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# Cyclotomic Spectra and Topological Cyclic Homology of $\mathbb{E}_\infty$-Ring Spectra

Master Thesis

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# Contents

# Abstract

This thesis introduces $\mathbb{E}_\infty$-ring spectra and cyclotomic spectra and their invariants topological Hochschild homology and topological cyclic homology respectively. The thesis is divided into four chapters: In the first we introduce the necessary $\infty$-categorical prerequisites. In the second we introduce the theory of symmetric monoidal $\infty$-categories, a $K$-theory machine analogous to Segals infinite loop space machine, and lastly topological Hochschild homology. In the third part we introduce the Tate construction and the Tate diagonal. In the last part we introduce different versions of cyclotomic spectra, and show in which cases these coincide.

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# Introduction

$K$-theory started as an algebraic invariant; as the Grothendieck group of isomorphism classes of finitely generated projective $R$-modules. This has since been ported to the realm of topology in the form of a spectrum, first by D. Quillen [41] using the plus-construction. It is an extremely delicate invariant which contains a lot of information about its input, be it ring or permutative category. The downside to having such a refined invariant as $K$-theory is that it is extremely hard to calculate, hence it became evident that approximations were needed. An important and classical approximation result is due to T. Goodwilie [1], which asserts that for $A$ a certain type of algebra there is a functorial trace map $\mathrm{tr} : K(A) \to HC^{-}(A)$. This trace map fits into the following cartesian square induced from a surjection $A \to B$ with nilpotent kernel,

$$\begin{array}{ccc} K(A) \otimes H\mathbb{Q} & \longrightarrow & HC^{-}(A \otimes \mathbb{Q}) \\ \downarrow{\scriptstyle \mathrm{tr}_{\mathbb{Q}}} & & \downarrow \\ K(B) \otimes H\mathbb{Q} & \longrightarrow & HC^{-}(B \otimes \mathbb{Q}). \end{array}$$

There is one immediate downside with this approximation, besides the rationality: $K(-)$ is an topological object, while $HC^{-}(-)$ is an algebraic object. Hence we would like a more general result which is non-rational, and which is given in terms of a *topological version* of cyclic homology. The topological version of cyclic homology was given by M. Bökstedt, W.C Hsiang and I. Madsen [31], and is called *topological cyclic homology*, and is denoted $\mathrm{TC}^{gen}(-)$. The domain of this functor is the $\infty$-category of *genuine cyclotomic spectra*, or the $\infty$-category of $\mathbb{E}_\infty$-ring spectra. The main issue with this definition, is that it was not model-independent: it relied on the fact that any cyclotomic spectrum or $\mathbb{E}_\infty$-ring spectrum has a lift to an orthogonal spectrum. Despite of this issue, topological cyclic homology has been an immensely fruitful invariant: as an example the desired analog of the above result was proved by B.I Dundas, T. Goodwillie and R. McCarthy [1], using this definition of topological cyclic homology, and genuine cyclotomic spectra. These results are our motivation for developing the theory of topological cyclic homology, cyclotomic spectra and $\mathbb{E}_\infty$-rings in the best and most computable way. To this end we shall mainly be introducing theory and results from the article [38] of T. Nikolaus and P. Scholze.

The fundamental framework of these ideas are that of $(\infty,1)$-categories, and are as many other ideas of modern stable homotopy theory inspired by G. Segal in [46]. This thesis begins with a brisk overview of the most prominents results for the most popular model for $(\infty,1)$-categories namely quasi-categories (also called $\infty$-categories) introduced by R.M. Vogt [11], and most completely described by J. Lurie in [22], building on the work of A. Joyal. These are merely prerequisites, and as such are taken for granted and are only included for easier referencing, see chapter 1. After the $\infty$-categorical prerequisites, we begin translating Segal's ideas to $\infty$-categories. Again a comprehensive treatments has been made also by J. Lurie, see [25]. One insight again due to J. Lurie is how to port the notion of commutative algebra objects of a symmetric monoidal category to $\infty$-categories. The insight is that to give a symmetric monoidal structure on an $\infty$-category $\mathscr{C}$, it is enough to give a a simplicial set $\mathscr{C}^{\otimes}$ and coCartesian fibration $p : \mathscr{C}^{\otimes} \to N(\Gamma)$ satisfying the Segal condition, such that the fiber $p^{-1}(\langle 1 \rangle) \simeq \mathscr{C}$. We shall consider this insight in more detail in chapter 2. Furthermore to give a commutative algebra object in the symmetric monoidal structure $p$ on $\mathscr{C}$, it is enough to give a section of $p$ with some mild conditions. If one is familiar with the 1-categorical definition of symmetric monoidal categories and their algebra objects, it is rather surprising that it is possible to package all the commutative diagrams of this definition *and* the coherence data afforded by the $\infty$-categories in question in such a concise way. Furthermore in the 1-categorical setting there is often associated a significant amount of work to see that the symmetric monoidal structure is compatible with the model structure. The first part of chapter 2 is devoted to describing this theory, namely to give the definition of $\mathbb{E}_\infty$-rings. Next we shall consider a version of $K$-theory which we shall define is for symmetric monoidal $\infty$-categories, and will be analogous to the classical definition of $K_0$. As we shall see $K$-theory gives rise to $\mathbb{E}_\infty$-ring spectra, analogous to the classical $K$-theory which gives rings. We introduce a closely related invariant called *topological Hochschild homology*, $\mathrm{THH}(R)$, which we introduce for $R$ an $\mathbb{E}_\infty$-ring spectrum. We shall briefly see its relation to the classical Hochschild homology, where we will see one of the philosophical reasons for insisting on the $\infty$-categorical framework in arithmetic, namely the vanishing of denominators - which is interpreted as the process of counting over $\mathbb{S}$ somehow remembers the different ways of counting contrary to counting over $\mathbb{Z}$.

Once we have described the theory of $\mathbb{E}_\infty$-rings, their topological Hochschild homology and indicated how to give their $K$-theory, we consider a functor, called the *Tate-construction*, $(-)^{tG} : \mathsf{Sp}^{BG} \to \mathsf{Sp}$, where $G$ is a nice group. The meaning of nice will depend on the context. This functor is introduced in chapter 3. The Tate construction and the fact that $\mathsf{Sp}$ is the universal stable $\infty$-category freely generated by the sphere spectrum $\mathbb{S}$, gives rise to a unique lax symmetric monoidal transformation $\Delta_p : \mathrm{id}_{\mathsf{Sp}} \to T_p$ see proposition 3.4.8. Here $T_p$ is the Tate diagonal which is an exact endofunctor on the $\infty$-category $\mathsf{Sp}$. The first main theorem of this thesis is the following.

**Theorem 1.** *Let $X \in \mathsf{Sp}$ be a bounded below spectrum. Then the map*

$$\Delta_p : X \to (X \otimes ... X)^{tC_p}$$

*exhibits $(X \otimes ... \otimes X)^{tC_p}$ as the p-completion of $X$.*

This is theorem 3.4.5. This result is a generalization of the Segal conjecture, as explained in example 3.3.6. Using the transformation $\Delta_p : \mathrm{id}_{\mathsf{Sp}} \to T_p$ it is possible to endow the topological Hochschild homology $\mathrm{THH}(R)$, for $R$ an $\mathbb{E}_\infty$-ring spectrum, with a cyclotomic structure, i.e. $\mathrm{THH}(R)^{tC_p}$ has the $\mathbb{T}/C_p \cong \mathscr{C}_p$-action, see proposition 4.2.9. In this chapter we also show the most fundamental technical lemma of the thesis namely the *Tate orbit lemma*, see theorem 3.3.4. This theorem does only hold for bounded below spectra as seen in example 3.3.8. This is roughly due to the non-availability of the main proof strategy of the thesis, namely utlizing the Eilenberg-MacLane spectrum version of the fundamental theorem of finitely generated abelian groups. This is essentially utilizing that the Postnikov-tower of bounded below spectra is far more well-behaved than their unbounded counterparts. This strategy is as follows: first show the desired result for the Eilenberg-MacLane spectrum $H\mathbb{F}_p$, then for $HM$ for $M$ a finitely generated abelian group, and then to reduce from bounded below spectra to $HM$. This is also the proof technique used to proof theorem 3.4.5. This is to a large extend why, all the main theorems of this thesis are only for bounded below spectra.

Once we have discussed the Tate construction, the Tate diagonal, and the Tate-lemmas, we begin the construction of the $\infty$-categories of cyclotomic spectra. We begin with the construction of the $\infty$-category of *naive cyclotomic spectra*, in which objects consist of a spectrum $X$ with a $\mathbb{T}$-action together with $\mathbb{T}$-equivariant maps $\varphi_p : X \to X^{tC_p}$ for all primes. This approach is due to T. Nikolaus and P. Scholze [38], which is the main source of this thesis. The construction of this $\infty$-category relies heavily on the notion of an lax equalizer, which we introduce in chapter 4 section 1. Analogous to genuine cyclotomic spectra, there is a version of topological cyclic homology of naive cyclotomic spectra. We introduce this version in chapter 4 section 2. Utilizing the cyclotomic structure of $\mathrm{THH}(R)$, for $R$ an $\mathbb{E}_\infty$-ring spectrum, we define $\mathrm{TC}(R) = \mathrm{TC}(\mathrm{THH}(R))$. It is possible to calculate $\mathrm{TC}(R)$ through an equalizer diagram, which for a bounded below $\mathbb{E}_\infty$-ring spectrum, takes the following especially aesthetically pleasing form,

$$\mathrm{TC}(R) \longrightarrow \mathrm{TC}^-(R) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(R)^\wedge_p,$$

see corollary 4.2.15, where $\mathrm{TP}(R)$ and $\mathrm{TC}^{-}(R)$ are related invariants. The form presented above is the $p$-typical version, there is also a "global version". That we have this formula is of computational importance.

Next we construct the $\infty$-category of *genuine cyclotomic spectra*, which objects consist roughly of a spectrum $X$ with a continuous $\mathbb{T}$-action together with homotopy-coherently compatible equivalences $X \simeq \Phi^{C_n} X$ for all $n \geq 1$. This is the classical approach to topological cyclic homology, though we present it $\infty$-categorical rather than model categorical. Here $\Phi^{C_n}$ is the *geometric fixed point functor*, which is one of three fixed point functors which we shall need in the description of these objects. Genuine cyclotomic spectra are constructed through orthogonal spectra, and as such we shall need many more or less classical results from equivariant homotopy theory. One of the most important notions from equivariant homotopy theory for us, is that of a Borel complete spectrum, i.e. one for which genuine fixed points and homotopy fixed points coincide. Furthermore we introduce the notion of topological cyclic homology of genuine cyclotomic spectra.

The first main theorem of this thesis and the formula above was for bounded below spectra and the theme that we can show remarkable relations once we assume our spectra to be bounded below continues. Once we have described these two different versions of cyclotomic spectra and their topological cyclic homology, we show that for bounded below genuine cyclotomic spectra the two formulas for topological cyclic homology coincide, i.e. the following theorem, which is the global version.

**Theorem 2.** *Let $X$ be a genuine cyclotomic spectrum such that the underlying spectrum is bounded below. There is is a canonical equalizer diagram*

$$\mathrm{TC}^{gen}(X) \longrightarrow X^{h\mathbb{T}} \underset{can}{\overset{\prod_{p\in\mathbb{P}}(\varphi_p^{h\mathbb{T}})}{\rightrightarrows}} \prod_{p\in\mathbb{P}}(X^{tC_p})^{h\mathbb{T}}.$$

*Hence we obtain an equivalence* $\mathrm{TC}^{gen}(X) \simeq \mathrm{TC}(X)$.

Here Borel completion and the Tate orbit lemma are key components in the comparison.

Furthermore in chapter 4 sections 4 and 5 we realize the $\infty$-category of genuine cyclotomic spectra as an $\infty$-category of fixed points of the coalgebra induced from the geometric fixed point functor $\Phi^{C_p} : \mathbb{T}\mathsf{Sp}_{\mathcal{F}} \to \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$. The theory of coalgebras of endofunctors and their fixed points are introduced in chapter 4 section 1. Through this recasting it is possible to show that there is an equivalence of $\infty$-categories of the full subcategories spanned by bounded below spectra, $\mathsf{CycSp}_{+}^{gen} \to \mathsf{CycSp}_{+}$. I.e. the following theorem, which is global version.

**Theorem 3.** *There exists a functor $\mathsf{CycSp}^{gen} \to \mathsf{CycSp}$ which induces an equivalence of $\infty$-categories of bounded below naive and genuine cyclotomic spectra, $\mathsf{CycSp}_{+}^{gen} \to \mathsf{CycSp}_{+}$.*

This is the last main theorem of the thesis, see theorem 4.6.11. We construct the functor $\mathsf{CycSp}^{gen} \to \mathsf{CycSp}$ explicitly.

# 1 $\infty$-Categorical Prerequisites

The entirety of this thesis is formulated in the language of $\infty$-*categories*. This is a setting for doing abstract homotopy theory introduced by by R.M. Vogt in [11], and further developed by J. Lurie and A. Joyal. We shall primarily use results contained in the books Higher Topos Theory and Higher Algebra both by J. Lurie these are [22] and [25] respectively. We shall use many results and notions concerning these, these are not necesarily the most important results from the theory, but those we shall use. This chapter should be seen as prerequisites, and as a warm-up to the actual content which begins in the next chapter. We shall assume the reader to be familiar with these results, and include them for completeness.

## 1.1 $\infty$-Categories

We begin with a definition of $\infty$-categories.

**Definition 1.1.1.** An $\infty$-*category* is a simplicial set $S$, which has all inner horn fillers. More explicitly, any map $f_0 : \Lambda^n_i \to S$ for $0 < i < n$ extends to an $n$-simplex $f : \Delta^n \to S$.

We view $\infty$-categories as a generalization of the theory of categories, by identifying $\mathscr{C}$ with its nerve $N(\mathscr{C})$, and using the following lemma.

**Lemma 1.1.2.** *The nerve of any category is an* $\infty$*-category.*

Recall that we have the Quillen-Kan model structure on $\mathsf{sSet}_*$, which yields a Quillen-equivalence of this model category, with the category of pointed compactly generated weak Hausdorff topological spaces equipped with the classical model structure. The fibrant objects of the Quillen-Kan model structure on $\mathsf{sSet}$ yields a class of important examples of $\infty$-categories.

**Example 1.1.3.** Any Kan complex is an $\infty$-category. In particular the singular simplicial complex of $X$, $\mathrm{Sing}(X)$, is an $\infty$-category, for every $X \in \mathsf{Top}_*$, where $\mathsf{Top}_*$ denotes the category of pointed compactly generated weak Hausdorff topological spaces.

In ordinary category theory we have that $\mathrm{Hom}(C, D) \in \mathsf{Set}$ for every $C$ and $D$ in a category $\mathscr{C}$, in some sense we think of every category as being enriched over $\mathsf{Set}$. In $\infty$-categories $\mathsf{Set}$ is replaced by the $\infty$-category of spaces.

**Definition 1.1.4.** *The* $\infty$*-category of spaces*, denoted $\mathcal{S}$ is defined as $N_\Delta(\mathsf{sSet}_{fc})$, where $\mathsf{sSet}_{fc}$ is the collection of (co)fibrant objects of $\mathsf{sSet}$ equipped with the Quillen-Kan model structure. From this point onwards we shall refer to $\mathsf{sSet}_{fc}$ as $\mathsf{Kan}$, because the (co)fibrant objects are the Kan complexes.

The enrichment over $\mathcal{S}$ for every $\infty$-category is the following theorem(which is contained in [22] chapter 1.), and the Joyal model structure we shall define shortly.

N infty cat **Theorem 1.1.5.** *Let* $\mathscr{C}$ *be a simplicial category for which the simplicial set* $\mathrm{Map}_{\mathscr{C}}(X, Y)$ *is a Kan complex for every pair of objects* $X, Y \in \mathscr{C}$. *In this case* $N(\mathscr{C})$ *is an* $\infty$*-category.*

Due to theorem 1.1.5 we define the $\infty$-category of $\infty$-categories as follows.

**Definition 1.1.6.** Let $\mathsf{cat}_\infty$ be the simplicial category which objects are $\infty$-categories, and which mapping spaces $\mathrm{Map}(\mathscr{C}, \mathscr{D})$ for $\mathscr{C}, \mathscr{D} \in \mathsf{cat}_\infty$ are the largest Kan complexes inside $\mathrm{Hom}(\mathscr{C}, \mathscr{D})$ as a simplicial category. We then define *the* $\infty$*-category of* $\infty$*-categories* as $\mathsf{Cat}_\infty := N_\Delta(\mathsf{cat}_\infty)$.

Recall that there is yet another model structure on $\mathsf{sSet}$, this is called the Joyal model structure which we will describe now:

**Theorem 1.1.7.** *There exists a left proper combinatorial model structure on* sSet *with the following properties:*

1. *A map $p : S \to S'$ of simplicial sets is cofibration, if it is a monomorphism.*
2. *A map $p : S \to S'$ is a categorial equivalence if the induced simplicial functor $\mathfrak{C}[S] \to \mathfrak{C}[S']$ is an equivalence of simplicial categories.*

*Moreover the adjoint functors $\mathfrak{C}$ and $N_\Delta$ functor (For a definition of these see chapter 1 of [22]), determine a Quillen equivalence between* sSet *and* $\mathsf{Cat}_\Delta$*, where* $\mathsf{Cat}_\Delta$ *are simplicially enriched categories.*

The collection of fibrant objects of the Joyal model structure are exactly the collection of $\infty$-categories.

fibrants **Proposition 1.1.8.** *Let* sSet *be the category of simplicial sets equipped with the Joyal model structure, then the fibrant objects are exactly the $\infty$-categories.*

As we can see above many of the notions in the theory of $\infty$-categories somehow related to model category theory, this is no coincidence. That this is so is the content of the following two results, which are A.3.7.6 and A.3.7.7 of [22].

A.3.7.6 **Proposition 1.1.9.** *The $\infty$-category $\mathscr{C}$ is presentable if and only if there exists a combinatorial simplicial model category $\mathscr{D}$ and an equivalence $\mathscr{C} \simeq N_\Delta(\mathscr{D}_{fc})$.*

A.3.7.7 **Proposition 1.1.10.** *Let $\mathscr{C}$ and $\mathscr{D}$ be combinatorial simplicial model categories. Then the underlying $\infty$-categories $N_\Delta(\mathscr{C}_{fc})$ and $N_\Delta(\mathscr{D}_{fc})$ are equivalent if and only if $\mathscr{C}$ and $\mathscr{D}$ can be joined by a chain of simplicial Quillen equivalences.*

Using the Joyal model structure it is possible to prove the $\infty$-categorical analog of the Grothendieck construction, called the Straightening/Unstraightening equivalence. To state this we need the notion of a coCartesian fibration, and their 1-categorical analog. These will play an important role throughout the thesis, which is because we shall need them for the definition of a symmetric monoidal $\infty$-category.

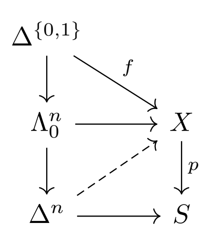
**Definition 1.1.11.** Let $p : X \to S$ be a map of simplicials sets. The map $p$ is an *inner fibration* if it has the right lifting property with respect to all inner horns. The map $p$ is a *categorical fibration* if it has the right lifting property with respect to all cofibrant categorical equivalences.

**Definition 1.1.12.** Let $p : X \to S$ be an inner fibration. An edge $f : \Delta^1 \to X$, say $x \to y$ is *p-coCartesian* if the induced map

$$X_{x/} \to X_{y/} \times_{S_{p(y)/}} S_{p(x)/}$$

is a trivial Kan fibration. Here $X_{x/}$ denotes the under-category.

*Remark* 1.1.13. An alternative, and perhaps a bit more tangeble definition is the following. The edge $f : \Delta^1 \to X$ is $p$-coCartesian if for all $n \geq 2$ there exists a dotted arrow in the following diagram rendering it commutative,




**Definition 1.1.14.** Let $p : X \to S$ be a map of simplicial sets. The map $p$ is a *coCartesian fibration* if it is an inner fibration and if there for all edges $f : x \to y \in S$ and all $\overline{x} \in X$ such that $p(\overline{x}) = x$ exists a $p$-coCartesian edge $\overline{f} : \overline{x} \to \overline{y}$ such that $p(\overline{y}) = y$.

$$\begin{CD} \overline{x} @>\overline{f}>> \overline{y} \\ @VpVV @VVpV \\ x @>f>> y \end{CD}$$

**Example 1.1.15.** Let $X \times S \to S$ be the projection associated to a cartesian product of simplicial sets. This is a coCartesian fibration.

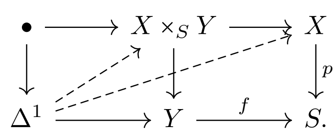
We begin with a few regularity results concerning coCartesian fibrations. Here both of the definitions will come in handy.

coCartbase **Lemma 1.1.16.** *Consider the following pullback diagram of simplicial sets,*

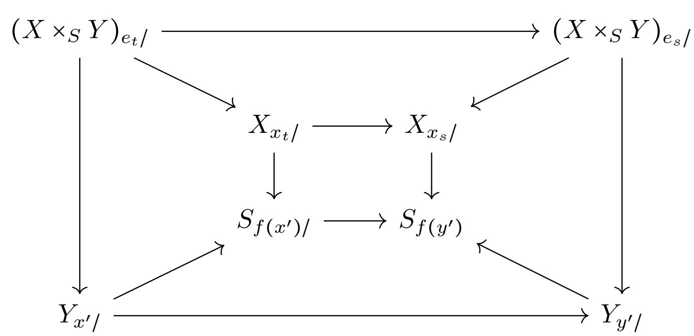
$$\begin{CD} X \times_S Y @>>> X \\ @Vp'VV @VVpV \\ Y @>f>> S. \end{CD}$$

*If $p$ is a coCartesian fibration, then so is $p'$, i.e. the collection of coCartesian fibrations is closed under base change.*

*Proof.* We begin by fixing some of notation. Consider the following diagram




The lower extension $\Delta^1 \to X$ exists by the alternative definition of $p$-coCartesian map, let this extension be denoted $x_s \to x_t$. Let the object which $\bullet \to X \times_S Y$ picks out be denoted $e_s$. Because the right hand square is a pullback the existence of the lower extension $\Delta^1 \to X$ implies the existence of the upper extension $\Delta^1 \to X \times_S Y$. Let the extension $\Delta^1 \to X \times_S Y$ be denoted $e_s \to e_t$. Lastly let the edge $\Delta^1 \to Y$ be denoted $x' \to y'$. We must show that the upper extension $\Delta^1 \to X \times_S Y$ is a $p$-coCartesian edge. Consider the following canonical diagram of simplicial sets



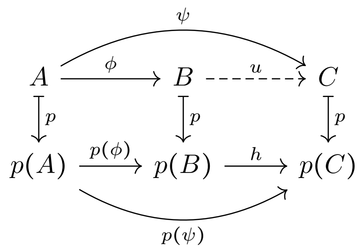
The lower extension $\Delta^1 \to X$ is a $p$-coCartesian morphism hence per. definition the inner square is a pullback square, which implies that the left and right squares are pullbacks. From this is follows that the outer square is a pullback, which implies that $\Delta^1 \to X \times_S Y$ is a $p$-coCartesian edge. ☐

We have the following slight variant of $p$-coCartesian edges.

**Definition 1.1.17.** Let $p : X \to S$ is an inner fibration. We say that an inner fibration is *locally $p$-coCartesian* if for every edge $\Delta^1 \to S$ the pullback $X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration.

Before we state the straightening/unstraightening equivalence, we introduce the 1-categorical versions of $p$-coCartesian maps and coCartesian fibrations, which we shall need later. We define them explicitly, even though it is a special case of the above definitions via the embedding of categories into $\infty$-categories.

**Definition 1.1.18.** Let $p : \mathscr{C} \to \mathscr{D}$ be a functor. An arrow $\phi : A \to B \in \mathscr{C}$ is called *Grothendieck $p$-coCartesian* if for any arrow $\psi : A \to C \in \mathscr{C}$ and any arrow $h : p(B) \to p(C) \in \mathscr{D}$ with $p(\phi) \circ h = p(\psi)$, there exists a unique arrow $u : B \to C \in \mathscr{C}$ such that $p(u) = h$ and $\phi \circ u = \psi$. Diagrammatically




**Definition 1.1.19.** Let $p : \mathscr{C} \to \mathscr{D}$ be a functor. The functor $p$ is called a *Grothendieck opfibration* if for all $x \in \mathscr{C}$ and $q : p(x) \to y$ in $\mathscr{D}$ there exists a $p$-coCartesian map $\overline{q} : x \to \overline{y}$ such that $p(\overline{q}) = q$.

For a proof of the following see remark 2.4.2.2 of [22].

opfib **Lemma 1.1.20.** *Let $p : \mathscr{C} \to \mathscr{D}$ be a Grothendieck opfibration, then $N(p) : N(\mathscr{C}) \to N(\mathscr{D})$ is a coCartesian fibration.*

We now state the Straightening/Unstraightening equivalence [22] Theorem 3.2.0.1, which we state in the version we shall need.

Straight **Theorem 1.1.21.** *Let $\mathscr{C}$ be an $\infty$-category. There is an equivalence of $\infty$-categories*

$$St : (\mathsf{Cat}_\infty)^{coCart}_{/\mathscr{C}} \to \mathsf{Cat}_\infty^{\mathscr{C}},$$
$$Un : \mathsf{Cat}_\infty^{\mathscr{C}} \to (\mathsf{Cat}_\infty)^{coCart}_{/\mathscr{C}}.$$

*The $\infty$-category $(\mathsf{Cat}_\infty)^{coCart}_{/\mathscr{C}}$ is the full subcategory of $(\mathsf{Cat}_\infty)_{/\mathscr{C}}$ spanned by the coCartesian fibrations over $\mathscr{C}$. Here $\mathsf{Cat}_\infty^{/\mathscr{C}}$ denotes the $\infty$-category of functors from $\mathscr{C}$ into $\mathsf{Cat}_\infty$.*

This theorem is extremely non-trivial, and equally useful, since it lets us (analogous to the Grothendieck construction) construct $\mathsf{Cat}_\infty$ valued functors, by constructing coCartesian fibrations over a $\infty$-category (in fact more general over simplicial sets).

We end this section with an perhaps equally important result. Analogous to ordinary category theory there is a notion of the Yoneda Lemma in the theory of $\infty$-categories.

introCons **Construction 1.1.22.** Let $K$ be a simplicial set, and set $\mathscr{C} = \mathfrak{C}[K]$. By construction $\mathscr{C}$ is a simplicial category, so $(X, Y) \mapsto \mathrm{Sing}\,|\mathrm{Hom}_{\mathscr{C}}(X, Y)|$ determines a simplicial functor $\mathscr{C}^{op} \times \mathscr{C} \to \mathsf{Kan}$. There exists a natural map $\mathfrak{C}[K^{op} \times K] \to \mathscr{C}^{op} \times \mathscr{C}$, composing these two maps we obtain a simplicial functor

$$\mathfrak{C}[K^{op} \times K] \to \mathsf{Kan}$$

Using the adjunction $(\mathfrak{C}, N)$, which holds by construction, we get a map of simplicial sets $K^{op} \times K \to N(\mathsf{Kan})$, which by the adjunction $(\mathsf{Fun}, \times)$ in $\mathsf{sSet}$, can be identified with

$$j : K \to \mathsf{Fun}(K^{op}, \mathcal{S}) \coloneqq \mathcal{P}(K).$$

We shall refer to $j$ as the *Yoneda embedding.*

The map $K^{op} \times K \to \mathcal{S}$ will be used later in the thesis. We now state the Yoneda lemma.

Yoneda **Proposition 1.1.23.** *Let $K$ be a simplicial set. Then the Yoneda embedding $j : K \to \mathcal{P}(K)$ is fully faithful.*

## 1.2 Adjoint functors

We begin this section with the following discussion on "structure vs. property", which is loosely based on talks given by Elden Elmanto and Tobias Bartel at "Topics in algebraic topology"-seminar at Copenhagen university, November 2018 and November 2017 respectively. As far as the author knows no written material from these talks exist. A mathematical object is usually defined by specifying an object in an $\infty$-category, equipped with *structure*, satisfying *properties*. We will throughout this thesis to use these terms in a more precise way, hence we shall define them. It will become apparent what this discussion has to do with adjoint functors soon.

**Definition 1.2.1.** Let $\mathscr{C}$ and $\mathscr{D}$ be (co)complete $\infty$-categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The functor $F$ is *(co)continuous* if it preserve (co)limits.

**Definition 1.2.2.** Let $\mathscr{C}$ and $\mathscr{D}$ be (co)complete. We shall say that an object of an $\infty$-category $C \in \mathscr{C}$ *has a property* $D$ if it belongs to a full subcategory $\mathscr{C}_0 \subseteq \mathscr{C}$ spanned by the objects satisfying the predicate $D$, i.e. there should exist a full and faithful functor $\mathscr{C}_0 \to \mathscr{C}$ such that $C$ is contained in the essential image. We shall say that an object $C \in \mathscr{C}$ *is a structure on* $D \in \mathscr{D}$, if there exists a continuous functor $S : \mathscr{C} \to \mathscr{D}$ such that there exists $S(C) \simeq D$, i.e. if $D$ is in the essential image of $C$ under the right adjoint functor $S$.

Hence when we write $Y$ has the structure of $X$, we are implicitely (or explicitly in some cases) *making a choice*, hence we are implicitly saying something not only about $Y$ but also about our choice. Often these choices are of no concern, because they give equivalent objects, but nonetheless we wish to emphasize that there is choice. Analogously when we say $Y$ has the property of $X$, we are saying something substantial about $Y$ and not $X$.

**Example 1.2.3.** Let $A \in \mathsf{Ab}$. We may define $A$ as the underlying set $A_0$, equipped with a group structure $A \times A \to A$, satisfying the equations $ab = ba$ for all $a, b \in A$. Here $A_0$ has the structure of $A$, because $F(A) = A_0$, under the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$, which is a right adjoint. Furthermore $\mathsf{Ab} \subseteq \mathsf{Grp}$ is a full subcategory, and hence being abelian is a property. Intuitively properties are intrinsic to the object, in some sense they are truth-values: either $ab = ba$ for all $a, b \in A$ is satisfied or not. Structures can be chosen from a set of possible different structures: there are many different group structures on a given set $A_0$.

Note that structure is given through a continuous functor, and such are, under certain regularity assumptions, the same as a right adjoint functor. This is the adjoint functor theorem, which is the result we use (by far) the most in this thesis.

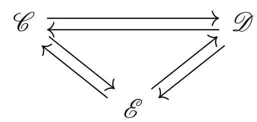
int functor theorem **Theorem 1.2.4.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable $\infty$-categories.*

*(1) The functor $F$ has a right adjoint if and only if it preserves colimits*

*(2) The functor $F$ has a left adjoint if and only if it is accessible and preserve limits. A functor is called accessible if its domain and codomain has filtered colimits and a set of compact objects generating them under colimits, and it preserve filtered colimits.*

We shall record a few important notions and lemmas relating to adjoint functors. We begin with a kind of 2-out-of-3-property for adjoint functors.

IncLeftAd **Lemma 1.2.5.** *Suppose we have inclusions $\mathscr{C} \subseteq \mathscr{D} \subseteq \mathscr{E}$ of $\infty$-categories, and both inclusions $\mathscr{C} \to \mathscr{E}$ and $\mathscr{D} \to \mathscr{E}$ are right adjoints. Then the inclusion $\mathscr{C} \to \mathscr{D}$ is a right adjoint, and we have the following commutative diagram, of adjunctions*



Note that if $\mathscr{C} \to \mathscr{E}$ and $\mathscr{D} \to \mathscr{E}$ have a certain property, then the right adjoint which arises in this fashion, must have the same property. We now turn our attention to the following special kind of adjoint functor which will be pervasive throughout the thesis.

**Definition 1.2.6.** Let $L : \mathscr{C} \to \mathscr{D}$ be a functor. Then $L$ is called a *localization* if it admits a right adjoint which is fully faithful.

Fully faithful adjoint functors can be described in the following way, which is proposition 5.2.7.4 of [22].

BorelLem **Lemma 1.2.7.** *Let $\mathscr{C}$ be an $\infty$-category. Let $L : \mathscr{C} \to \mathscr{C}$ be a functor with essential image $L\mathscr{C} \subseteq \mathscr{C}$. Then the following are equivalent,*

*(1) $L$ is a localization.*

*(2) When regarded as a functor $\mathscr{C} \to L\mathscr{C}$, $L$ is left adjoint to the inclusion $L\mathscr{C} \to \mathscr{C}$.*

*(3) There exists a natural transformation $\alpha : \mathscr{C} \times \Delta^1 \to \mathscr{C}$ from $\mathrm{id}_{\mathscr{C}} \to L$ such that, for every object $C \in \mathscr{C}$, the morphisms $L(\alpha(C)), \alpha(LC) : LC \to LLC$ of $\mathscr{C}$ are equivalences.*

Localizations which are conservative are in fact equivalences.

conservativelem **Lemma 1.2.8.** *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a conservative functor which admits a fully faithful right adjoint, then $F$ is an equivalence.*

*Proof.* Let $F_R$ be the right adjoint. Since $F_R$ is fully faithful, the counit $\varepsilon : F_R F \to \mathrm{id}_C$ is an equivalence. We show that the unit $\eta : \mathrm{id}_{\mathscr{D}} \to F F_R$ is also an equivalence. We have the following composite

$$F(c) \xrightarrow{F(\eta_{F(c)})} F F_R F(c) \xrightarrow{\varepsilon_{F(c)}} F(c)$$

is equivalent to $\mathrm{id}_{F(c)}$ for all $c \in \mathscr{C}$. This implies that $F(\eta_{F(c)})$ is an equivalence for all $c \in \mathscr{C}$, hence by conservativity of $F$ we have that $\eta_{F(c)}$ is an equivalence for all $c \in \mathscr{C}$. □

*Remark* 1.2.9. The dual statement, namely that $F$ admits a fully faithful left adjoint, is true and is proved with a precisely dual proof.

## 1.3 Stable ∞-Categories and Sp

In this thesis we are interested in algebraic invariants for studying topological problems. The right framework for doing algebra is the stable ∞-category of spectra Sp, we will try to justify this, while also stating a few results concerning general stable ∞-categories. All of these results are contained in the book Higher Algebra by J. Lurie [25]. We shall take the definition of a stable ∞-category for granted.

**Definition 1.3.1.** Let a functor $F : \mathscr{C} \to \mathscr{D}$ be a functor between stable ∞-categories. The functor is *exact* if it preserves fiber sequences.

*Remark* 1.3.2. A functor between stable ∞-categories is exact if and only if it preserves finite limits or colimits. This is proposition 1.1.4.1 of [25].

We now define the most important ∞-category of this thesis, namely the ∞-category of spectra. To make sense of this ∞-category we need the following preliminary definition.

**Definition 1.3.3.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between $\infty$-categories. If $\mathscr{C}$ admits pushouts, then we say that $F$ is *excisive* if $F$ carries pushout squares in $\mathscr{C}$ to pullback squares $\mathscr{D}$. The full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by the collection of excisive functors we denote by $\mathsf{Exc}(\mathscr{C},\mathscr{D})$. If $\mathscr{C}$ admits a final object $\bullet$, we will say that $F$ is *reduced* if $F(\bullet)$ is a final object of $\mathscr{D}$. The full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by the collection of reduced functors we denote by $\mathsf{Fun}_*(\mathscr{C},\mathscr{D})$. We denote $Exc_*(\mathscr{C},\mathscr{D}) \coloneqq \mathsf{Exc}(\mathscr{C},\mathscr{D}) \cap \mathsf{Fun}_*(\mathscr{C},\mathscr{D})$.

**Definition 1.3.4.** Let $\mathscr{C}$ be an $\infty$-category which admits finite limits. Then the $\infty$-category of spectrum objects, is defined and denoted as $\mathsf{Sp}(\mathscr{C}) \coloneqq \mathsf{Exc}_*(\mathcal{S}_*^{fin},\mathscr{C})$. We define $\mathsf{Sp} = \mathsf{Sp}(\mathcal{S})$.

Recall that we have a functor $\Sigma^\infty : \mathcal{S} \to \mathsf{Sp}$, and we obtain the *sphere spectrum*, denoted $\mathbb{S}$, as $\Sigma^\infty(S^0)$. The universal property of $\mathsf{Sp}$ says that $\mathsf{Sp}$ is the universal stable $\infty$-category which is freely generated under colimits by $\mathbb{S}$. This is the following theorem, which is corollary 1.4.4.6

UniSp **Theorem 1.3.5.** *Let $\mathscr{D}$ be a presetanble stable $\infty$-category. Then evaluation on the sphere spectrum induces an equivalence of $\infty$-categories,*

$$\theta : \mathsf{Fun}^{Lex}(\mathsf{Sp},\mathscr{D}) \to \mathscr{D}.$$

*Here $\mathsf{Fun}^{Lex}(\mathscr{C},\mathscr{D})$ is the full subcategory of $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ spanned by the collection of left adjoint functors.*

If this theorem is not enough to convince the reader that this is an important $\infty$-category, the following lemma should.

MapSpec **Lemma 1.3.6.** *Let $\mathscr{C}$ be a stable $\infty$-category. The functor $\mathrm{map}_{\mathscr{C}}(-,-) : \mathscr{C}^{op} \times \mathscr{C} \to \mathcal{S}$ lifts to a functor $\mathrm{Map}_{\mathscr{C}}(-,-) : \mathscr{C}^{op} \times \mathscr{C} \to \mathsf{Sp}$. We say that any stable $\infty$-category is enriched over $\mathsf{Sp}$.*

*Proof.* This follows from the universal property of the $\infty$-category of spectra, i.e. theorem 1.3.5 and [25] Corollary 1.4.2.23. The latter together with the Yoneda machinery described in the previous section, gives that for every $X \in \mathscr{C}^{\mathrm{op}}$, the left exact functor $\mathrm{Map}(X,-) : \mathscr{C} \to \mathcal{S}$ lifts uniquely to an exact functor $\mathrm{map}(X,-) : \mathscr{C} \to \mathsf{Sp}$. Considering $X$ as a variable, and using the universal property we obtain a functor

$$\mathscr{C}^{op} \to \mathsf{Fun}^{\mathrm{Lex}}(\mathscr{C},\mathcal{S}) \simeq \mathsf{Fun}^{\mathrm{Ex}}(\mathscr{C},\mathsf{Sp}).$$

This functor corresponds to a functor $\mathscr{C}^{op} \times \mathscr{C} \to \mathsf{Sp}$, through adjunction. □

The following is remark 1.4.2.25 of [25].

SpTower **Lemma 1.3.7.** *We may identify $\mathsf{Sp}(\mathscr{C})$ with the limit of the tower computed in the $\infty$-category of stable $\infty$-categories.*

$$\ldots \longrightarrow \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_*.$$

Lastly we relate the $\infty$-category of spectra to the 1-categorical notion of spectra. For a proof see 7.3.9 of [40].

SpToSp **Theorem 1.3.8.** *Let $\mathsf{Sp}^\Sigma$ be the category of symmetric spectra, and equip it with the stable model structure. For a definition of these see [43]. Then we have an equivalence $N_\Delta(\mathsf{Sp}^\Sigma_{fc}) \simeq \mathsf{Sp}$.*

This result also holds for the category of orthogonal spectra equipped with the stable model structure.

# 2 $\mathbb{E}_\infty$-Ring Spectra and their $K$-theory

In this chapter we shall give the $\infty$-categorical setting in which we shall define the $\infty$-categorical analog of algebra objects in a symmetric monoidal category. I.e. we shall introduce among other things symmetric monoidal $\infty$-categories, their algebra and module objects. The standard way to introduce these notions, which is also the most general, is through the theory of $\infty$-operads, where symmetric monoidal $\infty$-categories are a certain instances of $\mathscr{O}$-monoidal $\infty$-categories for $\mathscr{O}$ an $\infty$-operad, which is done in e.g. [25] chapter 2 and 3. We will diverge from this path to emphasize how these constructions are refinements of their classical counterparts. Hence one might think of the following as a brisk introduction to the theory, through instructive examples. We follow [20] section 3, [19] section 2, [21] chapter 1 and 4, [5], [24], and [25] chapter 5 and portions of chapter 7. We try to be as clear as possible about the relations to the classical theory as possible as we go along. The first two sections are loosely based on notes written for a talk given by the author at the "Topics in algebraic topology"-seminar at Copenhagen University, November 2018.

## 2.1 Symmetric Monoidal $\infty$-Categories

If we for a moment zoom-out and try to remember why we are doing all of this, we recall that we are trying to approximate $K$-theory. The version of $K$-theory that we shall be interested in, namely $K$-theory of symmetric monoidal $\infty$-categories is much inspired by G. Segals model for $K$-theory for permutative categories via his infinite loop space machinery [46]. This construction of $K$-theory relies heavily on the homotopy theory of (special) $\Gamma$-spaces, hence it is not surprising that we shall need Segals $\Gamma$-category.

**Definition 2.1.1.** For each $0 \le n$, we let $\langle n \rangle^\circ$ denote the set $\{1, 2, ..., n\}$ and $\langle n \rangle = \langle n \rangle^\circ_* := \{\bullet, 1, 2, ..., n\}$. In particular $\langle 0 \rangle = \{*\}$. We define a category $\Gamma$ as follows:

1. The objects of $\Gamma$ are the sets $\langle n \rangle$, where $0 \le n$, based at $\bullet$.
2. Given a pair of objects $\langle m \rangle$, $\langle n \rangle \in \Gamma$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ in $\Gamma$ is a basepoint preserving map of sets.

The following kind of morphisms of $\Gamma$ are extremely important.

**Definition 2.1.2.** We will call a morphism $f : \langle n \rangle \to \langle m \rangle$ in $\Gamma$ *inert* if, for each non-basepoint $i \in \langle n \rangle$, $f^{-1}\{i\}$ is a singleton.

rho **Example 2.1.3.** For every $n \le 0$, the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$ determined by $f^{-1}\{1\} = \{i\}$ is by definition inert. This map will play an important part later.

In the following we shall use the following notation. Let $p : K \to N(\Gamma)$ be a map of simplicial sets and let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $0 \le i \le n$ be the morphisms described in example 2.1.3. We denote the fibers $p^{-1}\{\langle n \rangle\} \subseteq K$ by $K_{\langle n \rangle}$. Assume further that $\rho^i$ for $0 \le i \le n$ are coCartesian edges, then note that the maps $\rho^i$ induce maps on the fibers:

$$\rho^i_! : K_{\langle n \rangle} \to K_{\langle 1 \rangle}.$$

**Definition 2.1.4.** A *symmetric monoidal $\infty$-category* is a simplicial set $\mathscr{C}^\otimes$ equipped with a coCartesian fibration $p : \mathscr{C}^\otimes \to N(\Gamma)$ such that the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $0 \le i \le n$, induce an equivalence of $\infty$-categories

$$\mathscr{C}^\otimes_{\langle n \rangle} \xrightarrow{(\rho^i_!)_{1 \le i \le n}} \prod_{1 \le i \le n} \mathscr{C}^\otimes_{\langle 1 \rangle}.$$

The last condition is often called *the Segal condtion.*

*Remark* 2.1.5. We could replace $\Gamma$ with $\Delta^{op}$, where $\Delta$ is the simplex category, in the above definition to obtain the notion of a *monoidal* $\infty$*-category.* Recall that there is a functor $\Delta^{op} \to \Gamma$, which takes $[n]$ to $\langle n \rangle$, thus every $\Gamma$-space has an underlying simplicial space. Utilizing this we see that all symmetric monoidal $\infty$-categories in particular are monoidal $\infty$-categories. We shall use these when describing the $\infty$-category of module objects over an $\mathbb{E}_\infty$-ring specrum, but in the following section we will mainly concern ourselves with symmetric monoidal $\infty$-categories, but all notions have a noncommutative analog, which will leave out. See [19] Section 1 for a discussion hereof.

*Remark* 2.1.6. Note that the fibers of $p : \mathscr{C}^{\otimes} \to N(\Gamma)$ are $\infty$-categories, because $p$ in particular is an inner fibration. In light of this we shall refer to the fiber $\mathscr{C} := \mathscr{C}^{\otimes}_{\langle 1 \rangle}$ as the underlying $\infty$-category of the symmetric monoidal $\infty$-category. Note that being a symmetric monoidal $\infty$-category is structure on the $\infty$-category. We shall often abuse terminology and say that $\mathscr{C}$ *is a symmetric monoidal* $\infty$*-category.*

The definition of a symmetric monoidal $\infty$-category is in direct analogy to the definition of $\Gamma$-categories(e.g. construction 7.28 of [43]). The coCartesian fibration condition in the definition of symmetric monoidal $\infty$-category encapsulates not only the data of the commutative diagrams in the definition of $\Gamma$-categories, but also higher coherence data.

**Definition 2.1.7.** Per. definition $\mathscr{C}^{\otimes}_{\langle 0 \rangle} \simeq (\mathscr{C}^{\otimes}_{\langle 1 \rangle})^0 \simeq \bullet$. Consider the unique map $\langle 0 \rangle \to \langle 1 \rangle$, it induces a functor $\mathscr{C}^{\otimes}_{\langle 0 \rangle} \to \mathscr{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathscr{C}$ which we identify with an object $1 \in \mathscr{C}$, which we call *the unit.* Consider the map $f : \langle 2 \rangle \to \langle 1 \rangle$. Since the maps $(\rho^i_!)_{i=\{1,2\}} : \mathscr{C} : \langle 2 \rangle^{\otimes} \to \mathscr{C}^{\otimes}_{\langle 1 \rangle} \times \mathscr{C}^{\otimes}_{\langle 1 \rangle}$ are equivalences, the map $f$ induces a functor

$$\begin{array}{ccc} \mathscr{C}^{\otimes}_{\langle 2 \rangle} & \xrightarrow{f_!} & \mathscr{C}^{\otimes}_{\langle 1 \rangle} \\ {\scriptstyle (\rho^i_!)_{i=\{1,2\}}}\downarrow & \nearrow{\scriptstyle \otimes} & \\ \mathscr{C}^{\otimes}_{\langle 1 \rangle} \times \mathscr{C}^{\otimes}_{\langle 1 \rangle} & & \end{array}$$

i.e. a functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$, which we denote $\otimes$ and refer to as a *tensor product.*

Note that there is associated a choice to the definition of the tensor product, since the image of the induced map of $\rho^1$ can be choosen to be the left hand factor of $\mathscr{C}^{\otimes}_{\langle 1 \rangle} \times \mathscr{C}^{\otimes}_{\langle 1 \rangle}$ or the right hand factor. These two choices are homotopic, so the choice is not significant, but nonetheless from this point onwards we shall assume that this choice has been made.

**Definition 2.1.8.** Let $p : \mathscr{C}^{\otimes} \to N(\Gamma)$ and $q : \mathscr{D}^{\otimes} \to N(\Gamma)$ be symmetric monoidal $\infty$-categories. Consider the map of simplicial sets $f : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$, and the following conditions.

(1) The diagram

$$\begin{array}{ccc} \mathscr{C}^{\otimes} & \xrightarrow{f} & \mathscr{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & N(\Gamma) & \end{array}$$

commutes.

(2) The map $f$ carries $p$-coCartesian maps to $q$-coCartesian maps.

(3) If $t$ is a $p$-coCartesian edge in $\mathscr{C}^{\otimes}$ such that $p(t)$ in $N(\Gamma)$ is inert, then $f(t)$ is a $q$-coCartesian edge and $q(f(t))$ is inert.

The map of simplicial sets $f$ is a *symmetric monoidal functor* if it satisfies condition (1) and (2). The full subcategory of $\mathsf{Fun}_{N(\Gamma)}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})$ spanned by the functors having these properties, we denote by $\mathsf{Fun}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})$. The map of simplicial sets $f$ is a *lax symmetric monoidal functor* if it satisfies condition (1) and (3). The full subcategory of $\mathsf{Fun}_{N(\Gamma)}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})$ spanned by these is denoted $\mathsf{Fun}_{\mathrm{lax}}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})$.

**Lemma 2.1.9.** *The collection of (lax) symmetric monoidal functors are closed under composition.*

*Proof.* Consider the (lax) symmetric monoidal functors $f : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ and $g : \mathscr{D}^{\otimes} \to \mathscr{E}^{\otimes}$. These fit into the following diagram

$$\begin{array}{ccccc} \mathscr{C}^{\otimes} & \xrightarrow{f} & \mathscr{D}^{\otimes} & \xrightarrow{g} & \mathscr{E}^{\otimes} \\ & \searrow^{p} & \downarrow{\scriptstyle q} & \swarrow_{r} & \\ & & N(\Gamma). & & \end{array}$$

This diagram obviously commutes per. assumption on the two triangle-diagrams of which it is composed. Hence the first property is preserved.

Note that $g \circ f$ takes $p$-coCartesian edges to $r$-coCartesian edges, because $f$ takes $p$-coCartesian edges to $q$-coCartesian edges and then $g$ takes $q$-coCartesian edges to $r$-coCartesian edges per. assumption. Hence the second property is preserved if it was assumed to hold for $f$ and $g$ to begin with.

Let $t$ be a $p$-coCartesian edge in $\mathscr{C}^{\otimes}$ such that $p(t)$ in $N(\Gamma)$ is inert, then $f(t)$ is a $q$-coCartesian edge and $q(f(t))$ is inert, furthermore $g(f(t))$ is a $r$-coCartesian edge and $r(g(f(t)))$ is inert. Hence the third property is preserved. ☐

*Remark* 2.1.10. We shall often abuse terminology and say that a functor $f : \mathscr{C} \to \mathscr{D}$ of $\infty$-categories is (lax) symmetric monoidal, when we in fact mean that it is the underlying map of fibers $f_{\langle 1 \rangle} : \mathscr{C}_{\langle 1 \rangle} \to \mathscr{D}_{\langle 1 \rangle}$ of a (lax) symmetric monoidal functor $f : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$.

The following example shows the usefulness of straightening/unstraightening.

**Example 2.1.11.** Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal $\infty$-category, and let $(\mathscr{C}^{\otimes})^{op}$ be its opposite category. $(\mathscr{C}^{\otimes})^{op}$ also carries a canonical structure of a symmetric monoidal $\infty$-category. This can be seen by considering the coCartesian fibration $p : \mathscr{C}^{\otimes} \to N(\Gamma)$ giving $\mathscr{C}^{\otimes}$ the structure of a symmetric monoidal $\infty$-category. Apply unstraightening to $p$ to obtain $\mathrm{Un}(p) : N(\Gamma) \to \mathsf{Cat}_{\infty}$. Postcomposing $(-)^{op} : \mathsf{Cat}_{\infty} \to \mathsf{Cat}_{\infty}$ with $\mathrm{Un}(p)$ and apply straightening to obtain a symmetric monoidal $\infty$-category $p' : (\mathscr{C}^{op})^{\otimes} \to N(\Gamma)$. This construction is in fact functorial, see [15] Appendix A.3.

YonedaSymMon **Example 2.1.12.** Let $p : \mathscr{C}^{\otimes} \to N(\Gamma)$ be a symmetric monoidal $\infty$-category and let $K$ be an arbitrary simplicial set. The projection $p_1 : N(\Gamma) \times K \to N(\Gamma)$, induces a map $(p_1)_* : N(\Gamma) \to \mathsf{Fun}(K, N(\Gamma))$ via adjunction. Consider the following pullback diagram

$$\begin{array}{ccc} \mathsf{Fun}(K, \mathscr{C}^{\otimes}) \times_{\mathsf{Fun}(K, N(\Gamma))} N(\Gamma) & \longrightarrow & \mathsf{Fun}(K, \mathscr{C}^{\otimes}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p \circ -} \\ N(\Gamma) & \xrightarrow{(p_1)_*} & \mathsf{Fun}(K, N(\Gamma)) \end{array}$$

It is elementary to show that the induced map

$$q : \mathsf{Fun}(K, \mathscr{C}^{\otimes}) \times_{\mathsf{Fun}(K, N(\Gamma))} N(\Gamma) \to N(\Gamma),$$

satisfies the Segal condition. Note that $q$ is a coCartesian fibration because $\mathsf{Fun}(K, -)$ preserve coCartesian fibrations, and the collection of coCartesian fibrations are closed under the base-change. Alternatively it follows directly from corollary 3.2.2.12 of [22]. We will denote the pullback by $\mathsf{Fun}(K,\mathscr{C})^{\otimes}$, and by the above reasoning the induced map $q : \mathsf{Fun}(K,\mathscr{C})^{\otimes} \to N(\Gamma)$ is a symmetric monoidal $\infty$-category.

In particular we see that the functor category $\mathsf{Fun}(\mathscr{C}^{op},\mathcal{S})$ admits an extension to a symmetric monoidal $\infty$-category $\mathsf{Fun}(\mathscr{C}^{op},\mathcal{S})^{\otimes}$. Using this symmetric monoidal structure it can shown that the Yoneda embedding $j : \mathscr{C}^{op} \to \mathsf{Fun}(\mathscr{C}^{op},\mathcal{S})$ admits a canonical refinement to a symmetric monoidal functor $(\mathscr{C}^{op})^{\otimes} \to \mathsf{Fun}(\mathscr{C}^{op},\mathcal{S})^{\otimes}$. See [7] section 2 and 3 for a proof of this result.

We now consider an example of a (lax) symmetric monoidal functor, and sketch our main way of realizing if a functor is (lax) symmetric monoidal.

OmegaIsLax **Example 2.1.13.** Consider the $\infty$-category of spectra, $\mathsf{Sp}$, this $\infty$-category has a canonical symmetric monoidal structure given by the smash product, which can be described in many ways, we shall describe one of these shortly. Likewise for the $\infty$-category of spaces $\mathcal{S}_*$, again given by the smash product. For now take the existence of these for granted. The functor $\Omega^\infty : \mathsf{Sp} \to \mathcal{S}_*$ which is the "0th space functor", is lax symmetric monoidal, we shall give more details for the reason behind this later. The functor $\Sigma^\infty : \mathcal{S}_* \to \mathsf{Sp}$ is symmetric monoidal.

*Remark* 2.1.14. We put "0th space functor" in quotations, because it is the usual incarnation of this functor given our definition of $\mathsf{Sp}$. It is nonetheless justified through lemma 1.3.7.

Example 2.1.13 is in fact a general fact, which holds more generally for $\infty$-operads and their functors, see corollary 7.3.2.7 of [25], in the case of symmetric monoidal $\infty$-categories the statement specializes to the following.

RadjointLax **Lemma 2.1.15.** *Suppose that $f : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ is a lax symmetric monoidal functor, such that for every $\langle n\rangle \in N(\Gamma)$ the induced map of fibers $f_{\langle n\rangle} : \mathscr{C}_{\langle n\rangle} \to \mathscr{D}_{\langle n\rangle}$ admits a right adjoint $g_{\langle n\rangle}$. Then $f$ admits a right adjoint $g$ relative to $N(\Gamma)$ which is lax symmetric monoidal.*

*Remark* 2.1.16. The dual is also true namely: if $g$ is symmetric monoidal, and each of its fibers admits a left adjoint, then it admits a left adjoint relative to $N(\Gamma)$ which is oplax symmetric monoidal.

As exemplified in the lemma above we have a great deal of appreciation for the information contained in the fibers of the coCartesian fibrations, hence the following definition should come as no surprise.

**Definition 2.1.17.** A symmetric monoidal functor $F : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ is called an *equivalence* of symmetric monoidal $\infty$-categories if and only if it induces an equivalence of the underlying $\infty$-categories $\mathscr{C}^{\otimes}_{\langle 1\rangle} \simeq \mathscr{D}^{\otimes}_{\langle 1\rangle}$.

Lets relate the $\infty$-categorical theory to the 1-categorical. In the following when we are discussing symmetric monoidal categories, we will supress the associator, left/right unit, and braiding from the notation. Furthermore we shall assume them all to be chosen beforehand.

SymMonCat **Theorem 2.1.18.** *Let $\mathscr{C}$ be a category equipped with the structure of a symmetric monoidal category $(\mathscr{C},\otimes,1)$, then $N(\mathscr{C})$ has the structure of a symmetric monoidal $\infty$-category.*

*Proof of theorem 2.1.18.* Our simplicial set $N(\mathscr{C})^{\otimes}$ will turn out to be the nerve of a certain category, which we will denote $\mathscr{D}$. The construction of $\mathscr{D}$ will be needed again later, hence we promote it to its own construction.

cons **Construction 2.1.19.**

1. The objects of $\mathscr{D}$ are finite sequences of objects $X_1,...,X_n \in \mathscr{C}$.
2. Given two objects $\{X_i\}_{1\le i\le m}$ and $\{Y_j\}_{1\le j\le n}$, a morphism between them is given by a map $\alpha : \langle m\rangle \to \langle n\rangle$ in $\Gamma$ together with a collection of morphisms

$$\{\bigotimes_{\alpha(i)=j} X_i \to Y_j\}_{1\le j\le n}$$

in $\mathscr{C}$, where $\otimes$ refers to the monoidal product of $\mathscr{C}$.

3. The composition is determined by the compositions of $\Gamma$ and $\mathscr{C}$.

Note that there is a forgetful functor $f : \mathscr{D} \to \Gamma$ which on objects is given by $\{X_i\}_{1\leq i\leq m} \mapsto \langle m\rangle$ and on morphisms as

$$\Big(\alpha, \{ \bigotimes_{\alpha(i)=j} X_i \to Y_j\}_{1\leq j\leq n}\Big) \mapsto \alpha.$$

It is elementary to show that $f$ is a Grothendieck opfibration. By lemma 1.1.20 $p := N(f) : N(\mathscr{D}) \to N(\Gamma)$ is a coCartesian fibration. Note that $f$ has the Segal condition, and that $N$ preserve limits (by virtue of being the right adjoint of the $(|-|, N)$-adjunction), hence $p$ also has the Segal condition. It is obvious from the construction of $\mathscr{D}$ that $N(\mathscr{D})_{\langle 1\rangle} \simeq N(\mathscr{C})$. □

This theorem will be our main source of examples of symmetric monoidal ∞-categories.

## 2.2 Algebra and Module objects of Symmetric Monoidal ∞-categories

Let $R$ be a commutative ring, then the category of finitely generated projective $R$-modules $\mathrm{Proj}_R \subseteq \mathsf{Mod}_R$ is a permutative category, hence Segals $K$-theory of permutative categories applies, and it recovers Quillens $K$-theory of $R$ see [39]. In the classical theory of symmetric monoidal categories, $R$ is a commutative algebra object in the symmetric monoidal category of abelian groups $\mathsf{Ab}$, and $\mathsf{Mod}_R$ is the collection of module objects over $R$ in $\mathsf{Ab}$. We will in this section define the analog of commutative algebra objects, and $\mathsf{Mod}_R$ for $R$ a commutative algebra object, and in the following section we will construct the analogous $K$-theory.

**Definition 2.2.1.** Let $p : \mathscr{C}^\otimes \to N(\Gamma)$ be a symmetric monoidal ∞-category. A *commutative algebra object* of $\mathscr{C}$ is a section of $p$, i.e. a map $A : N(\Gamma) \to \mathscr{C}^\otimes$ such that $p \circ A = \mathrm{id}_{N(\Gamma)}$, sending inert maps to $p$-coCartesian maps. We let $\mathsf{CAlg}(\mathscr{C})$ denote the full subcategory of $\mathrm{Map}_{N(\Gamma)}(N(\Gamma), \mathscr{C}^\otimes)$ spanned by the algebra objects of $\mathscr{C}^\otimes$. Given a commutative algebra object $A$, we call $A_{\langle 1\rangle} \in \mathscr{C}^\otimes$ the *underlying object of the commutative algebra object.*

Note that the commutativity is not a condition, it is *structure*: when we provide the section of the coCartesian fibration $p$ exhibiting $\mathscr{C}$ as a symmetric monoidal ∞-category we are choosing a homotopy $a \cdot b \simeq b \cdot a$, and all the higher homotopies, i.e. we are providing an infinite amount of data. This is the ingenious insight of J. Lurie and G. Segal mentioned in the introduction.

The following result is a consequence of [25] Corollary 3.2.3.5 and the Segal condition.

CAlgPresent **Theorem 2.2.2.** *Let $\mathscr{C}$ be a presentable symmetric monoidal ∞-category, where the monoidal product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is cocontinuous in each variable, then $\mathsf{CAlg}(\mathscr{C})$ is presentable.*

*Remark* 2.2.3. Consider the *trivial symmetric monoidal ∞-category* $\mathrm{id}_{N(\Gamma)} : N(\Gamma) \to N(\Gamma)$, and another symmetric monoidal ∞-category $p : \mathscr{C}^\otimes \to N(\Gamma)$, then note that algebra objects of $\mathscr{C}$ can be identified with lax symmetric monoidal functors $N(\Gamma) \to \mathscr{C}$, i.e. $\mathsf{CAlg}(\mathscr{C}) = \mathsf{Fun}_{\mathrm{lax}}(N(\Gamma), \mathscr{C}^\otimes)$.

By this remark together with the fact that the composition of lax symmetric monoidal functors is again a lax symmetric monoidal functor, we obtain the following lemma.

laxpresalg **Lemma 2.2.4.** *Let $p : \mathscr{C}^\otimes \to N(\Gamma)$ and $q : \mathscr{D}^\otimes \to N(\Gamma)$ be symmetric monoidal ∞-categories, and let $F : \mathscr{C}^\otimes \to \mathscr{D}^\otimes$ be a lax symmetric monoidal functor. Let $A : N(\Gamma) \to \mathscr{C}^\otimes$ be a commutative algebra object of $\mathscr{C}$. Then $F \circ A$ is a commutative algebra object of $\mathscr{D}$.*

1-catToinfty-Cat **Example 2.2.5.** Let $\mathscr{C}$ be a category equipped with the structure of a symmetric monoidal category $(\mathscr{C}, \otimes, 1)$, and let $\mathscr{C}$ have a simplicial combinatorial model structure compatible with the monoidal structure. Assume further that $\mathscr{C}$ is freely powered, for a definition of this see [25] 4.5.4.2. Now consider $\mathscr{C}$ as a symmetric monoidal $\infty$-category via example 2.1.18. Let $\mathscr{C}_{fc}$ denote the collection of fibrant-cofibrant objects of $\mathscr{C}$ in the model structure. Then $\mathsf{CAlg}(N(\mathscr{C}_{fc}))$ can be identified with the nerve of the category of commutative algebra objects of $\mathscr{C}$, $N_\Delta(\mathsf{CAlg}(\mathscr{C})_{fc})$. We will not prove this, for a proof see [21] theorem 4.3.22.

E_inftySpace **Example 2.2.6.** We may think of $\mathcal{S}_*$ as the nerve of the category of pointed compactly generated weak Hausdorff topological spaces, $N(\mathsf{Top}_*)$. The smash product $\wedge : \mathsf{Top}_* \times \mathsf{Top}_* \to \mathsf{Top}_*$ gives $\mathsf{Top}_*$ the structure of a symmetric monoidal category. Hence by theorem 2.1.18 we obtain a symmetric monoidal structure on $\mathcal{S}_*$. The commutative algebra objects of $\mathcal{S}_*$ are called $\mathbb{E}_\infty$-spaces. Recall the definition of special $\Gamma$-spaces, namely functors $A : \Gamma \to \mathsf{Top}_*$, satisfying the Segal condition. Special $\Gamma$-spaces are algebra objects of $\mathsf{Top}$ in the 1-categorical sense, see [6]. Now $\mathcal{S}_*$ is *not* freely powered, hence special $\Gamma$-spaces they are *not* models for $\mathbb{E}_\infty$-spaces. They are much a kin though, and as we shall se many definitions regarding $\mathbb{E}_\infty$-spaces will be inspired from special $\Gamma$-spaces.

*Remark* 2.2.7. Note that in the unpointed case, $\mathcal{S} = N_\Delta(\mathsf{Top})$ the cartesian product $\prod : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ on $\mathsf{Top}$ induces a symmetric monoidal structure on $\mathcal{S}$ in an analogous fashion.

As promised we will now describe one way to obtain a symmetric monoidal structure on $\mathsf{Sp}$. This is not the easiest way to obtain it, but it emphasises the relation to the classical theory. We will in the following section indicate the standard way of defining it.

**Example 2.2.8.** Consider the category of symmetric spectra $\mathsf{Sp}^\Sigma$ equipped with the structure of a symmetric monoidal category $(\mathsf{Sp}^\Sigma, \otimes, \mathbb{S})$, where the smash product is constructed through Day-convolution, see e.g. [29]. Equip $\mathsf{Sp}^\Sigma$ with the stable model structure, see e.g. [29]. Applying theorem 2.1.18 to the fibrant-cofibrant objects, $\mathsf{Sp}^\Sigma_{fc}$, we obtain a symmetric monoidal structure on the $\infty$-category $N(\mathsf{Sp}^\Sigma_{fc})$. Utilizing the following well known equivalence of theorem 1.3.8 we obtain a symmetric monoidal structure on $\mathsf{Sp}$. Furthermore $\mathsf{Sp}^\Sigma$ satisfies the assumptions of example 2.2.5. Hence we obtain the following string of equivalences

$$\mathsf{CAlg}(\mathsf{Sp}) \simeq \mathsf{CAlg}(N(\mathsf{Sp}^\Sigma_{fc})) \simeq N(\mathsf{CAlg}(\mathsf{Sp}^\Sigma)_{fc}).$$

Hence algebra objects of $\mathsf{Sp}^\Sigma$ represent algebra objects of $\mathsf{Sp}$. These algebra objects are exactly the $\mathbb{E}_\infty$-ring spectra. Here we could just as well have choosen the category of orthogonal spectra $\mathsf{Sp}^O$.

As mentioned the problem with this approach to giving the symmetric monoidal structure on $\mathsf{Sp}$ and defining $\mathbb{E}_\infty$-ring spectra is that it is not model independent; it relied on the choice of model structure, and even on the choice between orthogonal or symmetric spectra, which are not the only choices. We will sketch how to avoid this in the a later section, and as such we will relegate much of our discussion of $\mathbb{E}_\infty$-ring spectra until then.

We have now discussed algebra objects of symmetric monoidal $\infty$-categories, let us now discuss module objects. We will consider the non-commutative case, for a number of reasons. The first is that when compared with the commutative case or even the more general case of $\mathscr{O}$-monoidal $\infty$-categories for $\mathscr{O}$ an $\infty$-operad, the objects are significantly easier to write down explicitly, which makes the relation to the classical theory more apparent. Furthermore the definition is significantly easier to comprehend, and relies on only a fraction of auxillary results and notions compared to the methods mentioned above. Lastly for commutative algebra objects the commutative and non-commutative construction are, in a suitable sense, in fact equivalent. We shall follow parts of [21], [19] and [25].

**Definition 2.2.9.** Let $p : \mathscr{C}^\otimes \to N(\Delta)^{op}$ be a monoidal $\infty$-category. An *$\infty$-category left-tensored over $\mathscr{C}^\otimes$* is a inner fibration $q : \mathcal{M}^\otimes \to \mathscr{C}^\otimes$ with the following properties.

1. The composition $(p \circ q) : \mathcal{M}^{\otimes} \to N(\Delta)^{op}$ is a coCartesian fibration.

2. The map $q$ carries $(p \circ q)$-coCartesian edges of $\mathcal{M}^{\otimes}$ to $p$-coCartesian edges of $\mathscr{C}^{\otimes}$.

3. For each $n \geq 0$, the inclusion $i_n : \{n\} \subseteq [n]$ induces an equivalence of $\infty$-categories $(i_n)_! : \mathcal{M}^{\otimes}_{[n]} \to \mathscr{C}^{\otimes}_{[n]} \times \mathcal{M}^{\otimes}_{\{n\}}$, where $(p \circ q)^{-1}\{[n]\} =: \mathcal{M}^{\otimes}_{[n]}$.

**Proposition 2.2.10.** *A left-tensored $\infty$-category $p : \mathcal{M}^{\otimes} \to \mathscr{C}^{\otimes}$ over a symmetric monoidal category $q : \mathscr{C}^{\otimes} \to N(\Delta)^{op}$ induces a bifunctor on the underlying $\infty$-categories, $\otimes : \mathscr{C} \times \mathcal{M} \to \mathcal{M}$ well-defined up to homotopy. We will call the bifunctor a left action of $\mathscr{C}$ on $\mathcal{M}$.*

*Proof.* Condition (3) and the Segal condition, gives an equivalence of $\mathcal{M}^{\otimes}_{[n]} \simeq \mathscr{C}^n \times \mathcal{M}$. In particular we obtain an equivalence $(i_1)_! : \mathcal{M}^{\otimes}_{[1]} \to \mathscr{C} \times \mathcal{M}$. Now the coCartesian fibration $q$ induces from the inclusion $\{0\} \to [1]$ the desired bifunctor

$$\mathscr{C} \times \mathcal{M} \xrightarrow{\simeq} \mathcal{M}^{\otimes}_{[1]} \to \mathcal{M}^{\otimes}_{\{0\}} := \mathcal{M}.$$

We denote this functor by $\otimes : \mathscr{C} \times \mathcal{M} \to \mathcal{M}$ and call it the tensor product. Note that again there is associated a real choice with this tensor product, because it is induced by the inclusion $\{0\} \to [1]$. □

In fact one can say significantly more: the structure of $\mathcal{M}^{\otimes}_{[n]}$ for $n > 1$ ensures that the bifunctor $\otimes : \mathscr{C} \times \mathcal{M} \to \mathcal{M}$ is coherently associative. Decending to homotopy categories, one sees that the homotopy category $h\mathcal{M}$ is tensored over $h\mathscr{C}$ in the classical sense. See [19] Remark 2.1.2.

*Remark* 2.2.11. Let $\mathscr{C}$ be a monoidal $\infty$-category, then the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ exhibits $\mathscr{C}$ as left-tensored over itself, i.e. the functor induced in the above fashion agrees with the monoidal product on $\mathscr{C}$. This is a rather involved fact, hence we shall take it for granted, see [20] section 3.1.2 and [19] Example 2.1.3.

We shall need the notion of convex subsets of discrete sets with a total order.

**Definition 2.2.12.** Let $X$ be a totally ordered discrete set. A subset $Y \subseteq X$ is *convex* if and only if for all $a, b \in Y$, where $a < b$, the subset $\{x \in X : a < x < b\}$ is contained in $Y$.

**Definition 2.2.13.** A morphism $f : [m] \to [n]$ in $\Delta$ is *convex* if $f$ is injective and the image $\{f(0), ..., f(m)\} \subseteq [n]$ is a convex subset of $[n]$.

moduledef **Definition 2.2.14.** Let $p : \mathscr{C}^{\otimes} \to N(\Delta)^{op}$ be a monoidal $\infty$-category, and let $q : \mathcal{M}^{\otimes} \to \mathscr{C}^{\otimes}$ be an $\infty$-category equipped with a left action of $\mathscr{C}^{\otimes}$. A *left module object* of $\mathcal{M}$ is a functor $M : N(\Delta)^{op} \to \mathcal{M}^{\otimes}$ with the following properties,

1. The composition $q \circ M$ is an algebra object of $\mathscr{C}^{\otimes}$. In particular, $p \circ q \circ M$ is the identity on $N(\Delta)^{op}$.

2. Let $\alpha : [m] \to [n]$ be a convex map in $\Delta$ such that $\alpha(m) = n$. Then $M(\alpha)$ is a $(p \circ q)$-coCartesian morphism of $\mathcal{M}^{\otimes}$.

We let $L\mathsf{Mod}_{\mathcal{M}}$ denote the full subcategory of $\mathrm{Map}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{M}^{\otimes})$ spanned by the left module objects. Because we will be interested in the commutative case, we shall immediately drop the "$L$" from the notation $L\mathsf{Mod}_{\mathcal{M}}$ and write $\mathsf{Mod}_{\mathcal{M}}$.

**Example 2.2.15.** Let $p : \mathcal{M}^{\otimes} \to N(\Delta)^{op}$ be an $\infty$-category equipped with a left action of a monoidal $\infty$-category $\mathscr{C}^{\otimes}$, induced from $q : \mathcal{M}^{\otimes} \to \mathscr{C}^{\otimes}$. The functor $q$ induces a functor $q \circ - : \mathsf{Mod}_{\mathcal{M}} \to \mathsf{Alg}(\mathscr{C})$, via (1) of definition 2.2.14. If $R$ is an algebra object of $\mathscr{C}$, we let $\mathsf{Mod}_R(\mathcal{M})$ denote the fiber $(q \circ -)^{-1}\{R\}$. The functor $(q \circ -)$ is an inner fibration of simplicial sets, because it is post composition with an (in particular) inner fibration, hence $\mathsf{Mod}_R(\mathcal{M})$ is an $\infty$-category, which we will call the *$\infty$-category of left $R$-modules.* In the case $\mathcal{M} = \mathscr{C}$, i.e. $\mathscr{C}$ is left-tensored over itself, we shall often omit $\mathcal{M}$ from the notation, and simply write $\mathsf{Mod}_R$.

**Lemma 2.2.16.** *Consider $M \in \mathsf{Mod}_{\mathcal{M}}$ be a module object of $p : \mathcal{M}^{\otimes} \to N(\Gamma)$ which is equipped with a left action of $\mathscr{C}^{\otimes}$. Furthermore let $f : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ be a lax symmetric monoidal functor. Then $f(M) \in \mathsf{Mod}_{\mathcal{N}}$.*

This of courses specializes to the fact that if $A$ is an $R$-module then $f(A)$ is a $f(R)$-module.

As mentioned earlier there is an analogous construction for a symmetric monoidal $\infty$-category $\mathscr{C}$ and $R \in \mathsf{CAlg}(\mathscr{C})$, which is given in [21]. We follow [21] and denote this construction $c\mathsf{Mod}_R(\mathscr{C})$. The following theorem is proposition 5.6 of [21], and the construction of the functor $\theta$ is described in Notation 5.5.

**Proposition 2.2.17.** *Let $\mathscr{C}$ be a symmetric monoidal $\infty$-category, let $R \in \mathsf{CAlg}(\mathscr{C})$. Then there exists a functor $\theta : c\mathsf{Mod}_R(\mathscr{C}) \to \mathsf{Mod}_R(\mathscr{C})$, which is a trivial Kan fibration.*

Hence up to homotopy we may think of these two constructions to be equivalent, so from now on we shall not concern ourselves too much with specific constructions of $\mathsf{Mod}_R$.

## 2.3 K-theory and Symmetric Monoidal $\infty$-categories

There are many models for the $K$-theory spectrum for sufficiently nice $\infty$-categories. We shall consider a specific model which is heavily inspired by Segals infinite loop space machinery. In fact this model subsumes Segals $K$-theory for permutative categories [46]. Because of this it is also possible to recover the algebraic $K$-theory of rings.
This is not the most general $K$-theory, Waldhausens $K$-theory by way of the $S_\bullet$-construction works for pointed $\infty$-categories which admit pushouts, see [25] Remark 1.2.2.5. But this model is sufficiently general for our purposes. The up-shot is that it is rather easy to understand, is much reminiscent to the classic construction of $K_0$, and is very formal. This $K$-theory, which we shall call *Segals $K$-theory* following [5], is going to be a functor from the (yet to be defined) $\infty$-category of symmetric monoidal $\infty$-categories $\mathsf{SymMonCat}_\infty$ to connective spectra $\mathsf{Sp}_{\geq 0}$, i.e. spectra with vanishing negative homotopy groups. The main theorem of this section is that this $K$-theory has a canonical lax symmetric structure. Recall that in the classical construction of $K_0$ one formally adds inverses to the monoid of finitely generated projective $R$-modules, i.e. we group complete. The $\infty$-categorical analog of taking isomorphism classes, is to consider the maximal Kan complex in the $\infty$-category, i.e. the maximal $\infty$-groupiod in the $\infty$-category. In the following we think of $\mathcal{S}$ as $N_\Delta(\mathsf{Kan})$.

Core **Definition 2.3.1.** Consider the continuous inclusion $\mathcal{S} \to \mathsf{Cat}_\infty$. We call the left adjoint functor $(-)^{\simeq} : \mathsf{Cat}_\infty \to \mathcal{S}$, which assigns an $\infty$-category to the maximal Kan complex inside it $\mathscr{C}^{\simeq}$, the *core-functor.*

Before we give the definition of the $\infty$-category of symmetric monoidal $\infty$-categories we shall equip $\mathsf{Cat}_\infty$ with a symmetric monoidal structure.

Cat-infty **Example 2.3.2.** Recall that the $\infty$-category of $\infty$-categories $\mathsf{Cat}_\infty$ is defined as $N(\mathsf{cat}_\infty)$ where $\mathsf{cat}_\infty$ is the category which objects are $\infty$-categories and the maps are functors of $\infty$-categories, where the mapping spaces are taken to be the largest Kan complexes of the functor categories between $\infty$-categories. That there is a symmetric monoidal structure on $\mathsf{Cat}_\infty$ follows immediately from theorem 2.1.18, because $\mathsf{cat}_\infty$ has a symmetric monoidal structure, where the tensor product is the cartesian product.

Core-lax **Corollary 2.3.3.** *The core-functor $(-)^{\simeq} : \mathsf{Cat}^\infty \to \mathcal{S}$ is lax symmetric monoidal.*

*Proof.* By lemma 2.1.15 it suffices to show that the inclusion $\mathcal{S} \to \mathsf{Cat}_\infty$ is symmetric monoidal. Note that this inclusion preserves limits, in particular it preserves products, and since both symmetric monoidal structures were given through products, this functor is symmetric monoidal. □

Given the definition of symmetric monoidal $\infty$-categories, it is perhaps surprising that we define $\mathsf{SymMonCat}_\infty$ in the following way.

**Definition 2.3.4.** *The $\infty$-category of symmetric monoidal $\infty$-categories*, denoted $\mathsf{SymMonCat}_\infty$ is defined as $\mathsf{CAlg}(\mathsf{Cat}_\infty)$. A map in $\mathsf{SymMonCat}_\infty$ is called a *symmetric monoidal functor*, we denote the full subcategory of $\mathsf{Fun}_{\mathsf{Cat}_\infty}(\mathscr{C}, \mathscr{D})$ spanned by these $\mathsf{Fun}^{\otimes}(\mathscr{C}, \mathscr{D})$.

Before we continue with our discussion of Segals $K$-theory, we will take a slight detour discussing the definition of $\mathsf{SymMonCat}_\infty$. The upshot of this definition is that it is intuitive, and it mimics Segals original definition of $\Gamma$-categories (replace $\mathsf{Cat}_\infty$ with topologically (or simplicially) enriched categories). The downside is that this definition can be hard to work with, because it is hard to construct functors into $\mathsf{Cat}_\infty$. In the litterature this is remedied by the use of the *straightening/unstraightening equivalence*. The alternative point of view which this equivalence provides is essential for the theory of localization of $\infty$-categories, but since we shall take these results for granted we will not see its strength that clearly. We will now argue that we alternatively could have defined $\mathsf{SymMonCat}_\infty$ differently.

alt-defi-symmon **Proposition 2.3.5.** *The full subcategory of $(\mathsf{Cat}_\infty)^{coCart}_{/N(\Gamma)}$ spanned by the coCartesian fibrations over $N(\Gamma)$ with the Segal condition is categorically equivalent to $\mathsf{SymMonCat}_\infty$. The mapping spaces between symmetric monoidal $\infty$-categories $\mathscr{C}$ and $\mathscr{D}$, $\mathsf{Fun}^{\otimes}(\mathscr{C}, \mathscr{D})$ is equivalent to the full subcategory of $\mathsf{Fun}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})^{\simeq}$ spanned by the functors which carry $p$-coCartesian morphisms to $q$-coCartesian morphisms, such that*

$$\begin{array}{ccc} \mathscr{C}^{\otimes} & \longrightarrow & \mathscr{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & N(\Gamma) & \end{array}$$

*commutes, i.e. the symmetric monoidal functors.*

*Proof.* The first statement is essentially a consequence of the straightening/unstraightening equivalence: Consider a coCartesian fibration over $N(\Gamma)$, $p : \mathscr{C} \to N(\Gamma)$ which satisfies the Segal condition, i.e. the following map is an equivalence

$$\mathscr{C}_{\langle n \rangle} \xrightarrow{(\rho^i_!)_{1 \le i \le n}} \prod_{1 \le i \le n} \mathscr{C}_{\langle 1 \rangle}.$$

Applying straightening to $p$, we obtain a functor

$$\begin{aligned} \mathrm{St}(p) : N(\Gamma) &\to \mathsf{Cat}_\infty, \\ \langle n \rangle &\mapsto \mathscr{C}_{\langle n \rangle}. \end{aligned}$$

Hence $\mathrm{St}(p)$ induces a map

$$\mathscr{C}_{\langle n \rangle} \xrightarrow{(\mathrm{St}(p)(\rho^i))_{1 \le i \le n}} \prod_{1 \le i \le n} \mathscr{C}_{\langle 1 \rangle}.$$

It is clear that this map is an equivalence if the coCartesian fibration $p : \mathscr{C} \to N(\Gamma)$ satisfies the Segal condition, i.e. if $p$ is send to a symmetric monoidal $\infty$-category under St. Likewise if a functor $q : N(\Gamma) \to \mathsf{Cat}_\infty$ is a symmetric monoidal $\infty$-category, $\mathrm{Un}(q)$ satisfies the Segal condition. The last statement follows directly from the definitions. □

We shall use this theorem to realize that many functors are symmetric monoidal. Contrary to what we hinted at just before the theorem, we shall often realize functors as algebra morphisms in $\mathsf{Cat}_\infty$ and hence as symmetric monoidal functors via the above.

Lets now return to the discussion of Segals $K$-theory. As mentioned in example 2.2.6, special $\Gamma$-spaces model $\mathbb{E}_\infty$-spaces, this prompts us to define a notion of group-like $\mathbb{E}_\infty$-spaces, which should be modeled by group-like $\Gamma$-spaces (sometimes called very special $\Gamma$-spaces). There are many different defintions of what it means for a Γ-space $M$ to be group-like, the one which relates most directly to the definition below is the following: Let $M : \Gamma \to \mathsf{Top}$ be a Γ-space, it is said to be group-like if $\pi_0(M(\langle 1\rangle))$ is an abelian group, see [6]. Recall that the homotopy category associated to the model structure on special Γ-spaces established in [6], is equivalent to the homotopy category $\mathcal{H}$. Hence special Γ-spaces represent commutative algebra objects in $\mathcal{H}$, and thus $\mathbb{E}_\infty$-spaces have underlying commutative algebra objects of $\mathcal{H}$.

**Definition 2.3.6.** An $\mathbb{E}_\infty$-space $M$ is called *group-like* if the commutative algebra object of $\mathcal{H}$ underlying the $\mathbb{E}_\infty$-space $M$ is a group object. We write $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ for the full subcategory of the ∞-category $\mathsf{CAlg}(\mathcal{S})$ spanned by the group-like $\mathbb{E}_\infty$-spaces.

**Proposition 2.3.7.** *There is a functor $\mathcal{G} : \mathsf{CAlg}(\mathcal{S}) \to \mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$, which we call the group completion functor. This functor is characterized as being the left adjoint of the inclusion $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathsf{CAlg}(\mathcal{S})$.*

*Proof.* Consider the forgetful functor $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathsf{CAlg}(\mathcal{S})$. Note that both $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ and $\mathsf{CAlg}(\mathcal{S})$ are presentable and accessible by 2.2.2. Because of this limits of $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ and $\mathsf{CAlg}(\mathcal{S})$ are computed as the limits of the underlying objects. Therefore the forgetful functor is continuous and fully faithful. Now the adjoint functor theorem gives a left adjoint, which is the desired functor. □

*Remark* 2.3.8. Note that the existence of the group completion functor is completely formal, as opposed to the classical group completion map which is involved and certainly not formal, see [4].

GrpComp *Remark* 2.3.9. We shall in the next section show that the group completion functor is lax symmetric monoidal. For this we shall need to view it in a slightly more general setting than just spaces. Let $\mathscr{C}$ be an ∞-category with finite products. In this case the product defines a symmetric monoidal structure on $\mathscr{C}$. Furhtermore let $A : N(\Gamma) \to \mathscr{C}$ be a functor. The condition that $A \in \mathsf{CAlg}(\mathscr{C})$ is equivalent to the maps

$$A(\langle n\rangle) \xrightarrow{\prod_{1\le i\le n} A(\rho^i)} A(\langle 1\rangle)^n,$$

being equivalences and sending inert morphisms to coCartesian edges. If $\pi_0(A(\langle 1\rangle))$ is an abelian group, then $A$ is a *commutative group in* $\mathscr{C}$, the full subcategory of $\mathsf{Fun}(N(\Gamma),\mathscr{C})$ spanned by these we denote by $\mathsf{Grp}(\mathscr{C})$. We shall in the next section realize that there is a localizations $\mathsf{Pr}^L \to \mathsf{Pr}^L$ denoted $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}$ and $l_{\mathsf{CAlg}(\mathcal{S})}$, given by $\mathscr{C} \mapsto \mathsf{Grp}(\mathscr{C})$ and $\mathscr{C} \mapsto \mathsf{CAlg}(\mathscr{C})$ respectively. This will provide the symmetric monoidal structures for which $\mathcal{G}$ is lax symmetric monoidal. We refer the reader to theorem 2.4.10. Note that this strategy requires that we know that $\mathsf{Grp}(\mathscr{C})$ is presentable when $\mathscr{C}$ is presentable, this is true, but we shall take it for granted.

The following identification will supply the codomain of the $K$-theory functor.

GrpToSp **Proposition 2.3.10.** *There is an equivalence between the ∞-category of group-like $\mathbb{E}_\infty$-spaces and the ∞-category of connective spectra, $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathsf{Sp}_{\ge 0}$. This equivalence is lax symmetric monoidal.*

*Proof.* As mentioned group-like $\mathbb{E}_\infty$-spaces are modeled by group-like Γ-spaces. One of the main results of [6] is that there is a Quillen equivalence between the category of group-like Γ-spaces given the Friedlander-Bousfield model structure and the category of connective spectra given the stable model structure. Because they are Quillen equivalent combinatorial simplicial model categories, they give rise to equivalent ∞-categories, via proposition 1.1.10. □

Note we have not shown that the equivalence is lax symmetric monoidal, this follows from a model independent proof, which we will omit, see theorem 5.2.6.10 [25]. Of course this proof relies on a symmetric monoidal structure on $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$. We relegate the definition of this until the next section, as its construction will be analogous to the smash product symmetric monoidal structure on $\mathsf{Sp}$.. We are finally ready to define the $K$-theory spectrum of an symmetric monoidal ∞-category.

**Definition 2.3.11.** The $K$-theory spectrum $K(\mathscr{C})$ of a symmetric monoidal $\infty$-category $\mathscr{C}$ is the image of the composite functor:

$$K : \mathsf{SymMonCat}_\infty \xrightarrow{\mathsf{CAlg}((-)^{\simeq})} \mathsf{CAlg}(\mathcal{S}) \xrightarrow{\quad \mathcal{G} \quad} \mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \xrightarrow{\quad \simeq \quad} \mathsf{Sp}_{\geq 0}.$$

Where the first functor is precomposition with the core-functor of definition 2.3.1 under $\mathsf{CAlg}(-)$, the second is group-completion from proposition 2.3.9 and the last the equivalence from proposition 2.3.10.

This is theorem 1 of the introduction.

**Theorem 2.3.12.** *The $K$-theory functor is lax symmetric monoidal functor.*

*Proof.* We saw in corollary 2.3.3 that the core-functor was lax symmetric monoidal. Hence by lemma 2.2.4, we have that the corresponding functor $\mathsf{CAlg}((-)^{\simeq}) : \mathsf{SymMonCat} \to \mathsf{CAlg}(\mathcal{S})$ is lax symmetric monoidal. We postponed the proof of the fact that the group completion map $\mathcal{G}$ was lax symmetric monoidal to the next section. Hence we have shown the desired result. $\square$

**Example 2.3.13.** Consider the symmetric monoidal category of finitely generated projective $R$-modules $(\mathrm{Proj}_R, \otimes_R, R)$, we may view this as a symmetric monoidal $\infty$-category via 2.2.5. Then its $K$-theory, $K(\mathrm{Proj}_R)$ is equivalent to Quillens algebraic $K$-theory of $R$, see [35].

**Example 2.3.14.** Let $(\mathscr{C}, \otimes, 1)$ be a permutative category. In particular we may view $(\mathscr{C}, \otimes, 1)$ as a symmetric monoidal category, hence we may view it as a symmetric monoidal $\infty$-categoy via 2.2.5. Then $K(\mathscr{C})$ recovers Segal $K$-theory for permutative categories, see [35].

$K$-theory of $\mathbb{E}_\infty$-ring spectra is going to be analogous to this example, as we shall see later.

## 2.4 Symmetric Monoidal Structure on $\mathsf{Sp}$ and $\mathbb{E}_\infty$-Ring Spectra

As mentioned in the introduction we wish to consider $K$-theory of $\mathbb{E}_\infty$-ring spectra. These are commutative algebra objects of the smash product symmetric monoidal structure on the $\infty$-category of spectra $\mathsf{Sp}$. As seen in the previous sections we may readily construct a symmetric monoidal structure on $\mathsf{Sp}$ using the symmetric monoidal structure on $\mathsf{Sp}^\Sigma$, which of course was an unsatisfactory approach since it was not model independent. We will give a sketch of how to give a model independent description, before describing some properties of $\mathbb{E}_\infty$-ring spectra and module objects over them. This construction relies on the notion of a module objects of a symmetric monoidal $\infty$-category and propositon 2.3.5, hence it has been postponed until know. We follow [25] section 4.8.1 and 4.8.2.

Recall that there was a symmetric monoidal structure on the $\infty$-category of $\infty$-categories, see example 2.3.2. Let us now add the condition that the $\infty$-categories be presentable.

**Definition 2.4.1.** Let $\mathsf{Pr}^L$ be the subcategory of $\mathsf{Cat}_\infty$ which objects are presentable $\infty$-categories, and which morphisms are cocontinuous functors.

Let $\mathcal{P}(\mathscr{C})$ denote the $\infty$-category of presheaves on $\mathscr{C}$. Let $\mathrm{RFun}(\mathscr{C}, \mathscr{D})$ denote the full subcategory of $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ spanned by the functors $\mathscr{C} \to \mathscr{D}$ which admit left adjoints. In the case $\mathscr{C}$ and $\mathscr{D}$ are presentable, the adjoint functor theorem yields a simpler description, namely that a functor belongs to $\mathrm{RFun}(\mathscr{C}, \mathscr{D})$ if and only if it preserve limits and $\kappa$-filtered colimits for some regular cardinal $\kappa$. The following proposition is a summary of proposition 4.8.1.14, 4.8.1.14 and 4.8.1.16 of [25], which proofs rely heavily on the symmetric monoidal structure of $\mathsf{Cat}_\infty$ given above.

presentable1 **Proposition 2.4.2.** *The $\infty$-category $\mathsf{Pr}^L$ of presentable $\infty$-categories has a symmetric monoidal structure, where $\mathcal{P}(\Delta^0) \simeq \mathcal{S}$ is the unit object, and the tensor product of $\mathscr{C}$ and $\mathscr{D}$ is given by* $\mathrm{RFun}(\mathscr{C}^{op}, \mathscr{D})$.

SpIdem **Example 2.4.3.** Let $\mathscr{C} \in \mathsf{Pr}^L$. Recall that we define spectrum objects of an $\infty$-category $\mathscr{C}$ as $\mathsf{Sp}(\mathscr{C}) \coloneqq \mathsf{Exc}_*(\mathcal{S}_*^{fin}, \mathscr{C})$. Consider the tower of lemma 1.3.7, we have that $\mathsf{Sp}$ is given as the limit of this tower with $\mathscr{C}_*$ replaced by $\mathcal{S}_*$. Because of this we have the following equivalences

$$\mathscr{C} \otimes \mathsf{Sp} \simeq \mathrm{RFun}(\mathscr{C}^{op}, \mathsf{Sp}) \simeq \mathrm{holim}\,\mathrm{RFun}(\mathscr{C}^{op}, \mathcal{S}_*) \simeq \mathrm{holim}\,\mathscr{C}_* \simeq \mathsf{Sp}(\mathscr{C}).$$

Where the second to last equivalence follows from lemma 1.3.7. Note that if we take $\mathscr{C} = \mathsf{Sp}$ we obtain the existence of the following equivalence $\mathsf{Sp} \otimes \mathsf{Sp} \to \mathsf{Sp}$, which follows from $\mathsf{Sp}(\mathsf{Sp}) \simeq \mathsf{Sp}$. Analogous to $\mathsf{Sp}$ we have that $\mathsf{Sp}(\mathscr{C})$ is presentable and stable, therefore the functor $\Sigma^\infty : \mathcal{S} \to \mathsf{Sp}$ induces a map $\mathscr{C} \simeq \mathscr{C} \otimes \mathcal{S} \to \mathscr{C} \otimes \mathsf{Sp}$. Note that if we again take $\mathscr{C} = \mathsf{Sp}$, we may collect the above to obtain an equivalence $\mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$.

**Definition 2.4.4.** Let $\mathscr{C}$ be a symmetric monoidal $\infty$-category with unit object 1. We will say that a map $e : 1 \to E$ for $E \in \mathscr{C}$ is an *idempotent object of* $\mathscr{C}$ if the induced map

$$E \simeq E \otimes 1 \xrightarrow{\mathrm{id} \otimes e} E \otimes E$$

is an equivalence in $\mathscr{C}$.

*Remark* 2.4.5. By proposition 2.4.2 the $\infty$-category $\mathsf{Pr}^L$ has a symmetric monoidal structure in which $\mathcal{S}$ is the unit. This together with example 2.4.3, shows that $\Sigma^\infty : \mathcal{S} \to \mathsf{Sp}$ is an idempotent object of $\mathsf{Pr}^L$.

The following is a consequence of proposition 4.8.2.4 [25].

4.8.2.4 **Proposition 2.4.6.** *Let $\mathscr{C}$ be a symmetric monoidal $\infty$-category with unit object* 1 *and let $l_E : \mathscr{C} \to \mathscr{C}$ be the functor given by $\mathscr{C} \mapsto \mathscr{C} \otimes E$. The map $e : 1 \to E$ is an idempotent object of $\mathscr{C}$ if and only if $e$ induces a natural transformation $\alpha : \mathrm{id}_{\mathscr{C}} \to l_E$ which exhibits $l_E$ as localization functor on $\mathscr{C}$.*

We shall call the localization $l_E : \mathscr{C} \to \mathscr{C}$ associated to an idempotent object *the smashing localization on* $\mathscr{C}$. Hence we obtain that the idempotent object $\Sigma^\infty : \mathcal{S} \to \mathsf{Sp}$ gives rise to the smashing localization $l_{\mathsf{Sp}} : \mathsf{Pr}^L \to \mathsf{Pr}^L$. Following [25] we define $\mathsf{Pr}^L_\sigma$ as the localization of $\mathsf{Pr}^L$. Hence this exhibits $\mathscr{C} \otimes \mathsf{Sp}$ as a $\mathsf{Pr}^L_\sigma$-localization of $\mathscr{C}$ along $l_{\mathsf{Sp}}$. The following is proposition 4.8.2.7 of [25] specialized to our setting.

4.8.2.7 **Proposition 2.4.7.** *The localization $l_{\mathsf{Sp}}(\mathsf{Pr}^L) = \mathsf{Pr}^L_\sigma$ inherits a symmetric monoidal structure from $\mathsf{Pr}^L$. The unit of this symmetric monoidal structure is given as $l_{\mathsf{Sp}}(\mathcal{S}) \simeq \mathsf{Sp}$. Furthermore $\mathsf{Sp}$ obtains a unique structure of an object of $\mathsf{CAlg}(\mathsf{Pr}^L_\sigma)$, and hence in particular as an object of $\mathsf{CAlg}(\mathsf{Cat}_\infty)$.*

*Proof.* We will use the notation established in proposition 2.4.6. We need to show that $l_{\mathsf{Sp}}$ is compatible with the symmetric monoidal structure on $\mathsf{Pr}^L$. This means first of all that $l_{\mathsf{Sp}}$ is an *idempotent functor*, which means that for all $\mathscr{C} \in \mathsf{Pr}^L$ there exists an equivalence $\alpha_{\mathscr{C}} \simeq l_{\mathsf{Sp}}(\alpha_{\mathscr{C}}) : l_{\mathsf{Sp}}(\mathscr{C}) \to l_{\mathsf{Sp}}(l_{\mathsf{Sp}}(\mathscr{C}))$, and both $\alpha_{l_{\mathsf{Sp}}(\mathscr{C})}$ and $l_{\mathsf{Sp}}(\alpha_{\mathscr{C}})$ are localizations. This follows because $\mathsf{Sp}$ is stable an hence is its own stabilization. Furthermore it means that if $f : \mathscr{C} \to \mathscr{D}$ in $\mathsf{Pr}^L$ is a map such that $l_{\mathsf{Sp}}(f)$ is an equivalence, then so is $f \otimes \mathrm{id}_{\mathscr{E}} : \mathscr{C} \otimes \mathscr{E} \to \mathscr{D} \otimes \mathscr{E}$ for every $\mathscr{E} \in \mathsf{Pr}^L$. Note that if $l_{\mathsf{Sp}}(f) : \mathscr{C} \otimes \mathsf{Sp} \to \mathscr{D} \otimes \mathsf{Sp}$ is an equivalence, then so is $\mathscr{C} \otimes \mathscr{E} \otimes \mathsf{Sp} \to \mathscr{D} \otimes \mathscr{E} \otimes \mathsf{Sp}$, now the desired result holds by commuting the factors. This proves the first part of the proposition.

The second part follows because $l_{\mathsf{Sp}}$ is compatible with the symmetric monoidal structure on $\mathsf{Pr}^L$, hence the unit of $\mathsf{Pr}^L_\sigma$, is $l_{\mathsf{Sp}}(\mathcal{S}) \simeq \mathsf{Sp}$. We claim that because $\mathsf{Sp}$ is the unit of the symmetric monoidal structure on $\mathsf{Pr}^L_\sigma$ it canonically has the structure of an object of $\mathsf{CAlg}(\mathsf{Pr}^L_\sigma)$. Indeed the inclusion $(\mathsf{Pr}^L_\sigma)^\otimes \to (\mathsf{Pr}^L)^\otimes$ is a fully faithful embedding, which induces a fully faithful embedding of commutative algebra objects $\mathsf{CAlg}(\mathsf{Pr}^L_\sigma) \to \mathsf{CAlg}(\mathsf{Pr}^L)$, whose essential image is the collection of commutative algebra objects $A \in \mathsf{CAlg}(\mathsf{Pr}^L)$ such that $A \to A \otimes \mathsf{Sp}$ is an equivalence. Note that this holds for $\mathsf{Sp}$, hence $\mathsf{Sp}$ is a commutative algebra object of $\mathsf{Pr}^L_\sigma$. □

Hence $\mathsf{Sp}$ is a commutative algebra object of $\mathsf{Pr}^L_\sigma$ in particular a symmetric monoidal $\infty$-category by proposition 2.3.5.

**Corollary 2.4.8.** *There exists a symmetric monoidal structure on* $\mathsf{Sp}$ *such that* $\mathbb{S}$ *is the unit, and the tensor product* $\otimes : \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$ *is cocontinuous in each variable. Following [25] we shall call this symmetric monoidal structure the smash product symmetric monoidal structure.*

As an immediate consequence of this we have.

**Corollary 2.4.9.** *Note that any spectrum is an* $\mathbb{S}$*-module with respect to the smash product symmetric monoidal structure, because* $\mathbb{S}$ *is the unit. The functor* $\Sigma^\infty : \mathcal{S} \to \mathsf{Sp}$ *is symmetric monoidal.*

We will use the strategy that we employed to construct a symmetric monoidal structure on $\mathsf{Pr}^L_\sigma$ from the idempotent object $\mathcal{S} \to \mathsf{Sp}$, to show the following theorem, which is the missing component need to see that the $K$-theory functor is lax symmetric monoidal.

CAlgGrpSmash **Theorem 2.4.10.** *The group completion functor* $\mathcal{G}$ *is symmetric monoidal.*

*Proof.* Note that at this point we have not described the symmetric monoidal structure on neither $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ nor $\mathsf{CAlg}(\mathcal{S})$. We begin by describing the symmetric monoidal structure on $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$. We begin by showing that the functor $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})} : \mathsf{Pr}^L \to \mathsf{Pr}^L$ given by $\mathscr{C} \mapsto \mathsf{Grp}(\mathscr{C})$ is a smashing localization, see remark 2.3.9 for the definition of $\mathsf{Grp}(\mathscr{C})$. Notice that we have used suggestive notation, this is because the map $e : \mathcal{S} \to \mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ in fact is idempotent, and hence gives rise to the smashing localization $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}$ described above, but it is easier to show that it is a smashing localization.

Consider

$$\begin{aligned} l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathscr{C}) &\simeq l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathscr{C} \otimes \mathcal{S}) \\ &\simeq l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathrm{RFun}(\mathscr{C}^{op}, \mathcal{S})) \\ &\simeq \mathrm{RFun}(\mathscr{C}^{op}, l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathcal{S})) \\ &\simeq \mathscr{C} \otimes l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathcal{S}). \end{aligned}$$

Note that every equivalence, except the third follows from proposition 2.4.2. We now show the third seperately, which in fact holds a bit more general. We show that

$$\mathsf{Grp}(\mathrm{RFun}(\mathscr{C}, \mathscr{D})) \simeq \mathrm{RFun}(\mathscr{C}, \mathsf{Grp}(\mathscr{D}))$$

for $\mathscr{C}, \mathscr{D} \in \mathsf{Pr}^L$. Note that there is fully faithful inclusion, simply because being a commutative group in $\mathscr{C}$ is a property of the monoid structre,

$$\mathsf{Grp}(\mathrm{RFun}(\mathscr{C}, \mathscr{D})) \to \mathsf{Fun}(N(\Gamma), \mathsf{Fun}(\mathscr{C}, \mathscr{D})) \simeq \mathsf{Fun}(N(\Gamma) \times \mathscr{C}, \mathscr{D}).$$

Note that the essential image of this inclusion are those functors $F : N(\Gamma) \times \mathscr{C} \to \mathscr{D}$ for which each functor $F(-, C)$ is a commutative group in $\mathscr{D}$, and each $F(\langle n \rangle, -)$ is continuous. Note we also have a fully faithful inclusion,

$$\mathrm{RFun}(\mathscr{C}, \mathsf{Grp}(\mathscr{D})) \to \mathsf{Fun}(\mathscr{C}, \mathsf{Fun}(N(\Gamma), \mathscr{D})) \simeq \mathsf{Fun}(N(\Gamma) \times \mathscr{C}, \mathscr{D}).$$

Note that this inclusion has the same essential image. Now the identification follows because both functors are the restriction of the functor $\mathsf{Fun}(N(\Gamma) \times \mathscr{C}, \mathscr{D}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{D})$ given by fixing the first coordinate at $\langle 1 \rangle$.

Now if we apply proposition 4.8.2.7 of [25] to the localization $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}$, we obtain the localization $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathsf{Pr}^L)$ inherits a symmetric monoidal structure from $\mathsf{Pr}^L$, for which the unit is given as $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathcal{S}) \simeq \mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ and that $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \in \mathsf{CAlg}(l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}(\mathsf{Pr}^L)) \subset \mathsf{CAlg}(\mathsf{Cat}_\infty)$. In particular, via proposition 2.3.5, $\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ is a symmetric monoidal $\infty$-category, and the map $e : \mathcal{S} \to \mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ corresponding to the localization $l_{\mathsf{Grp}_{\mathbb{E}_\infty}(\mathcal{S})}$ is symmetric monoidal. The entirety of this proof also goes through for $l_{\mathsf{CAlg}(\mathcal{S})} : \mathsf{Pr}^L \to \mathsf{Pr}^L$, and hence the map $r : \mathcal{S} \to \mathsf{CAlg}(\mathcal{S})$ is symmetric monoidal. Now apply lemma 1.2.5, to obtain that the functor $\mathcal{G}$ is symmetric monoidal. □

As mentioned $\mathbb{E}_\infty$-ring spectra are, analogous to the 1-categorical theory of ring spectra, just commutative algebra objects in the $\infty$-category of spectra $\mathsf{Sp}$ equipped with the above symmetric monoidal structure.

**Definition 2.4.11.** An $\mathbb{E}_\infty$-ring spectrum is a commutative algebra object of the symmetric monoidal $\infty$-category of spectra $\mathsf{Sp}$. We let $\mathsf{CAlg}$ denote the $\infty$-category $\mathsf{CAlg}(\mathsf{Sp})$ of $\mathbb{E}_\infty$-rings.

*Remark* 2.4.12. The sphere spectrum is the unit of the smash product symmetric monoidal structure, and hence it aquires the structure of an $\mathbb{E}_\infty$-ring spectrum canonically. This $\mathbb{E}_\infty$-ring spectrum structure arises from the canonical maps of spaces $S^n \wedge S^m \to S^{n+m}$.

Lets now justify the name $\mathbb{E}_\infty$-ring spectrum.

**Proposition 2.4.13.** *Any $\mathbb{E}_\infty$-ring spectrum $R$ gives rise to a commutative ring object in the homotopy category of spaces $\mathcal{H}$.*

*Proof.* Consider the homotopy groups of $R$, $\{\pi_n R\}_{n\in\mathbb{Z}}$. Consider the direct sum $\pi_\bullet R = \bigoplus_{n\in\mathbb{Z}} \pi_n R$, this has the structure of a graded ring, through the following the following composite map

$$\pi_k R \times \pi_l R \longrightarrow \pi_{k+l}(R \otimes R) \xrightarrow{\mu_*} \pi_{k+l}(R)$$

Because $R$ is an $\mathbb{E}_\infty$-ring spectrum, we get a bit more, namely that $\pi_\bullet R$ is graded commutative. Consider $x \in \pi_n R$ and $y \in \pi_m R$, then we have $xy = (-1)^{nm} yx$, where the sign arises from the fact that the following composition

$$S^{n+m} \simeq S^n \otimes S^m \overset{\sigma}{\simeq} S^m \otimes S^n \simeq S^{n+m},$$

is given by the sign $(-1)^{nm}$. In particular $\pi_0 R$ has the structure of a commutative ring. Because $R$ is an $\mathbb{E}_\infty$-ring, its underlying space $X = \Omega^\infty R$ is an $\mathbb{E}_\infty$-space by proposition 2.2.4 and example 2.1.13, hence $R$ represents a commutative monoid in $\mathcal{H}$. □

Note that the above proposition could have been done without homotopy groups entering the picture. But then it would not be quite as easy to see that $\pi_\bullet R$ is graded commutative and that we, at this point, get the following result for free.

SphereCor **Corollary 2.4.14.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum, then $\pi_n(R)$ is a $\pi_0(R)$-module, for every $n \geq 0$.*

The following is a consequence of theorem 3.4.4.2 of [25].

RegResultsModR **Proposition 2.4.15.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. The $\infty$-category $\mathsf{Mod}_R$ is a presentable symmetric monoidal $\infty$-category.*

The following examples relate the classical theory of $R$-modules to the theory of module object over an $\mathbb{E}_\infty$-ring spectrum.

1-catRings **Example 2.4.16.** Every commutative ring $R$ can be seen as a commutative ring spectrum by identifying $R$ with its Eilenberg-MacLane spectrum $HR$. In this case, the $\infty$-category $\mathsf{Mod}_R$ has the following concrete algebraic description: its objects are chain complexes of 1-categorical $R$-modules, i.e. there is an equivalence $\mathsf{Mod}_R \simeq \mathcal{D}(R)$. Here $\mathcal{D}(R)$ is the derived $\infty$-category of $R$-modules, which is formed from $\mathsf{Ch}(R)$ via formally inverting quasi-isomorphisms. The

$\infty$-category $\mathcal{D}(R)$ inherits a symmetric monoidal structure from the symmetric monoidal structure on the 1-category of chain complexes of $R$, $\mathsf{Ch}(R)$, see [25] Remark 7.1.1.16. Let $\otimes_R$ be the symmetric monoidal product on $\mathcal{D}(R)$. See [25] theorem 7.1.2.13, for the details of the equivalence.

homol-to-homot **Example 2.4.17.** Recall that the Dold-Kan equivalence gives an equivalence between group homology, homology of chain complexes and homotopy groups of simplicial sets,

HomoToHmpty (2.1)
$$H_n(A, R) \simeq H_n(NA, R) \simeq \pi_n(A) \otimes R,$$

for $A$ an simplicial abelian group, and $N : \mathsf{sAb} \to \mathsf{Ch}(R)_{\geq 0}$. This fact has a version for unbounded chain complexes, and stably simplicial abelian groups, which are equivalent by an version of the Dold-Kan equivalence, see [13] Proposition 5.8. Both of these turn out to be Quillen-equivalent to the category of $HR$-module spectra for an Eilenberg-MacLane spectrum $R$, see [45]. Which through example 2.2.5 lifts (2.1) to a statement regarding $\infty$-categories. Hence if we wish to calculate homology groups of chain complexes of $R$-modules, we may instead calculate homotopy groups of the associated Eilenberg-MacLane spectrum.

SP-TO-MOD **Example 2.4.18.** The following example will become important later. Let $G$ be a group and let $G_+ = G \cup \{*\}$. Consider $G_+$ as a category with a single object, with a map for each $g \in G_+$. Then we define the classifying space of $G$ as $BG := N(G_+)$. Consider $G_+$ as a discrete space, then $G_+$ obviously is a $G$-space. Through example 2.2.5 the suspension spectrum $\Sigma^\infty G_+$ obtains the structure of an $\mathbb{E}_\infty$-ring spectrum. It was shown by M. Ando, A. Blumberg, and D. Gepner in [27] that there is an equivalence between the $\infty$-category of module spectra over $\Sigma^\infty G_+$ and the $\infty$-category of spectra with a $G$-action,

$$\mathsf{Mod}_{\Sigma^\infty G_+} \simeq \mathsf{Sp}^{BG}.$$

The result lifts to a Quillen equivalence which was shown by J. Lind and C. Malkiewich in [12]. The basic object of the following chapter are $G$-equivariant spectra. In light of this example we shall often talk about $G$-equivariant spectra as if they were modules with a $G$-action.

**Definition 2.4.19.** Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. We let $\mathsf{Mod}_R^{\mathrm{perf}}$ denote the smallest stable subcategory of $\mathsf{Mod}_R$ which contains $R$, regarded as a module over itself, and is closed under retracts. We will say that a $R$-module $M$ is *perfect* if it belongs to $\mathsf{Mod}_R^{\mathrm{perf}}$.

**Example 2.4.20.** Via 2.4.16 it can be shown that the objects of $\mathsf{Mod}_R^{\mathrm{perf}}$ are bounded chain complexes of finitely generated projective $R$-modules. The analog of proposition 2.4.15 also holds for $\mathsf{Mod}_R^{\mathrm{perf}}$, hence we can apply $K$-theory to it.

It is important to note that our $K$-theory will not give the correct $K$-theory as it would use the tensor product symmetric monoidal structure on $\mathsf{Mod}_R$, and thus we must use Waldhausens $K$-theory through the $S_\bullet$-construction. In that case the $K$-theory of an $\mathbb{E}_\infty$-ring spectrum is as follows.

**Definition 2.4.21.** Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. We set $K(R) = K(\mathsf{Mod}_R^{\mathrm{perf}})$.

**Example 2.4.22.** As mentioned if $R$ is a commutative ring, then the associated Eilenberg MacLane spectrum $HR$ is an $\mathbb{E}_\infty$-ring spectrum. Our $K$-theory and the one above, applied to $HR$, coincides with the classical $K$-theory spectrum of $R$.

Compared to a general $\mathbb{E}_\infty$-ring spectrum Eilenberg MacLane spectra are extremely well-behaved, e.g. they have homotopy concentrated in a single degree. So the above example really shows that we should expect the $K$-theory spectrum for a general $\mathbb{E}_\infty$-ring spectrum to be hard to compute, and even harder for symmetric monoidal $\infty$-categories which are not $\mathsf{Mod}_R$. Hence it would be nice to have related invariants, which should help us in these calculations. The next section is dedicated to one of these, namely topological Hochschild homology.

## 2.5 Topological Hochschild Homology of $\mathbb{E}_\infty$-rings.

In this section we will define topological Hochschild homology for $\mathbb{E}_\infty$-ring spectra. In the following chapter we will show that it admits the structure of a cyclotomic $\mathbb{E}_\infty$-ring spectrum. We will follow [33] and [16]. The construction of topological Hochschild homology in [38] is equivalent to the classical construction of topological Hochschild homology, (as an orthogonal cyclotomic spectrum [10]) see Section III.6. The construction which we present now is given in [33], which is also equivalent to the classical construction, in the case of $\mathbb{E}_\infty$-ring spectra.

**Proposition 2.5.1.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. The functor corepresented by $R$,*

$$\mathrm{Map}_{\mathsf{CAlg}}(R,-) : \mathsf{CAlg} \to \mathcal{S},$$

*admits a left adjoint, which we shall denote $R^{\otimes(-)} : \mathcal{S} \to \mathsf{CAlg}$.*

*Proof.* By Theorem 2.2.2, the $\infty$-category of $\mathbb{E}_\infty$-rings, $\mathsf{CAlg}$, is presentable. By [22] Proposition 5.5.2.7, functor out of a presentable $\infty$-category into spaces which is corepresented by an object preserve limits and is accessible, hence $\mathrm{Map}_{\mathsf{CAlg}}(R,-)$ is cocontinuous and accesible. Then the adjoint functor theorem immediately gives the desired result. □

THH **Definition 2.5.2.** Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. The topological Hochschild homology $\mathrm{THH}(R)$ of $R$ is defined by $\mathrm{THH}(R) := R^{\otimes \mathbb{T}}$. Where $\mathbb{T}$ is the circle group.

One of the philosophical ideas concerning algebra with the base-ring $\mathbb{S}$ rather than $\mathbb{Z}$ is that the counting process over $\mathbb{S}$ in some sense also remembers how to count, rather than just the result. We will now see an instance of this, exemplified by the difference between topological Hochschild homology of the Eilenberg-MacLane spectrum of $\mathbb{F}_p$ and the usual Hochschild homology of $\mathbb{F}_p$ as a $\mathbb{Z}$-module, which we write $\mathbb{F}_p/\mathbb{Z}$. The following discussion follows [9].

The following is a highly non-trivial and extremely important result due to M. Bökstedt, and is often called *Böksteds periodicity theorem.*

Bokstedt **Theorem 2.5.3.** *Fix a prime $p$. Then*

$$\pi_* \mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[x],$$

*where* $\deg(x) = 2$.

Let us compare it to the usual Hochschild homology. Let $R$ be a ring. Let $\otimes_R$ be the left derived of the symmetric monoidal product on $\mathcal{D}(R)$ mentioned in example 2.4.16.

**Definition 2.5.4.** Let $R$ be a $k$-module, then we define

$$HH(R/k) := R^{\otimes_k \mathbb{T}},$$

where $R^{\otimes_k(-)} : \mathcal{S} \to \mathcal{D}(k)$ is defined analogously to definition 2.5.2.

AltDefiHH *Remark* 2.5.5. Note that we have the following equivalences,

$$\begin{aligned} HH(R/k) &:= R^{\otimes_k \mathbb{T}} \\ &\simeq R^{\otimes_k (\Delta^1 \cup_{\partial\Delta^1} \Delta^1)} \\ &\simeq R^{\otimes_k \Delta^1} \otimes_{R^{\otimes_k \partial\Delta^1}} R^{\otimes_k \Delta^1} \\ &\simeq R \otimes_{R \otimes_k R} R. \end{aligned}$$

using the decomposition of $\mathbb{T}$ as a pushout, that $R^{\otimes_k(-)}$ is a left adjoint, hence commutes with limits, and that $\Delta^1$ is contractible, i.e.

$$R^{\otimes_k \partial\Delta^1} \simeq R^{\otimes_k \bullet} \otimes_k R^{\otimes_k \bullet} \simeq R \otimes_k R.$$

HHFP **Theorem 2.5.6.** *Fix a prime p. Consider $\mathbb{F}_p$ as a $\mathbb{Z}$-module, then*

$$HH(\mathbb{F}_p/\mathbb{Z}) \simeq \Gamma_{\mathbb{F}_p}\{x\},$$

*i.e. a divided polynomial algebra on a generator x, for which* $\deg(x) = 2$.

Recall that the divided polynomial algebra is defined as follows:

$$\Gamma_{\mathbb{F}_p}\{x\} := \mathbb{F}_p[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, ...],$$

and note $x^p = p!\frac{x^p}{p!} = 0$.

*Proof.* For the sake of brevity we will refrain from giving the details of the following equivalences. Using remark 2.5.5, we see that we want to calculate $\mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p} \mathbb{F}_p$. We begin by calculating the inner most derived tensor-product. We now resolve $\mathbb{F}_p$ as a commutative differentially graded algebra, i.e. utilize the following equivalence $\mathbb{F}_p \simeq \Lambda_{\mathbb{Z}}\{y\}$, where $\deg(y) = 1$ and $\delta(y) = p$. For a definition of the exterior algebra in this setting, i.e. over an object in the category $\mathsf{Ab}$ equipped with a symmetric monoidal structure $(\mathsf{Ab}, \otimes, \mathbb{Z})$, see [37]. Hence

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \Lambda_{\mathbb{Z}}\{y\} \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \Lambda_{\mathbb{F}_p}\{y\}.$$

where the degree of $y$ on the right hand side is 1, and the differential is trivial: $\delta(y) = 0$. Next we use the following equivalence,

$$\mathbb{F}_p \simeq \Lambda_{\mathbb{F}_p}\{y\} \otimes \Gamma_{\mathbb{F}_p}\{x\}$$

where $\deg(x) = 2$. The left hand side is free over $\Lambda_{\mathbb{F}_p}\{y\}$. So finally we have

$$\begin{aligned} HH(\mathbb{F}_p/\mathbb{Z}) &\simeq (\Lambda_{\mathbb{F}_p}\{y\} \otimes \Gamma_{\mathbb{F}_p}\{x\}) \otimes_{\Lambda_{\mathbb{F}_p}\{y\}} \mathbb{F}_p \\ &\simeq \Gamma_{\mathbb{F}_p}\{x\}. \end{aligned}$$

□

*Remark* 2.5.7. The denominators of the generators of $\Gamma_{\mathbb{F}_p}\{x\}$ is not an extraordinary phenomenon. They in fact arise for any commutative $k$-algebra over a commutative ring $k$, [9]. The denominators are encoding the $n!$ ways to count to $n$. In higher algebra, these denominators disappear because the process of counting is also remembered rather than just the result. That these disappear can be seen by comparing theorem 2.5.3 and theorem 2.5.6.

We end this chapter by showing that the map $R \to THH(R)$ of $\mathbb{E}_\infty$-ring spectra, induced from the map of spaces $* \to \mathbb{T}$, is initial in a certian sense.

**Definition 2.5.8.** Let $G$ be a group, and let $\mathscr{C}$ be an $\infty$-category. A $G$-equivariant object in $\mathscr{C}$ is a functor $BG \to \mathscr{C}$, where $BG$ is a classifying space of $G$. The $\infty$-category of $G$-equivariant objects in $\mathscr{C}$ is thusly the functor $\infty$-category $\mathscr{C}^{BG} := \mathsf{Fun}(BG, \mathscr{C})$.

The notion which we shall use is that of a $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring spectrum. The following proposition follows from the introduction of [38].

**Proposition 2.5.9.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. Then* $\mathrm{THH}(R)$ *admits the structure of a $\mathbb{T}$-equivariant object in* $\mathsf{CAlg}$*, i.e. it is an object of* $\mathsf{CAlg}^{B\mathbb{T}}$.

*Proof.* Consider the continuous action of $\mathbb{T}$ on $\mathbb{T}$ through left multiplication: $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$, which by adjunction gives the first of the following maps,

$$\mathbb{T} \to \mathrm{Map}_{\mathcal{S}}(\mathbb{T}, \mathbb{T}) \to \mathrm{Map}_{\mathsf{CAlg}}(\mathrm{THH}(R), \mathrm{THH}(R))$$

where the second map is induced from applying $R^{\otimes(-)}$. Consequently, the functor $\mathrm{THH} : \mathsf{CAlg} \to \mathsf{CAlg}$ refines to a functor $\mathrm{THH} : \mathsf{CAlg} \to \mathsf{CAlg}^{B\mathbb{T}}$. □

Hence if $R$ is an $\mathbb{E}_\infty$-ring spectrum, then $\mathrm{THH}(R)$ is a $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring spectrum equipped with a non-equivariant map $i : R \to \mathrm{THH}(R)$ of $\mathbb{E}_\infty$-ring spectra.

forgetTHH *Remark* 2.5.10. Note we have that the functor $(-)^{\otimes \mathbb{T}} : \mathsf{CAlg} \to \mathsf{CAlg}^{B\mathbb{T}}$ is left adjoint to the forgetful functor $U : \mathsf{CAlg}^{B\mathbb{T}} \to \mathsf{CAlg}$.

We end this chapter with the following fact due to J.McClure, R.Schwänzl and R.Vogt [33], which says that topological Hochschild homology is initial with these properties.

THHuniprop **Proposition 2.5.11.** *Let $R \in \mathsf{CAlg}$. If $R'$ is $\mathbb{T}$-equivariant spectrum equipped with a map $f : R \to R'$ of $\mathbb{E}_\infty$-rings, then there exists a unique $\mathbb{T}$-equivariant map $\overline{f} : \mathrm{THH}(R) \to R'$ of $\mathbb{E}_\infty$-ring spectra such that the following diagram in* $\mathsf{CAlg}$

$$\begin{array}{ccc} R & \xrightarrow{i} & \mathrm{THH}(R) \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & R' \end{array}$$

*commutes.*

*Proof.* A map $f : R \to R'$, is a map $R \to U(R')$, where $U$ is the forgetful functor of 2.5.10. Hence by the adjunction of 2.5.10 this map corresponds to the desired map $\overline{f} : THH(R) \coloneqq R^{\otimes \mathbb{T}} \to R'$ in $\mathsf{CAlg}^{B\mathbb{T}}$. □

# 3 The Tate construction

The aim of this chapter is to define the Tate-construction for a group and $\infty$-categories with (co)limits indexed by the classifying space of $G$. The Tate-construction will be pervasive throughout the thesis. We shall later see that the Tate-construction gives rise to a construction called the Tate-diagonal which has a strong connection to $p$-completion. We shall also see the Tate-diagonal admits a lax symmetric monoidal structure if $G$ is finite and the stable $\infty$-category is taken to be $\mathsf{Sp}$. Both of which will prove to be immensely powerful theorems.

## 3.1 The Norm-map

The Tate-construction will be the cofiber of an $\infty$-categorical analog of the norm-map. Recall that given a finite group $G$, and $M$ a $G$-module, the norm map $\mathrm{Nm}_G : M_G \to M^G$ defined by $\mathrm{Nm}_G(m) = \sum_{g \in G} g \cdot m$, lets us "splice" the group homology $H_*(G; M)$ and the group cohomology $H^*(G; M)$ together, to form *Tate-cohomology*, given by

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & 1 \leq i \\ \mathrm{coker}(\mathrm{Nm}_G) & i = 0 \\ \ker(\mathrm{Nm}_G) & i = -1 \\ H_{-i-1}(G, M) & i \leq -2 \end{cases}$$

The Tate-construction for a finite group $G$ will recover the Tate-cohomology when the input is an Eilenberg-MacLane spectrum, as we shall see later in this section.

Let us define the $\infty$-categorical analog of fixpoints and orbits. Recall we define (co)limits in diagram categories through [22] corollary 5.1.2.3.

**Definition 3.1.1.** Let $G$ be a group, and let $\mathscr{C}$ be an $\infty$-category. If $\mathscr{C}$ admits colimits indexed by $BG$, then we define the *homotopy orbits functor* by

$$-_{hG} : \mathscr{C}^{BG} \to \mathscr{C} : (F : BG \to \mathscr{C}) \mapsto \mathrm{colim}_{BG} F.$$

Dually, if $\mathscr{C}$ admits limits indexed by $BG$, then we define the *homotopy fixpoint functor* by

$$-^{hG} : \mathscr{C}^{BG} \to \mathscr{C} : (F : BG \to \mathscr{C}) \mapsto \lim_{BG} F.$$

Our main example is going to be $\mathscr{C} = \mathsf{Sp}$. In this case we shall use the following spectral sequences for calculations. See [3], for their definition.

SS-homotopy **Proposition 3.1.2.** *Let $G$ be a topological group and let $X$ be a $G$-equivariant spectrum, then there are spectral sequences,*

$$E^2_{s,t} = H_s(G, \pi_t(X)) \implies \pi_{t+s}(X_{hG}),$$
$$E_2^{s,t} = H^{-s}(G, \pi_t(X)) \implies \pi_{t+s}(X^{hG}).$$

*with differentials $d^r_{hG} : E^r_{s,t} \to E^t_{s-r,t+r-1}$ and $d_r^{hG} : E_r^{s,t} \to E_r^{s+r,t+r-1}$ respectively. I.e. the first is with homological Serre indexing, and the second is with cohomological Serre indexing.*

As is already evident the norm-map $\mathrm{Nm}_G : X_{hG} \to X^{hG}$ is going to be a natural transformation of functors $\mathscr{C}^{BG} \to \mathscr{C}$. Another property which we know that it should have is that it should specialize to the classical norm-map for Eilenberg-MacLane spectra.

ex-norm-cons **Example 3.1.3.** Let $f : X \to Y$ be any map of Kan complexes. Let $f^* : \mathscr{C}^Y \to \mathscr{C}^X$ be the induced pullback functor. If they exist let $f_!, f_* : \mathscr{C}^X \to \mathscr{C}^Y$ be the left and right adjoints resp. of $f^*$. Let $f : BG \to *$ be the projection to a point, then $f_!, f_* : \mathscr{C}^{BG} \to \mathscr{C}$ are given by $-_{hG}$ and $-^{hG}$ respectively.

We will use the notation of example 3.1.3 for the rest of this section.

**Definition 3.1.4.** A map $f : X \to Y$ of Kan complexes is a *relative finite groupoid* if the homotopy fibers of $f$ have finitely many connected components, each of which is equivalent to the classifying space of a finite group.

**Proposition 3.1.5.** *Let $\mathscr{C}$ be a pre-additive $\infty$-category admitting (co)limits indexed by BG for any finite group G, and let $f : X \to Y$ be a relative finite groupoid. Then there is a natural transformation* $\mathrm{Nm}_f : f_! \to f_*$ *of functors* $\mathscr{C}^X \to \mathscr{C}^Y$.

The following proof relies on an Hopkins-Lurie ambidexterity-type argument, see [28] for another example, in that we shall consider an ambidextrous adjunction, i.e. an, yet to be defined, adjunction $\delta_! \dashv \delta^* \dashv_*$ for which $\delta_! \simeq \delta_*$.

*Proof.* We begin by fixing some notation. Given a map $f : X \to Y$ of Kan complexes, consider its diagonal map $\delta : X \to X \times_Y X$. We will later argue that the above assumptions will secure the existence of an equivalence of functors $\delta_!, \delta_* : \mathscr{C}^X \to \mathscr{C}^{X \times_Y X}$, denoted $\mathrm{Nm}_\delta : \delta_! \to \delta_*$, for now assume it to be true. Let $p_0, p_1 : X \times_Y X \to X$ denote the projections onto the factors. Consider the following natural transformation given by

$$p_0^* \xrightarrow{\varepsilon} \delta_* \delta^* p_0^* \cong \delta_* \xrightarrow{\mathrm{Nm}_\delta^{-1}} \delta_! \cong \delta_! \delta^* p_1^* \xrightarrow{\eta} p_1^*,$$

Where $\varepsilon$ and $\eta$ is the unit and counit of the $\delta^* \dashv \delta_*$ and $\delta_! \dashv \delta^*$ adjunction respectively. From this map we obtain via adjunction a map $\mathrm{id}_{\mathscr{C}^X} \to p_{0*} p_1^*$. Consider the diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_0} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y. \end{array}$$

By [22] Lemma 6.1.6.3 this diagram is right adjointable, which means that the natural transformation $f^* f_* \to p_{0*} p_1^*$ is an equivalence. These two maps gives a map $\mathrm{id}_{\mathscr{C}^X} \to f^* f_*$ of functors $\mathscr{C}^X \to \mathscr{C}^Y$, which is adjoint to a natural transformation $f_! \to f_*$ of functors $\mathscr{C}^X \to \mathscr{C}^Y$.

Now lets argue that there exists an equivalence of functors $\mathscr{C}^X \to \mathscr{C}^{X \times_Y X}$, $\mathrm{Nm}_\delta : \delta_! \to \delta_*$, i.e. that the adjunction $\delta_! \dashv \delta^* \dashv_*$ is ambidextrous. We will argue "inductively" on the connectivity of $f$ and its diagonal $\delta$.

$(-1)$-truncated
: Assume $f$ is $(-1)$-truncated which is equivalent to $\delta : X \to X \times_Y X$ being an equivalence, which ensures that $\mathrm{Nm}_\delta$ exists and is an equivalence. Now [22] Proposition 6.1.6.7 applies and gives the existence of $\mathrm{Nm}_f$ and that it is an equivalence.

0-truncated
: Assume $f$ is 0-truncated with finite fibers, and consider the Mayer-Vitoris sequence on homotopy groups induced by $\delta$,

$$\ldots \longrightarrow \pi_n(X \times_Y X) \longrightarrow \pi_n(X) \oplus \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_{n-1}(X \times_Y X) \longrightarrow \ldots$$

From which we see that the diagonal is $(-1)$-truncated, and so the $(-1)$-truncated case applies and secures the existence of $\mathrm{Nm}_\delta$ and that it is an equivalence. Now proposition 6.1.6.12 of [22] applies and gives the existence of $\mathrm{Nm}_f$ and that it is an equivalence.

1-truncated

Under the assumptions above we may conclude that $f : X \to Y$, in particular, is 1-truncated. Consider the Mayer-Vietoris sequence on homotopy groups, analogously to the previous case we see that $\delta : X \to X \times_Y X$ is a 0-truncated map with finite fibers, hence $\mathrm{Nm}_\delta$ exists and is an equivalence by the 0-truncated case. Because we have assumed $\mathscr{C}$ to have (co)limits indexed by classifying spaces of finite groups, $f_!$ and $f_*$ exists, hence the above secures the existence of the desired norm map, $\mathrm{Nm}_f : f_! \to f_*$.

□

NormDefi **Definition 3.1.6.** Let $G$ be a finite normal subgroup of a topological group $H$, and let $f : BH \to B(H/G)$ be the projection. Let $\mathscr{C}$ be a preadditive $\infty$-category with (co)limits indexed by $BG$. Then there is a natural transformation $\mathrm{Nm}_f : f_! \to f_*$, which we shall call *the norm map for $G$ relative to $H$*.

As mentioned the Tate-construction will be the cofiber of the norm map, hence we shall need cofibers in our $\infty$-category $\mathscr{C}$. Therefore we require it to be stable.

**Definition 3.1.7.** Let $\mathscr{C}$ be a stable $\infty$-category with all (co)limits indexed by $BG$ for some finite group $G$. If $G$ is a normal subgroup of a topological group $H$, we write $-^{tG}$ for the functor

$$-^{tG} : \mathscr{C}^{BH} \to \mathscr{C}^{B(H/G)} : X \mapsto \mathrm{cofib}(\mathrm{Nm}_f : f_! X \to f_* X),$$

where $f : BH \to B(H/G)$ is the projection. This functor is called the *Tate construction.*

*Remark* 3.1.8. In the case where $G = H$ is a topological group, then we are in the situation of example 3.1.3, for which the Tate-construction is a functor

$$-^{tG} : \mathscr{C}^{BG} \to \mathscr{C} : X \mapsto \mathrm{cofib}(\mathrm{Nm}_f : X_{hG} \to X^{hG}).$$

In this case we will often write $\mathrm{Nm}_G$ instead of $\mathrm{Nm}_f$.

In the following sections we shall see that the Tate construction has a strong relation to $p$-completion, which is a universal functor, hence it feels rather akward that this is not a universal functor. This problem is remedied by abstracting the problem to a general Kan complex rather than the classifying space of a finite group. This is Theorem I.4.1 [38], which is originally due to J. R. Klein [14], applied to $S = BG$ for a finite group $G$. Using the notation of example 3.1.3, we have the following universal description of $-^{tG}$.

UniPropTate **Proposition 3.1.9.** *Let $f : BG \to *$ be the projection to the point. There is a unique initial functor $f_*^T : \mathsf{Sp}^{BG} \to \mathsf{Sp}$ with a natural transformation $-^{hG} \to f_*^T$ with the property that $f_*^T$ vanishes on compact objects, and the fiber of $-^{hG} \to f_*^T$ commutes with colimits.*

Note that $-^{tG}$ for $f : BG \to *$ has a natural transformation $-^{hG} \to -^{tG}$, since it is a cofiber of functor into $-^{hG}$. The compact objects of $\mathsf{Sp}^{BG}$ are those which are given by finite cones of certain $G$-equivariant spectra called induced spectra, we show that $-^{tG}$ vanish on these in proposition 3.1.12, in the case that $G$ is finite. Following [38] we will denote the subcategory of these $\mathsf{Sp}^{BG}_{ind}$. The last property, namely that the fiber $\mathrm{fib}(-^{hG} \to -^{tG}) \simeq -_{hG}$ commutes with colimits, holds because it is defined as a colimit. Hence $-^{tG}$ is in fact a universal functor.

We will for the remainder of this section prove a series of regularity results for the Tate construction. The following is an immediate consequence of the Tate construction being defined as a cofiber.

**Corollary 3.1.10.** *Let $\mathscr{C}$ be a stable $\infty$-category with all (co)limits indexed by BG for G a finite group, then the Tate-construction $-^{tG} : \mathscr{C}^{BG} \to \mathscr{C}$ is an exact functor.*

We now show that the Tate construction vanish on the generators of $\mathsf{Sp}^{BG}_{ind}$.

**Definition 3.1.11.** A $G$-equivariant spectrum of the form $Y \otimes \Sigma^\infty G_+$, where the $G$-action is given through left multiplication on the right hand factor, are called *induced $G$-spectra.*

InducedSpecTriv

**Proposition 3.1.12.** *If $G$ is a finite group seen as a discrete space, then $X^{tG} \simeq 0$ for $X$ an induced $G$-spectrum.*

*Proof.* Note that $X \otimes \Sigma^\infty G_+ \simeq \prod_{g\in G} X \simeq \bigoplus_{g\in G} X$ when $G$ is finite, because $\mathsf{Sp}$ is preadditive. Recall that a finite group is a filtered colimit of its cyclic subgroups. Hence we can limit ourselves to the case where $G$ is a cyclic group. The result now follows by the following observation, which ultimately is because $G$ is finite,

$$(X \otimes \Sigma^\infty G_+)_{hG} \simeq X \simeq (X \otimes \Sigma^\infty G_+)^{hG}.$$

For homotopy orbits it is clear, since it commutes with colimits and the $G$-action on $X$ is trivial since $G$ is cyclic. For homotopy fixpoint we can consider the homotopy fixpoint spectral sequence,

$$H^t(G, \pi_q(X \otimes \Sigma^\infty G_+)) \implies \pi_{q-t}(((X \otimes \Sigma^\infty G_+)^{hG})$$

Now note that $X \otimes \Sigma^\infty G_+$ can be seen as a $\Sigma^\infty G_+$-module via example 2.4.18. Consider the following isomorphisms

$$\begin{aligned}\pi_q(X \otimes \Sigma^\infty G_+) &\cong \pi_q(\prod_{g\in G} X)\\ &\cong \bigoplus_{g\in G} \pi_q(X)\\ &\cong \pi_q(X) \otimes_{\mathbb{Z}} \mathbb{Z}[G],\end{aligned}$$

which show that $\pi_q(X \otimes \Sigma^\infty G_+)$ is an induced $G$-module. Recall that for $G$ a finite group induced and coinduced $G$-modules coincide, and that group cohomology with coefficients in a coinduced $G$-module is concentrated in degree 0. Because of this there is only a single contributing factor to the total degree of the $E_2$-page, hence

$$\begin{aligned}\pi_q(((X \otimes \Sigma^\infty G_+)^{hG}) &\cong H^0(G, \pi_q(X \otimes \Sigma^\infty G_+))\\ &\cong (\pi_q(X \otimes \Sigma^\infty G_+))^G\\ &\cong \pi_q(\bigoplus_{g\in G} X)^G\\ &\cong (\bigoplus_{g\in G} \pi_q(X))^G\\ &\cong \pi_q(X).\end{aligned}$$

Hence there is an equivalence, $X \simeq (X \otimes \Sigma^\infty G_+)^{hG}$, therefore the norm map is an equivalence as desired. □

*Remark* 3.1.13. Note that cones are finite limits and are therefore preserved by the Tate construction, hence $X^{tG} \simeq 0$ for all $X \in \mathsf{Sp}^{BG}_{ind}$, and therefore $-^{tG}$ satisfies all the properties of proposition 3.1.9.

**Definition 3.1.14.** Let $X \in \mathsf{Sp}$. We say that $X$ is bounded below (above) if there exists some $n \in \mathbb{Z}$ such that $\pi_i(X) \simeq 0$ for $i < n$ ($i > n$). We denote the full subcategory of $\mathsf{Sp}$ spanned by the bounded below (above) $n \in \mathbb{Z}$ spectra by $\mathsf{Sp}_{\geq n}$ ($\mathsf{Sp}_{\leq n}$).

Recall that $\mathsf{Sp}$ has a $t$-structure $(\mathsf{Sp}_{\geq 0}, \mathsf{Sp}_{0\leq})$ consisting of the connective and coconnective spectra, for which the heart $\mathsf{Sp}^\heartsuit := \mathsf{Sp}_{\leq 0} \cap \mathsf{Sp}_{\geq 0}$ is canonically equivalent to $N(\mathsf{Ab})$ via the Eilenberg-MacLane functor. In particular $\mathsf{Sp}$ has truncations $\tau_{\leq n} : \mathsf{Sp} \to \mathsf{Sp}_{n\leq}$ and $\tau_{\geq n} : \mathsf{Sp} \to \mathsf{Sp}_{\geq n}$, which arise as left adjoints and right adjoints to the inclusions $\mathsf{Sp}_{n\leq} \to \mathsf{Sp}$ and $\mathsf{Sp}_{\geq n} \to \mathsf{Sp}$ respectively. It turns out that the Tate construction, homotopy orbits and fixpoints play nicely together with the truncations.

LI2.6 **Lemma 3.1.15.** *Let $Y$ be a spectrum with $G$-action for some finite group $G$. The natural maps*

$$Y^{hG} \to \lim_n (\tau_{\le n} Y)^{hG},$$
$$Y_{hG} \to \lim_n (\tau_{\le n} Y)_{hG},$$
$$Y^{tG} \to \lim_n (\tau_{\le n} Y)^{tG},$$

*are equivalences.*

Before we prove this result lets recall the following definition.

**Definition 3.1.16.** Let $n \ge 0$. We say that a spectrum is *$n$-connected* if $\pi_i(X)$ is trivial for $i < n$. We say that a map $f : X \to Y$ is *$n$-connected* if the homotopy fibers of $f$ are $n$-connected.

*Proof of 3.1.15.* Note that it is formal for homotopy fixpoints, because limits commute with each other. Note that for any $n$, the map $Y \to (\tau_{\le n} Y)$ is $n$-connected. By proposition 3.1.2 taking homotopy orbits only increases connectivity, hence the map $Y_{hG} \to (\tau_{\le n} Y)_{hG}$ is (atleast) $n$-connected. Passing to the limit $n \to \infty$ gives the result. □

Another important result, concerning bounded spectra is the following, which we shall need later in this chapter. We will not prove it.

fixabove **Lemma 3.1.17.** *Let $G$ be a finite group, and let $X_i \in \mathsf{Sp}^{BG}$ for each $i \in I$. Then the map*

$$(\bigoplus_{i \in I} X^{hG}) \to (\bigoplus_{i \in I} X)^{hG},$$

*is an equivalence if $\bigoplus_{i \in I} X_i$ is bounded above.*

Note the following corollary which follows immediately from the definition of the Tate construction. It is in fact this corollary that we are interested in.

**Corollary 3.1.18.** *Let $G$ be a finite group, and let $X_i \in \mathsf{Sp}^{BG}$ for each $i \in I$. Then the map*

$$(\bigoplus_{i \in I} X^{tG}) \to (\bigoplus_{i \in I} X)^{tG},$$

*is an equivalence if $\bigoplus_{i \in I} X_i$ is bounded above.*

We now argue that the Tate-construction recovers Tate-cohomology on Eilenberg-MacLane spectra. In some sense this result follows because Eilenberg-MacLane spectra are both bounded below and above.

tate-eilenburg **Proposition 3.1.19.** *Let $G$ be a finite group, and $M$ a $G$-module. Let $HM$ be the Eilenberg-MacLane spectrum of $M$, regarded as an object of $\mathsf{Sp}^{BG}$. Then $\pi_n(HM^{tG}) \cong \hat{H}^{-n}(G, M)$ for all integers $n$.*

*Proof.* Consider the following fiber sequence

$$HM_{hG} \xrightarrow{\mathrm{Nm}_G} HM^{hG} \longrightarrow HM^{tG}.$$

It yields a long-exact sequence in homotopy groups,

$$\ldots \longrightarrow \pi_n(HM_{hG}) \longrightarrow \pi_n(HM^{hG}) \longrightarrow \pi_n(HM^{tG}) \longrightarrow \ldots$$

Consider the two spectral sequences of proposition 3.1.2, from these we may conclude that $\pi_n(HM_{hG}) \cong H_n(G, M)$ and $\pi_n(HM^{hG}) \cong H^{-n}(G, M)$. This follows because there is only a single nonzero term in each total degree. Furthermore the map $\pi_0(\mathrm{Nm}_G) : \pi_0(HM_{hG}) \to \pi_0(HM^{hG})$ can be identified with the classical norm map. This can be seen by considering the comparison map between the two. From this we recover $\pi_n(HM^{tG}) \cong \hat{H}^{-n}(G, M)$ for all integers. □

Analogous to homotopy fixpoints and homotopy orbits there is a spectral sequence converging to the homotopy groups of the Tate construction. That this spectral sequence relates the Tate construction to Tate cohomology is in fact a consequence of proposition 3.1.19.

TateSS **Proposition 3.1.20.** *Consider $G$ a topological group and let $X \in \mathsf{Sp}^{BG}$, then there is a spectral sequence*

$$E_2^{t,q} = \hat{H}^{-q}(G, \pi_t(X)) \implies \pi_{q+t}(X^{tG}),$$

*with differentials $d_r^{tG} : E_r^{t,q} \to E_r^{t+r,q+r-1}$.*

*Proof.* We start by recalling a number of facts concerning $\mathsf{Sp}$. $\mathsf{Sp}$ is a stable $\infty$-category equipped with a $t$-structure, with sequential colimits compatible with the $t$-structure and homotopy groups preserve these. Because of this filtered objects in $\mathsf{Sp}$ make sense. Recall that filtered objects are triples consisting of a spectrum $X \in \mathsf{Sp}$, a sequential diagram $\hat{X} : N(\mathbb{Z}, <) \to \mathsf{Sp}$, and an equivalence $X \simeq \operatorname{colim}_{n\in\mathbb{Z}} \hat{X}$. Now consider the filtered object $X^{tG}$ of $\mathsf{Sp}$, where $G$ is finite and $X \in \mathsf{Sp}^{BG}$, the sequential diagram is the following:

$$\ldots \longrightarrow (\tau_{n\leq} X)^{tG} \longrightarrow (\tau_{n-1\leq} X)^{tG} \longrightarrow \ldots$$

Note that $X \simeq \operatorname{colim}_{n\in\mathbb{Z}} (\tau_{n\leq} X)^{tG}$ and that $\pi_i((\tau_{n\leq} X)^{tG}) \simeq 0$ for $i < n$. In this case we have from proposition 1.2.2.14 [25] a spectral sequence, called the *spectral sequence associated to the filtered object $X$*, which first page is:

$$E_1^{t,q} \simeq \pi_{p+q}(\operatorname{cofib}((\tau_{p+1\leq} X)^{tG} \to (\tau_{p\leq} X)^{tG})) \implies \pi_{q+t}(\operatorname{colim}_{n\in\mathbb{Z}} (\tau_{n-1\leq} X)^{tG}) \simeq \pi_{q+t}(X^{tG}),$$

That this is the first page follows from proposition 1.2.2.7 [25]. Now we have

$$\begin{aligned} \pi_{p+q}(\operatorname{cofib}((\tau_{p+1\leq} X)^{tG} \to (\tau_{p\leq} X)^{tG})) &\simeq \pi_{p+q}((\Sigma^p (H\pi_p(X))^{tG})) \\ &\simeq \pi_q((H\pi_p(X))^{tG}) \\ &\simeq \hat{H}^{-q}(G, \pi_p(X)). \end{aligned}$$

Where the first equivalence is the usual model for the cofiber: $\Sigma^n H\pi_n X$, and the second is suspending, and the last is proposition 3.1.19. Now collecting our results we obtain

$$E_1^{t,q} \simeq \hat{H}^{-q}(G, \pi_p(X)) \implies \pi_{p+q}(X^{tG}).$$

Up to reindexing this is the desired result. □

The idea to employ proposition 1.2.2.14 of [25] to obtain this spectral sequence was brought to the attention of the author by Jonas McCandless. The author has not been able to find a similiar proof in the litterature.

The following lemma is going to be a key ingredient in the proof of the Tate Lemmas, which are the key technical lemmas of this thesis.

LI2.8 **Lemma 3.1.21.** *Let $Y$ be a spectrum with $C_p$-action such that multiplication by $p$ is an isomorphism on $\pi_i Y$ for all $i \in \mathbb{Z}$. Then $Y^{tC_p} \simeq 0$.*

*Proof.* Because multiplication by $p$ is an isomorphism on $\pi_i Y$, both sides of the norm map become purely algebraic, and the norm map becomes the usual algebraic norm map, i.e. $\pi_i Y^{hC_p} \cong (\pi_i Y)^{C_p}$ and $\pi_i Y_{hC_p} \cong (\pi_i Y)_{C_p}$. This is a consequence of the homotopy fixed point spectral sequence. Now because multiplication by $p$ is an isomorphism, the norm map $(\pi_i Y)^{C_p} \to (\pi_i Y)_{C_p}$ is an isomorphism, which implies that the cofiber is 0, which shows the claim. □

From proposition 3.1.19 we see that there naturally is a ring structure on Tate cohomology. This is a consequence of the fact that the Tate-construction admits a canonical symmetric monoidal structure. The following is theorem I.3.1 of [38].

tatedialax **Theorem 3.1.22.** *The functor* $-^{tG} : \mathsf{Sp}^{BG} \to \mathsf{Sp}$ *admits a unique lax symmetric monoidal structure which makes the natural transformation* $-^{hG} \to -^{tG}$ *lax symmetric monoidal. More precisely the space consisting of all pairs of a lax symmetric monoidal structure on the functor* $-^{tG}$ *together with a lax symmetric monoidal refinement of the natural transformation* $-^{hG} \to -^{tG}$ *is contractible.*

## 3.2 $p$-Completion and the Tate-Construction

In the next section we will prove two very important lemmas, namely the "Tate-lemmas", also called the Tate orbit lemma and the Tate fixpoint lemma. A nice side effect of the lemmas needed for the Tate-lemmas is that there is a relation between $p$-completion of a spectrum and the Tate construction of $C_p$ under some mild conditions. This section is dedicated to giving this relation. We begin with giving a brief reminder on $p$-completion of spectra. We introduce $p$-completion in a slightly different fashion to avoid being dependent on a model for $\mathsf{Sp}$. We note that this process is equivalent to $p$-completion through Bousfield-localizations of any of the model categories category of spectra presenting $\mathsf{Sp}$, for which the canonical reference is A.K Bousfields orginal article on the subject [2].

Recall that the "usual" notion of $p$-completion of a spectrum $X$ is given as the colimit,

pcomp (3.1)
$$X_p^{\wedge} \coloneqq \operatorname{colim}_n(X \otimes S(C_{p^n})),$$

where $S(-) : \mathsf{Ab} \to \mathsf{Sp}$ is the *Moore spectrum.* The Moore spectrum of an abelian group $G$ is characterized by the following,

$$\begin{aligned} \pi_k(SG) &= 0 \quad \text{for } k < 0, \\ H_k(SG, \mathbb{Z}) &= 0 \quad \text{for } k > 0, \\ \pi_0(SG) &= G. \end{aligned}$$

MooreModel **Example 3.2.1.** Let $R$ be a $\mathbb{E}_\infty$-ring spectrum, and then consider $x \in \pi_0(R)$. The element $x$ corresponds to a map $S^0 \to R$, which we may smash with $R$, to obtain a map $R \to R \otimes R$, then using the multiplication on $R$ we obtain

$$R \longrightarrow R \otimes R \xrightarrow{m} R, \qquad (\text{composite } .x : R \to R)$$

The composite map is called the multiplication by $x$ map of $R$, and is denoted $x. : R \to R$. Note that if we consider $R = HM$ for $M$ a commutative ring, we obtain a multiplication by $p$ map of $HM$.

MooreModel **Definition 3.2.2.** Fix a prime $p$. Consider $\mathbb{S} \in \mathsf{CAlg}$, then we define the *p'th Moore spectrum* as,

$$\mathbb{S}/p \coloneqq \operatorname{cofib}(p. : \mathbb{S} \to \mathbb{S}).$$

There is the following equivalence $S(C_p) \simeq \mathbb{S}/p$, for detail see [47] Chapter IV. §2. Analogous we obtain a model for the Moore spectrum of $HM$, for $M$ a $p$-torsion free commutative ring,

$$HM/p \coloneqq \operatorname{cofib}(p. : HM \to HM).$$

*Remark* 3.2.3. The spectrum $\mathbb{S}/p$ is analogous to $C_p$ in abelian groups, but contrary to $C_p$ which inherits the commutative and associative structure from $\mathbb{Z}$, $\mathbb{S}/p$ does not inherit the $\mathbb{E}_\infty$-ring spectrum structure form $\mathbb{S}$, a proof of this can be found in [32] remark 4.3.

Now that we have familiarized ourselves a the Moore spectrum, lets see what it has to do with $p$-completion. For this we need the notion of a $E$-local object in a symmetric monoidal $\infty$-category. This is much analogous to $E$-local objects in the sense of [2].

**Definition 3.2.4.** Let $\mathscr{C}$ be a symmetric monoidal $\infty$-category. Let $E \in \mathscr{C}$ be an object. An object $Y \in \mathscr{C}$ is called *$E$-local* if $Y$ is equivalent to some object $X \in \mathscr{C}$ such that $X \otimes E \simeq 0$. Let $\mathscr{C}_E$ denote the full subcategory of $\mathscr{C}$ spanned by collection of $E$-local objects.

**Definition 3.2.5.** We define the $\infty$-category of *$p$-complete spectra*, denoted $\mathsf{Sp}_p^\wedge$, as the full subcategory $\mathsf{Sp}_{\mathbb{S}/p}$.

Now that we have a notion of $p$-complete spectra, we need a way to $p$-complete. Note that the inclusion $\mathsf{Sp}_p^\wedge \to \mathsf{Sp}$ is continuous.

**Definition 3.2.6.** We define *$p$-completion* as the left adjoint $L_{\mathbb{S}/p} : \mathsf{Sp} \to \mathsf{Sp}_p^\wedge$ of the inclusion $\mathsf{Sp}_p^\wedge \to \mathsf{Sp}$, afforded by the adjoint functor theorem.

It can be shown that this agrees with the usual $p$-completion, by showing it agrees with the $p$-completion of A.K. Bousfield [2] which can be calculated by formula (3.1).

As mentioned there is a strong relation between the Tate-construction of $C_p$ and $p$-completion. We shall use this result crucially in the coming sections. This result relies on the assumption that the spectrum be bounded below.

LI2.9 **Proposition 3.2.7.** *Let $X$ be a spectrum with a $C_p$-action which is bounded below. Then $X^{tC_p}$ is $p$-complete and equivalent to $(X_p^\wedge)^{tC_p}$.*

*Proof.* Since $(-)^{tC_p}$ is an exact functor it commutes with smashing with the Moore spectrum $\mathbb{S}/p$, which by definition makes the canonical map $X^{tC_p} \to (X_p^\wedge)^{tC_p}$ a $p$-adic equivalence. If $X$ is bounded below so is the $p$-completion. Therefore it suffices to prove that $X^{tC_p}$ is $p$-complete.

By 3.1.15 and the fact the limits of $p$-complete spectra are $p$-complete (this is lemma 1.8 of [2]), we can assume that $X$ is bounded above. We now argue that we may further restrict to Eilenberg-MacLane spectra. We shall use this strategy many times throughout this thesis.

Assume that $X$ is concentrated in the range 0 to $n$. We proceed by induction on $n$. For $n = 0$, $X$ is equivalent to an Eilenberg-MacLane spectrum. Assume that we've shown the desired result for $n-1$. Consider the fiber sequence associated to the map induced from the $n$-truncation $c_{n-1} : X \to \tau_{\leq n-1}X$,

$$\operatorname{fib}(c_{n-1}) \to X \to \tau_{\leq n-1}X,$$

A model for the fiber is $\Sigma^n H\pi_n X$. The claim follows from contemplating the long exact sequence in homotopy groups induced from the fiber sequence, together with the induction hypothesis. In this case we may use 3.1.19, to see that $\pi_t(HM^{tC_p}) \simeq \hat{H}^t(C_p, M)$. Note that the right hand side is $p$-torsion, hence $X^{tC_p}$ is itself $p$-power-torsion, and in particular $p$-complete. □

## 3.3 The Tate Lemmas

In this section we show the Tate lemmas. The Tate fixpoint lemma is going to be crucial later when we show an equivalence between the new and the old formula for topological cyclic homology. One could argue that these lemmas are perhaps the most crucial result of [38]. The proof of the Tate lemmas is going to be a reduction to first Eilenberg-MacLane spectra, and then to the "base case" of $H\mathbb{F}_p$. We follow section I.2 in [38].We begin by showing the base case. We shall need the following vanishing result for the Tate construction, it is Lemma I.2.5 of [38].

LI2.5 **Lemma 3.3.1.** *Let $\mathcal{D}(\mathbb{F}_p)^{BC_p}$ be the derived category of chain complexes with a $C_p$-action of $\mathbb{F}_p$-vector spaces. Consider $A \in \mathcal{D}(\mathbb{F}_p)^{BC_p}$, for which the only nonzero cohomology groups are $H^0(A, \mathbb{F}_p) = H^1(A, \mathbb{F}_p) = \mathbb{F}_p$. Assume that $A$ corresponds to a nonzero class in*

$$\pi_0(\operatorname{Map}_{\mathcal{D}(\mathbb{F}_p)^{BC_p}}(\mathbb{F}_p[-1], \mathbb{F}_p[1])) = H^2(C_p, \mathbb{F}_p) = \mathbb{F}_p.$$

*In this case $A^{tC_p} \simeq 0$.*

This result is going to be the technical input into the proof of the following lemma, which is the base case mentioned above. Recall that $H^i(C_p, \mathbb{F}_p) = H_i(C_p, \mathbb{F}_p) = \mathbb{F}_p$.

LI2.4 **Lemma 3.3.2.** *Let $X = H\mathbb{F}_p$ with trivial $C_{p^2}$ action. For any integer $i \geq 0$, we have*

tau1 (3.2)
$$(\tau_{[2i,2i+1]} X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0,$$

tau2 (3.3)
$$(\tau_{[-2i-1,-2i]} X^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

*Moreover*

$$(X^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0,$$
$$(X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

*Proof.* Consider $(H\mathbb{F}_p)_{hC_p}$, and $(H\mathbb{F}_p)^{hC_p}$. In the proof of proposition 3.1.19 we calculated the homotopy groups of these as the group homology $H_*(C_p, \mathbb{F}_p)$ and the group cohomology $H^*(C_p, \mathbb{F}_p)$ respectively. Now consider

$$A = (\tau_{[2i,2i+1]}((H\mathbb{F}_p)_{hC_p}))[-2i-1],$$
$$B = (\tau_{[-2i-1,-2i]}((H\mathbb{F}_p)^{hC_p}))[2i].$$

Here $A$ ($B$) is the $[2i, 2i+1]$ ($[-2i-1, -2i]$) truncation of the homotopy orbits (fixed points) of the Eilenberg-MacLane spectrum of $\mathbb{F}_p$ translated by $-2i-1$ ($2i$), so the homotopy groups are concentrated in degree 0 and 1. We have $\pi_0(A) = \pi_0(B) = \pi_1(A) = \pi_1(B) = \mathbb{F}_p$. Let

$$A_\bullet = (\tau_{[2i,2i+1]}((\mathbb{F}_p)_{hC_p})_\bullet)[-2i-1],$$
$$B_\bullet = (\tau_{[-2i-,-2i]}((\mathbb{F}_p)^{hC_p})_\bullet)[2i],$$

denote the chain complex which correspond to $A$ and $B$ respectively under the correspondence given in example 2.4.16. By remark 2.4.17 we have $H_0(A_\bullet, \mathbb{F}_p) = H_0(B_\bullet, \mathbb{F}_p) = H_1(A_\bullet, \mathbb{F}_p) = H_1(B_\bullet, \mathbb{F}_p) = \mathbb{F}_p$. The universal coefficient theorem gives $H^0(A_\bullet, \mathbb{F}_p) = H^0(B_\bullet, \mathbb{F}_p) = H^1(A_\bullet, \mathbb{F}_p) = H^1(B_\bullet, \mathbb{F}_p) = \mathbb{F}_p$.
We wish to apply lemma 3.3.1, and hence we wish to show that $A$ and $B$ correspond to a nonzero class in

$$\pi_0(\mathrm{Map}_{\mathcal{D}(\mathbb{F}_p)^{BC_p}}(\mathbb{F}_p[-1], \mathbb{F}_p[1])) = H^2(C_p, \mathbb{F}_p) = \mathbb{F}_p.$$

To see this we simply have to see that $A$ and $B$ are not $C_{p^2}/C_p$-equivariantly split.

Note that both $A$ and $B$ are $C_p \cong C_{p^2}/C_p$-equivariant. Furthermore recall that

$$\pi_n(H(\mathbb{F}_p)^{hC_p}) \cong H^{-n}(C_p, \mathbb{F}_p),$$

hence if we truncate we obtain an equivalence $A \simeq \tau_{[-1,0]} H(\mathbb{F}_p)^{hC_p} \simeq B$, hence we may instead show that $\tau_{[-1,0]} H(\mathbb{F}_p)^{hC_p}$ is not $C_{p^2}/C_p$-equivariantly split. Assume for contradiction that $\tau_{[-1,0]} H(\mathbb{F}_p)^{hC_p}$ is $C_{p^2}/C_p$-equivariantly split, and consider total degree 1 of the Hochschild–Serre spectral sequence,

$$H^i(C_{p^2}/C_p, H^j(C_p, \mathbb{F}_p)) \implies H^{i+j}(C_{p^2}, \mathbb{F}_p).$$

In this case the differential

$$H^0(C_{p^2}/C_p, H^1(C_p, \mathbb{F}_p)) \to H^2(C_{p^2}/C_p, H^0(C_p, \mathbb{F}_p)),$$

is zero, which implies that

$$H^0(C_{p^2}/C_p, H^1(C_p, \mathbb{F}_p)) \cong H^0(C_p, \mathbb{F}_p) \cong F_p,$$
$$H^1(C_{p^2}/C_p, H^0(C_p, \mathbb{F}_p)) \cong H^0(C_p, \mathbb{F}_p) \cong F_p,$$

survive to the $E_\infty$-page, and hence contribute two $\mathbb{F}_p$-factors to the total degree 1. From this we may conclude that $H^1(C_{p^2}, \mathbb{F}_p) \cong \mathbb{F}_p^2$, which is a contraction because $H^1(C_{p^2}, \mathbb{F}_p) \cong \mathbb{F}_p$. Hence we may conclude that (3.2) and (3.3) hold via lemma 3.3.1. Now we may splice together the chain complexes afforded for each $i$, use the correspondence of example 2.4.16 again, to obtain

$$(\tau_{\le 2i+1} X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0, \quad (\tau_{\ge -2i-1} X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0,$$

from which lemma 3.1.15 gives the remainder of the statement. □

Now that we have the base case we just have to reduce to the base case, from the Eilenberg-MacLane case.

LI2.7 **Lemma 3.3.3.** *Let $M$ be an abelian group with a $C_{p^2}$-action, and let $X = HM$ be the corresponding Eilenberg-MacLane spectrum with $C_{p^2}$-action. Then*

eq1 (3.4) $$(X^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0,$$

eq2 (3.5) $$(X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

*Proof.* We start with (3.4). We begin by showing the following claim: the functor $\mathsf{Ab} \to \mathsf{Sp}$ defined as

$$M \mapsto (HM^{hC_p})^{t(C_{p^2}/C_p)} = \mathrm{cofib}((HM^{hC_p})_{h(C_{p^2}/C_p)} \to HM^{hC_{p^2}}),$$

commutes with filtered colimits.

Homotopy orbits and cofibers are colimits, hence commute with all colimits. Therefore it is enough to show that $-^{hC_p}$ and $-^{hC_{p^2}}$ commutes with filtered colimits. Since $M$ is an abelian group we may realize it as a filtered colimit of finitely generated abelian groups, so let $M \cong \mathrm{colim}_i M_i$. By the homotopy fixpoints spectral sequence we obtain the following comparison map,

$$\begin{array}{ccc} H^t(C_p, \pi_q(HM)) & \Longrightarrow & \pi_{t+q} HM^{hC_p} \\ \downarrow & & \downarrow \\ \mathrm{colim}_i H^t(C_p, \pi_q(HM_i)) & \Longrightarrow & \mathrm{colim}_i \pi_{t+q} HM_i^{hC_p} \end{array}$$

Recall that $H^t(C_p, -)$ commutes with filtered colimits, which shows that the left hand comparison map,

$$H^t(C_p, M) \cong H^t(C_p, \lim_i M_i) \cong \lim_i H^t(C_p, M_i),$$

is an isomorphism, therefore $(-)^{hC_p}$ commutes with filtered colimits. Likewise for $(-)^{hC_{p^2}}$. Thus we may assume that $M$ is a finitely generated abelian group.

We now reduce further to torsion-free finitely generated abelian groups, by considering a two-term torsion-free resolution of $M$, and using exactness of $(-)^{t(C_{p^2}/C_p)}$. Hence by the fundamental theorem of finitely generated abelian groups, $M/pM$ is a finite-dimensional $\mathbb{F}_p$-vector space with $C_{p^2}$-action. It follows that the action of $C_{p^2}$ on $M/pM$ is trivial, and that $M/pM$ is a finite direct sum of copies of $\mathbb{F}_p$.

Now apply Lemma 3.3.2 to conclude that for $M/pM$ (3.4) is satisfied. We pass back to $M$ a finitely generated abelian group, by noting that

$$(HM^{hC_p})^{t(C_{p^2}/C_p)} \xrightarrow{p\cdot} (HM^{hC_p})^{t(C_{p^2}/C_p)} \longrightarrow (H(M/pM)^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

So the cofiber of multiplication by $p$ on $(HM^{hC_p})^{t(C_{p^2}/C_p)}$ is 0, therefore multiplication by $p$ is an automorphism. Thus, we can pass to the filtered colimit $M[\frac{1}{p}] := \operatorname{colim}_i M/p^i M$ along multiplication by $p$, and therefore assume that multiplication by $p$ is an isomorphism on $M$, which implies that the algebraic norm is an isomorphism. Therefore multiplication by $p$ is an isomorphism on $M$, which again means multiplication by $p$ is an isomorphism on $HM^{hC_p}$. Because of this lemma 3.1.21 applies and we obtain that (3.4) holds.

Next we deal with (3.5). Note that that $(HM_{hC_p})^{t(C_{p^2}/C_p)}$ is $p$-complete by lemma 3.2.7. We claim that,

$$(HM_{hC_p})^{t(C_{p^2}/C_p)} \otimes \mathbb{S}/p \simeq 0$$

I.e. that $(HM_{hC_p})^{t(C_{p^2}/C_p)}$ is $p$-adically equivalent to 0. Because $(HM_{hC_p})^{t(C_{p^2}/C_p)}$ is $p$-complete this will suffice to prove (3.5).

Again we may assume $M$ is $p$-torsion free. In this case we claim that:

$$(HM_{hC_p})^{t(C_{p^2}/C_p)} \otimes \mathbb{S}/p \simeq (H(M/pM)_{hC_p})^{t(C_{p^2}/C_p)}.$$

This follows from the fact that

$$\mathbb{S}/p \simeq \operatorname{cofib}(H\mathbb{F}_p \xrightarrow{p} H\mathbb{F}_p).$$

from which it is clear that $- \otimes \mathbb{S}/p$ commutes with both the Tate construction and homotopy orbits, because of exactness and homotopy orbits being a colimit respectively. Hence

$$\begin{aligned} HM \otimes \mathbb{S}/p &\simeq HM \otimes \operatorname{cofib}(H\mathbb{F}_p \xrightarrow{p} H\mathbb{F}_p) \\ &\simeq \operatorname{cofib}(HM \otimes H\mathbb{F}_p \xrightarrow{\mathrm{id} \otimes p} HM \otimes H\mathbb{F}_p) \\ &\simeq \operatorname{cofib}(HM \xrightarrow{p} HM) \simeq H(M/pM). \end{aligned}$$

Which shows the claim.

Consider a generator $\gamma \in C_{p^2}$, then $(\gamma - 1)^{p^2} = \gamma^{p^2} - 1 = 0$ in $M$. Hence $M$ can be written as a filtered colimit of a system

$$M \xrightarrow{\gamma} M \xrightarrow{\gamma} \dots$$

of length at most $p^2$, where each term has a trivial $C_{p^2}$-action. From this we can conclude that the $C_{p^2}$-action on $M$ is trivial. Thus $M$ is an $\mathbb{F}_p$-vector space with trivial $C_{p^2}$-action. Hence by lemma 3.3.2, we see that for all $i \geq 0$

$$(\tau_{[2i,2i+1]} HM_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0,$$

as this functor commutes with infinite direct sums, because $H(-)$, $(-)_{hC_p}$, $\tau_{[-,-]}(-)$, and $(-)^{tG}$ does by lemma 3.1.17, because $(\tau_{[2i,2i+1]} HM_{hC_p})$ is bounded above. Now this implies $(HM_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0$, because we can splice together the chain complexes afforded for each $i$ by the above, to obtain

$$(\tau_{\leq 2i+1} HM_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

We now obtain the result by applying lemma 3.1.15 to the above. □

At this point we are finally ready to state the Tate lemmas and give their (at this point) easy proofs.

TateOrbit **Theorem 3.3.4.** *[Tate Orbit Lemma] Let $X$ be a spectrum with a $C_{p^2}$-action. Assume that $X$ is bounded below. Then*

$$(X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

*Proof.* We prove that the norm map $X_{hC_{p^2}} \to (X_{hC_p})^{hC_p}$ is an equivalence. Consider the following diagram afforded by lemma 3.1.15

$$\begin{array}{ccc} X_{hC_{p^2}} & \longrightarrow & (X_{hC_p})^{hC_p} \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \lim_n(\tau_{\leq n}X)_{hC_{p^2}} & \longrightarrow & \lim_n(\tau_{\leq n}(\lim_m(\tau_{\leq m}X)_{hC_p}))^{hC_p}, \end{array}$$

which lets us further restrict to bounded $X$. Recall the strategy from the proof of lemma 3.2.7 to further reduce to Eilenberg-MacLane spectra. Lemma 3.3.3 gives the result for Eilenberg-MacLane spectra. □

An entirely analogous proof gives the Tate fixpoint lemma.

TateFixed **Theorem 3.3.5.** *[Tate Fixpoint Lemma] Let $X$ be a spectrum which is bounded above. Then*

$$(X^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

Before we move on we will show the necesity of $X$ being a bounded above spectrum in the Tate fixpoint lemma.

SegalConj **Example 3.3.6.** Consider $\mathbb{S}$ with the trivial $C_{p^2}$-action. Recall that $\mathbb{S}$ is not bounded above. We claim that

$$(\mathbb{S}^{hC_p})^{t(C_{p^2}/C_p)} \simeq \mathbb{S}_p^\wedge$$

In particular $(\mathbb{S}^{hC_p})^{t(C_{p^2}/C_p)}$ is not trivial, hence does not satisfy the Tate fixpoint lemma.

Consider the fiber sequence associated to the Tate construction

$$\mathbb{S}_{hC_p} \longrightarrow \mathbb{S}^{hC_p} \longrightarrow \mathbb{S}^{tC_p}.$$

Recall that the Segal conjecture for $C_p$, proved by W. Lin [18] and J. Gunawardena [8] for $p = 2$ and $p > 2$ respectively, states that $\mathbb{S}^{tC_p} \simeq \mathbb{S}_p^\wedge$, from this we see that the Tate construction for the sphere spectrum is $p$-complete. We claim that the map $\mathbb{S} \to \mathbb{S}^{tC_p}$ which gives rise to the desired equivalence is $C_{p^2}/C_p$-equivariant when we give $\mathbb{S}$ the trivial $C_{p^2}$-action. This follows from the fact that the map $\mathbb{S} \to \mathbb{S}^{tC_p}$ can be lifted to a map $\mathbb{S} \to (\mathbb{S}^{tC_p})^{h(C_{p^2}/C_p)}$, via the natural map

$$\mathbb{S} \to \mathbb{S}^{hC_p} \simeq (\mathbb{S}^{hC_p})^{h(C_{p^2}/C_p)} \to (\mathbb{S}^{tC_p})^{h(C_{p^2}/C_p)}.$$

From this it follows that the $C_{p^2}/C_p$-action on $\mathbb{S}^{tC_p}$ is trivial. Apply $-^{t(C_{p^2}/C_p)}$ to the above fiber sequence from which we obtain a fiber sequence (because the Tate construction is exact),

$$(\mathbb{S}_{hC_p})^{t(C_{p^2}/C_p)} \longrightarrow (\mathbb{S}^{hC_p})^{t(C_{p^2}/C_p)} \longrightarrow (\mathbb{S}^{tC_p})^{t(C_{p^2}/C_p)}.$$

Note that the first term is equivalent to 0 by theorem 3.3.4, because $\mathbb{S}$ is bounded below. Hence $(\mathbb{S}^{hC_p})^{t(C_{p^2}/C_p)}$ is equivalent to $(\mathbb{S}^{tC_p})^{t(C_{p^2}/C_p)}$ which is given by

$$(\mathbb{S}^{tC_p})^{t(C_{p^2}/C_p)} = (\mathbb{S}^{tC_p})^{tC_p} \cong \mathbb{S}^{tC_p} \cong \mathbb{S}_p^\wedge,$$

where we have used the Segal conjecture, and lemma 3.2.7.

In the next section we shall see a rather strong generalization of the Segal conjecture.

*Remark* 3.3.7. The argument showing that the $C_{p^2}/C_p$-action on $\mathbb{S}^{tC_p}$ is trivial will reappear later, when we define the cyclotomic sphere spectrum, where we have $\mathbb{T}$ in place of $C_{p^2}$.

As we shall see in the following example, we really need the bounded below assumption in the Tate orbit lemma.

BoundedBelowNeeded **Example 3.3.8.** Consider the functor $h\mathsf{Sp} \to \mathsf{Ab}$ given by $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$. Because $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, it is easy to see that this functor satisfies the Eilenberg-Steenrod axioms, and hence gives rise to a cohomology theory. Now using Brown representability we obtain a spectrum $I_{\mathbb{Q}/\mathbb{Z}}$ representing this cohomology theory. Now if we replace $\mathbb{Q}/\mathbb{Z}$ with $\mathbb{Q}$ we obtain rational singular cohomology, which is represented by the Eilenberg-MacLane spectrum $H\mathbb{Q}$. Note that the there is a map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, and hence by the Yoneda lemma and functoriality of Brown representability we obtain a map $H\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}$. Now consider the following fiber sequence

$$I_{\mathbb{Z}} \coloneqq \mathrm{fib}(H\mathbb{Q} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}) \longrightarrow H\mathbb{Q} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}.$$

The fiber $I_{\mathbb{Z}}$ is called the *Anderson dual of the sphere spectrum.* This spectrum in fact deserves this name because it is a *dualizing spectrum* of $\mathbb{S}$, see example 4.3.9 [23]. An important property of the dualizing spectra $K$ of a spectrum $S$, is that $\pi_k(K)$ are finitely generated $\pi_0(S)$-modules. Hence we have that $\pi_k(I_{\mathbb{Z}})$ are finitely generated $\mathbb{Z}$-modules. Given the definition of $I_{\mathbb{Q}/Z}$ as a spectrum represented by the cohomology theory described above, we see that the homotopy groups of $I_{\mathbb{Q}/\mathbb{Z}}$ are given as,

$$\pi_n(I_{\mathbb{Q}/\mathbb{Z}}) \simeq \begin{cases} 0 & \text{if } n > 0, \\ \mathbb{Q}/\mathbb{Z} & \text{if } n = 0 \\ \mathrm{Hom}(\pi_{-n}(\mathbb{S}), \mathbb{Q}/\mathbb{Z}) & \text{if } n < 0. \end{cases}$$

From these and the long exact sequence in homotopy the above fiber sequence induces we obtain the homotopy groups of $I_{\mathbb{Z}}$,

$$\pi_n(I_{\mathbb{Z}}) \simeq \begin{cases} 0 & \text{if } n > 0, \\ \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = -1 \\ \mathrm{Hom}(\pi_k(\mathbb{S}), \mathbb{Q}/\mathbb{Z}) & \text{if } n = -k-1 \text{ and } k > 0. \end{cases} .$$

I.e. it is bounded above, but not below. We show that this spectrum does not satisfy the Tate orbit lemma, when given the trivial $C_{p^2}$-action. Analogously the canonical map $I_{\mathbb{Z}} \to I_{\mathbb{Z}}^{tC_p}$ is $C_{p^2}/C_p$-equivariant. Consider the fiber sequence associated to the Tate construction, and apply $(-)^{t(C_{p^2}/C_p)}$

$$((I_{\mathbb{Z}})_{hC_p})^{t(C_{p^2}/C_p)} \longrightarrow ((I_{\mathbb{Z}})^{hC_p})^{t(C_{p^2}/C_p)} \longrightarrow ((I_{\mathbb{Z}})^{tC_p})^{t(C_{p^2}/C_p)}.$$

Now note that the middle term vanishes by the Tate fixpoint lemma, i.e. theorem 3.3.5. Hence contemplating the long exact sequence in homotopy which this fiber sequence induced we see that

$$\pi_{k+1}(((I_{\mathbb{Z}})_{hC_p})^{t(C_{p^2}/C_p)}) \cong \pi_k(((I_{\mathbb{Z}})^{tC_p})^{t(C_{p^2}/C_p)}).$$

Hence

$$\Sigma(((I_{\mathbb{Z}})_{hC_p})^{t(C_{p^2}/C_p)}) \simeq ((I_{\mathbb{Z}})^{tC_p})^{t(C_{p^2}/C_p)} \simeq I_{\mathbb{Z}}^{tC_p}.$$

Here the last equivalence is the residual action. Hence we are reduced to showing that $I_{\mathbb{Z}}^{tC_p}$ is non-trivial. Note that its homotopy groups are finitely generated. Contemplating the Tate spectral sequence, $E^{0,0}$-position is $\mathbb{Z}$, we argue that this survives to the $E_\infty$-page. Note that there are no non-trivial groups when $t > 0$, because $\pi_t(I_\mathbb{Z}) \simeq 0$ in this range, hence we should only be worried about the $\mathbb{Z}$ in $E^{0,0}$ being killed by morphisms into it, and not out of it. The only group which has a map into $E^{0,0}$ on the $E_r$-page is $E^{-r,-r+1}$, for $r > 2$. These are given as

$$\hat{H}^{r-1}(C_p, \pi_{-r}(I_\mathbb{Z})) \simeq H^{r-1}(C_p, \mathrm{Hom}(\pi_{r+1}(\mathbb{S}), \mathbb{Q}/\mathbb{Z})).$$

Note that these are finitely generated. In the case $r$ even, these cohomology groups are given as the $p$-torsion module $T \subseteq \mathrm{Hom}(\pi_{r+1}(\mathbb{S}), \mathbb{Q}/\mathbb{Z})$, and in particular there can not be an isomorphism $d_r^{tC_p} : T \to \mathbb{Z}$. In the case $r$ odd, these cohomology groups are given by the quotient of $\mathrm{Hom}(\pi_{r+1}(\mathbb{S}), \mathbb{Q}/\mathbb{Z})$ with $T$. Because $r > 2$, $\pi_{r+1}(\mathbb{S})$ are all finite, say $\pi_{r+1}(\mathbb{S}) \cong \bigoplus_{i=1}^n (C_{p_i^k})$, which follows from the structure theorem. Now

$$\mathrm{Hom}(\bigoplus_{i=1}^n (C_{p_i^k}), \mathbb{Q}/\mathbb{Z})) \cong \bigoplus_{i=1}^n \mathrm{Hom}(C_{p_i^k}, \mathbb{Q}/\mathbb{Z})) \cong \bigoplus_{i=1}^n (C_{p_i^k}).$$

Where the last isomorphism holds for all cyclic groups. Hence $E^{-r,-r+1}$ is

$$H^{r-1}(C_p, \mathrm{Hom}(\pi_{r+1}(\mathbb{S}), \mathbb{Q}/\mathbb{Z})) \simeq \bigoplus_{i=1}^n (C_{p_i^k})/T,$$

which can not be isomorphic to $\mathbb{Z}$. In total we have shown that $\pi_0(I_\mathbb{Z}^{tC_p}) \simeq \mathbb{Z}$, in particular $I_\mathbb{Z}^{tC_p} \not\simeq 0$, which shows the desired result.

So as it can be seen the bounded below assumption is essential, and can not be avoided.

## 3.4 The Tate Diagonal

In this section we shall define a natural transformation which is called the Tate diagonal. The existence of the Tate-diagonal is a rather deep fact which relies on a number of non-trivial results. The Tate diagonal has a strong relation to $p$-completion, this relation is going to generalize the Segal conjecture for $C_p$. Beside showing this relation, we shall also see that it admits an essentially unique lax symmetric monoidal structure, which is going to be crucial when we endow topological Hochschild homology with a cyclotomic structure. Throughout fix a prime $p$.

The Tate diagonal is going to be a natural transformation from $\mathrm{id}_{\mathsf{Sp}}$ to an exact functor $T_p$.

**Proposition 3.4.1.** *The functor $T_p : \mathsf{Sp} \to \mathsf{Sp}$ is given by*

$$X \mapsto (X \otimes ... \otimes X)^{tC_p}.$$

*Where $X \otimes ... \otimes X$ is a $p$-fold smash product, with the $C_p$-action given by cyclic permutation of the factors. $T_p$ is an exact functor.*

*Proof.* In general it suffices to check that $T_p$ commutes with extensions, but to show this we will need that $T_p$ preserves finite sums. Thus we calculate

$$\begin{aligned} T_p(X_0 \oplus X_1) &\simeq \left( \bigoplus_{(i_1,...,i_p) \in \{0,1\}^p} X_{i_1} \otimes ... \otimes X_{i_p} \right)^{tC_p} \\ &\simeq T_p(X_0) \oplus T_p(X_1) \oplus \bigoplus_{[i_1,...,i_p]} \left( \bigoplus_{(i_1,...,i_p) \in [i_1,...,i_p]} X_{i_1} \otimes ... \otimes X_{i_p} \right)^{tC_p} \end{aligned}$$

Where $[i_1, ..., i_p]$ is a set of representatives of orbits of the cyclic $C_p$-action on the set

$$S = \{0,1\}^p \setminus \{(0,...,0), (1,...,1)\}.$$

Since $p$ is a prime, these orbits have to contain all of $C_p$ without repetition, hence they are isomorphic to $C_p$. We conclude that each summand

$$\bigoplus_{(i_1,\ldots,i_p)\in[i_1,\ldots,i_p]} X_{i_1}\otimes\ldots\otimes X_{i_p}$$

is a $C_p$-equivariant spectrum which is induced from the trivial subgroup $*\subset C_p$. Note that we may conclude via proposition 3.1.12 that each of these summands vanish when the Tate construction is applied. Hence $T_p(X_0\oplus X_1)\simeq T_p(X_0)\oplus T_p(X_1)$. We now show that $T_p$ preserves extensions. Consider a fiber sequence $X_0\to X\to X_1$ in $\mathsf{Sp}$. Then we obtain the following fibration sequence

$$X_0\otimes\ldots\otimes X_0 \longrightarrow \bigoplus_{(i_1,\ldots,i_p)\in\mathbb{I}_1} X_{i_1}\otimes\ldots\otimes X_{i_p} \longrightarrow \ldots \longrightarrow X\otimes\ldots\otimes X \longrightarrow X_1\otimes\ldots\otimes X_1$$

Where for $1\le n\le p-1$, we define

$$\mathbb{I}_n=\{[i_1,\ldots,i_p]\in S/C_p|\sum_k^p i_k=n\}$$

The Tate-construction $-^{tC_p}$ is exact, hence each of the intermediate steps $\bigoplus_{(i_1,\ldots,i_p)\in\mathbb{I}_n} X_{i_1}\otimes\ldots\otimes X_{i_p}$ for $1\le n<p$ are trivial, hence we obtain a fiber sequence

$$(X_0\otimes\ldots\otimes X_0)^{tC_p}\longrightarrow (X\otimes\ldots\otimes X)^{tC_p}\longrightarrow (X_1\otimes\ldots\otimes X_1)^{tC_p}.$$

□

We denote the collection of exact functors from $\mathscr{C}\to\mathscr{D}$ by $\mathsf{Fun}^{\mathrm{Ex}}(\mathscr{C},\mathscr{D})$. By the above $T_p\in\mathsf{Fun}^{\mathrm{Ex}}(\mathsf{Sp},\mathsf{Sp})$, which is what will give rise to the Tate-diagonal.

**Proposition 3.4.2.** *For any $F\in\mathsf{Fun}^{Ex}(\mathsf{Sp},\mathsf{Sp})$ evaluation at the sphere spectrum $\mathbb{S}\in\mathsf{Sp}$ induces an equivalence,*

$$\mathrm{Map}_{\mathsf{Fun}^{Ex}(\mathsf{Sp},\mathsf{Sp})}(\mathrm{id}_{\mathsf{Sp}},F)\to\mathrm{Map}_{\mathsf{Sp}}(\mathbb{S},F(\mathbb{S}))=\Omega^\infty(F(\mathbb{S})).$$

*Here $\Omega^\infty:\mathsf{Sp}\to\mathcal{S}$ is the usual infinite delooping functor.*

*Proof.* We have an equivalence

$$\begin{aligned}\mathsf{Fun}^{\mathrm{Ex}}(\mathsf{Sp},\mathsf{Sp})&\to\mathsf{Fun}^{\mathrm{Lex}}(\mathsf{Sp},\mathcal{S})\\ F&\mapsto\Omega^\infty\circ F.\end{aligned}$$

Hence we have

$$\begin{aligned}\mathrm{Map}_{\mathsf{Fun}^{\mathrm{Ex}}(\mathsf{Sp},\mathsf{Sp})}(\mathrm{id}_{\mathsf{Sp}},F)&\simeq\mathrm{Map}_{\mathsf{Fun}^{\mathrm{Lex}}(\mathsf{Sp},\mathcal{S})}(\Omega^\infty(-),\Omega^\infty(F(-)))\\ &=\mathrm{Map}_{\mathsf{Fun}^{\mathrm{Lex}}(\mathsf{Sp},\mathcal{S})}(\mathrm{map}_{\mathsf{Sp}}(\mathbb{S},-),\mathrm{Map}_{\mathsf{Sp}}(\mathbb{S},F(-)))\\ &\simeq\mathrm{Map}_{\mathsf{Sp}}(\mathbb{S},F(\mathbb{S}))=\Omega^\infty(F(\mathbb{S})).\end{aligned}$$

Where the first equivalence is given by the equivalence above, and the second is given by the Yoneda lemma. □

Applying the above to $T_p:\mathsf{Sp}\to\mathsf{Sp}$, we obtain the following corollary.

Nat-Omega **Corollary 3.4.3.** *There is an equivalence,*

$$\mathrm{Map}_{\mathsf{Fun}^{Ex}(\mathsf{Sp},\mathsf{Sp})}(\mathrm{id}_{\mathsf{Sp}},T_p)\simeq\Omega^\infty(T_p(\mathbb{S})).$$

tate-dia **Definition 3.4.4.** The *Tate diagonal* is the natural transformation

$$\Delta_p : \mathrm{id}_{\mathsf{Sp}} \to T_p : X \to (X \otimes ... \otimes X)^{tC_p},$$

of functors $\mathsf{Sp} \to \mathsf{Sp}$ which under the equivalence 3.4.3 corresponds to the map

$$\mathbb{S} \to T_p(\mathbb{S}) \simeq \mathbb{S}^{tC_p},$$

which is the composition $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$. Note we have used that $\mathbb{S}$ is $\otimes$-unit in the above.

We will now show the relation between the Tate-diagonal and $p$-completion, which is the first main theorem of the thesis. This is theorem 1 of the introduction.

Tate-dia p-com **Theorem 3.4.5.** *Let $X \in \mathsf{Sp}$ be a bounded below spectrum. Then the map*

Deltap (3.6)
$$\Delta_p : X \to (X \otimes ... X)^{tC_p}$$

*exhibits $(X \otimes ... \otimes X)^{tC_p}$ as the p-completion of $X$.*

Note that this is a generalization of the Segal conjecture on $C_p$:

$$\Delta_p(\mathbb{S}) : \mathbb{S} \to T_p(\mathbb{S}) \simeq \mathbb{S}^{tC_p} \simeq \mathbb{S}_p^{\wedge}.$$

We shall limit ourselves to the case when $X$ has finitely generated homotopy groups, and sketch how to give the complete proof from here. We shall furthermore assume the following theorem due to S. Lunøe-Nielsen and J. Rognes, see [42].

LNR12 **Theorem 3.4.6.** *There is an equivalence*

$$\Delta_p : H\mathbb{F}_p \to T_p(H\mathbb{F}_p).$$

The proof of 3.4.5 is largely going to revolve around reducing the statement to the above case.

*Proof.* Because $\mathsf{Sp}$ is stable $\Sigma : \mathsf{Sp} \to \mathsf{Sp}$ is an equivalence, and hence via finitely many applications of $\Sigma$ we may assume that $X$ is connective. We claim both sides of (3.6) are the limit of the values at the $n$-truncation of $X$, $\tau_{\leq n}X$. It is clear that $\lim_n \tau_{\leq n}X \simeq X$, since $\pi_i(\tau_{\leq n}X) \cong \pi_i(X)$ for all $i \leq n$. For the right hand side we show that

$$(X \otimes ... \otimes X)_{hC_p} \to \lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)_{hC_p}$$

and

$$(X \otimes ... \otimes X)^{hC_p} \to \lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)^{hC_p}$$

are equivalences, because then we will obtain an equivalence of cofibers induced from the norm map:

$$\begin{array}{ccccc}
(X \otimes ... \otimes X)_{hC_p} & \xrightarrow{\mathrm{Nm}_{C_p}} & (X \otimes ... \otimes X)^{hC_p} & \longrightarrow & (X \otimes ... \otimes X)^{tC_p} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)_{hC_p} & \xrightarrow{\mathrm{Nm}_{C_p}} & \lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)^{hC_p} & \longrightarrow & \lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)^{tC_p}.
\end{array}$$

We claim we have an equivalence

LimEq (3.7)
$$X \otimes ... \otimes X \simeq \lim_n (\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X).$$

To see this consider the to $\lim_n(\tau_{\leq n}X \otimes ... \otimes \tau_{\leq n}X)$ associated Postnikov tower,

$$\dots \longrightarrow \tau_{\leq 1}X \otimes \dots \otimes \tau_{\leq 1}X \longrightarrow \tau_{\leq 0}X \otimes \dots \otimes \tau_{\leq 0}X$$

Now there is a comparison map for each $n \geq 0$:

1.1 (3.8)
$$X \otimes \dots \otimes X \to \tau_{\leq n}X \otimes \dots \otimes \tau_{\leq n}X,$$

We now argue that this is $n$-connected. Note that $c_n : X \to \tau_{\leq n}X$ is $n$-connected, hence the cofiber $\mathrm{cofib}(c_n)$ is $n$-connected. We wish to show that the composite

$$X \otimes X \xrightarrow{c_n \otimes \mathrm{id}_X} \tau_{\leq n}X \otimes X \xrightarrow{\mathrm{id}_{\tau_{\leq n}X} \otimes c_n} \tau_{\leq n}X \otimes \tau_{\leq n}X,$$

is $n$-connected. By contemplating the following distinguished triangle

$$\tau_{>n}X \longrightarrow X \longrightarrow \tau_{\leq n}$$

and using that smashing preserves colimits, we see that the following is also a distinguished triangle

$$X \otimes \tau_{>n}X \longrightarrow X \otimes X \xrightarrow{\mathrm{id}_X \otimes c_n} X \otimes \tau_{\leq n}X.$$

The cofiber of $\mathrm{id}_X \otimes c_n$ is $\Sigma(X \otimes \tau_{>n}X)$, which is $(n-1)$-connected, hence $\mathrm{id}_X \otimes c_n$ is $n$-connected. Analogously $c_n \otimes \mathrm{id}_{\tau_{\leq n}X}$ is $n$-connected, hence the composite is $n$-connected. Repeated use of this fact gives that the map (3.8) is $n$-connected. From this we have the following isomorphism for $i \leq n$,

$$\pi_i(X \otimes \dots \otimes X) \to \pi_i(\tau_{\leq n}X \otimes \dots \otimes \tau_{\leq n}X),$$

i.e. (3.8) is $n$-connected. This implies that we obtain the following isomorphism for all $i$,

$$\pi_i(X \otimes \dots \otimes X) \to \pi_i(\lim_n(\tau_{\leq n}X \otimes \dots \otimes \tau_{\leq n}X)).$$

Which shows (3.7). We now argue in much the same way that we also get an equivalence when we consider homotopy orbits. Like before we may argue that

(3.9)
$$(X \otimes \dots \otimes X)_{hC_p} \to (\tau_{\leq n}X \otimes \dots \otimes \tau_{\leq n}X)_{hC_p},$$

is $n$-connected, which give isomorphisms on homotopy groups up to $n$, which as before gives an isomorphism for all $i$ when passing to the limit. Hence we have shown the first equivalence. To see the second equivalence, we simply note that limits commute with each other, hence the homotopy fixpoints functor commutes with limits, therefore by (3.7) we are done.

We can now with out loss of generality assume $X$ to be bounded. We now argue that we may further restrict to $p$-torsion free Eilenberg-MacLane spectra. We start by restricting to Eilenberg-MacLane spectra using the strategy of proposition 3.2.7. Lets show that we can further restrict to $p$-torsion free Eilenberg-MacLane spectra. Consider an $p$-torsion free abelian group $A$, where multiplication by $p$ is an isomorphism, we argue that

$$(HA \otimes \dots \otimes HA)^{tC_p} \simeq 0.$$

To show this we see that the norm map $N_{C_p} : (HA)_{hC_p} \to (HA)^{hC_p}$ is an equivalence. Consider the homotopy orbit and homotopy fixpoint spectral sequences of proposition 3.1.2. Now because $C_p$ is a cyclic group of finite order, and $\pi_q HA$ is a $C_p$-module, the group (co)homology is well known, and we see that the spectral sequence degenerates at the second page, which means that we obtain

$$\pi_*(HA)_{hC_p} \cong \pi_*(HA)_{C_p},$$
$$\pi_*(HA)^{hC_p} \cong \pi_*(HA)^{C_p}.$$

So this becomes a completely algebraic problem. Because $A$ was $p$-torsion free multiplication by $p$ is an isomorphism, the algebraic norm map $\pi_*(HA)/C_p \to \pi_*(HA)^{C_p}$, given by $[x] \mapsto \sum_{g\in C_p} g.x$ is an isomorphism, which then shows that the norm map is an equivalence. Consider the following short exact sequence, where $A$ is the $p$-torsion in $M$,

$$0 \longrightarrow A \longrightarrow M \longrightarrow M/A \longrightarrow 0$$

Applying Eilenberg-MacLane spectra, we obtain a fiber sequence, and further applying $\Delta_p$, we obtain the following fiber sequence

$$(HA \otimes ... \otimes HA)^{tC_p} \longrightarrow (HM \otimes ... \otimes HM)^{tC_p} \longrightarrow (H(M/A) \otimes ... \otimes H(M/A))^{tC_p}$$

We obtain an equivalence

Hash (3.10) $$(HM \otimes ... \otimes HM)^{tC_p} \simeq (H(M/A) \otimes ... \otimes H(M/A))^{tC_p}.$$

By Lemma 3.2.7 the right hand side of (3.8), with $X = HM$ for $M$ $p$-torsion free, is $p$-complete, i.e. of the form on the right hand side of (3.10). Hence to finish the proof we need to show that $\Delta_p$ is a $p$-adic equivalence, which is equivalent to showing that it is an equivalence after smashing with the Moore-spectrum of $\mathbb{F}_p$. Using exactness, and the assertions of definition 3.2.2 we have

$$\begin{aligned} HM \otimes \mathbb{S}/p &\simeq HM \otimes \mathrm{cofib}(\mathbb{S} \xrightarrow{p} \mathbb{S}) \\ &\simeq \mathrm{cofib}(HM \otimes \mathbb{S} \xrightarrow{\mathrm{id}\otimes p} HM \otimes \mathbb{S}) \\ &\simeq \mathrm{cofib}(HM \xrightarrow{p} HM) \simeq H(M/pM). \end{aligned}$$

Note that the above calculation crucially uses that $M$ is $p$-torsion free, because if not $\mathrm{cofib}(HM \xrightarrow{p} HM)$ could have been 0. Again by exactness, and the above we also have

$$\begin{aligned} (HM \otimes ... \otimes HM)^{tC_p} \otimes \mathbb{S}/p &\simeq ((HM \otimes \mathbb{S}/p) \otimes ... \otimes (HM \otimes \mathbb{S}/p))^{tC_p} \\ &\simeq (H(M/pM) \otimes ... \otimes H(M/pM))^{tC_p} \end{aligned}$$

Hence we may equivalently show the following map is an equivalence,

$$H(M/pM) \longrightarrow (H(M/pM) \otimes ... \otimes H(M/pM))^{tC_p}.$$

Now because we've assumed that $X$ has finitely generated homotopy groups, we may deduce that $M/pM$ is a finite direct sum of copies of $\mathbb{F}_p$, hence we are done by exactness of $T_p$ and theorem 3.4.6. □

If we had not assumed that the homotopy groups of $X$ was finitely generated, $M/pM$ could have been an infinite direct sum of copies of $\mathbb{F}_p$. It is not clear that $T_p$ commutes with infinite direct sums, and it is (atleast to the author) suprising that a functor build from both a limit and a colimit should have such a property. We now sketch the proof of this claim.

*Sketch of proof.* We first restrict ourselves to the following truncation: for $0 \le n$, consider the functor

$$T_p^{(n)} : X \mapsto (\tau_{\le n}(X \otimes ... \otimes X))^{tC_p}.$$

Using lemma 3.1.15, the natural map $T_p \to \lim_n T_p^{(n)}$ is an equivalence. Furthermore, for $0 \le n$, it can be shown that $T_p^{(n)}$ commutes with infinite direct sums. Note that for each $n$, and each $i$, $\pi_i T_p^{(n)}(H\mathbb{F}_p)$ is finite, because $\tau_{\le n}(X \otimes ... \otimes X)$ only has finitely many non-zero homotopy groups, furthermore they are each finite. Because of this finiteness property we see that there are only finitely many non-zero terms of the $E^2$-page of the Tate spectral sequence, from proposition 3.1.20,

$$E_2^{t,q} = \hat{H}^{-q}(C_p, \pi_t \tau_{\le n}(H\mathbb{F}_p \otimes \dots \otimes H\mathbb{F}_p)) \implies \pi_{q+t}(\tau_{\le n}(H\mathbb{F}_p \otimes \dots \otimes H\mathbb{F}_p))^{tC_p}.$$

Because of this there are only finitely many contributions to each degree of $\pi_i T_p^{(n)}(H\mathbb{F}_p)$, which means that the directed system for the limit $\lim_n \pi_i T_p^{(n)}(H\mathbb{F}_p)$ becomes stationary, and thus we obtain an isomorphism

$$\pi_i T_p(H\mathbb{F}_p) \to \lim_n \pi_i T_p^{(n)}(H\mathbb{F}_p).$$

Which together with theorem 3.4.6 gives the following isomorphisms, also for each $i \in \mathbb{Z}$,

$$\pi_i H\mathbb{F}_p \to \pi_i T_p(H\mathbb{F}_p) \to \lim_n \pi_i T_p^{(n)}(H\mathbb{F}_p).$$

To finish the proof one invokes lemma III.1.8 of [38] concerning pro-isomorphisms which we will refrain from introducing, to see that the map

$$\pi_i H\mathbb{F}_p \to (\pi_i T_p^{(n)}(H\mathbb{F}_p))_n,$$

is a pro-isomorphism, which is a statement that passes to infinite direct sums, hence one obtains a pro-isomorphism for any $\mathbb{F}_p$-vector space $V$,

$$\pi_i HV \to (\pi_i T_p^{(n)}(HV))_n.$$

The right hand side is pro-constant, which implies that

$$\pi_i T_p^{(n)}(HV) \simeq \lim_n \pi_i T_p^{(n)}(HV) \simeq \pi_i HV.$$

Which finishes the sketch of proof. □

We now prove the Tate diagonal admits the structure of a lax symmetric monoidal transformation. The following is proposition III.3.1 of [38], for the proof we follow section 6 and 3 of [36].

LaxSymLem **Lemma 3.4.7.** *Let $\mathscr{C}$ be a symmetric monoidal $\infty$-category with unit $1 \in \mathscr{C}$. The functor* $\mathrm{Map}(-,1) : \mathscr{C}^{op} \to \mathcal{S}$ *admits a canonical lax symmetric monoidal refinement. With this refinement the functor* $\mathrm{Map}(-,1) : \mathscr{C}^{op} \to \mathcal{S}$ *is initial in* $\mathsf{Fun}_{lax}(\mathscr{C}^{op}, \mathcal{S})$.

*Proof.* Recall that the Yoneda embedding $\mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{op}, \mathcal{S})$ admits a symmetric monoidal structure by example 2.1.12. Because of this it sends the tensor unit of $\mathscr{C}$, 1 to the tensor unit $\mathrm{map}(-,1)$ in $\mathsf{Fun}(\mathscr{C}^{op}, \mathcal{S})$. By [25] Corollary 3.2.1.9 we have that $\mathrm{map}(-,1)$ canonically is an object of $\mathsf{CAlg}(\mathsf{Fun}(\mathscr{C}^{op}, \mathcal{S}))$, furthermore $\mathrm{map}(-,1)$ is initial in $\mathsf{CAlg}(\mathsf{Fun}(\mathscr{C}^{op}, \mathcal{S}))$. The following equivalence finishes the proof,

$$\mathsf{CAlg}(\mathsf{Fun}(\mathscr{C}^{op}, \mathcal{S})) \simeq \mathsf{Fun}_{\mathrm{lax}}(\mathscr{C}^{op}, \mathcal{S})$$

which is [7] proposition 2.12. □

Note that $T_p$ inherits the structure of a lax symmetric monoidal functor from theorem 3.1.22.

LaxSymMon **Proposition 3.4.8.** *There is a unique lax symmetric monoidal transformation*

$$\Delta_p : \mathrm{id}_{\mathsf{Sp}} \to T_p.$$

*The underlying transformation of functors is given by the transformation from definition 3.4.4.*

*Proof.* Note first that $T_p$ is lax monoidal because it is a composite of the lax symmetric monoidal functors $X \mapsto X \otimes ... \otimes X$ and $X \mapsto X^{tC_p}$. The identity functor $\mathrm{id}_{\mathsf{Sp}} : \mathsf{Sp} \to \mathsf{Sp}$ is initial in $\mathsf{Fun}^{\mathrm{Lex}}_{\mathrm{lax}}(\mathsf{Sp}, \mathsf{Sp})$. This follows from lemma 3.4.7, applied to $\mathsf{Sp}^{op}$, since $\mathrm{id}_{\mathsf{Sp}} \simeq \mathrm{map}_{\mathsf{Sp}}(\mathbb{S}, -)$. Hence there is a unique lax symmetric monoidal transformation $\mathrm{id}_{\mathsf{Sp}} \to T_p$. □

Recall from lemma 2.2.4 that application of lax symmetric monoidal endofunctors preserve the $\mathbb{E}_\infty$-ring spectrum structure. Hence we obtain the following corollary.

E-infty-lax **Corollary 3.4.9.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum with a $C_p$-action, then $R^{tC_p}$ admits the structure of an $\mathbb{E}_\infty$-ring spectrum. Furthermore the Tate diagonal $\Delta_p$ refines to a map of $\mathbb{E}_\infty$-ring spectra.*

Note that the fact that $R^{tC_p}$ admits the structure of an $\mathbb{E}_\infty$-ring spectrum actually follows from the Tate construction being lax symmetric monoidal, it is only the latter part that uses the above proposition.

# 4 Naive and Genuine Cyclotomic Spectra

In the chapter we will give the definitions of *naive cyclotomic spectra* and *genuine cyclotomic spectra*. Note in [38] naive cyclotomic spectra are simply called cyclotomic spectra. Naive cyclotomic spectra are going to be $\mathbb{T}$-equivariant objects $X \in \mathsf{Sp}$, together with $\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant maps $\varphi_p : X \to X^{tC_p}$, for each prime $p$. The latter are going to be a certain incarnation of orthogonal spectra equipped with a certain fixpoint functor, roughly speaking they are objects $X \in \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ together with homotopy coherent equivalences $\Phi^n : \Phi^{C_n} X \to X$ for $n \in \mathbb{N}$. The main result of this chapter is that the $\infty$-category of genuine cyclotomic spectra is equivalent to the $\infty$-category of naive cyclotomic spectra, when restricted to bounded below spectra. Furthermore the equivalence is rather explicit. We shall furthermore in this chapter define a yet another homotopy invariant called *topological cyclic homology* for both types of cyclotomic spectra. This invariant is closely related to topological Hochschild homology. The other result of this chapter is that the two notions of topological cyclic homology agrees when restricted to bounded below spectra. We begin by defining the $\infty$-category which shall contain the naive cyclotomic spectra and the first steps towards the equivalence mentioned above.

## 4.1 Lax equalizers and Coalgebras

This section will contain a general discussion of two $\infty$-categorical constructions, the first is the notion of a lax equalizer and the second, which relies on the first, is the notion of an $F$-coalgebra in an $\infty$-category. In the following sections we will specialize these constructions to the cases of interest. Lax equalizers will provide the setting in which we shall define the notion of naive cyclotomic spectra, and $F$-coalgebras will provide the setting in which we shall realize the equivalence between the naive and genuine cyclotomic spectra. We follow II.1 and II.5 of [38].

LaxEq **Definition 4.1.1.** Let $\mathscr{C}$ and $\mathscr{D}$ be $\infty$-categories, and let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. The lax equalizer of $F$ and $G$ is the simplicial set $\mathrm{LEq}(F,G)$ defined as the pullback in $\mathsf{sSet}$,

$$\begin{array}{ccc} \mathrm{LEq}(F,G) & \longrightarrow & \mathscr{D}^{\Delta^1} \\ \downarrow & & \downarrow (\mathrm{ev}_0, \mathrm{ev}_1) \\ \mathscr{C} & \xrightarrow{(F,G)} & \mathscr{D} \times \mathscr{D}. \end{array}$$

In particular, the objects of $\mathrm{LEq}(F,G)$ are given by pairs $(c,f)$, consisting of an object $c \in \mathscr{C}$ and a map $f : F(c) \to G(c)$ in $\mathscr{D}$. We shall define the mapping spaces shortly.

*Remark* 4.1.2. Note that $\mathrm{LEq}(F,G)$ is in fact an $\infty$-category, because the $(\mathrm{ev}_0, \mathrm{ev}_1) : \mathscr{D}^{\Delta^1} \to \mathscr{D} \times \mathscr{D}$ is an inner fibration via [22] Proposition 2.3.2.5, hence a fibration from [22] Proposition 2.4.6.5.

It turns out that $\mathrm{LEq}(F,G)$ inherits many properties that $\mathscr{C}$, $\mathscr{D}$ enjoy, much like how $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ inherits properties from $\mathscr{C}$ and $\mathscr{D}$. The properties are proved in [38] Chapter II Proposition 1.5. We will refrain from proving these, mainly because they to a large extend are corollaries of propositions contained in [22]. We collect them here for future reference.

Reg prop of LEq **Lemma 4.1.3.** *Consider the pullback diagram of 4.1.1.*

1. *If $\mathscr{C}$ and $\mathscr{D}$ are stable, and $F$ and $G$ are exact, then* $\mathrm{LEq}(F,G)$ *is stable, and the forgetful functor* $\mathrm{LEq}(F,G) \to \mathscr{C}$ *is exact.*

2. *If $\mathscr{C}$ is presentable, $\mathscr{D}$ is accessible, $F$ is cocontinuous, and $G$ is accessible, then* $\mathrm{LEq}(F,G)$ *is presentable, and the forgetful functor* $\mathrm{LEq}(F,G) \to \mathscr{C}$ *is cocontinuous.*

3. *If $p$ is a $K$-shaped diagram in* $\mathrm{LEq}(F,G)$*, which admits a limit in $\mathscr{C}$ along the forgetful functor, which is preserved by $G$, then $p$ admits a limit, and the forgetful functor preserves this limit.*

We shall in the following lemma give the mapping spaces of $\mathrm{LEq}(F,G)$ and show that the definition in fact makes sense.

mappingspacelax **Lemma 4.1.4.** *Consider the defining diagram for* $\mathrm{LEq}(F,G)$ *in 4.1.1.*

1. *This diagram is in fact a pullback diagram of $\infty$-categories.*

2. *Let $X, Y \in \mathrm{LEq}(F,G)$ be two objects, given by pairs $(c_X, f_X)$ and $(c_Y, f_Y)$. Then the mapping space is given by*

$$\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \simeq \mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \underset{(f_Y)_* F}{\overset{(f_X)^* G}{\rightrightarrows}} \mathrm{Map}_{\mathscr{D}}(F(c_X), G(c_Y))\Big).$$

*Moreover, a map $f : X \to Y$ in* $\mathrm{LEq}(F,G)$ *is an equivalence if and only if its image under the inclusion into $\mathscr{C}$ is an equivalence.*

*Proof.* The first part is in fact a result concerning the Joyal model structure, and classical results from model category theory. By proposition 1.1.8 $\mathscr{D}^{\Delta^1}$, $\mathscr{C}$ and $\mathscr{D} \times \mathscr{D}$ are fibrant objects of the Joyal model structure on $\mathsf{sSet}$. Then a sufficient condition for $\mathrm{LEq}(F,G)$ to be the pullback of the maps $(F,G) : \mathscr{C} \to \mathscr{D}$ and $(\mathrm{ev}_0, \mathrm{ev}_1) : \mathscr{D}^{\Delta^1} \to \mathscr{D} \times \mathscr{D}$, is that either of the morphisms are fibrations (This is [22] Proposition A.2.4.4). The latter is a fibration as we have already argued.

For the second part we make sense of the functor $\mathrm{Map}_{\mathrm{LEq}(F,G)} : \mathrm{LEq}(F,G)^{op} \times \mathrm{LEq}(F,G) \to \mathcal{S}$, where $\mathcal{S}$ through construction 1.1.22. At this point it is worth recalling the point of §2.2 of [22], namely that the simplicial category $\mathrm{Hom}^R_{\mathscr{C}}(X,Y) \in \mathsf{Cat}_\Delta$ represents the space $\mathrm{Map}_{\mathscr{C}}(X,Y) \in \mathcal{S}$, whenever $\mathscr{C}$ is an $\infty$-category, this is summarized in [22] section 1.2.2. Using this idea, we may pass from a "$\mathrm{Hom}^R$"-version of the relevant diagram to a "Map"-version. Consider the following pullback diagram of simplicial sets

$$\begin{array}{ccc}
\mathrm{Hom}^R_{\mathrm{LEq}(F,G)}(X,Y) & \longrightarrow & \mathrm{Hom}^R_{\mathscr{D}^{\Delta^1}}(f_X, f_Y) \\
\downarrow t & & \downarrow \\
\mathrm{Hom}^R_{\mathscr{C}}(c_X, c_Y) & \xrightarrow{(F,G)} & \mathrm{Hom}^R_{\mathscr{D}}(F(c_X), F(c_Y)) \times \mathrm{Hom}^R_{\mathscr{D}}(G(c_X), G(c_Y))
\end{array}$$

From part 1 we know the map $\mathscr{D}^{\Delta^1} \to \mathscr{D} \times \mathscr{D}$ is an inner fibration, which means by [22] Proposition 2.4.4.1 that the right hand vertical map is a Kan fibration, hence the square is a homotopy pullback square. Hence the following diagram is a pullback in $\mathcal{S}$,

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) & \longrightarrow & \mathrm{Map}_{\mathscr{D}^{\Delta^1}}(f_X, f_Y) \\
\downarrow t & & \downarrow \\
\mathrm{Map}_{\mathscr{C}}(c_X, c_Y) & \xrightarrow{(F,G)} & \mathrm{Map}_{\mathscr{D}}(F(c_X), F(c_Y)) \times \mathrm{Map}_{\mathscr{D}}(G(c_X), G(c_Y))
\end{array}$$

Denote by $E$ the equalizer given in the statement. We now build the desired equivalence, by using the universal property of the pullback. Using the usual method of writing an equalizer in terms of products and pullbacks, $E$ can be written as the pullback of the following diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times_{\mathrm{Map}_{\mathscr{D}}(F(c_X), G(c_Y))} \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \end{array}$$

Where the top right object is the pullback of the following diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) & & \xrightarrow{\quad t \quad} & & \\ & \overset{t'}{\dashrightarrow} \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times_{\mathrm{Map}_{\mathscr{D}}(F(c_X), G(c_Y))} \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \\ & \downarrow & & \downarrow (f_Y)_* F \\ \xrightarrow{\quad t \quad} & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) & \xrightarrow{(f_X)^* G} & \mathrm{Map}_{\mathscr{D}}(F(c_x), G(c_Y)) \end{array}$$

Using the map $t$ twice we get a map $t' : \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \to \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times_{\mathrm{Map}_{\mathscr{D}}(F(c_X), G(c_Y))} \mathrm{Map}_{\mathscr{C}}(c_X, c_Y)$. Using $t$ and $t'$, we get a map $\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \to E$, making $E$'s pullback diagram commute, i.e. the following diagram commutes,

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) & & \xrightarrow{\quad t' \quad} & & \\ & \dashrightarrow E & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times_{\mathrm{Map}_{\mathscr{D}}(F(c_X), G(c_Y))} \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \\ & \downarrow & & \downarrow \\ \xrightarrow{\quad t \quad} & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \times \mathrm{Map}_{\mathscr{C}}(c_X, c_Y) \end{array}$$

Using the inclusion $E \to \mathrm{Map}_{\mathscr{C}}(c_X, c_Y)$ and the following map $E \to \mathrm{Map}_{\mathscr{D}^{\Delta^1}}(f_X, f_Y)$ given by,

$$(t \in E) \longmapsto \begin{array}{ccc} F(c_X) & \xrightarrow{f_X} & G(c_X) \\ \downarrow F(t) & & \downarrow G(t) \\ F(c_Y) & \xrightarrow{f_Y} & G(c_Y), \end{array}$$

we obtain a map $E \to \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y)$, making the $\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y)$'s pullback diagram commute. By the universal property we get the desired mapping space.
For the last part of the statement, we note that if $f : X \to Y$ is a map in $\mathrm{LEq}(F,G)$ which becomes an equivalence in $\mathscr{C}$, then using the first part of the statement, we see that

$$f^* : \mathrm{Map}_{\mathrm{LEq}(F,G)}(Y,Z) \to \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Z),$$

is an equivalence for all $Z \in \mathrm{LEq}(F,G)$. The Yoneda lemma gives the desired result. □

We shall now turn our attention to coalgebras and fixed points for endofunctors. This is going to be the technical input in the $p$-local discussion of the equivalence between bounded below naive and genuine cyclotomic spectra.

**Definition 4.1.5.** Let $\mathscr{C}$ be an $\infty$-category and let $F : \mathscr{C} \to \mathscr{C}$ be an endofunctor. Then an $F$-*coalgebra* is given by an object $X \in \mathscr{C}$ together with a morphism $X \to FX$. A *fixpoint* of $F$ is an object $X \in \mathscr{C}$ together with an equivalence $X \to FX$. We define the $\infty$-category of $F$-coalgebra objects of $\mathscr{C}$, $\mathsf{CoAlg}_F(\mathscr{C})$ as the lax equalizer $\mathrm{LEq}(\mathrm{id}, F)$ of $\infty$-categories. The $\infty$-category $\mathsf{Fix}_F(\mathscr{C})$ is the full subcategory $\mathsf{Fix}_F(\mathscr{C}) \subseteq \mathsf{CoAlg}_F(\mathscr{C})$ spanned by the fixpoints. When $\mathscr{C}$ is clear from the context we will only write $\mathsf{CoAlg}_F$ and $\mathsf{Fix}_F$.

CoAlgCor **Corollary 4.1.6.** *Let $\mathscr{C}$ be a presentable $\infty$-category.*

1. *Let $F : \mathscr{C} \to \mathscr{C}$ be an accessible functor. Then the $\infty$-category $\mathsf{CoAlg}_F$ is presentable and the forgetful functor $\mathsf{CoAlg}_F \to \mathscr{C}$ is cocontinuous.*
2. *Let $F : \mathscr{C} \to \mathscr{C}$ be a cocontinuous functor. Then the $\infty$-category $\mathsf{Fix}_F$ is presentable and the forgetful functor $\mathsf{Fix}_F \to \mathscr{C}$ is cocontinuous. The inclusion $\iota : \mathsf{Fix}_F \subseteq \mathsf{CoAlg}_F$ is cocontinuous.*

*Proof.* The first statement follows directly from lemma 4.1.3 (2). For the second recall that morphisms in the $\infty$-category $\mathsf{Pr}^L$ are cocontinuous functors, hence $F$ is a morphism of $\mathsf{Pr}^L$. It is clear that we may realize $\mathsf{Fix}_F$ as the following equalizer of $\mathsf{Pr}^L$,

$$\mathsf{Fix}_F \longrightarrow \mathscr{C} \underset{F}{\overset{\mathrm{id}}{\rightrightarrows}} \mathscr{C}.$$

Hence $\mathsf{Fix}_F$ is presentable, and the forgetful functor $\mathsf{Fix}_F \to \mathscr{C}$ is equivalent to the induced map above, hence is cocontinuous. That the inclusion $\iota : \mathsf{Fix}_F \subseteq \mathsf{CoAlg}_F$ is cocontinuous follows from the above. □

Throughout the remainder of this section we shall assume that the $\infty$-category $\mathscr{C}$ is presentable and that the endofunctor $F : \mathscr{C} \to \mathscr{C}$ is cocontinuous. I.e. we wish the statement of corollary 4.1.6 (2) to be true. Note that the inclusion $\iota : \mathsf{Fix}_F \subseteq \mathsf{CoAlg}_F$ is cocontinuous and hence admits a right adjoint $R_\iota : \mathsf{CoAlg}_F \to \mathsf{Fix}_F$. This right adjoint induces an endofunctor $\iota R_\iota : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$. We shall need an explicit formula for $\iota R_\iota$. To this end we shall the following construction.

RbarCons **Construction 4.1.7.** Consider the following endofunctor $\overline{F} : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$ given on objects by

$$(\varphi : X \to FX) \mapsto (F\varphi : FX \to F^2X).$$

Note that there is a natural transformation $\mu : \mathrm{id} \to \overline{F}$. For further details of the construction of $\overline{F}$ see construction II.5.2 of [38]. The endofunctor $\overline{F}$ is cocontinuous because $F$ is cocontinuous, hence $\overline{F}$ admits a right adjoint $R_{\overline{F}} : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$. The adjoint of the natural transformation $\mu : \mathrm{id} \to \overline{F}$ is the transformation $\nu : R_{\overline{F}} \to \mathrm{id}$.

Given the map above we are able to describe the left adjoint of the inclusion $\iota : \mathsf{Fix}_F \to \mathsf{CoAlg}_F$. The left adjoint exists because being a fixed point of an endofunctor is a property of the endofunctor.

LeftAdFormula **Lemma 4.1.8.** *The inclusion $\iota : \mathsf{Fix}_F \to \mathsf{CoAlg}_F$, admits a left adjoint, $L_\iota : \mathsf{CoAlg}_F \to \mathsf{Fix}_F$, for which we have the following colimit formula*

$$\iota L_\iota \simeq \mathrm{colim}(\mathrm{id} \xrightarrow{\mu} \overline{F} \xrightarrow{\overline{F}\mu} \overline{F}^2 \xrightarrow{\overline{F}^2\mu} \overline{F}^3 \longrightarrow \ldots).$$

*Proof.* By proposition 5.2.7.4 of [22] a left adjoint as described in the lemma would fit into the following commutative diagram

$$\begin{array}{ccc} \mathsf{CoAlg}_F & \xrightarrow{\quad \mathbb{L} \quad} & \mathsf{Fix}_F \subseteq \mathsf{CoAlg}_F \\ & {\scriptstyle L_\iota} \searrow \quad \nearrow {\scriptstyle \iota} & \\ & \mathsf{Fix}_F & \end{array}$$

where $\mathbb{L} : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$ is a localization which essential image is $\mathsf{Fix}_F$. Proposition 5.2.7.4 of [22] gives equivalent conditions for recognising localizations, we use the third condition, and show that there exists a functor $\mathbb{L} : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$ such that there is a natural transformation $\mathrm{id} \to \mathbb{L}$ such that $\mathbb{L}(X) \to \mathbb{L}(\mathbb{L}(X))$ is an equivalence. We claim that the colimit of functors,

$$\mathbb{L} := \mathrm{colim}(\mathrm{id} \xrightarrow{\mu} \overline{F} \xrightarrow{\overline{F}\mu} \overline{F}^2 \xrightarrow{\overline{F}^2\mu} \overline{F}^3 \longrightarrow \ldots)$$

satisfies these properties. Note first that by definition of $\overline{F}$ the full subcategory $\mathsf{Fix}_F \subseteq \mathsf{CoAlg}_F$ can be described as those coalgebras $X$ for which the transformation $\mu_X : X \to \overline{F}(X)$ is an equivalence. Because $\overline{F}$ is cocontinuous it follows directly that $\mathbb{L}(X) \in \mathsf{Fix}_F$ for $X \in \mathsf{CoAlg}_F$. Furthermore it follows that $X \to \mathbb{L}(X)$ is an equivalence for $X \in \mathsf{Fix}_F$. Combining these two facts we have that $\mathbb{L}(\mathbb{L}(X)) \simeq \mathbb{L}(X)$, which exhibits $\mathbb{L}$ as a localization with $\mathsf{Fix}_F$ being the local objects. ☐

Now the following explicit description of $\iota R_\iota$ follows easily.

RightAdLim **Proposition 4.1.9.** *The endofunctor $\iota R_\iota : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$ is given by the limit of the following directed diagram of endofunctors,*

$$\ldots \longrightarrow R^3_{\overline{F}} \xrightarrow{R^2_{\overline{F}\nu}} R^2_{\overline{F}} \xrightarrow{R_{\overline{F}\nu}} R_{\overline{F}} \xrightarrow{\nu} \mathrm{id}\,.$$

*Proof.* The functor $\iota L_\iota : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$ is left adjoint to the functor $\iota R_\iota : \mathsf{CoAlg}_F \to \mathsf{CoAlg}_F$. Recall that a right adjoint to a colimit is the limit of the respective right adjoints. Using the notation of lemma 4.1.8, we $\iota L_\iota \simeq \mathbb{L}$ which was given by a colimit, so by construction 4.1.7 the claim now follows. ☐

We will also need a more explicit description of the right adjoint of $\overline{F} : \mathscr{C} \to \mathscr{C}$, denoted $R_{\overline{F}} : \mathscr{C} \to \mathscr{C}$. In the case of interest to us the functors $R_F$ and $F$ has a few extra properties, which we shall assume, to simplify the description of $R_{\overline{F}}$.

InducStart **Lemma 4.1.10.** *Assume that the counit $\varepsilon : FR_F \to \mathrm{id}_{\mathscr{C}}$ of the $F \dashv R_F$ adjunction is fully faithful. Assume that $F : \mathscr{C} \to \mathscr{C}$ preserves pullbacks. Consider the coalgebra $\varphi : X \to FX$, and consider the following pullback*

$$\begin{array}{ccc} X \times_{R_F FX} R_F X & \xrightarrow{s} & R_F X \\ \big\downarrow f & & \big\downarrow R_F(\varphi) \\ X & \xrightarrow{\eta_X} & R_F F X. \end{array}$$

*The natural transformation in the above diagram $\eta : \mathrm{id}_{\mathscr{C}} \to R_F F$ is the unit of the $F \dashv R_F$ adjunction. Then*

$$R_{\overline{F}}(\varphi : X \to FX) = (\overline{\varphi} : X \times_{R_F FX} R_F X \to F(X \times_{R_F FX} R_F X)).$$

*Whihc has a coalgebra structure induced from*

$$X \times_{R_F FX} R_F X \xrightarrow{f} X \xrightarrow{\varepsilon_X^{-1}} FR_F X \xrightarrow{\simeq} F(X \times_{R_F FX} R_F X).$$

*Where the last equivalence is induced from $F(s) : F(X \times_{R_F FX} R_F X) \to FR_F X$ and the fact that $F$ preserves pullbacks. Moreover $\overline{F} R_{\overline{F}} \to \mathrm{id}$ is an equivalence.*

*Proof.* We shall use the description of the mapping spaces of objects in a lax equalizer given in lemma 4.1.4. Let $\alpha : Z \to FZ$ be an arbitrary $F$-coalgebra, and let $\overline{\varphi} : X \times_{R_F FX} R_F X \to F(X \times_{R_F FX} R_F X) \simeq X$ be the $F$-coalgbra described in the statement of the lemma, then maps between them are given by

$$\mathrm{Map}_{\mathsf{CoAlg}_F}(Z, X \times_{R_F FX} R_F X) \simeq \mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Z, X \times_{R_F FX} R_F X) \rightrightarrows \mathrm{Map}_{\mathscr{C}}(Z, X)\Big).$$

Because $\mathrm{Map}_{\mathscr{C}}(Z,-)$ is continuous we have that the first term of the equalizer can be identified with

$$\mathrm{Map}_{\mathscr{C}}(Z,X)\times_{\mathrm{Map}_{\mathscr{C}}(Z,R_FFX)}\mathrm{Map}_{\mathscr{C}}(Z,R_FX).$$

The first of the maps into $\mathrm{Map}_{\mathscr{C}}(Z,X)$ in the equalizer under this identification is given by projection onto the second factor, while the other map is projection on the first fact and then postcomposing with the following map

$$\mathrm{Map}_{\mathscr{C}}(Z,R_FX)\xrightarrow{F}\mathrm{Map}_{\mathscr{C}}(FZ,FR_FX)\xrightarrow{(\varepsilon_X)_*}\mathrm{Map}_{\mathscr{C}}(FZ,X)\xrightarrow{\alpha^*}\mathrm{Map}_{\mathscr{C}}(Z,X).$$

The structure of $\mathrm{Map}_{\mathscr{C}}(Z,X\times_{R_FFX}R_FX)$ as the above pullback, gives us the following equivalences,

$$\begin{aligned}&\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Z,X\times_{R_FFX}R_FX)\rightrightarrows\mathrm{Map}_{\mathscr{C}}(Z,X)\Big)\\ &\simeq\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Z,R_FX)\rightrightarrows\mathrm{Map}_{\mathscr{C}}(Z,R_FFX)\Big)\end{aligned}$$

Next we apply $\overline{F}$ and the counit $\varepsilon$ to obtain

$$\begin{aligned}&\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Z,R_FX)\rightrightarrows\mathrm{Map}_{\mathscr{C}}(Z,R_FFX)\Big)\\ &\simeq\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(FZ,FR_FX)\rightrightarrows\mathrm{Map}_{\mathscr{C}}(FZ,FR_FFX)\Big)\\ &\simeq\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(FZ,X)\rightrightarrows\mathrm{Map}_{\mathscr{C}}(FZ,FX)\Big)\\ &\simeq\mathrm{Map}_{\mathsf{CoAlg}_F}(\overline{F}(Z),X).\end{aligned}$$

Collecting the above equivalences we obtain

$$\mathrm{Map}_{\mathsf{CoAlg}_F}(Z,X\times_{R_FFX}R_FX)\simeq\mathrm{Map}_{\mathsf{CoAlg}_F}(\overline{F}(Z),X)\simeq\mathrm{Map}_{\mathsf{CoAlg}_F}(Z,R_{\overline{F}}(X)),$$

which shows the desired result using Yoneda. We show that last equivalence by using the equivalence of the underlying objects

$$F(X\times_{R_FFX}R_FX)\xrightarrow{\simeq}FR_FX\xrightarrow{\varepsilon_X}X.$$

Which shows that the counit $\overline{F}R_{\overline{F}}\to\mathrm{id}$ is an equivalence on objects, i.e.

$$\begin{aligned}FR_{\overline{F}}(X\to FX)&=(F(\overline{\varphi}):F(X\times_{R_FFX}R_FX)\to F(F(X\times_{R_FFX}R_FX)))\\ &\simeq(\varphi:X\to FX).\end{aligned}$$

□

Using this description of $R_{\overline{F}}$, we may describe the individual steps of the limit in proposition 4.1.9.

ItePullbackRbar **Corollary 4.1.11.** *Assume that the counit $\varepsilon:FR_F\to\mathrm{id}_{\mathscr{C}}$ of the $F\dashv R_F$ adjunction is fully faithful. Assume that $F:\mathscr{C}\to\mathscr{C}$ preserves pullbacks. Consider the coalgebra $\varphi:X\to FX$, then the underlying object of the k-fold iteration $R_{\overline{F}}^kX$ is equivalent to*

$$R_F^kX\times_{R_F^kFX}R_F^{k-1}X\times_{R_F^{k-1}FX}\ldots\times_{R_FFX}X.$$

*The maps to the right are induced by the map of coalgebras $X\to FX$ and the maps to the left are induced by the unit $\eta:\mathrm{id}_{\mathscr{C}}\to R_FF$. The maps $(R_{\overline{F}}^k\nu)_X:R_{\overline{F}}^kX\to R_{\overline{F}}^{k-1}X$ can be described as forgetting the first factor.*

*Proof.* This follows by induction, where lemma 4.1.10 is the induction start. Furthermore the inductive step is analogous to the procedure in lemma 4.1.10 with more book keeping. Ultimately it works because both $R_F$ and $F$ preseve pullbacks. □

We are now done with the part of the discussion of endofunctors and coalgebras we shall need for the $p$-local part of the equivalence of bounded below naive and genuine cyclotomic spectra. We begin discussing the part needed for the global equivalence.

GlobalCoAlg **Definition 4.1.12.** Let $\{F_i : \mathscr{C} \to \mathscr{C}\}_{i\in\mathbb{N}}$ be countable collection of commuting endofunctors defining an action of the monoid $(\prod_{i=1}^{\infty}\mathbb{N}, +)$ on the $\infty$-category $\mathscr{C}$. I.e. the collection comes with chosen equivalences $F_i \circ F_j \simeq F_j \circ F_i$ for each $i, j \in \mathbb{N}$. Let $n \in \mathbb{N}$, then $F_{n+1}$ induces an endofunctor $c_{n+1} : \mathsf{CoAlg}_{F_n}(\mathscr{C}) \to \mathsf{CoAlg}_{F_n}(\mathscr{C})$ given by

CoAlgEq
$$(X \to F_nX) \mapsto (F_{n+1}X \to F_{n+1}F_nX \simeq F_nF_{n+1}X) \tag{4.1}$$

Where (4.1) is how the functor $c_{n+1}$ is defined on objects, on morphisms it is defined analogously to $\overline{F}$ in construction 4.1.7. This endofunctor restricts to an endofunctor of $\mathsf{Fix}_{F_n}(\mathscr{C}) \subseteq \mathsf{CoAlg}_{F_n}(\mathscr{C})$. This means we consider $F_{n+1}$-coalgebras in $F_n$-coalgebras. The collection of such objects we denote as

$$\mathsf{CoAlg}_{F_n,F_{n+1}}(\mathscr{C}) := \mathsf{CoAlg}_{F_{n+1}}(\mathsf{CoAlg}_{F_n}(\mathscr{C})).$$

We define inductively

$$\mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C}) := \mathsf{CoAlg}_{F_n}(\mathsf{CoAlg}_{F_1,\dots,F_{n-1}}(\mathscr{C})),$$
$$\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C}) := \lim_n \mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C}).$$

We define

$$\mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C}) \subseteq \mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}$$

to be the full subcategory consisting of the objects for which the morphisms $X \to F_i(X)$ are equivalences for all $i \in \mathbb{N}$.

*Remark* 4.1.13. Using proposition 4.2.4.4 of [22] we may strictify the action of the monoid $\prod_{i=1}^{\infty}\mathbb{N}$, i.e. we may assume that the equivalence in (4.1) is an equality. We shall make this assumption in the following.

The remainder of the section will be dedicated to showing the analogs for $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C})$ of the results we showed for $\mathsf{CoAlg}_F(\mathscr{C})$, $F$ and the maps related to $F$ previously in the section. Many of the results will follow by induction on $n$ and the fact that each intermediate step in the defining limit of $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C})$ satisfies the assumptions put on $\mathscr{C}$ to begin with. The first example is the first part of the following result which is analogous to corollary 4.1.6.

GlobalCoAlgCor **Corollary 4.1.14.** *Let $\mathscr{C}$ be a presentable $\infty$-category.*

1. *Let $F_i : \mathscr{C} \to \mathscr{C}$ for each $i \in \mathbb{N}$ be an accessible functor. Then the $\infty$-category $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}$ is presentable and the forgetful functor $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}} \to \mathscr{C}$ is cocontinuous.*

2. *Let $F_i : \mathscr{C} \to \mathscr{C}$ for each $i \in \mathbb{N}$ be an cocontinuous functor. Then the $\infty$-category $\mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}}$ is presentable and the forgetful functor $\mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}} \to \mathscr{C}$ is cocontinuous. The inclusion $\iota : \mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}} \subseteq \mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}$ is cocontinuous.*

*Proof.* The first part follows by the argument sketched above. We show the second part now. The action described in definition 4.1.12 for a single endomorphism $F$, is determined by application of $F$ corresponds to the action of 1 on $(\mathbb{N}, +)$. As in proof of corollary 4.1.6 we can realize $\mathsf{Fix}_F$, for a single endofunctor, as the following equalizer

$$\mathsf{Fix}_F(\mathscr{C}) \longrightarrow \mathscr{C} \underset{F}{\overset{\mathrm{id}}{\rightrightarrows}} \mathscr{C}.$$

Because 1 is the generator of $(\mathbb{N}, +)$, the action of 1 determines the entire action on $\mathbb{N}$, therefore the above equalizer is equivalent to $\mathscr{C}^{h\mathbb{N}}$, from which we have $\mathsf{Fix}_F(\mathscr{C}) \simeq \mathscr{C}^{h\mathbb{N}}$. Analogously the action described in definition 4.1.12 for a $F_1, ..., F_n$, is determined by application of $F_k$ corresponding to the action of $(0, ..., 1, ..., 0)$, where 1 is on the $k$'th factor, on $(\mathbb{N}^n, +)$. Therefore we have the equalizer

$$\mathsf{Fix}_{F_1,...,F_n}(\mathscr{C}) \longrightarrow \mathscr{C} \underset{\{F_k\}_{1\le k\le n}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathscr{C}.$$

which gives an equivalence $\mathsf{Fix}_{F_1,...,F_n}(\mathscr{C}) \simeq \mathscr{C}^{h(\prod_{i=1}^{\infty}\mathbb{N}}$. Passing to the limit we obtain

$$\mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C}) \simeq \mathscr{C}^{h(\prod_{i=1}^{\infty}\mathbb{N})}.$$

The desired result now follows because $\mathsf{Pr}^L$ is complete, since we have realized $\mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}}$ as a limit of presentable $\infty$-categories along left adjoint functors ($F_i$ admits right adjoints per. assumption). □

We now show the analog of lemma 4.1.10 and lemma 4.1.4 for $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}$.

L.II.5.8 **Lemma 4.1.15.** *Assume that each functor from the collection $\{F_i\}_{i\in\mathbb{N}}$ is accessible, and preserves the terminal object of $\mathscr{C}$. Assume that the objects $F_i(F_j(X))$ for $X \in \mathscr{C}$ are terminal for all distinct $i, j \in \mathbb{N}$.*

1. *Then there is a bijection,*

$$\left\{\begin{matrix}\textit{Isomorphism classes of}\\ \{F_i\}_{i\in\mathbb{N}}\textit{-coalgebra structures on } X.\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{Isomorphism classes of}\\ \textit{families of maps } \{\varphi_i : X \to F_i(X)\}_{i\in\mathbb{N}}.\end{matrix}\right\}$$

2. *Assume that $X$ has a coalgebra structure giving rise to morphisms $\{\varphi_i : X \to F_i(X)\}_{i\in\mathbb{N}}$ through part 1. Then for any $Y \in \mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C})$ the mapping space $\mathrm{Map}_{\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C})}(Y, X)$ is equivalent to the equalizer*

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y, X) \rightrightarrows \textstyle\prod_{i=1}^{\infty} \mathrm{Map}_{\mathscr{C}}(Y, F_i(X))\Big).$$

3. *The $\infty$-category $\mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}(\mathscr{C})$ is equivalent to the lax equalizer*

$$\mathrm{LEq}\Big(\mathscr{C} \underset{\{F_i\}_{i\in\mathbb{N}}}{\overset{\{\mathrm{id}\}_{i\in\mathbb{N}}}{\rightrightarrows}} \textstyle\prod_{i=1}^{\infty}\mathscr{C}\Big).$$

*Proof.* Note that part 3 follows from part 1 and 2 and lemma 4.1.4. We first show analogue of part 1 and 2 for finitely many endofunctors by induction on their number. The induction start, i.e. when there is a single endofunctor, follows from lemma 4.1.4.

Therefore assume that part 1 and 2 have been shown for $F_1, ..., F_n$, and assume we are given another endofunctor $F_{n+1}$ such that the finite analog of the assumptions hold for the collection $\{F_i\}_{0<i\le n+1}$. By induction we have that promoting an object $X \in \mathscr{C}$ to an object of $\mathsf{CoAlg}_{F_1,...,F_n}(\mathscr{C})$ is done by choosing maps $X \to F_i(X)$ for $1 \le i \le n$. Promoting an object $X$ of $\mathsf{CoAlg}_{F_1,...,F_n}(\mathscr{C})$ to an object of $\mathsf{CoAlg}_{F_1,...,F_{n+1}}(\mathscr{C})$ is done per. definition by choosing a map $X \to F_{n+1}(X)$. By virtue of being an object of $\mathsf{CoAlg}_{F_1,...,F_n}(\mathscr{C})$, the choosen map $X \to F_{n+1}(X)$ satisfies the hypothesis of the lemma, i.e. $F_i(F_{n+1}(X))$ is terminal for $1 \le i \le n$. Now by the inductive hypothesis of part 2, we deduce that a map $X \to F_{n+1}(X)$ in $\mathsf{CoAlg}_{F_1,...,F_n}(\mathscr{C})$ is

equivalent to a map $X \to F_{n+1}(X)$ in $\mathscr{C}$, which shows part 1 of the induction.

Consider the mapping space between any two coalgebras $X, Y \in \mathsf{CoAlg}_{F_1,\dots,F_{n+1}}(\mathscr{C})$, which by lemma 4.1.4 is given by

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C})}(Y,X) \rightrightarrows \mathrm{Map}_{\mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C})}(Y,F_{n+1}(X))\Big).$$

By the inductive hypothesis we have

$$\mathrm{Map}_{\mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C})}(Y,X) \quad \simeq \quad \mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y,X) \rightrightarrows \textstyle\prod_{i=1}^{n} \mathrm{Map}_{\mathscr{C}}(Y,F_i(X))\Big),$$

and similiarly

$$\mathrm{Map}_{\mathsf{CoAlg}_{F_1,\dots,F_n}(\mathscr{C})}(Y,F_{n+1}(X)) \;\simeq$$
$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y,F_{n+1}(X)) \rightrightarrows \textstyle\prod_{i=1}^{n} \mathrm{Map}_{\mathscr{C}}(Y,F_i(F_{n+1}(X)))\Big).$$

Now because $F_i(F_{n+1}(X))$ is terminal for all $1 \le i \le n$, we have from the above that

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y,F_{n+1}(X)) \rightrightarrows \textstyle\prod_{i=1}^{n} \mathrm{Map}_{\mathscr{C}}(Y,F_i(F_{n+1}(X)))\Big) \;\simeq \mathrm{Map}_{\mathscr{C}}(Y,F_{n+1}(X)).$$

Collecting these equivalences we see that the mapping space $\mathrm{Map}_{\mathsf{CoAlg}_{F_1,\dots,F_{n+1}}(\mathscr{C})}(Y,X)$ is given by

$$\mathrm{Eq}\Big(\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y,X) \rightrightarrows \textstyle\prod_{i=1}^{n} \mathrm{Map}_{\mathscr{C}}(Y,F_i(X))\Big) \rightrightarrows \mathrm{Map}_{\mathscr{C}}(Y,F_{n+1}(X))\Big).$$

It is elementary to show that this iterated equalizer is equivalent to

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathscr{C}}(Y,X) \rightrightarrows \textstyle\prod_{i=1}^{n+1} \mathrm{Map}_{\mathscr{C}}(Y,F_i(X))\Big).$$

Which shows the inductive step of part 2. The induction shows parts 1 and 2 for a finite number of endofunctors. Now the claim for countably many endofunctors follows by passing to the limit. $\square$

By corollary 4.1.14 the inclusion $\iota : \mathsf{Fix}_{\{F_i\}_{i\in\mathbb{N}}} \subseteq \mathsf{CoAlg}_{\{F_i\}_{i\in\mathbb{N}}}$ is cocontinuous, and hence admits a right adjoint $R_\iota$. Our goal is to study $R_\iota$ by factoring the inclusion $\iota$ into stepwise inclusions

$$\iota_n : \mathsf{Fix}_{F_1,\dots,F_n}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}) \subseteq \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_n,F_{n+1},\dots}),$$

and understanding their right adjoints $R_{\iota_n}$ for $n \in \mathbb{N}$, by using the result for a single endofunctor. For the treatment of these right adjoints, we shall need an analog of $\overline{F}_n$ for each intermediate step. We shall need the following isomorphism, which follows from simple unwinding of definitions.

FixIsoCoAlg **Corollary 4.1.16.** *There is an isomorphism of simplicial sets*

$$\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_n,F_{n+1},\dots}) \cong \mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})).$$

We now construct the global analog of the functor $\overline{F}$.

GlobalRbarCons **Construction 4.1.17.** Using construction 4.1.7, with $\mathscr{C}$ replaced by $\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})$, we may construct an endofunctor

$$\overline{F}_n : \mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})) \to \mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})).$$

Paired with the isomorphism from corollary 4.1.16 we obtain an endofunctor

$$\overline{F}_n : \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}) \to \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}).$$

By the arguments of construction 4.1.7 the endofunctor $\overline{F}_n$ is cocontinuous, hence it admits a right adjoint $R_{\overline{F}_n} : \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}) \to \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})$, which comes with a natural transformation $\nu_n : R_{\overline{F}_n} \to \mathrm{id}$.

By lemma 4.1.8 and lemma 4.1.9, we have the following result, for$F_n$ for each $n \geq 1$.

GlobalRightAdLim **Proposition 4.1.18.** *The endofunctor $\iota R_{\iota_n}$ is given by the limit of the following directed diagram of endofunctors,*

$$\dots \longrightarrow R^3_{\overline{F}_n} \xrightarrow{R^2_{\overline{F}_n}\nu_n} R^2_{\overline{F}_n} \xrightarrow{R_{\overline{F}_n}\nu_n} R_{\overline{F}} \xrightarrow{\nu_n} \mathrm{id}\,.$$

For the last remaining result, i.e. the analog of corollary 4.1.11, we shall make a few new assumptions on each $F_n : \mathscr{C} \to \mathscr{C}$.

obalItePullbackRbar **Corollary 4.1.19.** *Assume for all $n \in \mathbb{N}$ that $F_n : \mathscr{C} \to \mathscr{C}$ admits a fully faithful right adjoint $R_{F_n}$ such that for all $i \neq n$, the canonical morphism*

$$F_i R_{F_n} \to R_{F_n} F_i,$$

*which is adjoint to the morphism $F_n F_i R_{F_n} = F_i F_n R_{F_n} \simeq F_i$, is an equivalence. Assume further that each endofunctor of $\{F_i\}_{i\in\mathbb{N}}$ preserves pullbacks. Consider $X \in \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})$, then the underlying object of the k-fold iteration $R^k_{\overline{F}_n} X$ is equivalent to*

$$R^k_{F_n} X \times_{R^k_{F_n} FX} R^{k-1}_{F_n} X \times_{R^{k-1}_{F_n} FX} \dots \times_{R_{F_n} FX} X.$$

*The maps to the right are induced by the map of coalgebras $X \to F_n X$ and the maps to the left are induced by the unit $\eta_n : \mathrm{id}_{\mathscr{C}} \to R_{F_n} F_n$. The maps $(R^k_{\overline{F}_n} \nu_n)_X : R^k_{\overline{F}_n} X \to R^{k-1}_{\overline{F}_n} X$ can be described as forgetting the first factor.*

*Proof.* Consider the ∞-category $\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})$. This is by corollary 4.1.16 isomorphic to $\mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}))$. The endofunctor on this ∞-category $\underline{F}_n$, is the canonical extension of endofunctor $F_n : \mathscr{C} \to \mathscr{C}$ to $\mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}))$ through the procedure described in definition 4.1.12, more specifically in (4.1). By assumption $R_{F_n}$ commutes with all $F_i$ for $i \neq n$, this allow us to extend $R_{F_n}$ to $\mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}))$. We denote this extension by $\underline{R}_{F_n}$. Using the same argument we may extend the counit and unit, $\underline{\eta}_n : \mathrm{id} \to \underline{R}_{F_n}\underline{F}_n$ and $\underline{\varepsilon}_n : \underline{F}_n\underline{R}_{F_n} \to \mathrm{id}$. From which we deduce that $\underline{R}_{F_n}$ is right adjoint to $\underline{F}_n$. Note because $F_n F_i R_{F_n} = F_i F_n R_{F_n} \simeq F_i$ per. assumption, the non-extended counit is an equivalence, therefore the extended counit is an equivalence. By lemma 1.2.7 this is equivalent to $\underline{R}_{F_n}$ being fully faithful. Hence we may apply corollary 4.1.11 to compute the structure of the right adjoint $R_{\underline{\overline{F}}_n}$ to

$$\underline{\overline{F}}_n : \mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots})) \to \mathsf{CoAlg}_{F_n}(\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}))$$

This gives the desired formula for the underlying functor as well, because the forgetful functor

$$\mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}) \to \mathscr{C},$$

preserve pullbacks. This is because this forgetful functor preserves the limits which are preserved by all $F_i : \mathscr{C} \to \mathscr{C}$ for each $i \in \mathbb{N}$, which per. assumption was pullbacks. This follows from the intermediate inductive steps required for the proof of part 2 of corollary 4.1.14. □

Now we have described all of the results concerning coalgebra of endofunctors and their fixed points needed to give the equivalences advertised many times. We now proceed to define the first ∞-category of cyclotomic spectra.

## 4.2 Naive Cyclotomic Spectra and Topological Cyclic Homology

The ∞-category of naive cyclotomic spectra is going to be given as a lax equalizer of the certain functors. Besides defining this ∞-category we will also define topological cyclic homology of naive cyclotomic spectra, and see that $\mathrm{THH}(R)$ of a $\mathbb{E}_\infty$-ring spectrum has the structure of a cyclotomic spectrum. We shall see that topological cyclic homology of a cyclotomic spectrum $X$ is in fact is computable through an equalizer, and finally we will introduce an analog of the Frobenius map for $\mathbb{E}_\infty$-ring spectra. We follow [38] Section II.1, IV.1 and IV.2.

Consider the subgroup $C_{p^\infty} \subseteq \mathbb{T}$ consisting of $p$-power torsion, a model for this group is $\lim_n C_{p^n}$. Analogous to the sphere group there is an equivalence $C_{p^\infty}/C_p \cong C_{p^\infty}$ given by $c \mapsto c^p$. Note that $-^{tC_p} : \mathsf{Sp}^{BC_{p^\infty}} \to \mathsf{Sp}^{B(C_{p^\infty}/C_p)} \simeq \mathsf{Sp}^{BC_{p^\infty}}$.

CycSp **Definition 4.2.1.**

- The ∞-category of naive cyclotomic spectra is the lax equalizer of the functors $\mathsf{Sp}^{B\mathbb{T}} \to \prod_{p\in\mathbb{P}} \mathsf{Sp}^{B\mathbb{T}}$, which takes the Tate-construction with respect to each prime $p \in \mathbb{P}$ in each component, and the functor taking the identity in each component,

$$\mathsf{CycSp} := \mathrm{LEq}((\mathrm{id})_{p\in\mathbb{P}}, ((-)^{tC_p})_{p\in\mathbb{P}}).$$

- The ∞-category of naive p-cyclotomic spectra is the lax equalizer of the functors $-^{tC_p}$ : $\mathsf{Sp}^{BC_{p^\infty}} \to \mathsf{Sp}^{BC{p^\infty}}$, and the functor taking identity functor,

$$\mathsf{CycSp}_p := \mathrm{LEq}(\mathrm{id}, -^{tC_p}).$$

Hence the cyclotomic spectra are roughly $X \in \mathsf{Sp}^{B\mathbb{T}}$ together with $\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant maps $\varphi_p : X \to X^{tC_p}$ for all $p \in \mathbb{P}$.
The ∞-category of naive (p-)cyclotomic spectra has many desirable properties, again these are inherited from $(\mathsf{Sp}^{BC_{p^\infty}})$ $\mathsf{Sp}^{B\mathbb{T}}$ and its structure as a lax equalizer.

AltDefiCycSp *Remark* 4.2.2. An easy consequence of the definition of $\mathsf{CycSp}$ is that it can be identified with the following pullback

$$\mathsf{CycSp} \cong \mathsf{Sp}^{B\mathbb{T}} \times_{\prod_{p\in\mathbb{P}} \mathsf{Sp}^{BC_{p^\infty}}} \prod_{p\in\mathbb{P}} \mathsf{CycSp}_p.$$

CII1.7 **Proposition 4.2.3.** *The ∞-categories* $\mathsf{CycSp}$ *and* $\mathsf{CycSp}_p$ *are presentable and stable. Furthermore the forgetful functors* $\mathsf{CycSp} \to \mathsf{Sp}$, *and* $\mathsf{CycSp}_p \to \mathsf{Sp}$ *are conservative, exact and cocontinuous.*

*Proof.* We give the proof for $\mathsf{CycSp}$, the one for $\mathsf{CycSp}_p$ is analogous. It follows directly from lemma 4.1.3 using that

- $\mathsf{Sp}^{B\mathbb{T}}$ and $\prod_{p\in\mathbb{P}} \mathsf{Sp}^{B\mathbb{T}}$, are stable, accessible and presentable.
- $(\mathrm{id})_{p\in\mathbb{P}}$ is cocontinuous and $((-)^{tC_p})_{p\in\mathbb{P}}$ is accessible. In particular they are exact.
- The forgetful functor $\mathsf{Sp}^{B\mathbb{T}} \to \mathsf{Sp}$ is cocontinuous and conservative. In particular it is exact.

□

The ∞-category of naive (p-)cyclotomic spectra is also enriched over spectra. This follows directly from lemma 1.3.6.

**Corollary 4.2.4.** *The mapping objects of* $\mathsf{CycSp}$, $\mathrm{Map}_{\mathsf{CycSp}}(X, Y)$, *are mapping spectra, i.e.* $\mathrm{Map}_{\mathsf{CycSp}}(X, Y) \in \mathsf{Sp}$.

Note that the proof also gives the result for $\mathsf{CycSp}_p$.

As usual when one defines a category of certain spectra one wants a canonical version of the sphere spectrum.

**Example 4.2.5.** Consider the sphere spectrum $\mathbb{S}$ equipped with the trivial $\mathbb{T}$-action. Denote $\varphi_p$ by the canonical map given by the composite $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$. Recall that we have equivalences $\mathbb{S}^{h\mathbb{T}} \simeq (\mathbb{S}^{hC_p})^{h(\mathbb{T}/C_p)}$. Hence we have a natural map

$$\mathbb{S} \to \mathbb{S}^{h\mathbb{T}} \simeq (\mathbb{S}^{hC_p})^{h(\mathbb{T}/C_p)} \to (\mathbb{S}^{tC_p})^{h(\mathbb{T}/C_p)}$$

Hence we have lifted the map $\mathbb{S} \to \mathbb{S}^{tC_p}$ to a $\mathbb{T}$-equivariant map $\mathbb{S} \to (\mathbb{S}^{tC_p})^{h(\mathbb{T}/C_p)}$. This defines a cyclotomic spectrum because the $\mathbb{T}$-action on $\mathbb{S}$ is trivial. We will refer to this cyclotomic spectrum as the cyclotomic sphere and denote it by $\mathbb{S}$, or $\mathbb{S}^c$ when confusion may arise.

*Remark* 4.2.6. For every cyclotomic spectrum we get a $p$-cyclotomic spectrum by restriction. In particular we can consider $\mathbb{S}$ as a $p$-cyclotomic spectrum. We will denote this sphere spectrum as $\mathbb{S}^c_p$ if confusion may arise.

Finally we have the tools to define topological cyclic homology, and the related invariants.

**Definition 4.2.7.**

1. Let $(X, (\varphi_p)_{p\in\mathbb{P}})$ be a cyclotomic spectrum. The *topological cyclic homology* $\mathrm{TC}(X)$ is the mapping spectrum $\mathrm{map}_{\mathsf{CycSp}}(\mathbb{S}, X) \in \mathsf{Sp}$. The *topological negative cyclic homology* $\mathrm{TC}^{-}(X)$ is defined as $\mathrm{THH}(X)^{h\mathbb{T}}$. The *topological periodic cyclic homology* $\mathrm{TP}(X)$ is defined as $\mathrm{THH}(X)^{t\mathbb{T}}$.
2. Let $(X, \varphi_p)$ be a p-cyclotomic spectrum. The *p-typical topological cyclic homology* $\mathrm{TC}(X, p)$ is the mapping spectrum $\mathrm{map}_{\mathsf{CycSp}_p}(\mathbb{S}, X) \in \mathsf{Sp}$.

*Remark* 4.2.8. Note that we are considering the Tate construction of the non-finite group $\mathbb{T}$. This is the Tate construction given in proposition 3.1.9, which is Theorem I.4.1 of [38].

We also wish to define topological cyclic homology for $\mathbb{E}_\infty$-ring spectra, $R$. This will turn out to be the $\mathrm{TC}(\mathrm{THH}(R))$. This assumes that $\mathrm{THH}(R)$ has the structure of a cyclotomic spectrum, which we show now using the Tate-diagonal.

FrobeniusTHH **Proposition 4.2.9.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum, then* $\mathrm{THH}(R)$ *admits the structure of a cyclotomic spectrum, and we call the map $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ the Frobenius map.*

*Proof.* $R$ is an $\mathbb{E}_\infty$-ring spectrum, hence the $p$-fold smash product $R \otimes ... \otimes R$ equipped with $C_p$-action given by permutation is an induced object in $\mathsf{CAlg}^{BC_p}$. The functor $\mathsf{CAlg} \to \mathsf{CAlg}^{BC_p}$ given by $R \mapsto R \otimes ... \otimes R$ is left adjoint to the forgetful functor $\mathsf{CAlg}^{BC_p} \to \mathsf{CAlg}$. Let $\epsilon : R \to R \otimes ... \otimes R$ be the unit of this adjunction. Because $R$ is an $\mathbb{E}_\infty$-ring spectrum it has a multiplication map $m : R \otimes ... \otimes R \to R$ determined by $m \circ \epsilon = \mathrm{id}_R$. Using the map $i : R \to THH(R)$ we get an induced map

$$\theta := i \circ m : R \otimes ... \otimes R \to \mathrm{THH}(R).$$

Hence we get a map

$$R \xrightarrow{\Delta_p} (R \otimes ... \otimes R)^{tC_p} \xrightarrow{\theta^{tC_p}} \mathrm{THH}(R)^{tC_p},$$

of $\mathbb{E}_\infty$-ring spectra. By proposition 2.5.11, there exists a unique $\mathbb{T}$-equivariant map

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p},$$

of $\mathbb{E}_\infty$-ring spectra where $\mathrm{THH}(R)^{tC_p}$ has the residual action $\mathbb{T}/C_p \cong \mathbb{T}$-action. This map makes the following diagram of $\mathbb{E}_\infty$-ring spectra commute,

$$\begin{array}{ccc} R & \xrightarrow{i} & \mathrm{THH}(R) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi_p} \\ (R \otimes \dots \otimes R)^{tC_p} & \xrightarrow{\theta^{tC_p}} & \mathrm{THH}(R)^{tC_p}. \end{array}$$

□

**Definition 4.2.10.** Let $R$ be a $\mathbb{E}_\infty$-ring spectrum then topological cyclic homology is defined as $\mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R))$.

The amazing thing about topological cyclic homology is that it in fact is computable, and furthermore the proof is entirely formal

TC-formula0 **Proposition 4.2.11.** *1. Let $(X, (\varphi_p)_{p \in \mathbb{P}})$ be a cyclotomic spectrum. Then there is the following equalizer diagram,*

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \underset{can}{\overset{(\varphi_p^{h\mathbb{T}})_{p\in\mathbb{P}}}{\rightrightarrows}} \prod_{p \in \mathbb{P}} (X^{tC_p})^{h\mathbb{T}},$$

*where the maps are given by*

$$\varphi_p^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}},$$

*and,*

$$can : X^{h\mathbb{T}} \simeq (X^{hC_p})^{h(\mathbb{T}/C_p)} \simeq (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}.$$

*2. Let $(X, \varphi_p)$ be a p-cyclotomic spectrum. Then there is the following equalizer diagram,*

$$\mathrm{TC}(X,p) \longrightarrow X^{h\mathbb{T}} \underset{can}{\overset{\varphi_p^{hC_{p^\infty}}}{\rightrightarrows}} (X^{tC_p})^{hC_{p^\infty}},$$

*with the notation from the first part.*

*Proof.* By the universal property of $\mathsf{Sp}$, there is an equivalence $\mathsf{Fun}^{\mathrm{Ex}}(\mathsf{CycSp}, \mathsf{Sp}) \simeq \mathsf{Fun}^{\mathrm{Lex}}(\mathsf{CycSp}, \mathcal{S})$ by composition with $\Omega^\infty$, it suffices to check the formulas for the mapping space. This follows directly from lemma 4.1.4(2). □

We can improve slightly on the formula from part 1, when $X$ is a bounded below spectrum. We shall need the following technical lemma, which is a consequence of theorem 3.3.4. This is the first time we see the strength of the Tate orbit lemma. Here and in the following we identify $C_{p^n}/C_{p^m} \cong C_{p^{n-m}}$.

technicalresult0 **Lemma 4.2.12.** *Let $X \in \mathsf{Sp}^{BC_{p^n}}$ be a spectrum with $C_{p^n}$-action that is bounded below. Then the canonical morphism $X^{tC_{p^n}} \to (X^{tC_p})^{hC_{p^{n-1}}}$ is an equivalence.*

*Proof.* Consider $X_{hC_{p^{n-2}}}$ as a spectrum with $C_{p^2}$-action. Furthermore note that the norm map

$$(X_{hC_{p^{n-2}}})_{hC_{p^2}} \simeq X_{hC_{p^n}} \to (X_{hC_{p^{n-2}}})^{h(C_{p^2}/C_p)},$$

is an equivalence by theorem 3.3.4. By induction, we deduce that the norm map $X_{hC_{p^n}} \to (X_{hC_p})^{hC_{p^{n-1}}}$ is an equivalence. The norm map fits into a diagram

$$\begin{array}{ccccc} X_{hC_{p^n}} & \xrightarrow{\mathrm{Nm}_{C_{p^n}}} & X^{hC_{p^n}} & \longrightarrow & X^{tC_{p^n}} \\ \downarrow & & \downarrow & & \downarrow \\ (X_{hC_p})^{hC_{p^{n-1}}} & \xrightarrow{(\mathrm{Nm}_{C_p})^{hC_{p^{n-1}}}} & (X^{hC_p})^{hC_{p^{n-1}}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}} \end{array}$$

which commutes, because all the maps, besides the middle vertical map, in the left square are norm maps, and the right square are given through cofibers. Note that the two left most vertical maps are equivalences, hence induce an equivalence on cofibers. □

LI4.4 *Remark* 4.2.13. The natural map $(H\mathbb{Z})^{t\mathbb{T}} \to (H\mathbb{Z})^{tC_n}$, lifts to a map $(HM)^{t\mathbb{T}} \to (HM)^{tC_n}$ for $M$ torsion free.

The improved formula for $\mathrm{TC}(X)$ for a bounded below spectrum $X$ is a corollary of the following result.

TC-formula-lem **Proposition 4.2.14.** *If $X$ is a bounded below spectrum with a $\mathbb{T}$-action then $(X^{tC_p})^{h\mathbb{T}}$ is p-complete and the canonical morphism $X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$ exhibits $(X^{tC_p})^{h\mathbb{T}}$ as the p-completion of $X^{t\mathbb{T}}$.*

*Proof.* Consider $(X^{tC_p})^{h\mathbb{T}}$ and $(X^{tC_p})^{hC_{p^\infty}}$, these are both $p$-complete by proposition 3.2.7, hence the canonical morphism $(X^{tC_p})^{h\mathbb{T}} \to (X^{tC_p})^{hC_{p^\infty}}$ is an equivalence, because $C_{p^\infty}$ and $\mathbb{T}$ are $p$-adically equivalent. Note that because $C_{p^\infty}$ can be seen as a limit, we also have the following equivalence $(X^{tC_p})^{hC_{p^\infty}} \simeq \lim_n (X^{tC_p})^{hC_{p^n}}$.

Combining these equivalences we get that the following diagram

$$\begin{array}{ccc} X^{t\mathbb{T}} & \longrightarrow & (X^{tC_p})^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \lim_n (X^{tC_{p^n}}) & \longrightarrow & \lim_n (X^{tC_p})^{hC_{p^n}} \end{array}$$

commutes and by lemma 4.2.12 every object is equivalent, except for $X^{t\mathbb{T}}$. The left vertical map is defined via the maps from 4.2.13. We show that the left vertical map is a $p$-completion. Using lemma 3.1.15 we may assume $X$ to be bounded. Using the strategy from the proof of lemma 3.3.3 we may assume that $X = HM$ is an Eilenberg-MacLane spectrum, and that $M$ is torsion free. Note that the $\mathbb{T}$-action on $M$ is trivial. By inspection of the Tate spectral sequence we see that $\pi_*(HM^{t\mathbb{T}})$ agrees with the cohomology ring $H^{-*}(\mathbb{T}, M)$, i.e.

$$\pi_*(HM^{t\mathbb{T}}) \simeq M[u]/(u^2).$$

Again inspecting the Tate spectral sequence and using standard results about the group cohomology of $G$ a finite cyclic group with coefficients in a trivial $G$-module, we have

$$\pi_*(HM^{tC_{p^n}}) \simeq (M/p^n M)[u]/(u^2).$$

The maps in the limit diagram granting $\lim_n (X^{tC_{p^n}})$ are given by the projections $M \to M/p^n M$ coming from remark 4.2.13. □

Hence we obtain the following simplification of the formula for $\mathrm{TC}(X)$ for $X$ bounded below.

TC-formula **Corollary 4.2.15.** *Let $(X, (\varphi_p)_{p\in\mathbb{P}})$ be a cyclotomic spectrum where $X$ is bounded below. Then there is the following equalizer diagram,*

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \underset{can}{\overset{\varphi}{\rightrightarrows}} (X^{t\mathbb{T}})^\wedge_p,$$

*where can is as in proposition 4.2.11, and $\varphi$ is induced from $(\varphi^{h\mathbb{T}_p})_{p\in\mathbb{P}}$ and the equivalence of proposition 4.2.14. Note that if $R$ is a bounded below $\mathbb{E}_\infty$-ring spectrum, then this formula is especially aesthetically pleasing,*

$$\mathrm{TC}(R) \longrightarrow \mathrm{TC}^-(R) \underset{can}{\overset{\varphi}{\rightrightarrows}} (\mathrm{TP}(R))^\wedge_p, .$$

## 4.3 Genuine Cyclotomic Spectra and Borel Completion

In the following section we will compare the notion of topological cyclic homology which we have introduced above with the classical version. The classical version is for *genuine cyclotomic spectra*, which we will define in this section. The main theorem concerning these, is that if $X$ is a genuine cyclotomic spectrum whose underlying spectrum is bounded below, then the new and classical notions of topological cyclic homology agree. Our treatment of genuine cyclotomic spectra will be superficial. We follow [38] Section II.2 and II.3, and large portions of [44].

Recall that an orthogonal spectrum is a sequence of pointed topological spaces $X_n$ for $n \geq 0$, with pointed continuous actions of the orthogonal group in each level, and continuous structure maps $\sigma^m : X_n \wedge S^m \to X_{n+m}$. These structure maps are required to be $O(n) \times O(m)$-equivariant. We denote the associated category $\mathsf{Sp}^O$. The category $\mathsf{Sp}^O$ has a symmetric monoidal structure. We denote the monoidal product by $\wedge$, following [44]. Just like symmetric spectra, they are a model for the $\infty$-category of spectra $\mathsf{Sp}$, when equipped with the projective stable model structure. We will indicate what the cofibrations and stable equivalences are in the following.

OG-spec **Definition 4.3.1.** Let $G$ be a group. An *orthogonal $G$-spectrum* is an object in the functor category $G\mathsf{Sp}^O := \mathsf{Fun}(BG, \mathsf{Sp}^O)$.

The category $G\mathsf{Sp}^O$ inherits a symmetric monoidal structure from $\mathsf{Sp}^O$, where the smash product is the underlying one for $\mathsf{Sp}^O$ equipped with diagonal $G$-action, this we will also denote by $\wedge$.

To give the definition of genuine cyclotomic spectra we will need a certain functor $\Phi^G : G\mathsf{Sp}^O \to \mathsf{Sp}^O$. We will need a few auxillary constructions to give its definition. We follow [44] closely.

**Definition 4.3.2.** Let $X$ be an orthogonal $G$-spectrum. Let $V$ be an $n$-dimensional real vector space equipped with a scalar product. Let $\mathbb{L}(\mathbb{R}^n, V)$ be the space of linear isometries from $\mathbb{R}^n$ to $V$. The orthogonal group $O(n)$ acts on $\mathbb{L}(\mathbb{R}^n, V)$ by precomposition. We define the *value of $X$ on $V$*,

$$X(V) := \mathbb{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

i.e. as the coequalizer of the $O(n)$-action above,the diagonal action on $\mathbb{L}(\mathbb{R}^n, V)_+ \wedge X_n$ and $G$ act diagonally.

**Definition 4.3.3.** Let $X$ be an orthogonal $G$-spectrum. Let $\Phi^G X$, called the *geometric fixed points of $X$*, be the orthogonal spectrum whose $n$-th level is given by

$$(\Phi^G X)_n = X(\mathbb{R}^n \otimes \mathbb{R}[G])^G.$$

PhiSymMon *Remark* 4.3.4. The functor $\Phi^G : G\mathsf{Sp}^O \to \mathsf{Sp}^O$ has a natural lax symmetric monoidal structure, and when $X$ and $Y$ are cofibrant orthogonal $G$-spectra the map $\Phi^G(X) \wedge \Phi^G(Y) \to \Phi^G(X \wedge Y)$ is an equivalence. The cofibrations are those of the projective stable model structure due to M.A. Mandell and J.P. May [26], see theorem 4.2. The above equivalence is proposition 7.14 of [44].

**Definition 4.3.5.** Let $f : X \to Y$ be a map of orthogonal $G$-spectra. Then $f$ is an equivalence if for all subgroups $H \subseteq G$, the map $\Phi^H(X) \to \Phi^H(Y)$ is a stable equivalence of orthogonal spectra. We denote the collection of equivalences of orthogonal $G$-spectra by $W$. Here stable equivalences refers to $\overline{\pi_*}$-isomorphisms, i.e. morphisms which on homotopy groups induce isomorphisms.

It is straightforward to obtain the corresponding $\infty$-category.

GSpDefi **Definition 4.3.6.** Let $G$ be a finite group. The $\infty$-category of *genuine $G$-equivariant spectra* is the $\infty$-category $G\mathsf{Sp} \coloneqq N(G\mathsf{Sp}^O)[W^{-1}]$, i.e. the nerve of $G\mathsf{Sp}^O$ with the equivalences of orthogonal $G$-spectra inverted. For a more detailed describtion see construction 3.20 of [44]. Let $H \subseteq G$ be a subgroup, then the *geometric fixpoint functor* $\Phi^H : G\mathsf{Sp} \to \mathsf{Sp}$ is obtained from $\Phi^H : G\mathsf{Sp}^O \to \mathsf{Sp}^O$ by restricting to cofibrant orthogonal $G$-spectra and inverting equivalences of orthogonal $G$-spectra.

It can be shown that $\Phi^H : G\mathsf{Sp} \to \mathsf{Sp}$ has the same properties as its 1-categorical counterpart.

GSpPresent *Remark* 4.3.7. Equipping $\mathsf{Sp}^O$ with the stable model structure it becomes a combinatorial simplicial model category. The category $G\mathsf{Sp}^O$ inherits a simplicial model structure, this follows from proposition A.3.3.2 of [22]. Therefore the $\infty$-category $N(G\mathsf{Sp}^O)$ is presentable by proposition 1.1.9. From this and Duggers theorem, e.g. proposition 5.5.4.15 of [22], the localization $G\mathsf{Sp}$ is presentable.

**Corollary 4.3.8.** *The $\infty$-category $G\mathsf{Sp}$ has a symmetric monoidal structure inherited from the symmetric monoidal structure on $G\mathsf{Sp}^O$.*

*Proof.* Let $p : N(G\mathsf{Sp}^O)^\otimes \to N(\Gamma)$ be the symmetric monoidal $\infty$-category induced from the symmetric monoidal structure on $G\mathsf{Sp}^O$ via theorem 2.1.18. Let $L : N(G\mathsf{Sp}^O) \to N(G\mathsf{Sp}^O)$ be the localization associated to $G\mathsf{Sp}$ described above. Let $\{X_i \to Y_i\}_{i\in I}$ be morphisms in $N(G\mathsf{Sp}^O)$ such that the induced morphisms $\{L(X_i) \to L(Y_i)\}_{i\in I}$ are equivalences, then the map $L(\bigwedge_{i\in I} X_i) \to L(\bigwedge_{i\in I} Y_i)$ is an equivalence too. Let $L(N(G\mathsf{Sp}^O))^\otimes$ be as described in construction 2.1.19.Then it follows from proposition 2.2.1.9 of [25] that the restriction $p : L(N(G\mathsf{Sp}^O))^\otimes \to N(\Gamma)$ exhibits $L(N(G\mathsf{Sp}^O))^\otimes$ as a symmetric monoidal $\infty$-category with underlying $\infty$-category $L(N(G\mathsf{Sp}^O))$ which exactly is $N(G\mathsf{Sp}^O)[W^{-1}]$. □

ApproxOmega *Remark* 4.3.9. The localization described in definition 4.3.6 lets us assert that every orthogonal $G$-spectrum may be approximated, up to stable equivalence, by a $G$-$\Omega$-spectrum (for a definition see 3.18 of [44]). Because of this it will suffice to restrict a functor $G : G\mathsf{Sp}^O \to G\mathsf{Sp}^O$ to $G$-$\Omega$-spectra and then inverting stable equivalence to obtain a functor of $\infty$-categories $L : G\mathsf{Sp} \to G\mathsf{Sp}$.

We shall need another notion of fixed points.

**Definition 4.3.10.** Consider the fixed point functor of orthogonal $G$-spectra given by the $G$-fixed points of the $n$-th level, with restricted $O(n)$-action,

$$X_n \mapsto X_n^G.$$

Because the structure maps of $X$ are $G$-equivariant for the trivial $G$-action on $S^1$, they restrict to structure maps

$$\sigma_n^G : X_n^G \wedge S^1 = (X_n \wedge S^1) \longrightarrow X_{n+1}^G.$$

The fixed point functor does not preserve stable equivalences in general, but it does on $G$-$\Omega$-spectra. Hence taking the derived version of this functor we obtain a functor for all subgroups $H \subseteq G$

$$-^H : G\mathsf{Sp} \to \mathsf{Sp}.$$

We will call this functor the *genuine fixed point functor*.

We will now define the residual $G/H$-action for $X^H$.

ResAction **Definition 4.3.11.** Restricted to $G$-$\Omega$-spectra, the functor $G\mathsf{Sp}^O \to \mathsf{Sp}^O$ is given levelwise by

$$X_n \mapsto X_n^H,$$

i.e. through levelwise pointset $H$-fixpoints, which has an obvious $G/H$-action. Therefore the functor $G\mathsf{Sp}^O \to \mathsf{Sp}^O$ maps orthogonal $G$-$\Omega$-spectra to orthogonal $G/H$-$\Omega$-spectra. Furthermore it is compatible with composition $(X^H)^{G/H} = X^G$, because this is true for pointset fixpoints. In particular it preserve equivalences, and therefore we obtain a lax symmetric monoidal functor $G\mathsf{Sp} \to (G/H)\mathsf{Sp}$.

FixNatTrans *Remark* 4.3.12. Because the trivial representation embeds into $\mathbb{R}[G]$, there is a natural transformation $-^H \to \Phi^H$ of functors $G\mathsf{Sp} \to \mathsf{Sp}$ for all subgroups $H \subseteq G$. Furthermore both of these functors and the transformation are lax symmetric monoidal, see [44] proposition 7.13 and 7.14.

Note that there is an immediate problem with the definition of $G\mathsf{Sp}$, namely that $G$ is a finite group. This is a problem because we want to consider $\mathbb{T}$-equivariant objects and $C_{p^\infty}$-equivariant objects. This is remedied differently for each of the groups.

**Definition 4.3.13.**

1. The $\infty$-category of *genuine $C_{p^\infty}$-equivariant spectra* is the limit of the $\infty$-categories $C_{p^n}\mathsf{Sp}$ for varying $n$, along the forgetful functors $C_{p^n}\mathsf{Sp} \to C_{p^{n-1}}\mathsf{Sp}$.
2. Consider the category $\mathbb{T}\mathsf{Sp}^O$. Let $\mathcal{F}$ be the set of finite subgroups $C_n \subseteq \mathbb{T}$. A map $f : X \to Y$ in $\mathbb{T}\mathsf{Sp}^O$ is an $\mathcal{F}$-equivalence if the underlying map in $C_n\mathsf{Sp}^O$, induced from the projection to the $n$'th roots of unit $\mathbb{T} \to C_n$, is an equivalence for all finite subgroups $C_n \subseteq \mathbb{T}$. The $\infty$-category $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ of *$\mathcal{F}$-genuine $\mathbb{T}$-equivariant spectra* is obtained from inverting $\mathcal{F}$-equivalences in $N(\mathbb{T}\mathsf{Sp}^O)$.

Before we proceed with defining genuine ($p$-)cyclotomic spectra, we shall need a few technical results, which we shall take for granted.

N-action *Remark* 4.3.14. By inverting $\mathcal{F}$-equivalences, we obtain an action of the monoid $\prod_{i=1}^{\infty} \mathbb{N}$ on the $\infty$-category $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$. As a consequence hereof $\Phi^{C_p}$ and $\Phi^{C_{p'}}$ for distinct $p, p' \in \mathbb{P}$ commute up to coherent equivalence See [38] section II.3 for a the details hereof.

PhiEndo *Remark* 4.3.15. We shall take the definition of a residual $G/H$-action, for $H$ normal in $G$, on $\Phi^H(X)$ for granted, and refer the reader to section II.2 of [38]. The geometric fixpoint functor make sense for all finite subgroups of $C_{p^\infty}$ and $\mathbb{T}$, hence lifts to a functor $C_{p^\infty}\mathsf{Sp} \to (C_{p^\infty}/C_p)\mathsf{Sp} \simeq C_{p^\infty}\mathsf{Sp}$ and $\mathbb{T}\mathsf{Sp}_{\mathcal{F}} \to (\mathbb{T}/C_p)\mathsf{Sp}_{\mathcal{F}} \simeq \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$, via the $p$-th power map.

We shall take the following result for granted, it is proposition II.2.14 of [38]. It relates the geometric fixed points functor and the genuine fixed point functor.

PhiRight **Lemma 4.3.16.** *Let $G$ be a finite group, and $H \subseteq G$ be a normal subgroup. The functor $\Phi^H : G\mathsf{Sp} \to (G/H)\mathsf{Sp}$ has a fully faithful right adjoint $R_H : (G/H)\mathsf{Sp} \to G\mathsf{Sp}$. If $H \subseteq G$ and $H \notin N$, then $R_H(X)^N \simeq 0$ for all $X \in (G/H)\mathsf{Sp}$. The natural transformation $-^H \to \Phi^H$ of remark 4.3.12 is an equivalence on the image of $R_H$.*

*Remark* 4.3.17. By passing to the limit and using remark 4.3.15 we obtain a variant of lemma 4.3.16 for finite subgroups of $C_{p^\infty}$. There is also a variant for finite subgroups of $\mathbb{T}$, this case requires more work, i.e. tweaking the proof of the lemma 4.3.16. We shall take it for granted.

There is in fact a third (fourth counting the one introduced for $G$-equivariant spectra in chaper 3) type of fixed point functor.

InducesBorel **Definition 4.3.18.** Note that any equivalence of orthogonal $G$-spectra is an equivalence of the underlying orthogonal spectra, because $\Phi^{\{e\}}(X) \to \Phi^{\{e\}}(Y)$ in particular is an equivalence of the underlying spectra. Hence we obtain a functor $G\mathsf{Sp} \to \mathsf{Sp}^{BG}$, it can be shown that it is lax symmetric monoidal. Because of this there is yet another fixed point functor for genuine $G$-equivariant spectra, namely homotopy fixed points

$$-^{hH} : G\mathsf{Sp} \to \mathsf{Sp}^{BG} \to \mathsf{Sp},$$

which is post-composition with the forgetful functor.

**Definition 4.3.19.**

1. A *genuine $p$-cyclotomic spectrum* is an object $X \in C_{p^\infty}\mathsf{Sp}$ together with an equivalence $\Phi^{C_p}X \to X$ in $C_{p^\infty}\mathsf{Sp}$. The $\infty$-category of genuine $p$-cyclotomic spectra is the equalizer

$$\mathsf{CycSp}_p^{gen} \longrightarrow C_{p^\infty}\mathsf{Sp} \underset{\mathrm{id}}{\overset{\Phi^{C_p}}{\rightrightarrows}} C_{p^\infty}\mathsf{Sp}.$$

2. The $\infty$-category of *genuine cyclotomic spectra* is given as

$$\mathsf{CycSp}^{gen} = (\mathbb{T}\mathsf{Sp}_{\mathcal{F}})^{h(\prod_{i=1}^{\infty}\mathbb{N})}.$$

Here we take homotopy fixed points with respect to the action given in remark 4.3.14.

Hence genuine cyclotomic spectra are roughly speaking $X \in \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ together with homotopy coherently commutative equivalences $\Phi^n : \Phi^{C_n} X \to X$ for $n \in \mathbb{N}$.

We will need the following result, for a proof see proposition 2.1 of [17], which we will take for granted.

technicalresult2 **Proposition 4.3.20.** *Let $G$ be a cyclic group of $p$-power order. For $X \in G\mathsf{Sp}$ there is a natural fiber sequence*

$$X_{hG} \to X^G \to (\Phi^{C_p} X)^{G/C_p},$$

*of spectra where the second map is induced from the natural transformation $-^{C_p} \to \Phi^{C_p}$ of lax symmetric monoidal functors $G\mathsf{Sp} \to (G/C_p)\mathsf{Sp}$.*

Let $X$ be a genuine $p$-cyclotomic spectrum, then $X$ has $C_{p^n}$-fixpoints $X^{C_{p^n}}$ for all $n \geq 0$, and there are inclusions $F : X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ for all $n \geq 0$. Furthermore there is the following map

$$R : X^{C_{p^n}} \to (\Phi^{C_p} X)^{C_{p^{n-1}}} \simeq X^{C_{p^{n-1}}}.$$

which is the composite of the second map of the fiber sequence of proposition 4.3.20, and the structure equivalence $\Phi^{C_p} X \to X$. Both of these maps are part of the definition of $p$-typical topological cyclic homology for genuine $p$-cyclotomic spectra.

GenTC **Definition 4.3.21.** 1. Let $X$ be a genuine $p$-cyclotomic spectrum. We define *$p$-typical topological cyclic homology for genuine $p$-cyclotomic spectra* as

$$\mathrm{TC}^{gen}(X,p) := \lim_R \mathrm{Eq}\left( X^{C_{p^n}} \underset{F}{\overset{R}{\rightrightarrows}} X^{C_{p^{n-1}}} \right).$$

2. Let $X$ be a genuine cyclotomic spectrum, then we define *topological cyclic homology for genuine cyclotomic spectra* as

$$\begin{array}{ccc} \mathrm{TC}^{gen}(X) & \longrightarrow & X^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}^{gen}(X,p)_p^\wedge & \longrightarrow & \prod_{p\in\mathbb{P}} (X_p^\wedge)^{h\mathbb{T}}. \end{array}$$

Analogous to the naive case we have the following lemma, this is shown section 2 of [10].

**Lemma 4.3.22.** *Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. Then* $\mathrm{THH}(R)$ *has the structure of a genuine cyclotomic spectrum, i.e. we have an equivalence* $\Phi^{C_p}\mathrm{THH}(R) \simeq \mathrm{THH}(R)$.

Again analogously we have the following definition.

**Definition 4.3.23.** Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. Then we define $\mathrm{TC}^{gen}(R) := \mathrm{TC}^{gen}(\mathrm{THH}(R))$.

Before we begin to compare $\mathrm{TC}^{gen}$ with TC we shall need a certain functor called *Borel completion* and certain properties of it. The most important property is that on its image the fixed point functor and the homotopy fixed functor agrees. This is in some sense one of the most crucial lemmas in the comparison of $\mathrm{TC}^{gen}$ and TC. The Borel completion functor is induced from the functor introduced in definition 4.3.18. Both proposition 5.2.7.4 of [22] and lemma 1.2.7 will be central in the proof.

Borel **Proposition 4.3.24.** *The functor $G\mathsf{Sp} \to \mathsf{Sp}^{BG}$ admits a fully faithful right adjoint $B_G$ : $\mathsf{Sp}^{BG} \to G\mathsf{Sp}$, which we call Borel completion. It induces a homotopy fixed point functor $-^{hH} : G\mathsf{Sp} \to \mathsf{Sp}^{BG} \to \mathsf{Sp}$, for $H \subseteq G$ a subgroup. There is a natural transformation $-^{H} \to -^{hH}$ of lax symmetric monoidal functors. The full subcategory of spectra for which this is an equivalence is the essential image of $B_G$. We call these spectra Borel complete.*

*Proof.* Consider the orthogonal $G$-spectrum $\mathrm{Map}(X,Y)$ for $Y \in G\mathsf{Sp}^O$ and $X \in \mathsf{Fun}(G, \mathsf{Top})$, defined levelwise as

$$\mathrm{Map}(X,Y)_n = \mathrm{Map}(X, Y_n).$$

The $O(n)$-action is given through the action of $X_n$. The structure map is given by the composite

$$\mathrm{Map}(X,Y)_n \wedge S^1 \longrightarrow \mathrm{Map}(X, Y \wedge S^1)_n \xrightarrow{\mathrm{Map}(X,\sigma_n)} \mathrm{Map}(X,Y)_{n+1},$$

where the first map is given by sending $\phi \wedge t \in \mathrm{Map}(X,Y)_n \wedge S^1$ to the map $x \mapsto \phi(x) \wedge t$ for $x \in X$. The $G$-action on $\mathrm{Map}(X,Y)_n = \mathrm{Map}(X,Y_n)$ is given by $\phi_g(x) = g \cdot \phi(g^{-1}x)$. The space $\mathrm{Map}(X,Y)_n$ is pointed at the map sending $X$ to the basepoint of $Y_n$. The functors $\mathrm{Map}(X,-)$ and the functor $X \wedge -$, given in an analogous fashion to the above, are an adjoint pair on the level of orthogonal $G$-spectra

$$\mathrm{Map}(Z, \mathrm{Map}(X,Y)) \simeq \mathrm{Map}(X \wedge Z, Y).$$

Fix a contractible space $EG \in \mathsf{Top}$ with a free $G$-action. Consider the functor of 1-categories $L : G\mathsf{Sp}^O \to G\mathsf{Sp}^O$ given by

$$X_n \mapsto \mathbb{R}\,\mathrm{Map}(EG, X)_n.$$

Here $\mathbb{R}\,\mathrm{Map}(-,-)$ denotes the right derived mapping space. It can be shown that the functor $L$ preserves $G$-$\Omega$-spectra, see example 5.2 of [44]. Furthermore by proposition 5.4 of [44] $\mathbb{R}\,\mathrm{Map}(EG,-)$ sends equivalences to $G$-equivariant equivalences.

Now recall the usual definition of the homotopy fixed points of orthogonal $G$-spectra,

$$X^{hH} = \mathbb{R}\,\mathrm{Map}(EG, X)^H \simeq L(X)^H$$

Where $EG \in \mathsf{Top}_+$ is a contractible space equipped with a free $G$-action. Furthermore we have

BorelEq (4.2) $$L(X)^{hH} \simeq X^{hH}$$

because indeed,

$$\begin{aligned} L(X)^{hH} &= \mathbb{R}\,\mathrm{Map}(EG, X)^{hH} \\ &= \mathbb{R}\,\mathrm{Map}(EH, \mathbb{R}\,\mathrm{Map}(EG, X))^H \\ &\simeq \mathbb{R}\,\mathrm{Map}(EH \wedge EG, X)^H \\ &\simeq \mathbb{R}\,\mathrm{Map}(EG, X)^H \\ &= X^{hH}. \end{aligned}$$

By theorem 7.12 [44], $f : X \to Y$ is a stable equivalence of orthogonal $G$-spectra, if and only if for every subgroup $H \subseteq G$ the map $f^{hH} : X^{hH} \to Y^{hH}$ is a stable equivalence of orthogonal spectra. Hence $f : X \xrightarrow{\simeq} Y$ if and only if $f^{hH} : X^{hH} \xrightarrow{\simeq} Y^{hH}$ for all subgroups $H \subseteq G$ if and only if $L(f)^{hH} : L(X)^{hH} \xrightarrow{\simeq} L(Y)^{hH}$ for all subgroups $H \subseteq G$ if and only if $L(f) : L(X) \xrightarrow{\simeq} L(Y)$. It also follows that $L(X)^H \simeq L(X)^{hH}$ for all subgroups $H \subseteq G$, i.e. $L(X)$ is Borel complete. There is a natural transformation $L : \mathrm{id} \to T$ of functors $G\mathsf{Sp}^O \to G\mathsf{Sp}^O$, because there are natural maps compatible with the structure maps,

$$\begin{aligned} X_n &\to \mathbb{R}\,\mathrm{Map}(EG, X)_n, \\ x &\mapsto (c_x : e \mapsto x). \end{aligned}$$

Now via remark 4.3.9 we obtain a functor $L : G\mathsf{Sp} \to G\mathsf{Sp}$ of $\infty$-categories, by restricting $L$ to orthogonal $G$-$\Omega$-spectra, and inverting the stable equivalences. Therefore we get a natural transformation $\mathrm{id} \to L$ which satisfies (3) of proposition 5.2.7.4 of [22], this gives that the image of $L$ can be characterized as the full subcategory $G\mathsf{Sp}_B \subseteq G\mathsf{Sp}$ spanned by Borel-complete spectra. Because of this, it only remains to prove that the functor $G\mathsf{Sp}_B \to \mathsf{Sp}^{BG}$ is an equivalence of $\infty$-categories. Note that the functor $L : G\mathsf{Sp}^O \to G\mathsf{Sp}^O$ induces a functor $N(G\mathsf{Sp}^O) \to G\mathsf{Sp}_B$. The functor $L$ inverts all morphisms which are equivalences of the underlying spectra, by the argument given in definition 4.3.18. It follows from proposition 1.3.4.25 of [25] and theorem 9.2 of [30], that by inverting these in $N(G\mathsf{Sp}^O)$ we obtain $\mathsf{Sp}^{BG}$. Therefore we obtain a natural functor $B_G : \mathsf{Sp}^{BG} \to G\mathsf{Sp}_B$. By construction of $L$ and proposition 5.2.7.4 of [22] the functor described in 4.3.18 has a fully faithful right adjoint, which is $B_G : \mathsf{Sp}^{BG} \to G\mathsf{Sp}$, such that $G\mathsf{Sp} \to \mathsf{Sp}^{BG}$ factors over $L$. By lemma 1.2.7, the $\infty$-categories $G\mathsf{Sp}_B$ and $\mathsf{Sp}^{BG}$ are equivalent. □

BorelLax **Corollary 4.3.25.** *There is a lax symmetric monoidal structure on Borel completion* $B_G : \mathsf{Sp}^{BG} \to G\mathsf{Sp}$*, and a natural refinement of the adjunction map* $\mathrm{id} \to B_G$ *of endofunctors on* $G\mathsf{Sp}$ *to a lax symmetric monoidal transformation.*

*Proof.* This follows from example 2.1.13 because its left adjoint $G\mathsf{Sp} \to \mathsf{Sp}^{BG}$ is symmetric monoidal, which we stated in definition 4.3.18. □

## 4.4 Equivalence of TC

In this section we will show that if $X$ is a genuine cyclotomic spectrum whose underlying spectrum is bounded below, then the new and classical notions of topological cyclic homology agree. We follow II.4 of [38]. We begin by showing that the formula for the $p$-typical topological cyclic homology given in definition 4.3.21 agrees with the one given in proposition 4.2.11.

We shall need a version of homotopy orbits for genuine $G$-equivariant spectra. Let $R : \mathsf{Sp} \to \mathsf{Sp}^{BG}$ be the functor which equips a spectrum with the trivial $G$-action. This functor preserves limits and colimits.

**Definition 4.4.1.** We define the *genuine homotopy orbits functor* for genuine $G$-equivariant spectra $-_{hG} : G\mathsf{Sp} \to \mathsf{Sp}$ as the composite functor

$$G\mathsf{Sp} \to \mathsf{Sp}^{BG} \to \mathsf{Sp},$$

where the first functor is the one described in definition 4.3.18, and the second is the left adjoint of $R : \mathsf{Sp} \to \mathsf{Sp}^{BG}$ afforded by the adjoint functor theorem.

The following is lemma 13 of [34], we will refrain from proving it.

**Lemma 4.4.2.** *Let* $f : X \to Y$ *be a morphism in* $G\mathsf{Sp}$ *such that the underlying map in* $\mathsf{Sp}$ *is an equivalence. Then* $f_{hG} : X_{hG} \to Y_{hG}$ *is an equivalence.*

The following example will use the notation established in proposition 4.3.24.

BGEx **Example 4.4.3.** Consider $X \in G\mathsf{Sp}$, then there is a natural map $b : X \to B_G(X)$. We denote the (co)fibrant object of $G\mathsf{Sp}^O$ which represents $X$ by $X'$. Then by construction of $B_G$ we have $B_G(X) \simeq L(X) = \mathrm{Map}(EG, X)$ as described in the proof of proposition 4.3.24. Therefore the map $\overline{b} : X' \to \mathrm{Map}(EG, X')$ represents $b$ on the level of orthogonal $G$-spectra. The map $\overline{b}$ is an equivalence of underlying orthogonal spectra, because non-equivariantly we have $EG \simeq \bullet$, hence $\mathrm{Map}(EG, X') \simeq \mathrm{Map}(\bullet, X') \simeq X'$. Hence the underlying map of $b$ in $\mathsf{Sp}$ is an equivalence. Therefore $b_{hG} : X_{hG} \to B_G(X)_{hG}$ is an equivalence by the above lemma.

The following lemma is central, because it lets us relate the different kinds of fixed points, and through its proof also Borel completion. It is a consequence of proposition 4.3.20 and lemma 4.3.24.

bigdialem **Lemma 4.4.4.** *Let $X$ be a genuine $C_{p^n}$-equivariant spectrum. Assume that the underlying spectrum is bounded below. Then there is a natural pullback diagram of spectra for $n \geq 1$,*

$$\begin{array}{ccc} X^{C_{p^n}} & \xrightarrow{R} & (\Phi^{C_p} X)^{C_{p^{n-1}}} \\ \big\downarrow & & \big\downarrow \\ X^{hC_{p^n}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}}. \end{array}$$

*Proof.* Consider the map $X \to B_{C_{p^n}}(X)$ defined in proposition 4.3.24. From this map and proposition 4.3.20 we obtain a commutative diagram,

$$\begin{array}{ccccc} X_{hC_{p^n}} & \xrightarrow{N} & X^{C_{p^n}} & \longrightarrow & (\Phi^{C_p} X)^{C_{p^n}/C_p} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ B_{C_{p^n}}(X)_{hC_{p^n}} & \longrightarrow & B_{C_{p^n}}(X)^{C_{p^n}} & \longrightarrow & (\Phi^{C_p} B_{C_{p^n}}(X))^{C_{p^n}/C_p} \end{array}$$

By definition 4.3.11 we have

$$(\Phi^{C_p} X)^{C_{p^n}/C_p} \simeq (\Phi^{C_p} X)^{C_{p^{n-1}}}$$

from which it is evident that the map $X^{C_{p^n}} \to (\Phi^{C_p} X)^{C_{p^n}/C_p}$ is $R$. By proposition 4.3.24 we have

$$(B_{C_{p^n}}(X))^{C_{p^n}} \simeq (B_{C_{p^n}}(X))^{hC_{p^n}} \simeq (L(X))^{hC_{p^n}} \simeq X^{hC_{p^n}}.$$

The first equivalence is because the objects in the image of $B_{C_{p^n}}$ are Borel complete, the second equivalence is because the image of the localization $L$ is equivalent to the full subcategory of $C_{p^n}\mathsf{Sp}$ spanned by the Borel complete genuine $C_{p^n}$-spectra, and the last is (4.2).
By example 4.4.3 we have $B_{C_{p^n}}(X)_{hC_{p^n}} \simeq X_{hC_{p^n}}$. Now it follows from proposition 3.1.9 that the norm map $N$ descends to the norm map $\mathrm{Nm}_{C_{p^n}}$ described in definition 3.1.6 under $B_{C_{p^n}}$, therefore we have the following equivalence

$$(\Phi^{C_p} B_{C_{p^n}} X)^{C_{p^{n-1}}} \simeq X^{tC_{p^n}}.$$

By lemma 4.2.12, we have

$$X^{tC_{p^n}} \simeq (X^{tC_p})^{hC_{p^{n-1}}}.$$

Combining all of the assertions above we obtain the following commutative diagram

$$\begin{array}{ccccc} X_{hC_{p^n}} & \xrightarrow{N} & X^{C_{p^n}} & \xrightarrow{R} & (\Phi^{C_p} X)^{C_{p^{n-1}}} \\ \big\downarrow \simeq & & \big\downarrow & & \big\downarrow \\ X_{hC_{p^n}} & \xrightarrow{\mathrm{Nm}_{C_{p^n}}} & X^{hC_{p^n}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}}. \end{array}$$

That the right hand square is a pullback diagram follows from both of the horizontal rows being fiber sequences. □

*Remark* 4.4.5. Note that we asserted that $-^{tC_{p^n}}$ could be written as the composite

$$\mathsf{Sp}^{BC_{p^n}} \xrightarrow{B_{C_{p^n}}} C_{p^n}\mathsf{Sp} \xrightarrow{(\Phi^{C_p})^{C_{p^n}/C_p}} \mathsf{Sp},$$

of symmetric monoidal functor, i.e. the underlying spectrum of $(\Phi^{C_p} B_{C_{p^n}}(X))^{C_{p^n}/C_p}$ is $X^{tC_{p^n}}$. As consequence of theorem 3.1.22 this lax symmetric monoidal structure for a cyclic group of $p$-power order agrees with the usual lax symmetric monoidal structure afforded by theorem 3.1.22.

We record here the following corollary to show the strength of this lemma. This corollary will also play an important part in the equivalence between bounded below naive and genuine cyclotomic spectra.

BorelCor **Corollary 4.4.6.** *Let $X$ be a Borel complete object in $C_{p^\infty}\mathsf{Sp}$ whose underlying spectrum is bounded below. Then the object $\Phi^{C_p}X \in C_{p^\infty}\mathsf{Sp}$ is also Borel complete. In particular the canonical map*

$$\Phi^{C_p} B_{C_{p^\infty}} Y \xrightarrow{\simeq} B_{C_{p^\infty}}(Y^{tC_p}),$$

*is an equivalence for every bounded below spectrum $Y \in \mathsf{Sp}^{BC_{p^\infty}}$.*

*Proof.* By lemma 4.4.4 we obtain the following pullback square for $n \geq 1$, because $X$ is bounded below,

$$\begin{array}{ccc} X^{C_{p^n}} & \xrightarrow{R} & (\Phi^{C_p}X)^{C_{p^{n-1}}} \\ \downarrow & & \downarrow t \\ X^{hC_{p^n}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}}. \end{array}$$

By the proof of lemma 4.4.4, and the assumption that $X$ was Borel complete, we have that

$$(X^{tC_p})^{hC_{p^{n-1}}} \simeq (\Phi^{C_p} B_{C_{p^\infty}}(X))^{hC_{p^{n-1}}} \simeq (\Phi^{C_p}X)^{hC_{p^{n-1}}}.$$

Hence to show that $\Phi^{C_p}X$ is Borel complete, it suffices to show that the map $t$ is an equivalence, but it is because the left hand map is per. assumption on $X$. Note that if $Y \in \mathsf{Sp}^{BC_{p^\infty}}$ is bounded below, then so is $B_{C_{p^\infty}}Y \in C_{p^\infty}\mathsf{Sp}$, hence it gives rise to the above pullback square for $n \geq 1$, and therefore $\Phi^{C_p}B_{C_{p^\infty}}Y$ is Borel complete. Furthermore $\Phi^{C_p}B_{C_{p^\infty}}Y \simeq Y^{tC_p}$, collecting these pieces we have the desired equivalence. □

The following technical corollary is going to be important for the proof of the equivalence between the two formulas for topological cyclic homology. The following follows directly from lemma 4.4.4 and induction on $n$.

TCcor **Corollary 4.4.7.** *Let $X$ be a genuine $C_{p^n}$-equivariant spectrum. Assume that the spectra*

$$X, \Phi^{C_p}X, \Phi^{C_{p^2}}X, ..., \Phi^{C_{p^{n-1}}}X$$

*are bounded below. Then we have a diagram*

$$\begin{array}{ccccccc} X^{C_{p^n}} & \longrightarrow & & & & & \Phi^{C_{p^n}}X \\ \downarrow & & & & & & \downarrow \\ & & & & (\Phi^{C_{p^{n-1}}}X)^{hC_p} & \longrightarrow & (\Phi^{C_{p^{n-1}}}X)^{tC_p} \\ & & & & \downarrow & & \\ & & (\Phi^{C_{p^2}}X)^{hC_{p-2}} & \longrightarrow & \cdots & & \\ & & \downarrow & & & & \\ & (\Phi^{C_p}X)^{hC_{p^{n-1}}} & \longrightarrow ((\Phi^{C_p}X)^{tC_p})^{hC_{p^{n-2}}} & & & & \\ & \downarrow & & & & & \\ X^{hC_{p^n}} & \longrightarrow (X^{tC_p})^{hC_{p^{n-1}}} & & & & & \end{array}$$

*Which exhibits $X^{C_{p^n}}$ as an iterated pullback.*

Finally we are ready to give the $p$-local version main result of this section.

TCeqP **Theorem 4.4.8.** *Let $X$ be a genuine $p$-cyclotomic spectrum such that the underlying spectrum is bounded. There is is a canonical equalizer diagram*

$$\mathrm{TC}^{gen}(X,p) \longrightarrow X^{hC_{p^\infty}} \underset{can}{\overset{\varphi_p^{hC_{p^\infty}}}{\rightrightarrows}} (X^{tC_p})^{hC_{p^\infty}}.$$

*Hence we obtain an equivalence* $\mathrm{TC}^{gen}(X,p) \simeq \mathrm{TC}(X,p)$.

*Proof.* Note that since $\Phi^{C_p}X \simeq X$, all of the geometric fixpoints

$$X, \Phi^{C_p}X, \Phi^{C_{p^2}}X, ..., \Phi^{C_{p^{n-1}}}X$$

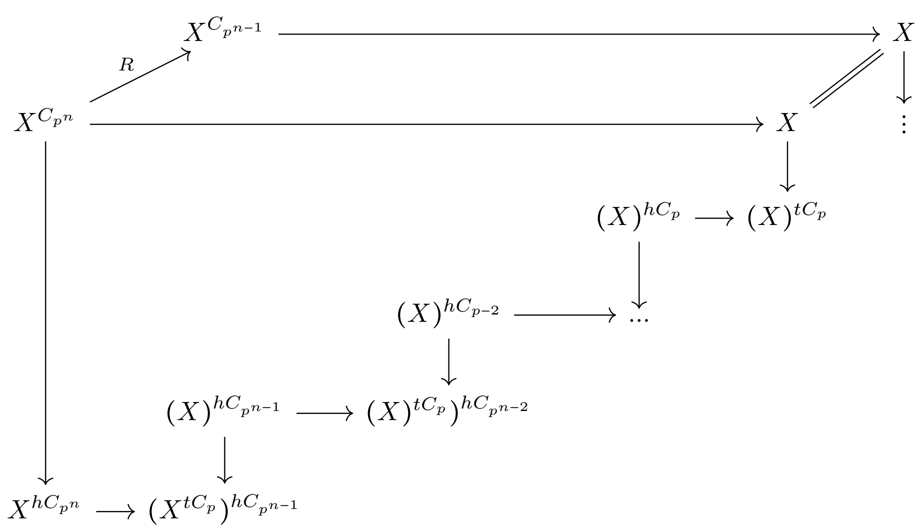
are bounded below, hence from corollary 4.4.7 we get an equivalence.

$$X^{C_{p^n}} \xrightarrow{\simeq} X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} X^{hC_{p^{n-1}}} \times ... \times_{X^{tC_p}} X.$$

We will refer to this equivalence as ($\square$). We claim that application of the map $R : X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ under the equivalence ($\square$) corresponds to "forgetting", i.e. projecting onto the point, the first factor of the iterated pullback. I.e.

$$\begin{array}{ccc} X^{C_{p^n}} & \xrightarrow{\simeq} & X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} X^{hC_{p^{n-1}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} ... \times_{X^{tC_p}} X \\ \downarrow R & & \downarrow \\ X^{C_{p^{n-1}}} & \xrightarrow{\simeq} & X^{hC_{p^{n-1}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} ... \times_{X^{tC_p}} X \end{array}$$

This may be realized by considering the following diagram induced from the defining diagram for the equivalence ($\square$), with the structure equivalences $\Phi^{C_p}X \simeq X$ applied and the map $R$,




From which we see that the exponent of $p$ in $X^{Cp^-}$ determines the shape diagram, and hence also when stop forming the iterated pullbacks. Thusly we have showed the claim concerning $R$ under the equivalence ($\square$).

We claim that under the equivalence ($\square$) application of the map $F : X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ corresponds to "forgetting" the last factor of the iterated pullback followed by application of projections maps $T_k : X^{hC_{p^k}} \to X^{hC_{p^{k-1}}}$ for $0 \le k < n$ factorwise. I.e. the effect of $F$ on the iterated pullback is,

$$
\begin{array}{c}
X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} X^{hC_{p^{n-1}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} \dots \times_{X^{tC_p}} X \\
\downarrow \\
T_n(X^{hC_{p^n}}) \times_{T_{n-1}((X^{tC_p})^{hC_{p^{n-1}}})} T_{n-1}(X^{hC_{p^{n-1}}}) \times_{T_{n-2}((X^{tC_p})^{hC_{p^{n-2}}})} \dots \times_{T_1((X^{tC_p})^{hC_p})} T_1(X^{hC_p}).
\end{array}
$$

Note that the "right" hand side is equal to

$$
X^{hC_{p^{n-1}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} \dots \times_{X^{tC_p}} X
$$

Because of the iterated pullback structure, the claim amounts to showing that each of the following squares commute,

$$
\begin{array}{ccc}
X^{C_{p^n}} & \xrightarrow{F} & X^{C_{p^{n-1}}} \\
\downarrow & & \downarrow \\
X^{hC_{p^n}} & \longrightarrow & X^{hC_{p^{n-1}}},
\end{array}
\qquad
\begin{array}{ccc}
X^{C_{p^n}} & \xrightarrow{F} & X^{C_{p^{n-1}}} \\
\downarrow & & \\
(\Phi^{C_{p^{n-1}}} X)^{C_{p^n}/C_{p^{n-1}}} & & \Big\downarrow \\
\downarrow & & \\
(\Phi^{C_{p^{n-1}}} X)^{hC_p} & \longrightarrow & (\Phi^{C_{p^{n-1}}} X)^{h\{e\}}.
\end{array}
$$

which they do because the lower vertical maps are maps induced from restriction of homotopy fixed points from a group to one of its subgroups.

According to the definition of $\mathrm{TC}^{gen}(X,p)$ we have to calculate the equalizer of $R$ and $F$, or equivalently the fiber of $R - F : X^{C_{p^n}} \to X^{C_{p^{n-1}}}$.

Consider the following square of products

$$
\begin{array}{ccc}
X^{hC_{p^n}} \times \dots \times X & \xrightarrow{R'-F'} & X^{hC_{p^{n-1}}} \times \dots \times X \\
\Big\downarrow \Omega_p - can & & \Big\downarrow \Omega_p - can \\
(X^{tC_p})^{hC_{p^{n-1}}} \times \dots \times X^{tC_p} & \xrightarrow{R''-F''} & (X^{tC_p})^{hC_{p^{n-2}}} \times \dots \times X^{tC_p}.
\end{array}
$$

In this diagram $R'$ forgets the first factor, $F'$ forgets the last and then projects $X^{hC_{p^k}} \to X^{hC_{p^{k-1}}}$. The map $R''$ forgets the first factor, and $F''$ forgets the last factor and then projects $(X^{tC_p})^{hC_{p^k}} \to (X^{tC_p})^{hC_{p^{k-1}}}$. The map $\Omega_p$ is the composite

$$
X^{hC_{p^n}} \times \dots \times X \xrightarrow{p} X^{hC_{p^{n-1}}} \times \dots \times X \xrightarrow{\prod_{0 \le k \le n} \varphi_p^{hC_{p^k}}} (X^{tC_p})^{hC_{p^{n-1}}} \times \dots \times X^{tC_p}
$$

where $p$ projects to the $n$ last factors, and the maps $\varphi_p^{hC_{p^k}} : X^{hC_{p^k}} \to (X^{tC_p})^{hC_{p^k}}$ are induced by homotopy fixed points from the cyclotomic structure map $X \to X^{tC_p}$ for $0 \le k \le n$. The map $can$ is the product of the canonical maps $can_k : X^{hC_{p^k}} \to (X^{tC_p})^{hC_{p^{k-1}}}$ for each $0 \le k \le n$.

Commutativity can be checked factorwise because the lower right edge of the diagram is a product. Thus it boils down to checking wether the effect of the application of $R' - F'$ and $R'' - F''$ is commutative for certains squares for $0 \le k \le n$. Note that for $0 < k < n$, the maps $R'$ and $R''$ has no effect and $F'$ and $F''$ reduce to the projections described above. Using that the canonical maps $can_k$ and $\varphi_p^{hC_{p^k}}$ for $0 \le k \le n$ commute with these projections it is elementary to show that $\Omega_p - can$ commutes with these projections. We split the rest into two seperate cases.

When we write $F'$, $F''$, $R'$ and $R''$ in the following diagrams we won't mean the actual maps, but rather their effect on the particular factor in question. Let $k = 0$. Then application of the maps $F'$ and $F''$ are the projections onto the point, application of $R'$ and $R''$ are identities, application of the map $can$ and $\Omega_p$ reduces to the canonical map $can_0 : X \to X^{tC_p}$ and the cyclotomic structure map $\varphi_p : X \to X^{tC_p}$. Hence the diagram reduces to

$$\begin{array}{ccc} X & \xrightarrow{F'} & \bullet \\ \downarrow & & \downarrow \\ X^{tC_p} & \xrightarrow{F''} & \bullet. \end{array}$$

which obviously commutes. Let $k = n$. Then application of the maps $F'$ and $F''$ are the projections described above, application of $R'$ and $R''$ are projections onto the point, application of $can$ and $\Omega_p$ reduces to the canonical maps $can_i : X^{hC_{p^i}} \to (X^{tC_p})^{hC_{p^{i-1}}}$ and the composite maps $p \circ \varphi_p^{hC_{p^i}} : X^{hC_{p^i}} \to (X^{tC_p})^{hC_{p^{i-1}}}$ for $i \in \{n, n-1\}$ respectively. Hence the diagram reduces to

$$\begin{array}{ccc} X^{hC_{p^n}} & \xrightarrow{R'-F'} & X^{hC_{p^{n-1}}} \\ \Big\downarrow{\scriptstyle (p\circ\varphi_p^{hC_{p^n}})-can_n} & & \Big\downarrow{\scriptstyle (p\circ\varphi_p^{hC_{p^{n-1}}})-can_{n-1}} \\ (X^{tC_p})^{hC_{p^{n-1}}} & \xrightarrow{R''-F''} & (X^{tC_p})^{hC_{p^{n-2}}}. \end{array}$$

Now because $p$ projects away from the first factor, the map $p \circ \varphi_p^{hC_{p^i}} : X^{hC_{p^i}} \to (X^{tC_p})^{hC_{p^{i-1}}}$ for $i \in \{n, n-1\}$ factor throught projections to the point, just like $R' - F'$ and $R'' - F''$ because $R'$ and $R''$ were projections to the point. The canonical maps $can_i$ for $i \in \{n, n-1\}$ commute with the projections described above. Because of this the square commutes.

Hence the square of products commutes. From left to right the fibers of the vertical maps are by construction $X^{C_{p^n}}$ and $X^{C_{p^{n-1}}}$, with the induced map given by $R - F$. The fiber of the upper horizontal map is $X^{hC_{p^n}}$ via the diagonal embedding, and likewise the fiber of the lower horizontal map is $(X^{tC_p})^{hC_{p^{n-1}}}$. Hence we obtain the following commutative diagram

$$\begin{array}{ccccc} & & X^{C_{p^n}} & \xrightarrow{R-F} & X^{C_{p^{n-1}}} \\ & & \downarrow & & \downarrow \\ X^{hC_{p^n}} & \longrightarrow & X^{hC_{p^n}} \times \dots \times X & \xrightarrow{R'-F'} & X^{hC_{p^{n-1}}} \times \dots \times X \\ \Big\downarrow{\scriptstyle \varphi_p - can} & & \Big\downarrow{\scriptstyle \varphi_p - can} & & \Big\downarrow{\scriptstyle \varphi_p - can} \\ (X^{tC_p})^{hC_{p^{n-1}}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}} \times \dots \times X^{tC_p} & \xrightarrow{R''-F''} & (X^{tC_p})^{hC_{p^{n-2}}} \times \dots \times X^{tC_p}. \end{array}$$

From commutativity we see that the fiber of $R - F : X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ is equivalent to the fiber of $\varphi_p - can : X^{hC_{p^n}} \to (X^{tC_p})^{hC_{p^{n-1}}}$. Finally if we take the limit over $n$, we obtain the result for $C_{p^\infty}$, and hence we get the desired result. □

Now using the $p$-typical result we may obtain the global result for genuine cyclotomic spectra. This is theorem 2 of the introduction.

TCMain **Theorem 4.4.9.** *Let $X$ be a genuine cyclotomic spectrum such that the underlying spectrum is bounded. There is is a canonical equalizer diagram*

$$\mathrm{TC}^{gen}(X) \longrightarrow X^{h\mathbb{T}} \underset{can}{\overset{\prod_{p\in\mathbb{P}}(\varphi_p^{h\mathbb{T}})}{\rightrightarrows}} \prod_{p\in\mathbb{P}} (X^{tC_p})^{h\mathbb{T}}.$$

*Hence we obtain an equivalence* $\mathrm{TC}^{gen}(X) \simeq \mathrm{TC}(X)$.

*Proof.* Consider the following commutative diagram, induced from the defining diagram of $\mathrm{TC}^{gen}(X)$ and the product over all primes $p$ of the $p$-completions of the fiber sequence corresponding to the equalizer diagram of theorem 4.4.8,

$$\begin{array}{ccccc} \mathrm{TC}^{gen}(X) & \longrightarrow & X^{h\mathbb{T}} & & \\ \downarrow & & \downarrow & & \\ \prod_{p\in\mathbb{P}} \mathrm{TC}^{gen}(X,p)_p^{\wedge} & \longrightarrow & \prod_{p\in\mathbb{P}}(X_p^{\wedge})^{h\mathbb{T}} & \longrightarrow & \prod_{p\in\mathbb{P}}((X_p^{\wedge})^{tC_p})^{hC_{p^\infty}}. \end{array}$$

By applying lemma 3.2.7 and using that $C_{p^\infty} \to \mathbb{T}$ is a $p$-adic equivalence, the natural maps

$$(X^{tC_p})^{h\mathbb{T}} \to ((X_p^{\wedge})^{tC_p})^{hC_{p^\infty}},$$

are equivalences for all primes $p$. Because of this we get an equivalence on the product of these over all primes. This equivalence fits into the above diagram in the following way

$$\begin{array}{ccccc} \mathrm{TC}^{gen}(X) & \longrightarrow & X^{h\mathbb{T}} & \xrightarrow{\prod_{p\in\mathbb{T}} \varphi_p^{h\mathbb{T}} - can} & \prod_{p\in\mathbb{P}}(X^{tC_p})^{h\mathbb{T}} \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \prod_{p\in\mathbb{P}} \mathrm{TC}^{gen}(X,p)_p^{\wedge} & \longrightarrow & \prod_{p\in\mathbb{P}}(X_p^{\wedge})^{h\mathbb{T}} & \longrightarrow & \prod_{p\in\mathbb{P}}((X_p^{\wedge})^{tC_p})^{hC_{p^\infty}} \end{array}$$

This equivalence, the lower line being a fiber sequence, and the left square being a pullback gives that the upper line is a fiber sequence and hence we obtain the desired equalizer diagram. □

## 4.5 The $p$-Local Equivalence of $\infty$-Categories of Cyclotomic Spectra

In this section we shall employ the result concerning coalgebras, endofunctors and their fixed points to show that there is an equivalence between naive cyclotomic spectra and genuine cyclotomic spectra when restricting to objects which underlying spectrum is bounded below. All of the assumptions we made in the section on lax equalizers and coalgebras was with the $\infty$-categories $C_{p^\infty}\mathsf{Sp}$ and $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$, and their associated endofunctors $\Phi^{C_p}$ for $p \in \mathbb{P}$, in mind. We begin by recasting these $\infty$-categories as fixed points of associated endofunctors. For this and the next section we will follow parts of section II.3, II.5 and II.6. As the title of the section indicates we will in this section work $p$-locally.

The cocontinuity of $\Phi^{C_p}$ on $C_{p^\infty}\mathsf{Sp}$ and $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ affords us with right adjoints, these play well together with the genuine fixed point functor. This is so is summarized in the following result which will be rather important in the proof of the equivalence in both the $p$-local and global setting.

FixFormula **Corollary 4.5.1.** *Let $R_{C_p} : C_{p^\infty}\mathsf{Sp} \to C_{p^\infty}\mathsf{Sp}$ and $R_{C_p} : \mathbb{T}\mathsf{Sp}_{\mathcal{F}} \to \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ be the right adjoint functors of $\Phi^{C_p}$ on $C_{p^\infty}\mathsf{Sp}$ and $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ respectively. We have the following formula*

$$(R_{C_p}X)^H \simeq \begin{cases} X^{H/C_p} & \text{if } C_p \subseteq H \\ 0 & \text{otherwise.} \end{cases}$$

*For $H$ be a finite subgroup of $C_{p^\infty}$ or $\mathbb{T}$ respectively.*

*Proof.* Because we are only considering genuine fixed points with a finite group, we immediately have that $(R_{C_p}X)^H \simeq 0$ for $C_p \notin H$ by lemma 4.3.16. If $C_p \subseteq H$, we have the following equivalences

$$R_{C_p}(X)^H \simeq (R_{C_p}(X)^{C_p})^{H/C_p} \simeq (\Phi^{C_p} R_{C_p}(X))^{H/C_p} \simeq X^{H/C_p},$$

where the first is a property of the residual action of genuine fixed points, see definition 4.3.11, the second is the second part of lemma 4.3.16, and the last is the fact that the right adjoint is fully faithful and hence by lemma **??** the counit is an equivalence. □

We now realize genuine $p$-cyclotomic spectra as fixed points of the endofunctor $\Phi^{C_p}$ on $C_{p^\infty}\mathsf{Sp}$.

CoAlgIncP **Lemma 4.5.2.** *There is an equivalence of $\infty$-categories,*

$$\mathsf{CycSp}_p^{gen} \simeq \mathsf{Fix}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp}),$$

*and the inclusion* $\iota : \mathsf{CycSp}_p^{gen} \subseteq \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$ *is cocontinuous.*

*Proof.* Recall that we had a model for the fixed points of an endofunctor $F$ on an $\infty$-category, as the equalizer of the identity and $F$, see the proof of 4.1.6. We stated in remark 4.3.15 that $\Phi^{C_p}$ was an endofunctor on $C_{p^\infty}\mathsf{Sp}$. From this it follows that $\mathsf{CycSp}_p^{gen} \simeq \mathsf{Fix}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$. The $\infty$-categories $C_{p^n}\mathsf{Sp}$ is presentable for all $n$, as argued in remark 4.3.7, therefore $C_{p^\infty}\mathsf{Sp}$ is to, by virtue of being a limit of presentable along continuous functors ($\mathsf{Pr}^L$ is complete). By lemma 4.3.16 that $\Phi^{C_p}$ was cocontinuous. Now as a consequence of this, and corollary 4.1.6 we obtain a cocontinuous inclusion $\iota : \mathsf{CycSp}_p^{gen} \subseteq \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$. □

The following will be a key input into the $p$-local equivalence mentioned before.

T.II.5.6 **Theorem 4.5.3.** *The inclusion* $\iota : \mathsf{CycSp}_p^{gen} \subseteq \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$ *admits a right adjoint* $R_\iota$ *such that the counit* $\iota R_\iota \to \mathrm{id}$ *of the adjunction induces an isomorphism of underlying non-equivariant spectra.*

*Proof.* By lemma 4.3.16 and remark 4.3.15 the endofunctor $\Phi^{C_p}$ admits a fully faithful right adjoint $R_{C_p} : (C_{p^\infty}/C_p)\mathsf{Sp} \simeq C_{p^\infty}\mathsf{Sp} \to C_{p^\infty}\mathsf{Sp}$. Using the notation of construction 4.1.7, $\Phi^{C_p}$ induces an endofunctor $\overline{\Phi}^{C_p}$ which also admits a right adjoint $R_{\overline{\Phi}^{C_p}} : \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp}) \to \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$. By corollary 4.5.1 we have that $(R_{C_p}(X))^{C_{p^m}} \simeq X^{C_{p^{m-1}}}$ for $1 \leq m$ and $(R_{C_p}(X))^{\{e\}} \simeq 0$. If we combine this with the formula of corollary 4.1.11 then we obtain that for $(X, \varphi : X \to \Phi^{C_p}X) \in \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$, the underlying object of $R^k_{\overline{\Phi}^{C_p}}(X)$ has genuine fixed points given by

$$(R^k_{\overline{\Phi}^{C_p}}(X))^{C_{p^m}} \simeq X^{C_{p^{m-k}}} \times_{(\Phi^{C_p}(X))^{C_{p^{m-k}}}} X^{C_{p^{m-k+1}}} \times \ldots \times_{(\Phi^{C_p}(X))^{C_{p^{m-1}}}} X^{C_{p^m}}.$$

Note that we in the equivalence have used that $(-)^{C_{p^m}}$ is continuous. Note that when $k > m$ this formula only depends on $m$. Therefore the limit

$$\ldots \longrightarrow (R^3_{\overline{\Phi}^{C_p}}(X))^{C_{p^m}} \xrightarrow{R^2_{\overline{\Phi}^{C_p}}\nu} (R^2_{\overline{\Phi}^{C_p}})^{C_{p^m}} \xrightarrow{R_{\overline{\Phi}^{C_p}}\nu} (R_{\overline{\Phi}^{C_p}}(X))^{C_{p^m}} \xrightarrow{\nu} X^{C_{p^m}},$$

which is afforded by proposition 4.1.9 stabilizes. We will refer to this directed system as $(\star)$. We have by lemma 4.1.10 that $\overline{\Phi}^{C_p} R_{\overline{\Phi}^{C_p}} \simeq \mathrm{id}$ as endofunctors of $\mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$. From this we have that by applying $\overline{\Phi}^{C_p}$ to the directed system

$$\ldots \longrightarrow R^2_{\overline{\Phi}^{C_p}}(X) \longrightarrow R_{\overline{\Phi}^{C_p}}(X) \longrightarrow X,$$

we obtain the directed system,

$$\ldots \longrightarrow R^2_{\overline{\Phi}^{C_p}}(X) \longrightarrow R_{\overline{\Phi}^{C_p}}(X) \longrightarrow X \longrightarrow \overline{\Phi}^{C_p}(X).$$

This directed system, analogous to the tower above, also stabilizes when genuine fixed points are applied. As a consequence $\overline{\Phi}^{C_p}$ commutes with the limit of the tower. Now lemma 4.1.3 (3) the limit of coalgebras is computed as the limit of underlying objects. Hence from proposition 4.1.9 we get that the limit of the tower $(\star)$ is given as the spectrum $R_\iota(X)^{C_{p^m}}$. Setting $m = 0$, we obtain that the canonical map of underlying non-equivariant spectra $(\iota R_\iota(X))^{\{e\}} \to X^{\{e\}}$ is an equivalence. □

Theorem 4.5.3 is an important input into the equivalence between bounded below naive and genuine $p$-cyclotomic spectra. We will now construct the $p$-local functor which will descend to an equivalence when restricted to bounded below spectra. Theorem 4.5.3 will in some sense provide half of the argumentation needed to see that the following functor restricts to an equivalence.

**Definition 4.5.4.** Let $(X, \Phi^{C_p})$ be a genuine $p$-cyclotomic spectrum. Consider the following composite

$$X \xrightarrow{\simeq} \Phi^{C_p} X \xrightarrow{(B_{C_{p^\infty}})_*} \Phi^{C_p} B_{C_{p^\infty}} X \xrightarrow{\simeq} X^{tC_p}.$$

Where the first map is the inverse of the structure map, the second is precomposition with Borel completion, and the last equivalence was shown in the proof of lemma 4.4.4. Per. construction of $C_{p^\infty}\mathsf{Sp}$ there is a natural functor

NatFun (4.3)
$$\eta : C_{p^\infty}\mathsf{Sp} \to \mathsf{Sp}^{BC_{p^\infty}}.$$

The map induced from the above composite under this functor is a $C_{p^\infty}$-equivariant map $X \to X^{tC_p}$, where the $C_{p^\infty}$-action on the right hand side is the residual $C_{p^\infty}/C_p$-action via the $p$th power map, see remark 4.3.15.

p-localFunctorProp **Proposition 4.5.5.** *The assignment described above defines a functor*

p-localFunctor (4.4)
$$\mathsf{CycSp}_p^{gen} \to \mathsf{CycSp}_p.$$

*Proof.* We will in fact construct a functor between $\infty$-categories which has the desired $\infty$-categories as subcategories, i.e. we will construct functors between the following lax equalizers instead of the corresponding equalizers,

LEq (4.5)
$$\mathrm{LEq}(\mathrm{id}, \Phi^{C_p}) \to \mathrm{LEq}(\mathrm{id}, -^{tC_p}).$$

Where the first lax equalizer is of functors $C_{p^\infty}\mathsf{Sp} \to C_{p^\infty}\mathsf{Sp}$ and the second is of functors $\mathsf{Sp}^{BC_{p^\infty}} \to \mathsf{Sp}^{BC_{p^\infty}}$. The natural functor (4.3) commutes with the first functor id in the first lax equalizer. There is a natural transformation between $\eta \circ \Phi^{C_p}$ and $-^{tC_p} \circ \eta$, which is given by passing to the underlying spectrum of the natural transformation $\Phi^{C_p} \to \Phi^{C_p} B_{C_{p^\infty}}$ and then invoking the identification of $-^{tC_p}$ and the underlying spectrum of $\Phi^{C_p} B_{C_{p^\infty}}$ (established in the proof of lemma 4.4.4). Per. definition of lax equalizers we get the desired map. ☐

Per. construction of the functor (4.4) and the results of the section concerning lax equalizers and coalgebras, the functor is cocontinuous and therefore admits a right adjoint functor. We will try to understand this right adjoint well enough to see that (4.4) induces an equivalence of subcategories of bounded below spectra. To this end we will factor (4.4) and examine the right adjoints of the factors.

p-localFactor **Corollary 4.5.6.** *The functor (4.4) can be factored as the following composite*

$$\mathsf{CycSp}_p^{gen} \xrightarrow{\iota} \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp}) \xrightarrow{U} \mathsf{CycSp}_p.$$

*In the above $\iota$ is the inclusion, and $U$ takes the underlying naive $p$-cyclotomic spectrum.*

*Proof.* First recall that $\mathsf{CycSp}_p^{gen} \simeq \mathsf{Fix}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$, hence there is an inclusion as claimed. Note that

$$\mathsf{CycSp}_p^{gen} \simeq \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp}) \overset{(4.5)}{\to} \mathsf{CoAlg}_{(-)^{tC_p}}(\mathsf{Sp}^{BC_{p^\infty}}) \simeq \mathrm{LEq}(\mathrm{id}, -^{tC_p}) \simeq \mathsf{CycSp}_p.$$

The claim now again follows from the fact that we constructed the functor (4.4) more generally on lax equalizers, i.e. as a the functor (4.5), together with the identification of $-^{tC_p}$ and the underlying spectrum of $\Phi^{C_p} B_{C_{p^\infty}}$. ☐

Both $\iota$ and $U$ admit right adjoints, because they both are cocontinuous. We have in the section on lax equalizers and coalgebras studied the right adjoint of $\iota$ extensively. We will now describe the right adjoint of $U$ which we denote $B$. The functor $B$ is in fact familiar to us when restricted to bounded below spectra. We fix some notation before showing this.

**Definition 4.5.7.** Let $\mathsf{CycSp}^{gen}_{p,+} \subseteq \mathsf{CycSp}^{gen}_p$, and $\mathsf{CycSp}_{p,+} \subseteq \mathsf{CycSp}_p$, denote the respective $\infty$-subcategories of objects whose underlying spectra are bounded below. Then the functor (4.4) restricts to a functor

$$\mathsf{CycSp}^{gen}_{p,+} \to \mathsf{CycSp}_{p,+}.$$

It turns out that the right adjoint $B : \mathsf{CycSp}_p \to \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$ is given by Borel completion when restricted to $\mathsf{CycSp}_{p,+}$.

L.II.6.2. **Lemma 4.5.8.** *Let $(X, \Phi_p : \Phi^{C_p}X \to X) \in \mathsf{CycSp}_{p,+}$. Then the right adjoint $B : \mathsf{CycSp}_p \to \mathsf{CoAlg}_{\Phi^{C_p}}(C_{p^\infty}\mathsf{Sp})$ of the forgetful functor $U$ is fully faithful and when applied to $(X, \Phi_p)$ is given by $B_{C_{p^\infty}}X \in C_{p^\infty}\mathsf{Sp}$ with the coalgebra structure given by postcomposition of the Tate construction,*

$$B_{C_{p^\infty}}\Phi_p : B_{C_{p^\infty}}X \to B_{C_{p^\infty}}(X^{tC_p}) \simeq \Phi^{C_p}(B_{C_{p^\infty}}X).$$

*Proof.* We have the equivalence from the coalgebra structure afforded by lemma 4.4.6. We will see that under the assumptions above $B_{C_{p^\infty}}X$ will satisfy the universal property of the mapping space in a lax equalizer for $B$. We established in lemma 4.1.4 (2) that for $(Y, \Phi_p^Y)$ an arbitrary $\Phi^{C_p}$-coalgebra the mapping space between $(Y, \Phi_p^Y)$ and $(B_{C_{p^\infty}}X, B_{C_{p^\infty}}\Phi_p)$ was given by the equalizer

$$\mathrm{Eq}\Big(\mathrm{Map}_{C_{p^\infty}\mathsf{Sp}}(Y, B_{C_{p^\infty}}X) \underset{(B_{C_{p^\infty}}\Phi_p)_*}{\overset{(\Phi_p^Y)_*\Phi^{C_p}}{\rightrightarrows}} \mathrm{Map}_{C_{p^\infty}\mathsf{Sp}}(Y, B_{C_{p^\infty}}(X^{tC_p}))\Big).$$

We show that this mapping space can be identified with the mapping space between the to $(Y, \Phi_p^Y)$ corresponding naive $p$-cyclotomic spectrum $(Y, \varphi_p^Y)$ and $(X, \Phi_p)$. But since the Borel completion functor $B_{C_{p^\infty}}$ is right adjoint to the functor $C_{p^\infty}\mathsf{Sp} \to \mathsf{Sp}^{BC_{p^\infty}}$ by proposition 4.3.24, the above mapping space can be identified with

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathsf{Sp}^{BC_{p^\infty}}}(Y, X) \underset{(\Phi_p)_*}{\overset{(\varphi_p^Y)_*((-)^{tC_p})}{\rightrightarrows}} \mathrm{Map}_{\mathsf{Sp}^{BC_{p^\infty}}}(Y, X^{tC_p})\Big).$$

Again from lemma 4.1.4 (2) this equalizer is the desired mapping space, granting the desired result. ☐

We are now ready to give the main result of the section, namely the $p$-local equivalence mentioned before.

T.II.6.3 **Theorem 4.5.9.** *The functor (4.4) $\mathsf{CycSp}^{gen}_p \to \mathsf{CycSp}_p$ induces an equivalence of $\infty$-categories of bounded below naive and genuine cyclotomic spectra, $\mathsf{CycSp}^{gen}_{p,+} \to \mathsf{CycSp}_{p,+}$.*

*Proof.* We show that the composite $U \circ \iota : \mathsf{CycSp}^{gen}_p \to \mathsf{CycSp}_p$ afforded by corollary 4.5.6, is an equivalence of $\infty$-categories, when restricted to the subcategories of bounded below spectra.

As discussed above both $U$ and $\iota$ have right adjoints, denoted $B$ and $R_\iota$ respectively, hence so does $U \circ \iota$. By lemma 1.2.8 we may instead show that $U \circ \iota$ is conservative and that it has a fully faithful right adjoint. Let $f : (X, \Phi_p^X) \to (Y, \Phi_p^Y)$ be a map of genuine $p$-cyclotomic spectra such that $(U \circ \iota)(f)$ is an equivalence in $\mathsf{CycSp}_p$. Now $(U \circ \iota)(f)$ is an equivalence if and only if $(U \circ \iota)(\Phi^{C_p}(f))$ is an equivalence if and only if $U(\Phi^{C_p}(f))$ is an equivalence. Note that $U(\Phi^{C_p}(f)) \simeq \Phi^e(f) \simeq f$. Hence $U \circ \iota$ is conservative.

Now it remains to show that the right adjoint is fully faithful, we do this by showing that the counit is an equivalence, i.e. that $U\iota RB((X,\Phi_p)) \to (X,\Phi_p)$ is an equivalence for $(X,\Phi_p) \in \mathsf{CycSp}_{p,+}$. By theorem 4.5.3 we have that $\iota R(B((X,\Phi_p))) \cong B((X,\Phi_p))$, and by lemma 4.5.8 we have that $UB((X,\Phi_p)) \simeq (X,\Phi_p)$, collecting these gives the desired result. □

## 4.6 The Global Equivalence of ∞-Categories of Cyclotomic Spectra

Almost every result in this section has a $p$-local analog which we proved in the previous section, but there are a few differences. The results of this section are far more cluttered, because we need the iterated coalgebra ∞-category rather than the just coalgebra ∞-category for a single endofunctor. We begin with a result for which the global version follows in much the same way as the $p$-local one (lemma 4.5.2).

CoAlgIncGlobal **Lemma 4.6.1.** *There is an equivalence of ∞-categories,*

$$\mathsf{CycSp}^{gen} \simeq \mathsf{Fix}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}}),$$

*and the inclusion* $\iota : \mathsf{CycSp}^{gen} \subseteq \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$ *is cocontinuous.*

*Proof.* Recall that we had a model for the fixed points of countably many coherently commuting endofunctors $\{F_i\}_{i\in\mathbb{N}}$ as the homotopy fixed points of the monoid $\prod_{i=1}^{\infty}\mathbb{N}$, see the proof of corollary 4.1.14. We argued in remark 4.3.14 that the collection $\{\Phi^{C_p}\}_{p\in\mathbb{P}}$ commuted coherently, and we argued in remark 4.3.15 that they were endofunctors on $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$. Hence we have an equivalence $\mathsf{CycSp}^{gen} \simeq \mathsf{Fix}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$. By arguments analogous to those applied to $C_{p^\infty}\mathsf{Sp}$, the ∞-category $\mathbb{T}\mathsf{Sp}_{\mathbb{F}}$ is presentable. Hence by corollary 4.1.14 we have that $\mathsf{Fix}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$ is presentable and we obtain a cocontinuous inclusion $\iota : \mathsf{CycSp}^{gen} \subseteq \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$. □

*Remark* 4.6.2. Note that we do not need the above equivalence to see that $\mathsf{CycSp}^{gen}$ is presentable. It follows easily from $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ being presentable, as then $\mathsf{CycSp}^{gen}$ is a limit along the continuous functor $(-)^{h\prod_{i=1}^{\infty}\mathbb{N}}$ of presentable ∞-categories.

We now proceed to show the analog of theorem 4.5.3 in the global case. Here we run into the first difference between the local and global settings, namely that it requires an argument to see that the assumptions of corollary 4.1.19 are satisfied, i.e. that the following lemma is true.

**Lemma 4.6.3.** *The canonical morphism* $\Phi^{C_p}R_{C_q} \to R_{C_q}\Phi^{C_p}$ *of endofunctors* $\mathbb{T}\mathsf{Sp}_{\mathcal{F}} \to \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ *is an equivalence for all primes* $p \neq q$.

*Proof.* Per. definition of equivalences in $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ it suffices to check that the morphism

$$\Phi^H(\Phi^{C_p}R_{C_q}) \to \Phi^H(R_{C_q}\Phi^{C_p})$$

is an equivalence for all objects $X \in \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ and all finite subgroups $H \subseteq \mathbb{T}$. By lemma 4.3.16 we may instead check that the morphism

com (4.6)
$$(\Phi^{C_p}R_{C_q}(X))^H \to (R_{C_q}\Phi^{C_p}(X))^H$$

is an equivalence for all objects $X \in \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ and all finite subgroups $H \subseteq \mathbb{T}$. Assume first that $C_q \notin H$. Then by lemma 4.3.16 we immediately get that $(R_{C_q}\Phi^{C_p})^H \simeq 0$. We now argue that $(\Phi^{C_p}R_{C_q}X)^H \simeq 0$. First $X$ is $\mathbb{S}$-module per. assumption. Recall that there is a lax symmetric monoidal transformation of lax symmetric monoidal functors $-^{C_p} \to \Phi^H$ by remark 4.3.12. In particular $\Phi^{C_q}$ is lax symmetric monoidal, hence from example 2.1.13 we have that $R_{C_q}$ is symmetric monoidal. Hence $R_{C_q}X$ is a $R_{C_q}\mathbb{S}$-module, and $\Phi^{C_p}R_{C_q}X$ is a $\Phi_{C_p}R_{C_q}\mathbb{S}$-module. Now consider the $p$-power map $\mathbb{T} \to \mathbb{T}$, and denote the preimage of $H$ by $\tilde{H}$, and denote the restriction by $p : \tilde{H} \to H$. Then we obtain a commutative square induced from $p$ and the structure map induced from $\Phi^{C_p}R_{C_q}X$ being a $\Phi_{C_p}R_{C_q}\mathbb{S}$-module,

$$\begin{array}{ccccc}
\Phi^{\tilde{H}}\Phi^{C_p}R_{C_q}\mathbb{S} & \longrightarrow & \Phi^{H}\Phi^{C_p}R_{C_q}\mathbb{S} & & \\
\downarrow & & \downarrow & & \\
\Phi^{\tilde{H}}\Phi^{C_p}R_{C_q}X & \longrightarrow & \Phi^{H}\Phi^{C_p}R_{C_q}X & \xrightarrow{\simeq} & (\Phi^{C_p}R_{C_q}X)^H.
\end{array}$$

We have the following equivalences

$$\Phi^{\tilde{H}}\Phi^{C_p}R_{C_q}\mathbb{S} \simeq \Phi^{C_p}\Phi^{\tilde{H}}R_{C_q}\mathbb{S} \simeq \Phi^{\tilde{H}}R_{C_q}\mathbb{S} \simeq (R_{C_q}\mathbb{S})^{\tilde{H}}.$$

The first is the fact that geometric fixed points commute, the second is because $C_p \subseteq \tilde{H}$ per. construction, and the third is by lemma 4.3.16. Therefore $(\Phi^{C_p}R_{C_q}X)^H$ is a $(R_{C_q}\mathbb{S})^{\tilde{H}}$-module. But note that $C_q \notin \tilde{H}$, from which we have $(R_{C_q}\mathbb{S})^{\tilde{H}} \simeq 0$, and therefore $(\Phi^{C_p}R_{C_q}X)^H \simeq 0$. Hence if $C_p \subset H$ then (4.6) is an equivalence, and both sides are 0, hence we have reduced to the case $H = C_p$. Therefore, again by 4.3.16 we are reduced to showing that the natural map $\Phi^{C_p}R_{C_q}X \to R_{C_q}\Phi^{C_q}X$ is an equivalence after applying $\Phi^{C_q}$. This is easy:

$$\Phi^{C_q}R_{C_q}\Phi^{C_p} \simeq \Phi^{C_p} \simeq \Phi^{C_p}\Phi^{C_q}R_{C_q} \simeq \Phi^{C_q}\Phi^{C_p}R_{C_q}.$$

Which follows from the fact that the counit of the $\Phi^{C_p} \dashv R_{C_p}$ adjunction is an equivalence.

In the situation $C_q \subseteq H$, the proof is slightly more straightforward. We have the following equivalences

$$(R_{C_q}\Phi^{C_p}X)^H \simeq (\Phi^{C_p}X)^{H/C_q} \simeq X^{(H/C_q)/C_p} \simeq X^{(H/C_p)/C_q} \simeq (R_{C_q}X)^{H/C_p} \simeq (\Phi^{C_p}R_{C_q}X)^H.$$

In the above string of equivalences, the first is by corollary 4.5.1, the second is lemma 4.3.16, the third is because $p \neq q$, the fourth is corollary 4.5.1, and the last is lemma 4.3.16. We leave out the checking that these equivalences are the canonical map. □

Just as in the local setting, this theorem will be a key input into the global equivalence.

T.II.5.13 **Theorem 4.6.4.** *The inclusion* $\iota : \mathsf{CycSp}^{gen} \subseteq \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$ *admits a right adjoint* $R_\iota$ *such that the counit* $\iota R_\iota \to \mathrm{id}$ *of the adjunction induces an isomorphism of underlying non-equivariant spectra.*

*Proof.* For convenience we start by fixing some notation. Let $\mathscr{C}$ denote the $\infty$-category $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$, and let $F_n := \Phi^{C_{p_n}}$ where $p_1, p_2, \dots$ is the list of primes. We will use the notation established in the first section of this chapter. We wish to prove that that the inclusion

$$\iota_n : \mathscr{C}_{n+1} := \mathsf{Fix}_{F_1,\dots,F_n}(\mathsf{CoAlg}_{F_{n+1},F_{n+2},\dots}) \subseteq \mathsf{Fix}_{F_1,\dots,F_{n-1}}(\mathsf{CoAlg}_{F_n,F_{n+1},\dots}) =: \mathscr{C}_n,$$

admits a right adjoint $R_{\iota_n}$ such that the counit of the adjunction induces an equivalence on underlying spectra. Now using corollary 4.1.19, and corollary 4.5.1 (and the notation established in theorem 4.5.3) we have

$$(R^k_{\overline{\Phi}^{C_p}})^{C_{p^m}\times C_r}X \simeq X^{C_{p^{m-k}}\times C_r} \times_{(\Phi^{C_p}X)^{C_{p^{m-k}}\times C_r}} \times \dots \times_{(\Phi^{C_p}X)^{C_{p^{m-1}}\times C_r}} X^{C_{p^m}\times C_r}$$

where $r$ is coprime to $p$. Analogous to the proof of theorem 4.5.3, if $m$ and $r$ are fixed, then for $k > m$ this formula only depends on $m$ and $r$. Therefore the limit of the following diagram

$$\dots \longrightarrow (R^3_{\overline{\Phi}^{C_p}}(X))^{C_{p^m}\times C_r} \xrightarrow{R^2_{\overline{\Phi}^{C_p}}\nu} (R^2_{\overline{\Phi}^{C_p}})^{C_{p^m}\times C_r} \xrightarrow{R_{\overline{\Phi}^{C_p}}\nu} (R_{\overline{\Phi}^{C_p}}(X))^{C_{p^m}\times C_r} \xrightarrow{\nu} X^{C_{p^m}\times C_r},$$

which is afforded by proposition 4.1.18 stabilizes. Using a totally analogous argument to the argument in the proof of theorem 4.5.3, we get that the limit of the above diagram commutes with $\Phi^{C_p}$. Therefore it is the underlying object of the limit in the $\infty$-category $\mathscr{C}_n$. Hence proposition 4.1.18 implies that this underlying object is the underlying object of the right adjoint $R_{\iota_n}X$.

Now lets specialize to $r = 0$ and $m = 0$, to obtain that the counit of the adjunction $\overline{F}_n \dashv R_{\overline{F}_n}$ induces an equivalence $(\iota_n R_{\iota_n} X)^{\{e\}} \simeq X^{\{e\}}$, for all $n \in \mathbb{N}$.

The right adjoint to the "total" inclusion

$$\iota : \lim_n(\mathscr{C}_n) = \mathsf{CycSp}^{gen} \subseteq \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}}) =: \mathscr{C}_1,$$

is the limit of the right adjoints

$$R_\iota = \lim\Big(\mathscr{C}_1 \xrightarrow{R_{\iota_1}} \mathscr{C}_2 \xrightarrow{R_{\iota_2}} \mathscr{C}_3 \xrightarrow{R_{\iota_3}} \ldots\Big)$$

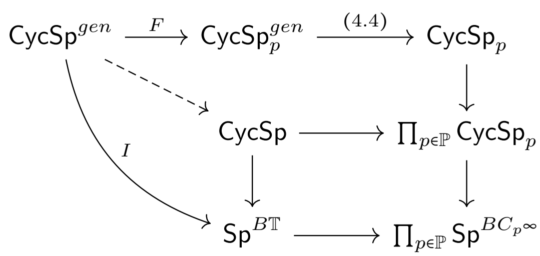
Note that for each $C_m \subseteq \mathbb{T}$, the geometric $C_m$-fixed points in this limit stabilize. Indeed if the prime factoring of $m$ is $p_1^{i_1} p_2^{i_2} \ldots p_k^{i_k}$, where $p := \max_{1\le i\le k}\{p_k\}$, and $p$ is the $t$'th prime in the list of primes, then the $C_m$-fixed points are stabilized in the $t$'th step in the above limit. Because of this the limit of the above, i.e. $R_\iota$, is preserved by $\Phi^{C_p}$, and hence we may calculate the limit on the underlying objects, analogously to the proof of theorem 4.5.3. $\square$

We will now construct the global functor which will descend to an equivalence when restricted to bounded below spectra. This construction follows from the local case, and the identitfication of $\mathsf{CycSp}$ given in remark 4.2.2.

**Proposition 4.6.5.** *There is a natural functor*

GlobalFunctor (4.7)
$$\mathsf{CycSp}^{gen} \to \mathsf{CycSp}.$$

*Proof.* For any $p \in \mathbb{P}$ we claim that we have the following commutative diagram,




Indeed for any $p$, we have a natural functor $F : \mathsf{CycSp}^{gen} \to \mathsf{CycSp}_p^{gen}$ induced from the functor $\mathbb{T}\mathsf{Sp}_{\mathcal{F}} \to C_{p^\infty}\mathsf{Sp}$ given by $(\Phi_p)_{p\in\mathbb{P}} \mapsto \Phi_p$ where $\Phi_p : \Phi^{C_p} X \to X$ is one of the structure equivalence associated to any genuine cyclotomic spectrum $X$. The map (4.4) is the functor given in the proposition above. The functor $I$ is induced from the map $\mathsf{CycSp}^{gen} \to \prod_{p\in\mathbb{P}} \mathsf{Sp}^{BC_{p^\infty}}$ by considering the underlying spectrum with $\mathbb{T}$-action of $X \in \mathbb{T}\mathsf{Sp}_{\mathcal{F}}$. These two maps give by universal property the dashed map. $\square$

Analogously to the previous section we have per. construction of the functor (4.7) and the results of the section concerning lax equalizers and coalgebras, that the functor is cocontinuous and therefore admits a right adjoint. Again we will try to understand the right adjoint to see that (4.7) is an equivalence through factoring and examining the right adjoints of the factors.

**Corollary 4.6.6.** *The functor (4.7) can be factored as the following composite*

$$\mathsf{CycSp}^{gen} \xrightarrow{\iota} \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}}) \xrightarrow{U} \mathsf{CycSp}.$$

*In the above $\iota$ is the inclusion, and $U$ takes the underlying naive cyclotomic spectrum.*

The proof is entirely analogous to the $p$-local case and hence we skip it.

Again we wish to understand the right adjoint of $U$ on bounded below naive cyclotomic spectra, which we denote $B$. Analogous to lemma 4.5.8 it is fully faithful and the underlying object of $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ is given by a Borel complete spectrum. This is however far more involved to show, and is the main difference between the $p$-local and the global setting. The problem arises because a genuine cyclotomic spectrum $X$ is equipped with equivalences $\Phi_n : X \simeq \Phi^{C_n} X$, for all $n \geq 1$, not just primes. We shall need the following lemma, which will help us deal with non-prime torsion groups, we will skip its proof, it can be found in [38] lemma II.6.7.

nonprimelem **Lemma 4.6.7.** *Let $G$ be a finite group which is not a $p$-group for some prime $p$. Let $X \in G\mathsf{Sp}$ be Borel complete, then $\Phi^G X \simeq 0$.*

L.II.6.6 **Lemma 4.6.8.** *Let $X$ be a Borel complete object in $\mathbb{T}\mathsf{Sp}_{\mathcal{F}}$ whose underlying spectrum is bounded below. Then for every $p$ the spectrum $\Phi^{C_p}$ is Borel complete.*

*Proof.* Again by the proof of lemma 4.4.4 we have that the underlying spectrum with $\mathbb{T}$-action of $\Phi^{C_p} X$ is given by the Tate construction $X^{tC_p} \in \mathsf{Sp}^{B\mathbb{T}}$. Now the unit of the adjunction afforded by proposition 4.3.24 gives a morphism $\Phi^{C_p} X \to B(X^{tC_p})$. We show that this map is an equivalence undr application of geometric fixed points for finite subgroups $H \subseteq \mathbb{T}$. We have in lemma 4.5.8 dealt with the cyclic $p$-groups. Therefore let $H$ be a subgroup of $\mathbb{T}$ which has $q$-torsion for $q \neq p$. In this case it follows from lemma 4.6.7 that $\Phi^H \Phi^{C_p} X \simeq \Phi^{\tilde{H}} X \simeq 0$, where $\tilde{H} = \{h \in \mathbb{T} | h^p \in H\}$.

The transformation $\mathrm{id} \to B$ gives that the map $\Phi^{C_p} X \to B(X^{tC_p})$ is a map of algebras. Recall that there is a lax symmetric monoidal transformation of lax symmetric monoidal functors $-^{C_p} \to \Phi^H$ by remark 4.3.12. In particular $\Phi^{C_q}$ is lax symmetric monoidal. Hence the map

$$\Phi^H \Phi^{C_p} X \to \Phi^H(B(X^{tC_p})),$$

is a map of algebras. The domain is zero by the above arguments, hence the codomain is also zero, for all Borel complete spectra. We have now shown the claim. ☐

As in the previous section we will use the following notation.

**Definition 4.6.9.** Let $\mathsf{CycSp}_+^{gen} \subseteq \mathsf{CycSp}^{gen}$ and $\mathsf{CycSp}_+ \subseteq \mathsf{CycSp}$ denote the respective $\infty$-subcategories of objects whose underlying spectra are bounded below. Then the functor (4.7) restrict to a functor

$$\mathsf{CycSp}_+^{gen} \to \mathsf{CycSp}_+.$$

We now prove the analog of lemma 4.5.8.

L.II.6.8 **Lemma 4.6.10.** *Let $X \in \mathsf{CycSp}_+$. Then the right adjoint of $U$, $B : \mathsf{CycSp} \to \mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$, is fully faithful and when applied to $X$ has underlying spectrum given by the Borel complete spectrum $B_{\mathbb{T}} X \in \mathbb{T}\mathsf{Sp}_{\mathbb{F}}$.*

*Proof.* By lemma 4.6.8 we have maps

$$B_{\mathbb{T}} X \to B_{\mathbb{T}}(X^{tC_p}) \simeq \Phi^{C_p} B_{\mathbb{T}} X,$$

for each $p \in \mathbb{P}$, which are induced from the structure maps. Note that $\Phi^{C_p}\Phi^{C_q} B_{\mathbb{T}} X \simeq 0$ for $p \neq q$ primes, by lemma 4.6.7, i.e. $\Phi^{C_p}\Phi^{C_q} B_{\mathbb{T}} X$ is terminal for $p \neq q$ and hence $B_{\mathbb{T}} X$ satisfies the assumptions of lemma 4.1.15. Thus by part one of the corollary $B_{\mathbb{T}} X$ has the structure of an object in $\mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$. Moreover by corollary 4.1.15 part 2, for any other object $Y$ in $\mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})$ the mapping space $\mathrm{Map}_{\mathsf{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathsf{Sp}_{\mathcal{F}})}(Y, B_{\mathbb{T}} X)$, is equivalent to

$$\mathrm{Eq}\Big(\mathrm{Map}_{\mathbb{T}\mathsf{Sp}_{\mathcal{F}}}(Y, B_{\mathbb{T}} X) \rightrightarrows \textstyle\prod_{p\in\mathbb{P}} \mathrm{Map}_{\mathbb{T}\mathsf{Sp}_{\mathcal{F}}}(Y, B_{\mathbb{T}}(X^{tC_p})\Big).$$

Which per. adjunction is equivalent to

$$\mathrm{Eq}\Big(\,\mathrm{Map}_{\mathbb{T}\mathsf{Sp}_{\mathcal{F}}}(UY, X) \rightrightarrows \textstyle\prod_{p\in\mathbb{P}} \mathrm{Map}_{\mathbb{T}\mathsf{Sp}_{\mathcal{F}}}(UY, X^{tC_p}\Big).$$

Which per. definition (or rather lemma 4.1.4) is equivalent to $\mathrm{Map}_{\mathsf{CycSp}}(UY, X)$. □

Finally we have the main theorem of this section, this is theorem 3 of the introduction.

T.II.6.3 **Theorem 4.6.11.** *The functor (4.7)* $\mathsf{CycSp}^{gen} \to \mathsf{CycSp}$ *induces an equivalence of* $\infty$*-categories of bounded below naive and genuine cyclotomic spectra,* $\mathsf{CycSp}^{gen}_{+} \to \mathsf{CycSp}_{+}$*.*

*Proof.* We show that the composite $U \circ \iota : \mathsf{CycSp}^{gen} \to \mathsf{CycSp}$ (4.7), is an equivalence of $\infty$-categories, when restricted to the subcategories of bounded below spectra.

As discussed above both $U$ and $\iota$ have right adjoints, denoted $B$ and $R_\iota$ respectively, hence so does $U \circ \iota$. We show that the composite $U \circ \iota$ is an equivalences. By the same argument as in the $p$-local setting we have that $U \circ \iota$ is conservative. Hence by lemma 1.2.8 it remains to show that the right adjoint is fully faithful. We do this analogously to the $p$-local case, i.e. by showing that the counit is an equivalence, i.e. that $U\iota RB((X, \Phi_p)) \to (X, \Phi_p)$ is an equivalence for $(X, \Phi_p) \in \mathsf{CycSp}_{p,+}$. By theorem 4.6.4 we have that $\iota R(B((X, \Phi_p))) \cong B((X, \Phi_p))$, and by lemma 4.6.10 we have that $UB((X, \Phi_p)) \simeq (X, \Phi_p)$, collecting these gives the desired result. □

$$\begin{array}{c} (M_1 : \text{Coffee} \longrightarrow \text{Theorems}) \\ \big\downarrow \\ (M_2 : \text{Coffee} \longrightarrow \text{Theorems}) \end{array}$$

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