frac{a_i}{a_{i+1}} \right)^{i(n-i) - 2n_{\sigma,i} - 2\varepsilon}$$

for $n_{\sigma,i} \in \mathbb{N}_0$ equal to the cardinality of the set

$$\{j = 1, \ldots, k \mid l_{\sigma(j)} \leq i \leq k_{\sigma(j)-1}\}.$$

Then $n_{\sigma,i} \leq k < \frac{n-1}{2}$ and therefore $\alpha_{i,\sigma,\varepsilon} := i(n-i) - 2n_{\sigma,i} - 2\varepsilon > 0$ for all $i = 1, \ldots, n-1$. We can now use the fact that the map $\theta^{-1} \colon A \to \mathbb{R}^{n-1}$ of the previous section, $\mathrm{pr}_i \circ \theta^{-1}(a) = \log t_i$, is a group isomorphism from A to the additive group $\mathbb{R}^{n-1}$ and the image measure $\nu = \theta_*^{-1} \mu_A$ is the Lebesgue measure:

$$\int_{A_{4/3}} \prod_{i=1}^{n-1} t_i(a)^{\alpha_{i,\sigma,\varepsilon}} \, d\mu_A(a) = \prod_{i=1}^{n-1} \int_{-\infty}^{\log(4/3)} \exp(\alpha_{i,\sigma,\varepsilon} x) \, dx < \infty.$$

This finishes our proof. $\square$

We will need a version of the Poincaré Lemma for logarithmic forms to prove the next proposition.

LEMMA 5.2.10 (Poincaré Lemma for Logarithmic Forms). Let $x \in \overline{X}/\Gamma$ with $V \subseteq \overline{X}/\Gamma$ a neighbourhood of x and let $k \geq 1$. If $\omega \in \Omega^k(V \cap X/\Gamma)$ has logarithmic growth near the boundary of V and $d\omega = 0$, then there is a neighbourhood $V' \subseteq V$ of x and a differential form $\omega' \in \Omega^{k-1}(V' \cap X/\Gamma)$ with logarithmic growth near the boundary of V' such that $\omega = d\omega'$ on V'.

Proof. If $x \in X/\Gamma$, this is just the standard Poincaré Lemma. So suppose $x \in \partial \overline{X}/\Gamma$. We may assume that ω has logarithmic growth on $V \cap (X/\Gamma)$ (we can just replace V by a neighbourhood of x, on which this holds true). Then there exist $\lambda > 4/3$, $\delta > 1/2$ such that $\pi^*\omega$ has logarithmic growth on $\pi^{-1}(V) \cap \mathfrak{S}_{(\lambda,\delta)}.C$.

For a given $y' \in \pi^{-1}(x) \cap \overline{\mathfrak{S}}_0.C$, let $\gamma \in C$ such that $y := y'.\gamma^{-1} \in \overline{\mathfrak{S}}_0$. Take a neighbourhood $U \subseteq \pi^{-1}(V).\gamma^{-1} \cap \overline{\mathfrak{S}}_{(\lambda,\delta)}$ of y for which $\pi|_U \colon U \to \pi(U)$ is a diffeomorphism. As $y \in \partial \overline{\mathfrak{S}}_0 \subseteq \partial X(B)_{[\mathrm{id}]_B}$, it must belong to the boundary component corresponding to some BUT P and id: Let P be a BUT defined by the partition $0 = l_0 < \cdots < l_k = n$ such that $y \in e(P)_{[\mathrm{id}]_P} \subseteq X(B)_{[\mathrm{id}]_B}$. Finally, let $a_y \in A$, $u_y \in N$ such that $a_y u_y \in N_P$ and $(0_P, a_y u_y)$ corresponds to y under the diffeomorphism $\overline{A}_P \times N_P \to X(P)_{[\mathrm{id}]_P}$, $(\overline{a}, u) \mapsto [\overline{a}, q_u]_{[\mathrm{id}]_P}$. Denote by $\overline{\varrho} \colon \overline{A} \times N \to X(B)_{[\mathrm{id}]_B}$, $(\overline{a}, u) \mapsto [\overline{a}, q_u]_{[\mathrm{id}]_B}$, the diffeomorphism extending ϱ. Then $\overline{\varrho}^{-1}(y) = (\overline{a}_y, u_y)$, where $\overline{a}_y = 0_P a_y$, so

$$
\begin{aligned}
(\overline{a}_y)_{l_j} &= 0 & &\text{for } j = 1, \dots, k-1, \\
0 < (\overline{a}_y)_i &= \frac{(a_y)_i}{(a_y)_{i+1}} < \lambda & &\text{for all } i \neq l_j, \\
|(u_y)_{ij}| &< \delta & &\text{for all } i < j.
\end{aligned}
$$

We consider a ball around the point $(\overline{a}_y, u_y)$: Let $t > 0$ such that

$$
U_t := \{(a, u) \in \overline{A} \times N \mid |a_i - (a_y)_i| < t, \ |u_{ij} - (u_y)_{ij}| < t\} \subseteq \overline{\varrho}^{-1}(U).
$$

Set $U_{(t)} := U_t \cap (A \times N)$. We pick a different set of coordinates on $A \times N$: Fix some $0 < t_0 < t$ and define

$$
\begin{aligned}
x_{l_j} \colon A &\longrightarrow \mathbb{R}, & x_{l_j}(a) &= \log t_{l_j}(a) - \log t_0 & &\text{for } j = 1, \dots, k-1, \\
x_i \colon A &\longrightarrow \mathbb{R}, & x_i(a) &= t_i(a) - \frac{(a_y)_i}{(a_y)_{i+1}} & &\text{for } i \neq l_j, \\
x_{ij} \colon N &\longrightarrow \mathbb{R} & x_{ij}(u) &= u_{ij} - (u_y)_{ij}, & &\text{for } i < j.
\end{aligned}
$$

Clearly, these form a coordinate system on $A \times N$. We see that $dx_i = d(\log t_i)$ are our chosen basis elements of TA^* for all $i = l_j$, $j = 1, \dots, k-1$, and for $i \neq l_j$, we see that $dx_i = dt_i = t_i d(\log t_i)$ and t_i is bounded on $\pi_A(U_{(t)})$. On $\pi_N(U_{(t)})$, $dx_{lk} = \sum_{i<j} f_{ij}\eta_{ij}$ for some $f_{ij} \in C^\infty(U_{(t)})$ and these f_{ij} must be bounded; conversely the η_{ij} can be written as a linear combination of the dx_{ij} with bounded coefficients on $\pi_N(U_{(t)})$. Pull the coordinates back to $A \times N$, keeping the same notation, fix an enumeration of the coordinates x_{ij} from $n, \dots, m$ and let $dx_\sigma = dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}$, $\sigma \in \Sigma_{k,m-k}$, denote the basis of $\Omega^k(U_{(t)})$ given by this coordinate system. For an arbitrary $\alpha \in \Omega^k(U_{(t)})$,

$$
\alpha = \sum f_\sigma \epsilon_\sigma = \sum g_\sigma dx_\sigma,
$$

and we see that the f_σ have logarithmic growth in the sense of Definition 5.2.3 Equation (5.1) if and only if the g_σ do as the dx_i which differ from the ϵ_i can be written as linear combinations of these with bounded coefficients and vice versa.

Let $\psi \colon A \times N \to \mathbb{R}^{n-1}_{>0} \times \mathbb{R}^{\frac{n(n-1)}{2}}$ denote the map $\psi(a, u) = ((x_i(a))_{i=1}^{n-1}, (x_i(u))_{i=n}^m)$; then

$$\psi(U_{(t)}) = \{x \in \mathbb{R}^m \mid x_{l_j} < \log(t/t_0) \text{ for all } j = 1, \ldots, k-1, \ |x_i| < t \text{ for all other } i\}.$$

In particular, $\psi(U_{(t)}) \subseteq \mathbb{R}^m$ is star-shaped with respect to the origin.

Let $z_1, \ldots, z_m$ denote the standard coordinates on $\mathbb{R}^m$ and $dz_1, \ldots, dz_m$ the induced basis of the cotangent space. Then for $\alpha = \sum g_\sigma dx_\sigma \in \Omega^k(U_t)$,

$$(\psi^{-1})^*\alpha = \sum g_\sigma \circ \psi^{-1} dz_\sigma = \sum h_\sigma dz_\sigma,$$

and the g_σ have logarithmic growth if and only if there exists a real polynomial p in $n-1$ variables such that

$$|h_\sigma(z)| = |g_\sigma \circ \psi^{-1}(z)| \leq |p(\check{z})| \qquad \text{for all } z \in \psi^{-1}(U_t) \subseteq \mathbb{R}^m,$$

where $\check{z} \in \mathbb{R}^{n-1}$ is given by $\check{z}_{l_j} = z_{l_j}$, $j = 1, \ldots, k-1$, and $\check{z}_i = \log(z_i)$ for all other i. For $i \neq l_j$, $\mathrm{pr}_i(\psi(U_{(t)}))$ is bounded, so the above condition is equivalent to the existence of a real polynomial p in $k-1$ variables such that

$$|h_\sigma(z)| \leq |p(z_{l_1}, \ldots, z_{l_{k-1}})| \qquad \text{for all } z \in \psi^{-1}(U_{(t)}) \subseteq \mathbb{R}^m. \tag{5.2}$$

We consider the standard homotopy operator on Euclidean space (see Appendix A.3)

$$H \colon \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m) \qquad \text{given by} \quad H(h_\sigma dz_\sigma) = \sum_{i=1}^k c_{\sigma^i} \, dz_{\sigma^i},$$

where $\sigma^i \in \Sigma_{k-1, m-k+1}$ is the permutation skipping $\sigma(i)$, i.e. $\sigma^i(j) = \sigma(j)$ for $j < i$ and $\sigma^i(j) = \sigma(j+1)$ for $j \geq i$. The coefficients $c_{\sigma^i} \in C^\infty(A \times N)$ are defined as

$$c_{\sigma^i}(z) = (-1)^{i-1} z_{\sigma(i)} \int_0^1 h_\sigma(zt) t^{k-1} \, dt.$$

On $\psi(U_{(t)})$, we know that $dH + Hd = \mathrm{id}$, so for $\alpha \in \Omega^k(\psi(U_{(t)}))$ satisfying $d\alpha = 0$, we have $\alpha = dH(\alpha)$. Thus, we need only show that H preserves logarithmic growth in the sense of Equation (5.2) above. Write

$$\alpha = \sum_{\sigma \in \Sigma_{k,m-k}} h_\sigma dz_\sigma, \quad \text{and} \quad H(\alpha) = \sum_{\tau \in \Sigma_{k-1, m-k+1}} c_\tau dz_\tau.$$

Then

$$c_\tau(z) = \sum_{\substack{\sigma \in \Sigma_{k,m-k} \\ \tau = \sigma^i}} c_{\sigma^i}(z) = \sum_{\substack{\sigma \in \Sigma_{k,m-k} \\ \tau = \sigma^i}} (-1)^{i-1} z_{\sigma(i)} \int_0^1 h_\sigma(zt) t^{k-1} \, dt.$$

If $|h_\sigma(z)| \leq |p(z_{l_1}, \ldots, z_{l_{k-1}})|$ for all $z \in \psi(U_{(t)})$, $\sigma \in \Sigma_{k,m-k}$ and some polynomial p, then as the integral of a polynomial is itself a polynomial and $|z_{\sigma(i)}|$ is bounded for all $i \neq l_j$, it is clear that there exists a real polynomial P in $k-1$ variables such that

$$|c_\tau(z)| \leq |P(z_{l_1}, \ldots, z_{l_{k-1}})| \qquad \text{for all } z \in \psi(U_{(t)}), \ \tau \in \Sigma_{k-1, m-k+1}.$$

To conclude, set $V' := \pi(\varrho(U_t).\gamma)$. We know that $(\psi^{-1})^*\varrho^*\gamma^*\pi^*\omega = dH((\psi^{-1})^*\varrho^*\gamma^*\pi^*\omega)$ on $\psi(U_{(t)})$, and $H((\psi^{-1})^*\varrho^*\gamma^*\pi^*\omega)$ has logarithmic growth in the sense of Equation (5.2). Then $\omega = d\omega'$ on $\pi(\varrho(U_{(t)}).\gamma)) = V' \cap X/\Gamma$ for

$$\omega' := \varphi(\gamma^{-1})^*(\varrho^{-1})^*\psi^*H((\psi^{-1})^*\varrho^*\gamma^*\pi^*\omega) \in \Omega^{k-1}(V' \cap X/\Gamma),$$

where $\varphi\colon \Omega^*(X)^\Gamma \to \Omega^*(X/\Gamma)$ is the inverse of π^*, and by the observations above, ω' has logarithmic growth near the boundary of V'. $\qquad\square$

For the proof of the following proposition we will use some basic sheaf theory (we refer to [21] and [8]). In fact, the proof is analogous to the proof of De Rham's Theorem using sheaf theory.

PROPOSITION 5.2.11. The inclusion $\Omega^*_{\log}(X/\Gamma) \hookrightarrow \Omega^*(X/\Gamma)$ is a quasi-isomorphism.

Proof. Let $\mathscr{U}(\overline{X}/\Gamma)$ denote the category of open sets on $\overline{X}/\Gamma$ (that is the category with objects open sets in $\overline{X}/\Gamma$ and morphisms the inclusions) and let $\mathbb{R}$-Vect denote category of real vector spaces. Then define $\mathscr{F}^k\colon \mathscr{U}(\overline{X}/\Gamma) \to \mathbb{R}$-Vect by $\mathscr{F}^k(\emptyset) = 0$ and for any non-empty open set $V \subseteq \overline{X}/\Gamma$, set

$$\mathscr{F}^k(V) := \{\omega \in \Omega^k(V \cap (X/\Gamma)) \mid \omega \text{ and } d\omega \text{ have logarithmic growth near the boundary of } V\}.$$

Clearly, an inclusion $V \subseteq U$ of open subsets of $\overline{X}/\Gamma$ induces a restriction map $\mathscr{F}^k(U) \to \mathscr{F}^k(V)$, and this correspondence preserves compositions and the identity, so $\mathscr{F}^k$ is a presheaf. It is easy to see that $\mathscr{F}^k$ satisfies the equaliser condition, so it is a sheaf.

Then $\mathscr{F}^*$ is a differential sheaf with the differential simply being exterior differentation of differential forms. Note that $\mathscr{F}^*(\overline{X}/\Gamma) = \Omega^*_{\log}(X/\Gamma)$ and $\mathscr{F}^*(X/\Gamma) = \Omega^*(X/\Gamma)$. We will prove that $\mathscr{F}^*$ is a fine resolution of the constant sheaf $\mathbb{R}_{\overline{X}/\Gamma}$; $\overline{X}/\Gamma$ is a compact Hausdorff space and thus paracompact, so the notion of a fine sheaf makes sense.

First, we prove that $\mathscr{F}^k$ is fine, i.e. that it possesses a "partition of unity": Let $f \in C^\infty(\overline{X}/\Gamma)$. For a given $\gamma \in C$, express $\varrho^*\gamma^*\pi^*df$ in terms of the basis $\{dt_i\}_{i=1}^{n-1} \cup \{\eta_{ij}\}_{i<j}$ on $A \times N$:

$$\varrho^*\gamma^*\pi^*df = \sum_{i=1}^{n-1} f_i dt_i + \sum_{i<j} f_{ij}\eta_{ij}, \quad f_i, f_{ij} \in C^\infty(A_{4/3} \times N_{1/2}).$$

As $\varrho^*\gamma^*\pi^*df$ extends smoothly to the compact set $\overline{A}_{4/3} \times N_{1/2}$ and the dt_i extend to a basis of the cotangent space on $\overline{A}$, the f_i and the f_{ij} must be bounded in absolute value. Now, expressing $\varrho^*\gamma^*\pi^*df$ in terms of our chosen basis $\{d(\log t_i)\}_{i=1}^{n-1} \cup \{\eta_{ij}\}_{i<j}$ as

$$\varrho^*\gamma^*\pi^*df = \sum_{i=1}^{n-1} f_i' \frac{dt_i}{t_i} + \sum_{i<j} f_{ij}'\eta_{ij}, \quad f_i', f_{ij}' \in C^\infty(A_{4/3} \times N_{1/2}),$$

we must have $f_i' = f_i t_i$ and $f_{ij}' = f_{ij}$ from which it follows that the f_i' and f_{ij}' are bounded in absolute value. In particular, both f and df have logarithmic growth at infinity. Then for any $V \in \mathscr{U}(\overline{X}/\Gamma)$, $\omega \in \mathscr{F}^*(V)$, both $f\omega$ and $d(f\omega) = df \wedge \omega + f d\omega$ belong to $\mathscr{F}(V)^*$ as the exterior product preserves logarithmic growth.

Now, let $\mathscr{U} = \{U_i\}$ be a locally finite open cover of $\overline{X}/\Gamma$ and let $\{\lambda_i\}$ be a partition of unity subordinate to $\mathscr{U}$. Define sheaf morphisms

$$\zeta_i\colon \mathscr{F}^* \to \mathscr{F}^* \quad \text{by} \quad \zeta_i(\omega) = \lambda_i\omega \quad \text{for } \omega \in \mathscr{F}^*(V),\, V \in \mathscr{U}(\overline{X}/\Gamma).$$

Then ζ_i is trivial on the complement of $\text{supp}\lambda_i$ for all i and $\sum_i \zeta_i = \text{id}_{\mathscr{F}^*}$. We conclude that $\mathscr{F}^*$ is indeed fine.

Now we prove that $\mathscr{F}^*$ is a resolution of $\mathbb{R}_{\overline{X}/\Gamma}$, where we take the inclusion as the augmentation map ϵ using that $\mathbb{R}_{\overline{X}/\Gamma}(U)$ consist of the locally constant functions on U.

We will use the fact that a sequence of sheaves over $\overline{X}/\Gamma$ with values in $\mathbb{R}$-Vect, $\mathscr{G}' \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{G}''$, is exact if and only if the sequence of stalks $\mathscr{G}'_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\psi_x} \mathscr{G}''_x$ is exact in $\mathbb{R}$-Vect for all $x \in \overline{X}/\Gamma$ (cf. [21, Theorem 5.85]). Recall that the stalk of a sheaf $\mathscr{G}$ at $x \in \overline{X}/\Gamma$ is the object $\mathscr{G}_x = \varinjlim_{x \in U} \mathscr{G}(U)$.

Let $x \in \overline{X}/\Gamma$ and consider the sequence

$$\cdots \xrightarrow{d_x} \mathscr{F}^{k-1}_x \xrightarrow{d_x} \mathscr{F}^k_x \xrightarrow{d_x} \mathscr{F}^{k+1}_x \xrightarrow{d_x} \cdots$$

Suppose $[\omega] \in \mathscr{F}^k_x$ is such that $0 = d_x[\omega] = [d\omega]$ and $\omega \in \mathscr{F}^k(V)$ for some neighbourhood V of x. Then there is a neighbourhood $x \in W \subseteq V$ such that $d\omega|_W = 0$. By the Poincaré Lemma for logarithmic forms (cf. Lemma 5.2.10), there is a neighbourhood $x \in U \subseteq W$ and a differential form $\omega' \in \mathscr{F}^{k-1}(U)$ such that $\omega|_U = d\omega'$. So $[\omega] = [\omega|_U] = [d\omega'] = d_x[\omega']$ in $\mathscr{F}^k_x$. As $d^2 = 0$, we conclude that the sequence is exact for $k > 0$.

Now, for $[f] \in \mathscr{F}^0_x$, $d_x[f] = 0$ if and only if f is locally constant at x. Hence, the sequence

$$0 \longrightarrow \mathbb{R}_{\overline{X}/\Gamma} \xrightarrow{\epsilon_x} \mathscr{F}^0_x \xrightarrow{d_x} \mathscr{F}^1_x \xrightarrow{d_x} \cdots$$

is exact at $\mathscr{F}^0_x$ and we conclude that $\mathscr{F}^*$ is a resolution of $\mathbb{R}_{\overline{X}/\Gamma}$.

Being a fine resolution of $\mathbb{R}_{\overline{X}/\Gamma}$, $\mathscr{F}^*(\overline{X}/\Gamma)$ calculates the sheaf cohomology of $\mathbb{R}_{\overline{X}/\Gamma}$ (cf. [21, Section 6.3]). So does the differential sheaf associated to the presheaf of singular cochains on $\overline{X}/\Gamma$ (cf. [8, I.7]), so by independence of the chosen resolution in computing derived functors, we get a canonical isomorphism $H_*(\Omega^*_{\log}(X/\Gamma)) \to H^*(\overline{X}/\Gamma)$, where $H^*(\overline{X}/\Gamma)$ denotes the singular cohomology of $\overline{X}/\Gamma$; on chain level, it is the map over $\text{id}_{\mathbb{R}_{\overline{X}/\Gamma}}$ given by the Comparison Theorem (cf. [21, Theorem 6.16]).

Let $\iota \colon X/\Gamma \to \overline{X}/\Gamma$ denote the inclusion. The de Rham sheaf $\Omega^*_{X/\Gamma}$ is a fine resolution of $\mathbb{R}_{X/\Gamma}$ and $\iota^*\mathscr{F}^* = \Omega^*_{X/\Gamma}$. The inverse image of the sheaf associated to the presheaf of singular cochains on $\overline{X}/\Gamma$ is exactly the sheaf associated to the presheaf of singular cochains on X/Γ. Finally, $\iota^*\mathbb{R}_{\overline{X}/\Gamma} = \mathbb{R}_{X/\Gamma}$. Consider the diagram below:

$$
\begin{array}{ccc}
H_*(\Omega^*_{\log}(X/\Gamma)) & \longrightarrow & H^*(\overline{X}/\Gamma) \\
\downarrow & & \downarrow \\
H_*(\Omega^*(X/\Gamma)) & \longrightarrow & H^*(X/\Gamma)
\end{array}
$$

The upper horizontal map is the isomorphism mentioned above, the vertical maps are the ones induced by the inclusion ι (on chain level, they are the maps over $\iota^* \colon \mathbb{R}_{\overline{X}/\Gamma} \to \mathbb{R}_{X/\Gamma}$ given by the Comparison Theorem) and the lower horizontal map is the de Rham isomorphism (on chain level, the map over $\text{id}_{\mathbb{R}_{X/\Gamma}}$ given by the Comparison Theorem). The diagram commutes as it commutes on chain level.

Now, ι is a homotopy equivalence, so the vertical map on the right is an isomorphism. Thus we conclude that $H_*(\Omega^*_{\log}(X/\Gamma)) \to H_*(\Omega^*(X/\Gamma))$ is an isomorphism, as desired. $\square$

We collect the above propositions in one theorem:

THEOREM 5.2.12. The subcomplex $\Omega^*_{\log}(X/\Gamma) \subseteq \Omega^*(X/\Gamma)$ satisfies:

   i). $j(\Omega^*(X)^G) \subseteq \Omega^*_{\log}(X/\Gamma)$, where $j \colon \Omega^*(X)^G \hookrightarrow \Omega^*(X)^\Gamma \cong \Omega^*(X/\Gamma)$.

   ii). The inclusion $\Omega^*_{\log}(X/\Gamma) \hookrightarrow \Omega^*(X/\Gamma)$ is a quasi-isomorphism.

   iii). $\Omega^k_{\log}(X/\Gamma) \subseteq \Omega^k_{(2)}(X/\Gamma)$ for all $k < \frac{n-1}{2}$.

6 | FINISHING OFF

In this final chapter, we show that the inclusion $\Omega^*(X)^{\mathrm{SL}_n(\mathbb{R})} \hookrightarrow \Omega^*(X)^{\mathrm{SL}_n(\mathbb{Z})}$ induces an isomorphism on cohomology in low degrees. To do this, we first review some results on harmonic and square integrable differential forms and briefly recap the definition of Lie algebra cohomology. Using a result of Borel and Garland, we prove a version of the Matsushima Vanishing Theorem applicable to our case which turns out to be last ingredient needed. We will see that our hard work in the previous chapter pays off: With the existence of the subcomplex of logarithmic forms and the Matsushima Vanishing Theorem, the fact that $\Omega^*(X)^{\mathrm{SL}_n(\mathbb{R})} \hookrightarrow \Omega^*(X)^{\mathrm{SL}_n(\mathbb{Z})}$ induces an isomorphism on cohomology in low degrees almost comes for free. Finally, we use this isomorphism to calculate the real cohomology of $\mathrm{SL}_n(\mathbb{Z})$ in low degrees using a clever little trick that enables us to consider the compact Lie group $\mathrm{SU}(n)$ instead of $\mathrm{SL}_n(\mathbb{R})$. Our calculations show that the cohomology stabilises and we can calculate the real cohomology of $\mathrm{SL}_\infty(\mathbb{Z})$.

6.1 PRELIMINARIES

To finish off, we need some results about harmonic and square integrable forms and we apply the theory of Lie algebra cohomology. Much of this is well-known and in any case it will take focus from the actual content of this project to introduce the theory in full formality and prove these results, so we opt to give a very brief overview, state the results needed and supply references.

HARMONIC FORMS AND SQUARE INTEGRABLE FORMS

We define the notion of a differential form being harmonic and review some important results.

Let M be a connected oriented complete Riemannian manifold of dimension n. Recall that $\Omega^*_{(2)}(M)$ denotes the space of square integrable forms on M with respect to the inner product $(-, -)_M$ induced by the metric tensor on M (cf. Section 1.3). Let $\| - \|_M$ denote the induced norm. We denote by $\langle -, - \rangle$ and $\| - \|$ the induced inner products and norms on $\Lambda^k T_x M^*$ for all $x \in M$, $k \in \mathbb{N}_0$.
Let $\star \colon \Omega^q(M) \to \Omega^{n-q}(M)$ denote the *Hodge star operator*: $\star \omega \in \Omega^{n-q}(M)$ is uniquely defined by the condition that

$$(\eta \wedge \star \omega)_x = (\eta_x, \omega_x)\mathrm{vol}_M \quad \text{for all } x \in M, \ \eta \in \Omega^q(M),$$

where the inner product on the right is the one on $\Lambda^q T_x M$ induced by the metric tensor on M and vol_M the induced volume element. The Hodge star $\star$ is invertible with inverse $\star^{-1} = (-1)^{q(n-q)}\star$.
Define the *codifferential* $\delta := (-1)^q \star^{-1} d\star \colon \Omega^q(M) \to \Omega^{q-1}(M)$ and the *Laplace-Beltrami operator* $\Delta := d\delta + \delta d \colon \Omega^q(M) \to \Omega^q(M)$.
By definition of $\star$, we have:

PROPOSITION 6.1.1. $(\alpha, \beta)_M = \int \alpha \wedge \star \beta$ for all $\alpha, \beta \in \Omega^q(M)$.

DEFINITION 6.1.2. A form $\omega \in \Omega^*(M)$ is said to be *harmonic* if $\Delta \omega = 0$. Let $\mathscr{H}^q_{(2)}(M)$ denote the space of square integrable harmonic forms.

As M is complete, we have the following result:

PROPOSITION 6.1.3 (Andreotti-Vesentini). A form $\omega \in \Omega^*_{(2)}(M)$ is harmonic if and only if $d\omega = \delta\omega = 0$.

Proof. See [12, Theorem 26]. $\qquad\square$

It follows that $(\mathscr{H}^*_{(2)}(M), d)$ is a chain complex with homology $\mathscr{H}^*_{(2)}(M)$.

PROPOSITION 6.1.4 (Kodaira). Any closed square integrable form $\omega \in \Omega^q_{(2)}(M)$ can be written uniquely as $\omega = \omega' + d\omega''$ for $\omega' \in \mathscr{H}^q_{(2)}(M)$ and $\omega'' \in \Omega^{q-1}(M)$.

Proof. See [12, Theorems 24 and 14]. $\qquad\square$

LEMMA 6.1.5. There exist compact sets $C_r \subseteq D_r$, smooth functions $\lambda_r \colon M \to [0,1]$, $r > 0$ and a constant $c > 0$ satisfying

   i) $M = \bigcup_{r>0} C_r$,

   ii) C_r contains the interior of C_s for $s < r$,

   iii) $\lambda_r(C_r) = 1$ and $\lambda_r(D_r^c) = 0$,

   iv) $\|d\lambda_r(x)\| \leq \frac{c}{r}$ for all $x \in M$.

Proof. See the proof of [12, Theorem 26] (see also [6, Lemma 1.2]). $\qquad\square$

The exterior derivative and the codifferential are adjoint in the following sense:

PROPOSITION 6.1.6. If $\alpha \in \Omega^q(M)$, $\beta \in \Omega^{q+1}(M)$ are such that $\alpha, \beta, d\alpha, \delta\beta$ are all square integrable, then

$$(d\alpha, \beta)_M = (\alpha, \delta\beta)_M.$$

Proof. Suppose first that one of the forms α or β has compact support. Then Stokes' Theorem and Proposition 6.1.1 yield

$$0 = \int d(\alpha \wedge \star\beta) = \int d\alpha \wedge \star\beta - \int (-1)^{q+1}\alpha \wedge d \star \beta$$

$$= (d\alpha, \beta) - \int \alpha \wedge \star(-1)^{q+1} \star^{-1} d \star \beta = (d\alpha, \beta) - \int \alpha \wedge \star\delta\beta = (d\alpha, \beta) - (\alpha, \delta\beta).$$

If neither α nor β has compact support, let λ_r, $r > 0$, as in Lemma 6.1.5. Then $\lambda_r\alpha$ is compactly supported for all $r > 0$, so by the above we have

$$(\lambda_r\alpha, \delta\beta) = (d(\lambda_r\alpha), \beta) = (d\lambda_r \wedge \alpha, \beta) + (\lambda_r d\alpha, \beta) \qquad \text{for all } r > 0.$$

As μ is inner regular and $M = \bigcup_{r\in\mathbb{R}} C_r$, $(\lambda_r\alpha, \delta\beta)$ and $(\lambda_r d\alpha, \beta)$ tend to $(\alpha, \delta\beta)$ and $(d\alpha, \beta)$, respectively, as $r \to \infty$. We need to show that $(d\lambda_r \wedge \alpha, \beta)$ tends to zero as $r \to \infty$.
Let $x \in M$ and take an orthonormal basis of T_xM^*; this gives rise to orthonormal basis of $\Lambda^k T_xM^*$ and the norm of an element in $\Lambda^k T_xM^*$ is the square root of the sum of the squares of its coefficients with respect to this basis. The coefficients of a wedge product are sums of products of a coefficient of the first factor and a coefficient of the second factor, multiplied by some sign, and each product occurs exactly once. Then the Cauchy-Schwarz inequality implies

that $\|\omega \wedge \omega'\| \leq \|\omega\| \|\omega'\|$ for all $\omega, \omega' \in \Lambda^* T_x M^*$. The Cauchy-Schwarz inequality for $(-, -)_M$ then yields

$$|(d\lambda_r \wedge \alpha, \beta)_M|^2 \leq \|\beta\|_M^2 \|d\lambda_r \wedge \alpha\|_M^2 = \|\beta\|_M^2 \int \|d\lambda_r(x) \wedge \alpha_x\|^2 d\mu$$

$$\leq \|\beta\|_M^2 \int \|d\lambda_r(x)\|^2 \|\alpha_x\|^2 d\mu \leq \frac{c^2}{r^2} \|\alpha\|_M^2 \|\beta\|_M^2$$

and we immediately get the desired. $\qquad\square$

PROPOSITION 6.1.7. If $\omega \in \mathscr{H}_{(2)}^*(M)$ satisfies $\omega = d\omega'$ for $\omega' \in \Omega_{(2)}^*(M)$, then $\omega = 0$.

Proof. Indeed, by Propositions 6.1.3 and 6.1.6, we have $(d\omega', d\omega')_M = (\omega', \delta\omega) = 0$ and thus $\omega = d\omega' = 0$. $\qquad\square$

Lastly, we need the following result of E. Cartan:

PROPOSITION 6.1.8. Any G-invariant differential form $\omega \in \Omega^*(X)^G$ is harmonic.

Proof. See for example [4, II §3]. $\qquad\square$

LIE ALGEBRA COHOMOLOGY

We briefly recap the definitions of Lie algebra cohomology and of relative Lie algebra cohomology for use in the following section. We refer to [4] for details.

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Given a real $\mathfrak{g}$-module V, the *Chevalley-Eilenberg chain complex* (C^*, d) is defined as

$$C^q = C^q(\mathfrak{g}, V) = \operatorname{Hom}_{\mathbb{R}}(\Lambda^q \mathfrak{g}, V)$$

with differential $d \colon C^q \longrightarrow C^{q+1}$ given by

$$df(x_0, \ldots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \ldots, \hat{x}_i, \ldots, x_q)$$

$$+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q)$$

for $f \in C^q$, $x_0, \ldots, x_q \in \mathfrak{g}$. The *Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in V is the homology of this complex, denoted by $H^q(\mathfrak{g}, V)$.

For a given $x \in \mathfrak{g}$, we have maps $i_x \colon C^q \to C^{q-1}$, $\theta_x \colon C^q \to C^q$ given by

$$(i_x f)(x_1, \ldots, x_{q-1}) = f(x, x_1, \ldots, x_{q-1}),$$

$$(\theta_x f)(x_1, \ldots, x_q) = x \cdot f(x_1, \ldots, x_q) + \sum_i f(x_1, \ldots, [x_i, x], \ldots, x_q)$$

for $f \in C^q$, $x_1, \ldots, x_q \in \mathfrak{g}$. The map i_x is the *interior product* and θ_x is related to the Lie derivative of differential forms.

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and let $C^q(\mathfrak{g}, \mathfrak{k}, V)$ denote the subspace of $C^q(\mathfrak{g}, V)$ consisting of the elements $f \in C^q(\mathfrak{g}, V)$ such that $i_x f = \theta_x f = 0$ for all $x \in \mathfrak{k}$. These subspaces are preserved by d and we define the *relative Lie algebra cohomology* of $\mathfrak{g}$ relative to $\mathfrak{k}$ with coefficients in V as the homology of the complex $(C^*(\mathfrak{g}, \mathfrak{k}, V), d)$. It is easy to show that

$$C^q(\mathfrak{g}, \mathfrak{k}, V) \cong \operatorname{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V)$$

with $\mathfrak{k}$ acting on $\Lambda^q(\mathfrak{g}/\mathfrak{k})$ by the adjoint representation, i.e.

$$x \cdot (\overline{x}_1 \wedge \cdots \wedge \overline{x}_q) = \sum_i \overline{x}_1 \wedge \cdots \wedge \overline{[x, x_i]} \wedge \cdots \wedge \overline{x}_q, \qquad x \in \mathfrak{k}, \, x_i \in \mathfrak{g},$$

where $\overline{y}$ denotes passing to the quotient. Then $C^q(\mathfrak{g}, \mathfrak{k}, V)$ is the subspace of $\mathrm{Hom}_{\mathbb{R}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V)$ consisting of the maps f satisfying

$$x \cdot f(\overline{x}_1, \ldots, \overline{x}_q) = \sum_i f(\overline{x}_1, \ldots, \overline{[x, x_i]}, \ldots, \overline{x}_q), \qquad x \in \mathfrak{k}, \, x_i \in \mathfrak{g}.$$

Assume now that $\mathfrak{g}$ is semisimple and let B denote the Killing form on $\mathfrak{g}$, i.e. $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $B(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$. Let (y_i) be a basis of $\mathfrak{g}$ and let (y_i') be the dual basis of $\mathfrak{g}$ with respect to B, that is, y_j' is such that $B(y_i, y_j') = \delta_{ij}$.

DEFINITION 6.1.9. The *Casimir element* is the element $C := \sum_i y_i y_i' \in \mathscr{U}(\mathfrak{g})$ in the universal enveloping algebra of $\mathfrak{g}$.

Remark 6.1.10. The Casimir element is independent of the choice of basis, and it belongs to the centre of $\mathscr{U}(\mathfrak{g})$.

Remark 6.1.11. There is an intricate relationship between the Casimir element and the Laplace-Beltrami operator, which is the reason why we will need the element in the following, but we do not need the explicit relation (see Kuga's Formula in for example [4, II Theorem 2.5]).

6.2   Matsushima's Vanishing Theorem

In this section, we prove a version of the Matsushima Vanishing Theorem. More specifically, we prove that in low enough degrees, a harmonic form on X/Γ is pulled back to a G-invariant form on X via the projection $X \to X/\Gamma$ for an appropriate torsion free subgroup $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$. We exploit a result by Borel and Garland which we state without proof. The fact that a harmonic form is pulled back to a G-invariant form is the last piece of the puzzle; with this we can finally prove that the inclusion $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^{\mathrm{SL}_n(\mathbb{Z})}$ induces an isomorphism on cohomology in low degrees.

Let $G = \mathrm{SL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ and $K = \mathrm{SO}(n)$ with Lie algebra $\mathfrak{k} = \mathfrak{so}(n)$. Let X be as in Section 2.1; we will identify X with $K \backslash G$. Finally, let Γ be a torsion free normal subgroup of $\mathrm{SL}_n(\mathbb{R})$ of finite index. Recall that G, G/Γ, X and X/Γ are complete orientable, Riemannian manifolds. We denote the projections as follows:

$$
\begin{array}{ccc}
G & \xrightarrow{\ \widetilde{\pi}\ } & G/\Gamma \\
\rho \downarrow & & \downarrow \widetilde{\rho} \\
X = K \backslash G & \xrightarrow[\ \pi\ ]{} & X/\Gamma
\end{array}
$$

A Result of Borel and Garland

After making the necessary introductions, we state the needed result of Borel and Garland.

First we define the G- and $\mathfrak{g}$-module structure on $C^\infty(G)$ and $C^\infty(G/\Gamma)$:
Left and right multiplication induce actions of G on $C^\infty(G)$: $g.f = f \circ L_{g^{-1}}$, $f.g = f \circ R_{g^{-1}}$ for $f \in C^\infty(G)$, $g \in G$. Likewise, left multiplication by G on G/Γ induces a left action of G on

$C^\infty(G/\Gamma)$. We identify $\mathfrak{g}$ with the right invariant vector fields on G and in turn with the induced vector fields on G/Γ, i.e. $x \in \mathfrak{g}$ is the vector field on G/Γ given by $x_{\widetilde{\pi}(g)} = D_g\widetilde{\pi} \circ D_e R_g(x)$. We denote by $\mathscr{L}_x \colon \Omega^*(-) \to \Omega^*(-)$ the Lie derivative with respect to the vector field $x \in \mathfrak{g}$, where the manifold in question is either G or G/Γ.

We consider $C^\infty(G)$ and $C^\infty(G/\Gamma)$ as G-modules with left multiplication as described above. As $\mathbb{R} \times G \to G$, $(t, g) \mapsto L_{\exp(tx)}(g)$, is the flow of the right invariant vector field $x \in \mathfrak{g}$, it is clear that the corresponding $\mathfrak{g}$-module structure on these vector spaces is given by the Lie derivative with respect to the right invariant vector fields.

Recall that the cotangent bundle of a Lie group is trivialisable via the map $T^*G \to \mathfrak{g}^* \times G$, $(T_gG)^* \ni x \mapsto (D_e R_g x, g)$, where R_g is right multiplication by G. This provides a trivialisation of the bundle $\Lambda^k T^*G$:

$$\varphi \colon \Lambda^k T^*G \to (\Lambda^k\mathfrak{g})^* \times G, \qquad (\Lambda^k T_gG)^* \ni f \mapsto (f \circ \Lambda^k D_e R_g, g).$$

This in turn allows us to make the following identification

$$\psi \colon \Omega^k(G) \longrightarrow \operatorname{Hom}_{\mathbb{R}}(\Lambda^k\mathfrak{g}, C^\infty(G)) = C^k(\mathfrak{g}, C^\infty(G)), \qquad \omega \mapsto (\varphi \circ \omega)^\sim,$$

where for $\eta \colon G \to (\Lambda^k\mathfrak{g})^* \times G$, we define $\widetilde{\eta} \colon \Lambda^k\mathfrak{g} \to C^\infty(G)$ as $\widetilde{\eta}(u)(g) = \operatorname{pr}_1(\eta(g))(u)$. More explicitly, ψ is given as follows:

$$(\psi\omega)(x_1, \ldots, x_k)(g) = \omega_g(D_e R_g(x_1), \ldots, D_e R_g(x_k)) \qquad \text{for all } x_i \in \mathfrak{g},\ g \in G,$$
$$(\psi^{-1}f)_g(v_1, \ldots, v_k) = f(D_g R_{g^{-1}}(v_1), \ldots, D_g R_{g^{-1}}(v_k))(g) \quad \text{for all } g \in G,\ v_i \in T_gG.$$

The invariant formula for the exterior derivative immediately shows that ψ is a chain isomorphism.

PROPOSITION 6.2.1. The composition $\psi \circ \rho^* \circ \pi^*$ induces a chain isomorphism

$$\Psi \colon \Omega^*(X/\Gamma) \longrightarrow C^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma)).$$

Proof. It is easy to see that the maps ι_x and θ_x on $C^*(\mathfrak{g}, C^\infty(G))$ correspond to ι_x and $\mathscr{L}_x$ on $\Omega^*(G)$. We know from Lemma 2.1.16 that a form $\omega \in \Omega^*(G)$ is the pullback of a differential form on X if and only if $\iota_x\omega = \mathscr{L}_x\omega = 0$ for all $x \in \mathfrak{k}$. We therefore immediately have a chain isomorphism

$$\psi \circ \rho^* \colon \Omega^*(X) \longrightarrow C^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G)).$$

Then it simply remains to show that $\psi \circ \rho^*$ restricted to $\Omega^*(X)^\Gamma$ induces the desired chain isomorphism. Let $\omega \in \Omega^k(G)$. For all $\gamma \in \Gamma$, we have

$$((\psi\omega)(x_1, \ldots, x_k).\gamma^{-1})(g) = (\psi\omega)(x_1, \ldots, x_k)(g\gamma) = \omega_{g\gamma}(D_e R_{g\gamma}(x_1), \ldots, D_e R_{g\gamma}(x_k))$$
$$= (R_\gamma^*\omega)_g(D_e R_g(x_1), \ldots, D_e R_g(x_k)) = (\psi R_\gamma^*\omega)(x_1, \ldots, x_k)(g)$$

for all $x_1, \ldots, x_k \in \mathfrak{g}$, $g \in G$. Hence, $\omega \in \Omega^k(G)^\Gamma$ if and only if $\psi\omega \in C^k(\mathfrak{g}, \mathfrak{k}, C^\infty(G)^\Gamma)$, where $\Omega^*(G)^\Gamma$, respectively, $C^\infty(G)^\Gamma$ denotes the right invariant differential forms, respectively, smooth maps on G. If $\omega = \rho^*\eta$, then as ρ^* is injective and ρ is equivariant, we see that $\omega \in \Omega^*(G)^\Gamma$ if and only if $\eta \in \Omega^*(X)^\Gamma$. It follows that $\psi \circ \rho^* \circ \pi^*$ is a chain isomorphism

$$\Omega^*(X/\Gamma) \longrightarrow C^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G)^\Gamma).$$

Finally, as $\widetilde{\pi} \colon G \to G/\Gamma$ is a local diffeomorphism, it is easy to see that the induced map $\widetilde{\pi}^* \colon C^\infty(G/\Gamma) \to C^\infty(G)$ yields an isomorphism $C^\infty(G/\Gamma) \cong C^\infty(G)^\Gamma$ which respects the G-module structure. $\qquad\square$

Remark 6.2.2. Explicitly, the isomorphism $\Psi\colon \Omega^*(X/\Gamma) \longrightarrow C^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma))$ is given by

$$(\Psi\omega)(x_1, \ldots, x_k)(\widetilde{\pi}(g)) = \omega_{\pi\rho(g)}(D_g\pi\rho \circ D_e R_g(x_1), \ldots, D_g\pi\rho \circ D_e R_g(x_k))$$

for $\omega \in \Omega^*(X/\Gamma)$, $x_1, \ldots, x_k \in \mathfrak{g}$, $g \in G$.

PROPOSITION 6.2.3. *Evaluating at the identity coset yields a chain isomorphism*

$$\Omega^*(X)^G \longrightarrow C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}),$$

where $\mathfrak{g}$ acts trivially on $\mathbb{R}$.

Proof. This is an immediate consequence of Lemma 2.1.16 and the invariant formula for the exterior derivative. $\qquad\square$

We exploit the following result due to Borel and Garland without proof (see [2] for details, in particular Proposition 5.6):

THEOREM 6.2.4 (Borel-Garland). *There exists a unitary $\mathfrak{g}$-module V satisfying*

  i) *V is a submodule of $C^\infty(G/\Gamma)$,*

 ii) *The Casimir element acts trivially on V,*

iii) *Ψ restricts to an isomorphism $\mathscr{H}^*_{(2)}(X/\Gamma) \to C^*(\mathfrak{g}, \mathfrak{k}, V)$.*

Remark 6.2.5. Being a unitary $\mathfrak{g}$-module means that V is an inner product space with inner product $(-, -)$ satisfying

$$(x \cdot v, w) + (v, x \cdot w) = 0, \qquad \text{for all } v, w \in V, \ x \in \mathfrak{g}.$$

Remark 6.2.6. V is a subspace of the separable Hilbert space $L^2(G/\Gamma)$ on which G acts by left multiplication. Therefore the completion of V has a countable orthonormal basis — we will use this in the following section.

MATSUSHIMA'S VANISHING THEOREM

We will show that π^* yields an isomorphism $\mathscr{H}^*_{(2)}(X/\Gamma) \longrightarrow \Omega^*(X)^G$ in low enough degrees. We follow the proof of Matsushima's Vanishing Theorem as given in [4] (Chapter II).

First, we need to set the scene. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ (cf. 1.1.2), i.e. $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ consists of the symmetric matrices with trace 0. Then the Killing form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $B(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y))$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Moreover

$$B(\mathfrak{k}, \mathfrak{p}) = 0, \qquad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}. \tag{6.1}$$

Set $m := \frac{1}{2}n(n+1) - 1$. Now, fix orthogonal bases $(x_i)_{1 \le i \le m}$ of $\mathfrak{p}$ and $(x_a)_{m+1 \le a \le n^2-1}$ of $\mathfrak{k}$ such that

$$B(x_i, x_i) = 1, \quad 1 \le i \le m, \qquad B(x_a, x_a) = -1, \quad m+1 \le a \le n^2 - 1.$$

From now on we always let i, j, k, l run from 1 to m and a, b run from $m+1$ to $n^2 - 1$. In terms of this basis, the Casimir element in $U(\mathfrak{g})$ is $C = \sum_i x_i^2 - \sum_a x_a^2$ (cf. Definition 6.1.9).

Let (ω^i) denote bases of $\mathfrak{p}^*$ dual to (x_i). As $\mathfrak{g}/\mathfrak{k}$ is canonically isomorphic to $\mathfrak{p}$, we may identify $C^q(\mathfrak{g}, \mathfrak{k}, V)$ with $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^q \mathfrak{p}, V)$, where $\mathfrak{k}$ acts on $\mathfrak{p}$ by the adjoint representation; in other words, we identify $C^q(\mathfrak{g}, \mathfrak{k}, V)$ with the space of $\mathbb{R}$-linear maps $f\colon \Lambda^q \mathfrak{p} \to V$ satisfying

$$x \cdot f(y_1, \dots, y_q) = \sum_{u=1}^{q} f(y_1, \dots, [x, y_u], \dots, y_q) \quad \text{for all } x \in \mathfrak{k}, \ y_i \in \mathfrak{p}. \tag{6.2}$$

Set

$$\omega^{j_1 \cdots j_q} := \omega^{j_1} \wedge \cdots \wedge \omega^{j_q} \qquad \text{for } 1 \le j_1, \dots, j_q \le m.$$

If the set $I = \{j_1, \dots, j_q\}$ is ordered, we write $\omega^I := \omega^{j_1 \cdots j_q}$. Then any element $\eta \in C^q(\mathfrak{g}, \mathfrak{k}, V)$ can be written uniquely as

$$\eta = \sum_I \eta_I \, \omega^I \qquad \text{with } \eta_I = \eta_{j_1 \cdots j_q} = \eta(x_{j_1}, \dots, x_{j_q}) \in V, \tag{6.3}$$

where $I = \{j_1, \dots, j_q\}$ runs through ordered subsets of $\{1, \dots, m\}$. We also define $\eta_{j_1 \cdots j_q}$ as above for unordered sets $\{j_1, \dots, j_q\} \subseteq \{1, \dots m\}$. Note that for any ordered $I = \{j_1, \dots, j_q\}$,

$$\omega^I = \mathrm{sign}\,\sigma \, \omega^{j_{\sigma(1)} \cdots j_{\sigma(q)}}, \qquad \eta_I = \mathrm{sign}\,\sigma \, \eta_{j_{\sigma(1)} \cdots j_{\sigma(q)}} \qquad \text{for all } \sigma \in \Sigma_q.$$

Define a symmetric bilinear form $L\colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ by

$$L(x, y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{p}}(x) \circ \mathrm{ad}_{\mathfrak{p}}(y)), \qquad x, y \in \mathfrak{k},$$

where $\mathrm{ad}_{\mathfrak{p}}(x) = \mathrm{ad}(x)|_{\mathfrak{p}} = [x, -] \colon \mathfrak{p} \to \mathfrak{p}$, $x \in \mathfrak{k}$. Let $B_{\mathfrak{k}}$ denote the Killing form on $\mathfrak{k}$; by 6.1, we see that

$$\mathrm{ad}(x) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{p}}(x) & 0 \\ 0 & \mathrm{ad}_{\mathfrak{k}}(x) \end{pmatrix} \qquad \text{for all } x \in \mathfrak{k}.$$

Hence, $B|_{\mathfrak{k}} = B_{\mathfrak{k}} + L$.

LEMMA 6.2.7. L is negative definite.

Proof. Let $x \in \mathfrak{k}$. Being skew-symmetric, x has purely imaginary eigenvalues; moreover it is diagonalisable by a unitary matrix. In particular, there exists a set of n linearly independent eigenvectors for x, say $u_1, \dots, u_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Consider the Kronecker products of these: $u_{kl} = u_k \otimes u_l^t$ with entry i, j equal to the product of the i'th entry of u_k and the j'th entry of u_l. By direct calculations, these matrices are seen to satisfy

$$x u_{kl} - u_{kl} x = (\lambda_k + \lambda_l) u_{kl} \qquad \text{for all } 1 \le k, l \le n,$$

and thus we conclude that $\mathrm{ad}(x)$ has purely imaginary eigenvalues. Applying $\mathrm{ad}_{\mathfrak{p}}(x)$ to the elements $E_{ij} - E_{ji} \in \mathfrak{p}$, we see that if $\mathrm{ad}_{\mathfrak{p}}(x)$ is identically zero, then x is a diagonal matrix and, being skew-symmetric, it must be zero. Therefore for any $x \neq 0$, we must have $\mathrm{ad}_{\mathfrak{p}}(x) \neq 0$, and thus $\mathrm{ad}_{\mathfrak{p}}$ has a non-zero eigenvalue, so we conclude that

$$L(x, x) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{p}}(x)^2) \le 0 \quad \text{for all } x \in \mathfrak{k}$$

with equality if and only if $x = 0$. $\square$

In view of 6.1, we can write

$$[x_i, x_j] = \sum_a c_{i,j}^a x_a, \qquad [x_a, x_i] = \sum_j c_{a,i}^j x_j, \quad \text{and} \quad [x_i, x_a] = \sum_j c_{i,a}^j x_j$$

for uniquely given constants $c_{i,j}^a$, $c_{a,i}^j$, $c_{i,a}^j \in \mathbb{R}$.

LEMMA 6.2.8. The constants above satisfy

$$c_{i,j}^a = -c_{j,i}^a, \qquad c_{a,i}^j = -c_{i,a}^j, \quad \text{and} \quad c_{i,j}^a = c_{a,j}^i.$$

Proof. The first two identities are clear. The third follows from invariance of the Killing form:

$$0 = B([x_i, x_j], x_a) + B(x_i, [x_a, x_j]) = -c_{i,j}^a + c_{a,j}^i.$$

$\square$

LEMMA 6.2.9. $L(x_a, x_b) = -\sum_{i,j} c_{i,j}^a c_{i,j}^b.$

Proof. First, we note that as $\mathrm{ad}_{\mathfrak{p}}(x_a)(x_j) = [x_a, x_j] = \sum_i c_{a,j}^i x_i$, the matrix of $\mathrm{ad}_{\mathfrak{p}}(x)$ in terms of the basis (x_i) of $\mathfrak{p}$ is the $(m \times m)$-matrix with entry i, j equal to $c_{a,j}^i$. Therefore

$$L(x_a, x_b) = \sum_{i,j} c_{a,j}^i c_{b,i}^j = \sum_{i,j} c_{i,j}^a c_{j,i}^b = -\sum_{i,j} c_{i,j}^a c_{i,j}^b.$$

$\square$

We assume from now on that the basis (x_a) of $\mathfrak{k}$ is orthogonal with respect to L (this is possible as $B_{\mathfrak{k}}$ and L can be simultaneously diagonalised).
Define constants

$$R_{ijkl} := -\sum_a c_{i,j}^a c_{k,l}^a,$$

and set

$$A := \min\{-L(x, x) \mid x \in \mathfrak{k},\ B(x, x) = -1\}.$$

Note that $0 < A < 1$ as both L and $B_{\mathfrak{k}}$ are negative definite. For $q \in \mathbb{N}$, define a real quadratic form F^q in m^2 variables by

$$F^q(\eta) = \frac{A}{2q} \sum_{i,j} \eta_{ij}^2 + \sum_{i,j,k,l} R_{ijkl}\, \eta_{il}\, \eta_{jk} \qquad \text{for } \eta = (\eta_{ij})_{i,j}.$$

It is immediate that if $p \le q$ and F^q is positive definite, then so is F^p.

DEFINITION 6.2.10. The *Matsushima constant* of $\mathfrak{g}$ is the number

$$m(\mathfrak{g}) := \max(\{0\} \cup \{q \mid F^q \text{ is positive definite}\}).$$

Remark 6.2.11. In [6], the Matsushima constant is denoted by $m(G)$.

We refer to [16, Theorem 4.1] for the calculation of Matsushima's constant:

PROPOSITION 6.2.12. $m(\mathfrak{g}) \ge \frac{n+1}{4}$.

We first prove a representation theoretic version of the Matsushima Vanishing Theorem which we can then apply to our case using Theorem 6.2.4. The proof follows that of [4] which in turn follows the structure of Matsushima's own version in [19].

THEOREM 6.2.13 (Matsushima Vanishing I). Let V be a unitary $\mathfrak{g}$-module on which the Casimir element acts trivially and such that its completion is separable. If $0 < q \leq \frac{n-1}{4}$, then any element $\eta = \sum_I \eta_I \, \omega^I \in C^q(\mathfrak{g}, \mathfrak{k}, V)$ satisfies $\eta_I \in V^{\mathfrak{g}}$ for all I, where we write η as in 6.3.

Proof. As in the above, i, j, k, l run from 1 to m, while a, b run through $m+1$ to $n^2 - 1$, and I, J run through all ordered subsets of $\{1, \ldots, m\}$ of q elements. In addition, we let j_i run through 1 to m for fixed i, and u run from 1 to q. It should in all cases be clear from the context what the indices are, this is just to make perfectly clear the conventions.

Let $(-, -)$ denote the inner product on V and $(e_r)_{r \in \mathbb{N}}$ the orthonormal basis of the completion of V. Recall that being unitary means that

$$(x \cdot v, w) + (v, x \cdot w) = 0 \qquad \text{for all } x \in \mathfrak{g}, \; v, w \in V.$$

Let $\eta = \sum_I \eta_I \, \omega^I \in C^q(\mathfrak{g}, \mathfrak{k}, V)$. To show that $\eta_I \in V^{\mathfrak{g}}$ for all I, we have to show that $x_i \cdot \eta_I = x_a \cdot \eta_I = 0$ for all i and a. As the Casimir element $C = \sum_i x_i^2 - \sum_a x_a^2 \in \mathscr{U}(\mathfrak{g})$ acts trivially on V, we have

$$0 = (C\eta_I, \eta_I) = \sum_i (x_i^2 \cdot \eta_I, \eta_I) - \sum_a (x_a^2 \cdot \eta_I, \eta_I) = \sum_a \|x_a \eta_I\|^2 - \sum_i \|x_i \eta_I\|^2. \tag{6.4}$$

It therefore suffices to show that $x_i \cdot \eta_I = 0$ for all i — this also follows readily from the fact that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

We define a real number, $\Phi(\eta)$, as follows

$$\Phi(\eta) := \frac{(q-1)!}{2} \sum_{i,j,I} \|[x_i, x_j] \cdot \eta_I\|^2 = \frac{1}{2q} \sum_{\substack{i,j \\ j_1, \ldots, j_q}} \|[x_i, x_j] \cdot \eta_{j_1 \cdots j_q}\|^2.$$

The second equality follows from the fact that $\|[x_i, x_j] \cdot \eta_I\| = \|[x_i, x_j] \cdot \eta_{j_{\sigma(1)} \cdots j_{\sigma(q)}}\|$ for all $\sigma \in \Sigma_q$, $I = \{j_1, \ldots, j_q\}$, so there are $q!$ occurances of $\|[x_i, x_j] \cdot \eta_I\|^2$ in the last sum.

We will write $\Phi(\eta)$ in two different ways in order to exploit the fact that the above defined quadratic form F^q is positive definite as $q \leq m(\mathfrak{g})$. First, using that $[x_i, x_j] = \sum_a c_{i,j}^a x_a$ and $L(x_a, x_b) = -\sum_{i,j} c_{i,j}^a c_{i,j}^b$, we get

$$\Phi(\eta) = \frac{(q-1)!}{2} \sum_{i,j,a,b,I} c_{i,j}^a c_{i,j}^b \, (x_a \cdot \eta_I, x_b \cdot \eta_I) = -\frac{(q-1)!}{2} \sum_{a,b,I} L(x_a, x_b) \, (x_a \cdot \eta_I, x_b \cdot \eta_I)$$

$$= -\frac{(q-1)!}{2} \sum_{a,I} L(x_a, x_a) \, \|x_a \cdot \eta_I\|^2,$$

where we use that the x_a are orthogonal with respect to L for the final equality. Then using the definition of A and 6.4, we get

$$\Phi(\eta) \geq \frac{A(q-1)!}{2} \sum_{a,I} \|x_a \cdot \eta_I\|^2 = \frac{A(q-1)!}{2} \sum_{i,I} \|x_i \cdot \eta_I\|^2 = \frac{A}{2q} \sum_{i, j_1, \ldots, j_q} \|x_i \cdot \eta_{j_1 \cdots j_q}\|^2. \tag{6.5}$$

Now, we use the equality $[x_i, x_j] = \sum_a c_{i,j}^a x_a$ on only one entry of the inner product in $\Phi(\eta)$ to get

$$
\begin{aligned}
\Phi(\eta) &= \frac{1}{2q} \sum_{\substack{i,j,a \\ j_1,\dots,j_q}} c_{i,j}^a \left(x_a \cdot \eta_{j_1 \cdots j_q}, [x_i, x_j] \cdot \eta_{j_1 \cdots j_q} \right) \\
&= \frac{1}{2q} \sum_{\substack{i,j,a \\ j_1,\dots,j_q}} \left(c_{i,j}^a \left(x_a \cdot \eta_{j_1 \cdots j_q}, x_i \cdot (x_j \cdot \eta_{j_1 \cdots j_q}) \right) + c_{j,i}^a \left(x_a \cdot \eta_{j_1 \cdots j_q}, x_j \cdot (x_i \cdot \eta_{j_1 \cdots j_q}) \right) \right) \\
&= \frac{1}{q} \sum_{\substack{i,j,a \\ j_1,\dots,j_q}} c_{i,j}^a \left(x_a \cdot \eta_{j_1 \cdots j_q}, x_i \cdot (x_j \cdot \eta_{j_1 \cdots j_q}) \right),
\end{aligned}
$$

where we use that $c_{i,j}^a = -c_{j,i}^a$. Invoking the fact that $\eta \colon \Lambda^q \mathfrak{p} \to V$ is a $\mathfrak{k}$-linear map (cf. 6.2), we see that

$$
\begin{aligned}
x_a \cdot \eta_{j_1 \cdots j_q} = x_a \cdot \eta(x_{j_1}, \dots, x_{j_q}) &= \sum_u \eta(x_{j_1}, \dots, [x_a, x_{j_u}], \dots, x_{j_q}) \\
&= \sum_{k,u} c_{a,j_u}^k \, \eta(x_{j_1}, \dots, x_k, \dots, x_{j_q}) \\
&= \sum_{k,u} (-1)^{u-1} c_{a,j_u}^k \, \eta(x_k, x_{j_1}, \dots, \hat{x}_{j_u}, \dots x_{j_q}),
\end{aligned}
$$

where in the second sum, x_k is in the u'th place. Then

$$
\begin{aligned}
q\Phi(\eta) &= \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} (-1)^{u-1} \left(\sum_a c_{i,j}^a c_{a,j_u}^k \right) \left(\eta_{k j_1 \cdots \hat{j}_u \cdots j_q}, x_i \cdot (x_j \cdot \eta_{j_1 \cdots j_q}) \right) \\
&= \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} (-1)^{u-1} R_{ijkj_u} \left(x_i \cdot \eta_{k j_1 \cdots \hat{j}_u \cdots j_q}, x_j \cdot \eta_{j_1 \cdots j_q} \right) \\
&= \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} R_{ijkj_u} \left(x_i \cdot \eta_{k j_1 \cdots \hat{j}_u \cdots j_q}, x_j \cdot \eta_{j_u j_1 \cdots \hat{j}_u \cdots j_q} \right),
\end{aligned}
$$

where we use the definition of the constants R_{ijkl} and that V is a unitary module. Note that for fixed k, l, $1 \le i_2, \dots, i_q \le m$, we have

$$
\eta_{k i_2 \cdots i_q} = \eta_{k j_1 \cdots \hat{j}_u \cdots j_q}, \quad \text{and} \quad \eta_{l i_2 \cdots i_q} = \eta_{j_u j_1 \cdots \hat{j}_u \cdots j_q},
$$

for all u and $j_1, \dots, j_q$ satisfying

$$
j_r = \begin{cases} i_{r+1} & r = 1, \dots, u-1, \\ l & r = u, \\ i_r & r = u+1, \dots, q. \end{cases}
$$

There are exactly q such tuples in the above sum, one for every $u = 1, \dots, q$. Hence,

$$
q\Phi(\eta) = q \sum_{\substack{i,j,k,l \\ j_2,\dots,j_q}} R_{ijkl} \left(x_i \cdot \eta_{k j_2 \cdots j_q}, x_j \cdot \eta_{l j_2 \cdots j_q} \right).
$$

Then, as $R_{ijkl} = -R_{ijlk}$, we get

$$\Phi(\eta) = -\sum_{\substack{i,j,k,l \\ j_2,\dots,j_q}} R_{ijkl} \left(x_i \cdot \eta_{lj_2\cdots j_q}, x_j \cdot \eta_{kj_2\cdots j_q} \right). \tag{6.6}$$

Combining 6.5 and 6.6, we obtain

$$\sum_{j_2,\dots,j_q} \left(\frac{A}{2q} \sum_{i,j} \|x_i \cdot \eta_{jj_2\cdots j_q}\|^2 + \sum_{i,j,k,l} R_{ijkl} \left(x_i \cdot \eta_{lj_2\cdots j_q}, x_j \cdot \eta_{kj_2\cdots j_q} \right) \right) \le \Phi(\eta) - \Phi(\eta) = 0. \tag{6.7}$$

We now use the orthonormal basis, $(e_r)_{r\in\mathbb{N}}$, of the completion of V. Each term in the above sum can then be written as

$$\frac{A}{2q} \sum_{i,j} \sum_{r\in\mathbb{N}} (x_i \cdot \eta_{jj_2\cdots j_q}, e_r)^2 + \sum_{i,j,k,l} R_{ijkl} \sum_{r\in\mathbb{N}} (x_i \cdot \eta_{lj_2\cdots j_q}, e_r) \cdot (x_j \cdot \eta_{kj_2\cdots j_q}, e_r)$$
$$= \sum_{r\in\mathbb{N}} \left(\frac{A}{2q} \sum_{i,j} (x_i \cdot \eta_{jj_2\cdots j_q}, e_r)^2 + \sum_{i,j,k,l} R_{ijkl} (x_i \cdot \eta_{lj_2\cdots j_q}, e_r) \cdot (x_j \cdot \eta_{kj_2\cdots j_q}, e_r) \right).$$

For all $r \in \mathbb{N}$, $j_2,\dots,j_q$, define

$$\xi^{r,j_2,\dots,j_q} := (\xi_{ij}^{r,j_2,\dots,j_q})_{i,j}, \qquad \text{where} \quad \xi_{ij}^{r,j_2,\dots,j_q} = (x_i \cdot \eta_{jj_2\cdots j_q}, e_r).$$

Then 6.7 reads

$$\sum_{j_2,\dots,j_q} \sum_{r\in\mathbb{N}} F^q(\xi^{r,j_2,\dots,j_q}) \le 0.$$

As $q \le \frac{n+1}{4} \le m(\mathfrak{g})$, F^q is positive definite, so the above inequality implies that $\xi^{r,j_2,\dots,j_q} = 0$ for all r and $j_2,\dots,j_q$. Then $x_i \cdot \eta_{jj_2\cdots j_q} = 0$ for all $i, j, j_2,\dots,j_q$, which exactly says that $x_i \cdot \eta_I = 0$ for all i and I, so $\eta_I \in V^{\mathfrak{g}}$ for all I. $\qquad\square$

The property we need is a corollary of the above:

COROLLARY 6.2.14 (Matsushima Vanishing II). If $\omega \in \mathscr{H}_{(2)}^q(X/\Gamma)$ and $q \le \frac{n+1}{4}$, then the pullback via π is G-invariant, i.e. $\pi^*\omega \in \Omega^q(X)^G$.

Proof. Let V be as in 6.2.4: This is a unitary $\mathfrak{g}$-module, the Casimir element acts trivially on it, and its completion is separable, so we can apply the above theorem to $C^*(\mathfrak{g}, \mathfrak{k}, V)$.
Let $\omega \in \mathscr{H}_{(2)}^q(X/\Gamma)$. Then by Theorem 6.2.4, $\Psi\omega \in C^q(\mathfrak{g}, \mathfrak{k}, V)$; write $\Psi\omega = \sum_I (\Psi\omega)_I \omega^I$ as in Equation (6.3). If $q = 0$, then $d(\Psi\omega) = \Psi(d\omega) = 0$ implying that $(\Psi\omega)_\emptyset = \Psi\omega \in V^{\mathfrak{g}}$. If $q > 0$, then Theorem 6.2.13 implies that $(\Psi\omega)_I \in V^{\mathfrak{g}}$ for all I. Using that elements are $\mathfrak{g}$-invariant if and only if they are G-invariant, i.e. $V^{\mathfrak{g}} = V^G$, we see that the coefficients are constant:

$$(\Psi\omega)_I = (\Psi\omega)_I \circ L_g \qquad \text{for all } g \in G.$$

Now, for any $g \in G$, we have

$$(\Psi\omega)_I(\widetilde{\pi}(g)) = (\rho^*\pi^*\omega)_g(D_e R_g(x_{j_1}), \dots, D_e R_g(x_{j_q})) = (R_g^*\rho^*\pi^*\omega)_e(x_{j_1}, \dots, x_{j_q}),$$

so $g \mapsto (R_g^*\rho^*\pi^*\omega)_e(x_{j_1}, \dots, x_{j_q})$ is constant on G for any fixed $I = \{j_1,\dots,j_q\}$. For fixed $g \in G$, we have

$$(R_g^*\rho^*\pi^*\omega)_h(D_e R_h(x_{j_1}), \dots, D_e R_h(x_{j_q})) = (R_{hg}^*\rho^*\pi^*\omega)_e(x_{j_1}, \dots, x_{j_q})$$
$$= (R_h^*\rho^*\pi^*\omega)_e(x_{j_1}, \dots, x_{j_q}) = (\rho^*\pi^*\omega)_h(D_e R_h(x_{j_1}), \dots, D_e R_h(x_{j_q}))$$

for all $h \in G$, $I = \{j_1, \ldots, j_q\}$. Moreover, for all $h \in G$ and $\{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n^2 - 1\}$, we have

$$(\rho^* \pi^* \omega)_h(D_e R_h(x_{i_1}), \ldots, D_e R_h(x_{i_q})) = 0,$$

if $i_u \geq m + 1$ for some $1 \leq u \leq q$ as $\iota_y(\rho^* \pi^* \omega) = 0$ for all $y \in \mathfrak{k}$ (cf. Lemma 2.1.16). Then, as $\{D_e R_h(x_i)\}_{i=1}^m \cup \{D_e R_h(x_a)\}_{a=m+1}^{n^2-1}$ forms a basis of $T_h G$, and a differential form is completely determined by its values on ordered subsets of distinct basis vectors at a given point, we conclude that $R_g^* \rho^* \pi^* \omega = \rho^* \pi^* \omega$. From ρ^* being injective and ρ being equivariant, we finally deduce that $\pi^* \omega \in \Omega^q(X)^G$. $\hfill \square$

COROLLARY 6.2.15. The map $\pi^* \colon \mathscr{H}_{(2)}^*(X/\Gamma) \to \Omega^*(X)^G$ exists in degrees $* \leq \frac{n+1}{4}$ and for $n \neq 3$ it is always an isomorphism. For $n = 3$, it is an isomorphism in degrees $* < \frac{n+1}{4}$.

Proof. Existence is a consequence of the above Corollary 6.2.14 and injectivity of π^* is immediate. Let $\omega \in \Omega^q(X)^G$ for some $q \leq \frac{n+1}{4}$. We know that ω is a harmonic form on X (cf. Proposition 6.1.8). Harmonic being a local condition, it is immediate that $\varphi\omega = (\pi^*)^{-1}\omega \in \Omega^q(X/\Gamma)$ is harmonic. For $n \neq 3$, $q < \frac{n-1}{2}$, so by Theorem 5.2.12 iii) we have $\varphi\omega \in \mathscr{H}_{(2)}^q(X/\Gamma)$. For $n = 3$, we have to require $q < \frac{n+1}{4} = 1$ in order to have $q < \frac{n-1}{2} = 1$. $\hfill \square$

Remark 6.2.16. In [6], Borel refers to the proof of Theorem 1 in the paper *On Betti Numbers of Compact, Locally Symmetric Riemannian Manifolds* by Yozô Matsushima ([19]) to prove that the harmonic square integrable forms on X/Γ are mapped to G-invariant forms. Given a harmonic square integrable form, he pulls it back to G/Γ and constructs sequences converging to the coefficients in $L^2(G/\Gamma)$; he then states that all the steps of Matsushima's argument follow through applied to the elements of the sequences. We have chosen to use the algebraic version of the Matsushima Vanishing Theorem as proved in [4] instead, exploiting first the result of Borel and Garland.

DEFINITION 6.2.17. For $n \in \mathbb{N}$, define a number $c(n)$ as $c(n) = \frac{n+1}{4}$ for $n \neq 3$ and $c(3) = 0$.

AND FINALLY...

Now we are ready to prove what we set out to, namely that the inclusion $\Omega^*(X)^G \to \Omega^*(X/\Gamma)$ is an isomorphism on cohomology in small degrees.

THEOREM 6.2.18. The map $j \colon \Omega^*(X)^G \hookrightarrow \Omega^*(X/\Gamma)$ induces an isomorphism on cohomology in degrees $* \leq c(n)$.

Proof. Consider the following commutative diagram in degrees $* \leq c(n)$ and the induced diagram on cohomology:

$$
\begin{array}{ccc}
\mathscr{H}_{(2)}^*(X/\Gamma) \longhookrightarrow \Omega_{(2)}^*(X/\Gamma) & \qquad & \mathscr{H}_{(2)}^*(X/\Gamma) \longrightarrow H_{(2)}^*(X/\Gamma) \\[2mm]
\pi^* \Big\downarrow \cong \qquad \Big\uparrow \sim & & \cong \Big\downarrow \qquad \Big\uparrow \cong \\[2mm]
\Omega^*(X)^G \longrightarrow \Omega_{\log}^*(X/\Gamma) & & H_*(\Omega^*(X)^G) \longrightarrow H_*(\Omega_{\log}^*(X/\Gamma)) \\[2mm]
\Big\uparrow \qquad \Big\uparrow \sim & & \Big\downarrow \qquad \Big\downarrow \cong \\[2mm]
\Omega^*(X)^\Gamma \xleftarrow[\pi^*]{\cong} \Omega^*(X/\Gamma) & & H_*(\Omega^*(X)^\Gamma) \xleftarrow[\cong]{} H^*(X/\Gamma)
\end{array}
$$

The map $j\colon \Omega^*(X)^G \to \Omega^*(X/\Gamma)$ is the composition of the lower left vertical map and the lower horizontal map and we know that the image of j is in $\Omega^*_{\log}(X/\Gamma)$ by 5.2.12 i), thus giving rise to the middle horizontal map. The lower right vertical map is a quasi-isomorphism by Theorem 5.2.12 ii). The top horizontal map is surjective on cohomology by Proposition 6.1.4, and the top left vertical map is an isomorphism by Corollary 6.2.15.

The top right vertical map $\Omega^*_{\log}(X/\Gamma) \hookrightarrow \Omega^*_{(2)}(X/\Gamma)$ exists by 5.2.12 iii), as $* < \frac{n-1}{2}$, and it induces an isomorphism on cohomology as $\Omega^*_{\log}(X/\Gamma) \hookrightarrow \Omega^*(X/\Gamma)$ is a quasi-isomorphism.

We wish to show that the middle horizontal map induces an isomorphism on cohomology; the lower right vertical map being a quasi-isomorphism, this will imply that j induces an isomorphism on cohomology. Surjectivity is clear by commutativity of the upper square. Injectivity is seen as follows: Assume $[\omega] \in H_*(\Omega^*(X)^G)$ is mapped to zero in $H_*(\Omega^*_{\log}(X/\Gamma))$. Then ω is a harmonic form satisfying $\omega = d\omega'$ for some $\omega' \in \Omega^*_{\log}(X/\Gamma) \subseteq \Omega^*_{(2)}(X/\Gamma)$. Hence, $\omega = 0$ by Proposition 6.1.7. $\square$

Remark 6.2.19. Note that in the above proof, we actually show that j induces an injective map on cohomology in degrees $* \leq \frac{n-1}{2}$.

By Theorem 2.3.7, we have proved Theorem 2.3.6 as we set out to several pages ago, namely that the inclusion $\Omega^*(X)^G \hookrightarrow \Omega^*(X)^{\mathrm{SL}_n(\mathbb{Z})}$ induces an isomorphism in degrees $* \leq \frac{n+1}{4}$ for $n \neq 3$ and $* < \frac{n+1}{4}$ for $n = 3$. Combining this with Proposition 2.3.5 and the fact that $\Omega^*(X)^G$ has trivial differential, we have:

THEOREM 6.2.20.

$$H^q(\mathrm{SL}_n(\mathbb{Z})) \cong H_q(\Omega^*(X)^G) = \Omega^q(X)^G \qquad \text{for all } q \leq c(n).$$

6.3 CALCULATING $H^*(\mathrm{SL}_n(\mathbb{Z}))$ IN LOW DEGREES

We will now put to use the results of our hard work and actually calculate the cohomology of $\mathrm{SL}_n(\mathbb{Z})$ in low degrees. We use the compact real form of $\mathfrak{sl}_n(\mathbb{R})$, allowing us to use the fact that the inclusion $\Omega^*(M)^H \hookrightarrow \Omega^*(M)$ is a quasi isomorphism when H is a compact Lie group and M a homogeneous H-space. We also consider the issue of stability.

Consider again the Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into a direct sum of skew-symmetric matrices with trace zero, $\mathfrak{k}$, and symmetric matrices with trace zero, $\mathfrak{p}$. Consider the subspace $i\mathfrak{p} \subseteq \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$; then the direct sum $\mathfrak{k} \oplus i\mathfrak{p}$ is a Lie algebra with the Lie bracket defined in the obvious way and it is in fact isomorphic to the special unitary Lie algebra

$$\mathfrak{k} \oplus i\mathfrak{p} = \{x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \mid x^* = -x\} \cong \{x \in \mathrm{M}_n(\mathbb{C}) \mid x^* = -x,\ \mathrm{tr}(x) = 0\} = \mathfrak{su}(n).$$

Recall that the special unitary group, $\mathrm{SU}(n) = \{g \in \mathrm{SL}_n(\mathbb{C}) \mid u^* = u^{-1}\}$, is a compact real Lie group. Recall also that for a compact Lie group H and homogeneous H-space M, the inclusion $\Omega^*(M)^H \hookrightarrow \Omega^*(M)$ is a quasi-isomorphism (see [23, Theorem 13.6.30] — the idea is to construct an inverse by averaging the translations of a closed differential form on M over H).

Write $G_u := \mathrm{SU}(n)$. Evaluating at the identity coset yields isomorphisms

$$\Omega^*(X)^G \xrightarrow{\cong} C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*(\mathfrak{g}/\mathfrak{k}), \mathbb{R}),$$

$$\Omega^*(K\backslash G_u)^{G_u} \xrightarrow{\cong} C^*(\mathfrak{k} \oplus i\mathfrak{p}, \mathfrak{k}, \mathbb{R}) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*((\mathfrak{k} \oplus i\mathfrak{p})/\mathfrak{k}), \mathbb{R}),$$

and the isomorphism $\mathfrak{p} \to i\mathfrak{p}$, $x \mapsto ix$, induces a vector space isomorphism and hence an isomorphism of chain complexes as both complexes have trivial differential:

$$\mathrm{Hom}_{\mathfrak{k}}(\Lambda^*(i\mathfrak{p}), \mathbb{R}) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*\mathfrak{p}, \mathbb{R}).$$

So we have the following sequence of chain complex isomorphisms:

$$\Omega^*(X)^G \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*(\mathfrak{g}/\mathfrak{k}), \mathbb{R}) \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*\mathfrak{p}, \mathbb{R}) \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*(i\mathfrak{p}), \mathbb{R})$$
$$\cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^*((\mathfrak{k} \oplus i\mathfrak{p})/\mathfrak{k}), \mathbb{R}) \cong \Omega^*(X\backslash G_u)^{G_u}.$$

Composing this with the quasi-isomorphism $\Omega^*(X\backslash G_u)^{G_u} \hookrightarrow \Omega^*(K\backslash G_u)$, we get

$$\Omega^*(X)^G \cong H_*(\Omega^*(X)^G) \cong H^*(K\backslash G_u).$$

The calculation of the cohomology of $K\backslash G_u$ can be done using spectral sequences (see [20, page 92] or [13, Proposition 7.2]):

PROPOSITION 6.3.1.

$$H^*(\mathrm{SO}(n)\backslash \mathrm{SU}(n), \mathbb{R}) \cong \Lambda^*\{ x_i \mid \deg(x_i) = 4i+1, \ i = 1, \ldots, \lfloor \tfrac{n+1}{2} \rfloor \}.$$

Combining the above results with Theorem 6.2.20, it follows directly that:

THEOREM 6.3.2.

$$H^q(\mathrm{SL}_n(\mathbb{Z})) \cong \Lambda^q\{ x_i \mid \deg(x_i) = 4i+1, \ i = 1, \ldots, \lfloor \tfrac{n+1}{2} \rfloor \}, \qquad \text{for any } q \leq c(n).$$

For a given n, the computable range is quite small. For $n = 2, 3$ for example, we only get information about the zero'th cohomology group, which we already knew. We also see, however, that the bound $c(n)$ tends to ∞ as $n \to \infty$, encouraging us to consider the question of stability. Consider the inclusion $f_n \colon \mathrm{SL}_n(\mathbb{R}) \hookrightarrow \mathrm{SL}_{n+1}(\mathbb{R})$. This induces maps

$$\mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \mathrm{SL}_{n+1}(\mathbb{Z}), \qquad X_n \to X_{n+1}, \quad \text{and} \quad X_n/\Gamma_n \to X_{n+1}/\Gamma_{n+1}$$

which we also denote by f_n. By tracing through the isomorphisms defining

$$H^q(\mathrm{SL}_n(\mathbb{Z})) \xrightarrow{\cong} \Omega^q(X_n)^{G_n},$$

we see that these commute with the maps f_n^*:

$$
\begin{array}{ccc}
H^q(\mathrm{SL}_n(\mathbb{Z})) & \xrightarrow{\ \cong\ } & \Omega^q(X_n)^{G_n} \\
{\scriptstyle f_n^*}\big\uparrow & & \big\uparrow{\scriptstyle f_n^*} \\
H^q(\mathrm{SL}_{n+1}(\mathbb{Z})) & \xrightarrow[\cong]{\ } & \Omega^q(X_{n+1})^{G_{n+1}}
\end{array}
$$

The only non-trivial observation to be done here is that the map $f_n \colon X_n/\Gamma_n \to X_{n+1}/\Gamma_{n+1}$ induces the map $f_n \colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_{n+1}(\mathbb{Z})$ on π_1, but this is immediate from the definition of the f_n. This implies that when interpreting the group cohomology in terms of these classifying spaces, the maps f_n^* on $H^*(\Gamma_n)$ and $H^*(X_n/\Gamma_n; \mathbb{R})$ are compatible.

The following result is due to H. Cartan (see [10, Exp. 16])

PROPOSITION 6.3.3. The sequence $(\Omega^q(X_n)^{G_n}, f_n^*)$ stabilises, i.e. given $q \geq 0$, there exists $n(q) \geq 2$ such that

$$\varprojlim \Omega^q(X_n)^{G_n} = \Omega^q(X_m)^{G_m} \cong \Lambda^q\{ x_i \mid \deg(x_i) = 4i+1, \ i = 1, \ldots, \lfloor \tfrac{m+1}{2} \rfloor \}$$

for all $m \geq n(q)$.

As a corollary, we have:

COROLLARY 6.3.4. The limit of $(\Omega^*(X_n)^{G_n}, f_n^*)$ is

$$\varprojlim \Omega^*(X_n)^{G_n} = \Lambda^*\{x_i \mid \deg(x_i) = 4i + 1,\, i \in \mathbb{N}\}.$$

Thus the real cohomology of $\mathrm{SL}_n(\mathbb{Z})$ stabilises:

THEOREM 6.3.5. For a given $q \geq 0$, the sequence $(H^q(\mathrm{SL}_n(\mathbb{Z})), f_n^*)$ stabilises, i.e. there exists $n'(q) \geq 2$ such that the composition $f_{m-1} \circ \cdots \circ f_{n'(q)} \colon \mathrm{SL}_{n'(q)}(\mathbb{Z}) \hookrightarrow \mathrm{SL}_m(\mathbb{Z})$ induces an isomorphism

$$H^q(\mathrm{SL}_m(\mathbb{Z})) \xrightarrow{\cong} H^q(\mathrm{SL}_{n'(q)}(\mathbb{Z})) \qquad \text{for all } m \geq n'(q).$$

THEOREM 6.3.6. The real cohomology of the stable special linear group, $\mathrm{SL}_\infty(\mathbb{Z})$, is

$$H^*(\mathrm{SL}_\infty(\mathbb{Z})) \cong \Lambda^*\{x_i \mid \deg(x_i) = 4i + 1,\, i \in \mathbb{N}\}.$$

Proof. Indeed, we have the following sequence of isomorphisms

$$\begin{aligned} H^*(\mathrm{SL}_\infty(Z)) &= H^*(\varinjlim \mathrm{SL}_n(\mathbb{Z})) \cong \varprojlim H^*(\mathrm{SL}_n(\mathbb{Z})) \\ &\cong \varprojlim \Omega^*(X_n)^{G_n} = \Lambda^*\{x_i \mid \deg(x_i) = 4i + 1,\, i \in \mathbb{N}\}. \end{aligned}$$

For any $q \geq 0$, we have isomorphisms $H^q(\mathrm{SL}_n(\mathbb{Z})) \cong \Omega^*(X_n)^{G_n}$ for all $n \geq 4q$ and they commute with the maps f_n^* defining the limit. This shows the second isomorphism. The first isomorphism is a consequence of the fact that homology commutes with limits, the fact that $\mathrm{Hom}_\mathbb{R}(-, \mathbb{R})$ takes colimits to limits and the Universal Coefficient Theorem. $\qquad\square$

A | Appendix

A.1 Proper Group Actions

In this section, we consider the notion of an action being proper. As the action to which we shall apply the following results is a right action, we shall consider only right actions, but it is immediate that all results hold for both right and left actions.

Definition A.1.1. Let G be a topological group acting on a space X. The action is *proper*, if the map $X \times G \to X \times X$, $(x, g) \mapsto (x.g, x)$, is proper, i.e. inverse images of compact sets are compact. We also say that G acts *properly* on X.
If G is discrete and acts properly on X, then we say that it acts *properly discontinuously* or that the action is *properly discontinuous*.

Proposition A.1.2. Let G be a topological group acting on a space X. If X is Hausdorff, then the following are equivalent:

1. The action, $X \curvearrowleft G$, is proper.

2. For any compacts $K_1, K_2 \subseteq X$, the set $\{g \in G \mid K_1 g \cap K_2 \neq \emptyset\}$ is compact.

3. For any compact $K \subseteq X$, the set $\{g \in G \mid Kg \cap K \neq \emptyset\}$ is compact.

Proof. Let $\alpha \colon X \times G \to X \times X$ denote the map $(x, g) \mapsto (x.g, x)$, let $\pi_X \colon X \times G \to X$, $\pi_G \colon X \times G \to G$ denote the projections, and let $\pi_i \colon X \times X \to X$ denote the projection onto the i'th coordinate, $i = 1, 2$. Note that for any compacts $K_1, K_2 \subseteq X$

$$\{g \in G \mid K_1 g \cap K_2 \neq \emptyset\} = \pi_G(\alpha^{-1}(K_2 \times K_1)). \tag{A.1}$$

Indeed, if $g \in G$ satisfies $K_1 g \cap K_2 \neq \emptyset$, then there exists $x_i \in K_i$, $i = 1, 2$, such that $x_1.g = x_2$. Hence, $\alpha(x_1, g) = (x_2, x_1)$. Conversely, if $g \in \pi_H(\alpha^{-1}(K_2 \times K_1))$, then there exists $x \in X$ such that $x.h \in K_2$ and $x \in K_1$, implying $x.g \in K_1 h \cap K_2$.
The implication (1) $\Rightarrow$ (2) follows immediately from A.1, and (2) $\Rightarrow$ (3) is obvious. For the implication (3) $\Rightarrow$ (1), let $F \subseteq X \times X$ be compact and set $K_1 := \pi_1(F)$, $K_2 := \pi_2(F)$, and $K := K_1 \cup K_2$. Then $F \subseteq K_1 \times K_2 \subseteq K \times K$, and thus

$$\alpha^{-1}(F) \subseteq \alpha^{-1}(K_1 \times K_2) \subseteq \alpha^{-1}(K \times K) \subseteq K \times A,$$

where $A := \{g \in G \mid Kg \cap K \neq \emptyset\}$. As X is Hausdorff, F is closed. We conclude that $\alpha^{-1}(F)$ is compact, being closed in a compact space. $\qquad\square$

Proposition A.1.3. For any topological group G, the action of G on itself by right multiplication is proper.

Proof. Let $K \subseteq G$ be a compact subset. Then $\{g \in G \mid Kg \cap K \neq \emptyset\} = K^{-1}K$ is compact being the image of $K^{-1} \times K \subseteq G \times G$ under multiplication. The action is proper by Proposition A.1.2. $\qquad\square$

Proposition A.1.4. Let G be a topological group acting on spaces X and Y. If $f \colon X \to Y$ is a proper surjective equivariant map and $X \curvearrowleft G$ is proper, then $Y \curvearrowleft G$ is proper.

Proof. Let $\alpha_X\colon X \times G \to X \times X$, $\alpha_Y\colon Y \times G \to Y \times Y$ denote the action maps described in Definition A.1.1. Let $K \subseteq X \times X$ be a compact subset and note that $\alpha_Y \circ (f \times \mathrm{id}) = (f \times f) \circ \alpha_X$. Then

$$\alpha_Y^{-1}(K) = (f \times \mathrm{id})(\alpha_X^{-1} \circ (f \times f)^{-1}(K))$$

as $f \times \mathrm{id}$ is surjective. By assumption, $(f \times f) \circ \alpha_X$ is proper, so $(f \times \mathrm{id})(\alpha_X^{-1} \circ (f \times f)^{-1}(C))$ is compact. We conclude that α_Y is proper. $\qquad\square$

COROLLARY A.1.5. *Let G be a topological group, $K \subseteq G$ a compact subgroup. Then the action of G on the coset space $K\backslash G$ by right multiplication is proper.*

Proof. The canonical projection $\pi\colon G \to K\backslash G$ is equivariant and surjective. To see that it is proper, allowing us to apply Propositions A.1.3 and A.1.4, let $F \subseteq K\backslash G$ be a compact subset. Let $W \subseteq G$ such that W contains exactly one representative of each coset contained in F. We claim that W is compact. Let $\{U_i\}_{i \in I}$ be an open covering of W. For every $i \in I$, set $V_i := KU_i$. The V_i are open by homogeneity of the topology on G. Then $\{\pi(V_i)\}_{i \in I}$ is an open cover of F as $\pi^{-1}(\pi(V_i)) = KV_i = V_i$ is open for all $i \in I$. Let $J \subseteq I$ be a finite subset such that $F \subseteq \bigcup_{i \in J} \pi(V_i)$. Then $W \subseteq \pi^{-1}(F) \subseteq \bigcup_{i \in J} V_i$, and since $W \cap V_i = W \cap U_i$ for all i by our choice of W, we conclude that $W \subseteq \bigcup_{i \in J} U_i$. Now, $\pi^{-1}(F) = \pi^{-1}(\pi(W)) = KW$ is compact as it is the image of the compact set $K \times W$ under multiplication. $\qquad\square$

PROPOSITION A.1.6. *A closed inclusion into a Hausdorff space is proper.*

Proof. Let $i\colon X \hookrightarrow Y$ be a closed inclusion with Y Hausdorff. Identify X with its image $i(X)$, which must be closed in Y by assumption. Let $K \subseteq Y$ be any compact subset; as Y is Hausdorff, K is closed, and $K \cap X$ is compact as it is closed in a compact space. $\qquad\square$

COROLLARY A.1.7. *If G is a Hausdorff topological group acting properly on a Hausdorff space, and H is a closed subgroup of G, then the inherited action of H on X is proper.*

Proof. This is a consequence of the previous proposition as $X \times H \to X \times G$ is a closed inclusion and thus the composite $X \times H \to X \times G \to X \times X$, $(x,h) \mapsto (x.h, x)$, is proper. $\qquad\square$

A.2 SMOOTH ACTIONS AND QUOTIENT SPACES

We are interested in group actions of Lie groups on smooth manifolds and will need the following results about smooth structures on quotients. We do not prove these results but refer to [18] for details.

PROPOSITION A.2.1. *Let M be a smooth manifold and G a Lie group acting smoothly on M (from the right). If the action $M \curvearrowleft G$ is proper and free, then the quotient space M/G can be equipped with a unique smooth structure such that the canonical projection $M \to M/G$ is a smooth submersion.*

Proof. [18, Theorem 21.10]. $\qquad\square$

PROPOSITION A.2.2. *Let G be a Lie group and H a closed subgroup of G. There is a unique smooth structure on the coset space $H\backslash G$ such that the canonical projection $G \to H\backslash G$ is a smooth submersion. With respect to this smooth structure, right multiplication of G on $H\backslash G$ is a smooth action.*

Proof. [18, Theorem 21.17]. $\qquad\square$

We have the following smooth version of the orbit-stabiliser theorem.

PROPOSITION A.2.3. Let M be a smooth manifold and G a Lie group acting smoothly on M (from the right). For a given $x \in M$, the stabiliser of x, G_x, is a closed subgroup of G and the orbit of x, $x.G$, is a submanifold of M. Moreover, the map $G \to M$, $g \mapsto x.g$, induces an equivariant diffeomorphism $G_x \backslash G \to x.G$, where the coset space $G_x \backslash G$ is equipped with the smooth structure described in A.2.2. In particular, if the action is transitive, then M is diffeomorphic to $G_x \backslash G$.

Proof. [18, Theorem 21.18]. $\square$

Remark A.2.4. A smooth manifold which is diffeomorphic to $H \backslash G$ for some Lie group G and closed subgroup $H \subseteq G$, is called a *homogeneous space*. Equivalently, it is a smooth manifold on which G acts smoothly and transitively.

A.3 STANDARD HOMOTOPY OPERATOR

We define the standard homotopy operator and prove that it is indeed a homotopy operator.

For $m \in \mathbb{N}$, let $e_1, \ldots, e_m$ be a basis of $\mathbb{R}^m$ and denote by $\epsilon_1, \ldots, \epsilon_m$ the dual basis of $(\mathbb{R}^m)^*$. Then $\epsilon_\sigma = \epsilon_{\sigma(1)} \wedge \cdots \wedge \epsilon_{\sigma(k)}$, $\sigma \in \Sigma_{k,m-k}$, form a basis of $\Omega^k(\mathbb{R}^m)$. Consider the map

$$H \colon \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m) \qquad \text{given by} \quad H(f\epsilon_\sigma) = \sum_{i=1}^k c_{\sigma^i} \, \epsilon_{\sigma^i}, \quad f \in C^\infty(\mathbb{R}^m), \, \sigma \in \Sigma_{k,m-k},$$

where $\sigma^i \in \Sigma_{k-1,m-k+1}$ is the permutation skipping $\sigma(i)$ (i.e. $\sigma^i(j) = \sigma(j)$ for $j < i$, and $\sigma^i(j) = \sigma(j+1)$ for $j \geq i$) and the coefficients $c_{\sigma^i} \in C^\infty(A \times N)$ are defined as

$$c_{\sigma^i}(x) = (-1)^{i-1} x_{\sigma(i)} \int_0^1 f(xt) t^{k-1} \, dt.$$

PROPOSITION A.3.1. On any subset $U \subseteq \mathbb{R}^m$ which is starshaped with respect to the origin, we have $dH + Hd = \mathrm{id}$ for $k \geq 1$.

Proof. First we see that

$$dH(f\epsilon_\sigma) = \sum_{i=1}^k dc_{\sigma^i} \wedge \epsilon_{\sigma^i} = \sum_{i=1}^k \sum_{j=1}^m \frac{\partial c_{\sigma^i}}{\partial x_j} \epsilon_j \wedge \epsilon_{\sigma^i} = \sum_{i=1}^k \sum_{\substack{j \neq \sigma^i(l) \\ l=1,\ldots,k-1}} \frac{\partial c_{\sigma^i}}{\partial x_j} \epsilon_j \wedge \epsilon_{\sigma^i}$$

$$= \sum_{i=1}^k \left((-1)^{i-1} \frac{\partial c_{\sigma^i}}{\partial x_{\sigma(i)}} \epsilon_\sigma + \sum_{\substack{j \neq \sigma(l) \\ l=1,\ldots,k}} \frac{\partial c_{\sigma^i}}{\partial x_j} \epsilon_j \wedge \epsilon_{\sigma^i} \right)$$

and

$$Hd(f\epsilon_\sigma) = H(df \wedge \epsilon_\sigma) = \sum_{\substack{j \neq \sigma(l) \\ l=1,\ldots,k}} H\left(\frac{\partial f}{\partial x_j} \epsilon_j \wedge \epsilon_\sigma \right) = \sum_{\substack{j \neq \sigma(l) \\ l=1,\ldots,k}} \left(\widetilde{c}_j \epsilon_\sigma + \sum_{i=1}^k \widetilde{c}_{j,\sigma^i} \epsilon_j \wedge \epsilon_{\sigma^i} \right),$$

where

$$\widetilde{c}_j = x_j \int_0^1 \frac{\partial f}{\partial x_j}(xt) t^k \, dt, \qquad \widetilde{c}_{j,\sigma^i} = (-1)^i x_{\sigma(i)} \int_0^1 \frac{\partial f}{\partial x_j}(xt) t^k \, dt.$$

For $j \neq \sigma(i)$, we have

$$\frac{\partial c_{\sigma^i}}{\partial x_j}(x) = (-1)^{i-1} x_{\sigma(i)} \int_0^1 \frac{\partial}{\partial x_j} f(xt) t^{k-1} \, dt = (-1)^{i-1} x_{\sigma(i)} \int_0^1 \frac{\partial f}{\partial x_j}(xt) t^k \, dt$$

and

$$\frac{\partial c_{\sigma^i}}{\partial x_{\sigma(i)}}(x) = (-1)^{i-1} \int_0^1 f(xt) t^{k-1} \, dt + (-1)^{i-1} x_{\sigma(i)} \int_0^1 \frac{\partial f}{\partial x_{\sigma(i)}}(xt) t^k \, dt.$$

We see that $\widetilde{c}_{j,\sigma^i} = -\frac{\partial c_{\sigma^i}}{\partial x_j}$, so

$$(dH + Hd)(f\epsilon_\sigma) = \sum_{i=1}^k (-1)^{i-1} \frac{\partial c_{\sigma^i}}{\partial x_{\sigma(i)}} \epsilon_\sigma + \sum_{\substack{j \neq \sigma(l) \\ l=1,\dots,k}} \widetilde{c}_j \epsilon_\sigma.$$

Finally, setting $h_x \colon I \to \mathbb{R}$, $h_x(t) = f(xt)$, for a given $x \in \mathbb{R}^m$, we have

$$\sum_{i=1}^k (-1)^{i-1} \frac{\partial c_{\sigma^i}}{\partial x_{\sigma(i)}}(x) + \sum_{\substack{j \neq \sigma(l) \\ l=1,\dots,k}} \widetilde{c}_j(x) = \sum_{j=1}^m x_j \int_0^1 \frac{\partial f}{\partial x_j}(xt) t^k \, dt + \sum_{i=1}^k \int_0^1 f(xt) t^{k-1} \, dt$$

$$= \int_0^1 \sum_{j=1}^m x_j \frac{\partial f}{\partial x_j}(xt) t^k \, dt + k \int_0^1 f(xt) t^{k-1} \, dt$$

$$= \int_0^1 t^k \frac{dh_x}{dt}(t) + k t^{k-1} h_x(t) \, dt = \int_0^1 \frac{d}{dt}(h_x(t) t^k) \, dt = [h_x(t) t^k]_0^1 = f(x).$$

We conclude that $(dH + Hd)(f\epsilon_\sigma) = f\epsilon_\sigma$ as desired. $\qquad\square$

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