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Symmetric arcs and homological stability for surfaces

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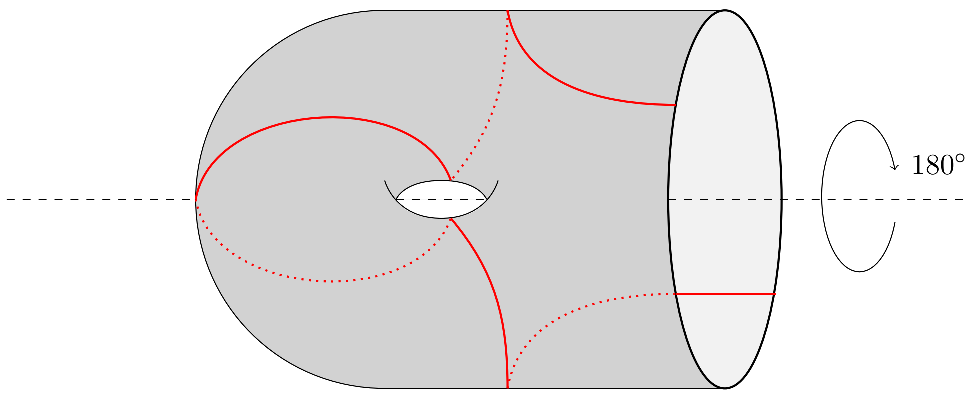

A symmetric arc on a surface with involution

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Abstract/resumé

English: Many classical sequences of groups G_n induce isomorphisms in group homology from some point depending on the homology degree. We cover a spectral sequence argument involving highly connected simplicial complexes X_n with G_n-actions that can be used to establish such results, including tools to show that the simplicial complexes are highly connected. We apply this to the sequence of braid groups and sequences of mapping class groups of surfaces with increasing genus or number of boundary components. Finally we apply it to a sequence of subgroups of mapping class groups of surfaces called symmetric mapping class groups whose elements commute with fixed involutions of the surfaces.

Dansk: Mange klassiske følger af grupper G_n inducerer isomorfier i gruppehomologi fra et vist punkt, der afhænger af homologigraden. Vi gennemgår et spektralfølgeargument, der indebærer højsammenhængende simplicielle komplekser X_n med G_n-virkninger, som kan bruges til at opnå sådanne resultater, herunder værktøjer til at vise, at de simplicielle komplekser er højsammenhængende. Vi anvender dette på følgen af fletningsgrupper samt følger af afbildningsklassegrupper på flader, hvor genus eller antallet af randkomponenter øges. Endelig anvender vi det på en følge af undergrupper af afbildningsklassegrupper på flader kaldet symmetriske afbildningsklassegrupper, hvis elementer kommuterer med bestemte involutioner på fladerne.

Introduction

A family of groups can often be arranged in a sequence $G_k \to G_{k+1} \to \cdots$. We say that such a sequence satisfies homological stability with slope β if there is some increasing function ϕ of slope β such that the induced maps $H_i(G_k) \to H_i(G_{k+1})$ are isomorphisms for all $i \leq \phi(k)$. Some examples of classical group sequences with this property are symmetric groups, automorphism groups of free groups, and general linear groups.

In Section 1.1 we will go through a general method for proving such results, originally due to Daniel Quillen. This method involves constructing for each n a simplicial complex X_n and an action of G_n on X_n that respects the simplicial structure. This action must be transitive on the vertices of X_n, and the stabilizer of a p-simplex must be conjugate to G_{n-r} for some $1 \leq r \leq p+1$. In some cases we always have $r = p+1$ so that the stabilizer of a p-simplex can be seen as going $p+1$ steps back in the sequence of groups. Besides a couple of other technical requirements, the complex X_n and the quotient X_n/G_n must be 'sufficiently highly connected' with respect to n. The last part, showing that the complexes are highly connected, is usually the most demanding one. In Section 1.2 we establish a number of tools that can be used to this end.

We will use Quillen's method on a number of group sequences to show that they satisfy homological stability. The first one is covered in Chapter 2. We consider the sequence $B_0 \to B_1 \to \cdots$, where B_n is the braid group on n strands. In order to define the simplicial complex X_n, we define B_n as the boundary fixing mapping class group of the 2-disk with n distinct marked points in the interior. For the spectral sequence argument we use the complex of tethers $Y_{n,1}$ in which a vertex is an isotopy class of arcs from a specified point in ∂D_2 to one of the marked points, and where a collection of isotopy classes of tethers spans a simplex if all the tethers end in different marked points and can be chosen to be disjoint except for their endpoints. A self-diffeomorphism of the marked 2-disk then takes systems of tethers to systems of tethers, thus defining the action $B_n \curvearrowright X_n$.

In Chapter 3 we cover homological stability for boundary fixing mapping class groups of surfaces. Given two surfaces with boundary, we can glue them together along a pair of boundary circles. Starting with some surface $S_{g,s}$ of genus g with s boundary components, we can glue on a copy of $S_{1,2}$ along one boundary component to obtain the surface $S_{g+1,s}$, or we can glue on a copy of $S_{0,3}$, also known as a pair of pants, along one boundary component to obtain the surface $S_{g,s+1}$. These two types of gluing give us two sequences of surfaces, one with increasing genus, and one with increasing number of boundary components. These sequences induce sequences of mapping class group which we will prove to be homologically stable with slope 2. We also show that the homology groups of mapping class groups of closed surfaces are independent of genus in the same range. Most of Chapter 3 follows the preprint [HV15b] which was published to arXiv in August 2015 but has been circulating in a less polished version for the last decade. Moreover, we import some methods and proofs from [HW07].

In 1985 John L. Harer published [Har85] in which he established a slope three function ϕ such that the i'th homology group of the mapping class group of a genus g surface with s boundary components is independent of g and r as long as $g \geq \phi(i)$. Since then, this result has been gradually improved. First Nikolai V. Ivanov improved the slope of ϕ to 2 in three different papers [Iva87, Iva89, Iva93]. In an unpublished preprint [Har93], Harer improved this slope to $\frac{3}{2}$, and the first complete proof of this is due to Søren K. Boldsen [Bol12].

Thus the stability theorem that we present in this thesis is not the strongest one known, but the complexes that we use are in some ways the natural ones, which is interesting in itself and also makes the arguments relatively simple. Moreover, the stabilizers of p-simplices correspond to moving $p + 1$ steps back in the group sequences, which is better than one could usually hope for.

While the first three chapters are based mainly on a preprint by Allen Hatcher and Karen Vogtmann [HV15b], Chapter 4 is the new contribution of this thesis. We study sequences of symmetric mapping class groups of surfaces. If S is a surface with boundary, and κ is an involution on S, i.e. a self-diffeomorphism of order 2, we can define a subgroup of the boundary fixing mapping class group of S consisting of elements that commute with κ. We can then glue pairs of pants, i.e. surfaces of genus 0 with 3 boundary components, with certain accompanying involutions onto S to get a sequence of surfaces $S = S_0 \to S_1 \to \cdots$ where each S_i comes with an involution κ_i. For each step in the sequence, the number of points that are fixed by the resulting involution is increased by one. Moreover, the genus is increased by one for every two steps. This sequence of surfaces with involutions induces a sequence of symmetric mapping class groups, and we prove that this sequence satisfies homological stability. The simplicial complexes that we use for the spectral sequence argument are built from systems of symmetric arcs between two chosen points $b_1, b_2 \in S_i$. Here a vertex is an isotopy class of arcs from b_1 to b_2 that are symmetric with respect to the involution κ, meaning that we must be able to choose an arc a in the isotopy class such that $\kappa(a) = a$. A system of symmetric arc classes then spans a simplex if the arcs can be chosen to be mutually disjoint. The involution κ_i then takes systems of symmetric arcs to systems of symmetric arcs, thus defining the group action used for the spectral sequence.

Chapter 1

Homological stability tools

This chapter is our toolbox. Here we will not prove any actual stability theorems, but we will set the stage for the theorems in the later chapters. We begin with the central spectral sequence argument. This requires some highly connected simplicial complexes, and the second section of this chapter consists of a range of tools to help us showing that such complexes are highly connected.

1.1 The spectral sequence argument

This section will form the backbone of our homological stability proofs throughout the thesis. Assume that we have a sequence of group inclusions

$$\cdots \to G_n \to G_{n+1} \to G_{n+2} \to \cdots . \tag{1.1}$$

After making some assumptions on this sequence we will prove that it is homologically stable of slope 2, i.e. that $H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism whenever $n > 2i + k$ for some constant k. We will also prove that it is surjective for $n = 2i + k$. We will prove this by induction using two spectral sequences. In one of these spectral sequence the homomorphism $H_i(G_{n-1}) \to H_i(G_n)$ will be a differential on the E^1 page.

Assume that for each n we have a simplicial complex X_n (see Appendix A.1 for a definition) and an action $G_n \curvearrowright X_n$ that respects the simplicial structure as well as the dimension of the simplices, i.e. it takes p-simplices to p-simplices for all p. This means that if we denote by X_n^p the set of p-simplices in X_n, we get an action $G_n \curvearrowright X_n^p$. The action $G_n \curvearrowright X_n$ should satisfy the following conditions:

(1) $G_n \curvearrowright X_n$ is transitive on the vertices of X_n, i.e. $G_n \curvearrowright X_n^0$ is transitive.
(2) For a p-simplex σ_p, the stabilizer $\mathrm{stab}(\sigma_p)$ fixes σ_p pointwise, and there is some $h \in G_n$ such that $\mathrm{stab}(\sigma_p) = hG_{n-r}h^{-1} \cong G_{n-r}$ for some r with $1 \le r \le p+1$. In particular if $p = 0$, then $\mathrm{stab}(\sigma_p) \cong G_{n-1}$.
(3) If e is an edge with vertices v and w, there is some $g \in G_n$ such that $gv = w$ and such that $gh = hg$ for all $h \in \mathrm{stab}(e)$.
(4) X_n and X_n/G_n are both sufficiently 'highly' connected.

In many cases the action will be transitive on simplices of any dimension. This will make X_n/G_n easier to describe, but is not strictly necessary, and we will in fact use the argument in the greater generality described here. The second condition means that we can see a simplex of X_n as something that 'undoes' the homomorphisms of (1.1). For instance when we prove that (1.1) is homologically stable for $G_n = \mathcal{M}_{n,s}$, the mapping class group of the surface $S_{n,s}$ of genus n with s boundary components, a p-simplex will be a constellation of arcs and curves that cuts $S_{n,s}$ into the surface $S_{n-p-1,s}$. The stabilizer is then isomorphic to the mapping class group of this cut up surface.

Remark 1.1. We assume for simplicity that X_n and X_n/G_n are highly connected, i.e. that their homotopy groups vanish until some degree (assumption 4), since this will be true in most of the cases that we investigate. However, we only need the homology groups to vanish until that degree since this is enough for the terms of high total degree to vanish in the spectral sequence that we will study.

Remark 1.2. By default we assume that the X_n's are simplicial complexes since this is true in most cases that we will encounter. However, we might as well assume that they are only Δ-complexes (semi-simplicial sets) as the whole spectral sequence argument works equally well in that setting.

1.1.1 Construction of the spectral sequence

We will now construct the two spectral sequences that we will use for the stability argument. For $G = G_n$, let E_*G be a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. Such a resolution exists by Lemma A.7. Let

$$\cdots \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} = \mathbb{Z} \to 0$$

be the augmented chain complex of $X = X_n$, i.e. $C_p = \mathbb{Z}[X^p]$ for $p \geq 0$, the free $\mathbb{Z}$-module with generators in the p-skeleton X^p. The differentials are induced by the face maps, i.e. $\partial_p = \sum_{i=0}^{p} (-1)^i d_i$ for $p \geq 0$, and ∂_0 is the augmentation homomorphism that maps each copy of $\mathbb{Z}$ by the identity to $C_{-1} = \mathbb{Z}$. Then the action $G \curvearrowright X$ makes C_* a complex of $\mathbb{Z}[G]$-modules with the action generated by $g\mathbb{Z}[\sigma] \xrightarrow{\cong} \mathbb{Z}[g\sigma]$. This means that we can form the tensor product over $\mathbb{Z}[G]$ to get a double complex

$$C_* \otimes_G E_*G := C_* \otimes_{\mathbb{Z}[G]} E_*G.$$

Filtering this complex horizontally, respectively vertically, gives rise to two spectral sequences both converging to the homology of the total complex $\mathrm{Tot}(C_* \otimes_G E_*G)$. For a construction of these spectral sequences, see e.g. [Wei94, Section 5.6].

Using the horizontal filtration, $E^1_{p,q}$ is formed by taking the p'th homology with respect to $C_* \otimes_G E_qG$. If we say that X is $c(X)$-connected, this means that C_* is exact until (and including) dimension $c(X)$. Thus since E_qG is free, $C_* \otimes_G E_qG$ is also exact in this range, so $E^2_{p,q} = 0$ for $p \leq c(X)$. In particular, the spectral sequence converges to 0 in total degrees $p+q \le c(X)$, so this also holds for the spectral sequence obtained by vertical filtration.

For the spectral sequence for the vertical filtration, we get $E^1_{p,q} = H_q(C_p \otimes_G E_*G) = H_q(G; C_p)$. We can compute this. Namely,

$$H_q(G; C_p) = H_q\left(G; \oplus_{\sigma \in X^p} \mathbb{Z}[\sigma]\right).$$

This can be expressed as

$$H_q\left(G; \oplus_{\sigma \in \Omega_p} \oplus_{\tau \in \operatorname{orb}(\sigma)} \mathbb{Z}[\tau]\right),$$

where Ω_p is a set of representatives, one for each orbit of p-simplices in X. By Theorem A.9, this is isomorphic to

$$H_q\left(G; \oplus_{\sigma \in \Omega_p} \oplus_{[g] \in G/\operatorname{stab}(\sigma)} \mathbb{Z}[g\sigma]\right) = H_q\left(G; \oplus_{\sigma \in \Omega_p} \oplus_{[g] \in G/\operatorname{stab}(\sigma)} g\mathbb{Z}[\sigma]\right),$$

where the action on $\mathbb{Z}[\sigma]$ is induced by the action on X. By Lemma A.11, this is isomorphic to

$$\bigoplus_{\sigma \in \Omega_p} H_q(G; \oplus_{g \in G/\operatorname{stab}(\sigma)} g\mathbb{Z}[\sigma]).$$

By Shapiro's Lemma (Lemma A.12), this is just

$$\bigoplus_{\sigma \in \Omega_p} H_q(\operatorname{stab}(\sigma); \mathbb{Z}[\sigma]). \tag{1.2}$$

Note that this is also true for $p = -1$ if we consider a (-1)-simplex to be empty, so the stabilizer is all of G, which means that we just get $H_q(G; \mathbb{Z})$. Also, by assumption (2), $\operatorname{stab}(\sigma) \cong G_{n-r}$ for some $r = r(\sigma)$ with $1 \le r(\sigma) \le \dim(\sigma) + 1$. Moreover $\operatorname{stab}(\sigma)$ fixes σ, so $\mathbb{Z}[\sigma]$ is just a copy of $\mathbb{Z}$. Thus we have

$$E^1_{p,q} \cong \bigoplus_{\sigma \in \Omega_p} H_q(G_{n-r(\sigma)}).$$

Since Ω_p has one element for each orbit, i.e. for each element of X/G, and since two elements in the same orbit have isomorphic stabilizers by Lemma A.10, we have

$$\bigoplus_{\sigma \in \Omega_p} H_q(\operatorname{stab}(\sigma)) \cong \bigoplus_{\operatorname{orb}(\sigma) \in X/G} H_q(\operatorname{stab}(\sigma)), \tag{1.3}$$

so the q'th row of the E^1 page is just the augmented chain complex of X/G with coefficients in the local system determined by $H_q(\operatorname{stab}(\sigma))$. The differentials of this complex, i.e. the d^1-differentials of the spectral sequence, can be described explicitly. Take a simplex $\sigma \in \Omega_p$ and consider the restriction of d^1 to the summand $H_q(\operatorname{stab}(\sigma))$

$$
\begin{array}{c|ccccccccc}
i & H_i(G_n) & \xleftarrow{d} & H_i(G_{n-1}) & \longleftarrow & H_i(G_{n-2}) & \longleftarrow & \cdots & & \\
i-1 & \cdots & \longleftarrow & H_{i-1}(G_{n-1}) & \longleftarrow & H_{i-1}(G_{n-2}) & \longleftarrow & H_{i-1}(G_{n-3}) & \longleftarrow & \cdots \\
i-2 & & & & & \cdots & \longleftarrow & H_{i-2}(G_{n-3}) & \longleftarrow & \cdots \\
 & & & & & & & & & \\
q=0 & H_0(G_n) & \longleftarrow & H_0(G_{n-1}) & \longleftarrow & H_0(G_{n-2}) & \longleftarrow & H_0(G_{n-3}) & \longleftarrow & \cdots \\
\hline
 & p=-1 & & 0 & & 1 & & 2 & & \cdots
\end{array}
$$

Figure 1.1: The E^1 page of the spectral sequence for a transitive action $G \curvearrowright X$

corresponding to σ. Let $\tau \in \Omega_{p-1}$ be an orbit representative of the i'th face $\partial_i\sigma$. Then $\tau = g^{-1}\partial_i\sigma g$ for some $g \in G$, and we have a homomorphism

$$d_i^1\colon H_q(\mathrm{stab}(\sigma)) \to H_q(\mathrm{stab}(\tau))$$

induced the group inclusion $\mathrm{stab}(\sigma) \to \mathrm{stab}(\partial_i\sigma)$ followed by conjugation with g. The d^1-differential is induced by the boundary map on the augmented chain complex and is therefore given by

$$d^1 = \sum_{i=0}^{p}(-1)^i d_i^1.$$

1.1.2 The general proof of homological stability

We denote the homomorphism $H_i(G_{n-1}) \to H_i(G_n)$ by d since we will see that it occurs as the differential $d\colon E^1_{0,i} \to E^1_{-1,i}$ in the second spectral sequence given above by vertical filtering. We will determine a linear function $\phi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $\phi(i) = 2i + k$ such that d is an isomorphism for $n > \phi(i)$ and a surjection for $n = \phi(i)$. The constant k will depend on the connectivity of the complexes X_n and the quotients X_n/G_n.

If the action $G_n \curvearrowright X_n$ is transitive on all simplices and not just on vertices, and if the stabilizer of a p-simplex is always conjugate to G_{n-p-1}, then the E^1 page of the spectral sequence looks like the picture in Figure 1.1. In general the action is not necessarily transitive on simplices of dimension 1 and above. In the general case the groups $H_*(G_{n-p})$ in column p for $p \geq 1$ will be replaced by direct sums of groups $H_*(G_{n-r})$ for $1 \leq r \leq *$.

The differential d is induced by the inclusion of the stabilizer of a vertex v into all of G_n. By assumption the stabilizer is isomorphic, by conjugation with some $g \in G_n$, to the image of G_{n-1} under the standard inclusion into G_n, i.e. the diagram

$$\begin{tikzcd}
\mathrm{stab}(v) \arrow[r, hook] \arrow[d, "c_g"] & G_n \arrow[d, "c_g"] \\
G_{n-1} \arrow[r, hook] & G_n
\end{tikzcd}$$

commutes, where c_g denotes conjugation with g. By Lemma A.13, the rightmost vertical map induces the identity on homology, so on homology the upper horizontal map can be identified with the lower one. Thus d is induced by the standard inclusion $G_{n-1} \hookrightarrow G_n$, so this is the homomorphism that we will study.

We will prove the homology stability by induction, starting with the trivial case $i = 0$. For the induction step, let $n = 2i + k$ and assume that $c(X_n/G_n) \geq i$ and that $c(X_n) \geq i - 1$ where $c(-)$ denotes connectivity. For the injectivity argument later we will ramp these assumptions up by 1. Furthermore assume that for any $j < i$, $H_j(G_{m-1}) \to H_q(G_m)$ is an isomorphism for $m > 2j+k$ and a surjection for $m = 2j+k$. We then want to show that

$$H_i(G_{n-1}) \to H_i(G_n)$$

is an isomorphism for $n > 2i + k$ and surjective for $n = 2i + k$.

We start with the surjectivity argument. By the connectivity of X we have $E^\infty_{p,q} = 0$ for $p+q \leq i-1$ by the earlier arguments. In particular, $E^\infty_{-1,i} = 0$, so if we can show that every differential $d^r\colon E^r_{r-1,i-r+1} \to E^r_{-1,r}$ is 0 for $r > 1$, then the only option for killing the $(-1, i)$ term is for d to be surjective. We will show this by showing that $E^r_{p,q} = 0$ whenever $p+q = i$ and $q < i$. In fact we will show that already on the E^2 page, $E^2_{p,q} = 0$ for such p and q.

So assume that $p + q = i$ and $q < i$. Recall from (1.3) that the q'th row of the E^1 page is the augmented chain complex of X_n/G_n with coefficients in the local system given by $H_q(\mathrm{stab}(\sigma))$, so $E^2_{p,q}$ is the $(p + q)$'th homology of X_n/G_n with coefficients in the same local system. We claim that this local system is isomorphic to the local system at $H_q(G_n)$, i.e. that for $q < i$ the inclusions of stabilizers into G_n induce isomorphisms

$$\bigoplus_{\sigma\in\Omega_p} H_q(G_{n-r(\sigma)}) \cong \bigoplus_{\sigma\in\Omega_p} H_q(\mathrm{stab}(\sigma)) \cong \bigoplus_{\sigma\in\Omega_p} H_q(G_n)$$

when $p + q < i$, that they induce surjections for $p + q = i$, and that the maps in the local coefficient system given by $H_q(\mathrm{stab}(\sigma))$ all reduce to the identity under these isomorphisms. Then

$$E^2_{p,q} = H_p(X_n/G_n; H_q(G_n)),$$

so if $i \leq c(X_n/G_n)$, then $E^2_{p,q}$ will be zero as long as $p + q = i$ and $q < i$.

Let us see how the coefficient groups can be replaced. Recall that the differentials on the E^1 page are induced by inclusion of groups followed by some conjugation. This means that they fit into commutative diagrams

$$\begin{tikzcd}
H_q(\mathrm{stab}(\sigma_p)) \arrow[r, "\partial_{i*}"] \arrow[d, "incl_*"] & H_q(\mathrm{stab}(\partial_i\sigma_p)) \arrow[r, "c_{g_0*}"] & H_q(\mathrm{stab}(\tau_{p-1})) \arrow[d, "incl_*"] \\
H_q(G_n) \arrow[rr, "c_{g_0*}"] & & H_q(G_n),
\end{tikzcd}$$

where c_{g_0} denotes conjugation by a suitable element $g_0 \in G_n$, and $incl$ denotes the group inclusions. The lower map is the identity by Lemma A.13. Thus if the vertical maps are isomorphisms, then the local system $\{H_q(\mathrm{stab}(\sigma))\}$ is isomorphic to the constant system at $H_q(G)$.

Since σ_p is a p-simplex, $\mathrm{stab}(\sigma_p)$ is conjugate by assumption to $G_{n-r(\sigma_p)}$ where $1 \leq r(\sigma_p) \leq p+1$. By the induction hypothesis we have an isomorphism $H_q(G_{n-r(\sigma_p)}) \cong H_q(G_n)$ if

$$n - r(\sigma_p) \geq 2q + k.$$

Since $n - r(\sigma_p) \geq n - p - 1$, this holds in particular if

$$n \geq p + 2q + k + 1 \quad \text{which is equivalent to} \quad n > p + 2q + k.$$

But

$$p + 2q = (p+q) + q = i + q < 2i,$$

so

$$p + 2q + k < 2i + k,$$

and $2i + k < n$ by assumption, so we get the result.

For **injectivity**, the argument is similar, but with an extra step. We now assume that $n > 2i + c$, that $c(X_n/G_n) \geq i+1$ and that $c(X_n) \geq i$. Then $E^\infty_{0,i} = 0$ since it has total degree $\leq i$. Thus, similarly to before, it suffices to show that all differentials $d^r\colon E^r_{r,i-r+1} \to E^r_{0,i}, r \geq 1$ are 0 since this will force $d\colon E^1_{0,i} \to E^1_{-1,i}$ to be injective as there are no non-trivial differentials out of $E^r_{0,i}$ when $r > 1$. Once again we will argue that all the terms $E^2_{r,i-r+1}$ are 0 for $r > 1$, i.e. that $E^2_{p,q} = 0$ if $p + q = i + 1$ and $q < i$. But this time we also need the differential $d^1\colon E^1_{1,i} \to E^0_{0,i}$ to be zero. Showing that the terms $E^2_{p,q}$ are 0 is done just like before, except we need to replace the condition that $i \leq c(X_n/G_n)$ with the condition that $i + 1 \leq c(X_n/G_n)$ since everything is shifted to the right by one.

We will now show that the differential $d^1\colon E^1_{1,i} \to E^1_{0,i}$ is the zero map. This will follow from assumption (3). $E^1_{1,i}$ is the direct sum of groups $H_i(\mathrm{stab}(e))$ where e runs over edges in X_n. The differential restricted to the summand $H_i(\mathrm{stab}(e))$ is induced by the map $\delta = d^1_1 - d^1_0$, where $d^1_i = c_{g_i} \circ \partial_i$, where g_i is the element such that the conjugation c_{g_i} by g_i takes $\mathrm{stab}(\partial_i e)$ to the stabilizer of the orbit representative of $\partial_i e$.

We can choose the orbit representatives such that g_0 is the identity element and such that g_1 is the element that exists by assumption (3), and which takes one vertex of e to the other and commutes with $\mathrm{stab}(e)$, see Lemma A.14. On the group level this gives a diagram

$$\begin{tikzcd} \mathrm{stab}(e) \arrow[r, hook] \arrow[d, "c_{g_1}=\mathrm{id}"'] & \mathrm{stab}(\partial_1 e) \arrow[r, "c_{g_1}"] & g_1 \,\mathrm{stab}(\partial_1 e) g_1^{-1} \arrow[d, equal] \\ \mathrm{stab}(e) \arrow[r, hook] & \mathrm{stab}(\partial_0 e) \arrow[r, "c_{g_0}=\mathrm{id}"] & \mathrm{stab}(\partial_0 e). \end{tikzcd} \tag{1.4}$$

The diagram commutes since either way around is just conjugation by g_1. The top row is d_1^1, and the bottom row is d_0^1, showing that $d_1^1 = d_0^1$, so $\delta = d_1^1 - d_0^1 = 0$. In particular, δ induces the zero map on homology.

1.1.3 Examples

Theorem 1.3 ([HV15b, Example 1.1]). *If X_n is $(n-3)$-connected and X_n/G_n is $(n-2)$-connected, then the homomorphism $d\colon H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > 2i+2$ and a surjection for $n = 2i+1$.*

Proof. The connectivity of X_n and X_n/G_n means that we want the inequalities $i \le n-2$ for surjectivity and $i-1 \le n-2$ for injectivity, or equivalently $n \ge i+2$ for surjectivity and $n \ge i+1$ for injectivity. Thus, for the stable range function $\phi(i) = 2i+k = 2i+1$, we want to have $2i+1 \ge i+2$ for all $i \ge 1$. By the induction argument in Section 1.1.2, we always have $\phi(i) \ge \phi(i-1)+2$, so it suffices to look at $i = 1$, i.e. we want $\phi(1) = 2+1 \ge 1+2 = 2+1$. This means that the optimal value of k is $k = 1$. □

Theorem 1.4. *If X_n is $\frac{n-2}{2}$-connected, and X_n/G_n is $(n-2)$-connected, then the homomorphism $d\colon H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > 2i+2$ and a surjection for $n = 2i+2$.*

Proof. For surjectivity we need the inequality $\frac{n-2}{2} \ge i-1$ which is equivalent to $n \ge 2i$, and we also need $n-2 \ge i$. For injectivity we need $i \le \frac{n-2}{2}$ which is equivalent to $n \ge 2i+2$, and we also need $n-2 \ge i+1$ or $n \ge i+2$. To satisfy these for $i = 1$, we need

$$2+k \ge 2, k \ge 1, 2+k \ge 4, 2+k \ge 3,$$

so the optimal value of k is $k = 2$. □

Theorem 1.5 ([HV15b, Example 1.2]). *If X_n is $\frac{n-3}{2}$-connected, and X_n/G_n is still $(n-2)$-connected, then the homomorphism $d\colon H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > 2i+2$ and a surjection for $n = 2i+2$.*

Proof. For surjectivity we need $i-1 \le c(X_n) = \frac{n-3}{2}$, which is equivalent to $2i-2 \le n-3$, which is again equivalent to $n \ge 2i+1$, and we still need $i \le n-2$. For injectivity we need $i \leq c(X_n) = \frac{n-3}{2}$, which is equivalent to $n \geq 2i+3$, and we still need $n \geq i+1$. To satisfy these for $i = 1$, we need

$$\phi(i) = 2i + c < 2i + 3,$$

so the optimal value of k is $k = 2$. □

Theorem 1.6. *If each X_n is $(n-2)$-connected, and each X_n/G_n is $(n-3)$-connected, then the homomorphism $d\colon H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > 2i+2$ and a surjection for $n = 2i+2$.*

Proof. For surjectivity we need $n \geq i+3$, and for injectivity we need $n \geq i+4$. To satisfy this for $i = 1$ we need $\phi(1) = 2 + k \geq 4$, so the optimal value of k is $k = 2$. □

We will also need the following stable homology result, where the setting is slightly different.

Theorem 1.7. *Let G_∞ be the direct limit of a sequence of group inclusions $i_n\colon G_n \hookrightarrow G_{n+1} \to \cdots$, and let $\lambda\colon G_\infty \to G_\infty$ be a self inclusion. Suppose that G_∞ acts on a contractible and infinite dimensional simplicial complex X_∞ such that:*

(1) $G_\infty \curvearrowright X_\infty$ is transitive on the vertices of X_∞.

(2) For a p-simplex σ_p, the stabilizer $\mathrm{stab}(\sigma_p)$ fixes σ_p pointwise, and the stabilizer $\mathrm{stab}(\sigma_p) \subset G_\infty$ is conjugate to the image of λ^{p+1}.

(3) If e is an edge of X_∞ with vertices v and w, there is a $g \in G_\infty$ such that $gv = w$ and such that $gh = hg$ for all $h \in \mathrm{stab}(e)$.

(4) X_∞ and X_∞/G_∞ are both contractible.

Then the induced homomorphism

$$\lambda_*\colon H_i(G_\infty) \to H_i(G_\infty)$$

is an isomorphism for all i.

Proof. All of the previous arguments used in Sections 1.1.1 and 1.1.2 can be used in this case, replacing the groups G_n and the complexes X_n by G_∞ and X_∞, and replacing the homomorphisms $G_n \to G_{n+1}$ by λ. In fact, some of the arguments are simpler since we no longer have to worry about connectivity. □

1.2 Connectivity tools

Whenever we have a simplicial complex X which we want to show is highly connected, it turns out that it is often convenient to embed it into a larger complex which is highly connected. In fact, we will usually have a string of multiple embeddings where the largest complex is contractible. Sometimes these embeddings can tell us that the smaller complex inherits some connectivity from the larger one, and sometimes they will even be homotopy equivalences. This section contains various ways to show this.

1.2.1 Bad simplex arguments

Let M be a smooth n-manifold with a finite triangulation, let X be a simplicial complex, let $f\colon M \to X$ be a simplicial map, and let Y be a subcomplex of X. We will show that under some conditions f can be homotoped to a map with image in Y that is constant on all simplices that already land inside Y. When M is some sphere S^k, this is can be used to show that the subcomplex Y is highly connected. The trick is then to decide which simplices are bad in the sense that they are far from mapping into Y and then deforming the restrictions of f to these simplices to eventually obtain a map with image in Y.

Definition 1.8. A set of simplices in $X \setminus Y$ may be called a set of **bad simplices** if it satisfies

(1) If a simplex σ in X has no bad faces (in which case we say that σ is *good*), it is contained in Y.
(2) If σ and τ are bad simplices that are faces of the same simplex, then the join $\sigma * \tau$ is also a bad simplex.

A simplex with no bad faces is called a **good simplex**, and the faces of a bad simplex need not be bad. Moreover, a simplex may be neither good nor bad. For a bad simplex σ, we say that a simplex $\tau \in \operatorname{link}(\sigma)$ is **good for** σ if any bad face of the join $\tau * \sigma$ is contained in σ. In that case, no face of τ is bad, so in particular τ is a good simplex and therefore is contained in Y. We denote by G_σ the subcomplex of $\operatorname{link}(\sigma)$ consisting of simplices that are good for σ.

Proposition 1.9 ([HV15b, Proposition 2.1])**.** *Let $f\colon M \to X$, Y and G_σ be as above. If G_σ is $(\dim(M) - \dim(\sigma) - 1)$-connected for all bad simplices σ, then f is homotopic to a map with image in Y by a homotopy that is constant on simplices whose images already lie in Y.*

Proof. Let a simplex μ in M be maximal such that $\sigma = f(\mu)$ is bad. Then we have $f(\operatorname{link}(\mu)) \cap \sigma = \emptyset$ by maximality of μ, so $f(\operatorname{link}(\mu)) \subset \operatorname{link}(\sigma)$. We claim moreover that $f(\operatorname{link}(\mu))$ is contained in G_σ. Assume towards a contradiction that is is not. Then there is a simplex $\nu \in \operatorname{link}(\mu)$ such that some face of $f(\nu) * \sigma$ is bad and is not contained in σ (note that $f(\nu) * \sigma$ is a simplex in X since $f(\nu) \subset \operatorname{link}(\sigma)$). If we choose ν minimally such that $f(\nu) * \sigma$ is bad, we know that such a bad face must be on the form $f(\nu) * \sigma_0$ for some face of σ. Then by property (2) the simplex

$$(f(\nu) * \sigma_0) * \sigma = f(\nu) * \sigma = f(\nu) * f(\mu) = f(\nu * \mu)$$

is bad since both $f(\nu) * \sigma_0$ and σ are bad. This is a contradiction since μ is maximal such that σ is bad.

We want to homotope f on $\operatorname{star}\mu$ such that the number of simplices with bad images is reduced. Note that since M is a smooth n-manifold, we can assume that M has a finite piecewise linear triangulation such that the link of a simplex that is not included in the boundary is a sphere, see Theorem A.4. In that case $\operatorname{star}(\mu) = \mu * \operatorname{link}(\mu) \cong D^{n-k+1}$,

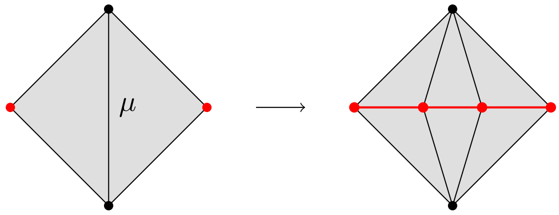


Figure 1.2: Retriangulation of star μ

and link(μ) is either contractible or homeomorphic to S^{n-k-1} where $n = \dim(M)$ and $k = \dim(\mu) \geq \dim(\sigma)$. In the case depicted in Figure 1.2, $n - k - 1 = 2 - 1 - 1 = 0$, so $S^{n-k-1} = S^0$.

Since $-k \leq -\dim(\sigma)$, G_σ is $(n - k - 1)$-connected by assumption, so the restriction of f to link(μ) can be extended to a map $g\colon D^{n-k} \to G_\sigma$ over the disk $D^{n-k} \subset \mathrm{star}(\mu)$, which we can assume is simplicial by Theorem A.5. Then we can retriangulate star(μ) as $\partial\mu * D^{n-k}$ and modify f to a new map $\bar{f}$ by defining it on the new triangulation as $f_{|\partial\mu} * g$, as indicated on Figure 1.2. In the figure, μ is the 1-simplex in the middle, link(μ) is the two red points, and $D^{n-k} = D^1$ is the red line going through the new triangulation. This new map $\bar{f}$ agrees with f on all simplices except μ. In particular it agrees with f on all simplices whose images already lie in Y. It is homotopic to f since it agrees with f on $\partial\,\mathrm{star}(\mu)$ and star(μ) is homeomorphic to D^{n-k+1}. The homotopy can be chosen to be constant on $\partial\,\mathrm{star}(\mu)$, in particular on all simplices whose images already lie in Y.

Since g maps into G_σ, any bad faces of $\bar{f}(\mathrm{star}(\mu))$ have to be contained in $f(\partial\mu)$. Thus we have passed to a situation where only proper faces of μ can have bad images, so we have reduced the number of simplices with bad images. Since the triangulation of M is finite, this process eventually ends such that no simplices of M have bad images. Thus by property 1 for bad simplices, the image of the resulting map is contained in Y. Since it is homotopic to f, we get the result. □

Corollary 1.10 ([HV15b, Corollary 2.2]). *Let Y be a subcomplex of an n-connected complex X, and suppose $X \setminus Y$ has a set of bad simplices satisfying (1) and (2) above. If G_σ is $(n - \dim(\sigma))$-connected for all bad simplices σ, then Y is n-connected.*

Proof. Let $g_0\colon S^i \to Y$ be a simplicial map with $i \leq n$. We want to extend g_0 to a map $g\colon D^{i+1} \to Y$ which agrees with g_0 on S^i. Since X is n-connected, we can extend g_0 to a map $f\colon D^{i+1} \to X$. Now use Proposition 1.9 with $M = D^{i+1}$. For any bad simplex σ, $\dim(D^{i+1}) - \dim(\sigma) - 1 = n + 1 - \dim(\sigma) - 1 = n - \dim(\sigma)$, so G_σ has connectivity $\dim(D^{i+1}) - \dim(\sigma) - 1$ by assumption. Thus f is homotopic to a map $g\colon D^{i+1} \to Y$ which agrees with f on S^i and therefore also with g_0. □

Let L be a finite set whose elements we call labels. Given a simplicial complex $X = (V, S)$, we can form a simplicial complex $X^L = (V^L, S^L)$ where $V^L = V \times L$, and where a simplex in S^L is the image of a map $\phi\colon \sigma \to V^L$ where $\sigma \in S$, and for all $v \in \sigma$,

$\phi(v) = (v, l)$ for some $l \in L$. This means that the simplices in X^L are the simplices from X with all possible labellings by elements of L, so for each k-simplex of X, there are $|L|^{k+1}$ k-simplices of X^L.

Corollary 1.11 ([HV15b, Corollary 2.3]). *Let $X = (V, S)$ be a simplicial complex and L a finite set of labels. If X is n-connected, and the link of each k-simplex in X is $(n-k-1)$-connected, then X^L is n-connected.*

Proof. Let $f\colon S^i \to X^L$ be a simplicial map with $i \leq n$. We want to extend f to a map $D^{i+1} \to X^L$. Fix an element $l_0 \in L$, and let Y be the subcomplex of X^L consisting of all simplices with vertices on the form (v, l_0) with $v \in V$. Then Y is isomorphic to X by just sending any (v, l_0) to v, and X is n-connected. We want to show that f is homotopic to a map with image in Y so that we can extend it over the disk since $Y \cong X$ is n-connected.

We say that a simplex in X^L is bad if all of its vertices have labels that are not l_0. Then a simplex with no bad faces in particular has no bad vertices and is therefore contained in Y, so condition (1) of Definition 1.8 is satisfied. The join of two bad simplices consists of the vertices of those two simplices, so no vertex of the join can be on the form (v, l_0), i.e. the join is also bad, and condition (2) is satisfied. If σ is a bad simplex, a simplex in $\operatorname{link}(\sigma)$ is good for σ if and only if it consists only of vertices on the form (v, l_0). This is due to the fact that a simplex τ in $\operatorname{link}\sigma$ contains a vertex (v, l') with $l' \neq l_0$ if and only if and only if $\tau * \sigma$ has a bad face consisting of the vertices of σ and (v, l') which are not contained in σ. Thus G_σ is isomorphic to $\operatorname{link}(\sigma') \subset X$, where σ' consists of all vertices v where $(v, l) \in \sigma$ for some l, i.e. σ' is the simplex in X corresponding to σ.

This means that G_σ is $(n - \dim(\sigma') - 1)$-connected for all bad simplices σ, so it is $(n - \dim(\sigma) - 1)$-connected since $\dim \sigma = \dim \sigma'$. Thus by Proposition 1.9, f is homotopic to a map with image in $Y \cong X$ and therefore extends over the disk. □

Lemma 1.12 (Coloring lemma, [HW07, Lemma 3.1]). *Assume that we have a triangulation of S^k (see Appendix A.1.1) with its vertices labeled by elements of a set L with at least $k+2$ elements. We say that a simplex in the triangulation is bad if each of the labels used for its vertices occurs at least twice. This labeled triangulation can be extended to a labeled triangulation of D_{k+1} whose only bad simplices lie in S^k, where the triangulation of S^k is a full subcomplex of that of D_{k+1}. Moreover, the labels of the vertices in the interior of D_{k+1} can be chosen to be contained in any subset of $E_0 \subset E$ with at least $k+2$ elements.*

Proof. We prove the lemma by induction on the number k, starting with the case $k = -1$. In that case $D_{k+1} = D_0$ is a point that can be labeled by any element of L_0. The set L_0 is non-empty since it has at least $k+2$ elements and $k+2 = 1$. Now assume that the lemma holds for all numbers less than k, and consider any triangulation of S^k labeled by L. Triangulate D_{k+1} by adding a vertex in the center and then coning off S^k to that vertex. Label the center vertex by any element of L_0. We will modify this triangulation without changing it on S^k until all bad simplices are contained in S^k.

If there are any bad simplices in D_{k+1} that are not contained in S^k, choose one such σ of maximal dimension p, i.e. such that σ is not contained in a bad simplex of dimension strictly larger than p. We must have $p > 0$ since σ is bad. Let $E_\sigma \subset E$ be the set of elements occurring as labels of σ. Since σ is not contained in S^k, or in other words the boundary of D_{k+1}, the link of σ is a $(k+1-p-1) = (k-p)$-sphere by Theorem A.4. The vertices of $\operatorname{link}\sigma$ are labeled by elements of $E \setminus E_\sigma$ since σ is bad of maximal dimension. Since $k-p < k$, we can use the induction hypothesis on $\operatorname{link}\sigma$ using labels from $E_0 \setminus E_\sigma \subset E \setminus E_\sigma$, where E_0 is any subset of E with at least $k+2$ elemtns. Namely, since $|E_\sigma| \leq p$, $|E_0 \setminus E_\sigma| \geq k+2-p = (k-p)+2$. This gives a triangulation of the disk $D^{k-p+1} \subset \operatorname{star}\sigma$ bounded by $\operatorname{link}\sigma = S^{k-p}$. Join this triangulation with $\partial\sigma$ to get a new triangulation of $\operatorname{star}\sigma$ as in the proof of Proposition 1.9. This triangulation agrees with the old one on simplices of $\partial \operatorname{star}\sigma$.

A simplex in the new triangulation must be of the form $\tau * \mu$ where τ is a face of σ and μ is a simplex in the disk D^{k-p+1}. If $\tau * \mu$ is bad, then μ must be empty since the set of labels of τ is disjoint from that of σ, and μ cannot be a bad simplex since then it would be contained in $S^{k-p} = \operatorname{link}\sigma$, contradicting the maximality of σ. This means that we have reduced the number of bad simplices of D_{k+1} of dimension p. Repeat the process until all bad simplices are contained in S^k.

To see that the fullness condition is still satisfied after retriangulating, note that the initial coning preserves fullness. If the vertices of a simplex $\tau * \mu$ as above in the new triangulation all lie in S^k, then μ must be contained in ∂D_{k-p+1} by the induction hypothesis on the sphere $S^{k-p} = \operatorname{link}\sigma$. Therefore, $\tau * \mu$ lies in the boundary of $\operatorname{star}\sigma$ where the triangulation has not been changed, so $\tau * \mu \subset S^k$ by assumption. $\square$

1.2.2 Poset complexes

Let P be a poset. One can form a simplicial complex with vertices in P and a k-simplex for each totally ordered chain $p_0 < \cdots < p_k$ in P. It is a simplicial complex since a face of a simplex corresponds to a totally ordered subchain. The **geometric realization** of P is then the geometric realization of this simplicial complex. A poset map (i.e. an order preserving function of posets) induces a simplicial map on the geometric realizations. When talking about topological properties of a poset or a poset map, we are talking about the geometric realization. In this section we establish tools to show that certain poset complex are highly connected as well as sufficient conditions for a poset map to be a homotopy equivalence.

Let $\phi\colon P \to Q$ be a poset map, and define the fiber $\phi_{\leq q}$ over an element of Q to be the sub-poset of P consisting of all $p \in P$ such that $\phi(p) \leq q$, and similarly define $\phi_{\geq q}$ as all $p \in P$ such that $\phi(p) \geq q$.

Lemma 1.13 (Quillen's Fiber Lemma, [HV15b, Proposition 2.5])**.** *Let $\phi\colon P \to Q$ be a poset map. If all fibers $\phi_{\leq q}$ are contractible or all fibers $\phi_{\geq q}$ are contractible, then ϕ is a homotopy equivalence on the geometric realizations.*

Proof. We can turn around the partial orders on P and Q without changing the geometric realizations or the corresponding map ϕ. Thus the two conditions are equivalent, so let us just assume that all $\phi_{\geq q}$ are contractible. We will construct $\psi\colon Q \to P$ inductively as follows. For each vertex q_0 in Q, we let $\psi(q_0)$ be any vertex in $\phi_{\geq q_0}$. This is possible since $\phi_{\geq q_0}$ is contractible and thus non-empty. We now extend ψ to edges. An edge in Q is a chain $q_0 < q_1$, and both $\psi(q_0)$ and $\psi(q_1)$ are in $\phi_{\geq q_0}$ since $\phi(p) \geq q_1$ implies $\phi(p) \geq q_0$, so $\phi_{\geq q_1} \subseteq \phi_{\geq q_0}$. Now let $\psi(q_0 < q_1)$ be any path inside $\phi_{\geq q_0}$ from $\psi(q_0)$ to $\psi(q_1)$ (which is possible since $\phi_{\geq q_0}$ is contractible and thus path-connected). Similarly, if we have defined ψ on $(n-1)$-simplices, then, given an n-simplex σ in Q, we have already defined ψ on the boundary of σ as a map into $\phi_{\geq \sigma_0}$ where σ_0 is the least vertex of σ under the partial ordering of Q, and we can then extend ψ to σ since $\phi_{\geq \sigma_0}$ is contractible and thus $(n-1)$-connected.

We claim that ψ is a homotopy inverse to ϕ. Indeed, $\phi\psi$ sends a simplex $q_0 < \cdots < q_k$ into $Q_{\geq q_0}$ (i.e. the elements of Q that are weakly greater than q_0) since ψ sends it into $\phi_{\geq q_0}$,. Moreover $Q_{\geq q_0}$ is contractible since it has a minimal element, so any simplex is a face of a simplex containing q_0 and thus there is a contraction from the whole space to q_0. This means that we can construct a homotopy from $\phi\psi$ to the identity inductively as follows:

Since $\phi\psi$ sends each vertex q_0 into $Q_{\geq q_0}$, and since each $Q_{\geq q_0}$ is contractible and contains q_0, the restriction of $\phi\psi$ to vertices is homotopic to the identity. Now assume that the restriction of $\phi\psi$ to $(n-1)$-simplices is homotopic to the identity by a homotopy that maps each simplex $\tau_0 < \cdots < \tau_n$ into $Q_{\geq \tau_0}$ at any stage, and fix such a homotopy F. Let $\sigma = (q_0 < \cdots < q_k)$ be an n-simplex in Q. Then the restriction of $\phi\psi$ to the boundary of σ (which is a sphere) is homotopic to the identity by the restriction of F to the boundary of σ. This has image inside the subcomplex $Q_{\geq q_0}$. Since $Q_{\geq q_0}$ is contractible and contains σ, this homotopy can be extended to a homotopy from the restriction of $\phi\psi_{|\sigma}$ to the identity on σ. This can be done continuously for all n-simplices since we only extend to the interiors of those n-simplices.

Similarly, $\psi\phi$ maps each simplex $p_0 < \cdots < p_k$ into the contractible subcomplex $\phi_{\geq \phi(p_0)}$ and so is homotopic to the identity by a similar homotopy. □

Remark 1.14. If X is a simplicial complex, we can consider the poset $\widehat{X}$ of simplices of X under face inclusion. The geometric realization of this poset can be identified with the barycentric subdivision of X. A simplicial map $\phi\colon X \to Y$ induces a poset map $\hat{\phi}\colon \widehat{X} \to \widehat{Y}$. Under the canonical homeomorphisms $X \cong \widehat{X}$ and $Y \cong \widehat{Y}$, the induced poset map $\hat{\phi}$ corresponds to the original map ϕ.

Lemma 1.15 ([HV15b, Lemma 2.6])**.** *Let $f\colon X \to Y$ be a simplicial map of simplicial complexes, let $\widehat{X}$ be the poset of simplices in X, and let $\widehat{Y}$ be the poset of simplices in Y. Let $\hat{f}\colon \widehat{X} \to \widehat{Y}$ be the induced poset map. Then for any simplex σ of Y we have the following:*

(1) $\hat{f}_{\leq\sigma}$ is homeomorphic to $f^{-1}(\sigma)$,
(2) $\hat{f}_{\geq\sigma}$ is homotopy equivalent to $\hat{f}^{-1}(\sigma)$,
(3) $\hat{f}^{-1}(\sigma)$ is homeomorphic to $f^{-1}(y)$ where y is the barycenter of σ.

Proof. For (i), $\hat{f}_{\leq\sigma}$ is the set of all simplices τ such that $f(\tau) \leq \sigma$, i.e. such that $f(\tau)$ is a face of σ. But this is equivalent to $f(\tau) \subseteq \sigma$, so $\hat{f}_{\leq\sigma}$ is just the barycentric subdivision of $f^{-1}(\sigma)$.

For (ii), $\hat{f}_{\geq\sigma}$ consists of all simplices τ such that σ is a face of $f(\tau)$. Therefore, since f is simplicial, some face of τ maps into σ. Let τ_σ be maximal such that this is the case. Then τ_σ is uniquely determined, since if some other face τ'_σ of τ maps to σ, then $\tau_\sigma * \tau'_\sigma$ is also a face of τ that maps to σ, and so by maximality $\tau_\sigma = \tau_\sigma * \tau'_\sigma$. The map $\phi\colon \hat{f}_{\geq\sigma} \to \hat{f}^{-1}(\sigma)$ that takes each τ to τ_σ is a poset map since $\tau \subseteq \tau'$ implies $\tau_\sigma \subseteq \tau'_\sigma$. For any τ such that $f(\tau) \subset \sigma$, consider

$$\phi_{\geq\tau} = \{\chi \subset X \mid \chi_\sigma \supset \tau\}.$$

Since $f(\tau) \subset \sigma$, $\tau_\sigma = \tau$, so this set has τ as minimal element and therefore is contractible, so ϕ is a homotopy equivalence by Lemma 1.13.

For (iii), note that $\hat{f}^{-1}(\sigma)$ is the subcomplex of $\widehat{X}$ consisting of all chains $\tau_0 < \cdots < \tau_n$ such that $f(\tau_i) = \sigma$ for all $i = 0, \ldots, n$. Assume that $\tau_0 < \cdots < \tau_n$ is a maximal such chain. Then $\nu_0 := f^{-1}(y) \cap \tau_0$ is a point. More generally each $\nu_i := f^{-1}(y) \cap \tau_i$ can be identified with an i-simplex in such a way that ν_i is a face of ν_j whenever $i \leq j$. Thus $f^{-1}(y)$ can be identified with $\hat{f}^{-1}(\sigma)$. $\square$

Corollary 1.16 ([HV15b, Corollary 2.7])**.** *Let $f\colon X \to Y$ be a simplicial map of simplicial complexes. If $f^{-1}(\sigma)$ is contractible for all closed simplices σ, or if $f^{-1}(y)$ is contractible for all barycenters y, then f is a homotopy equivalence.*

Proof. If $f^{-1}(\sigma)$ is contractible for all σ, then all lower fibers of $\hat{f}$ are contractible by Lemma 1.15 (1). Thus $\hat{f}$ is a homotopy equivalence by Lemma 1.13, and therefore f is a homotopy equivalence too by Remark 1.14.

If $f^{-1}(y)$ is contractible for all barycenters y, then all fibers $\hat{f}_{\geq\sigma}$ are contractible by Lemma 1.15 (2) and (3), so f is again a homotopy equivalence. $\square$

Definition 1.17. If X is a simplicial complex, we denote by $\hat{X}_m$ the **high-dimension subcomplexes** of the poset complex $\hat{X}$ consisting of all simplices of $\hat{X}$ whose vertices are simplices of X of dimension at least $m-1$. Moreover, we say that a simplicial complex is **weakly Cohen-Macaulay** of dimension n if it is $(n-1)$-connected and the link of any p-simplex is $(n-p-2)$-connected.

Lemma 1.18 ([HW07, Lemma 3.8])**.** *If X is weakly Cohen-Macaulay of dimension n, then $\hat{X}_m$ is $(n-m)$-connected.*

Proof. The proof of this is similar to the proof of Proposition 1.9. We proceed by induction on m. The base case $m = 1$ is true by assumption since $\hat{X}_1 = \hat{X}$ and X is $(n-1)$-connected. Now let $m > 0$, and let a map $f\colon S^k \to \hat{X}_m$ with $k \leq n-m$ be given. By the induction hypothesis we can extend f to a map $F\colon D_{k+1} \to \hat{X}_{m-1}$ that agrees with f on S^k since $\hat{X}_{m-1}$ is $(n-m+1)$-connected and $k \leq n-m < n-m+1$. We now say that a simplex of D_{k+1} is bad if F maps each of its vertices to a simplex of X with $m-1$ vertices. Then any bad simplex must be contained in the interior of D_{k+1}. Let σ

be a bad p-simplex, and assume that p is maximal, i.e. that there are no bad simplices of higher dimension. Then f must be constant on σ since a simplex in $\hat{X}_{m-1}$ is a string of strict inclusions of simplices of X, and f must map each vertex of σ to a simplex with exactly $m-1$ vertices. The link of σ is a $(k-p)$-sphere, so the restriction of f to $\operatorname{link}\sigma$ can be seen as a map

$$f_{\operatorname{link}\sigma}\colon S^{k-p} \to \widehat{\operatorname{link}_X(f(\sigma))},$$

where $\operatorname{link}_X(f(\sigma))$ is the link of $f(\sigma)$ as a simplex in X. This is $(n-m)$-connected by assumption since $f(\sigma)$ is an $(m-2)$-simplex of X. We have $k-p \leq k \leq n-m$, so $f_{\operatorname{link}\sigma}$ can be extended to a map

$$g_\sigma\colon D_{k-p+1} \to \widehat{\operatorname{link}_X(f(\sigma))}.$$

We rewrite $\operatorname{star}\sigma$ as $D_{k-p+1} * \partial\sigma$ like in Proposition 1.9, and we replace F on $\operatorname{star}\sigma$ by $(g_\sigma \cup f(\sigma)) * F$, where $g_\sigma \cup f(\sigma)$ is the map that takes a simplex τ to the join $g_\sigma(\tau) * f(\sigma)$. Since g_σ maps into $\widehat{\operatorname{link}_X(f(\sigma))}$, $g_\sigma(\tau) * f(\sigma)$ has at least m vertices, so it is a good simplex. This means that we has eliminated a bad simplex of maximal dimension. If we continue this process until there are no more bad simplices, we get a map $D_{k+1} \to \hat{X}_m$ that agrees with f on S^k, showing that $\hat{X}_m$ is $(n-m)$-connected. □

1.2.3 Fiber connectivity

Lemma 1.19 ([HV15b, Lemma 2.8]). *Let $f\colon X \to Y$ be a simplicial map of simplicial complexes. Suppose that Y is n-connected and that the fiber $f^{-1}(y)$ over the barycenter y of any k-simplex in Y is $(n-k)$-connected. Then X is n-connected.*

Proof. Let a map $g\colon S^i \to X$ with $i \leq n$ be given. Assume that g is simplicial. We want to extend g to a map $G\colon D^{i+1} \to X$. Consider the composition

$$h = fg\colon S^i \to Y.$$

Since Y is n-connected, we can extend h to a map

$$H\colon D^{i+1} \to Y.$$

We will use H to construct G inductively over k by defining it on the k'th skeleton of the barycentric subdivision D' of D^{i+1}.

First replace the complexes and maps by the simplicial posets and their corresponding poset maps. This means that we look at the following diagram.

$$\begin{tikzcd} \widehat{S^i} \arrow[r, "\widehat{g}"] \arrow[d, hook] & \widehat{X} \arrow[d, "\widehat{f}"] \\ \widehat{D}^{i+1} \arrow[r, "\widehat{H}"] & \widehat{Y} \end{tikzcd}$$

For the induction start, we want to define G on vertices such that it agrees with g on $\partial D'$, so let a vertex $\tau \subset D'$ be given, and view τ as a simplex in D^{i+1} (since vertices in D' correspond exactly to simplices in D^{i+1}) or a vertex of $\widehat{D}^{i+1}$. Since H is a simplicial map, $\sigma := H(\tau)$ has at most dimension $\dim(D^{i+1}) = i+1 \leq n+1$ (since we assumed $i \leq n$). By the assumption and Lemma 1.15, $\widehat{f}_{\geq \sigma}$ is $(n-(n+1)) = (-1)$-connected, so it is non-empty. Thus we can choose $x \in \widehat{f}_{\geq \sigma}$ and define $G(\tau) = x$. Note that if $\tau \subset \partial D'$, then $\sigma = H(\tau) = h(\tau) = fg(\tau)$, so $g(\tau) \in \hat{f}_{\geq \sigma}$. Thus we can choose x to be $g(\tau)$, i.e. we can make G agree with g on $\partial D'$.

For the induction step, assume that we have defined G on the $(k-1)$-skeleton of D' as well as on $\partial D'$. Let

$$\tau_0 < \cdots < \tau_k$$

be a k-simplex in D'. We want to define G on $\tau_0 < \cdots < \tau_k$. Let $\sigma_i = H(\tau_i)$. Then for any j and any simplex α in $\widehat{f}_{\geq \sigma_j}$ we have

$$\alpha \geq \sigma_j \geq \sigma_0,$$

for all j, so $\widehat{f}_{\geq \sigma_j} \subset \widehat{f}_{\geq \sigma_0}$ for all j. By construction G maps any face

$$\beta = (\sigma_0 < \cdots < \sigma_{i-1} < \sigma_{i+1} < \cdots < \sigma_k)$$

to $\widehat{f}_{\geq \beta} \subset \widehat{f}_{\geq \sigma_0}$ and thus maps the entire boundary of $\tau_0 < \cdots < \tau_k$ into $\widehat{f}_{\geq \sigma_0}$. Now, $\tau_0 < \cdots < \tau_k$ is a simplex in D', so it is a strictly increasing chain of simplices in D^{i+1}. Thus, since $\dim \tau_k \leq i+1$, we have $\dim(\tau_{k-1}) \leq i$, and so on. Continuing in this fashion, we get $\dim(\tau_0) \leq i+1-k$. Since H is a simplicial map, it cannot increase the dimension of a simplex, so

$$\dim(\sigma_0) = \dim(H(\tau_0)) \leq \dim(\tau_0) \leq i+1-k \leq n+1-k.$$

Note that $\widehat{f}_{\geq \sigma_0}$ is homotopy equivalent to $\widehat{f}^{-1}(\sigma_0)$ by Lemma 1.15 (2), and this is homeomorphic to $f^{-1}(y)$ where y is the barycenter of σ by Lemma 1.15 (3). By assumption the connectivity of $f^{-1}(y)$ is at least $n-(n+1-k) = k-1$, so $\hat{f}_{\geq \sigma_0}$ is at least $(k-1)$-connected as well. Since the boundary of $\tau_0 < \cdots < \tau_k$ is S^{k-1}, and since G maps $\tau_0 < \cdots < \tau_k$ into a $(k-1)$-connected subspace, we can extend G over the interior of $\tau_0 < \cdots < \tau_k$ such that the new extended map agrees with G on the boundary of $\tau_0 < \cdots < \tau_k$. Since D' is a simplicial complex and $\partial D'$ is a subcomplex, either the simplex $\tau_0 < \cdots < \tau_k$ is entirely inside $\partial D'$, or only (a part of) the boundary is. This means that the extended map indeed does agree with G on $\partial D'$ as well. □

1.2.4 Flowing into a subcomplex

Let X be a simplicial complex, and let $Y \subset X$ be a subcomplex. Suppose that we have a deformation retraction $F \colon X \times I \to X$ of X onto Y. Then for each $x \in X$, we get a path from x to a point in Y defined by $F(x,t)$, $t \in I$. We want to work backwards and

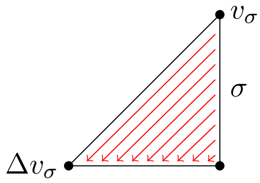


Figure 1.3: Flowing towards a subcomplex

instead create a family of paths and assemble them to a deformation retraction, a flow, onto Y. It will go as follows:

For each simplex $\sigma \subset X \setminus Y$, choose a preferred vertex $v_\sigma \subset \sigma$ and a simplex $\Delta v_\sigma \subset \mathrm{link}(v_\sigma)$ such that $\sigma * \Delta v_\sigma$ is a simplex in X. Now there is a straight line segment from v_σ to the barycenter of Δv_σ which travels inside of $\sigma * \Delta v_\sigma$. We then decompose $\sigma * \Delta v$ into line segments called flow lines that are parallel to this straight line, and we will deform the complex along the flow lines. See Figure 1.3 for an example when σ is a 1-simplex and Δv_σ is a vertex.

The flow lines should in some sense correspond to moving the point of $X \setminus Y$ closer to Y. We measure this using a complexity function. A complexity function is defined by first defining a function c from the vertices of X to $\mathbb{Z}_{\geq 0}$ such that $c(w) > 0$ for all vertices w in $X \setminus Y$. We can then extend c to all simplices of X by letting the value on a simplex be the sum of the values on its vertices. We will need to have a nice choice of preferred vertices and a complexity function such that the following lemma establishes a deformation retraction.

Lemma 1.20 ([HV15b, Lemma 2.9]). *Let Y be a subcomplex of a simplicial complex X with a complexity function c as described above. Suppose that for each vertex $v \in X \setminus Y$ there is a rule for associating a simplex $\Delta v \in \mathrm{link}(v)$, and that for each simplex σ that is not contained in Y there is a rule for picking a preferred vertex $v_\sigma \subset \sigma \subset X \setminus Y$ such that it always holds that*

(1) $\sigma * \Delta v_\sigma \in X$,
(2) $c(\Delta v) < c(v)$,
(3) if $v_\sigma \subset \tau \subset \sigma$, *then* $v_\tau = v_\sigma$.

Then Y is a deformation retract of X.

Proof. For each simplex σ in X that is not contained in Y we construct flow lines inside $\sigma * \Delta v_\sigma$ that are parallel to the line from v_σ to the barycenter of Δv_σ. In terms of barycentric coordinates in $\sigma * \Delta v_\sigma$ we are removing the coefficient of v_σ and equally distributing it among the vertices of Δv_σ while fixing the coefficients of the other vertices. This means that we have moved all the points of σ into $\sigma_{v_\sigma} * \Delta v_\sigma$ where σ_{v_σ} is the face of σ obtained by removing the vertex v_σ. Note that σ_{v_σ} has lower complexity than σ by condition (2), so we have moved the points "closer to Y".

If $\sigma_{v_\sigma} * \Delta v_\sigma$ is contained in Y, we stop, and if not we can repeat the process. Continuing in this way, all points of σ travel by a polygonal path in X into Y since the process eventually stops (when the complexity reaches 0 at the latest).

We want to do this to all of X simultaneously and continuously to get a deformation retraction onto Y. This means that the flow of any face of any simplex should be compatible with the flow of its "mother simplex". But this is ensured by condition (3). Thus we can define the flow by letting each point follow its destined polygonal path, but we need to do it continuously. Therefore we equip all simplices with the standard Euclidean metric and define the flow by letting all the points not in Y follow their polygonal paths at a constant speed such that they reach Y at exactly $t = 1$. $\square$

This concludes our toolbox chapter, and we are ready to move on and prove some actual homological stability results.

Chapter 2

Stability for braid groups

The classical braid group B_n on n strands can be included in B_{n+1} by adding a strand that is not intertwined with the others, producing a sequence $B_0 \to B_1 \to \dots$. We will show that this sequence is homologically stable. To do this we view B_n as the boundary fixing mapping class group $\mathcal{M}D(n)$ of a disk $D(n)$ with n marked points. More precisely we first form the group $\mathrm{Diff}^+(D_2 \text{ rel } S^1)$ of orientation preserving self-diffeomorphisms of D_2 fixing the boundary. We then take equivalence classes up to isotopy. The result is a group $\mathcal{M}D_2$ that we call the boundary fixing mapping class group of D_2. In fact, this group can be constructed for any orientable smooth surface; we will study these in the next chapter. Considering n distinct marked points in the interior of D_2 we then form the subgroup of $\mathcal{M}D_2$ consisting of isotopy classes of diffeomorphisms with some representative that permutes the marked points. This group is isomorphic to the braid group B_n as shown in [FM11, Section 9.1]. If x is some point in the interior of $D(n)$, any diffeomorphism of $D(n)$ can be isotoped to a diffeomorphism that fixes x. Thus we can embed $\mathcal{M}D(n)$ into $\mathcal{M}D(n+1)$ as the mapping classes fixing the $(n+1)$'st marked point, and this embedding corresponds to the inclusion $B_n \to B_{n+1}$ described above.

2.1 Complexes of tethers

We will construct a simplicial complex of so called tethers with a B_n-action. Consider as above the marked disk $D(n)$. Then choose distinct points $b_1, \dots, b_d$ on the boundary of D_2. A **tether** is an arc in D that connects a marked point p_i to a boundary point b_j such that it is disjoint from all the other boundary points and marked points. A **system of tethers** is a collection of tethers which are disjoint except at their endpoints such that no two tethers in the system are isotopic to each other; here we require isotopies to fix the endpoint p_i, and the other endpoint is allowed to move only along the boundary.

Let $X = X_{n,d}$ be the simplicial complex with one k-simplex for every isotopy class of systems of $k+1$ tethers. The faces of a simplex are the systems where some of the tethers have been removed. Here the isotopies must fix the endpoints of the tethers. If σ is a system of tethers, denote by $[\sigma]$ its isotopy class, and if ϕ is a diffeomorphism, denote by $[\phi]$ its isotopy class. The mapping class group presentation of the braid group then gives an action $B_n \curvearrowright X_{n,d}$ defined by $[\phi][\sigma] = [\phi\sigma]$. In the proof of the next theorem we will use the concept of normal form, described in the following.

Definition 2.1 (Normal form)**.** Let γ and δ be two systems of tethers. We may assume that γ is chosen in its isotopy class such that it only intersects δ transversally and does so in the minimal number of points. The minimality is equivalent to the requirement that there is no bigon, i.e. a pair consisting of an arc of a tether in γ and an arc of a tether in δ that together bound a disk in $S_{g,1}$ since any simply connected subsurface of $S_{g,1}$ is a disk. If this is the case, we say that γ is in **normal form** with respect to δ.

Theorem 2.2 ([HV15b, Theorem 3.1])**.** *The complex $X = X_{n,d}$ is contractible.*

Proof. Choose a fixed tether t. For any simplex $[\sigma]$ in X we will construct a rule for choosing a preferred vertex $[v_\sigma]$ of $[\sigma]$, and for each vertex $[v]$ of X an associated simplex $[\Delta v]$ in link$[v]$ such that we can use Lemma 1.20. The value of the complexity function on a system $[\sigma]$ in X is defined to be the total number of points in which the interiors (i.e. not endpoints) of tethers of σ intersect with t, assuming that σ is in normal form with respect to t.

For every tether s which intersects t at an interior point, we let x be the point of intersection of s and t which is closest to the boundary point b_i in terms of t, i.e. the first intersection point that t hits if we consider it as an arc starting in b_i. We now cut s into two arcs at x and move both of the new endpoints down along t to the boundary point b_i of t. This creates two new arcs that can be chosen such that they are disjoint from s except at their endpoints (by maintaining a sufficiently close distance). We define Δs to be the one of these two arcs whose other endpoint is at the marked point, so that Δs is a tether. Then $[\Delta s]$ has a lower complexity than $[s]$.

Now we want to fit this into the context of Lemma 1.20, so we let Y be the star of t (which is contractible), and for a simplex $[\sigma]$ in X we choose the preferred vertex $[v_\sigma]$ such that v_σ is the tether of σ containing the point in the intersection of t and σ which is closest to the boundary point b_i of t. Then $[\sigma] * [\Delta v_\sigma]$ is a simplex in X since we can choose Δv_σ to lie close enough to v_σ for it to not hit any of the other tethers, and since we choose the intersection point closest to the boundary, the part of Δv_σ that arises by following along t can also be chosen to be disjoint from σ. Thus condition (1) is satisfied. Moreover, the complexity has been decreased, and condition (3) is satisfied since a face is just a subcollection of tethers. This means that X is contractible by Lemma 1.20. $\square$

A system of tethers $\tau = \{t_1, \ldots, t_k\}$ is said to be **coconnected** if the complement $D \setminus \tau$ is connected. Note that τ is coconnected if and only if each t_i ends at a different marked point. Let $Y = Y_{n,d}$ denote the subcomplex of $X_{n,d}$ that consists of isotopy classes of coconnected tether systems. Note that in the following theorem we set $d = 1$ since this is all we need for the stability argument.

Theorem 2.3 ([HV15b, Theorem 3.2])**.** *The complex $Y = Y_{n,1}$ is contractible.*

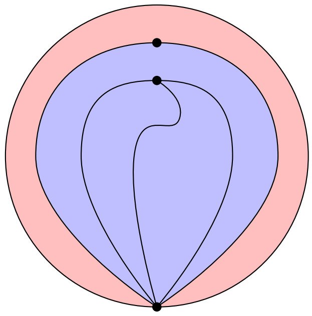


Figure 2.1: Inner and outer components of $D \setminus \sigma$

Proof. We will prove that Y is contractible by induction on the number n of marked points. If $n = 1$, then Y is already a single vertex. Now assume that $Y_{k,1}$ is contractible for all $k < n$. We will use Corollary 1.10, so we need to specify a set of bad simplices. We say that a simplex of $X = X_{n,1}$ is bad if each tethered marked point is hit by at least two of tethers. Then for every simplex $[\sigma]$ with no bad faces, $[\sigma]$ has no face with 2 or more tethers to a marked point, so it cannot itself have 2 or more tethers to any one marked point. Thus σ is coconnected and therefore $[\sigma]$ is contained in Y. Moreover, if $[\sigma]$ and $[\tau]$ are two bad faces of the same simplex, then they each connect at least two arcs to every marked point that they hit, so their join has to do it as well. This means that our set of bad simplices satisfies Definition 1.8. Thus it suffices by Corollary 1.10 to show that $G_{[\sigma]}$ is contractible for every bad simplex $[\sigma]$.

Let $[\sigma]$ be a bad simplex. Then a simplex $[\tau] \in \mathrm{link}([\sigma])$ is good for $[\sigma]$ if and only if any bad face of $[\tau] * [\sigma]$ is contained in $[\sigma]$, which means that if $[\chi] \subset [\tau] * [\sigma]$ has two tethers for each marked point it hits, then $[\chi] \subset [\sigma]$, or again equivalently τ consists of single tethers to marked point that are not touched by σ. Note that since we are working inside $X_{n,1}$ there is only one boundary point. Moreover, since $[\sigma]$ is bad, σ must be separating. This means that σ separates the disk in some inner components (marked in blue) and an outer component (marked in red), see Figure 2.1.

The inner components only touch the base point from the boundary, and the outer component touches the rest of the boundary. Moreover, since none of the tethers can be isotopic, there must be at least one marked point inside of each inner component since if not, then the two tethers that make up the loop that bounds said inner component would be isotopic. Also the outer component must contain at least one marked point since if not, then the two outer arcs would be isotopic. Moreover, a component can have at most $n - 1$ marked points. Now $G_{[\sigma]}$ is a join of complexes isomorphic to $Y_{n_i,1}$ with $n_i < n$ ranging over i, each built form tethers living inside one of these components. Since each $Y_{n_i,1}$ is contractible by the induction hypothesis, their join $G_{[\sigma]}$ is contractible as well, which is what we wanted. $\square$

2.2 Stability theorems

Theorem 2.4 ([HV15b, Theorem 3.3])**.** *The homomorphism $H_i(B_{n-1}) \to H_i(B_n)$ is an isomorphism for $n > 2i+1$ and a surjection for $n = 2i+1$.*

Proof. We want to use the spectral sequence argument for the action on $Y = Y_{n,1}$ which arises from viewing B_n as the mapping class group of a disk with n marked points. This means that we need to verify the four conditions on page 9.

Condition 1: Since any system of $k+1$ tethers in $Y_{n,1}$ is coconnected, we can move the tethers around each other in any way we like. Thus the action can take any k-simplex to any other k-simplex, i.e. it is transitive on simplices of all dimensions, and thus in particular on vertices. Note though that any isotopy must preserve the ordering of the tethers, i.e they cannot swap places.

Condition 2: Any system σ of $k+1$ tethers coming out of the single basepoint has a natural ordering determined by an orientation of the disk (e.g. by following a sufficiently small circular arc around the basepoint). This ordering must be preserved by any diffeomorphism that is the identity on the boundary of the disk. This means that if some diffeomorphism of the disk sends σ to itself, it must send each tether in σ to itself, and therefore the stabilizer of $[\sigma]$ fixes each vertex and thus fixes the whole simplex pointwise. Moreover, elements of the stabilizer are free to do whatever they want on the other vertices of the complex. Thus, since we are in the coconnected case and each tether therefore lands at a different marked point, stab($[\sigma]$) is exactly isomorphic to B_{n-k-1}. Actually stab($[\sigma]$) $= [\phi]^{-1} B_{n-k-1} [\phi]$ where B_{n-k-1} is considered as the subgroup of tethers to the marked points $p_1, \ldots, p_{n-k-1}$, and ϕ is a diffeomorphism that maps the marked points $p_1, \ldots, p_{n-k-1}$ to the marked points that are not hit by σ.

Condition 3: Let $[e]$ be an edge in Y, i.e. an isotopy class of systems consisting of two tethers v and w ending in marked points p_v and p_w. We can simply define a diffeomorphism of D supported in a small neighborhood of v and w that takes v to w and also takes p_w to p_v. By choosing the neighborhood sufficiently small we can ensure that this diffeomorphism commutes with stab($[e]$).

Condition 4: Since Y is contractible, we only need to show that Y/B_n is highly connected. Y consists of coconnected systems from one basepoint to n marked points, so there can be at most n tethers in a system. This makes Y an $(n-1)$-dimensional simplicial complex. Since $B_n \curvearrowright Y$ is transitive on k-simplices for all k, it puts all simplices of each dimension in one orbit. Thus Y/B_n can be seen as the quotient of Δ^{n-1} where all k-dimensional faces have been identified with each other for each k. This makes Y/B_n a Δ-complex (semi-simplicial set) with one k-simplex for each $k \leq n-1$. This means that the augmented cellular chain complex has a copy of $\mathbb{Z}$ in each dimension. The boundary maps are then alternating sums of the identity, so the chain complex look like the following.

$$\begin{aligned} \cdots \to C_{n-1}(Y/B_n) \xrightarrow{f} C_{n-2}(Y/B_n) \to \cdots \to C_2(Y/B_n) \\ \xrightarrow{\cong} C_1(Y/B_n) \xrightarrow{0} C_0(Y/B_n) \xrightarrow{\cong} \mathbb{Z} \to 0, \end{aligned}$$

We have $f = 0$ if n is even, and f is an isomorphism if n is odd. This means that $H_k(Y/B_n) = 0$ for all $k \leq n-2$, and $H_{n-1}(Y/B_n)$ is trivial if n is odd or $\mathbb{Z}$ if n is even. Now Y/B_n may not be $(n-2)$-connected, but the fact that the homology vanishes is sufficient by Remark 1.1. This means that all the conditions for the spectral sequence argument are satisfied, so the result follows by Theorem 1.3. $\square$

Theorem 2.5 ([HV15b, Theorem 3.4])**.** *If n is odd, the stabilization $H_i(B_{n-1}) \to H_i(B_n)$ is an isomorphism for all i. Moreover, if $n \geq 2$ (of any parity), then $H_i(B_n) = 0$ for $i \geq n$.*

Proof. We consider the same spectral sequence as we used in Theorem 2.4. Firstly, note that for any coconnected system of $k+1 \geq 2$ tethers there is a diffeomorphism of the disk that permutes the marked points and is supported in a small neighborhood of the tethers such that it takes any subset of k tethers to any other subset of k tethers, as long as it preserves their natural ordering. This commutes with the stabilizer of the union of the two sets of k tethers by making the neighborhood sufficiently small. Thus we can do an argument similar to what we did in Section 1.1.2. Namely all the terms of the boundary map $d^1\colon E^1_{p,q} = H_q(\mathrm{stab}(\sigma)) \to E^1_{p-1,q}$ are the same since each pair of two terms fits into a diagram like (1.4). Thus if p is odd, d^1 is a sum of an even number of equal terms, i.e. $p = 0$. If p is even, then d^1 is induced by the inclusion $\mathrm{stab}(\sigma) \cong B_{n-p-1} \hookrightarrow B_{n-p}$.

We prove the first statement by induction. For the base case $n = 1$, note that both B_0 and B_1 are trivial, so $H_i(B_0) \to H_i(B_1)$ is the zero map if $i > 0$ and the identity $\mathbb{Z} \to \mathbb{Z}$ if $i = 0$. Now for the induction step, assume that $H_i(B_{k-1}) \to H_i(B_k)$ is an isomorphism for all odd k with $k < n$ and for all i. Then the map $E^1_{p,q} \to E^1_{p-1,q}$ is the map $H_q(B_{n-p-1}) \to H_q(B_{n-p})$ induced by the inclusion. This can be rewritten as $H_q(B_{(n-2)-(p-2)-1}) \to H_q(B_{(n-2)-(p-2)})$, which by assumption is an isomorphism for all p even with $2 \leq p \leq n-1$. This means that the spectral sequence looks like the picture in Figure 2.2.

$$
\begin{array}{c|ccccccccc}
i & H_i(B_n) & \longleftarrow & H_i(B_{n-1}) & \xleftarrow{\;0\;} & H_i(B_{n-2}) & \xleftarrow{\;\cong\;} & \cdots & & \\
i-1 & \cdots & \longleftarrow & H_{i-1}(B_{n-1}) & \xleftarrow{\;0\;} & H_{i-1}(B_{n-2}) & \xleftarrow{\;\cong\;} & H_{i-1}(B_{n-3}) & \xleftarrow{\;0\;} & \cdots \\
i-2 & & & & & \cdots & \xleftarrow{\;\cong\;} & H_{i-2}(B_{n-3}) & \xleftarrow{\;0\;} & \cdots \\
 & & & & & & & & & \\
q=0 & H_0(B_n) & \longleftarrow & H_0(B_{n-1}) & \xleftarrow{\;0\;} & H_0(B_{n-2}) & \xleftarrow{\;\cong\;} & H_0(B_{n-3}) & \xleftarrow{\;0\;} & \cdots \\
\hline
 & p=-1 & & 0 & & 1 & & 2 & & \cdots
\end{array}
$$

Figure 2.2: The spectral sequence for the action $B_n \curvearrowright Y$

Since Y_{n-1} has dimension $n-1$, the last non-vanishing column of the spectral sequence is the $(n-1)$'st one. In fact $E^1_{n-1,0} = H_0(B_0)$ is the only non-zero term in this column since B_0 is the trivial group. Since n is odd, the d^1-differentials going out of this column must be isomorphisms. This means that every $E^1_{p,q}$ with $p \geq 1$ is either the domain or the target of an isomorphism and thus gets killed on the E^2-page. Thus no differential beyond the E^1-page can be non-zero. But the spectral sequence converges to 0, so in particular $E^\infty_{-1,i} = 0$, i.e. $d^1 \colon E^1_{0,i} \to E^1_{-1,i}$ which is the map $H_i(B_{n-1}) \to H_i(B_n)$, must be an isomorphism for any i.

We also prove the second statement by induction on n. The base case $H_1(B_1)$ is true since B_1 (like B_0) is the trivial group. Now assume that the statement holds for all $2 \leq k < n$. The diagonal of total degree $p+q = n-1$ in the spectral sequence contains the groups $H_j(B_j)$. By the induction hypothesis, these as well as the terms above them are 0 except for the column $p = -1$ which is not covered by the induction hypothesis. But any differential on the E^1-page or later that hits the $(-1,n)$'th term or above must come from above this diagonal and therefore must be 0, so since the spectral sequence converges to 0, $E^1_{-1,n} = H_n(B_n)$ must be 0 since it cannot be killed by anything. $\square$

Chapter 3

Stability for mapping class groups of surfaces

Any compact, connected and oriented smooth surface can be determined up to diffeomorphism by its genus and the number of boundary components, so let $S = S_{g,s}$ denote a compact orientable smooth surface of genus g with s boundary components. Denote by $\mathcal{M}_{g,s}$ the boundary fixing mapping class group of S as defined in the beginning of Chapter 2. One can then form homomorphisms $\mathcal{M}_{g,s} \to \mathcal{M}_{g+1,s}$. Moreover, for $s \geq 1$ there are homomorphisms $\mathcal{M}_{g,s} \to \mathcal{M}_{s+1}$, and finally there are homomorphisms $\mathcal{M}_{g,1} \to \mathcal{M}_{g,0}$. These homomorphisms will be described below, and the goal of this chapter is to prove that they induce isomorphisms on group homology H_i for g sufficiently large with respect to i. The main challenge in this is to prove that the simplicial complexes that we will use are highly connected. There are many complexes involved on the way, but in the end the complexes we are interested in are the complex of tethered chains and the complex of rooted curves. The connectivity of these is established in Theorem 3.27 and Theorem 3.13 respectively.

3.1 Curve Complexes

For the proofs of both Theorem 3.27 and Theorem 3.13 we first need to establish the connectivity of the complex $C^0(S)$ of coconnected curve systems stated in Theorem 3.10. A k-simplex of $C^0(S)$ is an isotopy class $[\sigma]$ of $k+1$ curves such that $S \setminus \sigma$ is connected. To show that this complex is highly connected, we first show that the complex $C(S)$ of all curve systems is highly connected. We do this by showing that it is homotopy equivalent to a complex of subsurfaces of S which is again homotopy equivalent to a certain complex of arcs on S.

We proceed in reverse order. An **arc system** on S is a set of disjoint embedded arcs a_i from points p_i to points q_i such that

(1) p_i and q_i are points in ∂S,
(2) if $i \neq j$, then a_i is not isotopic to a_j,

(3) no a_i is isotopic to an arc that is contained in ∂S,

where the isotopies are restricted such that the endpoints of the arcs must remain in ∂S at all times. Now choose a component $\partial_0 S$ of ∂S. We define a simplicial complex $A(S, \partial_0 S)$ with a k-simplex for each isotopy class of arc systems of $k+1$ arcs.

If α is a structure on S, e.g. an arc, a chain, a tethered chain etc., then $[\alpha]$ denotes the isotopy class that α represents.

Proposition 3.1 ([HV15b, Proposition 4.1])**.** *The simplicial complex $A(S, \partial_0 S)$ is contractible if it is non-empty, i.e. if there exists an arc that is not isotopic to an arc in ∂S, or equivalently if S is not a disk or an annulus.*

Proof. We will use Lemma 1.20 similarly to before, i.e. we will use surgery to create a flow into the star of some fixed arc class $[a]$. So fix an arc a inside S that starts and ends in $\partial_0 S$ and is not isotopic to an arc in ∂S. We define the complexity of an arc system that is in normal form with respect to a as the number of points in which it intersects a. Choosing an orientation of a, take an arc system σ that is in normal form with respect to a with non-empty intersection. Consider the point in $\sigma \cap a$ that is closest to the head of a. We cut the arc b that intersects a in this point in two at the intersection point and redirect the two new endpoints to the head of a such that the two new arcs only intersect a in the endpoint and such that two arcs do not intersect σ anywhere else. We can do this since we chose the intersection point that was closest to the head of a, so we can just follow a closely enough to not hit anything else. This means that we have replaced b by an arc system Δb of two arcs that meets a in one point fewer than b such that $[\Delta b]$ has lower complexity than $[b]$. Thus the three requirements in Lemma 1.20 are satisfied, showing that $A(S, \partial_0 S)$ is contractible. $\square$

In the proof Corollary 3.3 as well as several proofs later in this paper we will use the concept of cutting along arcs and curves. We describe the concept here for arc systems, but it can be defined analogously for the other structures that we will be using.

Remark 3.2 (Cutting along arcs)**.** Let α be an arc system on S with endpoints in $\partial_0 S$. Let N be a regular neighborhood of $\alpha \cup \partial_0 S$. By a regular neighborhood we mean a band $(0,1) \times (0,1)$ for each arc in α such that the arc lies in $\{\frac{1}{2}\} \times (0,1)$. We also include a band $[0,1) \times S^1$ around ∂_0 such that ∂_0 is $\{0\} \times S^1$. Assume that the bands intersect each other only in disks and that they do so a minimal number of times. See Figure 3.1 for an example of the closure a regular neighborhood of an arc. Now $S_\alpha := S \setminus N$ is a surface with boundary, and we say that S_α is the surface obtained from S by cutting along α. The two sides of a band in N around an arc a in α can be seen as two 'copies' of a. This fact is important in a lot of arguments in the rest of the paper as it allows us to do computations using Euler characteristic.

We say that an arc system in $A(S, \partial_0 S)$ is **at infinity** if its complement has some component which is neither a disk nor an annular neighborhood of a boundary component. Here a disk does not necessarily mean a closed or open one, i.e. it could be a closed disk with some parts of the boundary removed. We let $A_\infty(S, \partial_0 S)$ be the subcomplex consisting of arc systems at infinity.

Corollary 3.3 ([HV15b, Corollary 4.2]). *The complex $A_\infty(S, \partial_0 S)$ of arc systems at infinity is $(2g+s-5)$-connected.*

Proof. Note that $S_{g,s}$ has Euler characteristic $2-2g-s$, that the annulus has Euler characteristic 0, and that the disk has Euler characteristic 1. Consider an arc in S with endpoints in ∂_0. We can assume without loss of generality that the arc as well as its endpoints are already simplices in a triangulation of S. Then cutting along the arc adds two vertices and one edge, i.e. it increases the Euler characteristic by 1.

Since the disk and the annulus both have non-negative Euler characteristic, we must increase Euler characteristic by at least $2g+s-2$ in order to cut S in to disks and annuli. Thus, since each cut increases Euler characteristic by 1, we need at least $2g+s-2$ arcs to do so. This corresponds to a $(2g+s-3)$-simplex of $A(S, \partial S)$, so $A_\infty(S, \partial_0 S)$ contains at least the $(2g+2-4)$-skeleton of $A(S, \partial_0 S)$.

Any map $f\colon S^{2g+s-5} \to A_\infty(S, \partial_0 S)$ can be assumed to be simplicial by Theorem A.5. Moreover, since $A(S, \partial_0 S)$ is contractible, f can be extended to a map $F\colon D_{2g+s-4} \to A_\infty S, \partial_0 S$ that we can also assume to be simplicial. But then F maps into the $(2g+s-4)$-skeleton and thus into $A_\infty(S, \partial_0 S)$, showing that $A_\infty(S, \partial_0 S)$ is $(2g+s-5)$-connected. $\square$

For a surface $S = S_{g,s}$, we now define the **curve complex** $C(S)$ to be the simplicial complex in which a vertex is an isotopy class of simple closed curves in S which do not bound a disk and are not isotopic to a component of ∂S. A set of vertices in $C(S)$ span a simplex in $C(S)$ if we can choose curve representatives that are all pairwise disjoint. A collection of such curves is then called a **curve system**.

We now define the **subsurface complex** $S(S, \partial_0 S)$ as the poset of isotopy classes of compact connected subsurfaces S' of S such that S' contains $\partial_0 S$ and $\partial S' \setminus \partial S$ is a non-empty curve system in S. Note that it may contain parallel copies of the same curve, and that in particular no component of $\partial S' \setminus \partial S$ bounds a disk in S or is isotopic to a component of ∂S because $\partial S' \setminus \partial S$ must be a curve system.

Let $\alpha \in A(S, \partial_0 S)$ be an arc system. We can associate to α a subsurface $S(\alpha)$ of S by first taking the closure of a regular neighborhood N of $\alpha \cup \partial_0 S$ and then adding to it any components of $S \setminus N$ that are disks or annuli (with one boundary circle in ∂S). In this case the closure of a regular neighborhood is made up of one embedded $I \times I$ strip around each arc such that $I \times \{0\}$ and $I \times \{1\}$ are both contained in ∂S and such that $\{\frac{1}{2}\} \times I$ corresponds to the arc. These strips should be chosen small enough such that they intersect only in disks and do so a minimal number of times. See Figure 3.1 for an example of this when α is a single arc.

This means that $[\alpha] \notin A_\infty(S, \partial_0 S)$ if and only if $S(\alpha) = S$, so the simplices of $A_\infty(S, \partial_0 S)$ correspond exactly to systems $[\alpha]$ such that $S(\alpha) \subsetneq S$. Now let $\widehat{A}_\infty(S, \partial_0 S)$ denote the poset of simplices in $A_\infty(S, \partial_0 S)$. Then $[\alpha] \mapsto [S(\alpha)]$ defines a poset map $f\colon \widehat{A}_\infty(S, \partial_0 S) \to S(S, \partial_0 S)$ since $\alpha \subset \beta$ implies $S(\alpha) \subset S(\beta)$.

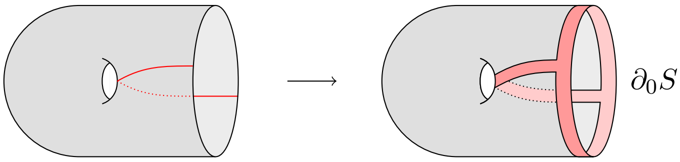


Figure 3.1: An arc α and its associated subsurface $S(\alpha)$

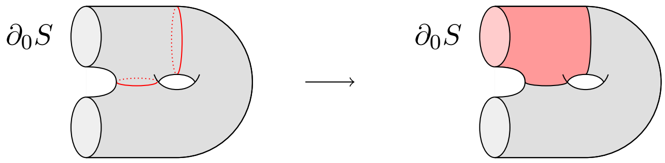


Figure 3.2: A curve system γ and its associated subsurface $S(\gamma)$

Proposition 3.4 ([HV15b, Proposition 4.3])**.** *The map $f\colon \widehat{A}_\infty(S, \partial_0 S) \to S(S, \partial_0 S)$ is a homotopy equivalence.*

Proof. We will use Quillen's Fiber Lemma, Lemma 1.13. If $[S'] \in S(S, \partial_0 S)$, then $f_{\leq [S']}$ consists of all classes of arc systems $[\alpha]$ such that $S(\alpha) \subset S'$, which is just $\widehat{A}(S', \partial_0 S)$ since $\partial_0 \subset S'$. Since no component of $\partial S'$ is allowed to bound a disk or to be isotopic to a component of ∂S, S' cannot be a disk or an annulus, so $\widehat{A}(S', \partial_0 S)$ is contractible by Proposition 3.1 since it is homeomorphic to $A(S', \partial_0 S)$. Thus Lemma 1.13 shows that f is a homotopy equivalence. $\square$

Let γ be a curve system on S. We want to define a subsurface $S(\gamma) \subset S$ that is related to γ. If necessary we can isotope γ such that it is disjoint from ∂S, and then we can choose a regular neighborhood $N(\gamma)$ of γ such that $N(\gamma)$ is disjoint from ∂S. Similarly to the case of arcs, a regular neighborhood of γ consists of a band $I \times S^1$ for each curve in γ such that the curve lies in $\{\frac{1}{2}\} \times S^1$ and such that the bands intersect in disks a minimal number of times.

We then denote by $S(\gamma)$ the component of $S \setminus N(\gamma)$ that contains $\partial_0 S$ as illustrated in Figure 3.2. We can choose the neighborhoods $N(\gamma)$ such that if $\gamma \subset \gamma'$, then $N(\gamma) \subset N(\gamma')$, so that $S(\gamma) \supset S(\gamma')$. Thus $[\gamma] \mapsto [S(\gamma)]$ defines a poset map $g\colon \widehat{C}(S) \to S(S, \partial_0 S)$. This poset map reverses the ordering, but this does not matter since we can just consider the reverse ordering on either $\widehat{C}(S)$ or $S(S, \partial_0 S)$ without changing it as a topological space.

Proposition 3.5 ([HV15b, Proposition 4.4])**.** *The map $g\colon \widehat{C}(S) \to S(S, \partial_0 S)$ is a homotopy equivalence.*

Proof. Let $[S'] \in S(S, \partial_0 S)$ be given. Then $g_{\geq [S']}$ consists of curve system classes $[\delta]$ where δ can be chosen such that $S' \subset S(\delta)$, i.e. such that $\delta \subset S \setminus S'$ since S' is connected and contains $\partial_0 S$. Here the curves in δ are allowed to be parallel to those in $\gamma(S') := \partial S' \setminus \partial S$. In particular $[\partial S' \setminus \partial S]$ is an element of $g_{\geq [S']}$ since $\partial S' \setminus \partial S$ can be isotoped to an appropriate curve system. The curve system class $[\gamma(S')]$ can be added to any curve system class in $g_{\geq [S']}$ since for $g_{\geq [S']}$ we must be able to choose curve representatives that do not cross $\partial S'$. Thus $g_{\geq [S']} = \text{star}([\gamma(S')])$, seen as the star of $[\gamma(S')]$ inside $g_{\geq [S']}$, which is contractible by (A.2). $\square$

Corollary 3.6 ([HV15b, Corollary 4.5]). *If $\partial S \neq \emptyset$, then $C(S)$ is $(2g+s-5)$-connected.*

Proof. By Proposition 3.5 and Proposition 3.4 we have homotopy equivalences

$$C(S) \simeq S(S, \partial_0 S) \simeq A_\infty(S, \partial_0 S),$$

and $A_\infty(S, \partial_0 S)$ is suitably connected by Corollary 3.3. $\square$

Corollary 3.7 ([HV15b, Corollary 4.6]). *If S has genus 0, then $C(S)$ is homotopy equivalent to a wedge of spheres of dimension $s-4$.*

Proof. If S has genus 0, then $C(S)$ is $(s-5)$-connected by Corollary 3.6. Thus, since $C(S)$ is $(s-4)$-dimensional and its $(s-5)$-skeleton is contractible, $C(S)$ must be homotopy equivalent to a wedge of spheres of dimension $s-4$. $\square$

3.1.1 Curves on closed surfaces

Lemma 3.8. *Let $S = S_{g,s}$, and let γ be a maximal curve system on S. Then*

- *If $g > 1$, γ cuts $S = S_{g,s}$ into pairs of pants, i.e. copies of $S_{0,3}$,*
- *If $g = 1$, then γ cuts S into pairs of pants if $s \geq 1$ or into a single cylinder if $s = 0$,*
- *If $g = 0$, then if $s \leq 3$, γ has to be empty, and if $s > 3$, then γ cuts S into pairs of pants.*

Proof. We will prove the first statement by a process of elimination. We need to show that γ cannot cut S into any of the following:

(1) A disk $S_{0,1}$,
(2) A cylinder $S_{0,2}$,
(3) Any $S_{0,s}$ with $s > 3$,
(4) Any $S_{g,s}$ with $g > 0$,

which will leave $S_{0,3}$ as the only remaining opportunity. (i) is impossible since at least one of the curves would have to be trivial. (ii) is impossible since two curves would have to be isotopic to each other, or one curve would have to be isotopic to a boundary component. (iii) is impossible since we could cut along a curve that separates two boundary components from the others and thus cuts $S_{0,s}$ into $S_{0,3}$ and $S_{0,s-1}$, so the curve system would not be maximal. Finally (iv) is impossible since we can cut $S_{g,s}$ along a non-separating curve to get $S_{g-1,s+2}$, so the curve system would not be maximal.

For the second statement, if $g = 1$ and $s = 0$, our only choice is to cut along a non-separating curve, so we get a cylinder. If $g = 1$ and $s \geq 1$, we must at least cut along a curve to get $S_{0,s+2}$ which must then be cut into pairs of pants as we already argued.

If $g = 0$ and $s \leq 3$, then any curve will be trivial or isotopic to a curve around a boundary component. If $g = 0$ and $s > 3$, we can cut around two of the boundary components to get $S_{0,3}$ and $S_{0,s-1}$. □

Theorem 3.9 ([HV15b, Theorem 4.7])**.** *Let $g \geq 1$, and let $\phi\colon C(S_{g,1}) \to C(S_{g,0})$ be the map induced by filling in the single boundary circle of $S_{g,1}$ with a disk. Then ϕ is a homotopy equivalence.*

Proof. If $g = 1$, then $C(S_{g,1})$ and $C(S_{g,0})$ both consist of a single curve system of one curve which cuts them into $S_{0,3}$ and $S_{0,2}$ respectively by Lemma 3.8, so ϕ is an isomorphism.

Now assume that $g > 1$. Then the dimension of $C(S_{g,1})$ is one greater than that of $C(S_{g,0})$ since we will need one extra curve to cut S into pairs of pants. Choose a maximal curve system δ on $S_{g,1}$. Then δ cuts $S_{g,1}$ into pairs of pants by Lemma 3.8. Let P denote the subsurface of $S_{g,1}$ corresponding to the pair of pants that contains the circle $\partial S_{g,1}$. Let d_1 and d_2 denote the two components of ∂P that are curves in δ.

We can isotope any curve system γ in $S_{g,1}$ to be in normal form with respect to δ. If γ is in normal form with respect to δ, then some arc of $\gamma \cap P$ must either cross P from d_1 to d_2 or enter in one d_i, go around $\partial S_{g,1}$ and go out again through the same d_i. Otherwise it would bound a circle together with an arc in d_i. An arc of the latter type that goes around $\partial S_{g,1}$ will be called a **return arc**.

Step 1, removing return arcs: Let c be a curve in normal form with respect to δ that contains at least one return arc. Let b be the innermost return arc, the one closest to $\partial S_{g,1}$. We can push b across $\partial S_{g,1}$ to get a new curve Δc with one return arc fewer than c. By staying sufficiently close to $\partial S_{g,1}$, we can assume that Δc is disjoint from c. Let $C_0 \subset C(S_{g,1})$ denote the subcomplex of curve complexes with no return arcs, and define a complexity function on $C(S_{g,1})$ by counting the number of return arcs. For any simplex $[\gamma]$ in $C(S_{g,1}) \setminus C_0$, we let c_γ be the curve in γ containing the innermost return arc of γ.

We now want to use the flow lemma, Lemma 1.20. Note that $[\Delta c_\gamma]$ is in the link of $[c_\gamma]$ since Δc_γ is disjoint from c_γ and $[\Delta c_\gamma] \cup [c_\gamma]$ is a 1-simplex in $C(S_{g,1})$. This is because we assumed γ to be in normal form with respect to δ so that it has the minimal amount of return arcs, so c_γ cannot be isotopic to Δc_γ which has fewer return arcs. Note that $[\gamma] * [\Delta c_\gamma]$ is also a simplex in $C(S_{g,1})$ since either Δc_γ is not isotopic to any other curve in γ, or $[\gamma] * [c_\gamma] = [\gamma]$. We can choose c_γ to be disjoint from γ since we chose the innermost return arc and therefore do not have to intersect anything. Since we have eliminated a return arc from c_γ, the complexity is now strictly lower. Moreover, if $[\gamma']$ is a face of $[\gamma]$ that contains $[c_\gamma]$, then c_γ will still contain the innermost return arc of γ', i.e. $[c_{\gamma'}] = [c_\gamma]$. Therefore Lemma 1.20 shows that C_0 is a deformation retract of $C(S_{g,1})$.

Step 2, filling $\partial S_{g,1}$ with a disk: There is a map $\psi\colon C(S_{g,1}) \to C(S_{g,0})$ induced by $i\colon S_{g,1} \hookrightarrow S_{g,0}$, the map obtained by attaching a disk to $\partial S_{g,1}$. Since the attached out disk gives more room for isotopy, some non-isotopic curves in $C(S_{g,1})$ may map to the same isotopy class in $C(S_{g,0})$. The map ψ restricts to a map $\phi\colon C_0 \to C(S_{g,0})$. We want to show that the induced poset map $\hat{\phi}\colon \hat{C}_0 \to \hat{C}(S_{g,0})$ is a homotopy equivalence. By Lemma 1.13 and Lemma 1.15 (2) it suffices to show that all fibers $\hat{\phi}^{-1}([\sigma])$, where σ is a curve system on $S_{g,0}$, are contractible. Note that these are in fact fibers since simplices are turned into vertices in the poset complex.

The map ϕ is surjective since moving a return arc across $\partial S_{g,1}$ does not change the isotopy type of a curve when viewed in $C(S_{g,0})$. So let $\sigma \in C(S_{g,0})$ be given, and choose $[\gamma]$ in C_0 such that $\phi([\gamma]) = [\sigma]$. We may assume without loss of generality that γ is in normal form with respect to δ. Note that the isotopy representative δ can be isotoped slightly to a curve representative δ' that is disjoint from δ. Then δ' in particular has no return arcs and thus is an element of C_0. This means that $\phi([\delta])$ is well-defined.

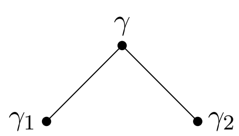
Since γ is in normal form with respect to δ and has no return arcs, it does not bound a disk together with δ when viewed as curve systems in $S_{g,0}$. Therefore $\sigma = i(\gamma)$ is in normal form with respect to $i(\delta)$. Assume first that $\gamma \cap P = \emptyset$. The curve system γ may contain a curve that is isotopic to d_1 or d_2. If a curve c is isotopic to d_1, and another curve c' is isotopic to d_2, then c and c' are not isotopic, but $\phi([c]) = \phi([c'])$ since d_1 and d_2 are isotopic when we are allowed to fill $\partial S_{g,1}$ with a disk. There are now four possibilities for the curve system γ, namely

(1) γ contains a curve that is isotopic to d_1, but not one isotopic to d_2,
(2) γ contains a curve that is isotopic to d_2, but not one isotopic to d_1,
(3) γ contains a curve that is isotopic to d_1, and another one that is isotopic to d_2,
(4) γ contains no curve that is isotopic to d_1 or d_2.

In the first case, we can add a curve that is isotopic to d_2 without changing the image under ϕ, and in the second case we can do the same, only now with d_1. Thus if γ satisfies any of the three first cases, we can assume without loss of generality that it is the third case, and then

$$\phi^{-1}([\sigma]) = \phi^{-1}([i(\gamma)]) = \phi^{-1}(\phi([\gamma])) = \{\gamma, \gamma_1, \gamma_2\},$$

where γ_1 is obtained from γ by removing the curve isotopic to d_2, and γ_2 is obtained by removing d_1. This means that $\hat{\phi}^{-1}([\sigma]) = \{[\gamma], [\gamma_1], [\gamma_2], [\gamma_1] < [\gamma], [\gamma_2] < [\gamma]\}$, i.e. it consists of three vertices connected by two edges, so it is homeomorphic to $I = [0,1]$ as indicated by the following picture, i.e. it is contractible.



Now assume that γ does not contain a curve that is isotopic to d_1 or d_2. Recall that γ does not intersect P. If c is a curve on $S_{g,1}$ such that $i(c)$ is isotopic to a curve in $i(\gamma)$, then either c is isotopic to a curve in γ, or c has a return arc. Thus the fiber $\phi^{-1}([\sigma])$ consists of the single element $[\gamma]$, so it is contractible.

Now assume that γ intersects P. Since γ has no return arcs, it must intersect P in arcs $a_1, \dots, a_k$ with $k \geq 1$ that enter through one d_i and exit through the other. In this case we do not have the problem above since if γ contained a curve isotopic to d_i, it would have to intersect itself. The arcs $a_1, \dots, a_k$ are contained in curves $c_1, \dots, c_k$, some of which may coincide since a curve may pass through P multiple times. We can choose the numbering such that the disk D that attaches to $\partial S_{g,1}$ lies in between the arcs a_k and a_1, and such that a_i is adjacent to a_{i+1} for all $i = 1, \dots, k$. By pushing a_1 across D, c_1 is converted to a curve c_1' which we can assume to be disjoint from c_1. Then $[\gamma_1] = [\gamma] \cup [c_1']$ is a simplex in C_0. Note that γ_1 will be equal to γ if c_1' is isotopic to a curve in γ. In any case the image is not changed, i.e. $\phi([\gamma_1]) = \phi([\gamma]) = [\sigma]$. Let $[\gamma_0] = [\gamma] \setminus [c_1']$ and $[\gamma_2] = [\gamma_1] \setminus [c_1]$. Then $[\gamma_1] \neq [\gamma]$ if and only if $\gamma_2 = \gamma$. We get inclusions

$$[\gamma_0] \subsetneq [\gamma_1] \supsetneq [\gamma_2],$$

i.e. a simplicial subcomplex of C_0 consisting of a line built from three vertices and two edges. Since we have pushed a_1 across D, a_2 is now adjacent to D, so we can push a_2 across D to get a new curve c_2' and new curve systems $[\gamma_3] = [\gamma_2] \cup [c_2']$ and $[\gamma_4] = [\gamma_3] \setminus [c_2]$. We can continue this process to get inclusions

$$[\gamma_0] \subsetneq [\gamma_1] \supsetneq [\gamma_2] \subsetneq [\gamma_3] \supsetneq [\gamma_4] \subsetneq [\gamma_5] \supsetneq \cdots.$$

Similiarly we could have worked the other way, starting by pushing a_k across D, to get inclusions in the other direction. Combining these, we get a doubly infinite string

$$\cdots \supsetneq [\gamma_{-4}] \subsetneq [\gamma_{-3}] \supsetneq [\gamma_{-2}] \subsetneq [\gamma_{-1}] \supsetneq [\gamma_0] \subsetneq [\gamma_1] \supsetneq [\gamma_2] \subsetneq [\gamma_3] \supsetneq [\gamma_4] \subsetneq \cdots.$$

Note that if we wanted to add two curves to γ without removing another and without changing the image under ϕ, we would get into trouble since the second curve would have to either intersect γ or be isotopic to the first added curve or a curve in γ. This is because we are in the complex without return arcs, so if we want to leave the image under ϕ unchanged, we should not change anything outside of P. This means that what we have done above covers all of $\widehat{\phi}^{-1}([\sigma])$ which is then homeomorphic to the real line, i.e. is contractible. $\square$

3.1.2 Complexes of coconnected curve systems

For a curve system γ on S, it might be the case that $S \setminus \gamma$ is connected, for example if γ is a non-trivial circle on the torus. In that case we say that γ is **coconnected**, and we let $C^0(S)$ denote the complex of coconnected curve systems on S.

To each curve system γ we associate an undirected graph $D(\gamma)$, called the **dual graph** of γ, with one vertex for each connected component of $S \setminus \gamma$. Moreover, for each curve c in γ there is an edge in $D(\gamma)$ between the vertices corresponding to the components on each 'side' of c. These two components are the same if c does not separate a component of $S \setminus (\gamma \setminus c)$, in which case the corresponding edge is a loop. This means that γ is coconnected if and only if $D(\gamma)$ consists of one vertex with a number of loops. Whether or not a curve is separating i stable under isotopy, so we can consider the graph $D([\gamma])$, which is isomorphic to $D(\gamma)$.

Theorem 3.10 ([HV15b, Theorem 4.8])**.** *The complex $C^0(S)$ is $\frac{g-3}{2}$-connected if S has genus g.*

Proof. The statement is vacuous for $g = 0$, so assume that $g \geq 1$. Note first that by Theorem 3.9 there is a homotopy equivalence $C(S_{g,1}) \simeq C(S_{g,0})$, so it suffices to prove the theorem for $s \geq 1$. Moreover, $C(S_{g,s})$ is $(2g-4)$-connected for any $s \geq 1$ by Corollary 3.6 since $g \geq 1$. This means that $C(S)$ is $(g-3)$-connected and therefore $\frac{g-3}{2}$-connected for $g \geq 2$. For $g = 1$, $C(S)$ is $\frac{g-3}{2}$-connected since it is non-empty. We will proceed by a bad simplex argument, using Corollary 1.10 with $C^0(S)$ as a subcomplex of $C(S)$. We define a set of bad simplices in $C(S) \setminus C^0(S)$ as the curve systems γ such that the dual graph $D([\gamma])$ has no loops. This means that a curve system γ represents a bad simplex if and only if each curve c in γ separates a component of $S \setminus (\gamma \setminus c)$.

If a curve system γ is not coconnected, then if necessary we can remove all curves c in γ that do not separate a component of the complement of the other curves. This results in a curve system $\gamma_0 \subset \gamma$ such that $D([\gamma_0])$ has no loops, i.e. $[\gamma_0]$ is a bad face of $[\gamma]$. By contraposition this shows that any curve system with no bad faces must be in $C^0(S)$, i.e. the first condition of Definition 1.8 is satisfied.

If $[\gamma_1]$ and $[\gamma_2]$ are bad faces of some $[\gamma]$, i.e. faces such that $D([\gamma_1])$ and $D([\gamma_2])$ have no loops, then $[\gamma_1] * [\gamma_2]$ has no loops by the following argument. Start with $[\gamma_1]$; the dual graph $D([\gamma_1])$ has no loops, and if we add a curve class $[c]$ to $[\gamma_1]$, none of the curves already in $[\gamma_1]$ will be turned into loops in the dual graph since we can only separate S into more pieces, not fewer. Similarly, adding curve classes to $[\gamma_2]$ does not create any loops in the dual graph. This means that $D([\gamma_1] * [\gamma_2])$ will not have any loops, so $[\gamma_1] * [\gamma_2]$ is a bad simplex. This means that the bad simplices also satisfy the second condition of Definition 1.8.

Now let $[\sigma]$ be a bad simplex in $C(S)$. The simplices that are good for $[\sigma]$ are the simplices $[\tau] \in \mathrm{link}([\sigma])$ such that any bad face of $[\tau] * [\sigma]$ is contained in $[\sigma]$, i.e. the curve systems $[\tau]$ such that if a subsystem $[\gamma]$ of $[\tau] * [\sigma]$ has no loops, then $[\gamma] \subset [\sigma]$. Thus each curve class $[c]$ in $[\tau]$, when added to $[\sigma]$, constitutes a loop in the resulting dual graph $D([\sigma] \cup [c])$, which is equivalent to the statement that $[c] \in C^0(S_i)$, where S_i is the component of the surface S_σ obtained by cutting S along σ that contains c. This means that $G_{[\sigma]}$ is the join $*_i\, C^0(S_i)$, where i ranges over all components of S_σ.

We want to proceed by induction on the lexicographically ordered pair (g, s), so we need to show that $(g_i, s_i) < (g, s)$, where $S_i = S_{g_i,s_i}$. Suppose that we cut S along some non-trivial curve. This may cut S into two components, or it might not. In any case, the total Euler characteristic of the surface(s) is preserved since we can do the cutting by adding one vertex and one edge. Now suppose that we cut S along a bad curve system. Then the Euler characteristic is still preserved. The total genus of all the components cannot be greater than g since that would require at least one of the cuts to add at least two additional components, which is impossible. Suppose now that the genus g_k of some S_k is not lower than g, i.e. $g_k = g$. Then $g_i = 0$ for all $i \neq k$, and since the total Euler characteristic is preserved, we have

$$\chi(S) = \sum_i \chi(S_i).$$

Let now $\dim \sigma = n-1$, and let c be the number of components S_i of S_σ. Then since each cut increases the total number of boundary components by 2, we have $\sum_i s_i = s + 2n$. Thus

$$2 - 2g - s = \sum_i (2 - 2g_i - s_i) = 2c - 2g - \sum_i s_i = 2c - 2g - s - 2n,$$

so that $2 = 2c - 2n$, i.e. $c = n+1$. But then since each S_i with $i \neq k$ must be S_{0,s_i} with $s_i \geq 3$, we have

$$s_k = \sum_i s_i - \sum_{i \neq k} s_i = s + 2n - \sum_{i \neq k} s_i \leq s + 2n - 3(c-1) = s + 2n - 3n = s - n,$$

which shows that $s_k < s$.

Now we know that for each i, either $g_i < g$ or $s_i < s$. Thus we can proceed by induction on the lexicographically ordered pair (g, s), the base case being $(1, 1)$. The base case holds since $C^0(S_{1,1})$ is non-empty, i.e. it has connectivity $-1 = \frac{g-3}{2}$.

Since cutting along a curve preserves Euler characteristic and adds 2 boundary components, it must either be separating or reduce genus by 1. This means that if $[\sigma]$ is a bad simplex of dimension k, then the total genus of S_σ is $g_\sigma = g - k - 1 + c - 1 = g - k + c - 2$, where c is the number of components of S_σ. By Lemma A.2, the join of a $\frac{g_1-3}{2}$-connected complex with a $\frac{g_2-3}{2}$-connected one has connectivity at least

$$\left(\frac{g_1 - 4}{2} + 2\right) + \left(\frac{g_2 - 4}{2} + 2\right) - 2 = \frac{g_1 + g_2 - 4}{2}. \tag{3.1}$$

Note that we have to round $\frac{g_i-3}{2}$ down to $\frac{g_i-4}{2}$ in case g_i is even. Successive use of (3.1) tells us that G_σ has connectivity at least $\frac{g_\sigma-2-c}{2}$. Thus it suffices to show that $\frac{g_\sigma-2-c}{2} \geq \frac{g-3}{2} - k$, which is equivalent to $g_\sigma \geq g - 2k + c - 1$. As argued above, $g_\sigma \geq g - k - +c - 2$, so the statement $g_\sigma \geq g - 2k + c - 1$ holds whenever $k \geq 1$.

We treat the case $k = 0$ specifically. In this case σ is a single separating curve, and we need to show that G_σ has connectivity $\frac{g-3}{2}$. If g is even, then $\frac{g-3}{2}$ is not an integer, so we can round down to $\frac{g-4}{2}$, so G_σ has the desired connectivity by (3.1). If g is odd, then one of the g_i's is odd while the other is even. Assume that g_1 is odd. Then by Lemma A.2, G_σ has connectivity

$$\left(\frac{g_1 - 3}{2} + 2\right) + \left(\frac{g_1 - 4}{2} + 2\right) - 2 = \frac{g_1 + g_2 - 3}{2} = \frac{g-3}{2},$$

which concludes the induction step. □

3.1.3 The complex of rooted curves

We will now establish the connectivity of the complex of rooted curves which we will use for the stabilization by boundary components. The complex of rooted curves is analogue to the complex Y^A of [HW07, Theorem 8.6], and the proof of connectivity is inspired by the proofs of that paper. Consider the surface $S_{g,s}$ with $s \geq 3$. Let x_0 be a point in one boundary component, and let x_1 be a point in another. A **rooted curve** is a pair (c, r) where c is a coconnected curve and r is an arc from x_0 to x_1 that intersects c transversally exactly once. The essential property of a rooted curve is that cutting along it reduces genus by one while preserving the number of boundary components. As usual, this can be verified by an Euler characteristic argument since the cut increases Euler characteristic by 2. We define a simplicial complex $C_R^0(S, x_0, x_1)$ in which an n-simplex is the isotopy class of a coconnected system of $n+1$ rooted curves, where the isotopies are required to fix the endpoints x_0 and x_1. Usually we will just write $C_R^0(S)$.

By gluing a copy of $S_{1,2}$ onto $S_{g,s}$ along a boundary component disjoint from x_1 and x_2, we obtain the surface $S_{g+1,s}$ as well as an inclusion $S_{g,s} \hookrightarrow S_{g+1,s}$. Let $S_{\infty,s}$ be the direct limit under these inclusions. The inclusions also induce inclusions $C_R^0(S_{g,s}) \hookrightarrow C_R^0(S_{g+1,s})$ as no rooted curves become trivial under the inclusions of surfaces. We then define the complex $C_R^0(S_{\infty,s})$ as the direct limit under the inclusions $C_R^0(S_{g,s}) \hookrightarrow C_R^0(S_{g+1,s})$. The objective of this subsection is to show that $C_R^0(S_\infty)$ is contractible. To do this, we first need to study the complex $C^0(S_{\infty,s})$ defined similarly to $C_R^0(S_\infty)$ as the direct limit under the inclusions $C^0(S_{g,s}) \hookrightarrow C^0(S_{g+1,s})$ induced by $S_{g,s} \hookrightarrow S_{g+1,s}$. For simplicity we write $S_\infty = S_{\infty,s}$ and $S_g = S_{g,s}$ for the remainder of this section.

Lemma 3.11. *The complex $C^0(S_\infty)$ is contractible.*

Proof. Let $k \geq 0$ be given, and let $f\colon S^k \to C^0(S_\infty)$ be any simplicial map. Since S^k is compact, the image of f must be compact and therefore contained in some $C^0(S_g)$. If we choose $g \geq k+2$, then $C^0(S_g)$ is k-connected by Theorem 3.10. Thus f is null-homotopic, showing that $C^0(S_\infty)$ is contractible by the Whitehead Theorem since it is connected. □

To show that $C_R^0(S_\infty)$ is contractible, we first embed it into a larger simplicial complex $MC_R^0(S_\infty)$ called the complex of **multi-rooted curves**. The vertices of $MC_R^0(S_{\infty,s})$ are the same as those of $C_R^0(S_{\infty,s})$, but there are more higher-dimensional simplices. Namely, a collection of rooted curves (c_i, r_i) span a simplex in $MC_R^0(S_{\infty,s})$ if the c_i's are pairwise disjoint or equal, and if there are regular neighborhoods A_i of the c_i's such that the following holds:

(1) $A_i = A_j$ when $c_i = c_j$ and $A_i \cap A_j = \emptyset$ when $c_i \neq c_j$.
(2) Each root r_i intersects $\cup_j A_j$ in a single arc crossing A_i from one component of ∂A_i to the other.
(3) The roots r_i only intersect each other at their endpoints or inside $\cup_j A_j$.

Inside $\cup_j A_j$, the roots are allowed to intersect without restrictions. For example a simplex in $MC_R^0(S_{\infty,s})$ can be constructed from a vertex (c_i, r_i) by adding modified

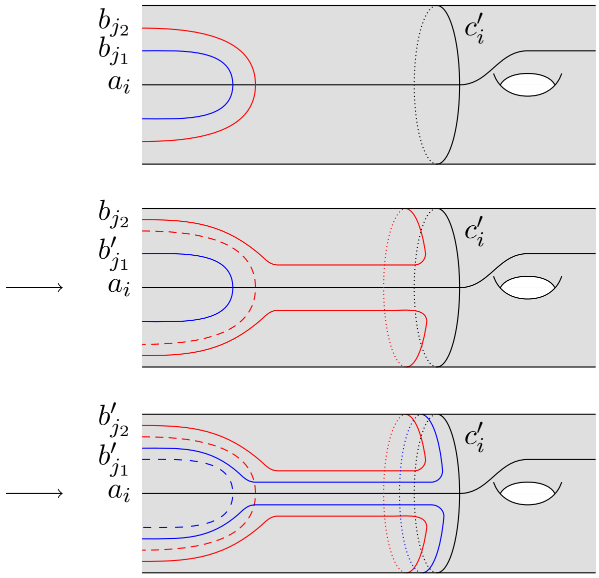


Figure 3.3: The rerouting in Lemma 3.12

roots obtained from r_i by applying any power of a Dehn twist around c_i. Note that a system of multi-rooted curves need not be coconnected.

Lemma 3.12. *The complex $MC^0_R(S_{\infty,s})$ is contractible.*

Proof. Consider the projection $MC^0_R(S_{\infty,s}) \to C^0(S_{\infty,s})$ that only remembers the curves c_i. The complex $C^0(S_{\infty,s})$ is contractible by Lemma 3.11. Thus, by Corollary 1.16 it suffices to show that the preimage $\pi^{-1}([\sigma])$ is contractible for any simplex $[\sigma]$ of $C^0(S_{\infty,s})$ represented by curves $c_1, \dots, c_k$.

Let a map $f\colon S^p \to \pi^{-1}([\sigma])$ be given. Then f has compact image $f(S^p) \subset MC^0_R(S_{g_0})$ for some $g_0 < \infty$. For each c_i choose two curves c'_i and c''_i in $S_\infty \setminus S_{g_0}$ such that $A = \cup_i(c_i \cup c'_i \cup c''_i)$ forms a coconnected curve system on S_{g_0}. This is possible since cutting along each curve reduces genus by 1, and we can choose $g_0 \geq 3k+3$. Choose a simplex $[\hat\sigma] = [(c_1, a_1), \dots, (c_k, a_k)]$ in $MC^0_R(S_\infty)$ such that each a_i intersects A transversely in exactly three points: Starting from x_o it first intersects c'_i, then c_i and then c''_i before reaching x_1. This is possible since A is coconnected.

We will use Lemma 1.20 with $Y = \mathrm{star}[\hat\sigma]$ and $X = f(S^p)$. Namely, we will construct a flow of $f(S^p)$ onto the $\mathrm{star}[\hat\sigma]$ to show that the image of f is contractible and therefore f is null-homotopic. Let $[\tau]$ be a simplex in the image of f represented by rooted curves $(d_1, b_1), \dots, (d_\ell, b_\ell)$. The complexity of $[\tau]$ is measured by the number of intersection points of τ with $\hat\sigma$, assuming that τ is in normal form with respect to $\hat\sigma$. We now divide each root a_i of $\hat\sigma$ into four arc segments,

(1) one from x_0 to c'_i,
(2) one from c'_i to c_i,

(3) one from c_i to c_i'', and
(4) one from c_i'' to x_1.

Starting with a_1 choose the intersection point between the first arc segment of a_1 and $\cup_j b_j$ that is closest to c_i'. If there is no such intersection point, choose the intersection point with the second arc segment of a_1 closest to c_i'. Then proceed similarly with the third and fourth segments, only with c_i'' instead of c_i'. Then proceed with a_2, and so on. If no intersection points are found in this way, τ is already contained in the star of $\hat{\sigma}$. If there are intersection points, we have a preferred vertex $[(d_j, b_j)]$. We then define a new vertex $\Delta[(d_j, b_j)]$ represented by a rooted curve (d_j, b_j'). The root b_j' is defined by following b_j closely from x_0 until close to a_i, then following a_i closely until close to c_i', then following c_i' closely to reach the other side of a_i, then following a_i closely until close to b_j again, and then finally following b_j back to x_1 as depicted in Figure 3.3. This reduces the number of intersection points with $\hat{\sigma}$, and the new root can be chosen to be disjoint from b_j. The third condition of Lemma 1.20 is also satisfied, so we get a flow into the star of $[\hat{\sigma}]$. □

Theorem 3.13. *The complex $C^0_R(S_{\infty,s})$ is contractible.*

Proof. Let a simplicial map $f\colon S^k \to C^0_R(S_\infty)$ be given. Since S^k is compact, the image of f is contained in $C^0_R(S_g)$ for some g. We want to show that for some g_1, f can be extended over the disk to a map $G\colon D_{k+1} \to C^0_R(S_{g_1})$ that is homotopic to f when restricted to the boundary sphere. We do this by first expanding f to a map $F\colon D_{k+1} \to \widehat{MC^0_R}(S_{g_0})_{k+2}$ for some g_0 and then modifying F to obtain the map G.

Step 1: Construction of $F\colon S^k \to \widehat{MC^0_R}(S_{g_0})_{k+2}$: Assume that f is simplicial with respect to some triangulation $\mathbb{T}_0$ of S^k. Let $\mathbb{T}_0'$ be the barycentric subdivision (poset complex) of $\mathbb{T}_0$. We want to construct a simplicial map F from a subdivision $\mathbb{T}_1$ of $\mathbb{T}_0'$ to $\widehat{MC^0_R}(S_{g_0})_{k+2}$; see Definition 1.17 (later we will extend over the disk). Moreover, the subdivision $\mathbb{T}_1$ and the map F must satisfy the following additional property: For any vertex v of $\mathbb{T}_1$ such that v lies in the interior of a simplex $(\sigma_0 < \cdots < \sigma_p)$ of $\mathbb{T}_0'$, $f(\sigma_0)$ is a **monic** subset of $F(v)$, meaning that every curve of $f(\sigma_0)$ intersects only one arc of $F(v)$.

We proceed by induction over the skeleta of $\mathbb{T}_0'$. For every vertex σ of $\mathbb{T}_0'$ such that $f(\sigma)$ is a p-simplex in $C^0_R(S_g)$, we extend $f(\sigma)$ to a $(k+1)$-simplex $F(\sigma) = f(\sigma) * \tau$ of $C^0_R(S_{g_0})$ for a suitable g_0. This is possible because $\dim f(\sigma) \leq k$ and we can choose $g_0 \geq k+2$. Then $F(\sigma)$ can be considered as a vertex of $\widehat{MC^0_R}(S)_{k+2}$, so we have defined F on vertices.

For the induction step, assume that F is defined on $\partial\tau$ for some p-simplex $\tau = (\sigma_0 < \cdots < \sigma)$ of $\mathbb{T}_0'$. We want to extend F over τ. Assume also that the monic subset property mentioned above is satisfied. Then for any vertex v of $\partial\tau$, $F(v)$ contains $f(\sigma_0)$ as a monic subset, i.e. every curve of $f(\sigma_0)$ intersects only one arc of $F(v)$. Thus we can cut S_{g_0} along $f(\sigma_0)$ to obtain a surface $S_{f(\sigma_0)}$, and the remaining rooted curves will not be affected by the cutting. This means that we can consider the restriction of F to $\partial\tau$ as a map $g_\tau\colon \partial\tau \to \widehat{MC^0_R}(S_{f(\sigma_0)})_{k+2-q}$, where q is the number of vertices in $f(\sigma_0)$ that we have 'removed' by cutting along them. By Lemma 3.12 and Lemma 1.18 we can replace g_0 by some $g_1 \geq g_0$ such that $\widehat{MA}^0_R(S_{f(\sigma_0)})_{k+2-q}$ is $(p-1)$-connected. Rooted curves in $S_{f(\sigma_0)}$ correspond to rooted curves in S_{g_1} that are disjoint from and not isotopic to rooted curves of $f(\sigma_0)$. Thus we can extend F over the interior of τ by extending in $\widehat{MA}^0_R(S_{f(\sigma_0)})_{k+2-q}$ and then pulling back to $\widehat{MA}^0_R(S_{g_1})_{k+2-q}$ and finally joining with $f(\sigma_0)$ to get a map into $\widehat{MA}^0_R(S_{g_1})_{k+2}$. The monic subset property is still satisfied since the extension was done with rooted curves disjoint from $f(\sigma_0)$.

Since the triangulation $\mathbb{T}_0'$ must be finite, this shows that we can modify f to a map $F\colon S^k \to \widehat{MC}^0_R(S_{g_1})_{k+2}$ for some sufficiently large g_1. We can then expand F over the disk simply by choosing g_1 large enough such that $\widehat{MC}^0_R(S_{g_1})_{k+2}$ is sufficiently highly connected. This gives a map $F\colon S^k \to \widehat{MC}^0_R(S_{g_1})_{k+2}$. Moreover, we can assume that F is simplicial with respect to a triangulation on D_{k+1} that restricts to the triangulation $\mathbb{T}_1$ of S^k.

Step 2: Construction of $G\colon D_{k+1} \to C^0_R(S_{g_1})$: We construct G by induction on the skeleta of $\mathbb{T}_1$ using a subdivision $\mathbb{T}_2$ of $\mathbb{T}_1$ with the extra property that if w is a vertex of $\mathbb{T}_2$ in the interior of a simplex $\tau = \{v_0, \dots, v_p\}$ of $\mathbb{T}_1$ with $F(v_0) \leq \cdots \leq F(v_p)$ in $\widehat{MC}^0_R(S_{g_1})_{k+2}$, then $G(w)$ is a vertex of $F(v_0)$. If $w \in S^k$ and w lies in the interior of a simplex $(\sigma_0 < \cdots < \sigma_q)$ of $\mathbb{T}_0'$, we moreover assume that $G(w)$ is a vertex of $f(\sigma_0)$. This is possible since $f(\sigma_0) \subset F(v_0)$ by the construction in step 1.

To define G on the 0-skeleton of $\mathbb{T}_1$ as well as on all of S^k, note that the additional property mentioned in the previous paragraph does not concern vertices of $\mathbb{T}_1$, so this leaves no obstruction to the definition of G. If w is a vertex of $\mathbb{T}_1$ that lies in the interior of a simplex σ of $\mathbb{T}_0$ in S^k, then $G(w)$ must lie in $f(\sigma)$, so set $G(w) = f(v)$ for some vertex v of σ. Since $\mathbb{T}_1$ is a subdivision of $\mathbb{T}_0$, we can extend G over S^k in this way, and the resulting map will be homotopic to f. This means that it only remains to define G on higher dimensional simplices in the interior of D_{k+1}.

Suppose that G is defined on the $(p-1)$-skeleton of $\mathbb{T}_1$. Let $\tau = \{v_0, \dots, v_p\}$ be a p-simplex of $\mathbb{T}_1$ that is not contained in S^k. For each vertex w in the triangulation $\mathbb{T}_2$ which exists on $\partial\tau$ by induction hypothesis, either $G(w)$ lies in the interior of a face of τ, or w is a vertex v_j of τ. In the former case we use the assumption on G, and in the latter case we already have $G(w) = f(w)$. Consider now the projection $\pi\colon MC^0_R(S_{g_1}) \to C^0(S_{g_1})$ that only remembers the curves c_i, and denote also by π the resulting projections on poset complexes. Let L be the set of vertices of $\pi F(v_p)$, and let L_0 be the set of vertices of $\pi F(v_0)$. Since $\partial\tau$ is a sphere, we can apply the Coloring Lemma, Lemma 1.12. This gives us an extension of the triangulation $\mathbb{T}_2$ over τ where vertices in the interior of τ are labeled by L_0, and bad simplices only occur in $\partial\tau$. For a vertex w in the interior of τ, we define $G(w)$ as any vertex in $F(v_0)$ mapping by π to w. For this definition to be valid, we need to check that for each simplex $\sigma = \{w_0, \dots, w_q\}$ in $\mathbb{T}_2$, $G(\sigma)$ is a simplex of $C^0_R(S_{g_1})$. We know that $G(\sigma)$ is already a simplex of $MC^0_R(S_{g_1})$ since it is a face of $g(v_p)$, so we only need to show that each $G(w_i)$ has a different associated curve. But G is defined such that if w_i or w_j or both are in the interior of D_{k+1}, and $i \neq j$, then

$\pi G(w_i) \neq \pi G(w_j)$. If $w_i, w_j \in S^k$, then we could have $\pi G(w_i) = \pi G(w_j)$, but in that case G has already been defined to map $\{w_i, w_j\}$ to a simplex of $C^0_R(S_{g_1})$. $\square$

3.2 Complexes of connecting arcs

Let $S = S_{g,s}$ with $s > 0$, and let P and Q be two disjoint 1-dimensional non-empty compact submanifolds of ∂S. Then P and Q consist of disjoint arcs and/or curves inside the boundary components of S. We consider isotopy classes of non-trivial arcs each with one endpoint in P and one in Q, where the arcs are not allowed to land in the endpoints of the components of P or Q. We let isotopies vary the endpoints as long as the endpoints stay inside P, respectively Q, and we say that an arc is trivial if it is isotopic to an arc inside ∂S which only touches $P \cup Q$ in its endpoints. These isotopy classes of arcs are the vertices in a complex $A(S,P,Q)$ in which a k-simplex has as vertices $k+1$ isotopy classes of arcs for which it is possible to choose representatives that are mutually disjoint.

A component of ∂S might contain arcs of P, arcs of Q, or arcs of both. We consider the components of $\partial S \setminus (P \cup Q)$. Such a component is either a component of ∂S or an arc between (possibly identical) components of $P \cup Q$ only touching $P \cup Q$ at its endpoints. If it goes between P and Q, we say that the arc is **mixed**, and if it goes from P to P or Q to Q, we say that it is **pure**. The number of pure arcs can be any integer $u \geq 0$, but the number of mixed arcs must be an even number $2t$ for $t \geq 0$. Moreover we consider the number s' of boundary components that intersect $P \cup Q$. The goal of this section is to prove the following theorem.

Theorem 3.14 ([HV15b, Theorem 5.1]). *The complex $A(S,P,Q)$ is $(4g+s+s'+t+u-6)$-connected.*

To prove this, we first need to take into account some special cases, which we will do in the following.

3.2.1 Contractible cases

The following lemmas give sufficient conditions for $A(S,P,Q)$ to be contractible, but this is not always the case. Namely, if $S = S_{0,3}$, a pair of pants, and P and Q each consist of a single component in the same boundary component, then $A(S,P,Q)$ is zero-dimensional since cutting along an arc separates S into two copies of $S_{0,2}$ which each inherit a part of P and a part of Q in the same boundary component, so there can be no more non-trivial arcs. $A(S,P,Q)$ is not contractible since it consists of infinitely many vertices. Namely, it has a vertex for each integer as indicated by Figure 3.5.

Lemma 3.15 ([HV15b, Lemma 5.2]). *If some component $\partial_0 S$ of ∂S intersects either P or Q, but not both, then $A(S,P,Q)$ is contractible.*

Proof. Let an arc $[a] \in A(S,P,Q)$ such that a has one end in $\partial_0 S$ (it must have the other end in another boundary component by assumption). We will do a surgery flow towards the star of the arc class $[a]$, i.e. we will use Lemma 1.20. So consider star$([a])$ as a

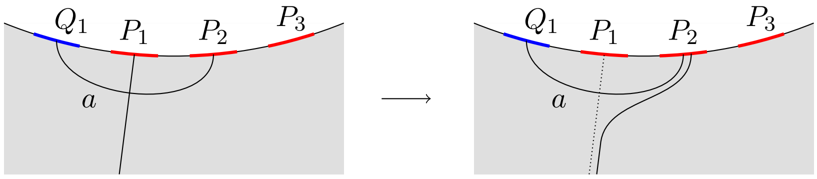


Figure 3.4: Surgery in Lemma 3.16

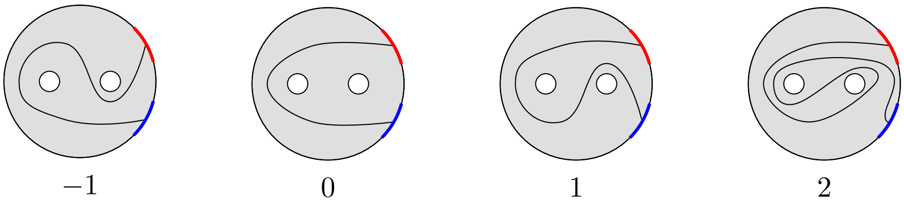


Figure 3.5: Elements of the complex $A(S_{0,3}, P, Q)$

subcomplex of $A(S, P, Q)$, and define a complexity function on $A(S, P, Q)$ that measures the number of times an arc must intersect a. If an arc a_0 intersects a, we cut it at the intersection point and redirect the two arc ends to the same component of $P \cup Q$ in $\partial_0 S$ that a lands in. This results in two arcs, one of which is a well-defined arc, i.e. it goes between P and Q, and it is non-trivial because it travels from some other component of ∂S to $\partial_0 S$. Moreover, if we always choose the intersection point of a curve system and a that is the closest to $\partial_0 S$, the conditions of Lemma 1.20 are satisfied, which shows that star($[a]$) is a deformation retract of $A(S, P, Q)$, i.e. $A(S, P, Q)$ is contractible by (A.2). □

Lemma 3.16 ([HV15b, Lemma 5.3])**.** *If some component $\partial_0 S$ of ∂S contains three adjacent arc components of P (or equivalently of Q), then $A(S, P, Q)$ is contractible.*

Proof. If $\partial_0 S$ contains no components of Q, then the statement is covered by Lemma 3.15. Thus we can label the consecutive components by P_1, P_2 and P_3 and assume that P_1 is adjacent to some component Q_1 of Q. Let a be a boundary-parallel arc from Q_1 to P_2. We want to use Lemma 1.20 by creating a surgery flow into star($[a]$). We will measure the complexity of an arc system class $[\sigma]$ by how many times σ must intersect a. The preferred arc is obtained by choosing the arc that intersects a the closest to P_2. We cut the arc and redirect the ends to P_2. One of the resulting arcs hits P_1 and is discarded. The other can be chosen to not intersect a, and it is non-trivial since P_2 is separated from Q on both sides by P_1 and P_3 respectively, so no arc landing in P_2 can be trivial. The complexity is reduced since we reduced the amount of intersection points with a by 1. The third condition of Lemma 1.20 is also satisfied, showing that star($[a]$) is a deformation retract of $A(S, P, Q)$, so $A(S, P, Q)$ is contractible by (A.2). □

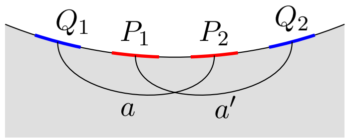


Figure 3.6: The setting of Lemma 3.17

3.2.2 Adding a component to P

Lemma 3.17 ([HV15b, Lemma 5.4])**.** *Suppose that some component of ∂S contains some component P_1 of P surrounded by components Q_1 and Q_2 of Q, possibly with $Q_1 = Q_2$. Now modify P by adding an extra component P_2 between P_1 and Q_2 to form P'. If $A(S, P, Q)$ is k-connected, then $A(S, P', Q)$ is $(k+1)$-connected.*

Proof. Let a be a boundary-parallel arc from Q_1 to P_2, and let a' be a boundary-parallel arc from P_1 to Q_2, as illustrated in Figure 3.6. Let A' be the subcomplex of $A(S, P', Q)$ consisting of arc system classes not containing $[a']$. Then by the same argument as in Lemma 3.16, A' is contractible since we can deform all of its arcs into $\operatorname{star}([a])$. We pick the intersection point closest to P_2 for our surgery, so that we can shift the ends of the arcs into P_2. The whole complex $A(S, P', Q)$ is the union $A' \cup \operatorname{star}([a'])$, and the intersection of these two complexes (which are both contractible) is $\operatorname{link}([a'])$. This means that $\operatorname{star}([a'])$ sits as a cone on top of the contractible complex A', so by collapsing A' inside $A(S, P', Q)$, we get the suspension of $\operatorname{link}([a'])$ since the suspension can always be obtained as the cone modulo the base. This means that the suspension of $\operatorname{link}([a'])$ is homotopy equivalent to $A(S, P', Q)$.

Now, a simplex in $\operatorname{link}([a'])$ consists of classes of arcs that are disjoint from a' and not isotopic to a'. Such arcs cannot land in P_2, and if they land in P_1, they cannot come from Q_2. But this is exactly the description of the smaller complex $A(S, P, Q)$. In conclusion,

$$A(S, P', Q) \simeq S(\operatorname{link}([a'])) \simeq S(A(S, P, Q)),$$

so the connectivity of $A(S, P', Q)$ is one higher than the connectivity of $A(S, P, Q)$. □

3.2.3 Filling with a disk

Lemma 3.18 ([HV15b, Lemma 5.5])**.** *Suppose that ∂S has some component $\partial_0 S$ that is disjoint from $P \cup Q$. Let S' be the surface obtained by attaching a disk along $\partial_0 S$. If $A(S, P, Q)$ is non-empty, then the connectivity of $A(S, P, Q)$ is one greater than the connectivity of $A(S', P, Q)$.*

Proof. By Lemma 3.15 we can assume that no component of ∂S intersects one of P or Q without intersecting the other. Since moreover we have assumed P and Q to be non-empty, we can choose adjacent arcs P_1 and Q_1 in some component $\partial_1 S \neq \partial_0 S$ of ∂S. Now consider arcs from P_1 to Q_1 that are non-trivial in S, but trivial in S'. For

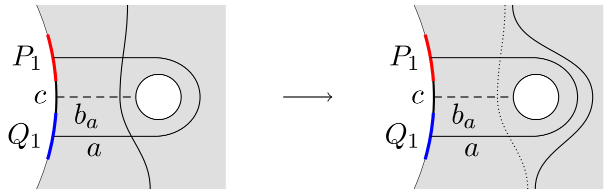


Figure 3.7: Pushing an arc across $\partial_0 S$

any choice of P_1 and Q_1, we call this kind of arcs **special arcs**. Let a be a special arc. Then a cuts from S an annulus which has $\partial_0 S$ as its inner boundary component, and whose outer boundary component is assembled from the arc a, a part of P_1, a part of Q_1 and an arc c_a in $\partial_1 S$ between P_1 and Q_1. Regardless of how P_1, Q_1 and a are chosen, there is an arc b_a from c_a to $\partial_0 S$. We can even choose b_a such that it is disjoint from a. Special arcs a are then in bijective correspondence with such arcs b_a since a special arc is always isotopic to an arc that travels closely around such an arc b_a and the boundary component $\partial_0 S$.

Since we have assumed $A(S, P, Q)$ to be non-empty, there exists a non-trivial arc and therefore also a special arc. Moreover, the fact that P_1 is adjacent to Q_1 ensures that two special arcs would either have to intersect, or one would have to be inside the annulus cut out by the other, in which case they would be isotopic.

Let $A' \subset A(S, P, Q)$ be the subcomplex consisting of arc systems with no special arcs, and let a be a special arc. By the argument in the previous paragraph, $\operatorname{link}(a) \subset A'$. We can use Lemma 1.20 to create a flow from A' into $\operatorname{link}(a)$. An arc system that is in normal form with respect to a and intersects a must also intersect b_{P_1,Q_1}, so we can choose the preferred arc to be the one that intersects b_{P_1,Q_1} the closest to $\partial_0 S$. This can then be pushed across $\partial_0 S$ as illustrated in Figure 3.7, and since it is non-special it will not become trivial. It will not become special either since then it would have been trivial in the first place. This flow satisfies the conditions of Lemma 1.20, so we can conclude that A' deformation retracts onto the link of a.

$A(S, P, Q)$ is the union of A' with the stars of all special arcs. Since different special arcs cannot appear in the same arc system, these stars are disjoint except for their links, and the links are included in A'. The complex $A' \cup \operatorname{star}(a)$ is contractible since A' deformation retracts onto $\operatorname{link}(a)$. If a is the only special arc, then $A' \cup \operatorname{star}(a) = A(S, P, Q)$, so $A(S, P, Q)$ is contractible. If there are other special arcs, we contract $A' \cup \operatorname{star}(a)$ inside of $A(S, P, Q)$ to see that $A(S, P, Q)$ is homotopy equivalent to a wedge sum of the suspensions of the links of special arcs. Note that with the usual meaning of suspension, this might not be the case, but we can fix that by regarding suspension as joining with S^0 which raises the connectivity by 1 also in the empty case.

Moreover, the link of a special arc a can be identified with the arc complex on the surface obtained by cutting S along a and removing the resulting annulus containing $\partial_0 S$. This complex is isomorphic to $A(S', P, Q)$. Thus $A(S, P, Q)$ is homotopy equivalent to a wedge of suspensions of $A(S', P, Q)$, so the connectivity is one higher. $\square$

3.2.4 Connectivity of $B(S, R)$

To prove Theorem 3.14, we will embed $A(S, P, Q)$ into a larger arc complex in which the conditions for being non-trivial are the same, but now the endpoints of the arcs are allowed to be anywhere in $P \cup Q$. This means that we don't have to specify the sets P and Q individually, so we just denote this complex by $B(S, P \cup Q)$, or just $B(S, R)$ for any 1-dimensional non-empty compact submanifold R of ∂S.

Lemma 3.19 ([HV15b, Lemma 5.6])**.** *If the complex $B(S, R)$ is non-empty, it is contractible except in the following two cases:*

(1) If S is a disk, then $B(S, R)$ is homotopy equivalent to S^{r-4}, where r is the number of arcs in R.

(2) If S is an annulus, and R is contained in one of the boundary components, then $B(S, R)$ is homotopy equivalent S^{r-2}.

Proof. Assume that $B(S, R) \neq \emptyset$. If some boundary component $\partial_0 S$ of S contains exactly one component R_1 of R, fix an arc b landing in R_1, and fix an endpoint p of this arc in R_1, since it might have both endpoints there. We will do a surgery flow towards star(b). For an arc system that intersects b, make a cut at the intersection point closest to p, and redirect the two resulting arcs to R_1. Since R_1 is the only component of R in $\partial_0 S$, at least one of the two new arcs is non-trivial since otherwise the original arc would have been trivial in the first place. This reduces the complexity (amount of intersection points with b) of the arc system by one, and defines a surgery flow into star(b).

We will reduce the general case to this scenario by an argument similar to the one that we used in Lemma 3.17. We claim that adding an arc component R_2 to R is equivalent to suspending $B(S, R)$ up to homotopy equivalence. Suppose that R' is obtained from R by adding a component R_2 next to an existing R_1. Denote by R_3 the arc next to R_1 opposite to R_2, and denote by R_4 the arc next to R_2 opposite to R_1. If the total number of arcs in $\partial_0 S \cap R'$ is 3, then $R_3 = R_4$, and if the number is four, then $R_3 = R_2$ and $R_4 = R_1$. Let now a be a boundary parallel arc from R_3 to R_2, and let a' be a boundary parallel arc from R_1 to R_4. Denote by B' the subcomplex of $B(S, R')$ consisting of arc systems not containing a'. Then by Lemma 1.20, B' deformation retracts onto star(a) by shifting arcs landing in R_1 over to R_2. The new arcs are non-trivial since if not, then the original arc would have to be trivial or isotopic to a'. This means that B' is contractible. Moreover we can view $B(S, R')$ as the union of B' with star(a'), and then star(a') is attached to B' along link(a'). By contracting B' inside $B(S, R')$, we see that $B(S, R')$ is homotopy equivalent to the suspension of link(a'), but link(a') is isomorphic to $B(S, R)$, so $B(S, R')$ is homotopy equivalent to the suspension of $SB(S, R)$.

This means that we can reduce to the case where some boundary component contains exactly one component of R. If this is the case, and if S is not a disk, or an annulus with R contained in one boundary component, then $B(S, R)$ is non-empty and therefore contractible by the argument in the beginning of the proof.

If S is a disk, then R must have at least four components for $B(S, R)$ to be non-empty. If it has exactly four, then $B(S, R) \cong S^0$, and the suspension argument gives the result.

If S is an annulus, and R is contained in one boundary circle, then R must have at least two components in order for $B(S,R)$ to be non-empty. If R has exactly two components, then again $B(S,R) \cong S^0$, and the rest follows by the suspension argument. □

Now follows a lemma that we will need for the proof of Theorem 3.14.

Lemma 3.20. *All maximal arc systems in $B(S,R)$, i.e. arc systems that are not strictly contained in any larger arc system, have the same number of arcs, provided that R intersects all boundary components of S.*

Proof. Consider a single arc a such that $[a]$ is a vertex in $B(S,R)$. Suppose that a is separating. Then both of its endpoints must in the same boundary component of S since otherwise the cutting along a reduces the number of boundary components by 1. Since moreover it increases Euler characteristic by one (since it adds two vertices and one edge), it cannot be separating as that would further increase Euler characteristic by 2. Thus cutting along a increases the total number of boundary components by one, so since a separating arc also increases Euler characteristic by one, we have

$$(2-2g-s)+1 = (2-2g_1)+(2-2g_2)-(s+1),$$

where g_1 and g_2 are the genera of the two components of the surface cut up by the separating arc. It follows that $g = g_1 + g_2$, i.e. a separating arc preserves the total genus.

Suppose now that a is non-separating. Then it still must increase Euler characteristic by one, so it may reduce s by one, or it may reduce g by one and increase s by one. It cannot reduce g by more than one since then it would have to increase s by three or more, which is impossible.

Now consider any arc system class $[\sigma]$ in $B(S,R)$. Since no boundary component of S is disjoint from R, the same holds for each component S_i of the surface obtained from S by cutting along σ and the set R_i inherited from R. This means that a maximal arc system must cut S into disks since if not, we could just take one more non-trivial arc that is not isotopic to and does not intersect the others, which either reduces the genus or the number of boundary components of the surface S_i that it lives in. Such a disk S_i contains a non-trivial arc if and only if R_i contains at least four arcs. Thus a maximal arc system must in fact cut S into disks S_i such that each R_i contains exactly three arcs. We denote a surface $S_{g,s}$ with a corresponding 1-manifold $R \subset \partial S_{g,s}$ by $S_{g,s}(n)$ if R has n components. Note that cutting along any arc divides the components of R adding two extra components. Considering the individual arcs, cutting along them can thus do the following three things:

(1) Reduce g by one, increase s by one and increase n by two.
(2) Reduce s by one and increase n by two.
(3) Separate into two surfaces, increase the total n by two and increase the total s by one.

We now claim that any maximal sequence of these operations will separate $S_{g,s}(n)$ into $n+4g+2s-4$ copies of $S_{g,s}(3)$. We prove this by induction on the lexicographically ordered triple (g,s,n). The base case $(0,1,3)$ is trivial since there are no non-trivial arcs. For the induction step, we consider each of the above operations separately. The first operation takes $S_{g,s}(n)$ to $S_{g-1,s+1}(n+2)$, which by the induction hypothesis separates into

$$n+2+4(g-1)+2(s+1)-4 = n+4g+2s-4$$

copies of $S_{0,1}(3)$, which coincides with the statement. The second operation takes $S_{g,s}(n)$ to $S_{g,s-1}(n+2)$ which again by the induction hypothesis separates into

$$n+2+4g+2(s-1)-4 = n+4g+2s-4$$

copies of $S_{0,1}(3)$. Finally, the third operation separates into two surfaces $S_{k,l}(m) + S_{g-k,s-l-1}(n-m+2)$ where $0 \leq k \leq g$, $1 \leq l \leq s$, and $3 \leq m \leq n-1$. Then $n-m+2 \leq n-1$, so the induction hypothesis applies to both surfaces. This means that these two surfaces together separate into

$$(m+4k+2l-4)+(n-m+2+4g-4k+2s-2l+s-4) = n+4g+2s-4$$

copies of $S_{0,1}(3)$. This shows that any maximal arc system cuts S into the same amount of disks, so the Euler characteristic is raised by the same amount, and therefore the amount of arcs must be the same as well. □

3.2.5 Connectivity of $A(S,P,Q)$

We are now ready to prove Theorem 3.14. We restate it here.

Theorem 3.14 ([HV15b, Theorem 5.1])**.** *The complex $A(S,P,Q)$ is $(4g+s+s'+t+u-6)$-connected.*

Proof. Let $c = 4g+s+s'+t+u-6$ denote the desired connectivity number. Recall that we assumed both P and Q to be non-empty. We will also want to assume that $A(S,P,Q)$ is non-empty. There are only four cases where $A(S,P,Q)$ is empty, and the third one actually covers two cases by interchanging P and Q. These four cases are

(1) $g=0$, $s=1$, $s'=1$, $t=1$, and $u=0$;
(2) $g=0$, $s=2$, $s'=1$, $t=1$, and $u=0$;
(3+4) $g=0$, $s=1$, $s'=1$, $t=1$, and $u=1$;
(5) $g=0$, $s=1$, $s'=1$, $t=2$, and $u=0$,

as illustrated in Figure 3.8. In these cases we can just determine the connectivity number, which in the respective cases is $c=-3$, $c=-2$, $c=-2$ and $c=-2$, so in all cases the statement is vacuous. Therefore we can now assume that $A(S,P,Q)$ is non-empty. We will prove the theorem by induction on the dimension (as a simplicial complex) of the complex $B(S,P\cup Q)$. Note that $A(S,P,Q)$ is a subcomplex of $B(S,P\cup Q)$ by definition.

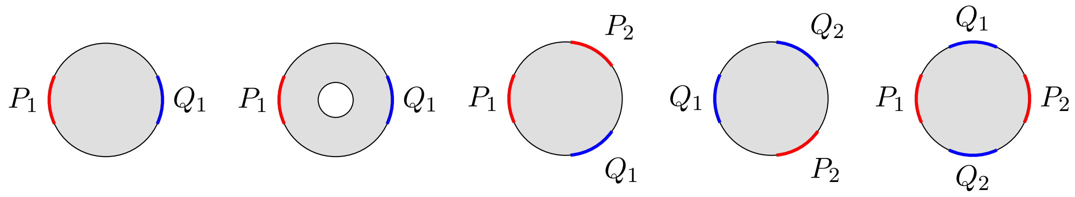


Figure 3.8: The cases where $A(S,P,Q)$ is empty

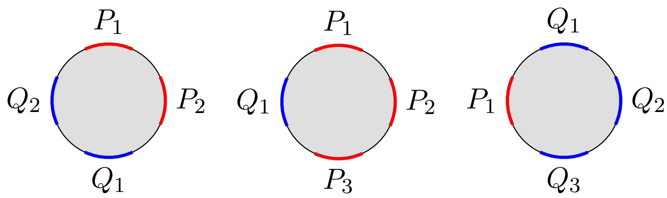


Figure 3.9: The cases where $A(S,P,Q)$ is empty and $B(S,P\cup Q)$ is non-empty and 1-dimensional

Therefore if the dimension of $B(S,P\cup Q)$ is -1, i.e. it is empty, then $A(S,P,Q)$ is empty as well, and these cases have already been covered. If $\dim B(S,P\cup Q)=0$, then $B(S,P\cup Q)$ only contains arc systems of one arc. This means that S must be 1) a disk, or 2) an annulus with $P\cup Q$ contained in one boundary component, since otherwise we could have two non-isotopic and non-trivial disjoint arcs. In the first case where S is a disk, $P\cup Q$ must have exactly 4 components for $B(S,P\cup Q)$ to be non-empty and 1-dimensional. There are three cases where this happens with $A(S,P,Q)$ non-empty. These are indicated in Figure 3.9.

In the second case, where S is an annulus with $P\cup Q$ contained in one boundary component, $P\cup Q$ must contain exactly two components for $B(S,P\cup Q)$ to be non-empty and 1-dimensional. In that case $A(S,P,Q)$ is empty, which we have already covered.

We will now proceed with the induction step, so assume that the theorem holds for all S',P',Q' with $\dim B(S',P'\cup Q')<\dim B(S,P\cup Q)$. By Lemma 3.15 and Lemma 3.16, $A(S,P,Q)$ is contractible whenever S has a boundary component touching one of P or Q, but not both, or whenever some boundary component contains three consecutive arcs of either P or Q, so the theorem holds in these cases. Thus we may assume that these are not the case. By Lemma 3.17, separating a component of P or Q in two and leaving a pure arc in between raises the connectivity of $A(S,P,Q)$ by one, but it also accordingly increases c by one, so we may assumme that there are no pure arc segments, i.e. that $u=0$. Moreover, Lemma 3.18 tells us that adding a boundary component not intersecting $P\cup Q$ also raises the connectivity of $A(S,P,Q)$ by one. Since this also increases c by one accordingly, we may assume that there are no boundary components not intersecting $P\cup Q$, i.e. that $s=s'$. Thus we can write

$$c=4g+s+s'+t+u-6=4g+2s+t-6,$$

or just

$$c = 2e + t - 2,$$

where $e = -\chi(S) = 2g - 2 + s$, the negative of the Euler characteristic of the surface S.

Remember that we assumed $A(S, P, Q)$ to be non-empty. We want to show that $A(S, P, Q)$ is $(2e+t-2)$-connected, so let $f : S^k \to A(S, P, Q)$ be given with $k \leq 2e+t-2$. We want to extend f to a map $D^{k+1} \to A(S, P, Q)$. In order to do this, we first extend f to a map $g : D^{k+1} \to B(S, P \cup Q)$ and then modify this map without changing its homotopy class, such that its image is contained in $A(S, P, Q)$. To show that we can extend f to such a map g, note that Lemma 3.19 says that $B(S, P \cup Q)$ is contractible in most cases, so we only have to consider the case when S is a disk since we have already taken care of the other case mentioned in Lemma 3.19. When S is a disk, $B(S, P \cup Q) \cong S^{2t-4}$ since $2t$ is the number of mixed arcs and therefore the number of components of $P \cup Q$ as there are no pure arcs. Thus $B(S, P \cup Q)$ is $(2t - 5)$-connected, and in this case

$$c = 2e + t - 2 = -2 + t - 2 = t - 4$$

which is less than or equal to $2t - 5$ since $t \geq 1$. In fact $t \geq 3$ since $A(S, P, Q)$ is non-empty.

We can assume that f is simplicial with respect to a piecewise linear triangulation of D^{k+1}. We will use a bad simplex argument to show that f can be homotoped to a map $D^{k+1} \to A(S, P, Q)$ that is simplicial with respect to a modified triangulation of D^{k+1}, and that this new map still extends the original map f on S^k.

We will introduce a definition of badness for an arc system in $B(S, P \cup Q)$, although it will not satisfy the original criteria for badness. To carry out the bad simplex argument, we will have to study what happens when we cut S along an arc system. Assuming that any arc representative is chosen such that its endpoints are in the interiors of the components of $P \cup Q$ that it touches, we will get a finite collection of surfaces S_i and corresponding collections P_i and Q_i of arcs in ∂S_i. If P_i or Q_i is empty, we say that S_i is a pure piece, and if they are both non-empty, we say that S_i is a mixed piece. Note that for any non-empty arc system, the dimension of each $B(S_i, P_i, Q_i)$ for mixed pieces is strictly less than the dimension of $B(S, P \cup Q)$ since at least one arc has taken up room from the others. Thus by the induction hypothesis $A(S_i, P_i, Q_i)$ is $(2e_i + t_i + u_i)$-connected, where e_i, t_i and u_i are the numbers for S_i corresponding to e, t and u for S. Note that the surfaces S_i may contain pure arcs, but cannot contain 'loose' boundary components, i.e. boundary components that don't intersect $P_i \cup Q_i$, hence the connectivity statement with u_i but without s'_i.

We now define bad vertices as the arc classes that have both endpoints in only one of P or Q since $A(S, P, Q)$ is a subcomplex of $B(S, P \cup Q)$ containing no such arcs. The naive definition of higher-dimensional bad simplices would then be the simplices all of whose vertices are bad. This however will not suffice in order to use Corollary 1.10. Namely, if $[\mu]$ is a maximal simplex in D^{k+1} such that the image $[\sigma] := f([\mu])$ consists of only bad arc classes, then by maximality link$[\mu]$ must map to good arc classes and

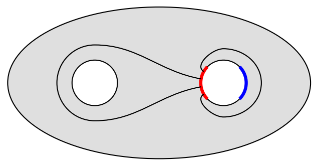


Figure 3.10: An example of why the naive definition of bad simplices does not work

therefore to vertices disjoint from $[\sigma]$. This means that link$[\mu]$ maps into the join of the complexes $A(S_i, P_i, Q_i)$ for the mixed pieces S_i. If this join were always $(k - \dim[\mu])$-connected, we could use Corollary 1.10, but it might not be. In fact it might even be empty, e.g. if $S = S_{0,3}$, and P and Q each have one component as illustrated in Figure 3.10 where P is drawn in red and Q is drawn in blue. Here $[\mu]$ is a 1-simplex, and the image consists of two arcs that are both bad. These two arcs cut S into three pieces, but two of the pieces are pure, and the third piece is the third example of $A(S, P, Q)$ being empty in Figure 3.8.

We can however solve the problem by employing a more strict version of badness. We say that an arc is bad if it has both its endpoints in one of P or Q, and we say that a simplex $[\sigma]$ is bad if all of its vertices are bad, and if no arc of σ has pure pieces on both sides. With this definition the 1-simplex illustrated in Figure 3.10 is no longer a bad simplex. Note that an arc of σ that does not separate a component of the complement of the other arcs of σ has the same piece on both sides, and so such a piece is not allowed to be pure. Moreover, if $[a]$ is a single bad vertex in $B(S, P \cup Q)$, then a can in fact never have pure pieces on both sides since then either P or Q would have to be empty.

This definition does not satisfy the original second criterion for badness. For example an edge might have two bad vertices but not be bad itself. The edge illustrated in Figure 3.10 is an example of this. We can however use a similar strategy again. We will retriangulate star$([\mu])$ to reduce what we will call the complexity. To define the complexity of a simplex μ, we first define the complexity of a pure piece S_i. The complexity of S_i is defined as the number of arcs in a maximal arc system in $B(S, P \cup Q)$.

For a simplex μ with bad image $[\sigma] := f(\mu)$ we then define the complexity as the ordered pair

$$(d([\sigma]), \dim \mu),$$

where $d([\sigma])$ is the sum of the complexities of all *pure* pieces of $S \setminus \sigma$. The complexities of simplices with bad images are then ordered lexicographically. Now assume that μ is a bad simplex in D^{k+1} of maximal complexity, i.e. the complexity (d, l) of μ is maximal among bad simplices. The restriction of f to link μ then maps to link$[\sigma]$ since if some vertex of link μ mapped into $[\sigma]$ it would contradict the maximality of μ. Moreover, any arc in the image of link μ can be seen as a vertex of $A(S_i, P_i, Q_i)$ for some S_i since otherwise it would contradict the maximality of $(d([\sigma]), \dim \mu)$. Therefore we denote by J the join of the complexes $A(S_i, P_i, Q_i)$ for the mixed pieces and $B(S_i, P_i \cup Q_i)$ for the pure pieces, such that link μ maps into J.

We will show that J is $(k-l)$-connected, but first let us assume that it is. Then note that $\operatorname{link}\mu \cong S^{k-l}$ by Theorem A.4 since the triangulation on D^{k+1} is piecewise linear, so we can extend $f_{|\operatorname{link}\mu}$ to a map $F\colon D^{k-l+1} \to J$. Like in the proof of Proposition 1.9 we modify f on $\operatorname{star}\mu$ by retriangulating $\operatorname{star}(\mu)$ to obtain a new map $f_{|\partial\mu} * F$. We now claim that a simplex with bad image under this new map on the new triangulation has strictly lower complexity than μ, i.e. if ν is such a simplex with bad image $[\tau]$, then either 1) $d([\tau]) < d([\sigma])$, or 2) $d([\tau]) = d([\sigma])$ and $\dim\nu < \dim\mu$. To see this, we note that ν must be the join of a simplex α in $\partial\mu$ with a simplex β in D^{k-l+1} such that $F(\beta) \subset J$. The image of $\alpha * \beta$ is $[\tau]$, which is a bad simplex, so each vertex of $F(\beta)$ is bad. Thus if β is *non-empty*, the vertices of $F(\beta)$ must be in some pure piece of the original $S \setminus \sigma$ since they are bad and therefore cannot be in any $A(S_i, P_i, Q_i)$. Therefore these arcs can only cut pure pieces into pure subpieces, so they most have pure pieces on both sides, i.e. $\sigma * F(\beta)$ cannot be a bad simplex. Thus since $[\tau]$ is bad, $[\tau]$ must be of the form $[\sigma_0] * F(\beta)$ for some proper face $[\sigma_0]$ of $[\sigma]$. This forces $d([\tau]) < d([\sigma])$ since the arcs of $[\sigma]$ are bad, so, disregarding $F(\beta)$, removing vertices can only eliminate pure pieces since they can only be merged with mixed pieces. Even if no pure pieces are eliminated, the non-emptiness of β ensures that at least one vacant space for an arc in the pure pieces is taken.

Conversely, assume that β is *empty*. Then ν is a simplex in $\partial\mu$, so $[\tau] \subset [\sigma]$. Thus we either have $d([\tau]) < d([\sigma])$ or $d([\tau]) = d([\sigma])$, but in the latter case we use the fact that $\dim\nu < \dim\mu$ since $\nu \subset \partial\mu$.

After a finite number of iterations of this procedure it will no longer be possible because there is a lower limit to complexity. At that point there are no more bad simplices, in particular no more bad vertices. This means that the image of the final map that we get contains no bad simplices, i.e. it is contained in $A(S, P, Q)$.

It remains to show that J is $(k-l)$-connected. The dimension of $[\sigma]$ is at most l, so since f is simplicial, $q-1 \le l$, where q is the number of arcs in σ. Thus $k-(q-1) \ge k-l$, so that it suffices to show that J is $k-(q-1)$-connected. Since $k \le 2e+t-2$, we can even get away with showing that the connectivity of J is at least $2e+t-2-(q-1) = 2e+t-q-1$. If there is a pure piece S_i which is not a disk, then $B(S_i, P_i \cup Q_i)$ is contractible by Lemma 3.19, so J is contractible. Thus we can assume that all pure pieces are disks, so that each $B(S_i, P_i \cup Q_i)$ is $(u_i - 5)$-connected by Lemma 3.19, where u_i is the number of (pure) arcs in the pure piece. By the induction hypothesis we may assume that each $A(S_i, P_i, Q_i)$ for mixed pieces is $(2e_i + t_i + u_i - 2)$-connected. By Lemma A.2, the connectivity of J is greater than or equal to

$$\sum_{\text{mixed pieces}} (2e_i + t_i + u_i) + \sum_{\text{pure pieces}} (u_i - 3) - 2. \tag{3.2}$$

Now let p denote the number of pure pieces of $S \setminus \sigma$. Then, since pure pieces have no mixed arcs, we can rewrite (3.2) as

$$2 \sum_{\text{mixed pieces}} e_i + \sum_i t_i + \sum_i u_i - 3p - 2. \tag{3.3}$$

Moreover, since the pure pieces are all disks, they have Euler characteristic 1. Thus, since cutting along an arc increases Euler characteristic by 1, we have

$$-\chi(S) = e = (\sum_i e_i) + q = \sum_{\text{mixed pieces}} e_i - p + q.$$

Cutting S along a bad arc only introduces pure arcs, not mixed ones, so $\sum t_i = t$. In fact it introduces two pure arcs, so $\sum_i u_i = 2q$ since we assumed $u = 0$. Therefore we can rewrite (3.3) as

$$2(e + p - q) + t + 2q - 3p - 2 = 2e + t - p - 2.$$

By the badness assumption, no arc of σ touches more than one pure piece, and each pure piece is a disk. Moreover each of these disks must have at least 3 arcs of σ in their boundary since S has no pure arcs by assumption, so that a pure piece, being a disk, must be constructed by joining 3 or more pieces of either P or Q with arcs of σ. Thus $3p \leq q$, so $q > p$ if $p > 0$. If $p = 0$ it is the case since $q \geq 1$ by assumption. In conclusion, the connectivity is greater than $2e + t - q - 1$ since the fact that $q > p$ implies that $-q - 1 \leq -p - 2$, and this was what we wanted. The connectivity $2e + t - q - 1$ is the one we wanted. □

Let $A^0(S, P, Q)$ denote the subcomplex of $A(S, P, Q)$ consisting of coconnected arc systems. The next result will be needed in the next section.

Proposition 3.21 ([HV15b, Proposition 5.7]). *If the arc components of P and Q alternate in ∂S, i.e. $u = 0$, then $A^0(S, P, Q)$ is $(2g + s' - 3)$-connected, where s' is the number of components of ∂S intersecting P or Q.*

Proof. Consider the surface S' obtained by attaching a disk to each boundary component of S that is disjoint from $P \cup Q$. Let $e = -\chi(S')$. Then $e - 1 = 2g + s' - 3$. Since $0 \leq 2g + s + t - 3$ in all cases except when $g = 0$ and $s = t = 1$, we have

$$2g + s' - 3 \leq 4g + s + s' + t - 6,$$

in those cases. In case $s = t = 1$, we have $s' = 1$ as well, so the connectivity number is $0 + 1 - 3 = -2$, i.e. the statement is vacuous. We will proceed by a bad simplex argument, so we need to define a set of bad simplices of $A(S, P, Q)$ such that the good simplices are contained in $A^0(S, P, Q)$. We say that a simplex in $A(S, P, Q)$ is bad if its dual graph contains no edges that are loops, i.e. all arcs separate the complement of the other arcs. Let $[\sigma]$ be a bad simplex in $A(S, P, Q)$. Then σ separates S into a collection of surfaces $S_1, \ldots, S_j$, where $j \geq 2$ since $[\sigma]$ is bad. We will proceed by induction on the lexicographically ordered triple (g, s, t) to show that S is $(e-1)$-connected. For the induction start we consider the cases where $g = 0$ and $s \leq 2$. If $s' = 1$, the statement is vacuous by the same argument as above. When $s' = 2$, we must at least have one arc between two boundary components. This arc will then will be coconnected as an arc system. Thus $A^0(S, P, Q)$ will be non-empty, which is sufficient since the connectivity number is $0 + 2 - 3 = -1$.

We want to use Corollary 1.10 to show the connectivity, so consider the complex $G_{[\sigma]}$ of simplices in link$[\sigma]$ that are good for $[\sigma]$, i.e. the simplices $[\tau]$ in link$[\sigma]$ such that any bad face of $[\tau] * [\sigma]$ is a face of $[\sigma]$. We want to show that

$$G_{[\sigma]} = \mathop{*}_i A^0(S_i, P_i, Q_i) =: A',$$

i.e. that $G_{[\sigma]}$ is the join of all the coconnected arc complexes of the surface components obtained from S by cutting along σ. These complexes $A^0(S_i, P_i, Q_i)$ can be seen as subcomplexes of $A(S, P, Q)$. Let us first show that $A' \subseteq G_{[\sigma]}$. It is obvious that $A' \subseteq$ link$[\sigma]$. Moreover, if $[\tau]$ is a simplex in A', then a bad face of $[\tau] * [\sigma]$ cannot contain a vertex of $[\tau]$ as such a vertex would correspond to a loop in the dual graph. This shows that $[\tau]$ is good for $[\sigma]$, i.e. $[\tau] \subseteq G_{[\sigma]}$.

Now let us show that $G_{[\sigma]} \subseteq A'$. We will show this by contraposition, so suppose that $[\tau]$ is not in A'. If even $[\tau]$ is not in link$[\sigma]$, then certainly $[\tau]$ is not in $G_{[\sigma]}$. Thus suppose that $[\tau]$ is in link$[\sigma]$, but not in A'. Then remove all vertices of $[\tau] * [\sigma]$ corresponding to loops in the dual graph of $[\tau] * [\sigma]$. Since we remove non-separating arcs, the constellation of complementary components does not change, so the resulting simplex is bad. But since $[\sigma]$ is already bad, we have only removed vertices of $[\tau]$. Thus we get a bad face of $[\tau] * [\sigma]$ of the form $[\tau_0] * [\sigma]$ for some face $[\tau_0]$ of $[\tau]$. But since $[\tau]$ is not in A', $[\tau_0]$ is non-empty, showing that $[\tau]$ is not good for $[\sigma]$.

This shows that $G_{[\sigma]} = *_i A^0(S_i, P_i, Q_i)$. For each i, $(g_i, s_i, t_i) < (g, s, t)$, and the induction hypothesis is that each $A^0(S_i, P_i, Q_i)$ is $(e_i - 1)$-connected, where $e_i = -\chi(S_i')$, and S_i' is the surface obtained from S_i by attaching a disk to each boundary component of S_i that is disjoint from $P_i \cup Q_i$. Thus by induction and Lemma A.2, the connectivity of $G_{[\sigma]}$ is at least

$$\sum_i (e_i - 1 + 2) - 2 = \sum_i e_i + j - 2,$$

but since cutting along an arc increases Euler characteristic by one, and the number e_i does not depend on the boundary components not intersecting P or Q, we get $e = \sum_i e_i + k + 1$ since $\dim[\sigma] = k$. Thus since equivalently $\sum_i e_i = e - k - 1$, the connectivity of $G_{[\sigma]}$ is at least

$$e - k - 1 + j - 2 = e - k + j - 3,$$

which is greater than or equal to $e - k - 1$ since $j \geq 2$. Since

$$(2g - s' - 3) - k = e - 1 - k = e - k - 1,$$

the hypothesis of Corollary 1.10 is satisfied, and we get the result. □

3.3 Chains and tethered chains

A **chain** in S is an ordered pair $c = (a, b)$ of simple closed curves embedded in S which intersect each other transversely in exactly one point. The transverse intersection condition means that the direction vectors of the two curves in the intersection point span a plane and not just a line. This means that the curves pass through each other instead of just touching. Let $Ch(S)$ denote the simplicial complex with one k-simplex for each isotopy class of systems of $k+1$ disjoint and pairwise non-isotopic chains. Moreover, if $s \geq 1$ we fix a point $p \in \partial S$. A **tethered chain** is then an ordered pair (c, t) of a chain $c = (a, b)$ and an arc t embedded in S that joins a point in $b \setminus a$ to p. Now let $TCh(S)$ denote the simplicial complex with one k-simplex for each isotopy class of systems of $k + 1$ tethered chains that are pairwise non-isotopic and disjoint except at p.

3.3.1 Chains and multichains

To show that $TCh(S)$ is highly connected we will first show that $Ch(S)$ is highly connected. To show that, we embed $Ch(S)$ into a larger complex $MCh(S)$ of **multichains**. Similarly to the complex of multi-rooted curves, it has the same vertices as $Ch(S)$ but many more higher dimensional simplices. Namely, a set of chains $c_{ij} = (a_i, b_{ij})$ forms a simplex in $MCh(S)$ if the curves a_i form a coconnected curve system in S and have disjoint annular neighborhoods $N(a_i)$ such that

(1) Each b_{ij} is disjoint from $N(a_k)$ if $i \neq k$, and each b_{ij} intersects $N(a_i)$ in an arc from one the boundary circles of $N(a_i)$ to the other. The curves b_{ij} can intersect as much as needed, as long as they only do so inside of $N(a_i)$.

(2) Outside of $\cup_i N(a_i)$, two different b_{ij}'s must be either disjoint or coincide completely, and their union must be a coconnected arc system in $S \setminus \cup_i N(a_i)$.

The complex $Ch(S)$ of chains is contained in $MCh(S)$ as the collections of chains (a_i, b_{ij}) such that there is only one b_{ij} for each a_i. Note that while $Ch(S)$ is finite-dimensional, $MCh(S)$ is not, since there is no bound on the number of b_{ij}'s that can be assigned to each a_i. For example, some b_{ij} can be distorted by multiple Dehn twists arbitrarily many times. Note that the isotopy classes of chains are inherited from $Ch(S)$, so we do not require the isotopies to be fixed on the boundaries $\partial N(a_i)$. This means that if some b_{ij} travels around a Dehn twist inside of $N(a_i)$, that Dehn twist can be pushed outside of $N(a_i)$ as long as the curves do not intersect outside of $N(a_i)$ and the other $N(a_k)$'s.

Proposition 3.22 ([HV15b, Proposition 6.1])**.** *The complex $MCh(S)$ of multichains is $(g-2)$-connected.*

Proof. Consider the map

$$f\colon MCh(S) \to C^0(S)$$

which forgets the $[b_{ij}]$'s and sends $[(a_i, b_{ij})]$ to $[a_i]$ for each i, j, and consider the map $\hat{f}$ on the poset complexes. Let $[\alpha] = \{[a_0], \ldots, [a_k]\}$ be a simplex in $C^0(S)$, and consider the fiber $F_\alpha = \hat{f}^{-1}([\alpha])$. By Lemma 1.19 it will suffice to show that F_α has connectivity $g - k - 2$.

To show this we first consider the subsurface S_α of S given by removing the neighborhoods $N(a_i)$, namely

$$S_\alpha = S \setminus (\cup_i \operatorname{int} N(a_i)) .$$

Then each $N(a_i)$ contributes with two boundary components of S_α which we will call a'_i and a''_i. Let now $[\gamma]$ be a simplex of $MCh(S)$, and let $\gamma_\alpha = \cup_{i,j} b_{ij} \cap S_\alpha$. Then γ_α is a coconnected arc system in S_α whose arcs each join some a'_i with the corresponding a''_i. This is due to the construction of $MCh(S)$.

Let now A_α denote the full poset of isotopy classes of coconnected arc systems in S_α, where each arc joins some a'_i to the corresponding a''_i. Because of the construction of $MCh(S)$ we have to allow isotopies that move the endpoints of the arcs as long as they stay inside ∂S_α. This means that an arc from a'_i to a''_i can be distorted by a Dehn twist around a curve parallel to a'_i or a''_i without changing its isotopy class. The isotopy classes under this definition of isotopy are exactly the isotopy classes inherited from $MCh(S)$.

We can now form the poset map $g\colon F_\alpha \to A_\alpha$ sending a simplex $[\gamma]$ to $[\gamma_\alpha]$. For some γ' to map to some σ, γ' must coincide with σ outside of $\cup_i N(a_i)$, but there is no restriction on γ' inside of $\cup_i N(a_i)$. This also means that any γ_0 coinciding with σ can be added to anything in $g^{-1}([\sigma])$. Thus any $[\gamma']$ in $g^{-1}([\sigma])$ lies inside a simplex in $g^{-1}([\sigma])$ containing $[\gamma_0]$, namely $[\gamma] \cup [\gamma_0]$. Thus all of $g^{-1}([\sigma])$ is contained in star$([\gamma_0])$, where $[\gamma_0]$ is seen as a simplex inside of $g^{-1}([\sigma])$, and we can contract all of $g^{-1}([\sigma])$ to $[\gamma_0]$, showing that g is a homotopy equivalence by Corollary 1.16, so it suffices to show that A_α is $(g-k-2)$-connected.

Now we introduce the general notation A_k for A_α since there are $k+1$ boundary pairs being joined by arcs, and since up to isomorphism $A_k = A_\alpha$ only depends on the dimension of $[\alpha]$. Removing the annular neighborhoods $N(a_i)$ from S is equivalent to cutting along the curves a_i. This reduces the genus by one for each a_i since the a_i's form a coconnected curve system. Thus S_α has genus $g_\alpha = g-k-1$, and we want to show that A_k is $(g_\alpha - 1)$-connected. We will proceed by induction on k:

For the induction start $k = 0$, note that A_0 is the poset of isotopy classes of coconnected arc systems joining one boundary component to another, where the isotopies are allowed to move the endpoints along the boundary. Since A_0 consists of arcs joining one component of ∂, which we will call P, to another, which we call Q, A_0 is isomorphic to $A^0(P,Q)$, so it has connectivity at least $(2g_\alpha + 2 - 3) = (2g_\alpha - 1)$, and $2g - 1 \geq g - 1$, which is what we wanted.

For the induction step, consider the poset map $f_0\colon A_k \to A_0$ that only remembers arcs going from a'_0 to a''_0. For each simplex $[\sigma]$ in A_0, the fiber $f_0^{-1}([\sigma])$ must contain all the arc classes of $[\sigma]$, but it cannot contain any other classes of arcs from a'_0 to a''_0. However for the other a_i's the arcs can be whatever they want, as long as they do not touch $N(a_0)$. Thus $f_0^{-1}([\sigma])$ can be seen as the complex A_{k-1} for the surface obtained from S_α by cutting along σ. Say that $[\sigma]$ has dimension m. Cutting along the $m+1$ arcs of σ can decrease genus by at most m since the first cut just joins two boundary components and thus cannot decrease genus as it increases Euler characteristic by 1.

Therefore we can use the induction hypothesis which shows that $f^{-1}([\sigma])$ is at least $(g_\alpha - m - 1)$-connected. Thus by Lemma 1.19 A_k is $(g_\alpha - 1)$-connected. □

Lemma 3.23. *A system of chains or a system of tethered chains is always non-separating, i.e. the surface obtained by cutting S along the system is connected.*

Proof. First consider a chain $c = (a, b)$ without a tether. Any one of the two curves, say a, must be non-separating since the other curve b intersects it transversally only once, so if a were separating, b would have to intersect it twice to get back to the same component of $S \setminus a$. Thus if we cut S along a, we get a connected surface S_a, and b becomes an arc between two different components of ∂S_a and therefore cannot be separating. This means that cutting S along the chain c gives a connected surface S_c, and a tether t for c then becomes an arc between two different components of S_c, so it cannot be separating.

Now we know that a system consisting of a single tethered chain is coconnected, so we can proceed by induction on the dimension (number of tethered chains). Assume that $[\sigma]$ is a system of tethered chains that is non-separating. Then for any vertex $[v]$ in $\text{link}[\sigma]$, v can be seen as a tethered chain in the surface S_σ obtained by cutting along σ, and so it must be non-separating by the above argument. This is what we wanted. □

Theorem 3.24 ([HV15b, Theorem 6.2])**.** *The complex $Ch(S)$ of chains in S is $\phi(g) = (g-3)/2$-connected, where we round down if the number is not an integer.*

Proof. It follows from Lemma 3.23 that a system of chains is always coconnected. Thus $Ch(S)$ embeds into $MCh(S)$ as the systems of multichains with only one b_{ij} for each a_i. We will prove the theorem by a bad simplex argument on the inclusion $Ch(S) \hookrightarrow MCh(S)$. We define the bad simplices to be the systems of with at least two b_{ij}'s for each a_i. Note that in particular a bad simplex must have dimension at least 1. For a bad simplex $[\sigma]$, $G_{[\sigma]}$ consists of simplices $[\tau]$ in $\text{link}[\sigma]$ such that any bad face of $[\tau] * [\sigma]$ is contained in $[\sigma]$. We can pick the representatives τ and σ such that the b_{ij}'s of τ do not coincide with or intersect any of the b_{ij}'s of σ. Thus τ and σ cannot share any a_i's since then there would be a bad simplex $[\tau_0] * [\sigma_0]$ with $[\tau_0]$ non-empty. Moreover, τ has to lie outside of all the $N(a_i)$'s obrained from σ, and no a_i of τ can have two associated b_{ij}'s. By definition any system of multichains is coconnected, so the surface S_σ obtained by removing the neighborhoods $N(a_i)$ from S and cutting along the b_{ij}'s is connected. In conclusion, $G_{[\sigma]} = Ch(S_\sigma)$.

If $[\sigma]$ has dimension 1, then the genus g_σ of S_σ cannot be less than $g - 2$. This is because cutting along σ adds 6 vertices and 4 edges, so Euler characteristic is increased by 2 while the number of boundary components is increased by 2. This leaves room for reducing genus by at most 2, and this is only if σ is non-separating. This number $g - 2$ cannot be improved to $g - 1$ since it might actually happen that the genus is reduced by two. For example the double chain illustrated in Figure 3.11 is non-separating, so it reduces genus by 2. In general if $\dim[\sigma] = k$, the genus can be decreased by at most $k + 1$, i.e. $g_\sigma \geq g - k - 1$.

We proceed by induction on g. For $g = 0$, $\frac{g-3}{2} = \frac{-3}{2}$ which rounds down to -2, so the statement is vacuous. For $g = 1$, $\frac{g-3}{2} = -1$, and $Ch(S)$ is indeed non-empty when

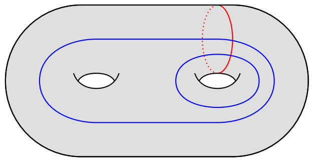


Figure 3.11: A double chain that reduces genus by 2

$g = 1$, so we can continue to the induction step. Let's analytically find out how we should define ϕ. If we manage to successfully use Corollary 1.10, we will get that $Ch(S)$ has the connectivity of $MCh(S)$, i.e. it is $(g-2)$-connected, so the function ϕ from the statement of the theorem must satisfy $\phi(g) \leq g - 2$.

The condition for using the corollary is that $Ch(S_\sigma)$ is $(\phi(g)-k)$-connected, so since $g_\sigma \geq g - k - 1$, it suffices to have $\phi(g-k-1) \geq \phi(g) - k$. For $k = 1$ this inequality is $\phi(g) - 1 \leq \phi(g-2)$ or equivalently $\phi(g) \leq \phi(g-2) + 1$. This means that ϕ needs to have slope $\leq \frac{1}{2}$.

Write ϕ as $\phi(g) = \frac{1}{2}g + B$ since we want ϕ to be affine. Then the condition $\phi(g) - k \leq \phi(g-k-1)$ becomes

$$\frac{1}{2}g + B - k \leq \frac{1}{2}(g-k-1) + B.$$

This is equivalent to $k \geq 1$ which is true since $[\sigma]$ is bad. If we set $B = \frac{-3}{2}$, we get the number from the theorem statement, so this works for $g = 0$ and $g = 1$ by the above. Also, the statement $\phi(g) \leq g - 2$ becomes

$$\frac{1}{2}g - \frac{3}{2} \leq g - 2 \quad \Leftrightarrow \quad \frac{-1}{2}g - \frac{3}{2} \leq -2 \quad \Leftrightarrow \quad -g - 3 \leq -4 \quad \Leftrightarrow \quad g \geq 1,$$

but we can just assume this since the statement is vacuous when $g = 0$. The number $B = \frac{-3}{2}$ is the best possible since setting $B = -1$ would mean that $Ch(S)$ would have to be non-empty for $g = 0$ which is not true. □

3.3.2 Tethered chains and multi-tethered chains

We will now define a new complex $MTCh(S)$ of what we call systems of multi-tethered chains. Choose points $p_1, \ldots, p_m \in \partial S$. The vertices of $MTCh(S)$ are then isotopy classes of tethered chains (c, t) on S where t attaches c to some p_k. An isotopy class of systems of tethered chains (c_i, t_i) then spans a simplex if the c_i's can be chosen to be either pairwise disjoint or equal, and the t_i's can be chosen to be disjoint from each other as well as the c_i's except for their endpoints. Note that there is no condition on coconnectedness, so although a system of tethered chains is coconnected by Lemma 3.23, a system of multi-tethered chains might not be.

Proposition 3.25 ([HV15b, Proposition 6.5]). *The complex $MTCh(S)$ is $\frac{g-3}{2}$-connected.*

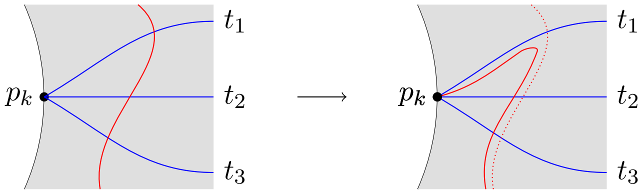


Figure 3.12: Surgery in Proposition 3.25

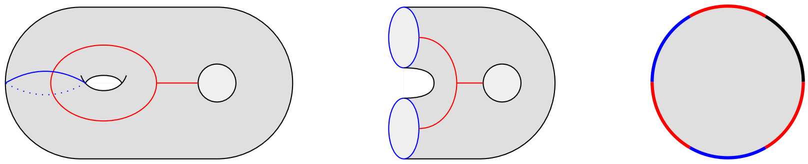


Figure 3.13: Cutting along a tethered chain

Proof. There is a forgetful map $MTCh(S) \twoheadrightarrow Ch(S)$ that forgets all the tethers since inside $MTCh(S)$ the c_i's are mutually disjoint or equal. Let $[\sigma]$ be a simplex in $Ch(S)$. The preimage over $[\sigma]$ by the mentioned forgetful map consists of all classes of systems of multi-tethers for σ. Choose such a system τ with exactly one tether t_i for each chain c_i in σ. We will do a surgery flow into the star of $[\tau]$, defining complexity as the number of intersection points with τ, assuming that σ is in normal form with respect to τ. To do so, let a system consisting of (c_i', t_i')'s be given such that for each i, $c_i' = c_j$ for some j. We begin at t_1 and choose the point of $(\cup t_i') \cap t_1$ that is closest to the point $p_k \in \partial S$ that t_1 lands in. Cut the intersecting tether t_i' at the intersection point, and redirect both pieces closely along t_1 to p_k. One of the resulting arcs is a tether attached to c_i'. This is illustrated in Figure 3.12. The other resulting arc is discarded. Continue the process until no tethers intersect t_1. Then continue with t_2, and so on. Then by Lemma 1.20, the star of the chosen system is a deformation retract of the preimage over σ, so the preimage is contractible. Thus $MTCh(S) \twoheadrightarrow Ch(S)$ is a homotopy equivalence by Corollary 1.16, so the connectivity follows from Theorem 3.24. $\square$

We now consider the subcomplex $TCh(S)$ of $MTCh(S)$ consisting of systems with only one tether to each chain, but where the tethers are allowed to connect to any p_i, though we will only use the special case with one $p_i = p_1$ for the spectral sequence argument. We first need the following lemma.

Lemma 3.26. *Cutting along a system of k tethered chains reduces genus by k and preserves the number of boundary components.*

Proof. We can consider any tethered chain together with the boundary component it is tethered to as consisting of 3 vertices and 4 edges, and then consider the rest of the surface (except for the other boundary components) as one face, as indicated by the first drawing in Figure 3.13.

Cutting along the blue curve increases this to 4 vertices, 5 edges and one face, and cutting along the rest of tethered chain (indicated in the second drawing with the result in the third drawing) increases this to 9 vertices, 8 edges and 1 face. This means that the Euler characteristic rises by 2. But the sides of the whole tethered chain together with the boundary component it is tethered to becomes just one boundary component in the new cut up surface, so the amount of boundary components is preserved. Thus the genus must decrease by one.

Any of the tethered chains in a system of tethered chains can be seen as a tethered chain in the (connected) surface obtained by cutting up S along all the other tethered chains, so the result follows by induction on k. □

This makes us able to establish the connectivity of $TCh(S)$.

Theorem 3.27 ([HV15b, Theorem 6.6]). *The complex $TCh(S)$ is $\frac{g-3}{2}$-connected.*

Proof. We will do a bad simplex argument on the inclusion $TCh(S) \subset MTCh(S)$. We define bad simplices to be those with at least two tethers for each chain. Let $[\sigma]$ be a bad simplex. The simplices $[\tau]$ in $G_{[\sigma]}$ good for $[\sigma]$ are usual tethered systems since if some chain of τ has more than one tether, then there is a bad face $[\tau_0] * [\sigma]$ of $[\tau] * [\sigma]$, where $[\tau_0]$ consists of the chosen chain and all of its (> 1) tethers. Moreover, for any $[\tau]$ in $G_{[\sigma]}$, τ is not allowed to intersect or coincide with any tethered chains of σ, so $G_{[\sigma]}$ is the join of the complexes $TCh(S_i)$ for each component S_i of the surface S_σ obtained by cutting up S along σ. We need to check that $G_{[\sigma]}$ has connectivity $\frac{g-3}{2}$ in order to use Corollary 1.10.

Cutting along a tethered chain reduces genus by one by Lemma 3.26, and then each additional tether can reduce genus by at most one since each cut increases Euler characteristic by 1 and either increases or decreases the number of boundary components by one. In case the number of boundary components is increased, cutting along the tether will either separate the surface or reduce its genus by 1, and if the number is decreased, neither of these happens. Thus cutting along a multi-tethered chain system can decrease genus by at most the number of tethers minus the number of separations. Let g_σ denote the genus of S_σ, understood as the sum of the genera of the components of S_σ. Then we have $g_\sigma \geq g - k - 1 + c - 1 = g - k + c - 2$, where k is the dimension of σ and c is the number of components of S_σ. Thus we can use the argument from the proof of Theorem 3.10 to conclude that G_σ is $(\frac{g-3}{2} - k)$-connected since $k \geq 1$ as σ is bad. □

3.4 Stability theorems

Let $S = S_{g,s}$ with $s \geq 1$. We will deal with the case $s = 0$ later. Let $\mathcal{M}_{g,s}$ be the boundary fixing mapping class group of $S_{g,s}$.

3.4.1 Genus stabilization

Theorem 3.28 ([HV15b, Theorem 7.1])**.** *For any $s \geq 1$ the homomorphism*

$$\alpha_* \colon H_i(\mathcal{M}_{g-1,s}) \to H_i(\mathcal{M}_{g,s})$$

induced by the map $\alpha \colon S_{g-1,s} \hookrightarrow S_{g,s}$ formed by gluing a copy of $S_{1,2}$ onto $S_{g,s}$ along a boundary circle is an isomorphism for $g > 2i+2$ and a surjection for $g = 2i+2$.

Proof. We consider the action of $\mathcal{M}_{g,s}$ on the complex $TCh(S)$ of tethered chains, and we want to show that it satisfies the conditions on page 9 for the spectral sequence argument, and then use Theorem 1.5.

Condition 1: Let σ and σ' be any two systems of k tethered chains on S. Then the surfaces S_σ and $S_{\sigma'}$ obtained by cutting S along σ and σ' respectively are both diffeomorphic to $S_{g-k,s}$. Thus they are homeomorphic to each other by some diffeomorphism $f \colon S_\sigma \to S_{\sigma'}$. We can assume right away that f fixes the boundary components not containing any instances of the tethering point p (cutting along a tether splits the tethering point into two points).

Now consider an oriented half-circle in S around p that is small enough such that each of the tethers of σ and σ' intersect it only once. These intersection points determine an ordering of the tethers of σ, respectively σ'. As long as we preserve this ordering, we can choose f such that the parts of the boundary of S_σ corresponding to the tethered chains of σ each map to corresponding parts of the boundary of $S_{\sigma'}$. This means that the homeomorphism f can be glued together to a homeomorphism $S \to S$ that takes σ to σ'.

Condition 2: Cutting along a tethered chain system σ of k tethered chains reduces genus by k and preserves s by Lemma 3.26. The stabilizer of $[\sigma]$ cannot non-trivially permute the vertices of $[\sigma]$ since it must preserve the ordering of the tethers. Also, while a diffeomorphism might 'distort' the tethered chains, this does not matter since we are working with isotopy classes. Thus we can view $\operatorname{stab}[\sigma]$ as the subgroup of the mapping class group of S of isotopy classes of homeomorphisms that fix ∂S as well as σ. Thus we can view $\operatorname{stab}[\sigma]$ as the boundary fixing mapping class group of the surface S_σ obtained by cutting S along σ. This means that $\operatorname{stab}[\sigma]$ is isomorphic to $\mathcal{M}_{g-k,s}$.

To see that $\operatorname{stab}[\sigma]$ is actually conjugate to $\mathcal{M}_{g-k,s} \subset \mathcal{M}_{g,s}$ under the inclusion that we use for the stabilization, note that there is some system of tethers σ' which 'undoes' α^k so that $\operatorname{stab}[\sigma'] = \mathcal{M}_{g-k,s}$. Since we already know that the action is transitive on simplices of each dimension, we can simply choose $[\phi] \in \mathcal{M}_{g,s}$ such that $[\phi][\sigma'] = [\sigma]$. Then Lemma A.14 tells us that

$$\operatorname{stab}[\sigma] = [\phi] \operatorname{stab}[\sigma'][\phi^{-1}] = [\phi]\mathcal{M}_{g-k,s}[\phi^{-1}].$$

Condition 3: Let $[e]$ be an edge in $TCh(S)$. Then e is a system of two tethered chains. Let $N(e)$ be the closure of a small neighborhood of e and the boundary component $\partial_0 S$ containing p. Then $N(e)$ is a surface of genus 2, and every tethered chain representative in $\operatorname{link}[e]$ can be chosen to be disjoint from $N(e)$. By the same transitivity argument as for Condition 2, there is a diffeomorphism ϕ that is supported in $N(e)$ and which fixes the boundary of $N(e)$ and maps one of the tethered chains to the other. We cannot interchange the two tethers, but this is not needed anyway. Let $[\psi] \in \mathrm{stab}[e]$ be given. Then ψ can be chosen such that it is supported in $S \setminus N(e)$. Thus, since ϕ is supported in $N(e)$, $\phi\psi = \psi\phi$. We conclude that $[\phi]$ commutes with $\mathrm{stab}[e]$.

Condition 4: Since the action of $\mathcal{M}_{g,s}$ on S is transitive in each dimension, the quotient space $TCh(S)/\mathcal{M}_{g,s}$ is the Δ-complex with precisely one cell in each dimension $\leq g-1$ since the maximal number of tethered chains is g. Thus it is $(g-2)$-connected as noted in the proof of Theorem 2.4. Moreover $TCh(S)$ is $(g-3)/2$-connected by Theorem 3.27, so the conditions of Theorem 1.5 are satisfied, and we get the result. □

3.4.2 Stabilization by boundary components

Consider the map $\alpha\colon S_{g,s} \hookrightarrow S_{g+1,s}$ of Theorem 3.28. This map can be seen as the composition $\eta\mu$ of two maps $\mu\colon S_{g,s} \to S_{g,s+1}$ and $\eta\colon S_{g,s+1} \to S_{g+1,s}$, where μ is formed by gluing a pair of pants $S_{0,3}$ onto $S_{g,s}$ along a boundary component, and η is formed by gluing a pair of pants onto $S_{g+1,s}$ along two boundary components. The goal of this section is to show that the sequence of homomorphisms on mapping class groups induced by the maps μ satisfies homological stability. The proof of this is inspired by the proof of [HW07, Theorem 1.8] as well as the method presented in [HVW06].

The compositions $\alpha\mu = \eta\mu^2$ and $\mu\alpha = \mu\eta\nu$ are then both obtained by gluing a copy of $S_{1,3}$ along one boundary component, so up to isotopy they are the same map. This means that we have a commutative diagram

$$\begin{tikzcd}
H_i(\mathcal{M}_{g,s}) \arrow[r, "\alpha_*"] \arrow[d, "\mu_*"] & H_i(\mathcal{M}_{g+1,s}) \arrow[r, "\alpha_*"] \arrow[d, "\mu_*"] & \cdots \arrow[r, "\alpha_*"] & H_i(\mathcal{M}_{g+k,s}) \arrow[r, "\alpha_*"] \arrow[d, "\mu_*"] & \cdots \\
H_i(\mathcal{M}_{g,s+1}) \arrow[r, "\alpha_*"] & H_i(\mathcal{M}_{g+1,s+1}) \arrow[r, "\alpha_*"] & \cdots \arrow[r, "\alpha_*"] & H_i(\mathcal{M}_{g+k,s+1}) \arrow[r, "\alpha_*"] & \cdots
\end{tikzcd}$$

This means that if μ_* is an isomorphism for some very large k, it must also be an isomorphism in the same range as α_*. In fact, it suffices to show that μ becomes an isomorphism after passing to the direct limit with respect to α_*, which is what we will do here. Let $\bar{\mu}_*\colon H_i(\mathcal{M}_{\infty,s}) \to H_i(\mathcal{M}_{\infty,s+1})$ denote the homomorphism of the direct limit induced by the homomorphisms μ_*. Note that this expression of the direct limit comes from the fact that homology commutes with direct limits.

First note that $\bar{\mu}_*$ is injective since μ always has a left inverse obtained by gluing a disk onto one of the boundary components of the copy of $S_{0,3}$ that has been glued on by μ. It remains to show that $\bar{\mu}_*$ is surjective. This follows if we can show that the direct limit homomorphism $\overline{\beta}_* = \overline{(\mu\eta)}_*$ obtained from the maps $\beta = \mu\eta$ is an isomorphism. Although β also increases genus by one, it is not isotopic to the map α since $\mu\eta$ glues on $S_{1,2}$ along two boundary components instead of just one. This also means that we cannot use the same simplicial complex as we did for α. The complex we will use in this case is the complex of rooted curves $C^0_R(S_{\infty,s})$ which is contractible by Theorem 3.13.

We will look at the maps α and β in a slightly different way. Namely, we consider $\alpha\colon S_{g,s} \to S_{g+1,s}$ as the map obtained by identifying a half-circle in the boundary of $S_{1,1}$ with a half-circle in the boundary of $S_{g,s}$, and we consider β as the map obtained by identifying a half-circle in each of the two boundary components of the cylinder $S_{0,2}$ with half-cirlces in two boundary components of $S_{g,s}$. Although these maps are not exactly the same as the original maps α and β, they induce the same homomorphisms on mapping class groups, which is why we let them keep their names. Let ϕ be a diffeomorphism $\beta(S_{g,s}) \to \alpha(S_{g,s})$, and let $\lambda = \phi \circ \beta$.

Theorem 3.29. *Let $\mathcal{M}_{\infty,s}$ be the direct limit of the sequence $\mathcal{M}_{g,s} \to \mathcal{M}_(g+1,s)$ induced by the maps α. For any $s \geq 2$ the homomorphism : $\bar{\lambda}_*\colon H_i(\mathcal{M}_{\infty,s}) \to H_i(\mathcal{M}_{\infty,s})$ induced by the maps $\lambda\colon S_{g,s} \hookrightarrow S_{g+1,s}$ is an isomorphism.*

Proof. We consider the action of $M_{\infty,s}$ on the complex $C^0_R(S_{\infty,s})$ of rooted curves, and we want to show that it satisfies the conditions necessary for Theorem 1.7.

Condition 1: We will show that the action is transitive on vertices. Let (c, r) and (c', r') be any two rooted curves on $S_{\infty,s}$. Since (c, r) and (c', r) are both compact as subspaces of $S_{\infty,s}$, they both lie inside some common $S_{g,s}$, so it suffices to show that the action of $\mathcal{M}_{g,s}$ on $C^0_R(S_{g,s})$ is transitive on vertices. The surfaces $S_{(c,r)}$ and $S_{(c',r')}$ obtained by cutting $S_{g,s}$ along (c, r) and (c', r') respectively are both diffeomorphic to $S_{g-k,s}$ by some diffeomorphism $f\colon S_{(c,r)} \to S_{(c',r')}$. As in the proof of Theorem 3.28, we can assume that f takes the parts of $\partial S_{(c,r)}$ corresponding to (c, r) to the corresponding parts in $S_{(c',r')}$, so that f can be glued together to a diffeomorphism of $S_{g,s}$ that takes (c, r) to (c', r').

Condition 2: The stabilizer of any simplex fixes the simplex pointwise since the orientations of the roots near x_1 and x_2 must be preserved. We need to show that the inclusion $\mathrm{stab}(\sigma_p) \hookrightarrow \mathcal{M}_{\infty,s}$ of the stabilizer of a p-simplex σ_p is conjugate to $\overline{\lambda}^{p+1}_*$. The stabilizer $\mathrm{stab}(\sigma_p)$ can be seen as the subgroup of $\mathcal{M}_{\infty,s}$ fixing the closure of a small regular neighborhood N of the union of σ_p and the two boundary components $\partial_1 S_\infty$ and $\partial_2 S_\infty$ it touches. The neighborhood N is a copy of $S_{p+1,4}$, two of whose boundary components lie in $S_{\infty,s}$.

The image of the map $\overline{\lambda}^{p+1}$ is the subgroup of $\mathcal{M}_{\infty,s}$ fixing the surfaces that have been glued on by $\overline{\lambda}^{p+1}$. For a small neighborhood M of these glued on surfaces and the two boundary components it touches, the closure $\overline{M}$ is also a copy of $S_{p+1,4}$, two of whose boundary components lie in $S_{\infty,s}$. Denote these boundary components by A and B.

Let now $\phi\colon S_\infty \to S_\infty$ be a diffeomorphism that maps $\overline{N}$ to $\overline{M}$ such that $\partial_0 S$ maps to A and $\partial_1 S$ maps to B. Then

$$\mathrm{im}(\mathrm{stab}[\sigma_p]) = \mathrm{im}([\phi^{-1}]\bar{\lambda}^{p+1}[\phi]).$$

Condition 3: Once again, it suffices by compactness to consider the case of finite genus g. Let e be a system of two rooted curves v and w. The closure of a small neighborhood of e and the two boundary components containing x_0 and x_1 is a copy of $S_{2,4}$, and the stabilizer of $[e]$ can be seen as the subgroup of $\mathcal{M}_{g,s}$ of classes of diffeomorphisms that fix this copy of $S_{2,4}$. By the same argument as for Condition 1, there is a diffeomorphism that takes v to w. Moreover, this can be chosen such that it is supported in $S_{2,4}$, i.e. it commutes with the stabilizer of $[e]$.

Condition 4: The complex $C^0_R(S_{\infty,s})$ is contractible by Theorem 3.13. To show that the quotient $Q_\infty = C^0_R(S_{\infty,s})/\mathcal{M}_{\infty,s}$ is contractible, we will show that for any g, the quotient $Q_g = C^0_R(S_{g,s})/\mathcal{M}_{g,s}$ is $(g-2)$-connected. While it is not a simplicial complex, Q_g is a Δ-complex since we can choose an ordering of the vertices of a simplex using the ordering of the arcs along a small half circle around x_0, say from 'left' to 'right'.

The action $\mathcal{M}_{g,s}$ is almost transitive in the sense that it can map a k-simplex σ to any other k-simplex σ' if the ordering of the roots at x_0 as compared to the ordering at x_1 coincides for the two simplices. This means that the ordering of the roots at x_1 uniquely determines a simplex of Q_g. This also means that the map $Q_{g-1} \to Q_g$ induced by the inclusion $C^0_R(S_{g-1,s}) \to C^0_R(S_{g,s})$ is injective, and we can extend it to the cone CQ_{g-1} by mapping the cone tip to an extra rooted curve whose root ends both lie to the right of all the other root ends at both x_0 and x_1. This is possible since genus has been increased by one.

Let a map $f\colon S^{g-2} \to Q_g$ be given. The image of the inclusion $Q_{g-1} \hookrightarrow Q_g$ contains the entire $(g-2)$-skeleton since we already have all possible orderings of $(g-1)$ arcs at x_1, so we can factor f through Q_{g-1}. Then we have the following commutative diagram, where the map $CS^{g-2} \to CQ_{g-1}$ is given by sending the cone point to the cone point and using the composition $S^{g-2} \to Q_{g-1} \hookrightarrow CQ_{g-1}$ on the base.

$$\begin{tikzcd}
 & S^{g-2} \arrow[dl, hook'] \arrow[d] \arrow[dr, "f"] & \\
D_{g-1} = CS^{g-2} \arrow[dr] & Q_{g-1} \arrow[r, hook] \arrow[d, hook] & Q_g \\
 & CQ_{g-1} \arrow[ur] &
\end{tikzcd}$$

This shows that the map $S^{n-2} \to Q_n$ extends over the disk, so Q_n is $(n-2)$-connected. $\square$

As explained above this gives the following result.

Theorem 3.30. *For any $s \geq 1$, the homomorphism*

$$\mu_*\colon H_i(\mathcal{M}_{g,s}) \to H_i(\mathcal{M}_{g,s+1})$$

is an isomorphism if $g > 2i+1$.

3.4.3 Closed surfaces

Let $Ch^{\pm}(S)$ denote the simplicial complex in which a vertex is an isotopy class of chains together with an orientation, and where a collection of oriented chains spans a simplex if the corresponding unoriented chains are all non-isotopic and span a simplex in $Ch(S)$, i.e. they form a system of chains.

Lemma 3.31. *The complex $Ch^{\pm}(S)$, like $Ch(S)$, is $\frac{g-3}{2}$-connected.*

Proof. We can consider the assignment of an orientation to a chain as the assignment of a label $+$ or $-$, where the $+$'s and $-$'s are assigned arbitrarily (but fixed once and for all) to each isotopy class of chains on S. We want to use Corollary 1.11. The link of a k-simplex $[\sigma]$ in the usual complex $Ch(S)$ can be identified with $Ch(S_\sigma)$, where S_σ is the surface obtained from S by cutting along σ. Any system of chains is coconnected, and cutting along each chain reduces genus by one. Thus S_σ has genus $g-k-1$, so $\operatorname{link}\sigma \cong Ch(S_\sigma)$ is $\frac{g-k-1-3}{2}$-connected. Thus it suffices to show that $\frac{g-k-4}{2} \geq \frac{g-3}{2}-k-1$. But this is equivalent to $g-k-4 \geq g-2k-5$ which is true for any $k \geq -1$. $\square$

Theorem 3.32 ([HV15b, Theorem 7.4])**.** *The stabilization $H_i(\mathcal{M}_{g,1}) \to H_i(\mathcal{M}_{g,0})$ induced by the map obtained by filling the single boundary circle of $S_{g,1}$ with a disk is an isomorphism for $g > 2i+2$ and a surjection for $g = 2i+2$.*

Proof. Our strategy for this proof is to show stability of the map $H_i(\mathcal{M}_{g-1,1}) \to H_i(\mathcal{M}_{g,0})$ and then factoring this as $H_i(\mathcal{M}_{g-1,1}) \to H_i(\mathcal{M}_{g,1}) \to H_i(\mathcal{M}_{g,0})$ to show stability for the latter map, which is the one from the statement of the theorem.

The complex $TCh(S)$ of tethered chains is not defined for $s=0$, so instead we will consider the modified version $Ch^{\pm}(S)$ of the complex $Ch(S)$ of chains. We do this because, while the action of $\mathcal{M}_{g,s}$ on the original complex $Ch(S)$ is transitive on simplices in each dimension, and the stabilizer of a simplex $[\sigma]$ of k chain classes contains $\mathcal{M}_{g-k,s+k}$, it does not necessarily fix $[\sigma]$ pointwise. This is due to two problems:

(1) Elements in $\operatorname{stab}[\sigma]$ may reverse the orientations of the two curves in a chain, as long as they are both reversed.
(2) Elements in $\operatorname{stab}[\sigma]$ may permute the chain classes in $[\sigma]$ since we are no longer restricted by the tethers.

To address the first problem we consider the complex $Ch^{\pm}(S)$. This complex is $(g-3)/2$-connected by Lemma 3.31. Using this complex instead of $Ch(S)$, we no longer have the problem with the stabilizers.

For the second problem we use a general construction of a Δ-complex $\Delta(Z)$ from any simplicial complex Z. The k-simplices of $\Delta(Z)$ are the maps $\Delta^k \to Z$ which are simplicial, i.e. they are linear expansions of maps that take vertices to vertices. Note that there is no condition that these maps are injective. The condition that the maps are simplicial ensures that the vertices of $\Delta(Z)$ correspond to those of Z, but since they do not have to be injective, $\Delta(Z)$ usually has far more higher-dimensional simplices. For instance when Z is just a 1-simplex, $\Delta(Z)$ contains four 1-simplices and a lot of higher-dimensional simplices too.

A simplicial map $Z \to Z'$ induces a map $\Delta(Z) \to \Delta(Z')$ by sending a map $\Delta^k \to Z$ to the composition $\Delta^k \to Z \to Z'$, making $\Delta(-)$ a functor. Let $\Delta(Z) \to Z$ be the projection defined by sending a map $\Delta^k \to Z$ to its image. This projection is natural since the diagram

$$\begin{array}{ccc} \Delta(Z) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \Delta(Z') & \longrightarrow & Z' \end{array}$$

commutes.

We will show that this projection induces an isomorphism on homology when Z is finite dimensional (which it will be in our case). We will show this by induction on the number of maximal simplices in Z, i.e. simplices that are not contained in a strictly larger simplex. For the induction start we consider the case where Z itself is a simplex. In this case $\Delta(Z)$ is contractible since every k-simplex $\Delta^k \to Z$ is contained in a $(k+1)$-simplex $\Delta^{k+1} \to Z$ where the additional vertex is mapped to a fixed vertex v of Z. This means that the whole complex $\Delta(Z)$ can be contracted to $\Delta^0 \to v$. For the induction step we use Mayer-Vietoris sequences on the 'decomposition' of Z into a closed simplex σ and the complement of its interior together with the five lemma to conclude that in the diagram below, the middle vertical map is an isomorphism.

$$\begin{array}{ccccccccc} H_n(\Delta(\partial\sigma)) & \longrightarrow & H_n(\Delta(Z\setminus \operatorname{int}\sigma)) \oplus H_n(\Delta(\sigma)) & \longrightarrow & H_n(\Delta(Z)) & \xrightarrow{\partial} & H_{n-1}(\Delta(\partial\sigma)) & \longrightarrow & H_{n-1}(\Delta(Z\setminus \operatorname{int}\sigma)) \oplus H_{n+1}(\Delta(\sigma)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(\partial\sigma) & \longrightarrow & H_n(Z\setminus \operatorname{int}\sigma) \oplus H_n(\sigma) & \longrightarrow & H_n(Z) & \xrightarrow{\partial} & H_{n-1}(\partial\sigma) & \longrightarrow & H_{n-1}(Z\setminus \operatorname{int}\sigma) \oplus H_{n+1}(\sigma) \end{array}$$

In the first, second and fourth squares, the horizontal maps are sums of homomorphisms induced by maps in Z, so since the projection $\Delta(-) \Rightarrow -$ is natural, and H_* is a functor, these squares commute. To see that the third square commutes, we look at the definition of the boundary maps ∂. A homology class $\beta \in H_n(Z)$ is given by a cycle $z \in C_n(Z)$. If needed, then after a barycentric subdivision, such a cycle can be given as a sum $z = x+y$ where x is a chain in σ, and y is a chain in $Z \setminus \operatorname{int}\sigma$. Since z is a cycle, $\partial z = \partial x + \partial y = 0$. Then $\partial\beta$ is defined as $\partial\beta = \partial x = -\partial y$. Now, a cycle in $H_n(Z)$ can be written as the homology class of the cycle $z = \sum n_\alpha(\Delta^k \to Z)$. Let $x = \sum n'_\alpha(\Delta^k \to Z)$ and $y = \sum n''_\alpha(\Delta^k \to Z)$. Then going horizontally in the third square and then vertically amounts to

$$\begin{aligned} \left[\sum n_\alpha(\Delta^k \to Z)\right] &= \left[\sum n'_\alpha(\Delta^k \to Z)\right] + \left[\sum n''_\alpha(\Delta^k \to Z)\right] \\ &\mapsto \partial\left[\sum n'_\alpha(\Delta^k \to Z)\right] \\ &\mapsto \partial\left[\sum n'_\alpha(\operatorname{im}(\Delta^k \to Z))\right], \end{aligned}$$

and going vertically and then horizontally amounts to

$$\begin{aligned} \left[\sum n_\alpha(\Delta^k \to Z)\right] &\mapsto \left[\sum n_\alpha(\operatorname{im}(\Delta^k \to Z))\right] \\ &= \left[\sum n'_\alpha(\operatorname{im}(\Delta^k \to Z))\right] + \left[\sum n''_\alpha(\operatorname{im}(\Delta^k \to Z))\right] \\ &\mapsto \partial\left[\sum n'_\alpha(\Delta^k \to Z)\right], \end{aligned}$$

so also the third square commutes. By the induction hypothesis the first, second, fourth and fifth vertical maps are isomorphisms, so $H_n(\Delta(Z)) \to H_n(Z)$ is an isomorphism by the five lemma.

Now, since $\mathcal{M}_{g,0}$ acts on $Ch^{\pm}(S)$, it acts on $\Delta(Ch^{\pm}(S))$ too. Namely let $[\phi] \in \mathcal{M}_{g,0}$, and let $f\colon \Delta^0 \to Ch^{\pm}(S)$ be a vertex in $\Delta(Ch^{\pm}(S))$. Then f maps $* = \Delta^0$ to some vertex v in $Ch^{\pm}(S)$. We define $[\phi]f$ to map Δ^0 to $[\phi]v$ and extend by linearity.

The complex $\Delta(Ch^{\pm}(S))$ is not a simplicial complex, but only a Δ-complex (semi-simplicial set). However, all the arguments of the spectral sequence argument still apply as noted in Remark 1.2.

Now we want to show that the quotient $\Delta(Ch^{\pm}(S))/\mathcal{M}_{g,0}$. The mapping class group $\mathcal{M}_{g,0}$ acts transitively on simplices of $Ch^{\pm}(S)$ in each dimension, and it can also permute the chains. Thus a class $[\Delta^k \to Ch^{\pm}(S)]$ in the quotient $\Delta(Ch^{\pm}(S))/\mathcal{M}_{g,0}$ is determined only by the number of distinct vertices (chain classes) it hits. This means that $\Delta(Ch^{\pm}(S))/\mathcal{M}_{g,0}$ is isomorphic to $\Delta(\Delta^{g-1})/\Sigma_g$, where the symmetric group Σ_g on g elements acts on Δ^{g-1} by permuting the vertices, since the complex $Ch^{\pm}(S)$ has dimension $g-1$.

The Δ-complex $\Delta(\Delta^{g-2})/\Sigma_{g-1}$ sits inside of $\Delta(\Delta^{g-1})/\Sigma_g$ as the classes of non-surjective maps since modding out by Σ_g makes sure that it does not matter which face of Δ^{g-1} a map lands in. Moreover, the $(g-2)$-skeleton of $\Delta(\Delta^{g-1})/\Sigma_g$ is contained in $\Delta(\Delta^{g-2})/\Sigma_{g-1}$ since $(g-2)$-simplices cannot arise from surjective maps. It then suffices to show that $\Delta(\Delta^{g-2})/\Sigma_{g-1}$ can be contracted inside of $\Delta(\Delta^{g-1})/\Sigma_g$.

The inclusion map $\Delta(\Delta^{g-2}) \hookrightarrow \Delta(\Delta^{g-1})$ induced by the inclusion $\Delta^{g-2} \hookrightarrow \Delta^{g-1}$ that skips a vertex v of Δ^{g-1} extends to a map on the cone, i.e. $C(\Delta(\Delta^{g-2})) \to \Delta(\Delta^{g-1})$ defined by sending the tip of the cone to the map $\Delta^0 \to v$. This map passes to a quotient map $C(\Delta(\Delta^{g-2})/\Sigma_{g-1}) \to \Delta(\Delta^{g-1})/\Sigma_g$ since if two maps represent the same class in $\Delta(\Delta^{g-2})/\Sigma_{g-1}$, they will also be identified in $\Delta(\Delta^{g-1})/\Sigma_g$, and joining with the cone tip preserves this identification. Thus the inclusion $\Delta(\Delta^{g-2})/\Sigma_{g-1} \hookrightarrow \Delta(\Delta^{g-1})/\Sigma_g$ extends over the cone and therefore is null-homotopic.

Now, although the third condition for the spectral sequence argument in Section 1.1 is satisfied for degenerate 1-simplices (maps from Δ^1 to a single vertex in $Ch^{\pm}(S)$), it is not satisfied in general since a non-degenerate 1-simplex maps to two disjoint chain classes, and any diffeomorphism that maps one chain to the other must be supported in a neighborhood that at least contains an arc a between the two chains. Such a diffeomorphism does not commute with the stabilizer of the two chains since some elements of that stabilizer may not fix a. However, we can still proceed with the argument for injectivity of the differential $d\colon H_i(\mathcal{M}_{g-1,1}) \to H_i(\mathcal{M}_{g,0})$ which is what we need this condition for. For a small neighborhood N of the two chains and the arc a between them, there is a diffeomorphism ϕ supported in N that takes one chain to the other, so the isotopy class of this diffeomorphism commutes with the subgroup E of the mapping class group that fixes the isotopy class of the two chains and the arc a. By Theorem 3.30 for s-stability the subgroup inclusion $\mathcal{M}_{g-2,1} \hookrightarrow \mathcal{M}_{g-2,2}$ from E to the stabilizer of the two chains is surjective whenever $g-2 \geq 2i+1$, i.e. $g > 2i+2$. The subgroup inclusion is the same as that we used for s-stabilization since in the stabilization we glued a pair of pants onto a single boundary component, and cutting along the arc a undoes such a map. Thus, denoting the system consisting of the two chains by e, we get a commutative diagram

$$\begin{tikzcd} H_i(E) \arrow[r, two heads] \arrow[d, "c_{g_1}=\mathrm{id}"] & H_i(\mathrm{stab}([e])) \arrow[d, "c_{g_1}"] \\ H_i(E) \arrow[r, two heads] & H_i(\mathrm{stab}([e])). \end{tikzcd}$$

This makes sure that the conjugation c_{g_1} by g_1 induces the identity on $H_i(\mathrm{stab}(e))$, so the argument centered around the diagram (1.4) on page 15 still holds, i.e. $H_i(\mathcal{M}_{g-1,1}) \to H_i(\mathcal{M}_{g,0})$ is injective for $g > 2i + 2$. The surjectivity argument did not use the third condition, so d is also surjective for $g \geq 2i + 2$ in this case.

Now we consider the factorization of $d\colon H_i(\mathcal{M}_{g-1,1}) \to \mathcal{M}_{g,0}$ into $H_i(\mathcal{M}_{g-1,1}) \to H_i(\mathcal{M}_{g,1}) \to H_i(\mathcal{M}_{g,0})$. The composition d is surjective for $g \geq 2i+2$, so the latter map in the factorization (from the statement of the theorem) is surjective in this case as well. Moreover we just showed that the composition is an isomorphism when $g > 2i + 2$, and the first map in the factorization is an isomorphism in this range as well by Theorem 3.28, so the latter map must also be an isomorphism in this range. □

Chapter 4

Stability for symmetric mapping class groups

For a smooth surface S, consider the group $\mathrm{Diff}^+(S)$ of orientation preserving diffeomorphisms of S onto itself. An order 2 element of $\mathrm{Diff}^+(S)$ is called an **involution**. For an involution κ on S, denote by $\mathrm{Diff}^+_\kappa(S)$ the centralizer of κ in $\mathrm{Diff}^+(S)$, i.e. the subgroup

$$\mathrm{Diff}^+_\kappa(S) = \{\phi \in \mathrm{Diff}^+(S) \mid \phi\kappa = \kappa\phi\} \leq \mathrm{Diff}^+_\kappa(S).$$

We then define the **κ-symmetric mapping class group** as the subgroup $\mathcal{M}_\kappa(S)$ of the boundary-fixing mapping class group $\mathcal{M}(S)$ consisting of all isotopy classes of diffeomorphisms with a representative in $\mathrm{Diff}^+_\kappa(S)$.

4.1 Involutions and the stabilization map

If S is a closed surface, it admits a certain involution called the **hyperelliptic involution**. This is defined by a 180 degree rotation around a central axis as depicted in Figure 4.1. If S is a surface with boundary, there is still a notion of hyperelliptic involution, although one must make a choice. Namely, a hyperelliptic involution ι is still defined by a 180 degree rotation, but it can now do two different things to a component $\partial_0 S$ of ∂S; it can rotate it around itself such that $\iota\partial_0 S = \partial_0 S$, or it can interchange it with another boundary component $\partial_1 S$ such that $\iota\partial_0 S = \partial_1 S$ and $\iota\partial_1 S = \partial_0 S$. For the type of parametrization of $S_{g,0} \subset \mathbb{R}^3$ indicated in Figure 4.1, $S_{g,0}$ has $2g+2$ intersection points with the x-axis (dashed line): 2 in each genus void and 1 in each end. Since the hyperelliptic involution revolves $S_{g,0}$ around the x-axis, these $2g+2$ points are the only ones that are fixed by ι.

In order to pass from closed surfaces to surfaces with boundary, we need to remove open disks $\mathring{D}_2$ from $S_{g,0}$. In order to still have a hyperelliptic involution as given above, we need to choose these disks in a certain way. Namely we need to choose them such that each disk is either rotated by ι onto itself or mapped to one of the other disks. For the type that rotates onto itself, the center must be a fixed point. Consequently,

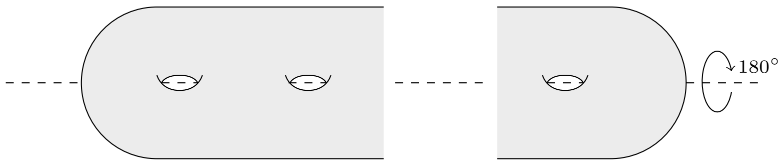


Figure 4.1: Hyperelliptic involution on a closed surface

there can be at most $2g + 2$ of that type. The only requirement for the second type is to find two disjoint antipodal disks, which we can do an arbitrary number of times. Thus we can have a hyperelliptic involution on any $S_{g,s}$, but only $2g+2$ of the boundary components can be of the type that are rotated by ι. For non-negative integers a and b with $a + 2b = s$, we denote by $\mathcal{M}_H^{a,b}(S_{g,s})$ the hyperelliptic mapping class group where the hyperelliptic involution rotates a of the boundary components and interchanges b pairs of boundary components. For $a \geq 1$, we will show that the sequence

$$\mathcal{M}_H^{a,b}(S_{g,s}) \to \mathcal{M}_H^{a-1,b+1}(S_{g,s+1}) \to \mathcal{M}_H^{a,b}(S_{g+1,s}) \to \cdots$$

obtained by successively gluing on pairs of pants in a certain way is homologically stable. In fact, we will show this in a greater generality where the starting surface is any surface with an involution that rotates at least one of the boundary components. The stabilization will depend on the number of fixed points, which is described in the following lemma.

Lemma 4.1. *Let κ be an involution on S. If S has genus g, the set of κ-fixed points, i.e. those $x \in S$ such that $\kappa(x) = x$, is discrete, i.e. it consists only of isolated points.*

Sketch of proof, see e.g. [FM11, Section 7.1.2]. Let x be any point that is fixed by κ. First choose any Riemannian metric h on S. We consider $h' = h + \kappa^*h$. This is a Riemannian metric since h and κ^*h are both Riemannian metrics, and it is non-zero since h and κ^*h are positive definite. We have $\kappa^*h' = \kappa^*h + \kappa^*\kappa^*h = \kappa^*h + h = h'$, so κ is an isometry with respect to h'. Moreover, a neighborhood of x is a disk which is either fixed, reflected or rotated by κ. Since κ is an isometry, the first case would force κ to be trivial, which is not the case since it is an involution. The second case would mean that κ was orientation reversing which leaves us with the third case, which shows that x is an isolated fixed point. $\square$

Remark 4.2. In fact, a calculation using the Lefschetz fixed point theorem shows that there can be at most $2g + 2$ fixed points, but we do not need this fact here.

Consider a surface S with an involution κ, and assume that there is at least one component $\partial_0 S$ of S that is rotated by κ. We create a sequence of surfaces by successively gluing pairs of pants onto S. A pair of pants $S_{0,3}$ has only one type of *hyperelliptic* involution, namely one that rotates one of the boundary components and interchanges the two others. This hyperelliptic involution has one fixed point. Let $\partial_0 S_{0,3}$ be the boundary component that is rotated by the hyperelliptic involution. We can glue $S_{0,3}$

onto S along $\partial_0 S_{0,3}$ and $\partial_0 S$. This gives us a new surface S_1 with an involution κ_1. Instead of the boundary component $\partial_0 S$, S_1 has two boundary components and an additional fixed point. We can now proceed with gluing a new copy of $S_{0,3}$ onto the resulting surface, only now we glue along the interchanging boundary components. This gives us a surface S_2 with involution κ_2. S_2 has one more fixed point than S', its genus is one greater, and the interchanging boundary components have been replaced by a rotating one again. Thus we can repeat the process to get a surface S_3 with involution κ_3, and so on. Our goal is to show that the sequence

$$\mathcal{M}_\kappa(S) = \mathcal{M}_{\kappa_0}(S_1) \to \mathcal{M}_{\kappa_1}(S_1) \to \mathcal{M}_{\kappa_2}(S_2) \to \mathcal{M}_{\kappa_3}(S_3) \to \cdots \tag{4.1}$$

is stable in group homology. Here we let $\kappa_0 = \kappa$ and $S_0 = S$ for notational purposes.

Remark 4.3. Let S be a surface with an involution κ. In the following we will consider the quotient S/κ obtained by identifying each $x \in X$ with $\kappa(x)$. For $x \in S$, let $\bar{x}$ denote the image in S/κ. If x is not a fixed point, then $\bar{x}$ has a small neighborhood which is an open smooth disk and whose preimage in S consists of two open smooth disks. However, if x is a fixed point, then $\bar{x}$ is a cone point of order 2, i.e. a small neighborhood of $\bar{x}$ comes from wrapping a disk in S around itself by a 180 degree rotation. The quotient has a smooth structure except for those cone points, so it can be viewed as a smooth surface with marked points, one for each κ-fixed point. A diffeomorphism of S/κ that permutes the marked points then lifts to a diffeomorphism of S that commutes with κ.

4.2 The complex of symmetric arcs

In order to prove that the sequence (4.1) is homologically stable, we can no longer use the same complexes as we did in the previous chapter. For example transitivity now fails for tethered chains. What we need to do is to create a complex of symmetric structures, i.e. the vertices must be structures that in some way commute with the chosen involution. An arc complex satisfying this must consist of symmetric arcs, i.e. arcs a satisfying $\kappa(a) = a$.

Consider a set $\Delta = \{b_1, b_2\} \subset \partial S$ with $\kappa(b_1) = b_2$. The precise complex we will use is the complex $A^0_\kappa(S, \Delta)$ in which a k-simplex is an isotopy class of systems of $k+1$ arcs from b_1 to b_2 that are pairwise non-isotopic and disjoint except for their endpoints. The isotopies must fix the endpoints of the arcs, and an arc may not be isotopic to an arc whose interior is contained in $\partial S \setminus \Delta$, i.e. a line segment of ∂S between two adjacent points of Δ. Moreover, for any simplex $[\sigma]$ in $A^0_\kappa(S, \Delta)$ we require that:

(1) σ can be chosen such that each arc a of σ is symmetric, i.e. $\kappa(a) = a$, and
(2) σ is non-separating, i.e. $S \setminus \sigma$ is connected.

The second requirement ensures that if we choose σ to be symmetric, then the surface obtained by cutting S along σ is a connected surface with an involution determined by κ.

An isotopy class of $k+1$ symmetric arcs spans a k-simplex in $A^0_\kappa(S, \Delta)$ if the arcs are pairwise non-isotopic and the arcs can be chosen such that they are disjoint. Note that a symmetric arc must consist of two half arcs meeting at a point on S that is fixed by κ. Since there are at most $2g+2$ such fixed points, the complex $A^0_\kappa(S,\Delta)$ has dimension at most $2g+1$. Note also that a symmetric arc cannot pass through two different κ-fixed points since then it would contain a symmetric closed curve through those two points. We still work with usual non-restrictive isotopy classes of arcs, so if a is a symmetric arc, its isotopy class is going to contain non-symmetric arcs, but usually we will choose a symmetric representative.

To prove that $A^0_\kappa(S,\Delta)$ is highly connected we will proceed by induction. For the induction argument we will make use of the complexes of symmetric arcs on surfaces cut up by symmetric arc systems. When cutting along a symmetric arc, its endpoints in Δ will be split up into four new endpoints. Thus we will have to look at the more general complex where Δ is not neccessarily a set of two points, but may contain any even number of points as long as $\kappa(\Delta)=\Delta$. To show that $A^0_\kappa(S,\Delta)$ is highly connected, we will embed it into the larger complex $A_\kappa(S,\Delta)$ of arc systems that may be separating. This will again be embedded into an even larger complex $A^\cap_\kappa(S,\Delta)$ in which the arcs are allowed to intersect in the κ-fixed points. We start with the largest complex. First we will find out when the two largest complexes are non-empty.

Proposition 4.4. *The complexes $A^\cap_\kappa(S,\Delta)$ and $A_\kappa(S,\Delta)$ are non-empty if S contains at least one point that is fixed by κ and S is not a disk with $|\Delta|=2$.*

Proof. This proof holds for both complexes since in either case we just need the existence of one symmetric arc. If S is a disk with $|\Delta|=2$, any arc with endpoints in Δ is trivial, so we have to leave that example out of account, hence the exception in statement of the proposition. If S is a disk with $|\Delta|>2$, the complex $A_\kappa(S,\Delta)$ is indeed non-empty since an arc is trivial only if it is isotopic to an arc whose interior lies inside $\partial S\setminus\Delta$. If S is a cylinder $S_{0,2}$ and ∂S consists of two components that are interchanged by κ then $A_\kappa(S,\Delta)$ is certainly non-empty. If ∂S consists of two components that are rotated by κ, then there are no κ-fixed points, so that case is not relevant. If $S=S_{0,3}$, the presence of the two additional boundary components makes sure that there are non-trivial arcs, so $A_\kappa(S,\Delta)$ is non-empty. Similarly $A_\kappa(S,\Delta)$ is nonempty when $S=S_{0,s}$ for all higher s.

If S has genus $g\geq 1$, then $A_\kappa(S,\Delta)$ is non-empty by the following argument: There is a symmetric arc since the quotient S/κ is connected, and an arc in S/κ corresponds to a symmetric arc on S. The question is whether this symmetric arc is non-trivial. For a symmetric arc to be trivial, it must be isotopic to an arc in ∂S. Therefore it must separate S into two components, one of which is a disk. But since the arc is symmetric, the other component must be a disk as well. This means that S is a disk since it can be obtained by gluing together two disks along a single arc in each of their boundaries, contradicting the fact that S has positive genus. A simple example of a non-trivial arc when S has genus 1 and κ is a hyperelliptic involution is given in Figure 4.2, and a slightly more complex one can be seen on the front page. □

Proposition 4.5. *The complex $A^\cap_\kappa(S,\Delta)$ is contractible whenever it is non-empty.*

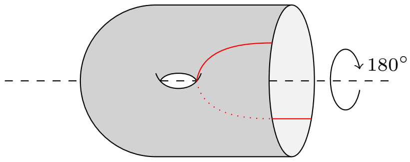


Figure 4.2: Non-trivial symmetric arc for $g = 1$

Proof. Suppose that $A_\kappa^\cap(S, \Delta)$ is non-empty and fix a symmetric arc a. We will use Lemma 1.20 and do a surgery flow of $A_\kappa^\cap(S, \Delta)$ into the star of $[a]$, i.e. we set $X = A_\kappa^\cap(S, \Delta)$ and $Y = \mathrm{star}[a]$. We define the complexity $c(\sigma)$ of a system σ of symmetric arcs by first putting it in normal form with respect to a and then counting the number of intersection points with a except the κ-fixed point p_a that a passes through. Note that we can put σ in normal form with respect to a and still retain a system of symmetric arcs. Namely, any bigon D is either symmetric such that $\kappa(D) = D$, or it has a 'twin' D' such that $\kappa(D') = D$.

If $[\sigma]$ is a simplex in $A_\kappa^\cap(S, \Delta)$ but not in the star of $[a]$, then σ intersects a in at least one pair of points $p, \kappa(p) \subset a$ that are not fixed under κ. Perform surgery on the arc v_σ that intersects a closest to the boundary. This results in a partition of v_σ into three sub-arcs, one of which is the 'middle' arc from p to $\kappa(p)$. Redirect both of the endpoints of this middle arc along a and away from p_a so that they land in the two boundary components. If we do this symmetrically with respect to κ in both ends, we get a symmetric arc Δv_σ which only intersects σ in the κ-fixed point that v_σ passes through, i.e. $[\Delta v_\sigma]$ is in the link of $[\sigma]$. The number of intersection points is reduced by at least two, so the complexity is reduced. The third condition of Lemma 1.20 is also satisfied, so this defines a surgery flow, and we get the result. □

We will pass to the smaller complex where the arcs are only allowed to intersect at their endpoints. We lose some connectivity, but we retain enough in order to get the stability results.

Proposition 4.6. *If S has k κ-fixed points, the complex $A_\kappa(S, \Delta)$ is $(k-2)$-connected when S is not a disk with $|\Delta| = 2$.*

Proof. We proceed by induction on the number k of κ-fixed points. For the case $k = 0$ the statement is vacuous, but for $k = 1$ the statement is that $A_\kappa(S, \Delta)$ is non-empty, which is true by Proposition 4.4. The base case is therefore $k = 1$. We will do the induction step with a bad simplex argument using Corollary 1.10 for the inclusion $A_\kappa(S, \Delta) \hookrightarrow A_\kappa^\cap(S, \Delta)$. We say that a simplex $[\sigma]$ in $A_\kappa^\cap(S, \Delta)$ is bad if, regardless of the isotopy class, each arc of σ intersects at least one of the others, i.e. if every κ-fixed point that is hit by σ is hit by at least two distinct arcs of σ. Note that a bad simplex has dimension at least 1.

The complex $G_{[\sigma]}$ of simplices good for $[\sigma]$ is then built from disjoint symmetric arcs in $S \setminus \sigma$ that are not isotopic to each other or to any arcs of σ. This can be viewed as the join $*_i\, A_{\kappa_i}(S_i, \Delta_i)$ where the S_i's are components of the surface S_σ obtained by cutting S along σ, where Δ_i is the set of boundary points inherited from Δ, and where κ_i is the involution inherited from κ. For this to make sense we need to discard the components that are not closed under the inherited hyperelliptic involution since if σ is separating, there might be some components that are mapped by the inherited hyperelliptic involution to each other and not to themselves. These components have no symmetric arc complex since that requires a hyperelliptic involution. Also, they cannot contain any fixed points. Note that none of the the S_i's can be a disk with $|\Delta_i| = 2$ since only a trivial arc could create that situation, so the induction hypothesis applies to all of the S_i's.

In order to use Corollary 1.10, we need to show that $G_{[\sigma]}$ is $(g-2-n)$-connected, where n is the dimension of $[\sigma]$. Say that each component S_i has k_i fixed points. Then for each i we must have $k_i < k$ since at least one fixed point has been eliminated by cutting along σ. Thus, the induction hypothesis holds, so for each i, $A_{\kappa_i}(S_i, \Delta_i)$ is (k_i-2)-connected. Each arc of σ hits at most one fixed point, and at least one fixed point is hit twice. Thus, since the discarded components of S_σ contain no fixed points, we have $\sum_i k_i \geq k - n$. Moreover, by Lemma A.2 the connectivity of $G_\sigma \cong *_i\, A_{\kappa_i}(S_i, \Delta_i)$ is at least

$$\sum_i (k_i - 2 + 2) - 2 = \sum_i k_i - 2 \geq k - n - 2,$$

so by Corollary 1.10 $A_\kappa(S, \Delta)$ is $(k-2)$-connected. □

We are now ready to prove the connectivity of the complex $A^0_\kappa(S, \Delta)$ of coconnected symmetric arcs.

Proposition 4.7. *If $A_\kappa(S, \Delta)$ is non-empty, the complex $A^0_\kappa(S, \Delta)$ is $(k-3)$-connected, where k is the number of κ-fixed points of S.*

Proof. Let a be a symmetric arc on any surface. If a is separating, then because it is symmetric, it must separate S into two components that are interchanged by κ. Thus these components cannot contain any κ-fixed points. This means that a separating arc system must consist of at least k arcs. Thus $A^0_\kappa(S, \Delta)$ contains the entire $(k-2)$-skeleton of $A_\kappa(S, \Delta)$. Let a map $f\colon S^{k-3} \to A^0_\kappa(S, \Delta)$ be given. Then since $A_\kappa(S, \Delta)$ is $(k-2)$-connected, f extends to a map $\bar{f}\colon D_{k-2} \to A^0_\kappa(S, \Delta)$. By Theorem A.5 we can assume that $\bar{f}$ is simplicial, so that it maps into the $(k-2)$-skeleton of $A_\kappa(S, \Delta)$ and therefore into $A^0_\kappa(S, \Delta)$. □

4.3 The stability theorem

Recall the sequence

$$\mathcal{M}_{\kappa_0}(S_0) \to \mathcal{M}_{\kappa_1}(S_1) \to \mathcal{M}_{\kappa_2}(S_2) \to \mathcal{M}_{\kappa_3}(S_3) \to \cdots \tag{4.2}$$

defined on page 77. We now want to show that this sequence stabilizes in homology by putting it into the context of the spectral sequence argument of Section 1.1. The result we need to prove is the following.

Theorem 4.8. *The homomorphism $H_i(M_{\kappa_{i-1}}(S_{i-1})) \to H_i(M_{\kappa_i}(S_i))$ induced from the sequence* (4.2) *is surjective for $i \geq 2k+2$ and an isomorphism for $i \geq 2k+3$.*

Proof. We need to assign to each $\mathcal{M}_{\kappa_i}(S_i)$ a simplicial complex X_i with an action $\mathcal{M}_{\kappa_i}(S_i) \curvearrowright X_i$. If i is even, S_i has a component $\partial_0 S$ that is rotated by κ_i. Let Δ_i consist of two points in $\partial_0 S_i$ such that $\kappa_i(\Delta_i) = \Delta_i$. Then let $X_i = A^0_{\kappa_i}(S_i, \Delta_i)$, and assign to $\mathcal{M}_\kappa(S)$ the simplicial complex X_0 consisting of just one vertex. If i is odd, we have instead two boundary components $\partial_0 S_i$ and $\partial_1 S_i$ that are interchanged by κ_i. Then for a point $x \in \partial_0 S_i$ we let $\Delta_i = \{x, \kappa_i(x)\}$ and define X_i in the same way as before.

An element $[\phi]$ of $M_{\kappa_i}(S_i)$ then acts on a simplex $[\sigma]$ of X_i simply by $[\phi][\sigma] = [\phi(\sigma)]$. We have to show that the four conditions of section 1.1 are satisfied.

<u>Condition 1</u>: We will show that the action $\mathcal{M}_\kappa(S) \curvearrowright A^0_\kappa(S)$ is transitive on simplices of any dimension for any surface S with involution κ. Let σ and σ' be two coconnected systems of $k+1$ symmetric arcs on $S_{g,s}$. Let $\pi\colon S \to S/\kappa$ denote the quotient map. By Remark 4.3, both $\pi(\sigma)$ and $\pi(\sigma')$ are systems of $k+1$ arcs from a point in $\partial(S/\kappa)$ to $k+1$ distinct marked points (cone points). Since we can move these arcs around each other in any way we like, we can simply choose a diffeomorphism of S/κ (seen as a marked smooth surface) that permutes the marked points and takes $\pi(\sigma)$ to $\pi(\sigma')$. This lifts to a diffeomorphism of S that commutes with κ and takes σ to σ'.

<u>Condition 2</u>: Let $i \geq 1$ be given. For any $[\phi] \in \mathcal{M}_{\kappa_i}(S_i)$, ϕ must fix the boundary ∂S_i. Therefore it must preserve the ordering of the arcs given by a small half-circle around a point of Δ_i. Therefore, if σ is a coconnected system of symmetric arcs on S_i and $[\phi]$ is an element of stab$[\sigma]$, then $[\phi]$ must fix each vertex of $[\sigma]$, i.e. it fixes $[\sigma]$ pointwise. This means that stab$[\sigma]$ is the subgroup of $\mathcal{M}_{\kappa_i}(S_i)$ of elements $[\phi]$ such that $[\phi]$ fixes $[\sigma]$. This is exactly the group $\mathcal{M}_{\kappa_\sigma}(S_\sigma)$ where S_σ is the surface obtained by cutting along σ, and κ_σ is the resulting involution inherited from κ_i. If σ has dimension k, then S_σ is a copy of S_{i-k-1}, and stab$[\sigma] \cong \mathcal{M}_{\kappa_{i-k-1}}(S_{i-k-1})$.

The stabilizer of $[\sigma]$ is conjugate to $\mathcal{M}_{\kappa_{i-k-1}}(S_{i-k-1})$ since the action is transitive on simplices of each dimension by the same argument as we used in the proof of Theorem 3.28.

<u>Condition 3</u>: Let $[e]$ be an edge in X_i. Then e consists of two symmetric arcs a and a' in S_i. Now pass to the quotient by the quotient map $\pi\colon S_i \to S_i/\kappa_i$. Then $\pi(a)$ and $\pi(a')$ are two arcs to two different marked points on S_i. A small neighborhood N of these two arcs is a disk with two marked points. We can easily choose a diffeomorphism ϕ of S_i/κ_i with support in this neighborhood which interchanges these two marked points and which takes $\pi(a)$ to $\pi(a')$. Then ϕ lifts to a symmetric diffeomorphism $\hat{\phi}$ on S_i that takes a to a'. Since ϕ is supported in N, $[\hat{\phi}]$ commutes with the stabilizer of $[e]$.

<u>Condition 4</u>: The action $M_{\kappa_i}(S_i) \curvearrowright X_i$ is transitive on simplices of each dimension, and since S has no κ-fixed points, X_i is $(i-1)$-dimensional, so $X_i/M_{\kappa_i}(S_i)$ is the quotient of Δ^{i-1} identifying all simplices of each dimension. Thus the homology of $X_i/M_{\kappa_i}(S_i)$ vanished up to degree $(n-2)$ by the same argument as in the proof of Theorem 2.4, which is enough for Condition 4 since we already know from Proposition 4.7 that X_i is $(i-3)$-connected. Thus the result follows from Theorem 1.6. □

Prospects for further work

The key property of a symmetric arc a is that a small neighborhood of a is a copy of $S_{0,3}$ with a hyperelliptic involution. This made us able to prove that the sequence obtained from attaching such copies of $S_{0,3}$ is homologically stable. However, the involution on the initial surface was not necessarily a hyperelliptic one. It may be possible to prove similar results for sequences obtained by gluing surfaces with more exotic involutions. In our case the stabilization was closely related to the number of fixed points. Perhaps we just need to know that the surface that we glue on has a fixed point in order to be able to prove a stability theorem.

In our proof of Theorem 4.8 we did not use the number of fixed points of the initial surface. If this surface has fixed points, we might be able to obtain a slightly better stability result. Although it would not improve the slope of the stable range function, it could shift it down.

The homology of symmetric mapping class groups of closed hyperelliptic surfaces does not stabilize; see e.g. [Kaw97]. Theorem 4.8 can be used to show that in the non-closed case they do stabilize nonetheless. It would be interesting to understand on a deeper level what it is that makes our sequences stabilize in comparison with the closed case.

Appendix A

Auxiliary lemmas

A.1 Simplicial complexes

Definition A.1. An abstract simplicial complex $X = (V, S)$ is an ordered tuple consisting of a finite set V called the set of vertices and a set S of non-empty subsets of V caled the simplices of X such that

(i) For all $v \in V$, $\{v\} \in S$,
(ii) If $\sigma \in S$ and $\emptyset \neq \sigma' \subseteq \sigma$, then $\sigma' \in S$.

A **subcomplex** of X is a simplicial complex $X' = (V', S')$ such that $V' \subset V$ and $S' \subset S$. The **full subcomplex** of X spanned by some $V' \subset V$ is the subcomplex (V', S_V) of X where S_V consists of all simplices of X with vertices in V'. For simplicity we will sometimes identify a full subcomplex by the spanning V'. A **map of simplicial complexes** $X = (V, S) \to X' = (V', S')$ is a function $f\colon V \to V'$ such that $f(\sigma)$ is a simplex of X' if σ is a simplex of X.

Let σ be a simplex in X. We will typically just write $\sigma \in X$. If a simplex σ has $n+1$ elements for some $n \geq -1$, we say that the dimension of σ is n, and we write $\dim(\sigma) = n$. The **closure** $\mathrm{Cl}(\sigma)$ is the full subcomplex spanned by σ. If B is a set of simplices in X, the closure $\mathrm{Cl}(B)$ of B is the full subcomplex complex spanned by the union of all elements of B. If $\sigma' \subset \sigma$ is non-empty, we say that σ' is a **face** of σ, and if moreover $\sigma' \neq \sigma$, we say that σ' is a **proper face** of σ. The **star** $\mathrm{star}(\sigma)$ of a simplex σ in X is the subcomplex consisting of simplices τ in X such that some face of σ is a face of τ. The star of a set B of simplices in X is the union of the stars of each simplex in B. The **link** $\mathrm{link}(\sigma)$ of a simplex σ in X is $\mathrm{Cl}(\mathrm{star}(\sigma)) \setminus \mathrm{star}(\mathrm{Cl}(\sigma))$, and the link of a set B of simplices is $\mathrm{Cl}(\mathrm{star}(B)) \setminus \mathrm{star}(\mathrm{Cl}(B))$. For two simplices σ and τ, their **join** $\sigma * \tau$ is the simplex with vertices $\sigma \cup \tau$. This is now always a simplex of X, but $\mathrm{star}\,\sigma$ is the subcomplex of σ consisting of simplices τ such that $\sigma * \tau \in X$, and $\mathrm{link}\,\sigma$ is the full subcomplex of $\mathrm{star}\,\sigma$ spanned by the vertices disjoint from σ. Thus these can be joined, and

$$\mathrm{star}(\sigma) = \sigma * \mathrm{link}\,\sigma. \tag{A.1}$$

Moreover, joining is associative, so we can unambiguously join more that two spaces at once.

From a simplicial complex X one can form a topological space called the **geometric realization** of X, which is usually also denote by X. This is a cell complex where each n-cell is a copy of the standard n-simplex Δ^n. The standard n-simplex is the convex hull of all unit vectors in $\mathbb{R}^{n+1}$, and the faces of Δ^n are the convex hulls of subsets of the set of unit vectors and thus are standard simplices of lower dimensions. The standard n-simplex is homeomorphic to the n-disk D_n, and its boundary is homeomorphic to the $(n-1)$-sphere S^{n-1}. The space X is defined inductively over its skeleta. The 0-skeleton is the discrete space of all 0-simplices of X, and the n-skeleton is formed from the $(n-1)$-skeleton as follows: For each n-simplex σ we glue a copy of the standard n-simplex along its faces to the $(n-1)$-cells corresponding to the faces of σ.

If $f\colon X \to X'$ is a map of simplicial complexes, then f induces a continuous map of the geometric realizations by mapping vertices to vertices and expanding linearly. In terms of geometric realizations a join $\sigma_0 * \cdots * \sigma_n$ is homeomorphic to the topological join of the relizations of the complexes. Thus a point in $\sigma_0 * \cdots * \sigma_n$ be described as a weighted sum

$$t_0 x_1 + \cdots + t_n x_n,$$

where each $t_i \in I$ and $x_i \in \sigma_i$ and $\sum_i t_i = 1$. We have the following lemma for the connectivity of joins, due to Milnor.

Lemma A.2 ([Mil56, Lemma 2.3])**.** *Let $A_0, \ldots, A_n$ be non-empty spaces such that for each i, A_i is $(c_i - 2)$-connected. Then the join $A_0 * \cdots * A_n$ is $((\sum_{i=0}^n c_i) - 2)$-connected.*

Remark A.3. We sometimes talk about topological properties of a simplicial complex X, e.g. that X is n-connected. In those cases we are referring to the geometric realization of X.

A.1.1 Triangulations

For a manifold M we say that M admits a triangulation if M is the geometric realization of a simplicial complex. The following is a consequence of a theorem first proved by Whitehead in [Whi40]. It boils down to the fact that any smooth manifold has a so called piecewise linear triangulation.

Theorem A.4. *Any smooth n-manifold M has a triangulation in which the link of any k-simplex σ is contractible if σ is included in ∂M or homeomorphic to S^{n-k-1} if not.*

Given a simplex σ of a simplicial complex X, we can subdivide σ by removing it from X but instead adding an extra vertex and then joining that vertex with $\partial\sigma$. We say that Y is a subdivision of X if Y can be obtained from X by doing so a number of times. The geometric realization of Y is then homeomorphic to X. The following theorem dates back to J.W. Alexander [Ale15], and a modern proof can be found in Spanier's book [Spa66, section 3.4].

Theorem A.5. *Let X and X' be simplicial complexes, and let f be a (not neccesarily simplicial) map of their geometric realizations. Then f is homotopic to a simplicial map $f\colon Y \to X'$ for some subdivision of X.*

For some of the proofs of Section 1.2 we will need the fact that the star of any simplex is contractible. Moreover, we need to be able to retriangulate the star of a simplex in a certain way. The following lemma will help us in that regard

Lemma A.6 ([Hud69, Lemma 1.13])**.** *Assume that we have triangulations of S^m, S^n, D_m, and D_n and that these can all be joined together. Then*

(1) $D_m * D_n$ is homeomorphic to D_{m+n+1},
(2) $D_m * S^n$ is homeomorphic to D_{m+n+1}, and
(3) $S^m * S^n$ is homeomorphic to S^{m+n+1}.

Using (A.1) and Theorem A.4 this means that for any k-simplex σ in a piecewise linear triangulation of an n-manifold we have

$$\operatorname{star}(\sigma) = \sigma * \operatorname{link}(\sigma) \cong D_k * S^{n-k-1} \cong D_n. \tag{A.2}$$

A.2 Lemmas from algebra

Lemma A.7. *Let G be a group. There is always a free resolution*

$$\cdots \to E_2G \xrightarrow{\phi_1} E_1G \xrightarrow{\phi_0} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules, where ϵ is the augmentation map that sends $1 \in \mathbb{Z}[G]$ to $1 \in \mathbb{Z}$.

Proof. The kernel of ϵ is a submodule of $\mathbb{Z}[G]$, so it is the homomorphic image of some $\phi_0\colon E_1G \to \mathbb{Z}[G]$ where E_1G is a free module. Then the kernel of ϕ_0 is the homomorphic image of $\phi_1\colon E_2G \to E_1G$, where E_2G is also free, and so on. □

Definition A.8. Let X be a set with a G-action, and let $x \in X$. The **stabilizer subgroup** $\operatorname{stab}(x)$ of x is the smallest subgroup of G that fixes x pointwise.

Theorem A.9 (Orbit-stabilizer Theorem)**.** *Let X be a set with a group action for a group G. For each $x \in X$, let $G/\operatorname{stab}(x)$ denote the set of left cosets of $\operatorname{stab}(x)$ in G, and let $[g]$ denote $g\operatorname{stab}(x)$ for $g \in G$. Then the map*

$$\phi\colon \operatorname{orb}(x) \to G/\operatorname{stab}(x)$$

given by $\phi(gx) = [g]$ is a bijection. Moreover, the set $\{gx \mid [g] \in G/\operatorname{stab}(x)\}$ is well-defined and equal to $\operatorname{orb}(x)$.

Proof. To show that ϕ is well-defined, let $g, h \in G$ be given such that $[g] = [h]$. Since $1 \in \operatorname{stab}(x)$, $g \in [g]$, so there is some $a \in \operatorname{stab}(x)$ such that $ha = g$. But then

$$\phi([g]) = gx = hax = hx = \phi([h]).$$

The map ϕ is obviously surjective. To see that it is injective, assume that $g, h \in G$ and $gx = hx$. Then $h^{-1}gx = g^{-1}hx = x$, so $h^{-1}g, g^{-1}h \in \operatorname{stab}(x)$. Thus

$$[g] \subseteq [gg^{-1}h] = [h] \subseteq [hh^{-1}g] = [g].$$

The last statement follows simply because $\operatorname{stab}(x)$ acts trivially on x. $\square$

Lemma A.10. *If x and y are in the same orbit of X, then* $\operatorname{stab}(x)$ *is conjugate to* $\operatorname{stab}(y)$.

Proof. Let $g \in G$ such that $y = gx$, and let $h \in \operatorname{stab}(y)$. Then $hy = hgx = gx$, so $g^{-1}hgx = g^{-1}gx = x$ i.e. $g^{-1}hg \in \operatorname{stab}(x)$, and vice versa, if $g^{-1}hg \in \operatorname{stab}(x)$, then $h \in \operatorname{stab}(y)$. Thus $\operatorname{stab}(y) = g^{-1} \operatorname{stab}(x) g$. $\square$

Lemma A.11. *Let G be a group, and for each α in some index set A, let M_α be a G-module. Then there are isomorphisms*

$$H_*(G; \oplus_\alpha M_\alpha) \cong \bigoplus_\alpha H_*(G; M_\alpha).$$

Proof. This is just due to the fact that tensor product and homology both respect direct sums. I.e.

$$\begin{aligned} H_*(G; \oplus_\alpha M_\alpha) &= H_*(F \otimes_G \oplus_\alpha M_\alpha) \cong H_*(\oplus_\alpha (F \otimes_G M_\alpha)) \\ &\cong \oplus_\alpha H_*(F \otimes_G M_\alpha) = \oplus_\alpha H_*(G, M_\alpha), \end{aligned}$$

where F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. $\square$

Lemma A.12 (Shapiro, [Wei94, Lemma 6.3.2])**.** *If H is a subgroup of G, and M is an H-module, then there are isomorphisms*

$$H_*(H; M) \cong H_*(G, \operatorname{Ind}_H^G M) \cong H_*(G; \oplus_{g \in G/H} gM),$$

where G/H denotes a set of representatives of left cosets of H in G.

Lemma A.13 ([Wei94, Theorem 6.7.8])**.** *Any automorphism $G \to G$ that is given by conjugation with some $g \in G$ induces the identity on group homology.*

Lemma A.14. *Let G be a group, and let X be a set with an action $G \curvearrowright X$. Let $g_1, g, g' \in G$, and let v, w be subsets of X. If $g_1 v = w$, then* $\operatorname{stab}(w) = g_1 \operatorname{stab}(v) g_1^{-1}$*, and if $gv = g'v = w$, then the conjugations c_g and $c_{g'}$ induce the same map $H_*(\operatorname{stab}(v)) \to H_*(\operatorname{stab}(w))$ on group homology.*

Proof. Assume that $g_1 v = w$, and let $x \in \operatorname{stab}(w)$. We want to show that $\operatorname{stab}(w) = g_1 \operatorname{stab}(v) g_1^{-1}$, so we need to show that $g_1^{-1} x g_1 \in \operatorname{stab}(v)$. This follows since

$$g_1^{-1} x g_1 v = g_1^{-1} x w = g_1^{-1} w = g_1^{-1} g_1 v = v.$$

Now let $y \in \mathrm{stab}(v)$. We want to show that $g_1 y g_1^{-1} \in \mathrm{stab}(w)$. This follows since

$$g_1 y g_1^{-1} w = g_1 y g_1^{-1} g_1 v = g_1 y v = g_1 v = w.$$

For the second part, assume that $gv = g'v = w$. We want to show that the homomorphism $c_g c_{g'^{-1}} \colon \mathrm{stab}(w) \to \mathrm{stab}(w)$ induces the identity on homology, since then

$$c_g = c_g g_{g'^{-1}} c_{g'}$$

induces the same map as $c_{g'}$ on homology. But this follows since

$$gg'^{-1} w = gg'^{-1} g' v = gv = w,$$

so $gg'^{-1} \in \mathrm{stab}(w)$, and thus conjugation by gg'^{-1} induces the identity by Lemma A.13.
$\square$

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