FUSION SYSTEMS AND THEIR CLASSIFYING SPACES

ISABELLE LAUDE

Advisor: Jesper Grodal

Handed-in: 27-05-2013

Thesis for the Master degree in Mathematics.
Department of Mathematical Sciences, University of Copenhagen

Speciale for cand.scient graden i matematik.
Institut for matematiske fag, Københavns Universitet

Abstract

The goal of this thesis is to demonstrate that the geometric realization of the central linking system associated to a saturated fusion system can be seen as the classifying space of the fusion system. This is done by showing several theorems that generalize the case of the classifying space of a finite group. Most notably is a generalized Martino–Priddy conjecture. Furthermore we show the existence of a central linking for saturated fusion systems over finite p-groups with small p-rank.

Resume

Målet med dette speciale er at vise at den geometriske realisation af det centrale linking system for et mættet fusionsystem kan betragtes som det klassificerende rum for fusion sytemet. Dette bliver gjort ved at vise flere sætninger, som generaliserer tilfældet af det klassificerende rum af en endelig gruppe. Her bemærkes især en generaliseret udgave of Martino–Priddy formodningen. Desuden vises eksistensen af et centralt linking system for mættede fusionssystemer over endelige p-grupper med lav p-rang.

Contents

Abstract . . . 3
Resume . . . 3
1. Introduction . . . 5
2. Fusion system and associated linking systems . . . 6
3. The classifying space is p-good. . . . 8
4. A homotopy decomposition of the classifying space . . . 11
5. The centralizer p-local finite group . . . 15
5.1. Centralizer fusion systems and elementary abelian p-groups . . . 17
5.2. The centralizer decomposition . . . 19
6. Obstruction theory and higher limits . . . 22
6.1. An obstruction theory for central linking systems . . . 22
6.2. Higher limits and $\mathcal{O}^c(\mathcal{F})$ . . . 27
6.3. Projective summands in the Steinberg complex . . . 32
6.4. Consequences . . . 36
7. The mapping space $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ . . . 40
7.1. Homotopy colimit, p-completion and mapping spaces . . . 40
7.2. Mapping space and p-local finite groups . . . 45
8. The cohomology ring of fusion systems . . . 51
8.1. The cohomology ring Noetherian . . . 51
8.2. $R_{\mathcal{L}}$ is an F-isomorphism . . . 54
8.3. Characteristic $(S \times S)$-set for a fusion system . . . 55
8.4. The quotient fusion system . . . 60
8.5. Lannes T-functor and the cohomology ring of a fusion system . . . 68
8.6. Main Theorem on the cohomology ring of fusion systems . . . 73
9. A topological characterization of p-local finite groups. . . . 80
References . . . 87

1. Introduction

For a finite group G, a prime p and Sylow-p-subgroup S the study of the G-conjugation action on the subgroups of S has been of interest to group theorists for over a century. We call this fusion, which is a term attributed to Brauer.

Two groups G and G' have homotopy equivalent classifying spaces BG and BG' if and only if they are isomorphic. The question is then whether there is a topological space which is the analog to the classifying space in connection with fusion of a group. The Martino–Priddy conjecture indicated that the Bousfield–Kan p-completion of BG has the wanted properties, as it states that two groups have the same fusion structure if and only if their p-completed classifying spaces are homotopic. It was later proved in [9]. The proof was based on the study of a category $\mathcal{L}_S^c(G)$ called the central linking system, which was shown to describe many of the properties of $BG_p^\wedge$.

Lluis Puig showed how to generalize the idea of the fusion of a Sylow-p-subgroup in G by not only considering conjugation maps between subgroups but also allowing injective group homomorphisms. He called this a Frobenius category, but we will use the more common notion of a fusion system. The generalization of the classifying space for an abstract fusion system will be in the form a category $\mathcal{L}$ called the central linking system, which is a generalization of $\mathcal{L}_S^c(G)$. The main theorem of this thesis is a generalized version of the Martino–Priddy conjecture. It states that the isomorphism class of $(S, \mathcal{F}, \mathcal{L})$ depends only of the homotopy type of the p-completion of $|\mathcal{L}|$, hence $|\mathcal{L}|$ can be considered to be the classifying space of the fusion system. The central part of the proof is similar to [9], a description of $[BQ, |\mathcal{L}|_p^\wedge]$ for a finite p-group Q, which corresponds to the result $\mathrm{Rep}(Q, S) = [BQ, BS]$ for finite groups.

We also study the cohomology ring of a fusion system, which is defined abstractly in terms of the cohomology ring of the objects. Our other main theorem is that when the saturated fusion system has a central linking system, the cohomology rings of the fusion system and of the p-completed classifying space are isomorphic and Noetherian.

Note that it is not clear from the definition that a central linking system for a fusion system exists. This was proved by Chermak in [12] for saturated fusion systems over a finite p-group using the classification of finite simple groups. We will take a different approach by forming an obstruction theory for the existence of a central linking system in terms of higher limits of a functor over an orbit category. We then use methods developed in [18] for computing these limits to prove the existence and uniqueness of the central linking system of saturated fusion systems over finite p-groups with p-rank strictly less that p^3 respectively p^2.

The thesis will depend heavily on Bousfield–Kan's p-completion [7] in connection with Lannes' T-functor [26]. The knowledge of basic module theory, homological algebra and algebraic topology will be assumed. We use the convention that p is always a prime number, unless stated otherwise.

Note that thesis is mainly based on [10] but also parts of [18].

2. Fusion system and associated linking systems

In this chapter contains the basic definitions of fusions system and their associated central linking systems.

Note that for any group G and subgroups $H, H' \subseteq G$, we let $\mathrm{Hom}_G(H, H')$ be the elements of $\mathrm{Hom}(H, H')$ of the form $c_g(x) = gxg^{-1}$ for some $g \in G$.

Definition 2.1. *A fusion system $\mathcal{F}$ over a finite p-group S is a category with objects the subgroups of S, such that the morphism-set satisfies*

- *For every $P, Q \subseteq S$ we have that $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$.*
- *If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, then the restriction of φ to the image is an element of $\mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$.*

The standard example of a fusion system is $\mathcal{F}_S(G)$, where $S \in \mathrm{Syl}_p(G)$ for a finite group G and the objects are the subgroups of S and morphisms are given by conjugation with elements from G.

Definition 2.2. *For a fusion system $\mathcal{F}$ over a p-group S, we say that two subgroups $P, Q \subseteq S$ are $\mathcal{F}$-conjugate, if there exists an isomorphism in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$.*

We call $Q \subseteq S$ fully normalized in $\mathcal{F}$ if $|N_S(Q)| \geq |N_S(P)|$ for any $P \subseteq S$ which is $\mathcal{F}$-conjugate to Q. Similarly we call $Q \subseteq S$ fully centralized in $\mathcal{F}$ if $|C_S(Q)| \geq |C_S(P)|$ for any $P \subseteq S$ which is $\mathcal{F}$-conjugate to Q.

Definition 2.3. *A fusion system $\mathcal{F}$ over a p-group S is saturated, if*

- *Any fully normalized $Q \subseteq S$ is also fully centralized and*
$$\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q)).$$
- *If $P \subseteq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that $\varphi(P)$ is fully centralized, then there exists $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$, where*
$$N_\varphi = \{x \in N_S(P) \mid \varphi \circ c_x \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$
such that $\tilde{\varphi}|_P = \varphi$.

Definition 2.4. *For a $\mathcal{F}$ fusion system over a p-group S, we say that $P \subseteq S$ is $\mathcal{F}$-centric if $C_S(P') = \mathrm{Z}(P')$ for any P', which is $\mathcal{F}$-conjugate to P. We let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ on the $\mathcal{F}$-centric elements.*

Definition 2.5. *Let $\mathcal{F}$ be a fusion system over a p-group S. A central linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$, where the objects are the $\mathcal{F}$-centric subgroups of $\mathcal{F}$ together with a functor $\pi\colon \mathcal{L} \to \mathcal{F}$ and for any $\mathcal{F}$-centric subgroup P a monomorphism $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ satisfying the following axioms:*

(A) *The functor π is the identity on objects and for any pair of $\mathcal{F}$-centric subgroups P and Q the map $\pi\colon \mathrm{Mor}_{\mathcal{L}}(P, Q) \to \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the orbit map for the free $\mathrm{Z}(P)$-action on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ defined for $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and $g \in \mathrm{Z}(P)$ by setting $g \cdot \varphi = \varphi \circ \delta_P(g)$.*

(B) *For any $g \in P$ where P is an $\mathcal{F}$-centric subgroup, we have that $\pi(\delta_P(g)) = c_g$.*

(C) *For any* $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ *and* $g \in P$ *the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(\varphi))"] \\ P \arrow[r, "\varphi"] & Q \end{tikzcd}$$

If $\mathcal{L}$ *is a central linking system associated to* $\mathcal{F}$, *we call* $(S,\mathcal{F},\mathcal{L})$ *a p-local finite group. Furthermore we call* $|\mathcal{L}|$ *the classifying space of the fusion system* $\mathcal{F}$.

Definition 2.6. *An isomorphism of p-local finite groups* $(S,\mathcal{F},\mathcal{L})$ *and* $(S',\mathcal{F}',\mathcal{L}')$ *is a triple* $(\xi_S,\xi_{\mathcal{F}},\xi_{\mathcal{L}})$, *where* $\xi_S\colon S \to S'$ *is a group isomorphism and* $\xi_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}'$ *and* $\xi_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}'$ *are isomorphisms of categories, such that for any subgroup* $P \subseteq S$ *we have that* $\xi_S(P) = \xi_{\mathcal{F}}(P) = \xi_{\mathcal{L}}(P)$ *when defined. Furthermore* $\pi' \circ \xi_{\mathcal{L}} = \xi_{\mathcal{F}} \circ \pi$ *and* $\delta_{\xi_S(P)} \circ \xi_S|_P = \xi_{\mathcal{L}} \circ \delta_P$ *for any* $\mathcal{F}$*-centric subgroup* P *of* S.

3. The classifying space is p-good.

For any finite p-group S, we have that BS is p-good by [3, III 1.4 Proposition 1.10]. In this chapter we show the corresonding result for p-local finite groups.

Lemma 3.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, and assume P, Q and R are $\mathcal{F}$-centric subgroups of S. Then the following holds:*

(a) *For any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ let $\tilde{\psi} \in \pi_{Q,R}^{-1}(\psi)$ and $\widetilde{\psi\varphi} \in \pi_{P,R}^{-1}(\psi\varphi)$. Then there exists a unique $\tilde{\varphi} \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, such that $\tilde{\psi} \circ \tilde{\varphi} = \widetilde{\psi\varphi}$ and furthermore we have $\pi_{P,Q}(\tilde{\varphi}) = \varphi$.*

(b) *If $\tilde{\varphi}, \tilde{\varphi}' \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ satisfies that $c_x \circ \pi_{P,Q}(\tilde{\varphi}) = \pi_{P,Q}(\tilde{\varphi}')$ for some $x \in Q$, then there is a unique element $g \in Q$, such that $\delta_Q(g) \circ \tilde{\varphi} = \tilde{\varphi}'$.*

Proof. For part (a) let $\alpha \in \pi_{P,Q}^{-1}(\varphi)$. Then $\pi_{P,R}(\widetilde{\psi\varphi}) = \pi_{P,R}(\tilde{\psi} \circ \alpha)$, so axiom (A) implies that there exists $g \in \mathrm{Z}(P)$, such that $\widetilde{\psi\varphi} = \tilde{\psi} \circ \alpha \circ \delta_P(g)$. By (B) we have that $\pi_P(\delta_P(g)) = c_g = \mathrm{id}_P$, hence $\tilde{\varphi} = \alpha \circ \delta_P(g)$ satisfies the requirements. Assume that $\tilde{\varphi}' \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ is another morphism such that $\tilde{\psi} \circ \tilde{\varphi}' = \widetilde{\psi\varphi}$. As ψ is injective, this implies that $\pi_{P,Q}(\tilde{\varphi}') = \varphi = \pi_{P,Q}(\alpha \circ \delta_P(g))$. Then by (A) there exists $h \in \mathrm{Z}(P)$ such that $\tilde{\varphi}' = \alpha \circ \delta_P(gh)$. Now $\delta_P(h)$ acts trivially on $\widetilde{\psi\varphi}$, and as the action is free, we conclude that $h = 1$, and thus $\tilde{\varphi}' = \alpha \circ \delta_P(g)$.

Set $\varphi = \pi_{P,Q}(\tilde{\varphi})$. For (b) the assumptions and axiom (B) imply that

$$\pi_{P,Q}(\delta_Q(x) \circ \tilde{\varphi}) = \pi_{P,Q}(\tilde{\varphi}'),$$

hence by (A) there exists a $g \in \mathrm{Z}(P)$ such that $\delta_Q(x) \circ \tilde{\varphi} \circ \delta_P(g) = \tilde{\varphi}'$. Thus by (C) we conclude that $\delta_Q(x\varphi(g)) \circ \tilde{\varphi} = \tilde{\varphi}'$ and this proves the existence part. Assume that $g, g' \in Q$ are elements such that $\delta_Q(g) \circ \tilde{\varphi} = \tilde{\varphi}' = \delta_Q(g') \circ \tilde{\varphi}$. Then by applying π we conclude that $c_g = c_{g'}$ on $\varphi(P)$, so $g'^{-1}g \in C_S(\varphi(P)) \subseteq \varphi(P)$, as P is $\mathcal{F}$-centric. Then $g'^{-1}g = \varphi(h)$ for some $p \in P$, and $\tilde{\varphi} = \delta_Q(g'^{-1}g) \circ \tilde{\varphi} = \tilde{\varphi} \circ \delta_P(h)$ by (C). By part (a) we conclude that $\delta_P(h) = 1_P$. As δ_P is injective, we get that $h = 1$ and thus $g = g'$. $\square$

Definition 3.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and P a $\mathcal{F}$-centric subgroup. We define $\theta_P \colon \mathcal{B}(P) \to \mathcal{L}$ to be the functor given by $*_P \mapsto P$ and $g \mapsto \delta_P(g)$ for all $g \in P$.*

Proposition 3.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $|\mathcal{L}|$ is p-good and the composite*

$$S \xrightarrow{\pi_1(|\theta_S|)} \pi_1(|\mathcal{L}|) \xrightarrow{\pi_1(\phi_{|\mathcal{L}|})} \pi_1(|\mathcal{L}|_p^\wedge)$$

is surjective. Here ϕ is the natural transformation from p-completion.

Proof. For each $\mathcal{F}$-centric subgroup P of S, we choose an $\iota_P \in \mathrm{Mor}(P, S)$ lifting the inclusion of P into S such that $\iota_S = 1_S$. Then for any inclusion $P \subseteq Q$ of $\mathcal{F}$-centric subgroups there exists by Lemma 3.1 (a) a unique morphism $\iota_P^Q \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ such that $\iota_Q \circ \iota_P^Q = \iota_P$. Note that $\mathcal{L}$ is connected. For any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ let γ_φ be the path in $|\mathcal{L}|$ from P to Q. Let the vertex S be the base-point for $|\mathcal{L}|$ and define $\omega\colon\ \mathrm{Mor}(\mathcal{L}) \to \pi_1(|\mathcal{L}|)$ as sending any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ to the class of the loop $\gamma_{\iota_Q}\gamma_\varphi\gamma_{\iota(P)}^{-1}$ in $|\mathcal{L}|$. Then for $\varphi \in \mathrm{Hom}_F(P, Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ we have that $\omega(\psi \circ \varphi) = \omega(\psi)\omega(\varphi)$. Furthermore for an inclusion $P \subseteq Q$ of $\mathcal{F}$-centric subgroups we have that $\omega(\iota_P^Q) = 1$. As γ_{ι_S} is the constant path, we see that $\omega(\iota_P) = 1$

for any $\mathcal{F}$-centric subgroup P. By cellular approximation [19, Theorem 4.8] we have that any loop at S in $|\mathcal{L}|$ is homotopic to a finite composition of loops of the form $\omega(\varphi)$ for $\varphi \in \operatorname{Mor}(\mathcal{L})$. In particular $\operatorname{im}(\omega)$ generates $\pi_1(|\mathcal{L}|)$. Alperin's fusion theorem for saturated fusion systems [10, A.10] implies that every morphism in $\mathcal{F}$ is a composition of automorphism of fully normalized $\mathcal{F}$-centric subgroups and inclusions. Hence every morphism of $\mathcal{L}$ can be expressed as automorphisms of fully normalized $\mathcal{F}$-centric subgroups together with ι_P and ι_P^Q. As ω is trivial on the ι's and respects composition, we conclude that $\pi_1(|\mathcal{L}|)$ is generated by $\omega(\operatorname{Aut}_{\mathcal{L}}(Q))$ where Q is a fully normalized $\mathcal{F}$-centric subgroup of S.

Let P be a fully normalized $\mathcal{F}$-centric subgroup of S. By Lemma 3.1 part (a) there exists for any $x \in N_S(P)$ a unique $\varphi_x \in \operatorname{Aut}_{\mathcal{L}}(P)$ such that $\iota_P \circ \varphi_x = \delta_S(x) \circ \iota_P$ and $\pi(\varphi_x) = c_x$. For $x, y \in N_S(P)$ we have that $\iota_P \circ \varphi_x \circ \varphi_y = \delta_S(xy) \circ \iota_P$ hence the uniqueness part gives that $\varphi_{xy} = \varphi_x \circ \varphi_y$. If $\varphi_x = \varphi_y$ for some $x, y \in N_S(P)$, we have by construction that $\iota_P = \delta_S(x^{-1}y) \circ \iota_P$. As $\iota_P = \delta_S(1) \circ \iota_P$ the uniqueness part of Lemma 3.1 (b) implies that $1 = x^{-1}y$. Thus the map $\varphi_- \colon N_S(P) \to \operatorname{Aut}_{\mathcal{L}}(P)$ is an injective group homomorphism. Let $N_{\mathcal{L}}(P)$ denote the image. We observe that $\omega(N_{\mathcal{L}}(P)) \subseteq \omega(\delta_S(S))$ and $\omega \circ \delta_S(-)$ corresponds to $\pi_1(|\theta_S|)$.

As P is fully normalized we have that $\operatorname{Aut}_S(P) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(P))$. Furthermore $\operatorname{Aut}_S(P) = N_S(P)/C_S(P) = N_S(P)/\operatorname{Z}(P)$ as P is $\mathcal{F}$-centric. By Axiom (A) for $\mathcal{L}$ we have that $\operatorname{Aut}_{\mathcal{L}}(P)/\operatorname{Z}(P) \cong \operatorname{Aut}_{\mathcal{F}}(P)$, hence a Sylow-$p$-subgroup of $\operatorname{Aut}_{\mathcal{L}}(P)$ has the same order as $N_S(P)$, so $N_{\mathcal{L}}(P)$ is a Sylow-p-subgroup of $\operatorname{Aut}_{\mathcal{L}}(P)$. Thus $N_{\mathcal{L}}(P)$ generates $\operatorname{Aut}_{\mathcal{L}}(P)$ together with elements of order prime to p. Since $\omega(N_{\mathcal{L}}(P)) \subseteq \omega(\delta_S(S))$, we conclude that $\pi_1(|\mathcal{L}|)$ is generated by $\pi_1(|\theta|)(S)$ and the subgroup K generated by all elements prime to p. As conjugation preserves order of the elements, we have that K is normal in $\pi_1(|\mathcal{L}|)$, and so $\pi_1(|\theta|)$ maps surjectively onto $\pi_1(|\mathcal{L}|)/K$. In particular $\pi = \pi_1(|\mathcal{L}|)/K$ is a finite p-group.

By the universal coefficient theorem [19, Theorem 3A.3] in connection with the fact that $\operatorname{H}_0(K, \mathbb{Z}) = \mathbb{Z}$, we conclude that $\operatorname{H}_1(K; \mathbb{F}_p) = \operatorname{H}_1(K) \otimes \mathbb{F}_p = K/[K,K] \otimes \mathbb{F}_p$. The abelianization $K/[K,K]$ of K is also generated by elements of order prime to p. Let k be such a generator. Then there exists $n, m \in \mathbb{Z}$ such that $1 = m|k| + np$, and hence for a $x \in \mathbb{F}_p$ we have

$$k \otimes x = (m|k| + np)k \otimes x = (m|k|k) \otimes x + k \otimes (npx) = 0.$$

As $K/[K,K] \otimes \mathbb{F}_p$ are generated by such elements we conclude that $H_1(K; \mathbb{F}_p) = 0$, i.e. K is p-perfect.

Let X be the cover of $|\mathcal{L}|$ with fundamental group K. Since the object and morphism sets of $\mathcal{L}$ are finite, we have that $|\mathcal{L}|$ has a finite skeleton. Thus the same holds for X, so by cellular homology we have that $\operatorname{H}_i(X; \mathbb{F}_p)$ is a finite p-group for every i. As $\pi_1(X)$ is p-perfect, [7, Proposition VII 3.2] implies that X is p-good and $X_p^\wedge$ is simply connected. As K is normal in $\pi_1(|\mathcal{L}|)$ we have that $X \to |\mathcal{L}|$ is a normal covering space by [19, Proposition 1.39] and thus a principal π-bundle. As π is finite we can consider the homotopy fibration $X \to |\mathcal{L}| \to B(\pi)$. It has connected fiber and furthermore both π and $\operatorname{H}_i(X; \mathbb{F}_p)$ are a finite p-groups for every i. So by the mod-R-fiber lemma and the following example [7, II 5.1 and 5.2], we conclude that $X_p^\wedge \to |\mathcal{L}|_p^\wedge \to B(\pi)_p^\wedge$ is a homotopy fibration. By [3, III 1.4 Proposition 1.10] we have that $B(\pi)$ is p-complete, so $\phi_{B(\pi)}$ is a homotopy equivalence. Hence $B(\pi)_p^\wedge$ has fundamental group π. Since $X_p^\wedge$ is simple connected, we have that $\pi_1(X_p^\wedge) = 0$. The long exact sequence in homotopy for the fibration now implies that $\pi_1(|\mathcal{L}|^\wedge_p) \cong \pi$. As ϕ is a natural transformation between the identity and p-completion, we have that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\phi_X"'] & {|\mathcal{L}|} \arrow[r] \arrow[d, "\phi_{|\mathcal{L}|}"'] & B(\pi) \arrow[d, "\phi_{B(\pi)}"'] \\ X^\wedge_p \arrow[r] & (|\mathcal{L}|^\wedge_p) \arrow[r] & B(\pi)^\wedge_p \end{tikzcd}$$

As the long exact sequence in homotopy for a fibration is natural, we get the following commutative diagram.

$$\begin{tikzcd} \pi_1(|\mathcal{L}|) \arrow[r] \arrow[d, "\pi_1(\phi_{|\mathcal{L}|})"'] & \pi \arrow[d, "\pi_1(\phi_{B(\pi)}) = \mathrm{id}_\pi"] \\ \pi_1(|\mathcal{L}|^\wedge_p) \arrow[r, "\cong"] & \pi \end{tikzcd}$$

The upper horizontal map is the quotient map of $\pi_1(|\mathcal{L}|)$ onto $\pi_1(|\mathcal{L}|)/K$, so the same is true for $\pi_1(\phi_{|\mathcal{L}|})$. Thus by the above the map $\pi_1(\phi_{|\mathcal{L}|}) \circ \pi_1(|\theta|)$ is surjective.

As X is p-good, we have that $\mathrm{H}_*(\phi_X)\colon \mathrm{H}_*(X;\mathbb{F}_p) \to \mathrm{H}_*(X^\wedge_p;\mathbb{F}_p)$ is an isomorphism. Hence $X^\wedge_p \to |\mathcal{L}|^\wedge_p \to B(\pi)^\wedge_p$ satisfy the conditions on the mod-$\mathbb{F}_p$-fiber lemma so we have that $(X^\wedge_p)^\wedge_p \to (|\mathcal{L}|^\wedge_p)^\wedge_p \to (B(\pi)^\wedge_p)^\wedge_p$ is fibration, and the diagram

$$\begin{tikzcd} X^\wedge_p \arrow[r] \arrow[d, "(\phi_X)^\wedge_p"'] & {|\mathcal{L}|^\wedge_p} \arrow[r] \arrow[d, "(\phi_{|\mathcal{L}|})^\wedge_p"'] & B(\pi)^\wedge_p \arrow[d, "(\phi_{B(\pi)})^\wedge_p"'] \\ (X^\wedge_p)^\wedge_p \arrow[r] & (|\mathcal{L}|^\wedge_p)^\wedge_p \arrow[r] & (B(\pi)^\wedge_p)^\wedge_p \end{tikzcd}$$

commutes as p-completion is a functor. They induce maps between the two long exact sequences in homotopy corresponding to the fibrations. As X is p-good we have that ϕ_X is a mod-p-equivalence. Similarly $B(\pi)$ is p-complete, so $\phi_{B(\pi)}$ is a homotopy equivalence and in particular a mod p-equivalence. Thus $(\phi_X)^\wedge_p$ and $(\phi_{B(\pi)})^\wedge_p$ are homotopy equivalences by [7, Lemma I.5.5.]. In particular both $\pi_*(\phi_{X^\wedge_p})$ and $\pi_*(\phi_{B(\pi)^\wedge_p})$ are isomorphisms. The 5-lemma now implies that $\pi_*((\phi_{|\mathcal{L}|})^\wedge_p)$ is an isomorphism. As both $|\mathcal{L}|$ are $|\mathcal{L}|^\wedge_p$ are connected CW-complexes, we conclude by Whiteheads Theorem [19, Theorem 4.5] that $(\phi_{|\mathcal{L}|})^\wedge_p$ is a homotopy equivalence, so by [7, Lemma I.5.5] $\phi_{|\mathcal{L}|}$ is a mod-p-equivalence and thus $|\mathcal{L}|$ is p-good. $\square$

4. A homotopy decomposition of the classifying space

Definition 4.1. *The orbit category of a fusion system $\mathcal{F}$ on a p-group S is the category with objects the subgroups of S and*

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) = \mathrm{Inn}(Q) \setminus \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

For any subcategory $\mathcal{F}_0$ of $\mathcal{F}$ let $\mathcal{O}(\mathcal{F}_0)$ be the full subcategory of $\mathcal{O}(\mathcal{F})$ on the objects of $\mathcal{F}_0$, and set $\mathcal{O}^c(\mathcal{F}) = \mathcal{O}(\mathcal{F}^c)$.

Observe that for $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ and elements $q \in Q$ and $r \in R$ we have that $(c_r \circ \psi) \circ (c_q \circ \varphi) = c_{r\psi(q)} \circ (\psi \circ \varphi)$. Thus the composition is well-defined on $\mathcal{O}(\mathcal{F})$, and hence it is a category.

For any $g \in P$ we have that $g \in \mathrm{Mor}(\mathcal{B}P)$ gives a natural transformation between $\mathrm{id}_{\mathcal{B}P}$ and the conjugation map c_g, hence by [33, Proposition 2.1] we have that $B(c_g)$ is homotopic to the identity on BP. Thus $P \mapsto BP$ induces a well-defined functor from $\mathcal{O}^c(\mathcal{F})$ to hoTop. The following proposition provides a homotopy decomposition of the classifying space in terms of this functor.

Proposition 4.2. *Let $(\mathcal{F},\mathcal{L},S)$ be a p-local finite group and let $\tilde{\pi}\colon \mathcal{L} \to \mathcal{O}^c(\mathcal{F})$ be the functor defined by $\tilde{\pi}(P) = P$ and mapping $\tilde{\pi}(\varphi)$ to the class of $\pi(\varphi) \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$. Let $\tilde{B}\colon \mathcal{O}^c(\mathcal{F}) \to \mathsf{Top}$ be the left homotopy Kan extension to the constant functor $*\colon \mathcal{L} \to \mathsf{Top}$. Then $\tilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$ and $|\mathcal{L}| \simeq \mathrm{hocolim}_{P\in\mathcal{O}^c(\mathcal{F})} \tilde{B}(P)$.*

Let $\mathcal{L}_0$ be a full subcategory of $\mathcal{L}$ and let $\mathcal{F}_0$ be the full subcategory of $\mathcal{F}^c$ with $\mathrm{Ob}(\mathcal{F}_0) = \mathrm{Ob}(\mathcal{L}_0)$. Then $|\mathcal{L}_0| \simeq \mathrm{hocolim}_{P\in\mathcal{O}(\mathcal{F}_0)} \tilde{B}(P)$.

Proof. By [23, Chapter 5 and (4.3)] we have that the left homotopy Kan extension for a constant functor is given in the form of an over-category[1]. Hence for any $P \in \mathcal{F}^c$ we have that $\tilde{B}(P) = |\tilde{\pi} \downarrow P|$, where $\tilde{\pi} \downarrow P$ is the category with $\mathrm{Ob}(\tilde{\pi} \downarrow P) = \{(Q,\alpha) \mid Q \in \mathcal{F}^c, \alpha \in \mathrm{Rep}_{\mathcal{F}}(Q,P)\}$ and

$$\mathrm{Mor}_{\tilde{\pi}\downarrow P}((Q,\alpha),(R,\beta)) = \{\psi \in \mathrm{Mor}_{\mathcal{L}}(Q,R) \mid \beta \circ \tilde{\pi}(\psi) = \alpha\}.$$

For any morphism $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,P')$ we have that $\tilde{B}(\varphi) = |\tilde{\pi} \downarrow \varphi|$, where $\tilde{\pi} \downarrow \varphi\colon \tilde{\pi} \downarrow P \to \tilde{\pi} \downarrow P'$ is given by $(Q,\alpha) \mapsto (Q,\varphi\circ\alpha)$ and $\psi \mapsto \psi$.

By [23, Theorem 5.5] we have that

$$\underset{\mathcal{O}^c(\mathcal{F})}{\mathrm{hocolim}}\, \tilde{B} \simeq \underset{\mathcal{L}}{\mathrm{hocolim}}(*),$$

and as $|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{L}}(*)$ we obtain the desired isomorphism. Similar arguments apply for any full subcategory $\mathcal{L}_0$ of $\mathcal{L}$ and the restriction of $\tilde{\pi}$ to $\mathcal{L}_0$, which is a functor $\tilde{\pi}_0\colon \mathcal{L}_0 \to \mathcal{O}(\mathcal{F}_0)$. If $\tilde{B}_0$ is the left homotopy Kan extension of $*\colon \mathcal{L}_0 \to \mathsf{Top}$ over $\tilde{\pi}_0$, then $\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F}_0)} \tilde{B}_0 \simeq |\mathcal{L}_0|$ and $\tilde{B}_0 = |\tilde{\pi}_0 \downarrow (-)|$.

Let $P \in \mathcal{F}^c$. Then for any $g \in P$ we have that $\pi(\delta_P(g)) = c_g \in \mathrm{Inn}(P)$, so $\tilde{\pi}(\delta_P(g)) = \mathrm{id}_P$ in $\mathcal{O}^c(\mathcal{F})$. Thus $\delta_P(P) \subseteq \mathrm{Aut}_{\tilde{\pi}\downarrow P}(P,\mathrm{id})$. Let $\mathcal{B}'(P)$ be the subcategory of $\tilde{\pi} \downarrow P$ with one object (P,id_P) and morphisms $\delta_P(P)$. As δ_P is injective, we conclude that $|\mathcal{B}'(P)| \simeq BP$. We will now prove that $|\mathcal{B}'(P)|$ is in fact a deformation retract of $|\tilde{\pi} \downarrow P|$. For this we pick a section $\tilde{\sigma}\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L})$

[1]In the terminology of [23] we have that $F_{h^*}(*) = B((\tilde{\mathcal{F}}^e)^{op}, \hat{\mathcal{L}}, *) = B(\tau \downarrow (\tilde{\mathcal{F}}^e)^{op})$, so $F_{h^*}(*)(E) = |\tau \downarrow E|$. The theorem 5.5 states, that $\mathrm{hocolim}_{\hat{\mathcal{L}}}(*) \cong \mathrm{hocolim}_{(\tilde{\mathcal{F}}^e)^{op}} F_{h^*}(*)$.

of $\tilde{\pi}$, such that $\tilde{\sigma}(\mathrm{id}_P) = 1_P$ for any $P \in \mathcal{F}^c$. Consider $\psi \in \mathrm{Mor}_{\tilde{\pi}\downarrow P}((Q,\alpha),(R,\beta))$. Then

$$\tilde{\pi}(\tilde{\sigma}(\alpha)) = \alpha = \beta \circ \tilde{\pi}(\psi) = \tilde{\pi}(\tilde{\sigma}(\beta) \circ \psi)$$

in $\mathrm{Rep}_{\mathcal{F}}(Q,P)$. By Lemma 3.1 (b) this implies that there exists a unique $g_\psi \in P$ such that $\delta_P(g_\psi) \circ \tilde{\pi}(\tilde{\sigma}(\alpha)) = \tilde{\pi}(\tilde{\sigma}(\beta)) \circ \psi$. Then clearly $g_{1_P} = 1$ and for $\psi \in \mathrm{Mor}_{\tilde{\pi}\downarrow P}((Q,\alpha),(R,\beta))$ and $\varphi \in \mathrm{Mor}_{\tilde{\pi}\downarrow P}((R,\beta),(S,\gamma))$ we have the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[r, "\psi"] \arrow[d, "\tilde{\sigma}(\alpha)"'] & R \arrow[r, "\varphi"] \arrow[d, "\tilde{\sigma}(\beta)"'] & S \arrow[d, "\tilde{\sigma}(\gamma)"'] \\
P \arrow[r, "\delta_P(g_\psi)"] & P \arrow[r, "\delta_P(g_\varphi)"] & P
\end{tikzcd}$$

The uniqueness of $g_{\psi\circ\varphi}$ implies that it is equal to $g_\psi g_\varphi$. Thus we have a well-defined functor $\Psi\colon \tilde{\pi} \downarrow P \to \mathcal{B}'(P)$ by setting $\Psi(Q,\alpha) = (P, \mathrm{id}_P)$ and $\Psi(\varphi) = \delta_P(g_\varphi)$. As $\tilde{\sigma}(\mathrm{id}_P) = 1_P$ we have that the restriction of Ψ to $\mathcal{B}'(P)$ is the identity. For any $(Q,\alpha) \in \tilde{\pi} \downarrow P$ we have that $\tilde{\sigma}(\alpha)$ is a morphism in $\tilde{\pi} \downarrow P$ from (Q,α) to (P,id_P), and the defining property of $\Psi(\varphi)$ for a $\varphi \in \mathrm{Mor}_{\tilde{\pi}\downarrow P}((Q,\alpha),(R,\beta))$ implies that $\tilde{\sigma}(-)$ is a natural transformation from $\mathrm{id}_{\tilde{\pi}\downarrow P}$ to $\mathrm{incl} \circ \Psi$, where incl is the inclusion of $\mathcal{B}'(P)$ into $\tilde{\pi} \downarrow P$. By [33, Proposition 2.1] we have that the geometric realizations $\mathrm{id}_{|\tilde{\pi}\downarrow P|} = |\,\mathrm{id}_{\tilde{\pi}\downarrow P}\,|$ and $|\,\mathrm{incl} \circ \Phi| = \mathrm{incl}_{|\mathcal{B}'(P)| \hookrightarrow |\tilde{\pi}\downarrow P|} \circ |\Phi|$ are homotopic. Since Φ is the identity on $\mathcal{B}'(P)$, we conclude that $|\tilde{\pi} \downarrow P|$ is a deformation retract of $|\mathcal{B}'(P)|$. In particular we have that $\tilde{B}(P) = |\tilde{\pi} \downarrow P| \simeq |\mathcal{B}'(P)| \simeq BP$.

For any $P \in \mathcal{F}_0$ we have that $\mathcal{B}'(P)$ is a subcategory of $\tilde{\pi}_0 \downarrow P$, and by the above argument the inclusion $|\mathcal{B}'(P)| \hookrightarrow \tilde{B}_0(P)$ is a homotopy equivalence. As the inclusion $|\mathcal{B}'(P)| \hookrightarrow \tilde{B}(P)$ is the composition $|\mathcal{B}'(P)| \hookrightarrow \tilde{B}_0(P) \hookrightarrow \tilde{B}(P)$, we conclude that $\tilde{B}_0(P) \hookrightarrow \tilde{B}(P)$ is a homotopy equivalence for any $P \in \mathcal{F}_0$. Thus we get that

$$|\mathcal{L}_0| \simeq \underset{\mathcal{O}^c(\mathcal{F}_0)}{\mathrm{hocolim}}\, \tilde{B}_0 \simeq \underset{\mathcal{O}^c(\mathcal{F}_0)}{\mathrm{hocolim}}\, \tilde{B}.$$

Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and let $\bar{\varphi}$ be the class in $\mathrm{Rep}_{\mathcal{F}}(P,Q)$. Consider the functors $\mathcal{B}'(P) \to \tilde{\pi} \downarrow Q$ given by $F_1 = (\tilde{\pi} \downarrow \bar{\varphi}) \circ \mathrm{incl}$ and $F_2 = \mathrm{incl} \circ \mathcal{B}'(\varphi)$, where $\mathcal{B}'(\varphi)\colon \mathcal{B}'(P) \to \mathcal{B}'(Q)$ is the functor $\mathcal{B}(\varphi)$ under the natural identification of $\mathcal{B}'(P)$ with $\mathcal{B}P$ and $\mathcal{B}'(Q)$ with $\mathcal{B}Q$. Thus they are exactly the functors given by

$$F_1(P,\mathrm{id}) = (P,\bar{\varphi}), \quad F_1(\delta_P(g)) = \delta_P(g),$$

$$F_2(P,\mathrm{id}) = (Q,\mathrm{id}), \quad F_2(\delta_P(g)) = \delta_Q(\varphi(g)).$$

As π is surjective on morphisms, we may pick a $\tilde{\varphi} \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\pi(\tilde{\varphi}) = \varphi$. Then $\tilde{\varphi}$ is a morphism in $\tilde{\pi} \downarrow Q$ from $(P,\bar{\varphi})$ to (Q,id), and by (C) for $\mathcal{L}$ we have that the following diagram commutes for any $g \in P$:

$$\begin{tikzcd}
P \arrow[r, "\tilde{\varphi}"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\varphi(g))"] \\
P \arrow[r, "\tilde{\varphi}"] & Q
\end{tikzcd}$$

By comparing definitions we see, that this implies that $\tilde{\varphi}\colon F_1(P,\mathrm{id}) \to F_2(P,\mathrm{id})$ gives a natural transformation from F_1 to F_2. Thus by [33, Proposition 2.1] we have that $|F_1|$ and $|F_2|$ are homotopic, so we conclude that

$$\begin{array}{ccc} |\mathcal{B}'P| & \xrightarrow{|\,\mathrm{incl}\,|} & \tilde{B}(P) \\ \big\downarrow{\scriptstyle |\mathcal{B}'(\varphi)|} & & \big\downarrow{\scriptstyle \tilde{B}(\bar{\varphi})} \\ |\mathcal{B}'(Q)| & \xrightarrow{|\,\mathrm{incl}\,|} & \tilde{B}(Q) \end{array}$$

commutes up to homotopy. We remark that $|\,\mathrm{incl}\,|$ is a homotopy equivalence, hence we conclude that $\tilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$. $\square$

This Proposition implies, that if there exists a central linking system associated to a saturated fusion system $\mathcal{F}$ on a p-group S, then there exists a lifting of the homotopy functor $P \mapsto BP$ on $\mathcal{O}^c(\mathcal{F})$. In fact the converse is also true, as seen in the following proposition.

Proposition 4.3. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. If there exists a lifting of the homotopy functor $P \mapsto BP$ on $\mathcal{O}^c(\mathcal{F})$ to* Top*, then there exists a central linking system associated to $\mathcal{F}$.*

Proof. Let $\tilde{B}\colon \mathcal{O}^c(\mathcal{F}) \to \mathsf{Top}$ be a lifting of $P \mapsto BP$. Then $\tilde{B}$ is a functor together with a homotopy class $\eta_P \in [BP, \tilde{B}P]$ representing a homotopy equivalence for any $P \in \mathcal{O}^c(\mathcal{F})$ such that the following diagram (D) commutes in hoTop for any $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$:

$$\begin{array}{ccc} BP & \xrightarrow{\eta_P} & \tilde{B}(P) \\ \big\downarrow{\scriptstyle B(\varphi)} & & \big\downarrow{\scriptstyle [\tilde{B}(\bar{\varphi})]} \\ BQ & \xrightarrow{\eta_Q} & \tilde{B}(Q) \end{array}$$

For each $P \in \mathcal{F}^c$ we now choose a map $\hat{\eta}_P\colon BP \to \tilde{B}(P)$, such that $[\hat{\eta}_P] = \eta_P$. Let $*_P \in \tilde{B}(P)$ be the image of the base-point in BP. Then $\hat{\eta}_P$ induces a map from $\pi_1(BP,*) = P$ to $\pi_1(\tilde{B}(P),*_P)$ which we will denote γ_P. As $\hat{\eta}_P$ is a homotopy equivalence, we conclude that γ_P is an isomorphism of groups. We now define a category $\mathcal{L}$ with objects $\mathcal{F}^c$ and

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = \{(\varphi,u) \mid \varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q), u \in \pi_1(\tilde{B}(Q); \tilde{B}(\varphi)(*_P), *_Q)\},$$

where $\pi_1(\tilde{B}(Q); \tilde{B}(\varphi)(*_P), *_Q)$ is the set of homotopy classes of paths from $\tilde{B}(\varphi)(*_P)$ to $*_Q$ in $\tilde{B}(Q)$. The composition is defined for $(\varphi,u) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and $(\psi,v) \in \mathrm{Mor}_{\mathcal{L}}(Q,R)$ by the following equation

$$(\psi,v) \circ (\varphi,u) = (\psi \circ \varphi, v \cdot \tilde{B}(\psi)_*(u)).$$

We let $\pi\colon \mathcal{L} \to \mathcal{F}^c$ be the identity on objects and for $(\varphi,u) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ we set $\pi(\varphi,u)$ to be

$$P \xrightarrow{\gamma_P} \pi_1(\tilde{B}(P),*_p) \xrightarrow{\varphi_*} \pi_1(\tilde{B}(Q),\tilde{(}\varphi)(*_P)) \xrightarrow{u_*} \pi_1(\tilde{B}(Q),*_Q) \xrightarrow{\gamma_Q^{-1}} Q$$

For any $P \in \mathcal{F}^c$ we set $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ to be $p \mapsto (\mathrm{id}_P, \gamma_P(p))$.

It is straightforward to check that the given construction is in fact a central linking system associated to $\mathcal{F}$, the definition of π being the main difficulty. We will show only some of the properties here.

Note that for a specific choice of representative $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ of $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ we have that the homotopy from the diagram (D) provides a path $u_{\tilde{\varphi}}$ from $\tilde{B}(\varphi)(*_p)$ to $*_Q$. Furthermore the homotopy implies that $(u_{\tilde{\varphi}})_* \circ \tilde{B}(\varphi)_* \circ \gamma_P = \gamma_Q \circ \tilde{\varphi}$, and thus $\pi(\varphi,u) = c_{\gamma_Q^{-1}(u \cdot u_{\tilde{\varphi}}^{-1})} \circ \tilde{\varphi}$. Note that π is clearly surjective. If $\pi(\varphi,u) = \pi(\psi,v)$ for $(\varphi,u),(\psi,v) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, we see that $\varphi = \psi \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ and for a representative $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ we have that

$$\gamma_Q^{-1}(u_{\tilde{\varphi}} \cdot v^{-1} \cdot u \cdot u_{\tilde{\varphi}}) \in N_Q(\tilde{\varphi}(P)) = \mathrm{Z}(\tilde{\varphi}(P)) = \tilde{\varphi}(\mathrm{Z}(P)).$$

If $\gamma_Q^{-1}(u_{\tilde{\varphi}} \cdot v^{-1} \cdot u \cdot u_{\tilde{\varphi}}) = \tilde{\varphi}(g)$ for $g \in \mathrm{Z}(P)$, then the above remarks imply that $\tilde{B}(\varphi)(\gamma_P(\tilde{\varphi}(g))) = v^{-1} \cdot u$, so $(\varphi,v) \circ \delta_p(g) = (\varphi,u)$.

For any $g \in P$ we have that $\pi(\delta_P(g))$ is the map:

$$h \mapsto \gamma_P^{-1}(\gamma_P(g) \cdot \gamma_P(h) \cdot \gamma_P(g)^{-1}) = c_g(h).$$

For $(\varphi,u) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and $g \in P$ the diagram corresponding to property (C) is the following:

$$\begin{tikzcd}
P \arrow[r, "{(\varphi,u)}"] \arrow[d, "{(\mathrm{id}_P, \gamma_P(g))}"'] & Q \arrow[d, "{(\mathrm{id}_Q, u \cdot \tilde{B}(\varphi)_*(\gamma_P(g)) \cdot u^{-1})}"] \\
P \arrow[r, "{(\varphi,u)}"] & Q
\end{tikzcd}$$

The first components of the compositions are both φ, while the second are

$$u \cdot \tilde{B}(\varphi)_*(\gamma_P(g)), \quad u \cdot B(\varphi)_*(\gamma_P(g)) \cdot u^{-1} \cdot \tilde{B}(\mathrm{id}_Q)_*(u).$$

As $\tilde{B}(\mathrm{id}_Q) = \mathrm{id}_{\tilde{B}(Q)}$ we have that $\tilde{B}(\mathrm{id}_Q)_*(u) = u$, so the diagram commutes. $\square$

5. The centralizer p-local finite group

We will now define and study the centralizer p-local finite group and use this to provide another homotopy decomposition of the classifying space of a fusion system, which later on will have an important role in the study of the cohomology ring of fusion systems.

Lemma 5.1. *Let F be a saturated fusion system over S. Let $P, Q \in \mathcal{F}$ and $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P, Q)$. If P is fully centralized, then there exists a morphism $C_S(\varphi) \in \operatorname{Hom}_{\mathcal{F}}(C_S(Q), C_S(P))$.*

Proof. Set $\psi := \varphi^{-1} \in \operatorname{Hom}_{\mathcal{F}}(\varphi(P), P)$. Then $\psi(\varphi(P)) = P$, which is fully centralized in $\mathcal{F}$. As $\mathcal{F}$ is saturated, there exists a $\bar{\psi} \in \operatorname{Hom}_{\mathcal{F}}(N_\psi, S)$ which is an extension of ψ. Directly from the definition we get that $C_S(\varphi(P)) \subseteq N_\psi$ and $\bar{\psi}(C_S(\varphi(P))) \subseteq C_S(P)$. As $C_S(Q) \subseteq C_S(\varphi(P))$, we have that the restriction $\bar{\psi}|_{C_S(Q)}$ is in $\operatorname{Hom}_{\mathcal{F}}(C_S(Q), C_S(P))$. □

Definition 5.2. *Let $\mathcal{F}$ be a fusion system over a finite p-group S. For any $Q \subseteq S$, which is fully centralized in $\mathcal{F}$, let $C_{\mathcal{F}}(Q)$ be the category with objects the subgroups of $C_S(Q)$ and for $P, P' \in C_S(Q)$ setting $\operatorname{Mor}_{C_{\mathcal{F}}(Q)}(P, P')$ to be*

$$\{\varphi \in \operatorname{Hom}_{\mathcal{F}}(P, P') \mid \exists \bar{\varphi} \in \operatorname{Hom}_{\mathcal{F}}(PQ, P'Q), \bar{\varphi}|_P = \varphi, \bar{\varphi}|_Q = \mathrm{id}_Q\}.$$

Note that as $P \subseteq C_S(Q)$ we have that PQ is in fact a subgroup of S.

It follows easily from the above definition that if $\mathcal{F}$ is a fusion system over S and $Q \subseteq S$ is fully centralized in $\mathcal{F}$, then $C_{\mathcal{F}}(Q)$ is a fusion system over $C_S(Q)$. A special case of [10, Proposition A.6] is that $C_{\mathcal{F}}(Q)$ is saturated when the original fusion system is saturated. The analog of the central linking system is described in the following.

Definition 5.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. For any $Q \subseteq S$, which is fully centralized in $\mathcal{F}$, let $C_{\mathcal{L}}(Q)$ be the category with objects the subgroups of $C_S(Q)$ which are $C_{\mathcal{F}}(Q)$-centric and for any two such subgroups P and P' setting $\operatorname{Mor}_{C_{\mathcal{L}}(Q)}(P, P')$ to*

$$\{\varphi \in \operatorname{Mor}_{\mathcal{L}}(PQ, P'Q) \mid \pi(\varphi)(P) \leq P', \pi(\varphi)|_Q = \mathrm{id}_Q\}.$$

Proposition 5.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and a Q a subgroup of S which is fully centralized in $\mathcal{F}$. Then the following holds:*

(a) *A subgroup $P \subseteq C_S(Q)$ is $C_{\mathcal{F}}(Q)$-centric if and only if $\mathrm{Z}(Q) \leq P$ and PQ is $\mathcal{F}$-centric, and in this case $\mathrm{Z}(P) = \mathrm{Z}(PQ)$.*
(b) *The category $C_{\mathcal{L}}(Q)$ is a central linking system associated to $C_{\mathcal{F}}(Q)$.*

Proof. Let $P \subseteq C_S(Q)$. Assume that PQ is $\mathcal{F}$-centric and $\mathrm{Z}(Q) \leq P$. As PQ is $\mathcal{F}$-centric we conclude that

$$C_{C_S(Q)}(P) = C_S(P) \cap C_S(Q) = C_S(PQ) = Z(PQ).$$

Since $P \subseteq C_S(Q)$ we have that for all $p, \bar{p} \in P$ and $q, \bar{q} \in Q$ the following holds:

$$(pq)^{-1}\bar{p}\bar{q}(pq) = q^{-1}p^{-1}\bar{p}\bar{q}pq = (p^{-1}\bar{p}p)(q^{-1}\bar{q}q).$$

This implies that $\mathrm{Z}(PQ) = \mathrm{Z}(P)\,\mathrm{Z}(Q)$. Since $\mathrm{Z}(Q) \subseteq P$, we see that $\mathrm{Z}(P)\,\mathrm{Z}(Q) \subseteq P$ and the above relation with $\bar{q} = 1$ implies that $\mathrm{Z}(P)\,\mathrm{Z}(Q) \subseteq C_S(P)$ and hence $\mathrm{Z}(P)\,\mathrm{Z}(Q) \subseteq \mathrm{Z}(P)$. As the other inclusion is trivial, we conclude that

$$\mathrm{Z}(P) = \mathrm{Z}(P)\,\mathrm{Z}(Q) = \mathrm{Z}(PQ) = C_{C_S(Q)}(P).$$

Let $P' = \varphi(P)$ for a $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')$. According to the definition of $C_{\mathcal{F}}(Q)$ there exists a $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, P'Q)$ such that $\bar{\varphi}|_P = \varphi$ and $\bar{\varphi}|_Q = \mathrm{id}_Q$. As $\mathrm{Z}(Q) \leq Q$ we see that

$$P' = \varphi(P) = \bar{\varphi}(P) \geq \bar{\varphi}(\mathrm{Z}(Q)) = \mathrm{Z}(Q).$$

As $\bar{\varphi}(PQ) = P'Q$, we conclude that $P'Q$ is also $\mathcal{F}$-centric. Thus the properties of P used in the above calculations also hold for P', hence for any P' which is $C_{\mathcal{F}}(Q)$-conjugate to P we have $C_{C_S(Q)}(P') = \mathrm{Z}(P')$, i.e. P is $C_{\mathcal{F}}(Q)$-centric.

Now assume that $P \subseteq C_S(Q)$ is $C_{\mathcal{F}}(Q)$-centric. Let $q \in \mathrm{Z}(Q) = Q \cap C_S(Q)$. As $P \subseteq C_S(Q)$ we have that q commutes with all elements in P. Since $P \subseteq C_S(Q)$ is $C_{\mathcal{F}}(Q)$-centric we see that

$$q \in C_{C_S(Q)}(P) = \mathrm{Z}(P) \subseteq P,$$

hence $\mathrm{Z}(Q) \subseteq P$. To show that PQ is $\mathcal{F}$-centric choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$. Observe that this is not always a homomorphism in $C_{\mathcal{F}}(Q)$, since it is not required to be the identity on Q. The first step is then to modify φ into such a map. For this set $Q' = \varphi(Q)$ and $\psi := \varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(Q', Q)$. Since the image of ψ is Q, which is fully centralized in the saturated fusion system $\mathcal{F}$, there exists a $\bar{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, S)$ such that $\bar{\psi}|_{Q'} = \varphi^{-1}$. According to the definitions $C_S(Q')Q' \subseteq N_\psi$ and $\bar{\psi}(C_S(Q')Q') \subseteq \bar{\psi}(C_S(Q'))\bar{\psi}(Q') \subseteq C_S(Q)Q$. Hence we can consider $\bar{\psi}$ as a map in $\mathrm{Hom}_{\mathcal{F}}(C_S(Q')Q', C_S(Q)Q)$. Note that as $PQ \subseteq C_S(Q)Q$, we get that $\varphi(PQ) \subseteq C_S(\varphi(Q))\varphi(Q) = C_S(Q')Q'$, so the map $\varphi' := \bar{\psi} \circ \varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, C_S(Q)Q)$ is well-defined. It satisfies that $\varphi'|_Q = \bar{\psi}|_{\varphi(Q)} \circ \varphi|_Q = \mathrm{id}_Q$, so $\varphi'|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, C_S(Q))$. As P is assumed to be $C_{\mathcal{F}}(Q)$-centric we have that $C_{C_S(Q)}(\varphi'(P)) \subseteq \varphi'(P)$. Thus

$$\begin{aligned} C_S(\varphi'(PQ)) &= C_S(\varphi'(P)Q) = C_S(\varphi'(P)) \cap C_S(Q) \\ &= C_{C_S(Q)}(\varphi'(P)) \subseteq \varphi'(P) \subseteq \varphi'(PQ). \end{aligned}$$

Since $\varphi' = \bar{\psi} \circ \varphi$, the inclusions above implies that

$$\bar{\psi}(C_S(\varphi(PQ))) \subseteq C_S(\varphi'(PQ)) \subseteq \varphi'(PQ) = \bar{\psi}(\varphi(PQ)).$$

As $\bar{\psi}$ is injective, we deduce that $C_S(\varphi(PQ)) \subseteq \varphi(PQ)$. As this holds for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$, we conclude that PQ is $\mathcal{F}$-centric.

By part (a) we have that PQ is $\mathcal{F}$-centric whenever P is $C_{\mathcal{F}}(Q)$-centric, so the morphism sets of $C_{\mathcal{L}}(Q)$ are well-defined. Composition of morphisms is induced from $\mathcal{L}$, since if $\varphi \in \mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P, P')$ and $\psi \in \mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P', P'')$, then $\psi \circ \varphi \in \mathrm{Mor}_{\mathcal{L}}(PQ, P''Q)$ with $\pi(\psi \circ \varphi)(P) = \pi(\psi)(\pi(\varphi)(P)) \subseteq \pi(\psi)(P') \subseteq P''$ and $\pi(\psi \circ \varphi)|_Q = \pi(\psi)|_Q \circ \pi(\varphi)|_Q = \mathrm{id}_Q$. Since $1_{PQ} \in \mathrm{Mor}_{\mathcal{L}}(PQ, PQ)$ satisfies $\pi(1_{PQ}) = \mathrm{id}_{PQ}$, we have that $1_{PQ} \in \mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P, P)$ for any $C_{\mathcal{F}}(Q)$-centric subgroup P. Let this be the identity morphism in $C_{\mathcal{F}}(Q)$. With the given composition and identities it follows directly from the fact that $\mathcal{L}$ is a category that $C_{\mathcal{L}}(Q)$ is a category. Let $\pi\colon C_{\mathcal{L}}(Q) \to C_{\mathcal{F}}(Q)$ be given for any $P, P' \subseteq C_S(Q)$ which are $C_{\mathcal{F}}(Q)$-centric and $\varphi \in \mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P, P')$ by setting $\pi(P) = P$ and $\pi(\varphi) = \pi(\varphi)|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')$. As $\pi\colon \mathcal{L} \to \mathcal{F}$ is a functor, the same holds for $\pi\colon C_{\mathcal{L}}(Q) \to C_{\mathcal{F}}(Q)$. For a $C_{\mathcal{F}}(Q)$-centric subgroup P, consider $p \in P$. Then $\pi(\delta_{PQ}(p)) = c_p \in \mathrm{Aut}_{\mathcal{F}}(PQ)$. As $P \in C_S(Q)$, we have $c_p|_Q = \mathrm{id}_Q$. Clearly we also have that $c_p(P) = P$, hence $\delta_{PQ}(p) \in \mathrm{Aut}_{C_{\mathcal{L}}(Q)}(P)$. Thus by setting $\delta_P\colon P \to \mathrm{Aut}_{C_{\mathcal{L}}(Q)}(P)$ to be $p \mapsto \delta_{PQ}(p)$ we get a well-defined monomorphism. With this definition we have that $\pi(\delta_P(p)) = c_p|_P \in \mathrm{Aut}_{C_{\mathcal{F}}(Q)}(P)$ for any $C_{\mathcal{F}}(Q)$-centric subgroup P, hence property (B) holds.

Consider any $P, P' \subseteq C_S(Q)$ which are $C_{\mathcal{F}}(Q)$-centric. As the action of $\mathrm{Z}(P)$ on $\mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P, P')$ is induced by the action of $\mathrm{Z}(PQ)$ on $\mathrm{Mor}_{\mathcal{L}}(PQ, P'Q)$, is it free with orbit set

$$\begin{aligned}
\mathrm{Mor}_{C_{\mathcal{L}}(P,P')}/Z(P) &= \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(PQ, P'Q) \mid \pi(\varphi)|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')\}/\,\mathrm{Z}(P) \\
&= \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(PQ, P'Q) \mid \pi(\varphi)|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')\}/\,\mathrm{Z}(PQ) \\
&\cong \{\varphi \in \mathrm{Mor}_{\mathcal{F}}(PQ, P'Q) \mid \varphi|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')\} \\
&\cong \mathrm{Mor}_{C_{\mathcal{F}}(Q)}(P, P')
\end{aligned}$$

as $\mathrm{Z}(P) = \mathrm{Z}(PQ)$ and any $\varphi \in \mathrm{Mor}_{\mathcal{F}}(PQ, P'Q)$ such that $\varphi|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P')$ is determined by its values on P. Hence property (A) holds for $C_{\mathcal{L}}(Q)$. For property (C) consider $\varphi \in \mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P, P')$. Then for $g \in P \subseteq PQ$ the diagram

$$\begin{tikzcd}
PQ \arrow[r, "\varphi"] \arrow[d, "\delta_{PQ}(g)"'] & P'Q \arrow[d, "\delta_{PQ}(\pi(\varphi)(g))"] \\
PQ \arrow[r, "\varphi"] & P'Q
\end{tikzcd}$$

commutes and with the given definitions it is exactly the diagram corresponding to $C_{\mathcal{L}}(Q)$. Hence we conclude that (C) holds, and thus $C_{\mathcal{L}}(Q)$ is a central linking system associated to $C_{\mathcal{F}}(Q)$. □

5.1. **Centralizer fusion systems and elementary abelian p-groups.** When Q is an abelian group, we have that $\mathrm{Z}(Q) = Q$, hence the conditions required to be $C_{\mathcal{F}}(Q)$-centric from the previous proposition is simply to contain Q. We will now study the centralizer p-local finite group in the case of a fully centralized elementary abelian subgroup.

Definition 5.5. *For a fusion system $\mathcal{F}$ over a p-group S, let $\tilde{\mathcal{F}}^e$ be the full subcategory of $\mathcal{F}$ on the objects which are the nontrivial elementary abelian p-subgroups of S. Similarly let $\mathcal{F}^e$ be the full subcategory of $\mathcal{F}$ on the objects which are the nontrivial elementary abelian p-subgroups of S which are fully centralized in $\mathcal{F}$.*

Lemma 5.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then for $E, E' \in \mathcal{F}^e$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E', E)$ the homomorphism $C_S(\varphi) \in \mathrm{Hom}_{\mathcal{F}}(C_S(E), C_S(E'))$ from Lemma 5.1 give rise to functors $C_{\mathcal{F}}(\varphi)\colon C_{\mathcal{F}}(E) \to C_{\mathcal{F}}(E')$ and $C_{\mathcal{L}}(\varphi)\colon C_{\mathcal{L}}(E) \to C_{\mathcal{L}}(E')$ satisfying $\pi \circ C_{\mathcal{L}}(\varphi) = C_{\mathcal{F}}(\varphi) \circ \pi$.*

Proof. Let $E, E' \in \mathrm{Ob}(\mathcal{F}^e)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E', E)$. Define $\bar{\varphi} := C_S(\varphi) \in \mathrm{Hom}_{\mathcal{F}}(C_S(E), C_S(E'))$ using the map from Lemma 5.1. Observe that $E' = \bar{\varphi}(\varphi(E')) \subseteq \bar{\varphi}(E)$.

Let $P \subseteq C_S(E)$. Then $\bar{\varphi}(P) \subseteq C_S(E')$. A $\psi \in \mathrm{Hom}_{C_{\mathcal{F}}(E)}(P, P')$ has an extension $\bar{\psi} \in \mathrm{Hom}_{\mathcal{F}}(PE, P'E)$, which is the identity on E. Then $\bar{\varphi} \circ \psi \circ \bar{\varphi}^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\bar{\varphi}(P), \bar{\varphi}(P'))$ and as $E' \subseteq \bar{\varphi}(E)$ we have an extension $\bar{\varphi} \circ \bar{\psi} \circ \bar{\varphi}^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\bar{\varphi}(P)E', \bar{\varphi}(P')E')$ which is the identity on E'. Hence by setting $C_{\mathcal{F}}(\varphi)(P) = \bar{\varphi}(E)$ and $C_{\mathcal{F}}(\varphi)(\psi) = \bar{\varphi} \circ \psi \circ \bar{\varphi}^{-1}$ we get a well-defined functor from $C_{\mathcal{F}}(E)$ to $C_{\mathcal{F}}(E')$.

Let $P \subseteq C_S(E)$. Then by Proposition 5.4 we have $P \in C_{\mathcal{L}}(E)$ if and only if $E \subseteq P$ and $P \in \mathcal{L}$. Consider a $P \in C_{\mathcal{L}}(E)$. Then $\bar{\varphi}(P) \subseteq C_S(E')$ for which $E' \subseteq \bar{\varphi}(E) \subseteq \bar{\varphi}(P)$, and as $\bar{\varphi}$ is a morphism in $\mathcal{F}$, we also have that $\bar{\varphi}(P) \in \mathcal{L}$. By the above this implies that $\bar{\varphi}(P) \in C_{\mathcal{L}}(E')$

For any $P \in C_{\mathcal{L}}(E)$, we have that $P, \bar{\varphi}(P) \in \mathcal{L}$, so we can choose $\beta_P^{-1} \in \mathrm{Mor}_{\mathcal{L}}(\bar{\varphi}(P), P)$ such that $\pi(\beta_P^{-1}) = \bar{\varphi}^{-1}$. Lemma 3.1(a) implies that there exists $\beta_P \in \mathrm{Mor}_{\mathcal{L}}(P, \bar{\varphi}(P))$ such that $\pi(\beta_P) = \bar{\varphi}$ and $\beta_P^{-1} \circ \beta_P = 1_P$. Let $P, Q \in C_{\mathcal{L}}(E)$ and consider $\alpha \in \mathrm{Mor}_{C_{\mathcal{L}}(E)}(P, Q)$. By the definition $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and $\pi(\alpha)|_E = \mathrm{id}_E$. Then $\beta_Q \circ \alpha \circ \beta_P^{-1} \in \mathrm{Mor}_{\mathcal{L}}(\bar{\varphi}(P), \bar{\varphi}(Q))$, such that

$$\pi(\beta_Q \circ \alpha \circ \beta_P^{-1})|_{E'} = \pi(\beta_Q) \circ \pi(\alpha) \circ \pi(\beta_P^{-1})|_{E'} = \bar{\varphi} \circ \pi(\alpha) \circ \bar{\varphi}^{-1}|_{E'} = \mathrm{id}_{E'}$$

since $\bar{\varphi}^{-1}(E') \subseteq E$. Thus $\beta_Q \circ \alpha \circ \beta_P^{-1} \in \mathrm{Mor}_{C_{\mathcal{L}}(E')}(\bar{\varphi}(P), \bar{\varphi}(Q))$. Hence by mapping $P \to \bar{\varphi}(P)$ and $\alpha \to \beta_Q \circ \alpha \circ \beta_P^{-1}$ we get a functor from $C_{\mathcal{L}}(E)$ to $C_{\mathcal{L}}(E')$, which satisfies $\pi_{E'} \circ C_{\mathcal{L}}(\varphi) = C_{\mathcal{F}}(\varphi) \circ \pi_E$. □

Note that the functors from the previous Lemma do not in general satisfy $C_{\mathcal{L}}(\varphi) \circ C_{\mathcal{L}}(\psi) = C_{\mathcal{L}}(\psi \circ \varphi)$ for $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E, E')$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(E', E'')$, and therefore it is not a functor from $(\mathcal{F}^e)^{op}$ to Cat. We will instead consider a category, where this is the case.

Definition 5.7. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. For a $E \in \mathrm{Ob}(\tilde{\mathcal{F}}^e)$ we define $\bar{C}_{\mathcal{L}}(E)$ to be the category with*

$$\mathrm{Ob}(\bar{C}_{\mathcal{L}}(E)) = \{(P, \alpha) \mid P \in \mathrm{Ob}(\mathcal{L}), \alpha \in \mathrm{Hom}_{\mathcal{F}}(E, \mathrm{Z}(P))\}$$

and

$$\mathrm{Mor}_{\bar{C}_{\mathcal{L}}(E)}((P, \alpha), (Q, \beta)) = \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q) \mid \pi(\varphi) \circ \alpha = \beta\},$$

and the composition of morphisms and the identity morphism is given by those in $\mathcal{L}$.

Lemma 5.8. *Let $E \in \mathcal{F}^e$ where $(S, \mathcal{F}, \mathcal{L})$ is a p-local finite group. The functor $F\colon C_{\mathcal{L}}(E) \to \bar{C}_{\mathcal{L}}(E)$ defined by $F(P) = (P, \iota_{E \to \mathrm{Z}(P)})$ and $F(\varphi) = \varphi$ is an equivalence of categories and hence induces a homotopy equivalence $|C_{\mathcal{L}}(E)| \to |\bar{C}_{\mathcal{L}}(E)|$.*

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $E \in \mathcal{F}^e$. A $P \in C_{\mathcal{L}}(E)$ satisfies that $P \in \mathcal{L}$ and $E \subseteq P \leq C_S(E)$. The last condition implies that $E \subseteq \mathrm{Z}(P)$, so the inclusion $\iota_{E \to \mathrm{Z}(P)} \in \mathrm{Hom}_{\mathcal{F}}(E, \mathrm{Z}(P))$, and hence $(P, \iota_{E \to \mathrm{Z}(P)}) \in \bar{C}_{\mathcal{L}}(E)$. By definition of $C_{\mathcal{L}}(E)$ we have that $\varphi \in \mathrm{Mor}_{C_{\mathcal{L}}(E)}(P, Q)$ is a morphism in $\mathcal{L}$ such that $\pi(\varphi)|_E = \mathrm{id}_E$. This implies that $\pi(\varphi) \circ \iota_{E \to \mathrm{Z}(P)} = \iota_{E \to \mathrm{Z}(Q)}$, so $\varphi \in \mathrm{Mor}_{\bar{C}_{\mathcal{L}}(E)}((P, \iota_{E \to \mathrm{Z}(P)}), (Q, \iota_{E \to \mathrm{Z}(Q)}))$. From this we conclude that F is a well-defined functor. For any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ the condition $\pi(\varphi)|_E = \mathrm{id}_E$ is equivalent with $\pi(\varphi) \circ \iota_{E \to \mathrm{Z}(P)} = \iota_{E \to \mathrm{Z}(Q)}$, thus F induces an isomorphism on the morphism sets. Consider $(P, \alpha) \in \bar{C}_{\mathcal{L}}(E)$. By Lemma 5.1 there exists a $C_S(\alpha) \in \mathrm{Hom}_{\mathcal{F}}(C_S(\mathrm{Z}(P)), C_S(E))$. As $P \subseteq C_S(\mathrm{Z}(P))$, we have $P' := C_S(\alpha)(P) \subseteq C_S(E)$. As P' is $\mathcal{F}$-conjugate with P, we have that P' is $\mathcal{F}$-centric. Since $\alpha(E) \subseteq P$ and $C_S(\alpha)$ is an extension of α^{-1} we get that $E = C_S(\alpha)(\alpha(E)) \subseteq P'$. Hence $P' \in C_{\mathcal{L}}(P)$. Choose $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, P')$ such that $\pi(\varphi) = C_S(\alpha)$. As $C_S(\alpha)|_{\alpha(E)} = \alpha^{-1}|_{\alpha(E)}$ we get

$$\pi(\varphi) \circ \alpha = C_S(\alpha) \circ \alpha = \iota_{E \to \mathrm{Z}(P')},$$

hence $\varphi \in \mathrm{Mor}_{\bar{C}_{\mathcal{L}}(E)}((P, \alpha), F(E'))$ and is an isomorphism. Thus F is an equivalence of categories and therefore induces a homotopy equivalence on the geometric realizations by [3, III Corollary 2.2(b)]. □

Lemma 5.9. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $\bar{C}_{\mathcal{L}}(-)\colon (\tilde{\mathcal{F}}^e)^{op} \to \mathsf{Cat}$ is a functor.*

Proof. Let $E, E' \in \mathrm{Ob}(\mathcal{F}^e)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E', E)$. Define $\bar{C}_{\mathcal{L}}(\varphi)\colon \bar{C}_{\mathcal{L}}(E) \to \bar{C}_{\mathcal{L}}(E')$ by $\bar{C}_{\mathcal{L}}(\varphi)(P,\alpha) = (P, \alpha\circ\varphi)$ for $(P,\alpha) \in \mathrm{Ob}(\bar{C}_{\mathcal{L}}(E))$ and $\bar{C}_{\mathcal{L}}(\varphi)(\psi) = \psi$ for $\mathrm{Mor}_{\bar{C}_{\mathcal{L}}(E)}((P,\alpha),(Q,\beta))$. Then $\bar{C}_{\mathcal{L}}(\varphi)$ is a functor and it follows easily from the definitions that $\bar{C}_{\mathcal{L}}(-)$ is a functor from $(\tilde{\mathcal{F}}^e)^{op}$ to Cat. □

5.2. **The centralizer decomposition.**

Theorem 5.10. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $S \neq 1$. Then the map*

$$\operatorname*{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)| \to |\mathcal{L}|$$

induced by the forgetful functor $F\colon \bar{C}_{\mathcal{L}}(E) \to \mathcal{L}$ given by $F(P,\alpha) = P$ and $F(\varphi) = \varphi$ for any $E \in \tilde{\mathcal{F}}^e$ is a homotopy equivalence.

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $S \neq 1$. First we consider the category $\hat{\mathcal{L}}$ with objects the set of pairs (P, E) where $P \subseteq S$ such that P is $\mathcal{F}$-centric and E is a nontrivial elementary abelian subgroup of $\mathrm{Z}(P)$. For $(P,E),(P',E') \in \hat{\mathcal{L}}$ we set

$$\mathrm{Mor}_{\hat{\mathcal{L}}}((P,E),(P',E')) = \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,P') \mid E' \subseteq \pi(\varphi)(E)\}.$$

With this definition there is clearly a functor $T\colon \hat{\mathcal{L}} \to \mathcal{L}$ given by $T(P,E) = P$, and $T(\varphi) = \varphi$. To define a functor $S\colon \mathcal{L} \to \hat{\mathcal{L}}$ we set for any nontrivial subgroup P of S $\mathcal{E}(P)$ to be the subgroup of generated by the elements of order p in $\mathrm{Z}(P)$. Note that as P is a nontrivial p-group we have that $1 \neq \mathcal{E}(P)$ and it is elementary abelian. Now for $P \in \mathcal{L}$ we have that $(P, \mathcal{E}(P)) \in \hat{\mathcal{L}}$ and for any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,P')$ we remark since P and P' are $\mathcal{F}$-centric that

$$\pi(\varphi)(\mathrm{Z}(P)) \geq \mathrm{Z}(\pi(\varphi)(P)) = C_S(\pi(\varphi)(P)) \geq C_S(P') = \mathrm{Z}(P'),$$

so by considering elements of order p, we conclude that $\pi(\varphi)(\mathcal{E}(P)) \geq \mathcal{E}(P')$. Thus by setting $S(P) = (P, \mathcal{E}(P))$ and $S(\varphi) = \varphi$ we get a well-defined functor $S\colon \mathcal{L} \to \hat{\mathcal{L}}$. With the above definition $T\circ S = \mathrm{id}_{\mathcal{L}}$ and $S\circ T\colon \hat{\mathcal{L}} \to \hat{\mathcal{L}}$ is given $(P,E) \mapsto (P, \mathcal{E}(P))$. By definition $E \subseteq \mathcal{E}(P)$ for any elementary abelian subgroup $E \subseteq \mathrm{Z}(P)$. So for any $(P,E) \in \hat{\mathcal{L}}$ we have $1_P \in \mathrm{Mor}_{\hat{\mathcal{L}}}((P,\mathcal{E}(P)),(P,E))$ and this morphism gives a natural transformation from $S\circ T$ to $\mathrm{id}_{\hat{\mathcal{L}}}$ as the following diagram commutes:

$$\begin{tikzcd}
(P,\mathcal{E}(P)) \arrow[r, "1_P"] \arrow[d, "\varphi"'] & (P,E) \arrow[d, "\varphi"'] \\
(P',\mathcal{E}(P')) \arrow[r, "1_{P'}"] & (P',E')
\end{tikzcd}$$

This implies that $|S|\colon |\mathcal{L}| \to |\hat{\mathcal{L}}|$ is a homotopy equivalence.

For $(P,E) \in \hat{\mathcal{L}}$ we have that E is a non-trivial elementary abelian p-group and for $\varphi \in \mathrm{Mor}_{\hat{\mathcal{L}}}((P,E),(P',E'))$ we have that $\pi(\varphi)^{-1} \in \mathrm{Hom}_{\mathcal{F}}(E',E)$. Thus by setting $\tau(P,E) = E$ and $\tau(\varphi) = \pi(\varphi)^{-1}$ we get a well-defined functor $\tau\colon \hat{\mathcal{L}} \to (\tilde{\mathcal{F}}^e)^{op}$. By [23, Theorem 5.5] we have that $\mathrm{hocolim}_{\hat{\mathcal{L}}}(*) \cong \mathrm{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\tau \downarrow E|$, as $E \mapsto |\tau\downarrow E|$ is the left homotopy Kan extension of the trivial functor over τ. We now want to construct a natural transformation between the functors $|\bar{C}_{\mathcal{L}}(-)|$ and $|\tau\downarrow -|$, which are both functors $(\tilde{\mathcal{F}}^e)^{op} \to \mathsf{Top}$. For this pick an $E \in \tilde{\mathcal{F}}^e$. Then the category $\tau \downarrow E$ has objects $\{(P, E', \alpha) \mid (P, E') \in \hat{\mathcal{L}}, \alpha \in \mathrm{Hom}_{\mathcal{F}}(E, E')\}$ and morphisms

$$\mathrm{Mor}_{\tau\downarrow E}((P, E', \alpha), (Q, F', \beta)) = \{\varphi \in \mathrm{Mor}_{\hat{\mathcal{L}}}((P, E'), (Q, F')) \mid \pi(\varphi)^{-1} \circ \beta = \alpha\}.$$

By comparing definitions we see that by setting $F_E(P, \alpha) = (P, \alpha(E), \alpha)$ and $F_E(\varphi) = \varphi$ we get a well-defined functor F_E from $\bar{C}_{\mathcal{L}}(E)$ to $\tau \downarrow E$. Similarly, setting $G_E(P, E', \alpha) = (P, \alpha)$ and $G_E(\varphi) = \varphi$ gives a well-defined functor from $\tau \downarrow E$ to $\bar{C}_{\mathcal{L}}(E)$. Note that $G_E \circ F_E = \mathrm{id}_{\bar{C}_{\mathcal{L}}(E)}$ and there is a natural transformation from $\mathrm{id}_{\tau\downarrow E}$ to $F_E \circ G_E$ given by $\mathrm{id}_P \colon (P, E, \alpha) \to (P, E, \alpha)$. From this we conclude that $|G_E| \colon |\tau \downarrow E| \to |\bar{C}_{\mathcal{L}}(E)|$ is a homotopy equivalence.

Consider any $\psi \in \mathrm{Mor}_{\mathcal{F}}(E', E)$. For $(P, E'', \alpha) \in \tau \downarrow E$, we have that

$$\begin{aligned} G_{E'}((\tau \downarrow \psi)(P, E'', \alpha)) &= G_{E'}(P, E'', \alpha \circ \psi) = (P, \alpha \circ \psi) \\ &= \bar{C}_{\mathcal{L}}(\psi)(P, \alpha) = \bar{C}_{\mathcal{L}}(\psi)(G_E(P, \alpha)). \end{aligned}$$

As the involved functors act as the identity on morphisms we conclude that $G_{E'} \circ (\tau \downarrow \psi) = \bar{C}_{\mathcal{L}}(\psi) \circ G_E$ as functors from $\tau \downarrow E$ to $\bar{C}_{\mathcal{L}}(E')$. Hence G_- is a natural transformation from $\tau \downarrow (-)$ to $\bar{C}_{\mathcal{L}}(-)$, and so it induces a homomorphism $\mathrm{hocolim}(|G_-|) \colon \mathrm{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\tau \downarrow E| \to \mathrm{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)|$. As $|G_E|$ is a homotopy equivalence for any $E \in \tilde{\mathcal{F}}^e$, we furthermore have by [17, IV Proposition 1.9] that $\mathrm{hocolim}(|G_-|)$ is a homotopy equivalence.

From the results above we get

$$|\mathcal{L}| \cong |\hat{\mathcal{L}}| \cong \operatorname*{hocolim}_{\hat{\mathcal{L}}}(*) \cong \operatorname*{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\tau \downarrow E| \cong \operatorname*{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)|.$$

This equivalence is induced by the following functors, where $(P, \alpha) \in \bar{C}_{\mathcal{L}}(E)$:

$$(P, \alpha) \xrightarrow{F_E} (P, \alpha(E), \alpha) \longrightarrow (P, \alpha(E)) \xrightarrow{T} P.$$

□

Corollary 5.11. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $S \neq 1$. Then the map*

$$\operatorname*{hocolim}_{E\in(\mathcal{F}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)| \to |\mathcal{L}|$$

induced by the forgetful functor $F \colon C_{\mathcal{L}}(E) \to \mathcal{L}$ given by $F(P, \alpha) = P$ and $F(\varphi) = \varphi$ for $E \in \mathcal{F}^e$ is a homotopy equivalence.

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $S \neq 1$. Let $\iota \colon \mathcal{F}^e \to \tilde{\mathcal{F}}^e$ be the inclusion functor. For $\tilde{E} \in \tilde{\mathcal{F}}^e$ there exists an $E \in \mathcal{F}^e$ and an isomorphism $\varphi \in \mathrm{Mor}_{\mathcal{F}}(E, \tilde{E})$. In the category $(\iota \downarrow \tilde{E})$ the pair (E, φ) is a terminal object, as for any $(E', \alpha) \in (\iota \downarrow \tilde{E})$ the diagram

$$\begin{tikzcd} E' \arrow[r, "\alpha"] \arrow[d, "\alpha \circ \varphi^{-1}"'] & \tilde{E} \arrow[d, "\mathrm{id}"] \\ E \arrow[r, "\varphi"] & \tilde{E} \end{tikzcd}$$

is in $\mathcal{F}$ and commutes. Since $(\tilde{E} \downarrow \iota^{op}) = (\iota \downarrow \tilde{E})^{op}$, the category $(\tilde{E} \downarrow \iota^{op})$ has an initial object and thus $|\tilde{E} \downarrow \iota^{op}|$ is contractible. As this holds for any $\tilde{E} \in \tilde{\mathcal{F}}^e$, the functor $\iota^{op} \colon (\mathcal{F}^e)^{op} \to (\tilde{\mathcal{F}})^{op}$ is right cofinal, and hence the map

$\operatorname{hocolim}_{E\in(\mathcal{F}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)| \to \operatorname{hocolim}_{E\in(\tilde{\mathcal{F}}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)|$ induced by $(P,\alpha)\mapsto(P,\alpha)$ is a homotopy equivalence by [22, 19.6.7]. The result now follows directly from the above theorem. $\square$

6. Obstruction theory and higher limits

Consider $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$. As both Q and $\varphi(P)$ are $\mathcal{F}$-centric, we conclude that $\mathrm{Z}(Q) = C_S(Q) \subseteq C_S(\varphi(P)) = \mathrm{Z}(\varphi(P)) = \varphi(\mathrm{Z}(P))$. Note for any $g \in \mathrm{Z}(Q)$ and $h \in Q$ we have that $(c_h \circ \varphi)^{-1}(g) = \varphi^{-1}(c_{h^{-1}}(g)) = \varphi^{-1}(g)$. This implies that the following is well-defined.

Definition 6.1. *Let $\mathcal{F}$ be a fusion system on a p-group S. We define the functor $\mathcal{Z}_{\mathcal{F}} \colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathsf{Ab}$ by $\mathcal{Z}_{\mathcal{F}}(P) = \mathrm{Z}(P)$ and for $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ we set $\mathcal{Z}_{\mathcal{F}}(\varphi)$ to be the composition*

$$\mathrm{Z}(Q) \xrightarrow{\text{incl}} \mathrm{Z}(\varphi(P)) \xrightarrow{\varphi^{-1}} \mathrm{Z}(P)$$

6.1. **An obstruction theory for central linking systems.** The next proposition gives an obstruction theory for the existence of a central linking system in terms of higher limits of this functor.

Proposition 6.2. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then there exists an element $\eta(\mathcal{F}) \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$ such that $\mathcal{F}$ has an associated centric linking system if and only if $\eta(\mathcal{F}) = 0$. If such a central linking system exists, them the group $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$ acts freely and transitively on the set of isomorphism classes of central linking systems associated to $\mathcal{F}$.*

Proof. By [3, Proposition 5.3] we conclude that the higher limits of $\mathcal{Z}_{\mathcal{F}}$ can be computed via the normalized chain complex for $\mathcal{Z}_{\mathcal{F}}$, i.e.

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(\mathcal{Z}_{\mathcal{F}}) \cong \mathrm{H}^i(C^*(\mathcal{O}^c(\mathcal{F}); \mathcal{Z}_{\mathcal{F}}), d),$$

where the chain complex is defined as $C^n(\mathcal{O}^c(\mathcal{F}); \mathcal{Z}_{\mathcal{F}}) = \prod_{P_0 \to \cdots \to P_n} \mathcal{Z}_{\mathcal{F}}(P_0)$ for any n. Note that we consider $\omega \in C^n(\mathcal{O}^c(\mathcal{F}); \mathcal{Z}_{\mathcal{F}})$ as a map sending a sequence $(P_0 \to \cdots \to P_n) \in \mathcal{N}(\mathcal{O}^c(\mathcal{F}))_n$ to a element in $\mathrm{Z}(P_0)$ satisfying that if $(P_0 \to \cdots \to P_n)$ contains an identity morphism, then the image is $1 \in \mathrm{Z}(P_0)$. The differential is given by

$$\begin{aligned} d(\omega(P_0 \to^{\varphi} P_1 \cdots \to P_{n+1}) = &\sum_{i=1}^{n+1} (-1)^i \omega(P_0 \to \cdots \to \hat{P}_i \to \cdots \to P_{n+1}) \\ &+ \mathcal{Z}_{\mathcal{F}}(\varphi)\omega(P_1 \to \cdots \to P_n). \end{aligned}$$

We first construct an element $[u] \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$ and prove that the construction is independent of the choices made. Let $\sigma \in \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{F}^c)$ be a section, which sends identity morphisms to identity morphisms. We use the notation $\sigma(\varphi) = \tilde{\varphi}$. For each pair $P, Q \in \mathcal{F}^c$ we set $X(P,Q) = Q \times \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and define a map $\pi^{P,Q}_{\sigma} \colon X(P,Q) \to \mathrm{Hom}_{\mathcal{F}}(P,Q)$ by $\pi^{P,Q}_{\sigma}(g,\varphi) = c_g \circ \tilde{\varphi}$. For each pair of maps $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ we have that $\tilde{\psi} \circ \tilde{\varphi}$ and $\widetilde{\psi\varphi}$ determine the same class in $\mathcal{O}^c(\mathcal{F})$, hence there exists a $t(\varphi,\psi) \in R$ such that $\tilde{\psi} \circ \tilde{\varphi} = c_{t(\varphi,\psi)} \circ \widetilde{\psi\varphi}$. Since σ sends identity morphisms to identity morphisms, we may require that $t(\varphi,\psi) = 1$ if either ψ or φ is the identity. We now define maps $* \colon X(Q,R) \times X(P,Q) \to X(P,R)$ by

$$(h,\psi) * (g,\varphi) = (h \cdot \tilde{\psi}(g) \cdot t(\varphi,\psi), \psi\varphi).$$

For each triple of objects $P, Q, R \in \mathcal{F}^c$ we can consider the diagram:

$$\begin{tikzcd} X(Q,R) \times X(P,Q) \arrow[r, "*"] \arrow[d, "\pi_\sigma^{Q,R} \times \pi_\sigma^{P,Q}"'] & X(P,R) \arrow[d, "\pi_\sigma^{P,R}"] \\ \mathrm{Hom}_{\mathcal{F}}(Q,R) \times \mathrm{Hom}_{\mathcal{F}}(P,Q) \arrow[r, "\circ"] & \mathrm{Hom}_{\mathcal{F}}(P,R) \end{tikzcd}$$

For any $((h,\psi),(g,\varphi)) \in X(Q,R) \times X(P,Q)$ we have that

$$\begin{aligned} \pi_\sigma^{Q,R}(h,\psi) \circ \pi_\sigma^{P,Q}(g,\varphi) &= c_h \circ \tilde{\psi} \circ c_g \circ \tilde{\varphi} = c_{h\tilde{\psi}(g)} \circ \tilde{\psi} \circ \tilde{\varphi} \\ &= c_{h\tilde{\psi}(g)t(\varphi,\psi)} \widetilde{\psi \circ \varphi} = \pi_\sigma^{P,R}((h,\psi) * (g,\varphi)), \end{aligned}$$

hence the diagram commutes. By the commutativity of the diagram we have for any triple of composable maps $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$, $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ and $\chi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(R,T)$ that $\pi_\sigma^{P,T}$ agrees on $((1,\chi)*(1,\psi))*(1,\varphi)$ and $(1,\chi)*((1,\psi)*(1,\varphi))$. If $(g,\alpha),(g',\alpha) \in X(P,R)$ satisfy that $\pi_\sigma^{P,R}(g,\alpha) = \pi_\sigma^{P,R}(g,\alpha)$, then $c_{g^{-1}g'} = \mathrm{id}_{\varphi(P)}$ and thus $g^{-1}g' \in C_S(\tilde{\varphi}(P)) = \mathrm{Z}(\tilde{\varphi}(P))$. If we set $u = \tilde{\varphi}^{-1}(g^{-1}g') \in \mathrm{Z}(P)$, we see that $(g,\varphi) = (g',\varphi) * (u, \mathrm{id}_P)$, since $t(\mathrm{id}_P, \varphi) = 1$. We remark that u is the unique element with this property. As the second components of $((1,\chi)*(1,\psi))*(1,\varphi)$ and $(1,\chi)*((1,\psi)*(1,\varphi))$ agree, we conclude that there exists a unique $u_{\sigma,t}(\varphi,\psi,\chi) \in \mathrm{Z}(P)$ such that

$$((1,\chi) * (1,\psi)) * (1,\varphi) = [(1,\chi) * ((1,\psi) * (1,\varphi))] * (u_{\sigma,t}(\varphi,\psi,\chi), \mathrm{id}_P)$$

and by the definition of $*$ we conclude that

$$\widetilde{\chi\psi\varphi}(u_{\sigma,t}(\varphi,\psi,\chi)) = t(\psi\varphi,\chi)^{-1} \cdot \tilde{\chi}(t(\varphi,\psi))^{-1} \cdot t(\psi,\chi) \cdot t(\varphi,\chi\psi).$$

For any $g \in P, h \in Q, k \in R$ the first component of $((k,\chi) * (h,\psi)) * (g,\varphi)$ respectively $(k,\chi) * ((h,\psi) * (g,\varphi))$ is

$$g = k\tilde{\chi}(h)t(\psi,\chi)\widetilde{\chi\psi}(g)t(\varphi,\chi\psi)$$

$$g' = k\tilde{\chi}(h)t(\psi,\chi)\widetilde{\chi\psi}(g)t(\psi,\varphi)^{-1}\tilde{\chi}(t(\varphi,\psi))t(\psi\varphi,\chi)$$

while the second is $\chi\psi\varphi$. We remark that $(g')^{-1}g = u_{\sigma,t}(\varphi,\psi,\chi)$, hence

$$((k,\chi) * (h,\psi)) * (g,\varphi) = [(k,\chi) * ((h,\psi) * (g,\varphi))] * (u_{\sigma,t}(\varphi,\psi,\chi), \mathrm{id}_P) \tag{1}$$

as well. Note the requirements on t imply that $\widetilde{\chi\psi\varphi}(u_{\sigma,t}(\varphi,\psi,\chi)) = 1$ when at least one of the morphisms is the identity. As $\widetilde{\chi\psi\varphi}$ is injective, we conclude that in this case $u_{\sigma,t}(\varphi,\psi,\chi) = 1$, hence $u_{\sigma,t} \in C^3(\mathcal{O}^c(\mathcal{F}), \mathcal{Z}_{\mathcal{F}})$. If

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} R \xrightarrow{\chi} T \xrightarrow{\omega} U$$

is a sequence of morphisms in $\mathcal{O}^c(\mathcal{F})$, then by the definition of the differential on $C^*(\mathcal{O}^c(\mathcal{F}), \mathcal{Z}_{\mathcal{F}})$ we have that $du(\varphi,\psi,\chi,\omega)$ is the element

$$u(\psi\varphi,\chi,\omega)^{-1} \cdot u(\varphi,\chi\psi,\omega) \cdot u(\varphi,\psi,\omega\chi)^{-1} \cdot u(\varphi,\psi,\chi) \cdot \tilde{\varphi}^{-1}(u(\psi,\chi,\omega))$$

in $\mathrm{Z}(P)$. By computing the image of each of the five components under $\sigma(\omega\chi\psi\varphi)$ using the defining property of t and the definition of $u_{\sigma,t}$ to compare the factors, we conclude that $\sigma(\omega\chi\psi\varphi)(du(\varphi,\psi,\chi,\omega)) = 1$. As $\sigma(\omega\chi\psi\varphi)$ is injective, we conclude that $du(\varphi,\psi,\chi,\omega) = 1$ and hence $[u] \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$.

We now want to show that the class $[u]$ is independent of the particular choice of t. If for each pair of morphisms $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$, we have made another choice of elements $t'(\varphi,\psi)$ in R, such that t' satisfies the same conditions as t, then we have that $c_{t(\varphi,\psi)} = c_{t'(\varphi,\psi)}$ on $\widetilde{\psi\varphi}(P)$. By similar arguments as before this implies that there exists a $c(\varphi,\psi) \in \mathrm{Z}(P)$ such that $t'(\varphi,\psi) = t(\varphi,\psi)\cdot \widetilde{\psi\varphi}(c(\varphi,\psi))$. As t and t' is the neutral element when either ψ or φ is the identity and $\widetilde{\psi\varphi}$ is injective, we have that $c(\varphi,\psi) = 1$ when either of the maps is the identity, hence $c \in C^2(\mathcal{O}^c(\mathcal{F});\mathcal{Z}_\mathcal{F})$. Let $u' = u_{\sigma,t'} \in C^3(\mathcal{O}^c(\mathcal{F});\mathcal{Z}_\mathcal{F})$ be defined in the same way as before. We then get the following relation between t' and u':

$$\widetilde{\chi\psi\varphi}(u_{\sigma,t'}(\varphi,\psi,\chi)) = t'(\psi\varphi,\chi)^{-1}\cdot \tilde{\chi}(t'(\varphi,\psi))^{-1}\cdot t'(\psi,\chi)\cdot t'(\varphi,\chi\psi).$$

By inserting the relation $t'(\varphi,\psi) = t(\varphi,\psi)\cdot\widetilde{\psi\varphi}(c(\varphi,\psi))$ into the above and using the fact that u and t satisfy a similar relation we conclude that:

$$\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi)\cdot\tilde{\varphi}^{-1}(c(\psi,\chi))\cdot c(\varphi,\chi\psi)) = \widetilde{\chi\psi\varphi}(c(\varphi,\psi)\cdot c(\psi\varphi,\chi)\cdot u'(\varphi,\psi,\chi))$$

As $\widetilde{\chi\psi\varphi}$ is injective, we conclude that the above relation still holds without the $\widetilde{\chi\psi\varphi}$. This is a relation between elements in the abelian group $\mathrm{Z}(P)$, so by comparing with the definition of the differential, we conclude that $u^{-1}\cdot u' = dc$. Hence $[u] = [u']$ in $\varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_\mathcal{F})$.

Assume that $\sigma'\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{F})$ is another section sending identity morphism to identity morphisms. Set $\tilde{\varphi}' = \sigma'(\varphi)$. As both σ,σ' are sections, we can for every $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ choose a $g_\varphi \in Q$ such that $\tilde{\varphi} = c_{g_\varphi}\tilde{\varphi}'$ and $g_{\mathrm{id}_P} = 1$ for any $P \in \mathcal{F}^c$. Let $t(\varphi,\psi) \in R$ for any $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ be a choice of elements corresponding to the section σ as before. By defining

$$t'(\varphi,\psi) = \tilde{\psi}'(g_\varphi)^{-1}\cdot g_\psi^{-1}\cdot t(\varphi,\psi)\cdot g_{\psi\varphi}$$

we get an element of R, such that $\tilde{\psi}'\tilde{\varphi}' = c_{t'(\varphi,\psi)}\widetilde{\psi\varphi}'$ and t' is the neutral element when one of the morphisms is the identity. Consider the map $F\colon X(P,Q) \to X(P,Q)$ given by $F(h,\varphi) = (hg_\varphi,\varphi)$. Then $\pi^{P,Q}_\sigma = \pi^{P,Q}_{\sigma'}\circ F$. Let $*$ and $*'$ be the compositions defined by t and t'. Then $F((h,\psi)*(g,\varphi)) = F(h,\psi)*'F(g,\varphi)$. As $F(g,\mathrm{id}_P) = (g,\mathrm{id}_P)$ and F is a bijection, we conclude that the relation (1) holds for $*'$ as well with the same u. Thus $u_{\sigma',t'} = u_{\sigma,t}$, so the class $[u] \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_\mathcal{F})$ does not depend on the choice of section.

We will now prove that the class $[u]$ vanishes exactly when there exists a central linking system associated to $\mathcal{F}$. We first assume that $[u] = 0$ and construct a central linking system associated to $\mathcal{F}$. By assumption there exists a $c \in C^2(\mathcal{O}^c(\mathcal{F});\mathcal{Z}_\mathcal{F})$ such that $dc = u$. If $u = u_{\sigma,t}$ we may set $t'(\varphi,\psi) = t(\varphi,\psi)\widetilde{\psi\varphi}(c(\varphi,\psi)^{-1})$ for any $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$. As $c(\varphi,\psi) \in \mathrm{Z}(P)$ this will be another choice of t corresponding to the section σ, and by the above calculations we have that $u_{\sigma,t}^{-1}\cdot u_{\sigma,t'} = dc^{-1}$. As $u_{\sigma,t} = dc$, we conclude that $u_{\sigma,t'} = 1$. Let $\mathcal{L}$ be the category with objects $\mathcal{F}^c$ and morphism set $X(P,Q)$, where the composition is defined using t'. By the definition of the composition we see that $(1,\mathrm{id}_P)$ is a neutral element. Combining this with (1) and the fact that $u_{\sigma,t'} = 1$, we conclude that the composition is associative. Let $\pi\colon \mathcal{L} \to \mathcal{F}^c$ be given by the identity on objects and $\pi = \pi_\sigma$ for morphisms. Then $\pi(1,\mathrm{id}_p) = \mathrm{id}_P$ and by a previous result $\pi((h,\psi)*(g,\varphi)) = \pi(h,\psi)\circ\pi(g,\varphi)$, hence π is a functor. For $P \in \mathcal{F}^c$ let $\delta_P\colon P \to X(P,P)$ be the map $\delta_p(g) = (g,\mathrm{id}_p)$. We then have that δ_P is a monomorphism of groups. Consider a $\varphi \in \mathrm{Mor}_{\mathcal{F}^c}(P,Q)$. Let $\bar\varphi$ be the corresponding class in $\mathcal{O}^c(\mathcal{F})$. As σ is section, there exists a $g \in Q$, such that $\varphi = c_g \circ \sigma(\bar\varphi)$. The element $(g,\bar\varphi) \in X(P,Q)$ and $\pi(g,\bar\varphi) = \varphi$, so π is surjective on morphisms. Assume that the elements $(g,\varphi),(h,\psi) \in X(P,Q)$ satisfy that $\pi(g,\varphi) = \pi(h,\psi)$. Then $\varphi = \psi$ and thus $g^{-1}h \in \mathrm{Z}(\tilde\varphi(P))$. Let $u = \tilde\varphi^{-1}(g^{-1}h) \in \mathrm{Z}(P)$. Then $(g,\varphi) = (h,\varphi) * (u,\mathrm{id}_p) = (h,\varphi) * \delta_p(u)$. As $\pi(\delta_p(g)) = c_g$ for any $g \in P$, we conclude that π is in fact the orbit map for the $\mathrm{Z}(P)$-action on $X(P,Q)$ induced by δ_P. For any $(g,\varphi) \in X(P,Q)$ and $h \in P$ we have that $(g,\varphi) = (g,\varphi) * \delta_p(h)$ implies that $\tilde\varphi(h) = 1$ and thus $h = 1$. So the action of $\mathrm{Z}(P)$ on $X(P,Q)$ is free. For any $(g,\varphi) \in X(P,Q)$ and $h \in P$ we also have that

$$(g,\varphi) * \delta_P(h) = (g \cdot \tilde\varphi(h),\varphi) = ((c_g \circ \tilde\varphi)(h),\mathrm{id}_Q) * (g,\varphi) = \delta_Q(\pi(g,\varphi)) * (g,\varphi)$$

Thus $\mathcal{L}$ is a central linking system associated to $\mathcal{F}$. Note that the map

$$\tilde\sigma\colon \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q) \to \mathrm{Mor}_{\mathcal{L}}(P,Q)$$

given by $\tilde\sigma(\varphi) = (1,\varphi)$ is a section, which lifts σ and we have for any $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ that $\tilde\sigma(\psi) \circ \tilde\sigma(\varphi) = \delta_R(t'(\varphi,\psi)) \circ \tilde\varphi(\psi\varphi)$.

Now assume that there exists a central linking system $\mathcal{L}$ associated to $\mathcal{F}$. We want to prove that this implies that $[u] = 0$. Let $\sigma\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{F}^c)$ be a section and $\tilde\sigma\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L})$ be a lift of this. We assume that both sections send identity morphisms to identity morphisms. Let $G\colon X(P,Q) \to \mathrm{Mor}_{\mathcal{L}}(P,Q)$ where $(g,\psi) \mapsto \delta_Q(g) \circ \tilde\sigma(\psi)$. Consider a $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and let ψ be the class of $\pi(\varphi)$ in $\mathcal{O}^c(\mathcal{F})$ and $\tilde\psi = \tilde\sigma(\psi)$. Then $\pi(\tilde\psi)$ and $\pi(\varphi)$ determines the same class in $\mathcal{O}^c(\mathcal{F})$, so by Lemma 3.1 (b) there exists a unique element $g \in Q$, such that $\varphi = \delta_Q(g) \circ \tilde\psi$. Then $G(g,\psi) = \varphi$, so G is surjective. Assume for $(g,\psi),(g',\psi') \in X(P,Q)$, that $\delta_Q(g) \circ \tilde\sigma(\psi) = \delta_Q(g') \circ \tilde\sigma(\psi')$. As $\tilde\sigma$ is a section, we deduce that $\psi = \psi'$, and thus by the above uniqueness result we have $g = g'$, hence G is injective as well. By Lemma 3.1 (b) we also conclude that for any pair of morphisms $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ there exists a $t(\varphi,\psi) \in R$ such that

$$\tilde\sigma(\psi) \circ \tilde\sigma(\varphi) = \delta_R(t(\varphi,\psi)) \circ \tilde\sigma(\varphi\psi).$$

Then t satisfies the previous conditions with the section σ, hence we can define a multiplication on the sets $X(P,Q)$ using t. Then the map G is easily seen to respect the multiplication. As $\mathcal{L}$ is a category, the multiplication in $\mathcal{L}$ is associative, hence the same holds in $X(P,Q)$, so by the equation (1) we conclude that $(r,\chi\psi\varphi) * (u_{\sigma,t}(\varphi,\psi,\chi),\mathrm{id}_P) = (r,\chi\psi\varphi)$ for any triple of composable maps in $\mathcal{O}^c(\mathcal{F})$. By the definition of the multiplication combined with the fact that maps in $\mathcal{F}$ are injective, we conclude that $u_{\sigma,t}(\varphi,\psi,\chi) = 1$. Hence the class $[u] \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$ is trivial.

For a proof of the second part of the proposition we consider two central linking systems associated to $\mathcal{F}$, denoted $\pi_i\colon \mathcal{L}_i \to \mathcal{F}$ for $i = 1,2$. Let $\sigma\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{F})$ be a section as before and let $\tilde\sigma_i\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L}_i)$ be corresponding lifts. As in the previous paragraph we can for any pair of morphism $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ choose unique $t_i(\varphi,\psi) \in R$ such that

$$\tilde\sigma_i(\psi) \circ \tilde\sigma_i(\varphi) = \delta^i_R(t_i(\varphi,\psi)) \circ \tilde\sigma_i(\varphi\psi)$$

for $i = 1,2$. Furthermore we have that t_i is compatible with σ and the corresponding $u_{\sigma,t_i} = 1$. By a previous part we conclude that there exists a $c(\varphi,\psi) \in \mathrm{Z}(P)$ with

$$t_2(\varphi,\psi) = t_1(\varphi,\psi)\widetilde{\psi\varphi}(c(\varphi,\psi))$$

and $dc = u_{\sigma,1}^{-1}u_{\sigma,2} = 1$, so $[c] \in \varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$. Assume that $\tilde{\sigma}'_i$ for $i = 1$ or 2 is another lift of the section σ to $\mathcal{L}_i$. Then for any $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ we have that $\pi_i(\tilde{\sigma}'_i(\varphi)) = \sigma(\varphi) = \pi_i(\tilde{\sigma}_i(\varphi))$, hence by condition (A) for $\mathcal{L}_i$ there exists a unique element $w(\varphi) \in \mathrm{Z}(P)$ such that $\tilde{\sigma}'_i(\varphi) = \tilde{\sigma}_i(\varphi) \circ \delta^i_P(w(\varphi))$. As $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ agree on the identity morphisms, we conclude that $w(\mathrm{id}_p) = 1$ for any $P \in \mathcal{F}^c$, hence $w \in C^1(\mathcal{O}^c(\mathcal{F}), \mathcal{Z}_{\mathcal{F}})$. Let t'_i be defined similar to t_i. Then for any pair of morphisms $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ we have that

$$\begin{aligned}
\delta^i_R(\tilde{\psi}(w(\psi))\tilde{\psi}\tilde{\varphi}(w(\varphi)t_i(\varphi,\psi))\tilde{\sigma}_i(\psi\varphi)) &= \delta^i_R(\tilde{\psi}(w(\psi))\tilde{\psi}\tilde{\varphi}(w(\varphi))\tilde{\sigma}_i(\psi)\tilde{\sigma}_i(\varphi)) \\
&= \delta^i_R(\tilde{\sigma}_i(\psi)\delta^i_Q(w(\psi))\tilde{\sigma}_i(\varphi)\delta^i_P(w(\varphi))) \\
&= \delta^i_R(\tilde{\sigma}'_i(\psi)\tilde{\sigma}'_i(\varphi)) = \delta^i_R(t'_i(\varphi,\psi)\tilde{\sigma}'(\psi\varphi)) \\
&= \delta^i_R(t'_i(\varphi,\psi)\widetilde{\psi\varphi}(w(\psi\varphi))\tilde{\sigma}(\psi\varphi))
\end{aligned}$$

By the uniqueness part of Lemma 3.1 (b) we conclude that

$$\tilde{\psi}(w(\psi))\tilde{\psi}\tilde{\varphi}(w(\varphi)t_i(\varphi,\psi) = t'_i(\varphi,\psi)\widetilde{\psi\varphi}(w(\psi\varphi)$$

and thus by the relation $c_{t_i(\varphi,\psi)} \circ \widetilde{\psi\varphi} = \tilde{\psi} \circ \tilde{\varphi}$ we conclude

$$\begin{aligned}
t_i(\varphi,\psi)^{-1}t_i(\varphi,\psi) &= c^{-1}_{t_i(\varphi,\psi)}(\tilde{\psi}(w(\psi)) \cdot \tilde{\psi}\tilde{\varphi}(w(\varphi))) \cdot \widetilde{\psi\varphi}(w(\psi\varphi))^{-1} \\
&= \widetilde{\psi\varphi}(\tilde{\varphi}^{-1}(w(\psi)) \cdot w(\varphi) \cdot w(\psi\varphi)^{-1}) = \widetilde{\psi\varphi}(dw).
\end{aligned}$$

A change of $\tilde{\sigma}_i$ will only change c by a coboundary and so it does not change the class $[c] \in \varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$. Hence for any fixed section σ there corresponds a unique class $[c] \in \varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}})$ to any pair of central linking systems $\mathcal{L}_1$ and $\mathcal{L}_2$ associated to $\mathcal{F}$.

Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two isomorphic central linking systems associated to $\mathcal{F}$. Let $F\colon \mathcal{L}_1 \to \mathcal{L}_2$ be the functor corresponding to the isomorphisms. Let $\tilde{\sigma}_1\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L}_1)$ be a lift of the section σ. Then $\tilde{\sigma}_2 = F \circ \tilde{\sigma}_1\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L}_2)$ and $\pi_2 \circ \tilde{\sigma}_2 = \pi_2 \circ F \circ \tilde{\sigma}_1 = \pi_1 \circ \tilde{\sigma}_1 = \sigma$. For any $P \in \mathcal{F}^c$ we have that $\tilde{\sigma}_2(\mathrm{id}_P) = F(\tilde{\sigma}_1(\mathrm{id}_P)) = F(1_P) = 1_P$, since $F(P) = P$. Thus $\tilde{\sigma}_2$ is a lift of σ as well. We remark that

$$\tilde{\sigma}_2(\psi)\tilde{\sigma}_2(\varphi) = F(\tilde{\sigma}_1(\psi)\tilde{\sigma}_1(\varphi)) = F(\delta^i_R(t_1(\varphi,\psi))\tilde{\sigma}_1(\psi\varphi)) = \delta^2_R(t_1(\varphi,\psi))\tilde{\sigma}_2(\psi\varphi)$$

for any pair of morphisms $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$, hence $t_1 = t_2$.

Conversely assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two central linking systems associated to $\mathcal{F}$, for which there exists lifts of the section σ called $\tilde{\sigma}_i$ for $i = 1,2$ such that $t_1 = t_2$. Let $G_i\colon X(P,Q) \to \mathrm{Mor}_{\mathcal{L}_i}(P,Q)$ for $i = 1,2$ be the bijection given by $(g,\varphi) \mapsto \delta_Q(g)\tilde{\sigma}_i(\varphi)$. By definition $*$ by $t_1 = t_2$ we get a category with object set $\mathcal{F}^c$ and morphism set $X(P,Q)$ such that G_i and G_i^{-1} for $i = 1,2$ all are functors. Thus $F = G_2 \circ G_1^{-1}\colon \mathcal{L}_1 \to \mathcal{L}_2$ is a well-defined functor, which is the identity on objects and bijective on morphism sets. As $\tilde{\pi}_i \circ G_i = \pi_\sigma$ and $G_i(p,\mathrm{id}_P) = \delta^i_P(p)$ for $p \in P$, it follows that F commutes with $\tilde{\pi}_i$ and δ^i_P, and is a isomorphism of central linkings systems.

We conclude that two linking systems associated to $\mathcal{F}$ are isomorphic if and only if there exists lifts of the section σ such that $t_1 = t_2$. If $t_1 = t_2$, then by the definition of c we conclude that $[c] = 0$. If $c = dw$, then by setting $\tilde{\sigma}'_1(\varphi) = \tilde{\sigma}_1(\varphi)w(\varphi)$

and letting t'_i be the elements corresponding to this section, then for any pair of morphisms $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ we have that

$$t'_1(\varphi,\psi) = t_1(\varphi,\psi)\widetilde{\psi\varphi}(dw(\varphi,\psi)) = t_2(\varphi,\psi).$$

Thus two linking systems are isomorphic if and only if $[c]=0$.

Let $\mathcal{L}$ be a central linking system and $\tilde{\sigma}$ a lift of the section σ. Let $[c] \in \varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_\mathcal{F})$ be any class. If t is the choice of elements corresponding to $\tilde{\sigma}$, we can for any $\varphi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(Q,R)$ set $t'(\varphi,\psi) = t(\varphi,\psi)\widetilde{\psi\varphi}(c(\varphi,\psi))$. Then t' is compatible with σ and $u_{\sigma,t'} = dc \cdot u_{\sigma,t} = 1$ as $\mathcal{L}$ is a category. Similar to before there exists a central linking system $\mathcal{L}_c$ associated to $\mathcal{F}$ with a section $\tilde{\sigma}_c$ such that the corresponding elements are exactly t'. If we replace $\mathcal{L}$ with an isomorphic linking system or choose another representative for c the above arguments imply that the change in t' will be by a coboundary, so the resulting linking system will be isomorphic to $\mathcal{L}_c$. Thus we have a well-defined action by $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_\mathcal{F})$ on the set of isomorphism classes of central linking systems by setting $[c][\mathcal{L}] = [\mathcal{L}_c]$. The above results implies that this action is free and transitive. □

6.2. **Higher limits and $\mathcal{O}^c(\mathcal{F})$.** The above proposition shows that the ability to compute higher limits of functors over orbit categories is very useful in the study of fusion systems. We recall the following definition of graded groups $\Lambda^*(\Gamma; M)$, which are central in the study of higher limits.

Definition 6.3. *Let Γ be a finite group and M a $\mathbb{Z}_{(p)}[\Gamma]$-module. Let $\mathcal{O}_p(\Gamma)$ be the full subcategory of $\mathcal{O}(\Gamma)$ with object set the orbit Γ/P where P is a p-subgroup of Γ. Let $\mathbb{Z}_{(p)}$-mod be the category of additive functors from $\mathbb{Z}_{(p)} \to \mathsf{Ab}$. Note that this is in bijection with the category of $\mathbb{Z}_{(p)}$-modules. We define the functor $F_M \colon \mathcal{O}_p(\Gamma)^{op} \to \mathbb{Z}_{(p)}$-$\mathsf{mod}$ by $F_M(\Gamma/1) = M$ and $F_M(\Gamma/P) = 0$ for $P \neq 1$. The morphism set $\mathrm{Aut}_{\mathcal{O}_p(\Gamma)}(\Gamma/1) = \Gamma$, so we define the functor $F_M(\gamma)$ by the γ^{-1} action on M. We then set*

$$\Lambda^*(\Gamma; M) = \varprojlim_{\mathcal{O}_p(\Gamma)}{}^{*}(F_M).$$

The following proposition is an example on how useful these groups are, since it shows that for certain higher limits over the orbit category of a fusion system, we may replace the category by $\mathcal{O}_p(\Gamma)^{op}$ for a suitable group Γ.

Proposition 6.4. *Let $\mathcal{F}$ be a saturated fusion system over S. Let $\Phi \colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}$-$\mathsf{mod}$, which vanishes except on the isomorphism class of some $\mathcal{F}$-centric subgroup $Q \subseteq S$. Then*

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{*}(\Phi) \cong \Lambda^*(\mathrm{Out}_\mathcal{F}(Q); \Phi(Q)).$$

Proof. For an $\mathcal{F}$-centric subgroup Q we have that $\mathrm{Inn}(Q) \cong Q/\mathrm{Z}(Q)$, hence given two isomorphic $\mathcal{F}$-centric subgroups Q and Q' we have that $\mathrm{Out}_\mathcal{F}(Q) \cong \mathrm{Out}_\mathcal{F}(Q')$. We also have that $\Phi(Q) \cong \Phi(Q')$ and the action of $\mathrm{Out}_\mathcal{F}(Q)$ on $\Phi(Q)$ corresponds to the action of $\mathrm{Out}_\mathcal{F}(Q')$ on $\Phi(Q')$ under this isomorphism. Thus the groups on the right hand side only change up to isomorphism when replacing Q by Q', hence we can without loss of generality assume that Q is fully normalized in $\mathcal{F}$. As $\mathcal{F}$ is saturated we have that $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(Q))$ and thus $\mathrm{Out}_S(Q) \in \mathrm{Syl}_p(\mathrm{Out}_\mathcal{F}(Q))$. To simplify the notation we set $\Gamma = \mathrm{Out}_\mathcal{F}(Q)$ and $\Gamma_p = \mathrm{Out}_S(Q)$.

Note that the proposition now concerns comparing higher limits of functors over the categories $\mathcal{O}^c(\mathcal{F})$ and $\mathcal{O}_p(\Gamma)$. Let $\mathcal{O}_{\Gamma_p}(\Gamma)$ be the full subcategory of $\mathcal{O}_p(\Gamma)$ on the orbits Γ/Γ', where $\Gamma' \subseteq \Gamma_p$. By Sylows theorems we have that every p-subgroup Γ'' of Γ is Γ-conjugate to a $\Gamma' \subseteq \Gamma_p$, and if $\Gamma''^g = \Gamma'$ then the map $h\Gamma'' \mapsto hg^{-1}\Gamma'$ is a Γ-isomorphism from Γ/Γ'' to Γ/Γ'. Hence the categories $\mathcal{O}_{\Gamma_p}(\Gamma)$ and $\mathcal{O}_p(\Gamma)$ are equivalent. For any category $\mathcal{C}$ let the category of finite formal sums be denoted $\mathcal{C}_{\amalg}$. Then by the orbit decomposition we see that $\mathfrak{Set}_p(\Gamma) = \mathcal{O}_P(\Gamma)_{\amalg}$ is the category of finite Γ-sets, where all isotropy subgroups are p-groups, and the morphisms are Γ-maps. As the inclusion $i\colon \mathcal{O}_{\Gamma_p}(\Gamma) \to \mathcal{O}_p(\Gamma)$ is an equivalence of categories, we have that the same holds for the inclusion $i\colon \mathcal{O}_{\Gamma_p}(\Gamma)_{\amalg} \to \mathfrak{Set}_p(\Gamma)$. Let $s\colon \mathfrak{Set}_p(\Gamma) \to \mathcal{O}_{\Gamma_p}(\Gamma)_{\amalg}$ be an inverse.

We now want to define a functor $\bar{\alpha}\colon \mathcal{O}_{\Gamma_p}(\Gamma)_{\amalg} \to \mathcal{O}^c(\mathcal{F})_{\amalg}$. We set

$$\bar{\alpha}(\Gamma/\Gamma') = N_S^{\Gamma'}(Q) = \{x \in N_S(Q) \mid [c_x] \in \Gamma'\}$$

for any $\Gamma' \subseteq \Gamma_p = \mathrm{Out}_S(Q)$. Note that $\Gamma' = \{[c_x] \mid x \in N_S^{\Gamma'}(Q)\}$. We have that $Q \subseteq N_S(Q)$ and for any $g \in Q$ we see that $[c_g] = 1 \in \Gamma'$, hence $N_S^{\Gamma'}(Q)$ contains the $\mathcal{F}$-centric subgroup Q, and is therefore $\mathcal{F}$-centric. A Γ-map $f\colon \Gamma/\Gamma' \to \Gamma/\Gamma''$ is determined by $[\varphi] \in \Gamma$ such that $f(\Gamma') = [\varphi]\Gamma''$. The conditions for Γ-maps imply that $[\varphi]^{-1}\Gamma'[\varphi] \subseteq \Gamma''$. Then for every $x \in N_s^{\Gamma'}(Q)$ we have that $[\varphi^{-1} \circ c_x \circ \varphi] \in \Gamma''$. In particular $\varphi^{-1} \circ c_x \circ \varphi \in \mathrm{Aut}_S(Q)$. As Q is fully normalized this ensures the existence of an extension $\tilde{\varphi}^{-1}\colon \mathrm{Hom}_{\mathcal{F}}(N_S^{\Gamma'}(Q), S)$ of φ^{-1}. Then $c_{\tilde{\varphi}^{-1}(x)} = \varphi^{-1} \circ c_x \circ \varphi$ for any $x \in N_S^{\Gamma'}(S)$, so the image of $\tilde{\varphi}^{-1}$ is contained in $N_S^{\Gamma''}(Q)$. A different choice of φ corresponding to f may only differ with an element in $[c_g] \in \Gamma''$, where $g \in N_S^{\Gamma''}(Q)$. Then extension of $(\varphi \circ c_g)^{-1}$ is $c_g^{-1} \circ \tilde{\varphi}^{-1}$, so the Γ-map f defines a unique class $[\tilde{\varphi}^{-1}] \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(N_S^{\Gamma'}(Q), N_S^{\Gamma''}(Q))$. We set $\bar{\alpha}(f) = [\tilde{\varphi}^{-1}]$. The identity Γ-map corresponds to the identity in Γ and for Γ-maps $f\colon \Gamma/\Gamma' \to \Gamma/\Gamma''$ and $f'\colon \Gamma/\Gamma'' \to \Gamma/\Gamma'''$ corresponding to $[\varphi]$ and $[\varphi']$ we see that the composition $f' \circ f$ corresponds to $[\varphi \circ \varphi']$. From this we conclude that $\bar{\alpha}$ is a functor.

Our next goal is a functor $\beta\colon \mathcal{O}^c(\mathcal{F})_{\amalg} \to \mathfrak{Set}_p(\Gamma)$. For any $P \in \mathcal{F}^c$ we have a well-defined Γ action on $\mathrm{Rep}_{\mathcal{F}}(Q, P)$ be setting $[\varphi] \cdot [\psi] = [\psi \circ \varphi^{-1}]$ for $[\psi] \in \mathrm{Rep}_{\mathcal{F}}(Q, P)$ and $[\varphi] \in \Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$. Assume for $[\psi] \in \mathrm{Rep}_{\mathcal{F}}(Q, P)$ we have that $[\varphi] \in \Gamma$ lies in the isotropy subgroup of $[\psi]$. Then $[\psi \circ \varphi^{-1}] = [\psi]$ in $\mathrm{Rep}_{\mathcal{F}}(Q, P)$, so there exists an $x \in P$ such that $\psi \circ \varphi^{-1} = c_x \circ \psi$. Then for any $n > 0$ we have that $\psi \circ \varphi^{-n} = c_{x^n} \circ \psi$. As P is a finite p-group we have that $|x| = p^m$, so the equation for $n = p^m$ is $\psi \circ \varphi^{-p^m} = \psi$. As ψ is injective, we conclude that $\varphi^{-p^m} = \mathrm{id}_Q$, so $[\varphi] \in \Gamma$ has p order. Hence all the isotropy subgroups are p-subgroups of Γ. Note that for any $[\chi] \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ the map $[\varphi] \mapsto [\chi \circ \varphi]$ is a Γ-map from $\mathrm{Rep}_{\mathcal{F}}(Q, P)$ to $\mathrm{Rep}_{\mathcal{F}}(Q, P')$, hence we get a well-defined functor $\beta\colon \mathcal{O}^c(\mathcal{F})_{\amalg} \to \mathfrak{Set}_p(\Gamma)$ by setting $\beta(P) = \mathrm{Rep}_{\mathcal{F}}(Q, P)$ and $\beta([\chi]) = [\chi] \circ -$.

We will now construct an isomorphism

$$\mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(\bar{\alpha}(\Gamma/\Gamma'), P) \to \mathrm{Mor}_{\mathfrak{Set}_p(\Gamma)}(i(\Gamma/\Gamma'), \beta(P)) \tag{2}$$

which is natural in $\Gamma/\Gamma' \in \mathcal{O}_{\Gamma_p}(\Gamma)$ and $P \in \mathcal{O}^c(\mathcal{F})$. Fix a $\Gamma' \subseteq \Gamma_p$ and $P \in \mathcal{O}^c(\mathcal{F})$. As before $Q \subseteq N_S^{\Gamma'}(Q, P)$, so we may consider the restriction

$$\mu\colon\ \mathrm{Rep}_{\mathcal{F}}(N_S^{\Gamma'}(Q), P) \to \mathrm{Rep}_{\mathcal{F}}(Q, P).$$

By [10, Lemma A.8] we have that $\mu[\varphi] = \mu[\varphi']$ if there exists a $y \in \mathrm{Z}(Q)$ such that $\varphi = \varphi' \circ c_y$. Then

$$[\varphi] = [\varphi'] \circ [c_y] = [c_{\varphi'(y)}] \circ [\varphi'] = [\varphi'],$$

hence μ is injective. As before we have Γ action on the set $\mathrm{Rep}_{\mathcal{F}}(Q, P)$. For any $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(N_S^{\Gamma'}(Q), P)$ and $[c_x] \in \Gamma'$ we have that

$$[c_x] \cdot \mu[\varphi] = [\varphi|_Q \circ c_{x^{-1}}] = [c_{\varphi(x^{-1})} \circ \varphi] = [\varphi]$$

as $x \in N_S^{\Gamma'}(G)$. Thus we see that $\mathrm{im}(\mu) \subseteq \mathrm{Rep}_{\mathcal{F}}(Q, P)^{\Gamma'}$. Now we consider a $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(Q, P)^{\Gamma'}$ and $x \in N_S^{\Gamma'}(Q)$. Then $[c_{x^{-1}}] \in \Gamma'$ and

$$[\varphi] = [c_{x^{-1}}] \cdot [\varphi] = [\varphi \circ c_x]$$

so $\varphi c_x \varphi^{-1} \in \mathrm{Inn}(P)|_{\varphi(Q)} \subseteq \mathrm{Aut}_S(\varphi(Q))$. As Q is fully normalized, it is fully centralized. Since Q is $\mathcal{F}$-centric, the same holds for $\varphi(Q)$ and $C_S(Q) = \mathrm{Z}(Q) \cong \mathrm{Z}(\varphi(Q)) = C_S(\varphi(Q))$, hence $\varphi(Q)$ is fully centralized as well. As $N_S^{\Gamma'}(Q) \subseteq N_\varphi$ we conclude that there exists an extension $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S^{\Gamma'}(Q), S)$ of φ. For $x \in N_S^{\Gamma'}(Q)$ we have that $\varphi \circ c_x = c_{\tilde{\varphi}(x)} \circ \varphi$, so by the above $[\varphi] = [c_{\tilde{\varphi}(x)} \circ \varphi]$ in $\mathrm{Rep}_{\mathcal{F}}(Q, P)$. This implies that there exists a $y \in P$ such that $\tilde{\varphi}(x)y \in C_S(\varphi(Q)) \subseteq \varphi(Q) \subseteq P$. Hence the image of $\tilde{\varphi}$ is contained in P, so we may consider $[\tilde{\varphi}] \in \mathrm{Rep}_{\mathcal{F}}(N_S^{\Gamma'}(Q), P)$. Then $\mu[\tilde{\varphi}] = [\varphi]$, so μ is a bijection between $\mathrm{Rep}_{\mathcal{F}}(N_A^{\Gamma'}(Q), P)$ and $\mathrm{Rep}_{\mathcal{F}}(Q, P)^{\Gamma'}$.

For any $x \in \mathrm{Rep}_{\mathcal{F}}(Q, P)^{\Gamma'}$ the map $f_x \colon \Gamma/\Gamma' \to \mathrm{Rep}_{\mathcal{F}}(Q, P)$ given by $f_x(\gamma\Gamma') = \gamma \cdot x$ is a well-defined Γ-map, and conversely for every Γ-map $f \colon \Gamma/\Gamma' \to \mathrm{Rep}_{\mathcal{F}}(Q, P)$ we have that $f(\Gamma') \in \mathrm{Rep}_{\mathcal{F}}(Q, P)^{\Gamma'}$, hence we have established a bijection

$$\mu_0 \colon \mathrm{Rep}_{\mathcal{F}}(N_S^{\Gamma'}(Q), P) \cong \mathrm{Map}_\Gamma(\Gamma/\Gamma', \mathrm{Rep}_{\mathcal{F}}(Q, P))$$

by $[\varphi]$ by mapping to $\gamma\Gamma' \mapsto \gamma \cdot [\varphi|_Q]$. This bijection is clearly natural in the second variable with respect to morphisms $\mathrm{Rep}_{\mathcal{F}}(P, P')$, since this is simply post-composition. To see that it natural in the first variable consider $f \in \mathrm{Mor}_{\mathcal{O}_{\Gamma_p}(\Gamma)}(\Gamma/\Gamma', \Gamma/\Gamma'')$ given by $f(\Gamma') = [\varphi]\Gamma''$ and $[\psi] \in \mathrm{Rep}_{\mathcal{F}}(N_S^{\Gamma'}(Q), P)$, then $\mu_0([\psi]) \circ f$ is given by $\Gamma' \mapsto [\psi|_Q \circ \varphi^{-1}]$ while $\mu \circ \bar{\alpha}(f)$ is given by $\Gamma' \mapsto [(\psi \circ \tilde{\varphi}^{-1})|_Q]$. As $\tilde{\varphi}$ is an extension of $\varphi \in \mathrm{Aut}(Q)$ the result follows.

As i is an equivalence of categories we have a bijection

$$\mathrm{Mor}_{\mathfrak{Set}_p(\Gamma)}(i(s(X)), \beta(P)) \cong \mathrm{Mor}_{\mathfrak{Set}_p(\Gamma)}(X, \beta(P)) \tag{3}$$

which is natural in the first variable. It is also natural in the second variable as before. We set $\alpha = \bar{\alpha} \circ s \colon \mathfrak{Set}_p(\Gamma) \to \mathcal{O}^c(\mathcal{F})_{\amalg}$. Then we get directly from (2) and (3) that the functors α and β are adjoint.

For any category $\mathcal{C}$ we let $\mathcal{C}$-mod be the category of functors $\mathcal{C}^{op} \to \mathsf{Ab}$. Any functor F from $\mathcal{C}$ to $\mathcal{C}'$ will induce a functor from $\mathcal{C}'$-$\mathsf{mod} \to \mathcal{C}$-$\mathsf{mod}$ by pre-composition with F^{op}. Thus we have functors $\alpha^* \colon (\mathcal{O}^c(\mathcal{F})_{\amalg})$-$\mathsf{mod} \to \mathfrak{Set}_p(\Gamma)$-$\mathsf{mod}$ and $\beta^* \colon \mathfrak{Set}_p(\Gamma)$-$\mathsf{mod} \to (\mathcal{O}^c(\mathcal{F})_{\amalg})$-$\mathsf{mod}$ induced by α and β. As α and β are adjoint, the same holds for α^* and β^*. The category $\mathcal{C}_{\amalg}$-mod contains a subcategory of functors F satisfying that $F(\coprod_i c_i) = \oplus_i F(c_i)$ and similarly for morphisms and natural transformations. This subcategory is equivalent to $\mathcal{C}$-mod. As both functors α and β on a formal sum is defined as the formal sum of the images, we see that α^* and β^* are functors between the subcategories of this form. Thus using this equivalence we get adjoint functors between $\mathcal{O}_p(\Gamma)$-mod and $\mathcal{O}^c(\mathcal{F})$-$\mathsf{mod}$. If

$$F_1 \xrightarrow{\tau} F_2 \xrightarrow{\sigma} F_3$$

is an exact sequence in $\mathcal{O}^c(\mathcal{F})$-mod and $X \in \mathcal{O}_p(\Gamma)$ with $\alpha(X) = \coprod_i P_i$, then $F_j(\alpha(X)) = \oplus_i F_j(P_i)$ and

$$F_1(P_i) \xrightarrow{\tau(P_i)} F_2(P_i) \xrightarrow{\sigma(P_i)} F_3(P_i)$$

is exact for all i. As $\tau(\alpha X) = \oplus_i(\tau(P_i))$, we get that the sequence

$$F_1 \circ \alpha \xrightarrow{\tau \circ \alpha} F_2 \circ \alpha \xrightarrow{\sigma \circ \alpha} F_3 \circ \alpha$$

is exact as well, hence α^* preserves exact sequences. A similar argument holds for β^*. As β^* is a functor it thus preserves split exact sequences, and as injective objects can be characterized by split exact sequences, we conclude that β^* sends injectives to injectives.

Consider $\alpha^*\Phi \in \mathcal{O}_p(\Gamma)$-mod. As $s(\Gamma/1) = \Gamma/1$ we have that

$$\alpha(\Gamma/1) = \bar{\alpha}(\Gamma/1) = N_S^1(Q) = \{x \in N_S(Q) \mid c_x \in \mathrm{Inn}(Q)\} = Q,$$

hence $\alpha^*\Phi(\Gamma/1) = \Phi(Q)$. Observe that $\mathrm{Aut}_{\mathcal{O}_p(\Gamma)}(\Gamma/1) = \Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ and $\alpha([\varphi]) = [\varphi^{-1}]$. The same holds for α^{op} so we see that $\alpha^*\Phi([\varphi]) = \Phi([\varphi^{-1}])$, which is the action of $[\varphi]$ on $\Phi(Q)$. If $\Gamma' \subseteq \Gamma$ is a non-trivial p-subgroup, then $s(\Gamma/\Gamma') = \Gamma/\Gamma''$ where Γ'' is isomorphic to Γ'. In particular $\Gamma'' \neq 1$, so there exists an $x \in N_S^{\Gamma''}(Q)$ so $c_x \notin \mathrm{Inn}(Q)$, i.e. $x \notin Q$. Then $Q \subsetneq N_S^{\Gamma''}(Q)$, so $N_S^{\Gamma''}(Q)$ is not $\mathcal{F}$-conjugate to Q. Hence we have that $\alpha^*\Phi(\Gamma/\Gamma') = \Phi(N_S^{\Gamma''}(Q)) = 0$. Thus we conclude that $\alpha^*\Phi = F_{\Phi(Q)}$, where we consider $\Phi(Q)$ as a $\mathbb{Z}_{(p)}[\Gamma]$-module.

Let $\underline{\mathbb{Z}}$ be the constant functor on $\mathcal{O}^c(\mathcal{F})^{op}$ sending all objects to $\mathbb{Z}$ and all morphisms to the identity on $\mathbb{Z}$. As α sends objects of $\mathcal{O}_p(\Gamma)$ to objects in $\mathcal{O}^c(\mathcal{F})$ and not a formal sum, we see that $\alpha^*(\underline{\mathbb{Z}})$ is the constant functor on $\mathcal{O}_p(\Gamma)^{op}$. For any $D \in \mathcal{C}$-mod, where $\mathcal{C}$ is either $\mathcal{O}^c(\mathcal{F})$ or $\mathcal{O}_p(\Gamma)$ we have that

$$\varprojlim_{\mathcal{C}}{}^0(D) \cong \mathrm{Hom}_{\mathcal{C}\text{-mod}}(\underline{\mathbb{Z}}, D).$$

Let I_* be a injective resolution of $\alpha^*\Phi$. Then we may compute the higher limits as the cohomology of $\mathrm{Hom}_{\mathcal{O}_p(\Gamma)\text{-mod}}(\alpha^*(\underline{\mathbb{Z}}), I_*)$. As β^* respects exact sequences and sends injectives to injectives we have that $\beta^* I_*$ is an injective resolution of $\beta^*(\alpha^*\Phi)$, so we can compute the higher limits of $\beta^*(\alpha^*\Phi)$ as the cohomology of $\mathrm{Hom}_{\mathcal{O}^c(\mathcal{F})\text{-mod}}(\underline{\mathbb{Z}}, \beta^* I_*)$. As the adjunction between α and β is natural in both entries, the same holds for the induced functors α^* and β^* between $\mathcal{O}_p(\Gamma)$-mod and $\mathcal{O}^c(\mathcal{F})$-mod, hence the adjunction induces an isomorphism between the chain complexes $\mathrm{Hom}_{\mathcal{O}_p(\Gamma)\text{-mod}}(\underline{\mathbb{Z}}, \beta^* I_*)$ and $\mathrm{Hom}_{\mathcal{O}^c(\mathcal{F})\text{-mod}}(\alpha^*\underline{\mathbb{Z}}, I_*)$. Thus we see that for any i,

$$\begin{aligned}\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(\beta^*\alpha^*\Phi) &\cong \mathrm{H}^i(\mathrm{Hom}_{\mathcal{O}^c(\mathcal{F})\text{-mod}}(\underline{\mathbb{Z}}, \beta^* I_*)) \cong \mathrm{H}^i(\mathrm{Hom}_{\mathcal{O}_p(\Gamma)\text{-mod}}(\alpha^*\underline{\mathbb{Z}}, I_*) \\ &\cong \varprojlim_{\mathcal{O}_p(\Gamma)}{}^i(\alpha^*\Phi) = \Lambda^i(\Gamma; \Phi(Q)).\end{aligned}$$

It is therefore sufficient to prove that $\beta^*(\alpha^*\Phi) \cong \Phi$.

For any $P \subseteq S$ which is $\mathcal{F}$-centric we choose $m \in \mathbb{N}$ and $\varphi_i \in \mathrm{Rep}_{\mathcal{F}}(Q, P)$ for $1 \leq i \leq m$ such that $\mathrm{Rep}_{\mathcal{F}}(Q, P) = \coprod_{i=1}^m \Gamma \cdot \varphi_i$ is a orbit decomposition with respect to the Γ-action. For any $1 \leq i \leq m$ let Γ_{φ_i} be the isotropy subgroup of φ_i. Then the orbit $\Gamma \cdot \varphi_i$ is isomorphic to $\Gamma/\Gamma_{\varphi_i}$. For any $[\psi] \in \mathrm{Out}_P(\bar{\varphi}_i(Q))$, where $\bar{\varphi}_i$

is a representative for the class φ_i. We see that

$$[\bar{\varphi}_i^{-1} \circ \psi \circ \bar{\varphi}_i] \cdot [\bar{\varphi}_i] = [\psi^{-1} \circ \bar{\varphi}_i] = [\bar{\varphi}_i]$$

as ψ is an element of $\mathrm{Inn}(P)$, hence $\varphi_i^{-1}(\mathrm{Out}_P(\bar{\varphi}_i(Q)))\varphi_i \subseteq \Gamma_{\varphi_i}$. Assume that $\bar{\varphi}_i(Q)$ is a proper subgroup of P. Then by results of finite p-groups we have that there exists $x \in N_P(\bar{\varphi}(Q))\backslash\bar{\varphi}_i(Q)$. As Q is $\mathcal{F}$-centric we conclude that $C_S(\bar{\varphi}_i(Q)) = \bar{\varphi}_i(Q)$, so we have that $c_x \notin \mathrm{Inn}(\bar{\varphi}_i(Q))$. Then the class $[c_x] \in \mathrm{Aut}_P(\bar{\varphi}_i(Q))$ is non-trivial, so $\mathrm{Aut}_P(\bar{\varphi}_i(Q)) \neq 1$ and similarly we have that $\varphi_i^{-1}(\mathrm{Out}_P(\bar{\varphi}_i(Q)))\varphi_i \neq 1$. Thus we conclude that $\Gamma_{\varphi_i} \neq 1$.

Assume that P is not $\mathcal{F}$-conjugate to Q. Then $\bar{\varphi}_i(Q) \neq P$ for all $1 \leq i \leq m$, so $\Gamma_{\varphi_i} \neq 1$ for all $1 \leq i \leq m$. By the previous results we have that $\alpha^*\Phi(\Gamma \cdot \varphi_i) \cong \alpha^*\Phi(\Gamma/\Gamma_{\varphi_i}) = 0$, and therefore $\beta^*(\alpha^*(\Phi(P))) = 0$. Assume that P is $\mathcal{F}$-conjugate to Q, and let $\varphi \in \mathrm{Rep}_{\mathcal{F}}(Q,P)$. Then for any $\psi \in \mathrm{Rep}_{\mathcal{F}}(Q,P)$ we have that $\psi^{-1} \circ \varphi \in \Gamma$ and $(\psi^{-1} \circ \varphi) \cdot \varphi = \psi$, hence $\mathrm{Rep}_{\mathcal{F}}(Q,P) = \Gamma \cdot \varphi$. Furthermore for any $\psi \in \Gamma$ we have that $\varphi = \psi \cdot \varphi = \varphi \circ \psi^{-1}$ implies that $\psi = 1$ as φ is a bijection. Hence $\mathrm{Rep}_{\mathcal{F}}(Q,P)$ consists of only one free orbit and thus we have that $s(\mathrm{Rep}_{\mathcal{F}}(Q,P)) = \Gamma/1$. Then the natural isomorphism between $\mathrm{id}_{\mathcal{O}_p(\Gamma)}$ and $i \circ s$ will provide an isomorphism i_P between $\mathrm{Rep}_{\mathcal{F}}(Q,P)$ and $\Gamma/1$, which is natural with respect to $\beta(\mathrm{Rep}_{\mathcal{F}}(P,P'))$ if P' is $\mathcal{F}$-conjugate to Q. We choose for any P which is $\mathcal{F}$-conjugate to Q a $\varphi_P \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ such that $i_P(\varphi_P) = 1_\Gamma$. Then for any $\psi \in \mathrm{Rep}_{\mathcal{F}}(P,P')$ where both P and P' are $\mathcal{F}$-conjugate to Q the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathcal{F}}(Q,P) \arrow[r, "i_P"] \arrow[d, "\beta(\psi)"'] & \Gamma \arrow[d, "s(\beta(\psi))"'] \\
\mathrm{Rep}_{\mathcal{F}}(Q,P') \arrow[r, "i'_P"] & \Gamma
\end{tikzcd}$$

The commutativity of the diagram combined with the fact that the i's are isomorphisms implies that $s(\beta(\psi))\colon \Gamma \to \Gamma$ is the Γ-map given by $1 \mapsto (\varphi_P^{-1} \circ \psi^{-1} \circ \varphi_{P'}) \cdot 1$. We now see that $\alpha(\beta(\psi)) = \varphi_{P'}^{-1} \circ \psi \circ \varphi_P$, so the following diagram commutes in $\mathcal{O}^c(\mathcal{F})$:

$$\begin{tikzcd}
\alpha(\beta(P)) = Q \arrow[r, "\varphi_P"] \arrow[d, "\alpha(\beta(\psi))"'] & P \arrow[d, "\psi"'] \\
\alpha(\beta(P')) = Q \arrow[r, "\varphi'_P"] & P'
\end{tikzcd}$$

By applying Φ to the corresponding diagram in the opposite category, we get isomorphisms $\Phi(P) \to \beta^*(\alpha^*(\Phi(P)))$ for any P $\mathcal{F}$-conjugate to Q, which are natural with respect to morphisms in $\mathrm{Rep}_{\mathcal{F}}(P,P')$, if P' is $\mathcal{F}$-conjugate to Q. As both $\beta^*(\alpha^*(\Phi))$ and Φ are trivial for any $P \in \mathcal{F}^c$ not $\mathcal{F}$-conjugate to Q the uniqueness of module morphisms, where either the source or target is trivial, implies that this can be extended to a natural isomorphism of functors Φ and $\beta^*(\alpha^*(\Phi))$. $\square$

6.3. **Projective summands in the Steinberg complex.** We will in this chapter analyze the projective summands of the Steinberg complex to give some simple requirement for the vanishing of Λ^*. We consider the following complex:

Definition 6.5. *For any finite group G let $\mathcal{S}_p(G)$ be the category of p-subgroups where the morphisms are inclusions. Then we set $\mathrm{st}_*(G)$ called the Steinberg complex of G to be the reduced normalized simplical chain complex $\tilde{C}_*(|\mathcal{S}_p(G)|, \mathbb{Z}_{(p)})$.*

A reformulation of a special case of [18, Theorem 1.1.] is that for any $\mathbb{Z}_{(p)}$-module M we have that $\Lambda^i(G, M) \cong \mathrm{H}^{i-1}(\mathrm{Hom}_G(\mathrm{st}_*(G), M)$. Thus, to compute the higher limits of the form Λ^* we will investigate the structure of the Steinberg complex. By [35, Theorem 2.7.1] we conclude that $\mathrm{st}_*(G; \mathbb{Z}_p) = D_* \oplus P_*$, where D_* is a $\mathbb{Z}_p[G]$-split acyclic complex and P_* is a complex of projective $\mathbb{Z}_p[G]$-modules. The ring of p-adic integer $\mathbb{Z}_p$ is a complete discrete valuation ring. When G is a finite group, we have that the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[G]$ is a finitely generated $\mathbb{Z}_p$-module, hence Krüll-Schmidts Theorem holds for $\mathbb{Z}_p[G]$-modules by [13, Theorem 6.12]. This also holds for bounded chain complexes of $\mathbb{Z}_p[G]$-modules. So there exists a unique minimal P_* with this property. We denote this $\tilde{\mathrm{st}}_*(G; \mathbb{Z}_p)$.

Note that $\mathrm{H}^{i-1}(\mathrm{Hom}_G(\mathrm{st}_*(G), M)) = \mathrm{H}^{i-1}(\mathrm{Hom}_G(\tilde{\mathrm{st}}_*(G), M))$ for any $\mathbb{Z}_p[G]$-module M, so for our purpose we need to investigate which projective modules may occur in P_*. The following lemma will be central in this regard.

Recall that for a R-module M the socle of M, writtenSoc(M), is the sum of all semi-simple submodules of M. Dually the radical of M, written $\mathrm{Rad}(M)$, is the intersection of all submodules with semi-simple quotients, and we have that the head of M is $M/\mathrm{Rad}(M)$. Any $f\colon M \to N$ of R-modules satisfies that $f(\mathrm{Soc}(M)) \subseteq \mathrm{Soc}(N)$ and $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(N)$. When M is semi-simple we have that $\mathrm{Soc}(M) = M$ so we note that $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_R(M, \mathrm{Soc}(N))$ whenever M is semi-simple. Similar if M is semi-simple then $\mathrm{Rad}(M) = 0$, so for any $f\colon N \to M$ we have that $f(\mathrm{Rad}(N)) = 0$, and thus we get a bijection $\mathrm{Hom}_R(N, M) = \mathrm{Hom}_R(N/\mathrm{Rad}(N), M)$.

In our case R will be group-rings $k[G]$, where k is a field and G is a finite group. Let P_S be the projective cover over $k[G]$ of a simple nontrivial $k[G]$-module S. Then both the socle and head of P_S is isomorphic to S by [5, Proposition 3.1.2]. As a simple module T is semi-simple as well the above remarks implies that $\mathrm{Hom}_G(T, P_S) = \mathrm{Hom}_G(T, S)$ and $\mathrm{Hom}_G(P_S, T) = \mathrm{Hom}_G(S, T)$. As both S and T are simple, a $k[G]$-morphism between them is either an isomorphism or trivial, so we conclude that $\mathrm{Hom}_G(T, P_S)$ and $\mathrm{Hom}_G(T, P_S)$ are nontrivial if and only if $T \cong S$ and in this case we can identify it with $\mathrm{End}_G(T)$.

Lemma 6.6. *Let k be a field of characteristic p, and assume that H is a subgroup of a finite group G. Let S be a simple $k[G]$-module and let T be a simple $k[H]$-module. Let P_S and P_T be the respective projective covers. Then $[\mathrm{End}_G S : k]$ times the multiplicity of P_S in $T \uparrow_H^G$ equals $[\mathrm{End}_H T : k]$ times the multiplicity of P_T in $S \downarrow_H^G$*

Proof. As the field k is a complete discrete valuation ring, the Krüll-Schmidt Theorem holds for the group ring $k[G]$ by [13, Theorem 6.12]. Thus every $k[G]$-module has a unique decomposition as a finite direct sum of indecomposable modules. Let $\Omega_0(M)$ be the direct sum of all the non-projective indecomposable components. Then $\Omega_0(M)$ does not contain any projective summands, and $M = \Omega_0(M) \oplus P$, where P is projective. By [34, Chapter 14.3. Corollary 1] we have that the indecomposable projective $k[G]$-modules are exactly the projective covers of simple $k[G]$-modules. Furthermore, [34, Chapter 14.3. Proposition 41] implies that two projective covers are isomorphic if and only if the corresponding simple modules are isomorphic.

Write $S\downarrow_H^G = \Omega_0(S\downarrow_H^G) \oplus (\oplus_{\tilde{S}} m_{\tilde{S}} P_{\tilde{S}})$, where $\tilde{S}$ are distinct simple non-trivial $k[H]$-modules and $m_{\tilde{S}}$ is the multiplicity of the projective cover $P_{\tilde{S}}$. Then the co-kernel of the inclusion $\operatorname{Hom}_H(T, \Omega_0(S\downarrow_H^G)) \hookrightarrow \operatorname{Hom}_H(T, S\downarrow_H^G)$ can be identified with $\operatorname{Hom}_H(T, \oplus_{\tilde{S}} m_{\tilde{S}} P_{\tilde{S}})$. As noted for $\tilde{S} \neq T$ we have that $\operatorname{Hom}_H(T, P_{\tilde{S}}) = 0$, so we see that

$$
\begin{aligned}
\dim_k \operatorname{Hom}_H(T, \oplus_{\tilde{S}} m_{\tilde{S}} P_{\tilde{S}}) &= \dim_k \operatorname{Hom}_H(T, m_T P_T) = \sum_{m_T} \dim_k \operatorname{Hom}_H(T, P_T) \\
&= m_T \dim_k \operatorname{Hom}_H(T, T) = m_T \dim_k \operatorname{End}_H(T)
\end{aligned}
$$

Thus we conclude that the multiplicity m_T of P_T in $S\downarrow_H^G$ times $[\operatorname{End}_H(T) : k]$ is the dimension of co-kernel of $\operatorname{Hom}_H(T, \Omega_0(S\downarrow_H^G)) \hookrightarrow \operatorname{Hom}_H(T, S\downarrow_H^G)$ over k. By a similar argument we conclude that the multiplicity of P_S in $T\uparrow_H^G$ times $[\operatorname{End}_G(S) : k]$ is the k-dimension of co-kernel of $\operatorname{Hom}_G(\Omega_0(T\uparrow_H^G), S) \hookrightarrow \operatorname{Hom}_G(T\uparrow_H^G, S)$. By Frobenius reciprocity [5, Proposition 3.3.1] we have that $\operatorname{Hom}_G(T\uparrow_H^G, S) \cong \operatorname{Hom}_H(T, S\downarrow_H^G)$, so the result follows if $\operatorname{Hom}_G(\Omega_0(T\uparrow_H^G), S) \cong \operatorname{Hom}_H(T, \Omega_0(S\downarrow_H^G))$, since we get a bijection between the co-kernels. The way will be through the stable module category.

Let $\alpha \in \Omega_0(T\uparrow_H^G) \to S$ be a $k[G]$-map that factors trough a projective module P, i.e. there exist $k[G]$-maps $\alpha' \colon \Omega_0(T\uparrow_H^G) \to P$ and $\beta \colon P \to S$, such that $\alpha = \beta \circ \alpha'$. Assume that α is non-trivial. Then β must be non-trivial as well. Since S is simple, we conclude that β is surjective. Let $\psi : P_S \to S$ be the epimorphism from the definition of projective cover. As P is projective there exists a $\tilde{\psi} \colon P \to P_S$, such that $\psi \circ \tilde{\psi} = \beta$. By setting $\tilde{\alpha} = \tilde{\psi} \circ \alpha' \colon \Omega_0(T\uparrow_H^G) \to P_S$ we see that α factors through P_S via $\tilde{\alpha}$ and the essential homomorphism ψ. As α is non-trivial we have that α is surjective. Assume that $\tilde{\alpha}$ is not surjective, then $\operatorname{im}(\tilde{\alpha}) \neq P_S$ so by definition of essential homomorphism we conclude that $\operatorname{im}(\alpha) = \operatorname{im}(\psi \circ \tilde{\alpha}) \neq S$ and a contradiction arises since α is surjective. Hence we conclude that α is trivial.

Let $f, g \in \operatorname{Hom}_G(T\uparrow_H^G, S)$ and assume that they agree on the non projective summand $\Omega_0(T\uparrow_H^G)$. Then $f - g$ is trivial on $\Omega_0(T\uparrow_H^G)$, so using the projection onto the projective summand, we conclude that $f - g$ factors trough a projective module. Similarly let $f, g \in \operatorname{Hom}_G(T\uparrow_H^G, S)$ such that $f - g$ factors trough a projective module, then the same holds for $f - g$ restricted to $\Omega_0(T\uparrow_H^G)$. So by the above $f = g$ on $\Omega_0(T\uparrow_H^G)$. Thus f and g determine the same class in $\underline{\operatorname{Hom}}_G(T\uparrow_H^G, S)$ if and only if they agree on $\Omega_0(T\uparrow_H^G)$, hence we have a bijection between $\underline{\operatorname{Hom}}_G(T\uparrow_H^G, S)$ and $\operatorname{Hom}_G(\Omega_0(T\uparrow_H^G), S)$. A dual argument gives a bijection from $\underline{\operatorname{Hom}}_H(T, S\downarrow_H^G)$ to $\operatorname{Hom}_H(T, \Omega_0(S\downarrow_H^G))$. The result now follows by Frobenius reciprocity in the stable module category [1, page 74]. □

We know that the ring of p-adic integer $\mathbb{Z}_p$ is a local ring with factor ring $\mathbb{F}_p$. By Proposition [34, Proposition 42, 4.14], we now conclude that there is a bijection between projective $\mathbb{F}_p[G]$-modules and projective $\mathbb{Z}_p[G]$-modules. Using the correspondence and the above lemma, we get:

Proposition 6.7. *Let P_S be the projective module over $\mathbb{Z}_p[G]$ corresponding to the projective cover over $\mathbb{F}_p[G]$ of a simple $\mathbb{F}_p[G]$-module S. Assume that P_S appears as a summand in $\tilde{\mathrm{st}}_m(G;\mathbb{Z}_p)$. Then the following holds:*

(1) *No elements of order p in G act trivially on S.*
(2) *There exists an elementary abelian p-subgroup V of G with $\operatorname{rk} V \geq m+1$ such that $S\downarrow^S_{C_S(V)}$ contains the projective cover $P_{\mathbb{F}_p}$ of the trivial $\mathbb{F}_p[C_S(V)]$-module $\mathbb{F}_p$. Furthermore, $\dim_{\mathbb{F}_p} S \geq |C_S(V)|_p \geq p^{m+1}$.*

Proof. As $\tilde{\mathrm{st}}_m(G)$ is a G-set, the orbit decomposition implies that $\tilde{\mathrm{st}}_m(G;\mathbb{Z}_p) = \bigoplus_i \mathbb{Z}_p[G/G_{\sigma_i}]$ for some $\sigma_i \in |\mathcal{S}_p(G)|_m$. The projective cover over $\mathbb{F}_p[G]$ of S is irreducible, so by [34, Corollary 1 4.14] we have that P_S is irreducible. As P_S is a summand of $\tilde{\mathrm{st}}_m(G;\mathbb{Z}_p)$, there exists some $\sigma \in |\mathcal{S}_p(G)|_m$ such that P_S is a direct summand of $\mathbb{Z}_p[G/G_\sigma]$. Then the projective cover $\tilde{P}_S = P_S/pP_S$ of the simple $\mathbb{F}_p[G]$-module S is a direct summand of $\mathbb{F}_p[G/G_\sigma]$.

Set $K = \ker(G \to \mathrm{Aut}(S))$ and assume that there exists an element of order p in K. We have that σ corresponds to a chain $P_0 \subseteq \cdots \subseteq P_m$ of p-subgroups of G. Let P be a Sylow-p-subgroup of G containing the chain. A Sylow-p-subgroup of K is maximal among the groups $\tilde{P} \cap K$, where $\tilde{P} \in \mathrm{Syl}_p(G)$. As G acts transitively on $\mathrm{Syl}_p(G)$ by conjugation and K is normal in G, we see that the groups $\tilde{P} \cap K$ all agree. In particular $P \cap K \in \mathrm{Syl}_p(K)$. As there exists element of order p in K we conclude that $P \cap K \neq 1$. Then $P \cap K$ is a normal subgroup of a p-group, so it has nontrivial intersection with the center $\mathrm{Z}(P)$. Let $g \in K \cap \mathrm{Z}(P)$ be non-trivial. Then $g \in N_G(P_i)$ for $0 \leq i \leq m$, so $g \in \bigcap_{i=1}^m N_G(P_i) = G_\sigma$. Let $\mathbb{F}_p$ be the trivial $\mathbb{F}_p[G_\sigma]$-module. In particular it is a simple module. As both S and $\mathbb{F}_p$ are nontrivial modules, we have that both $[\mathrm{End}_G(S);\mathbb{F}_p]$ and $[\mathrm{End}_{G_\sigma}(\mathbb{F}_p);\mathbb{F}_p]$ are non-zero. Furthermore the induced representation $\mathbb{F}_p \uparrow^G_{G_\sigma} = \mathbb{F}_p[G/G_\sigma]$, so the multiplicity of $\tilde{P}_S$ as a summand in $\mathbb{F}_p[G/G_\sigma]$ is non-zero. Lemma 6.6 now implies that the multiplicity of the projective cover $P_{\mathbb{F}_p}$ of $\mathbb{F}_p$ in $S\downarrow^G_{G_\sigma}$ is non-trivial. Then $S\downarrow^G_{G_\sigma}$ contains a projective summand F, which is a direct summand of a free $\mathbb{F}_p[G_\sigma]$-module. As $\mathbb{F}_p[G_\sigma]\downarrow^{G_\sigma}_{\langle g\rangle} \cong \bigoplus_{[G_\sigma:\langle g\rangle]} \mathbb{F}_p[\langle g\rangle]$, we conclude that $F\downarrow^{G_\sigma}_{\langle g\rangle}$ is a direct summand of a free $\mathbb{F}_p[\langle g\rangle]$-module and thus projective. So restriction $S\downarrow^G_{\langle g\rangle}$ contains the projective summand $F\downarrow^{G_\sigma}_{\langle g\rangle}$. Since $g \in K$ the action of $\langle g\rangle$ on $S\downarrow^G_{\langle g\rangle}$ is trivial, hence the action on every direct summand is trivial. But $F\downarrow^{G_\sigma}_{\langle g\rangle}$ is a direct summand of a free $\mathbb{F}_p[\langle g\rangle]$-module, so the $\langle g\rangle$-action is free as well. Hence $\langle g\rangle = 1$ and a contradiction arises.

Let $\mathcal{A}_p(G)$ be the category of non-trivial elementary abelian p-subgroups of S with inclusion. By [31, Proposition 2.1] the inclusion $|\mathcal{A}_p(G)|$ to $|\mathcal{S}_p(G)|$ is a homotopy equivalence. Furthermore it respects the G-action, so the chain complexes $\tilde{C}_*(|\mathcal{A}_p(G)|;\mathbb{Z}_{(p)})$ and $\mathrm{st}_*(G)$ are G-homotopy equivalent. Thus $\tilde{C}_*(|\mathcal{A}_p(G)|;\mathbb{Z}_{(p)}) \otimes \mathbb{Z}_p$ and $\mathrm{st}_*(G;\mathbb{Z}_p)$ are $\mathbb{Z}_p[G]$-homotopy equivalent. Let f_* be a chain homotopy equivalence. Then $\mathrm{H}_*(f)$ is an isomorphism.

The [35, Theorem 2.7.1] holds for $\tilde{C}_*(|\mathcal{A}_p(G)|;\mathbb{Z}_{(p)}) \otimes \mathbb{Z}_p$ as well. Let $\tilde{P}_*$ be the minimal projective module, such that $\tilde{C}_*(|\mathcal{A}_p(G)|;\mathbb{Z}_{(p)}) \otimes \mathbb{Z}_p$ is the direct sum of $\tilde{P}_*$ and an acyclic split complex. Then the induced map $H_*(\tilde{P}_*) \to \mathrm{H}_*(\tilde{\mathrm{st}}_*(G;\mathbb{Z}_p))$ from f_* must be an isomorphism, since the other two summand are acyclic. An argument similar to [6, Lemma 5.17.1.] gives a splitting of $\tilde{P}_* = C'_* \oplus P'_*$ and $\tilde{\mathrm{st}}_*(G;\mathbb{Z}_p) = D'_* \oplus Q'_*$, where P'_* and Q'_* are exact sequences of projective modules and the restriction of f from $C'_* \to D'_*$ is an isomorphism. As $\tilde{P}_*$ and $\tilde{\mathrm{st}}_*(G;\mathbb{Z}_p)$ are projective chain-complexes, the same is true for C'_* and D'_*. The minimality requirements on $\tilde{P}_*$ and $\tilde{\mathrm{st}}_*(G;\mathbb{Z}_p)$ imply that both P'_* and Q'_* are trivial, so we conclude that $\tilde{P}_*$ and $\tilde{\mathrm{st}}_*(G;\mathbb{Z}_p)$ are isomorphic. By the uniqueness part of the Krüll-Schmidt theorem we conclude that P_S is a projective summand of $\tilde{P}_*$ as well.

Similar to before, we see that there exists some $\sigma \in |\mathcal{A}_p(G)|_m$ such that P_S is a direct summand of $\mathbb{Z}_p[G/G_\sigma]$. Let σ be given by the non-degenerate m-simplex $P_0 \subseteq P_1 \cdots \subseteq P_m$ in $\mathcal{A}_p(G)$. As this contains m strict inclusions between non-trivial elementary abelian p-groups, we conclude that $V = P_m$ is an elementary abelian p-subgroup of G with $\operatorname{rk} V \geq m+1$. Note that $V \subseteq C_S(V)$ so $|C_S(V)|_p \geq p^{m+1}$. By an argument similar to the one used in (a), we see that $S \downarrow^G_{G_\sigma}$ contains the projective cover $P_{\mathbb{F}_p}$ of the trivial $\mathbb{F}_p[G_\sigma]$-module $\mathbb{F}_p$ as a direct summand. As $C_G(V) \subseteq N_G(P_i)$ for $1 \leq i \leq m$ we have that $C_G(V) \subseteq \bigcap_{i=1}^m N_G(P_i) = G_\sigma$. Then the further restriction $S \downarrow^G_{C_G(V)}$ contains the projective module $P_{\mathbb{F}_p} \downarrow^{G_\sigma}_{C_G(V)}$ as a direct summand.

By the above we have that if P is a projective $\mathbb{F}_p[Q]$-module, where Q is finite group, then $P \downarrow^P_R$ is a projective $\mathbb{F}_p[R]$-modules for $R \in \mathrm{Syl}_p(Q)$. For a finite p-group R the ring $\mathbb{F}_p[R]$ is local, hence the only projective $\mathbb{F}_p[R]$-modules are the free ones by [25, Theorem 2], and thus $P \downarrow^P_R = \mathbb{F}_p[R]^n$ for some n. Then $\dim_{\mathbb{F}_p}(\mathbb{F}_p[R]) = |R|$ is a divisor of $\dim_{\mathbb{F}_p}(P \downarrow^P_R)$. Note that $\dim_{\mathbb{F}_p}(P \downarrow^P_R) = \dim_{\mathbb{F}_p} P$.

Combining these results we conclude that

$$\dim_{\mathbb{F}_p} S \geq \dim_{\mathbb{F}_p} P_{\mathbb{F}_p} \geq |C_G(V)|_p \geq p^{m+1}.$$

$\square$

For any finite group G and any field k, we have that $k[G]$ is a finitely generated k-algebra, so $k[G]$ is Artinian. Then every finitely generated $k[G]$-module M is both Artinian and Noetherian. The Jordan-Hölder theorem implies it has a finite filtration with simple quotients. For a simple $k[G]$-module S and an finitely generated $k[G]$-module M we observe that if none of the simple sub-quotients from the filtration are isomorphic to S, then there exist no non-trivial elements of $\mathrm{Hom}_G(S, M)$. Then by induction on the length of the filtration with simple quotients we conclude for finitely generated $k[G]$-modules N and M that if $\mathrm{Hom}_G(N, T) = 0$ for every simple sub-quotient in the filtration of M, then we have that $\mathrm{Hom}_G(N, M) = 0$.

Corollary 6.8. *Let G be a finite group and M a finitely generated $\mathbb{Z}_p$-module. Assume that there exists a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ of M such that for each $1 \leq i \leq m$ we have that either*

(1) $\ker(G \to \mathrm{Aut}(M_i/M_{i-1}))$ *has order divisible by p.*
(2) *The module M_i/M_{i-1} is generated over $\mathbb{Z}_p$ by strictly less than p^k elements, or*

Then $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M) = 0$ *and* $\Lambda^k(G; M) = 0$.

Proof. First note that as $\tilde{\mathrm{st}}_{k-1}(G)$ is a projective $\mathbb{Z}_p$-module, we have that the functor $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), -)$ is exact. Then for any $1 \leq i \leq m$ by applying this functor to the short exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ we get the exact sequence

$$\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M_{i-1}) \to \mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M_i) \to \mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M_i/M_{i-1}).$$

Since the filtration is finite, we deduce that it is sufficient to show that for all $1 \leq i \leq m$ we have $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M_i/M_{i-1}) = 0$.

Let $1 \leq i \leq m$ and set $N = M_i/M_{i-1}$. As M is a Noetherian module, we have that N is a finitely generated $\mathbb{Z}_p$-module. Consider the exact sequence

$$N \xrightarrow{\cdot p} N \longrightarrow N/pN \longrightarrow 0$$

of $\mathbb{Z}_p$-modules. Then

$$\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N) \xrightarrow{\cdot p} \mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N) \longrightarrow \mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N/pN) \longrightarrow 0$$

is also exact. Assume that $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N/pN) = 0$. As G is a finite group, we have that $\tilde{\mathrm{st}}_{k-1}(G)$ is a finitely generated $\mathbb{Z}_p$-module. Since N is also a finitely generated $\mathbb{Z}_p$-module, we conclude that it is true for $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N)$ as well. Note that $\mathbb{Z}_p$ is a commutative local ring with maximal ideal $p\mathbb{Z}_p$. The above exact sequence implies that $(p\mathbb{Z}_p)\,\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N) = \mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N)$. By Nakayama's Lemma we now see that $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N) = 0$. So it is sufficient to show that $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), N/pN) = 0$. Set $\bar{N} = N/pN$.

By Proposition 6.7 we have that the projective summands in $\tilde{\mathrm{st}}_{k-1}(G)$ are of the form P_S where $\bar{P}_S = P_S/pP_S$ is the projective cover of a simple $\mathbb{F}_p[G]$-module S. We now consider $\mathrm{Hom}_G(P_S, \bar{N})$. Assume that $\mathrm{Hom}_G(P_S, \bar{N}) \neq 0$. For any $f \in \mathrm{Hom}_G(P_S, \bar{N})$ we have that $f(pP_S) \subseteq p(\bar{N}) = 0$, hence we get a bijection between $\mathrm{Hom}_G(P_S, \bar{N})$ and $\mathrm{Hom}_G(\bar{P}_S, \bar{N})$, so $\mathrm{Hom}_G(\bar{P}_S, \bar{N}) \neq 0$. Note that both $\bar{P}_S$ and $\bar{N}$ have a structure as $\mathbb{Z}_p/(p\mathbb{Z}_p)[G] = \mathbb{F}_p[G]$-modules. Furthermore they are both finitely generated as $\mathbb{F}_p[G]$-modules as well. By the previous observation this implies that there exists a simple sub-quotient of $\bar{N}$ called T, such that $\mathrm{Hom}_G(\bar{P}_S, T) \neq 0$. As T is simple we conclude that $0 \neq \mathrm{Hom}_G(\mathrm{Soc}(\bar{P}_S), T) = \mathrm{Hom}_G(S, T)$ and thus $S \cong T$ as $\mathbb{F}_p[G]$-modules. If (1) holds then $\ker(G \to \mathrm{Aut}(N))$ contains elements of order p, thus the same is true when we replace N with $\bar{N}$ and any sub-quotient T of N/pN. So $\ker(G \to \mathrm{Aut}(S))$ has an element of order p. If (2) holds, then N is generated over $\mathbb{Z}_p$ by strictly less than p^k elements, so $\bar{N}$ is generated by strictly less than p^k over $\mathbb{F}_p$. As this is true for any sub-quotient of $\bar{N}$, we conclude

$$\dim_{\mathbb{F}_p}(S) \leq \dim_{\mathbb{F}_p}(N) < p^k.$$

Both conditions give a contradiction in relation with Proposition 6.7, so we conclude that $\mathrm{Hom}_G(P_S, \bar{N}) = 0$. As this holds for any projective summand of $\tilde{\mathrm{st}}_{k-1}(G)$, we get that $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), \bar{N}) = 0$ and the result follows by the previous reductions.

As noted we have that $\Lambda^k(G, M) \cong \mathrm{H}^{k-1}(\mathrm{Hom}_G(\mathrm{st}_*(G), M))$, which is a quotient of $\mathrm{Hom}_G(\tilde{\mathrm{st}}_{k-1}(G), M)$, we conclude that it is trivial as well. □

6.4. **Consequences.** As $\mathcal{O}^c(\mathcal{F})$ is a finite category, we will now show when a functor $\mathcal{O}^c(\mathcal{F}) \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ vanishes in terms of the groups Λ, as described in the following corollary.

Corollary 6.9. *Let $\mathcal{F}$ be a saturated fusion system over a p-group and*

$$F\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be a functor. If for some i we have that $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(P), F(P)) = 0$ for all $P \in \mathcal{F}^c$, then $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} F = 0$.

Proof. Let $F\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-mod}$ be a functor and let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be the $\mathcal{F}$-conjugacy classes in $\mathcal{O}^c(\mathcal{F})$ ordered after the size of the subgroups they contain, i.e. if $P \in \mathcal{P}_j$ and $P' \in \mathcal{P}_k$ with $j \le k$, then $|P| \le |P'|$. We now set for $1 \le j \le n$ $F_j\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-mod}$ to be given by $F_j(P)$ is $F(P)$ if $P \in \bigcup_{k=1}^{j} \mathcal{P}_k$ and 0 otherwise, and for $\psi \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ with both $P, P' \in \bigcup_{k=1}^{j} \mathcal{P}_k$ we set $F_j(\psi) = F(\psi)$ and else the 0-morphism. Note that for a $\psi \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ either P and P' are $\mathcal{F}$-conjugate or $|P| < |P'|$. Thus the ordering of the $\mathcal{F}$-conjugacy classes implies that if $\psi \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ and $P \notin \bigcup_{k=1}^{j} \mathcal{P}_k$, that $P' \notin \bigcup_{k=1}^{j} \mathcal{P}_k$. Then F_j as defined above preserves composition of functors and thus it is a functor itself. Let for $1 \le j \le n$ $\tilde{F}_j\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-mod}$ be the restriction of F to the conjugacy class $\mathcal{P}_j$. Note that $\tilde{F}_1 = F_1$. By Proposition 6.4 we have that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} \tilde{F}_j \cong \Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q); \tilde{F}_j(Q))$ for any $Q \in \mathcal{P}_j$. By the assumptions we conclude that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} \tilde{\mathcal{F}}_j = 0$ for every $1 \le j \le n$.

For every $1 < j \le n$ we have a short exact sequence of functors $0 \to F_{j-1} \to F_j \to \tilde{F}_j \to 0$. From the long exact sequence of higher limits we get that for every i there is an exact sequence

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(F_{j-1}) \to \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(F_j) \to \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(\tilde{F}_j),$$

hence $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})}(F_j) = 0$ if the other two are zero. As both $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} \tilde{F}_j = 0$ for all j and $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} F_1 = 0$, we conclude that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})} \tilde{F}_n = 0$. But $F_n = F$, so the result follows. □

Definition 6.10. *A category $\mathcal{C}$ has bounded limits at a prime p if there exists an integer d, such that for every functor $\Phi\colon \mathcal{C}^{op} \to \mathbb{Z}_{(p)}\text{-mod}$ we have that $\varprojlim^i(\Phi) = 0$ for $i > d$.*

Corollary 6.11. *Let $\mathcal{F}$ be a saturated fusion system. Then the category $\mathcal{O}^c(\mathcal{F})$ has bounded limits at p.*

Proof. For any finite group G let $\mathcal{E}(G)$ be the maximal rank of an elementary abelian subgroup of G. Set $N = \max\{\mathcal{E}(\mathrm{Out}_{\mathcal{F}}(P)) \mid P \in \mathcal{F}^c\}$. Then N is a integer. We will now prove that for any functor $F\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-mod}$ we have that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})}(F) = 0$ for $i > N$. By Corollary 6.9 it is sufficient to show that for every $Q \in \mathcal{F}^c$ and $\mathbb{Z}_{(p)}[\mathrm{Out}_{\mathcal{F}}(Q)]$-module M, we have that $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q); M) = 0$. A reformulation of [18, Theorem 1.1] is with the notation from the previous chapter

$$\Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q), M) \cong \mathrm{H}^{i-1}(\mathrm{Hom}_{\mathrm{Out}_{\mathcal{F}}(Q)}(\tilde{\mathrm{st}}_*(\mathrm{Out}_{\mathcal{F}}(Q)), M)).$$

Every elementary abelian subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ has rank at most N. So for $i \ge N$ there exists no elementary abelian subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ of rank $i+1$ and thus by Proposition 6.7 we have that $\tilde{\mathrm{st}}_i(\mathrm{Out}_{\mathcal{F}}(Q))$ has no indecomposable projective summands. Since $\tilde{\mathrm{st}}_i(\mathrm{Out}_{\mathcal{F}}(Q))$ is projective, we conclude that $\tilde{\mathrm{st}}_i(\mathrm{Out}_{\mathcal{F}}(Q)) = 0$. Then the isomorphism implies that $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q), M) = 0$ for any $i > N$. □

Corollary 6.12. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. If $\mathrm{rk}_p(S) < p^3$, then there exists a central linking system associated to $\mathcal{F}$. If $\mathrm{rk}_p(S) < p^2$ there exists a unique central linking system associated to $\mathcal{F}$.*

Proof. Assume that $\mathrm{rk}_p(S) < p^n$ for a $n \in \mathbb{N}$. Let $Q \in \mathcal{F}^c$. Since $\mathrm{Z}(Q)$ is a p-subgroup of S, we have that $\mathrm{rk}_p(\mathrm{Z}(Q)) < p^n$. Thus $\mathrm{Z}(Q)$ is generated by strictly less than p^n elements as a module over $\mathbb{Z}_{(p)}$. Corollary 6.8 implies that $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q), \mathrm{Z}(Q)) = 0$ for all $i \geq n$. As $\mathcal{Z}_{\mathcal{F}}(Q) = \mathrm{Z}(Q)$, Lemma 6.9 now implies that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}}) = 0$ for all $i \geq n$. If $n = 3$, then $\varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}}) = 0$, so the class $\eta(\mathcal{F})$ from Proposition 6.2 is zero, and thus there exists a central linking system associated to $\mathcal{F}$. If $n = 2$, then $\varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}}) = 0$ and $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}_{\mathcal{F}}) = 0$, so Proposition 6.2 implies that there exists a unique central linking system associated to $\mathcal{F}$. □

Definition 6.13. *Let $\mathcal{F}$ be a fusion system over S. Then a subgroup $P \subseteq S$ is $\mathcal{F}$-radical, if $\mathrm{Out}_{\mathcal{F}}(P)$ contains no normal nontrivial p-subgroups.*

Lemma 6.14. *Let $\mathcal{F}$ be saturated fusion system over a finite group S. Let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be a full subcategory containing all $\mathcal{F}$-radical subgroups of S. Furthermore assume that $\mathcal{F}_0$ is closed under taking $\mathcal{F}$-centric over-groups, in the sense that if $P \subseteq P' \subseteq S$ with $P \in \mathrm{Ob}(\mathcal{F}_0)$ and $P' \in \mathrm{Ob}(\mathcal{F}^c)$, then $P' \in \mathrm{Ob}(\mathcal{F}_0)$. Let $i\colon \mathcal{O}(\mathcal{F}_0) \to \mathcal{O}^c(\mathcal{F})$ be the inclusion. Then for any functor $F\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ the inclusion induces an isomorphism $\varprojlim^*_{\mathcal{O}^c(\mathcal{F})} F$ to $\varprojlim^*_{\mathcal{O}(\mathcal{F}_0)} F \circ i^{op}$.*

Proof. For any $F\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ let $F_0\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be given by $F_0(P) = F(P)$, if $P \in \mathrm{Ob}(\mathcal{F}_0)$ and zero otherwise and similarly for morphisms. The over-group condition implies that F_0 respects the composition and thus is a functor itself. The inclusion of $\mathcal{O}(\mathcal{F}_0)$ into $\mathcal{O}^c(\mathcal{F})$ gives an isomorphism of $\varprojlim^*_{\mathcal{O}^c(\mathcal{F})} F_0$ and $\varprojlim^*_{\mathcal{O}(\mathcal{F}_0)} F \circ i^{op}$. Similarly let $F/F_0\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be given by $F_0(P) = F(P)$, if $P \in \mathrm{Ob}(\mathcal{F}^c) \setminus \mathrm{Ob}(\mathcal{F}_0)$ and zero otherwise and similarly for morphisms. As $\mathrm{Ob}(\mathcal{F}^c) \setminus \mathrm{Ob}(\mathcal{F}_0)$ is closed under taking under groups, this implies that F/F_0 respects the composition and is a functor. Note for any $P \in \mathrm{Ob}(\mathcal{F}^c) \setminus \mathrm{Ob}(\mathcal{F}_0)$ we have that $\mathrm{Out}_{\mathcal{F}}(P)$ is not p-reduced. By [24, Proposition 6.1.(ii)] we see that $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P)) = 0$, so using Lemma 6.9 we conclude $\varprojlim^*_{\mathcal{O}^c(\mathcal{F})}(F/F_0) = 0$. By construction we have a short exact sequence of functors $0 \to F_0 \to F \to F/F_0 \to 0$. The long exact sequence for the higher limits combined with $\varprojlim^*_{\mathcal{O}^c(\mathcal{F})}(F/F_0) = 0$ implies that $\varprojlim^*_{\mathcal{O}^c(\mathcal{F})}(F_0) \cong \varprojlim^*_{\mathcal{O}^c(\mathcal{F})} F$, and the lemma follows from the above. □

Corollary 6.15. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a full subcategory containing all $\mathcal{F}$-radical $\mathcal{F}$-centric subgroups of S. Furthermore assume that $\mathcal{L}_0$ is closed under taking over-groups. Then the inclusion $|\mathcal{L}_0| \subseteq |\mathcal{L}|$ is a mod p equivalence.*

Proof. Let $\mathcal{F}_0$ be the full subcategory of $\mathcal{F}^c$ on the objects $\mathrm{Ob}(\mathcal{L}_0)$. By Proposition 4.2 we have decompositions $|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B})$ and $|\mathcal{L}_0| \simeq \mathrm{hocolim}_{\mathcal{O}(\mathcal{F}_o)}(\tilde{B})$ where $\tilde{B}$ is a lift of the homotopy functor $P \mapsto BP$. Then for all $P \in \mathcal{F}^c$ we have that $\mathrm{H}^*(\tilde{B}P; \mathbb{F}_p) \cong \mathrm{H}^*(P; \mathbb{F}_p)$. As both $\mathcal{L}$ and $\mathcal{L}_0$ are finite categories the spectral sequence for cohomology of the homotopy colimit [7, XII.4.5] gives spectral sequences $E_r^{ij}(\mathcal{L})$ and $E_r^{ij}(\mathcal{L}_0)$ converging to $\mathrm{H}^{i+j}(|\mathcal{L}|; \mathbb{F}_p)$ respectively $\mathrm{H}^{i+j}(|\mathcal{L}_0|; \mathbb{F}_p)$ such that

$$E_2^{ij}(\mathcal{L}) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(\mathrm{H}^j(-; \mathbb{F}_p)), \quad E_2^{ij}(\mathcal{L}_0) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_0)}{}^i(\mathrm{H}^j(-; \mathbb{F}_p)).$$

The inclusion of categories $i\colon \mathcal{O}(\mathcal{F}_0) \to \mathcal{O}^c(\mathcal{F})$ induces a morphism of spectral sequences $i_r\colon E_r^{ij}(\mathcal{L}) \to E_r^{ij}(\mathcal{L}_0)$, such that $i_\infty\colon E_\infty^{ij}(\mathcal{L}) \to E_\infty^{ij}(\mathcal{L}_0)$ corresponds to the map induced by $i^*\colon \mathrm{H}^{i+j}(|\mathcal{L}|) \to \mathrm{H}^{i+j}(|\mathcal{L}_0|)$ on the successive quotients of their filtrations.

We may consider $\mathrm{H}^*(-;\mathbb{F}_p)\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$. Lemma 6.14 implies that $i_2\colon E_2^{i,j}(\mathcal{L}) \to E_2^{i,j}(\mathcal{L}_0)$ is an isomorphism. Then $i_r\colon E_r^{i,j}(\mathcal{L}) \to E_r^{i,j}(\mathcal{L}_0)$ is an isomorphism for any $r \geq 2$, and thus $i_\infty\colon E_\infty^{i,j}(\mathcal{L}) \to E_\infty^{i,j}(\mathcal{L}_0)$ is an isomorphism. By Corollary 6.11 we conclude that $E_2^{i,j}(\mathcal{L})$ has only finitely many non-zero columns. By the construction of spectral sequences we see, that $E_r^{i,j}(\mathcal{L})$ has only finitely many non-zero columns. The isomorphism i_2 implies that the same holds for $E_r^{i,j}(\mathcal{L}_0)$ for $r \geq 2$, hence the filtrations of $\mathrm{H}^{i+j}(|\mathcal{L}_0|;\mathbb{F}_p)$ and $\mathrm{H}^{i+j}(|\mathcal{L}|;\mathbb{F}_p)$ are both finite. Successive elements in the filtration of $\mathrm{H}^{i+j}(|\mathcal{L}|;\mathbb{F}_p)$ together with their quotients form a short exact sequence and the restriction of i^* induces a map to the corresponding short exact sequence of $\mathrm{H}^{i+j}(|\mathcal{L}_0|;\mathbb{F}_p)$. On the quotients i^* agrees with i_∞ and thus they are isomorphisms. As both filtrations start with the trivial group and the filtrations are finite, using the 5-lemma a finite number of times implies that $i^*\colon \mathrm{H}^{i+j}(|\mathcal{L}|;\mathbb{F}_p) \to \mathrm{H}^{i+j}(|\mathcal{L}_0|;\mathbb{F}_p)$ is an isomorphism. $\square$

7. The mapping space $\mathrm{Map}(BQ, |\mathcal{L}|_p^\wedge)$

In this chapter will provide a description of homotopy classes of $\mathrm{Map}(BQ, |\mathcal{L}|_p^\wedge)$ for any p-local finite group $(\mathcal{F}, \mathcal{L}, S)$ and p-group Q in form of a bijection with a orbit set of $\mathrm{Hom}(Q, S)$. This can be seen as an extension of the classical bijection $\mathrm{Rep}(Q, S) = [BQ, BS]$. The central idea will be results giving conditions under which the homotopy colimit and mapping space with respect to BQ commute and afterward use the existence of the homotopy decomposition of $|\mathcal{L}|$.

Remark that all cohomology in this chapter is with coefficients in $\mathbb{F}_p$.

7.1. Homotopy colimit, p-completion and mapping spaces.

Lemma 7.1. *Let p be a prime number, and set $V = \mathbb{Z}/p$. Let $\mathcal{C}$ be a finite category having bounded limits at p. Consider a functor $F\colon \mathcal{C} \to V$-$\mathsf{Spaces}$, such that for any $c \in \mathcal{C}$ both $F(c)$ and the homotopy fix points $F(c)^{hV}$ are p-complete spaces with finite mod p cohomology in each degree. Then there is a homotopy equivalence:*

$$[\operatorname*{hocolim}_{\mathcal{C}}(F(-)^{hV})]_p^\wedge \to [(\operatorname*{hocolim}_{\mathcal{C}}(F(-)))_p^\wedge]^{hV}.$$

Proof. Set $X = \mathrm{hocolim}_{\mathcal{C}}(F(-))$ and $Z = \mathrm{hocolim}_{\mathcal{C}}(F(-)^{hV})$. As

$$X = \left(\coprod_{n \geq 0} \coprod_{c_0 \to \cdots \to c_n} F(c_0) \times \Delta^n \right) / \sim$$

we have a filtration of X by the i'th skeleton

$$X_i = \left(\coprod_{n=0}^{i} \coprod_{c_0 \to \cdots \to c_n} F(c_0) \times \Delta^n \right) / \sim .$$

For all $c \in \mathcal{C}$ we have that $F(c)$ is a topological space with an action of the group V, so the same holds for X and X_i for all i. Furthermore the inclusion $X_{i-1} \to X_i$ is a map in V-Spaces. So we can consider the space $EV \times X_i$ with the diagonal V-action and the orbit set $(X_i)_{hV} = (EV \times X_i)/V$. Then $\cdots \subseteq (X_i)_{hV} \subseteq (X_{i+1})_{hV} \subseteq \cdots$ is a filtration of X_{hV}. By definition $\mathrm{H}^*_V(X_i) = \mathrm{H}^*((EV \times X_i)/V)$ so by a staircase diagram of the long exact sequences for the pairs $((X_{i+1})_{hV}, (X_i)_{hV})$ similar to the one on [20, page 3], we get an exact couple $C(X_{hV})$ of the form:

$$\begin{tikzcd}
\bigoplus_i \mathrm{H}^*_V(X_i) \arrow[rr] & & \bigoplus_i \mathrm{H}^*_V(X_i) \arrow[dl] \\
& \bigoplus_i \mathrm{H}^*_V(X_{i+1}, X_i) \arrow[ul] &
\end{tikzcd}$$

For any V-Spaces X we have that the collapsing map $X \to *$ is a V-map and it induces a $\mathcal{K}$-morphism from $\mathrm{H}^*(V) = \mathrm{H}^*_V(*)$ to $\mathrm{H}^*_V(X)$. The composition of the collapsing map with a V-map is the collapsing map of the source, hence we way consider the functor $H^*_V\colon V$-$\mathsf{Spaces} \to \mathrm{H}^*V \setminus \mathcal{K}$, where $\mathrm{H}^*V \setminus \mathcal{K}$ is the under-category $\mathrm{H}^*V \downarrow \mathrm{id}_{\mathcal{K}}$. Let H^*V-$\mathcal{U}$ be the category of unstable modules X over the Steenrod algebra $\mathcal{A}_p$, which are equipped with the structure of a H^*V-module induced by an $\mathcal{A}_p$-linear map $\mathrm{H}^*V \otimes X \to X$. The morphisms are both $\mathcal{A}_p$ and H^*V-morphisms. As noted in [26, 4.4.1] the forgetful functor $\mathcal{K} \to \mathcal{U}$ induces a functor $\mathrm{H}^* V \setminus \mathcal{K} \to \mathrm{H}^* V\text{-}\mathcal{U}$. The exact couple thus is in $\mathrm{H}^*(V)\text{-}\mathcal{U}$, in the sense that it consists of unstable modules over the Steenrod algebra, which also have a structure as a $\mathrm{H}^*(V)$-module and the maps are $\mathrm{H}^*(V)$-module morphisms. Let $E_r^{**}(X_{hV})$ be the induced spectral sequence by this exact couple. Each page of $E_r^{**}(X_{hV})$ corresponds to an exact couple, where both modules are a direct sum of modules in $\mathrm{H}^*(V)\text{-}\mathcal{U}$ and the corresponding restriction of the morphisms are in $\mathrm{H}^*(V)\text{-}\mathcal{U}$ as well. So each column in of $E_r^{**}(X_{hV})$ and the differential d_r are in $\mathrm{H}^* V\text{-}\mathcal{U}$.

Similarly let Z_i be the corresponding i'th skeleton of Z. Then we have an exact couple $C(Z)$

$$\begin{tikzcd} \bigoplus_i \mathrm{H}^*(Z_i) \arrow[rr] & & \bigoplus_i \mathrm{H}^*(Z_i) \arrow[dl] \\ & \bigoplus_i \mathrm{H}^*(Z_{i+1}, Z_i) \arrow[ul] & \end{tikzcd}$$

Let $E_r^{**}(Z)$ be the induced spectral sequence.

Let $c \in \mathcal{C}$. Then $F(c)^{hV} = \hom_V(EV, F(c))$, so we have a map $\Psi_c \colon EV \times F(c)^{hV} \to F(c)$ by $(x, \varphi) \mapsto \varphi(x)$. By giving $F(c)^{hV}$ the trivial V-action, we get that this is a map between V-Spaces, such that for any $v \in V$, $x \in EV$ and $\varphi \in F(c)^{hV}$ it satisfies

$$\Psi_c(v(x,\varphi)) = \Psi_c(vx, \varphi) = \varphi(vx) = v\varphi(x) = v\Psi_c(x,\varphi).$$

Thus Ψ_c is an equivariant map. For any $\psi \in \mathrm{Mor}_{\mathcal{C}}(c, c')$, we have that

$$F(\psi)^{hV} \colon \hom_V(EV, F(c)) \to \hom_V(EV, F(c'))$$

is composition with $F(\psi)$, so the following diagram commutes:

$$\begin{tikzcd} EV \times F(c)^{hV} \arrow[r, "\Psi_c"] \arrow[d, "\mathrm{id} \times F(\psi)^{hV}"'] & F(c) \arrow[d, "F(\psi)"'] \\ EV \times F(c')^{hV} \arrow[r, "\Psi_{c'}"] & F(c') \end{tikzcd}$$

This implies that Ψ induces a well-defined map $EV \times Z_i \to X_i$ for any $i \geq 0$, and if we let V act trivially on Z_i, this map will be equivariant as well. Thus it induces an equivariant map $EV \times Z_i \to EV \times X_i$ by $(x, \varphi) \mapsto (x, \Psi(x, \varphi))$, that factors though to the orbit spaces $BV \times Z_i = (EV \times Z_i)/V \to (EV \times X_i)/V = (X_i)_{hV}$ and we remark the map commutes the inclusions of the filtrations, i.e. the diagram commutes for all i:

$$\begin{tikzcd} BV \times Z_{i-1} \arrow[r] \arrow[d] & X_{i-1} \arrow[d] \\ BV \times Z_i \arrow[r] & X_i \end{tikzcd}$$

We note that this implies that we have a map of pairs $((X_i)_{hV}, (X_{i-1})_{hV}) \mapsto (BV \times Z_i, BV \times Z_{i-1})$. Since we are taking cohomology with coefficients in a field, we thus have maps $\mathrm{H}^*_V(X_i) \to H^*(BV) \otimes \mathrm{H}^*(Z_i)$ as well as $\mathrm{H}^*_V(X_i, X_{i-1}) \to H^*(BV) \otimes \mathrm{H}^*(Z_i, Z_{i-1})$ that commute with the associated long exact sequence in cohomology. We remark that all these modules and maps are in $\mathrm{H}^* V \downarrow \mathrm{id}_{\mathcal{K}}$. By the adjunction involving Lannes Fix-functor [26, Theorem 4.6.3.1.] they are adjoint to morphisms $\mathrm{Fix}(\mathrm{H}^*_V(X_i)) \to \mathrm{H}^*(Z_i)$ and $\mathrm{Fix}(\mathrm{H}^*_V(X_i, X_{i-1})) \to \mathrm{H}^*(Z_i, Z_{i-1})$ and as the adjunction is natural it commutes with the morphisms originating from the long exact sequence. Since Fix is exact by [26, Theorem 4.6.1.1.], we have that $\mathrm{Fix}(C(X_{hV}))$ is also an exact couple. As the maps in the exact couples are derived from the long exact sequence in cohomology for pair, we thus get a map between the two exact couples $\mathrm{Fix}(C(X_{hV}))$ and $C(Z)$. This induces a morphism between the induced spectral sequences. Since Fix is exact the spectral sequence corresponding to $\mathrm{Fix}(C(X_{hV}))$ is $\mathrm{Fix}(E_r^{**}(X_{hV}))$, where we have applied Fix to each column of $E_r^{**}(X_{hV})$.

As the filtration of the homotopy colimit used to define the two exact couples are the same as in [7, XII.4.5] we have that the E_2-pages are identified as follows:

$$E_2^{j*}(X_{hV}) \cong \varprojlim_{\mathcal{C}}{}^j H^*_V(F(-)), \quad E_2^{j*}(Z) \cong \varprojlim_{\mathcal{C}}{}^j H^*(F(-)^{hV})$$

As $\mathcal{C}$ has finite limits at p, we see that both spectral sequences have only a finite number of nonzero columns on the E_2, and hence for every E_r with $r \geq 2$. Thus both spectral sequences are convergent and the E_∞-page has only finitely many non-zero columns. Note that they converge to $\mathrm{H}^*_V(X)$ and $\mathrm{H}^*(Z)$ respectively. Then $\mathrm{Fix}(E_r^{**}(X_{hV}))$ converges to $\mathrm{Fix}(\mathrm{H}^*_V(X))$.

We consider the map $\Phi\colon \mathrm{Fix}(E_r^{**}(X_{hV})) \to E_r^{**}(Z)$ of spectral sequences, which is induced by the adjoint of Ψ. Let $c \in \mathcal{C}$. The map $\Psi_c\colon EV \times F(c)^{hV} \to F(c)$ induces a map $\tilde{\Psi}_c\colon (F(c)^{hV})^\wedge_p \to (F(c)^\wedge_p)^{hV}$ as in [26, 4.3.2]. It is the adjoint to the upper composition in the following diagram

$$\begin{tikzcd}
EV \times (F(c)^{hV})^\wedge_p \arrow[r, "\psi_{EV} \times \mathrm{id}"] & EV^\wedge_p \times (F(c)^{hV})^\wedge_p \arrow[r, "a"] & (EV \times F(c)^{hV})^\wedge_p \arrow[r, "(\Psi_c)^\wedge_p"] & F(c)^\wedge_p \\
EV \times F(c)^{hV} \arrow[u, "\mathrm{id} \times \psi_{(F(c))^{hV}}"] \arrow[rrr, "\Psi_c"] & & & F(c) \arrow[u, "\psi_{F(c)}"']
\end{tikzcd}$$

where ψ the map from the natural transformation corresponding to p-completion and a is the map from [7, I 7.2]. As a is compatible with the triple structure of p-completion, we have that $a \circ (\psi_{BV} \times \psi_{(F(c))^{hV}}) = \psi_{BV \times F(c)^{hV}}$. So the above diagram commutes as ψ is a natural transformation. By assumption $F(c)$ and $F(c)^{hV}$ are p-complete and BV is p-complete by [3, III 1.4 Proposition 1.10]. Hence $\psi_{F(c)}$, $\psi_{F(c)^{hV}}$ and ψ_{BV} are all homotopy equivalences. Likewise a is a homotopy equivalence by [7, I 7.2], hence the adjoint $\tilde{\Psi}_c$ is a homotopy equivalence if and only if it holds for the adjoint of Ψ_c. The adjoint of Ψ_c from $F(c)^{hV}$ to itself is the identity and therefore clearly a homotopy equivalence, and thus the same is true for $\tilde{\Psi}_c$. As $F(c)$ and $F(c)^{hV}$ have finite mod p cohomology, [26, Theorem 4.9.1.] implies that the induced map $\mathrm{Fix}\, \mathrm{H}^*_V(F(c)) \to \mathrm{H}^*(F(c)^{hV})$ is an isomorphism. As this isomorphism is natural in c, we get that $E_2^{j*}(Z) \cong \varprojlim{}^j_{\mathcal{C}} \mathrm{Fix}\, H^*_V(F(-))$. As Φ

is defined by Ψ_c, we have that $\Phi_2\colon \operatorname{Fix}(E_2^{j*}(X_{hV})) \to E_2^{j*}(Z)$ corresponds under this isomorphism to the natural map:

$$\operatorname{Fix} \varprojlim_{\mathcal{C}}{}^j H_V^*(F(-)) \to \varprojlim_{\mathcal{C}}{}^j \operatorname{Fix} H_V^*(F(-)).$$

Since $\mathcal{C}$ is a finite category the bar resolution to compute higher limits is a finite product of spaces in each degree [3, III Proposition 5.3.]. Since Fix is exact it commutes with cohomology and finite products, so the map Φ_2 is an isomorphism for all j. This implies that Φ_r is an isomorphism for all $r \geq 2$. Both spectral sequences converge, so Φ_∞ is well-defined and also an isomorphism.

Similarly we have that Ψ induces a map $EV \times X \to Z$, which by adjointness induces a map $\Phi\colon \operatorname{Fix}\mathrm{H}_V^*(X) \to \mathrm{H}^*(Z)$. This preserves the filtrations from the spectral sequences, and the induced map on the quotients are exactly Φ_∞. Both E_∞-pages have only a finite number of nonzero columns, so the filtrations of $\operatorname{Fix}\mathrm{H}_v^i(X)$ and $\mathrm{H}^i(Z)$ are finite for each i. Let $0 = F_0 \subseteq \cdots \subseteq F_n = \operatorname{Fix}\mathrm{H}_v^i(X)$ and $0 = F_0' \subseteq \cdots \subseteq F_n' = \mathrm{H}^i(Z)$ be the filtrations. Then $\Phi\colon F_0 \to F_0'$ is an isomorphism, and by applying the 5-lemma to

$$\begin{tikzcd}
0 \arrow[r] & F_i \arrow[r] \arrow[d, "\Phi"'] & F_{i+1} \arrow[r] \arrow[d, "\Phi"'] & F_{i+1}/F_i \arrow[r] \arrow[d, "\Phi_\infty"'] & 0 \\
0 \arrow[r] & F_i' \arrow[r] & F_{i+1}' \arrow[r] & F_{i+1}'/F_i' \arrow[r] & 0
\end{tikzcd}$$

we conclude that $\Phi\colon F_i \to F_i'$ is an isomorphism for all i. In particular we have that $\Phi\colon \operatorname{Fix}\mathrm{H}_V^*(X) \to \mathrm{H}^*(Z)$ is an isomorphism.

By assumption $F(c)^{hV}$ has finite mod p cohomology in each degree for all $c \in \mathcal{C}$. Since $\mathcal{C}$ is finite, we conclude that each entry of $E_2(Z)$ is a finite dimensional vector space over $\mathbb{F}_p$. As the following pages are formed by taking quotients of the previous one, we see that this is true for all $E_r(Z)$ where $r \geq 2$. In particular it holds for $E_\infty(Z)$. So for any i the quotients of the finite filtration of $\mathrm{H}^i(Z)$ are finite dimensional vector spaces over $\mathbb{F}_p$, and thus the same holds for $\mathrm{H}^i(Z)$. Then by [26, Theorem 4.9.1.] we get that $Z_p^\wedge \to (X_p^\wedge)^{hV}$ is a homotopy equivalence, which with our notation is exactly the statement of the lemma. $\square$

Proposition 7.2. *Let p be a prime and Q a p-group. Let $\mathcal{C}$ be a finite category having bounded limits at p. Consider a functor $F\colon \mathcal{C} \to \mathsf{Top}$, such that for any $c \in \mathcal{C}$ and $Q_0 \subseteq Q$ the space $\operatorname{Map}(BQ_0, F(c))$ is p-complete with finite mod p cohomology in each degree. Then the natural map*

$$[\operatorname*{hocolim}_{\mathcal{C}}(\operatorname{Map}(BQ, F))]_p^\wedge \to [\operatorname{Map}(BQ, (\operatorname*{hocolim}_{\mathcal{C}}(F))_p^\wedge)]$$

is a homotopy equivalence, where $\operatorname{Map}(BQ, F)\colon \mathcal{C} \to \mathsf{Top}$ is the functor given by $c \mapsto \operatorname{Map}(BQ, F(c))$ and $\varphi \mapsto \varphi \circ (-)$

Proof. We have that $|Q| = p^n$ for some $n \geq 0$. The statement will be proven by induction on n. If $n = 0$ then $BQ = *$, so $\operatorname{Map}(BQ, X) = X$ for any space X, hence both the considered spaces are $(\operatorname{hocolim}_{\mathcal{C}}(F))_p^\wedge$ and the natural map is the identity, which is a homotopy equivalence. Assume that $n > 0$ and the statement holds for $n-1$. As the finite p-group Q is solvable, there exists a normal subgroup Q_0, such that $[Q,Q_0]=p$. As $BQ_0 = EQ/Q_0$ and $|Q_0| = p^{n-1}$ the induction hypothesis implies that the natural map

$$f_0\colon [\operatorname*{hocolim}_{\mathcal{C}}(\mathrm{Map}(EQ/Q_0,F))]^\wedge_p \to [\mathrm{Map}(EQ/Q_0,(\operatorname*{hocolim}_{\mathcal{C}}(F))^\wedge_p)]$$

is a homotopy equivalence. Let $V = Q/Q_0$. The action of Q on EQ induces an action of V on EQ/Q_0. This in turn induces an action of V on the two spaces above, which makes f_0 an equivariant map. By definition the homotopy fix-points space is $X^{hV} = \mathrm{Map}_V(EV,X)$ for any V-space X, so an equivariant homotopy equivalence between two V-spaces induces by post-composition a homotopy equivalence between the homotopy fixed point spaces. In particular f_0 induces a homotopy equivalence between $([\mathrm{hocolim}_{\mathcal{C}}(\mathrm{Map}(EQ/Q_0,F))]^\wedge_p)^{hV}$ and $(\mathrm{Map}(EQ/Q_0,(\mathrm{hocolim}_{\mathcal{C}}(F))^\wedge_p))^{hV}$.

For any space Y we have that $\mathrm{Map}(BQ_0,Y)^{hV} = \mathrm{holim}_{\mathcal{B}V}\mathrm{Map}(BQ_0,Y)$. By [7, Proposition XII 4.1] we conclude that

$$\mathrm{Map}(BQ_0,Y)^{hV} = \operatorname*{holim}_{\mathcal{B}V}\mathrm{Map}(BQ_0,Y) \simeq \mathrm{Map}(\operatorname*{hocolim}_{\mathcal{B}V} BQ_0, Y).$$

We want to prove that in fact $\mathrm{hocolim}_{\mathcal{B}V} BQ_0 \simeq BQ$. Let $\pi\colon \mathcal{B}Q \to \mathcal{B}V$ be the projection. Then by [23, Theorem 5.5] we have that

$$BQ \simeq \operatorname*{hocolim}_{\mathcal{B}Q}(*) \cong \operatorname*{hocolim}_{\mathcal{B}V} |\pi\downarrow(-)|.$$

In this case the overcategory $\pi \downarrow o_V$ is the category with objects $v\in V$, and morphisms $v\to v'$ are elements $q\in Q$ such that $v = v'\cdot\pi(q)$ in V. The full subcategory on the object 1 has a morphism set isomorphic to Q_0, so we may identify this with $\mathcal{B}Q_0$. Let $\sigma\colon V\to Q$ be a section, such that $\sigma(1)=1$. Comparing definitions we get a well-defined functor $G\colon \pi\downarrow o_V \to \mathcal{B}Q_0$ by setting $v\mapsto o_Q$ and $q\colon v\to v'$ is mapped to $\sigma(v')\cdot q\cdot \sigma(v)^{-1}\in Q_0$. This is the identity on $\mathcal{B}Q_0$, and $\sigma(v)\colon v\to 1$ gives a natural transformation from $\mathrm{id}_{\pi\downarrow o_V}$ to $\mathrm{incl}\circ G$. By considering the geometric realization we conclude that BQ is a deformation retract of $|\pi\downarrow o_V|$, and thus $\mathrm{hocolim}_{\mathcal{B}V}|\pi\downarrow(-)| \simeq \mathrm{hocolim}_{\mathcal{B}V} BQ_0$, hence we see that $\mathrm{Map}(BQ_0,Y)^{hV}\simeq \mathrm{Map}(BQ,Y)$.

For any $c\in\mathcal{C}$ we have that $\mathrm{Map}(EQ/Q_0,F(c))$ is a V-space by the induced action from Q. Then for any $\varphi\in \mathrm{Mor}_{\mathcal{C}}(c,c')$ post-composition with $F(\varphi)$ will be a V-map. Hence we have a functor $F'\colon \mathcal{C}\to V\text{-}\mathsf{Spaces}$ given by $F'(c) = \mathrm{Map}(EQ/Q_0,F(c))$ and $F'(\varphi) = F(\varphi)\circ(-)$. For any $c\in\mathcal{C}$ we have that

$$F'(c) = \mathrm{Map}(BQ_0,F(c)),\quad F'(c)^{hV}\simeq \mathrm{Map}(BQ_0,F(c))^{hV}\simeq \mathrm{Map}(BQ,F(c)),$$

which are p-complete spaces with finite mod p cohomology in each degree by assumption. Since $V=\mathbb{Z}/p$ we can apply Lemma 7.1 to this functor, resulting in a homotopy equivalence

$$[\operatorname*{hocolim}_{\mathcal{C}}(\mathrm{Map}(BQ,F))]^\wedge_p \simeq [(\operatorname*{hocolim}_{\mathcal{C}}\mathrm{Map}(EQ/Q_0,F))^\wedge_p]^{hV}.$$

Thus we conclude that

$$\begin{aligned}
[\operatorname*{hocolim}_{\mathcal{C}}(\mathrm{Map}(BQ,F))]^\wedge_p &\simeq [(\operatorname*{hocolim}_{\mathcal{C}}\mathrm{Map}(EQ/Q_0,F))^\wedge_p]^{hV}\\
&\simeq (\mathrm{Map}(EQ/Q_0,(\operatorname*{hocolim}_{\mathcal{C}}(F))^\wedge_p))^{hV}\\
&\simeq \mathrm{Map}(BQ,(\operatorname*{hocolim}_{\mathcal{C}}(F))^\wedge_p).
\end{aligned}$$

□

7.2. **Mapping space and p-local finite groups.** We will now apply the previous results to the case of p-local finite groups.

Proposition 7.3. *Let p be a prime. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and Q a finite p-group. Define a category $\mathcal{L}_Q$ by setting* $\mathrm{Ob}(\mathcal{L}_Q) = \{(P, \alpha) \mid P \in \mathcal{F}^c, \alpha \in \mathrm{Hom}(Q, P)\}$ *and*

$$\mathrm{Mor}_{\mathcal{L}_Q}((P, \alpha), (P', \alpha')) = \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, P') \mid \alpha' = \pi(\varphi) \circ \alpha \in \mathrm{Hom}(Q, P')\}.$$

Let $\Phi\colon \mathcal{L}_Q \times \mathcal{B}(Q) \to \mathcal{L}$ be given by

$$\Phi((P, \alpha), o_Q) = P, \quad \Phi(\varphi\colon (P, \alpha) \to (P', \alpha'), x) = \varphi \circ \delta_P(\alpha(x))$$

Then Φ is a well-defined functor and the adjoint to $|\Phi|$ after p-completion

$$|\Phi|'\colon |\mathcal{L}_Q|^\wedge_p \to \mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$$

is a homotopy equivalence.

Proof. By property (C) for the p-local finite group, we conclude that for any $(\varphi, x) \in \mathrm{Mor}_{\mathcal{L}_Q}((P, \alpha), (P', \alpha')) \times Q$ we have that

$$\varphi \circ \delta_P(\alpha(x)) = \delta_{P'}(\pi(\varphi)(\alpha(x))) \circ \varphi = \delta_{P'}(\alpha'(x)) \circ \varphi,$$

so for $\varphi \in \mathrm{Mor}_{\mathcal{L}_Q}((P, \alpha), (P', \alpha')), \varphi' \in \mathrm{Mor}_{\mathcal{L}_Q}((P', \alpha'), (P'', \alpha''))$ and $x, x' \in Q$ we have

$$\begin{aligned}\Phi(\varphi' \circ \varphi, x'x) &= \varphi' \circ \varphi \circ \delta_P(\alpha(x'x)) = \varphi' \circ \delta_P(\alpha'(x')) \circ \varphi \circ \delta_P(\alpha(x)) \\ &= \Phi(\varphi', x') \circ \Phi(\varphi, x)\end{aligned}$$

We see that $1_{((P,\alpha),o_Q)} = (1_P, 1)$ and since $\Phi(1_P, 1) = 1_P \circ \delta_P(\alpha(1)) = 1_P$, we conclude that Φ is a well-defined functor.

Consider the functor $\tilde{\pi}\colon \mathcal{L} \to \mathcal{O}^c(\mathcal{F})$, which is π composed with the projection onto the orbit category. Define $\tilde{\pi}_Q\colon \mathcal{L}_Q \to \mathcal{O}^c(\mathcal{F})$ to be the functor given by $\tilde{\pi}_Q(P, \alpha) = P$ and $\tilde{\pi}_Q(\varphi) = \tilde{\pi}(\varphi)$. Let $\tilde{B}_Q, \tilde{B}\colon \mathcal{O}^c(\mathcal{F}) \to \mathsf{Top}$ be the left homotopy Kan extensions of the constant functor $*\colon \mathcal{L} \to \mathsf{Top}$ over $\tilde{\pi}$ respectively the left homotopy Kan extension of $*\colon \mathcal{L}_Q \to \mathsf{Top}$ over $\tilde{\pi}_Q$. Then $\tilde{B} = |\tilde{\pi} \downarrow (-)|$ and $\tilde{B}_Q = |\tilde{\pi}_Q \downarrow (-)|$. By [23, Theorem 5.5] we have that $|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B})$ and $|\mathcal{L}_Q| \simeq \mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}_Q)$.

Now we consider the diagram

$$\begin{tikzcd} \mathcal{L}_Q \times \mathcal{B}(Q) \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{L} \arrow[d, "\tilde{\pi}"] \\ \mathcal{L}_Q \arrow[r, "\tilde{\pi}_Q"] & \mathcal{O}^c(\mathcal{F}) \end{tikzcd}$$

where pr_1 is the projection onto the first component. Both ways map $((P, \alpha), o_Q)$ to P, and for $(\varphi, x) \in \mathrm{Mor}_{\mathcal{L}_Q}((P, \alpha), (P', \alpha')) \times Q$ we have that

$$\begin{aligned}\tilde{\pi}(\Phi(\varphi, x)) &= \tilde{\pi}(\varphi \circ \delta_P(\alpha(x))) = (\pi(\varphi) \circ c_{\alpha(x)}) \,\mathrm{Inn}(P) \\ &= \pi(\varphi) \,\mathrm{Inn}(P) = \tilde{\pi}_Q \circ \mathrm{pr}_1(\varphi, x),\end{aligned}$$

hence the diagram commutes. As $\tilde{\pi}_Q \circ \mathrm{pr}_1 \downarrow (-)$ can be identified with $\tilde{\pi}_Q \downarrow (-) \times \mathcal{B}(Q)$, the left homotopy Kan extension to $\tilde{\pi}_Q \circ \mathrm{pr}_1$ over the constant functor is exactly $\tilde{B}_Q(-) \times BQ$. The commutativity of the diagram implies that Φ

induces a natural transformation from $\tilde{\pi}_Q \circ \mathrm{pr}_1 \downarrow (-)$ to $\tilde{\pi} \downarrow (-)$. We denote this $\Phi' \colon \tilde{B}_Q(-) \times BQ \to \tilde{B}$. The adjoint map $\tilde{\Phi} \colon \tilde{B}_Q \to \mathrm{Map}(BQ, \tilde{B})$ is then also a natural transformation. We consider the diagram:

$$\begin{tikzcd}
(\operatorname*{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}_Q))^{\wedge}_p \arrow[r, "\operatorname{hocolim}(\tilde{\Phi})^{\wedge}_p"] \arrow[d] & (\operatorname*{hocolim}_{\mathcal{O}^c(\mathcal{F})} \mathrm{Map}(BQ, \tilde{B}))^{\wedge}_p \arrow[r, "\omega"] & \mathrm{Map}(BQ, \operatorname*{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B})^{\wedge}_p) \arrow[d] \\
|\mathcal{L}_Q|^{\wedge}_p \arrow[rr, "|\Phi|'"] & & \mathrm{Map}(BQ, |\mathcal{L}|^{\wedge}_p)
\end{tikzcd}$$

For any functor $F \colon \mathcal{C} \to \mathcal{C}'$ and $c' \in \mathcal{C}'$ there is a functor $F \downarrow c' \to \mathcal{C}$ that forgets the structure from $\mathcal{C}'$. Denote this functor pr_F. For any $P \in \mathrm{Ob}(F_0)$ the upper composition is induced by the map

$$\tilde{B}_Q(P) \times BQ \xrightarrow{\Phi'} \tilde{B}(P) \xrightarrow{|\,\mathrm{pr}_{\tilde{\pi}}\,|} |\mathcal{L}|$$

while the lower is induced by

$$\tilde{B}_Q(P) \times BQ \xrightarrow{|\,\mathrm{pr}_{\tilde{\pi}_Q}\,| \times \mathrm{id}} |\mathcal{L}_Q| \times BQ \xrightarrow{|\Phi|} |\mathcal{L}|.$$

These maps are identical, so the diagram commutes. By Proposition 4.2 we have that for any $P \in \mathcal{F}^c$ the space $\tilde{B}(P)$ is homotopy equivalent to BP. Thus for any $Q_0 \subseteq Q$ we have that $\mathrm{Map}(BQ_0, \tilde{B}(P)) \simeq \mathrm{Map}(BQ_0, BP)$. By [9, Proposition 2.1] we have that the components of $\mathrm{Map}(BQ_0, BP)$ are in bijection with $\mathrm{Rep}(Q_0, P)$ and for any $\rho \in \mathrm{Hom}(Q_0, P)$ we also have that $\mathrm{Map}(BQ_0, BP)_{B\rho} \simeq BC_P(\rho(Q_0))$. As P is a finite p-group the space $BC_P(\rho(Q_0))$ is p-complete and has finite mod p cohomology in each degree. Then $\mathrm{Map}(BQ_0, BP)$ is an finite disjoint union of p-complete spaces with finite mod p cohomology in each degree, so the same is true for $\mathrm{Map}(BQ_0, BP)$ and thus also for $\mathrm{Map}(BQ_0, \tilde{B}P)$. We have that $\mathcal{O}^c(\mathcal{F})$ is a finite category and by Corollary 6.11 it has bounded limits at p. Then Proposition 7.2 implies that ω is a homotopy equivalence. As the vertical map in the diagram are p-completion of homotopy equivalences, they are in particular p-completion of p-equivalences. Thus by [7, Lemma I 5.5.] they are homotopy equivalence. We conclude by commutativity of the diagram that $|\Phi|'$ is a homotopy equivalence if $\operatorname{hocolim} \tilde{\Phi}$ is. As we are working the category of simplicial sets and maps we have by [17, IV Proposition 1.9] that this holds if $\tilde{\Phi}(P)$ is a homotopy equivalence for any $P \in \mathcal{F}^c$.

Let $\tilde{\sigma} \colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \to \mathrm{Mor}(\mathcal{L})$ be a section, such that $\tilde{\sigma}(\mathrm{id}_P) = 1_P$ for any $P \in \mathcal{F}^c$. In the proof of Theorem 4.2 we constructed a functor $\Psi \colon \tilde{\pi} \downarrow P \to \mathcal{B}'(P)$ for any $P \in \mathcal{F}^c$, where $\mathcal{B}'(P)$ is the full subcategory of $\tilde{\pi} \downarrow P$ on the object (P, id_P) by setting $\Psi(R, \chi) = (P, \mathrm{id}_P)$ and $\Psi(\varphi) = \delta_P(g_\varphi)$. Furthermore the map $\tilde{\sigma}(\chi) \colon (R, \chi) \to (P, \mathrm{id})$ is a natural transformation from $\mathrm{id}_{\tilde{\pi}\downarrow P} \to \mathrm{incl} \circ \Psi$, so $BP \simeq |\mathcal{B}'(P)| \subseteq |\tilde{\pi} \downarrow P|$ is a deformation retract. We will now make a similar construction on $\tilde{\pi}_Q \downarrow P$. Note that $\tilde{\pi}_Q \downarrow P$ is the category with objects (R, α, χ) where $R \in \mathcal{F}^c$, $\alpha \in \mathrm{Hom}(Q, R)$ and $\chi \in \mathrm{Rep}_{\mathcal{F}}(R, P)$, and morphisms

$$\mathrm{Mor}_{\tilde{\pi}_Q \downarrow P}((R, \alpha, \chi), (R', \alpha', \chi')) = \{\varphi \in \mathrm{Mor}_{\mathcal{L}}(R, R') \mid \alpha' = \pi(\varphi) \circ \alpha, \chi = \chi' \circ \tilde{\pi}(\varphi)\}.$$

Let $\mathcal{B}'_Q(P)$ be the full subcategory on the objects (P, α, id) where $\alpha \in \mathrm{Hom}(Q, P)$. Let $(R, \alpha, \chi) \in \tilde{\pi}_Q \downarrow P$. As $\pi(\tilde{\sigma}(\chi)) \in \mathrm{Hom}_{\mathcal{F}}(R, P) \subseteq \mathrm{Hom}(R, P)$ we have that

$(P, \pi(\tilde{\sigma}(\chi))\circ\alpha, \mathrm{id}) \in \mathcal{B}'_Q(P)$. Let $\varphi \in \mathrm{Mor}_{\tilde{\pi}_Q\downarrow P}((R,\alpha,\chi),(R',\alpha',\chi'))$. By definition $g_\varphi \in P$ is the unique element such that $\tilde{\sigma}(\chi')\circ\varphi = \delta_p(g_\varphi)\circ\tilde{\sigma}(\chi)$. Then

$$\pi(\delta_P(g_\varphi))\circ\pi(\tilde{\sigma}(\chi))\circ\alpha = \pi(\tilde{\sigma}(\chi'))\circ\pi(\varphi)\circ\alpha = \pi(\tilde{\sigma}(\chi'))\circ\alpha'$$

and as $\pi(\delta_p(g_\varphi)) \in \mathrm{Inn}(P)$, we conclude that $\delta_p(g_\varphi)$ is a morphism from $(P, \pi(\tilde{\sigma}(\chi))\circ \alpha, \mathrm{id})$ to $(P, \pi(\tilde{\sigma}(\chi'))\circ\alpha', \mathrm{id})$. Thus we can define a retraction functor $\Psi_Q\colon \tilde{\pi}_Q \downarrow P \to \mathcal{B}'_Q(P)$ by setting $\Psi_Q(R,\alpha,\chi) = (P, \pi(\tilde{\sigma}(\chi))\circ\alpha, \mathrm{id})$ and $\Psi_Q(\varphi) = \delta_P(g_\varphi)$. As $\tilde{\sigma}$ is a section we see that $\tilde{\sigma}(\chi)$ will be a map from (R,α,χ) to $(P, \pi(\tilde{\sigma}(\chi))\circ\alpha, \mathrm{id})$ and the definition of g_φ implies, that this is in fact a natural transformation $\mathrm{id}_{\tilde{\pi}_Q\downarrow P} \to \mathrm{incl}\circ\Psi_Q$. Hence $|\mathcal{B}'_Q(P)| \subseteq |\tilde{\pi}_Q \downarrow P|$ is a deformation retract.

We remark that $\Phi((P,\alpha), o_Q) = P$, so the induced map on the over-categories will map $\mathcal{B}'_Q(P)\times\mathcal{B}Q$ into $\mathcal{B}'(P)$. The adjoint map after geometric realization $\tilde{\Phi}_0(P)\colon |\mathcal{B}'_Q(P)| \to \mathrm{Map}(BQ, |\mathcal{B}'(P)|)$ makes the following diagram commute:

$$\begin{tikzcd}
{|\mathcal{B}'_Q(P)|} \arrow[r, "\tilde{\Phi}_0(P)"] \arrow[d, "\mathrm{incl}"'] & \mathrm{Map}(BQ, |\mathcal{B}'(P)|) \arrow[d, "\mathrm{incl}\circ(-)"'] \\
\tilde{B}_Q(P) \arrow[r, "\tilde{\Phi}(P)"] & \mathrm{Map}(BQ, \tilde{B}(P))
\end{tikzcd}$$

As both inclusions are homotopy equivalences, it is sufficient to show that the same holds for $\tilde{\Phi}_0(P)$.

Consider two objects (P,α,id) and (P,α',id) in $\mathcal{B}'_Q(P)$. A morphism between these is a $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,P)$ such that $\alpha' = \pi(\varphi)\circ\alpha$ and $\tilde{\pi}(\varphi) = \mathrm{id}$. Thus $\pi(\varphi) \in \mathrm{Inn}(P)$, hence α and α' are conjugate in P. Similarly if $\alpha' = c_p\circ\alpha$ for some $p \in P$, then $\delta_P(p)$ is a morphism between (P,α,id) and (P,α',id) in $\mathcal{B}'_Q(P)$. Thus there exists a morphism between (P,α,id) and (P,α',id) in $\mathcal{B}'_Q(P)$ if and only if α and α' agree in $\mathrm{Rep}(Q,P)$. The connected components of $\mathcal{B}'_Q(P)$ are thus in bijection with the set $\mathrm{Rep}(Q,P)$. Let $\alpha \in \mathrm{Hom}(Q,P)$. As the maps from $(P, c_p\circ\alpha, \mathrm{id})$ to (P,α,id) are isomorphisms, we see that the connected component of $\mathcal{B}'_Q(P)$ containing (P,α,id) deformation retracts onto the full subcategory on the object (P,α,id). By the above we have that an automorphism of (P,α,id) is a $\varphi \in \mathrm{Aut}_{\mathcal{L}}(P)$ such that $\pi(\varphi) = c_p$ for some $p \in P$ and $\alpha = c_p\circ\alpha$. The first condition implies that $\varphi = \delta_P(pp')$ for some $p' \in \mathrm{Z}(P)$ while he second condition implies that $p \in C_P(\alpha(Q))$. As $\mathrm{Z}(P) \subseteq C_P(\alpha(Q))$ we conclude that the automorphism set is exactly $\delta_P(C_P(\alpha(Q)))$. Since δ_P is injective, we thus have that full subcategory on the object (P,α,id) is isomorphic to $\mathcal{B}C_P(\alpha(Q))$. As this applies to all the connected components of $\mathcal{B}'_Q(P)$, we conclude that

$$|\mathcal{B}'_Q(P)| \simeq \coprod_{\alpha\in\mathrm{Rep}(Q,P)} BC_P(\alpha(Q)).$$

Hence it is sufficient to show that $\tilde{\Phi}_0$ restricted to $\coprod_{\alpha\in\mathrm{Rep}(Q,P)} BC_P(\alpha(Q))$ is a homotopy equivalence.

We remark that $|\mathcal{B}'(P)| \simeq BP$ and the restriction of $\Phi'\colon \mathcal{B}C_P(\alpha(Q))\times\mathcal{B}Q \to \mathcal{B}P$ for any $\alpha \in \mathrm{Hom}(Q,P)$ under the given identification satisfies that

$$\Phi'(p,x) = \Phi(\delta_P(p),x) = \delta_P(p)\circ\delta_P(\alpha(x)) = p\cdot\alpha(x).$$

It thus corresponds to $\mathrm{incl} \circ \alpha \colon C_P(\alpha(P)) \times Q \to P$. By [9, Proposition 2.1] the adjoint map $BC_P(\alpha(Q)) \to \mathrm{Map}(BQ, BP)_{B\alpha}$ is a homotopy equivalence. The Proposition also implies that the connected components of $\mathrm{Map}(BQ, BP)$ are in bijection with $\mathrm{Rep}(Q, P)$, then the map $\coprod_{\alpha \in \mathrm{Rep}(Q,P)} BC_P(\alpha(Q)) \to \mathrm{Map}(BQ, BP)$ induced by $\mathrm{incl} \circ \alpha$ ranging over $\alpha \in \mathrm{Rep}(Q, P)$ is a homotopy equivalence and it corresponds to the restriction of $\tilde{\Phi}_0$. From this the proposition follows. □

We will now give a description of $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ and some of its connected components. The case of the components may be considered as a generalization of the result $BC_P(\alpha(Q)) \simeq \mathrm{Map}(BQ, BP)_{B\alpha}$ for any $\alpha \in \mathrm{Rep}(Q, P)$.

Theorem 7.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let $\theta \colon \mathcal{B}S \to \mathcal{L}$ be the functor induced by δ_S and let $f = \phi_{|\mathcal{L}|} \circ |\theta| \colon BS \to |\mathcal{L}|^\wedge_p$, where ϕ is the natural transformation from p-completion. Then the following holds for any p-group Q.*

(a) *Each map $BQ \to |\mathcal{L}|^\wedge_p$ is homotopic to $f \circ B\rho$ for some $\rho \in \mathrm{Hom}(Q, S)$.*

(b) *Given any two $\rho, \rho' \in \mathrm{Hom}(Q, S)$, then $f \circ B\rho$ and $f \circ B\rho'$ are homotopic as maps $BQ \to |\mathcal{L}|^\wedge_p$ if any only if there exists $\chi \in \mathrm{Hom}_{\mathcal{F}}(\rho Q, \rho' Q)$ such that $\rho' = \chi \circ \rho$.*

(c) *For each $\rho \in \mathrm{Hom}(Q, S)$ such that $\rho Q \in \mathcal{F}^c$, the adjoint to the composite*

$$B\,\mathrm{Z}(\rho Q) \times BQ \xrightarrow{\mathrm{incl} \cdot B\rho} BS \xrightarrow{f} |\mathcal{L}|^\wedge_p$$

is a homotopy equivalence $B\,\mathrm{Z}(\rho Q) \to \mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{f \circ B\rho}$.

(d) *The evaluation map $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{triv} \to |\mathcal{L}|^\wedge_p$ is a homotopy equivalence.*

Proof. Using the notation from Proposition 7.3, we have that the map $|\Phi|' \colon |\mathcal{L}_Q|^\wedge_p \to \mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ is a homotopy equivalence. For $(P, \alpha) \in \mathcal{L}_Q$ the image of $|\Phi'|$ on the vertex corresponding to (P, α) in $|\mathcal{L}_Q|^\wedge_p$ is the map

$$BQ \xrightarrow{B\alpha} BP \xrightarrow{|\theta_P|} |\mathcal{L}| \xrightarrow{\phi_{|\mathcal{L}|}} |\mathcal{L}|^\wedge_p$$

where $\theta_P \colon \mathcal{B}P \to \mathcal{L}$ is the functor induced by δ_P. By property (C) for the linking system, we conclude that any lift of the inclusion $i \colon P \to S$ to $\mathcal{L}$ will give rise to a natural transformation from $\theta_P \to \theta \circ \mathcal{B}i$. Thus we have that $|\theta_P| \simeq |\theta| \circ Bi$. Hence the above map is homotopic to $f \circ B(i \circ \alpha)$ where $i \circ \alpha \in \mathrm{Hom}(Q, S)$. A map from (P, α) and (P', α') in $\mathcal{L}_Q$ corresponds to a $\chi \in \mathrm{Hom}_{\mathcal{F}}(\alpha(Q), \alpha'(Q))$ such that $\alpha = \chi \circ \alpha$. Two vertises (P, α) and (P', α') in $\mathcal{L}_Q$ are connected in $|\mathcal{L}_Q|$ if and only if there exists a chain of maps connecting them in $\mathcal{L}_Q$. Note that direction of the maps may alternate, but the composition of the corresponding χ_i's or the inverse of χ_i in case of opposite direction will give a $\chi \in \mathrm{Hom}_{\mathcal{F}}(\alpha Q, \alpha' Q)$ such that $\alpha = \chi \circ \alpha$.

Let $\varphi \in \mathrm{Hom}(BQ, |\mathcal{L}|^\wedge_p)$. As $|\Phi|'$ is a homotopy equivalence, we have that $|\Phi|'$ is a bijection on the connected components, so the connected component of φ contains the image of a connected of $|\mathcal{L}_Q|^\wedge_p$. This connected component of $|\mathcal{L}_Q|^\wedge_p$ contains some vertex of the form (P, α), thus the connected component of φ in $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ contains $|\Phi|'(P, \alpha)$. Then $\varphi \simeq |\Phi|'(P, \alpha) \simeq f \circ B(i \circ \alpha)$, where $i \circ \alpha \in \mathrm{Hom}(Q, S)$. For any two $\rho, \rho' \in \mathrm{Hom}(Q, S)$, we have that $f \circ B(i \circ \alpha) \simeq f \circ B(i \circ \alpha)$ if and only if (P, α) and (P', α') are connected in $\mathcal{L}_Q$, so part (b) follows as well.

Let $\rho \in \mathrm{Hom}(Q, S)$, such that $\rho Q \in \mathcal{F}^c$. Then $(\rho Q, \rho) \in \mathcal{L}_Q$ and let $(\mathcal{L}_Q)_{(\rho Q, Q)}$ be the connected components containing $(\rho Q, \rho)$. Let $\sigma \colon \mathrm{Mor}(\mathcal{F}) \to \mathrm{Mor}(\mathcal{L})$ be a section such that $\sigma(\mathrm{id}_{\rho Q}) = 1_{\rho Q}$. For any $(P, \alpha) \in (\mathcal{L}_Q)_{(\rho Q, Q)}$ there exists a $\chi_{(P,\alpha)} \in \mathrm{Hom}_{\mathcal{F}}(\rho Q, \alpha Q)$ such that $\chi_{(P,\alpha)} \circ \rho = \alpha$. Assume that $\chi_{(\rho Q,\rho)} = \mathrm{id}_{\rho Q}$. For any morphism $\varphi \in (P, \alpha) \to (P', \alpha')$ in $(\mathcal{L}_Q)_{(\rho Q,Q)}$, we have that $\pi(\varphi) \circ \chi_{(P,\alpha)} = \chi_{(P',\alpha')}$ on ρQ. Let $\psi = \chi^{-1}_{(P',\alpha')} \circ \pi(\varphi) \circ \chi_{(P,\alpha)}$. By Lemma 3.1 (a) there exists a unique $\tilde{\psi} \in \mathrm{Aut}_{\mathcal{L}}(\rho Q)$ such that

$$\begin{tikzcd}
P \arrow[r, "\varphi"] & P' \\
\rho Q \arrow[u, "\sigma(\chi_{(P,\alpha)})"] \arrow[r, "\tilde{\psi}"] & \rho Q \arrow[u, "\sigma(\chi_{(P',\alpha')})"']
\end{tikzcd}$$

commutes with $\pi(\tilde{\psi}) = \psi$. Comparing definitions we get that $\tilde{\psi} \in \mathrm{Aut}_{\mathcal{L}_Q}(\rho Q, \rho)$. Thus we get a well-defined retraction functor R from $(\mathcal{L}_Q)_{(\rho Q,Q)}$ to the full subcategory on the object $(\rho Q, \rho)$ using the above construction on the morphisms. Similar to previous cases $\sigma(\chi_{(P,\alpha)})$ will be a natural transformation from $\mathrm{incl} \circ R$ to $\mathrm{id}_{(\mathcal{L}_Q)_{(\rho Q,Q)}}$, so $B\,\mathrm{Aut}_{\mathcal{L}_Q}(\rho Q, \rho)$ will be a deformation retract of $|(\mathcal{L}_Q)_{(\rho Q,Q)}|$. Then by property (A) for $\mathcal{L}$ we have that

$$\mathrm{Aut}_{\mathcal{L}_Q}(\rho Q, \rho) = \{\varphi \in \mathrm{Aut}_{\mathcal{L}}(\rho Q) \mid \pi(\varphi) = \mathrm{id}_{\rho Q}\} = \delta_{\rho Q}(\mathrm{Z}(\rho Q)).$$

As $\delta_{\rho Q}$ is injective we conclude that $|(\mathcal{L}_Q)_{(\rho Q,Q)}|$ has the homotopy type of $B\,\mathrm{Z}(\rho Q)$. Then the component of $|\mathcal{L}_Q|^\wedge_p$ containing $(\rho Q, \rho)$ has the homotopy type of $(B\,\mathrm{Z}(\rho Q))^\wedge_p$ which is the same as $B\,\mathrm{Z}(\rho Q)$, since $\mathrm{Z}(\rho Q)$ is a finite p-group so the space $B\,\mathrm{Z}(\rho Q)$ is p-complete [3, III 1.4 Proposition 1.10]. The restriction of $|\Phi|'$ to the component containing $(\rho Q, \rho)$ is a homotopy equivalence onto the component containing $|\Phi|'(\rho Q, \rho)$, which is exactly $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{f \circ B\rho}$. We note that Φ on the full subcategory of $\mathcal{L}_Q$ on the object $(\rho Q, \rho)$ is

$$BZ(\rho Q) \times BQ \xrightarrow{\mathrm{incl} \cdot B(\rho)} B\rho Q \xrightarrow{|\theta_{\rho Q}|} |\mathcal{L}| \xrightarrow{\phi_{|\mathcal{L}|}} |\mathcal{L}|^\wedge_p$$

which is homotopic to the map stated in part (c), so the results follows.

The vertises $(P, 1)$ for $P \in \mathcal{F}^c$ lie in the same component of $\mathcal{L}_Q$. For any $(P, \alpha) \in \mathcal{L}_Q$ there exists a path to $(P, 1)$ in $\mathcal{L}_Q$ if there is a $\chi \in \mathrm{Hom}(1, \alpha(Q))$ such that $\alpha = \chi \circ 1 = 1$. Hence $(P, 1)$ for $P \in \mathcal{F}^c$ constitutes a connected component of $\mathcal{L}_Q$. This component is equivalent to $\mathcal{L}$ and the restriction of Φ onto $\mathcal{L} \times \mathcal{B}Q$ is just the projection onto the first component. Thus the adjoint map $|\Phi|'$ on this component is the map from $x \in |\mathcal{L}|^\wedge_p$ to the constant map at x in $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$, which lies in $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{\mathrm{triv}}$, thus by the above this is homotopy equivalence from $|\mathcal{L}|^\wedge_p$ to $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{\mathrm{triv}}$. We note that this has the evaluation map as an inverse. $\square$

Definition 7.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. For any finite p-group Q, $\rho, \rho' \in \mathrm{Hom}(Q, S)$ are $\mathcal{F}$-conjugate if there exists a $\chi \in \mathrm{Hom}_{\mathcal{F}}(\rho Q, \rho' Q)$ such that $\rho' = \chi \circ \rho$. This is an equivalence relation and we denote the set of equivalence classes* $\mathrm{Rep}(Q, \mathcal{L})$.

Observe that as all conjugation maps are $\mathcal{F}$-morphisms, this also induces an equivalence relation on $\mathrm{Rep}(Q, P)$. So we may call elements of $\mathrm{Rep}(Q, P)$ $\mathcal{F}$-conjugate. With this definition we get the following reformulation:

Corollary 7.6. *For a p-local finite group* $(S, \mathcal{F}, \mathcal{L})$ *and finite p-group* Q*, the map* $\mathrm{Rep}(Q, \mathcal{L}) \to [BQ, |\mathcal{L}|^{\wedge}_{p}]$ *given by* $\rho \in \mathrm{Rep}(Q, \mathcal{L})$ *is mapped to* $\phi_{|\mathcal{L}|} \circ |\theta| \circ B\rho$ *is a homotopy equivalence.*

8. The cohomology ring of fusion systems

In this chapter we will define the cohomology ring for a fusion system, and prove that it is Noetherian. Furthermore in the case of a p-local finite group, it is isomorphic to the cohomology ring of the p-completion of its classifying space. In the following chapter cohomology will always be with $\mathbb{F}_p$-coefficients.

Definition 8.1. *For a fusion system $\mathcal{F}$ over a p-group S, we define the cohomology ring as*
$$\mathrm{H}^*(\mathcal{F}) = \varprojlim_{\mathcal{O}(\mathcal{F})} \mathrm{H}^*(-).$$
We note that it is a subring of $\mathrm{H}^*(BS)$.

Observe that for any finite group Q and $q \in Q$ the element q gives a natural transformation between the identity functor on $\mathcal{B}Q$ and $\mathcal{B}c_q$, so $c_q^* = \mathrm{id}$ on $\mathrm{H}^*(Q)$. Hence $\mathrm{H}^*(-)$ is a functor on $\mathcal{O}(\mathcal{F})^{op}$, so the above limit exists. Furthermore we have that the category $\mathcal{O}(\mathcal{F}_S(S))$ has a unique morphism between each pair of elements, and as S is a maximal element, we see that $\mathrm{H}(\mathcal{F}_S(S)) = \mathrm{H}^*(BS)$.

Lemma 8.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $\theta\colon \mathcal{B}S \to \mathcal{L}$ be the functor induced by $\delta_S\colon S \to \mathrm{Mor}_{\mathcal{L}}(S)$. Let $P \subseteq S$ and $i_P\colon P \to S$ be the inclusion. Then the collection of maps $i_P^* \circ |\theta|^*\colon \mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(P)$ induces a map $R_{\mathcal{L}}\colon \mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(\mathcal{F})$.*

Proof. For $P, Q \in \mathcal{F}$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ it follows by Theorem 7.4 that $\phi_{|\mathcal{L}|} \circ |\theta| \circ B(i_Q \circ \varphi)$ and $\phi_{|\mathcal{L}|} \circ |\theta| \circ Bi_P$ are homotopic as maps from BP to $|\mathcal{L}|^\wedge_p$. As $|\mathcal{L}|$ is p-good, we have that $\phi_{|\mathcal{L}|}^*$ is an isomorphism, so we conclude that $\varphi^* \circ i_Q^* \circ |\theta|^* = i_P^* \circ |\theta|^*$ as maps $\mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(P)$ and thus induces a map to the limit over $\mathcal{O}(\mathcal{F})$. $\square$

The main theorem is then that $R_{\mathcal{L}}$ is an isomorphism.

8.1. The cohomology ring Noetherian.

In this section we will prove that the cohomology ring for any fusion system over a finite group is Noetherian. The central tool will be F-isomorphisms as defined in [29, Chapter 3].

In the following $\mathcal{E}(S)$ will for any p-group S denote the set of elementary abelian subgroups of S.

Proposition 8.3. *Let $\mathcal{F}$ be a fusion system over a p-group S. Let $\bar{\mathcal{F}}^e$ denote the full subcategory of $\mathcal{F}$ over the elementary abelian subgroups of S. Then the restriction*
$$\lambda_{\mathcal{F}}\colon \mathrm{H}^*(\mathcal{F}) \to \varprojlim_{\bar{\mathcal{F}}^e} \mathrm{H}^*(-)$$
is an F-isomorphism.

Proof. Let $\mathcal{F}$ be a fusion system over a p-group S. Let Q be a p-group and $\mathcal{F}_Q^e$ be the full subcategory of $\mathcal{F}_Q(Q)$ over $\mathcal{E}(Q)$ and consider the restriction $\lambda_Q\colon \mathrm{H}^*(Q) \to \varprojlim_{\mathcal{F}_Q^e} \mathrm{H}^*(-)$. By comparing definitions the map of [29, Theorem 6.2] with X equal to a point is exactly λ_Q, so we have that λ_Q is an F-isomorphism. As $\mathcal{F}_S^e$ is subcategory of $\bar{\mathcal{F}}^e$ we have that $\ker(\lambda_{\mathcal{F}}) \subseteq \ker(\lambda_S)$. Since λ_S is an F-isomorphism, the kernel consists of nilpotent elements, so the same holds for the kernel of $\lambda_{\mathcal{F}}$.

Now consider $\mathbf{x} = (x_E)_{E \in \bar{\mathcal{F}}^e} \in \varprojlim_{\bar{\mathcal{F}}^e} \mathrm{H}^*(-)$. We will now prove that there exists $k > 0$ such that $\mathbf{x}^{p^k} \in \mathrm{im}(\lambda_{\mathcal{F}})$. Let $P \subseteq S$. Then $\mathbf{x}_P = (x_E)_{E \subseteq P} \in \varprojlim_{\mathcal{F}_P^e} \mathrm{H}^*(-)$.

Since λ_P is an F-isomorphism, there exists $k_P > 0$ such that $(\mathbf{x}_P)^{p^{k_P}} \in \operatorname{im}(\lambda_P)$. By definition of λ_P it means that there exists $y_P \in \mathrm{H}^*(P)$ such that $\varphi^*(y_P) = (x_E)^{p^{k_P}}$ for any elementary abelian $E \subseteq P$ and $\varphi \in \operatorname{Hom}_P(E, P)$. Since S is a finite group, it has only finitely many subgroups. Hence for $k = \max\{k_P \mid P \subseteq S\}$ we can for any $P \subseteq S$ replace y_P by $(y_P)^{p^{k-k_P}}$ and k_P by k in the above and the statement still holds.

Let $\psi \in \operatorname{Hom}_{\mathcal{F}}(P, Q)$. For an elementary abelian subgroup $E \subseteq P$ the restriction $\tilde{\psi}\colon E \to \psi(E)$ is a morphism in $\bar{\mathcal{F}}^e$, so $\tilde{\psi}^*(x_{\psi(E)}) = x_E$. Consider $\psi^*(y_Q) \in \mathrm{H}^*(P)$. As $\psi(E) \subseteq Q$ is elementary abelian we see that

$$\begin{aligned} i_E^*(\psi^*(y_Q)) &= (\psi \circ i_E)^*(y_Q) = (i_{\psi(E)} \circ \tilde{\psi})^*(y_Q) = \tilde{\psi}^*(i_{\psi(E)}^*(y_Q)) \\ &= \tilde{\psi}^*((x_{\psi(E)})^{p^k}) = (x_E)^{p^k} = i_E^*(y_P). \end{aligned}$$

Hence $\lambda_P(\psi^*(y_Q)) = \lambda_P(y_P)$. As λ_P is an F-isomorphism, we have that $\psi^*(y_Q) - y_P$ is nilpotent, in particular there exists $m \geq 0$ such that $0 = (\psi^*(y_Q) - y_P)^{p^m}$. As we are in characteristic p, we conclude $\psi^*(y_Q)^{p^m} = y_P^{p^m}$. Since there are only finitely many morphisms in $\mathcal{F}$, we can choose m sufficiently large, such that $\psi^*(y_Q^{p^m}) = y_P^{p^m}$ for all $P, Q \subseteq S$ and $\psi \in \operatorname{Mor}_{\mathcal{F}}(P, Q)$. Then $\mathbf{y} = (y_P^{p^m})_{P \subseteq S} \in \mathrm{H}^*(\mathcal{F})$ with

$$\lambda_{\mathcal{F}}(\mathbf{y}) = (y_E^{p^m})_{E \in \bar{\mathcal{F}}^e} = (x_E^{p^{m+k}})_{E \in \bar{\mathcal{F}}^e} = \mathbf{x}^{p^{m+k}}.$$

□

Lemma 8.4. *Consider a commutative diagram of commutative rings of characteristic p of the form*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \Big\downarrow{\scriptstyle \lambda_A} & & \Big\downarrow{\scriptstyle \lambda_B} \\ A' & \longrightarrow & B' \end{array}$$

where the horizontal maps are inclusion and both λ_A and λ_B are F-isomorphisms. If the extension $A' \to B'$ is integral, then the same holds for the extension $A \to B$.

Proof. Let $b \in B$. We want to produce a monic polynomial in $A[x]$ having b as a root. As $\lambda_B(b) \in B'$ and the extension $A' \subseteq B'$ is integral, there exists a monic polynomial $f \in A'[x]$ having $\lambda_B(b)$ as a root. Since f only has finitely many coefficients, there exists a $k > 0$ such that $a^{p^k} \in \operatorname{im}(\lambda_A)$ for every coefficient in a of f. As A is is a commutative ring of characteristic p, we have that the polynomial $f^{p^k} \in A'[x]$ has coefficient of the form a^{p^k}, where a is a coefficient of f. Hence there exists a monic polynomial $\tilde{f} \in A[x]$ with $\lambda_A(\tilde{f}) = f^{p^k}$. By commutativity of the diagram, this implies that

$$\lambda_B(\tilde{f}(b)) = \lambda_A(\tilde{f})(\lambda_B(b)) = f^{p^k}(\lambda_B(b)) = f(\lambda_B(b))^{p^k} = 0.$$

As λ_B is an F-isomorphism we have that $\tilde{f}(b)$ is nilpotent. Hence there exists $N > 0$, such that $\tilde{f}(b)^N = 0$. Then $\tilde{f}^N \in A[x]$ is monic with b as a root. □

Proposition 8.5. *Let $\mathcal{F}$ be a fusion system over a p-group S. Then the ring* $\mathrm{H}^*(\mathcal{F})$ *is Noetherian and the inclusion* $\mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(BS)$ *makes* $\mathrm{H}^*(BS)$ *a finitely generated* $\mathrm{H}^*(\mathcal{F})$*-module.*

Proof. Consider $\mathcal{F}$ a fusion system over a finite p-group S. Let n be the maximal rank of an element in $\mathcal{E}(S)$ and set $V = (\mathbb{Z}/p)^n$. Then $\mathrm{GL}(V)$ acts on V and induces an action on $\mathrm{H}^*(BV)$. We now define a map $\tau\colon\ \mathrm{H}^*(BV)^{\mathrm{GL}(V)} \to \prod_{E\in\mathcal{E}(S)} \mathrm{H}^*(BE)$ by choosing monomorphisms $\psi_E\colon E \to V$ for all $E \in \mathcal{E}(S)$ and setting $\tau(x) = (\psi_E^*(x))_{E\in\mathcal{E}(S)}$. If for $E \in \mathcal{E}(S)$ we consider any monomorphism $\tilde{\psi}_E\colon E \to V$, then $\tilde{\psi}_E(E)$ and $\psi_E(E)$ are two subspaces of the finite dimensional $\mathbb{F}_p$-vector space V of the same dimension, so the isomorphism between them induced by $\tilde{\psi}_E(E)$ and $\psi_E(E)$ can be extended to all of V. Hence there exists $\varphi \in \mathrm{GL}(V)$ such that $\psi_E = \varphi \circ \tilde{\psi}_E$. For $x \in \mathrm{H}^*(BV)^{\mathrm{GL}(V)}$ we have that $\varphi^*(x) = x$ and therefore $\psi_E^*(x) = \tilde{\psi}_E^*(\varphi^*(x)) = \tilde{\psi}_E^*(x)$. So τ does not depend on the particular choice of ψ_E.

Any map $\varphi \in \mathrm{Mor}_{\bar{\mathcal{F}}^e}(E, \tilde{E})$ is injective, so $\psi_{\tilde{E}}\circ\varphi$ and ψ_E are two monomorphisms from E to V. By the above we get that $\psi_E^*(x) = \varphi^*(\psi_{\tilde{E}}^*(x))$ for $x \in \mathrm{H}^*(BV)^{\mathrm{GL}(V)}$, which implies that $\tau(x) \in \varprojlim_{\bar{\mathcal{F}}^e} \mathrm{H}^*(-)$. Note that there exists $E \in \mathcal{E}(S)$ such that E has the same rank as V. For this particular E the map ψ_E is an isomorphism, so the same holds for ψ_E^*. So we see that the map τ is injective.

Let $\mathrm{H}^{ev}(BV)$ be the elements of $\mathrm{H}^*(BV)$ of even degree. We have by [19, Thm page 500] that $\mathrm{H}^*(\mathbb{Z}/p; \mathbb{F}_p)$ equals $\mathbb{F}_p[x] \otimes \Lambda_{\mathbb{F}_p}[y]$ for some $y \in \mathrm{H}^1(\mathbb{Z}/p; \mathbb{F}_p)$ and $x \in \mathrm{H}^2(\mathbb{Z}/p; \mathbb{F}_p)$ for p an odd prime while $\mathrm{H}^*(\mathbb{Z}/2; \mathbb{F}_2)$ equals $\mathbb{F}_2[x]$ for a $x \in \mathrm{H}^1(\mathbb{Z}/p; \mathbb{F}_p)$. As the classifying spaces of finite groups are CW-complexes we have by the Kunneth formula [19, Theorem 3.16] that $\mathrm{H}^*(BV)$ is a finite tensor product of $\mathrm{H}^*(\mathbb{Z}/p)$ over $\mathbb{F}_p$. So in both cases $\mathrm{H}^{ev}(BV)$ is a finitely generated polynomial algebra over $\mathbb{F}_p$. In the theory of commutative rings one of the basic results is that for an integral ring extension $B \subseteq A$ where A is finitely generated B-algebra, we have that A is a finitely generated B-module [4, Proposition 5.1]. For $B = \mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$ and $A = \mathrm{H}^{ev}(BV)$ for an $x \in \mathrm{H}^{ev}(BV)$ the polynomial $\prod_{\sigma\in\mathrm{GL}(V)}(t - \sigma(x))$ is monic, has x as a root and coefficients in $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$, so we conclude that $\mathrm{H}^{ev}(BV)$ is a finitely generated $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$-module. Then by [2, Theorem 1] we conclude that $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$ is a finitely generated $\mathbb{F}_p$-algebra. By Hilbert's basis Theorem this implies that the ring $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$ is Noetherian.

For each $E \in \mathcal{E}(S)$ we have a chosen an injective map $\psi_E\colon E \to V$. The fundamental theorem of finite abelian groups implies that $V = \psi_E(E) \oplus V'$ for some subgroup V' of V. Hence there exists a group homomorphism $\varphi_E\colon V \to E$ given by $(\psi_E(g), v') \mapsto g$ satisfying $\varphi_E \circ \psi_E = \mathrm{id}_E$. Then $\psi_E^* \circ \varphi_E^* = \mathrm{id}_{\mathrm{H}^*(E)}$, so ψ_E^* is surjective. In particular $\psi_E^*\colon\ \mathrm{H}^{ev}(BV) \to \mathrm{H}^{ev}(BE)$ is surjective, so the image of the generating set for $\mathrm{H}^{ev}(BV)$ as a $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$-module, will be a generating set for $\mathrm{H}^{ev}(BE)$ as a $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$-module, where the module structure is induced by ψ_E^*. As $\mathcal{E}(S)$ is a finite set we get that $\prod_{E\in\mathcal{E}(S)} \mathrm{H}^{ev}(BE)$ is finitely generated as a $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$-module by the map τ.

As $\operatorname{im}\tau \subseteq \varprojlim_{E\in\bar{\mathcal{F}}^e} \mathrm{H}^*(BE)$, the map τ can be see as the composition

$$\begin{tikzcd}
\mathrm{H}^*(BV)^{\mathrm{GL}(V)} \arrow[r, "\tau"] & \varprojlim_{E\in\bar{\mathcal{F}}^e} \mathrm{H}^*(BE) \arrow[r] & \varprojlim_{E\in\mathcal{F}_S^e} \mathrm{H}^*(BE) \arrow[r] & \prod_{E\in\mathcal{E}(S)} \mathrm{H}^*(BE) \\
& \mathrm{H}^*(\mathcal{F}) \arrow[u, "\lambda_{\mathcal{F}}"] \arrow[r] & \mathrm{H}^*(BS) \arrow[u, "\lambda_S"'] &
\end{tikzcd}$$

where all other horizontal maps are inclusions and the notation agrees with the one used in the proof of Proposition 8.3. As $\prod_{E\in\mathcal{E}(S)} \mathrm{H}^{ev}(BE)$ is finitely generated as a $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$-module, every element of $\prod_{E\in\mathcal{E}(S)} \mathrm{H}^{ev}(BE)$ is integral over $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$. In particular every element of the submodule $\varprojlim_{E\in\mathcal{F}_S^e} \mathrm{H}^*(BE)$ is integral over $\mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}$. Let $\omega \in \varprojlim_{E\in\mathcal{F}_S^e} \mathrm{H}^*(BE)$ be a root of the monic non-trivial polynomial $f \in \mathrm{H}^{ev}(BV)^{\mathrm{GL}(V)}[x]$. The map τ is injective, so the polynomial $\tau(f)$ will be a monic non-trivial polynomial with coefficients in $\varprojlim_{E\in\bar{\mathcal{F}}^e} \mathrm{H}^{ev}(BE)$ having ω as a root. So every element in $\varprojlim_{E\in\mathcal{F}_S^e} \mathrm{H}^{ev}(BE)$ is also integral over $\varprojlim_{E\in\bar{\mathcal{F}}^e} \mathrm{H}^{ev}(BE)$. By Proposition 8.3 both the maps $\lambda_{\mathcal{F}}$ and λ_S are F-isomorphisms. For any of the above rings the subring of elements of even degree is a commutative ring of characteristic p. By Lemma 8.4 used on the square in the diagram restricted to the elements of even degree, we conclude that the extension $\mathrm{H}^{ev}(\mathcal{F}) \subseteq \mathrm{H}^{ev}(BS)$ is integral.

As S is a finite group and $\mathbb{F}_p$ is Noetherian [15, Corollary 6.2] implies that $\mathrm{H}^*(BS)$ is a finitely generated $\mathbb{F}_p$-algebra. Hence is it also a finitely generated $\mathrm{H}^{ev}(\mathcal{F})$-algebra. For any $\omega \in \mathrm{H}^*(BS)$ of odd degree, we have that $\omega^2 \in \mathrm{H}^{ev}(BS)$, so there exists monic $f \in \mathrm{H}^{ev}(\mathcal{F})[x]$ with $f(\omega^2) = 0$. In particular $f(x^2) \in \mathrm{H}^{ev}(\mathcal{F})[x]$ will be monic with ω as a root, so the extension $\mathrm{H}^{ev}(\mathcal{F}) \subseteq \mathrm{H}^*(BS)$ is integral. From this we conclude that $\mathrm{H}^*(BS)$ is a finitely generated $\mathrm{H}^{ev}(\mathcal{F})$-module and hence also a finitely generated $\mathrm{H}^*(\mathcal{F})$-module.

We have that $\mathrm{H}^*(BS)$ is finitely generated as a $\mathbb{F}_p$-algebra. The elements of even degree will be generated by the generators of even degree and in the case of $p = 2$ also by the all products of two generators of odd degree. In all cases $\mathrm{H}^{ev}(BS)$ is a finitely generated $\mathbb{F}_p$-algebra. As the extension $\mathrm{H}^{ev}(\mathcal{F}) \subseteq \mathrm{H}^{ev}(BS)$ is integral, this implies that $\mathrm{H}^{ev}(\mathcal{F})$ is a finitely generated $\mathbb{F}_p$-algebra. Hilbert basis theorem then implies that the ring $\mathrm{H}^{ev}(\mathcal{F})$ is Noetherian. The ring $\mathrm{H}^*(BS)$ is a finitely generated $\mathrm{H}^{ev}(\mathcal{F})$-module, and therefore Noetherian. As $\mathrm{H}^*(\mathcal{F})$ is a submodule of $\mathrm{H}^*(BS)$ this implies that $\mathrm{H}^*(\mathcal{F})$ itself is Noetherian. □

8.2. $R_{\mathcal{L}}$ is an F-isomorphism.

Lemma 8.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then there exists a spectral sequence converging to $\mathrm{H}^*(|\mathcal{L}|)$, where the columns of the E_1-page are finite products of $\mathrm{H}^*(BP)$ for $P \subseteq S$, and the E_2-page has only finitely many non-zero columns.*

Furthermore the homomorphism $R_{\mathcal{L}}\colon \mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(\mathcal{F})$ is an F-isomorphism and it makes $\mathrm{H}^*(\mathcal{F})$ into a finitely generated $\mathrm{H}^*(|\mathcal{L}|)$-module.

Proof. By Proposition 4.2 it follows that $|\mathcal{L}| \simeq \operatorname{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B}$, where $\tilde{B}\colon \mathcal{O}^c(\mathcal{F}) \to$ Top satisfies $\tilde{B}P \simeq BP$. Using the spectral sequence for the cohomology of the homotopy colimit [7, XII.4.5] we get a spectral sequence converging to $\mathrm{H}^*(|\mathcal{L}|)$ with

$$E_1^{nk} \cong \prod_{P_0 \subsetneq \cdots \subsetneq P_n \subseteq S} \mathrm{H}^k(BP_0), \quad E_2^{nk} \cong \varprojlim_{O^c(\mathcal{F})}{}^n \mathrm{H}^k(B(-)).$$

As S is a finite group, there exists only finitely many strictly increasing chains of subgroup, so the products on the E_1-page are all finite. Since $\mathcal{F}$ is saturated, it follows from Corollary 6.11, that $\mathcal{O}^c(\mathcal{F})$ has bounded limits at p and therefore by considering the functor $\mathrm{H}^*(\tilde{B}(-))\colon \mathcal{O}^c(\mathcal{F})^{op} \to \mathbb{Z}_{(p)}$-$\mathsf{mod}$ we get that the E_2-page has only finitely many non-zero columns.

The spectral sequence for the cohomology of the projection $\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}) \to |\mathcal{O}^c(\mathcal{F})|$ uses the same filtration of the homotopy colimit as the spectral sequence for the cohomology of hocolim used above. So the two spectral sequences agree and by [8, IV 6.5] they are multiplicative.

As $\mathcal{F}$ is saturated, it follows by Alperin's fusion theorem for saturated fusion systems [10, Theorem A.10] that every morphism in $\mathcal{O}(\mathcal{F})$ is the restriction of a morphism in $\mathcal{O}^c(\mathcal{F})$, so $\mathrm{H}^*(\mathcal{F}) \cong \lim_{\mathcal{O}^c(\mathcal{F})} \mathrm{H}^*(-)$. Thus we have that $E_2^{0*} \cong \mathrm{H}^*(\mathcal{F})$. The spectral sequence converges to $\mathrm{H}^*(|\mathcal{L}|)$ and for a k we consider the largest submodule F_1^k in the resulting filtration different from $\mathrm{H}^k(|\mathcal{L}|)$. Then the isomorphism $\mathrm{H}^k(|\mathcal{L}|)/F_1^k \to E_\infty^{0,k} \subseteq \mathrm{H}^k(\mathcal{F})$ is exactly $R_{\mathcal{L}}$. So $E_\infty^{0,k} = R_{\mathcal{L}}(\mathrm{H}^k(|\mathcal{L}|))$ and $F_1^k = \ker(R_{\mathcal{L}}) \cap \mathrm{H}^k(|\mathcal{L}|)$. Hence the image of $R_{\mathcal{L}}$ is generated by the permanent cycles.

Now consider a $x \in F_1^k = \ker(R_{\mathcal{L}}) \cap \mathrm{H}^k(|\mathcal{L}|)$. As E_∞ has only finitely many nonzero columns, we have that $E_\infty^{*N} = 0$ for all $N \geq M$ for some large M. Using the notation from [20, Chapther 1.2] we have $0 = F_{kM}^{kM} = \cdots = F_M^{kM}$, since all their quotients are zero. Using the multiplicative structure we have $x^M \in F_M^{kM}$. So we conclude that $x^M = 0$ and hence nilpotent. As any element in $\ker(R_{\mathcal{L}})$ is a finite sum of homogeneous elements, we conclude that the kernel consists of nilpotent elements.

Consider $x \in \mathrm{H}^k(\mathcal{F}) = E_2^{0k}$ and assume that x is not a permanent cycle. Then there exists $r \geq 2$ such that $d_r(x) \neq 0$. Let $r \geq 2$ be the smallest integer such that this holds. If both p are k are odd, we have that $x^2 = 0$, so $x^p = 0$ and hence is a permanent cycle. Assume that either p or k is even. Then by the Leibnitz formula for the differential we get that $d_r(x^p) = px^{p-1}d_r(x) = 0$, so $d_N(x^p) = 0$ for all $2 \leq N \leq r$. By iterating the process we get that there exists a $M > 0$ such $d_r(x^{p^M}) = 0$ for all r less than the number of non-zero columns of the spectral sequence. Then all higher differentials are trivially zero, so we get that x^{p^M} is a permanent cycle and so $x^{p^M} \in \mathrm{im}(R_{\mathcal{L}})$. As we are working in characteristic p, this implies that for every finite sum of homogeneous elements, there exists a $M > 0$, such that the p^M-power of the element is in $\mathrm{im}(R_{\mathcal{L}})$.

By Proposition 8.5 $\mathrm{H}^*(BS)$ is finitely generated as both a $\mathrm{H}^*(\mathcal{F})$-module and as $\mathbb{F}_p$-algebra, so we have that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as a $\mathbb{F}_p$-algebra. Let $\{x_1, \ldots, x_n\}$ be a set of homogeneous algebra generators for $\mathrm{H}^*(\mathcal{F})$. Then by the above there exists $m_i > 0$, such that $x_i^{m_i} \in \mathrm{im}(R_{\mathcal{L}})$. Hence we get that the finite set $\{x_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m_i\}$ will generate $\mathrm{H}^*(\mathcal{F})$ as a $\mathrm{im}(R_{\mathcal{L}})$-module. □

8.3. **Characteristic $(S\times S)$-set for a fusion system.** In the chapter we will focus on the characteristic $(S \times S)$-set for a saturated fusion system $\mathcal{F}$ over a p-group S. Such a $(S \times S)$-set always exists and gives rise to a left inverse of the inclusion $\mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(BS)$. Note that in the literature one traditionally considers the corresponding (S, S)-biset. As there is a bijective correspondence between (S, S)-bisets and $(S \times S)$-sets and all proofs are done for the associated $(S \times S)$-sets, we will choose only to work with $(S \times S)$-sets.

Lemma 8.7. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $\mathcal{H}$ be a set of subgroups of S which is closed under taking subgroups and $\mathcal{F}$-conjugacy. Let Ω_0 be a S-set with the property that if $P, Q \subseteq S$ are $\mathcal{F}$-conjugate and not elements of $\mathcal{H}$, then $|\Omega_0^P| = |\Omega_0^Q|$. Then there exists a S-set Ω containing Ω_0, such that $|\Omega^P| = |\Omega^Q|$ for each pair of $\mathcal{F}$-conjugate subgroups $P, Q \subseteq S$ and if $P \notin \mathcal{H}$, then*

$\Omega^P = \Omega_0^P$. *In particular we have for all* $P \subseteq S$ *and* $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ *that* Ω *considered as a* P*-set via restriction and via* α *are isomorphic.*

Proof. The proof will be by induction on the number of $\mathcal{F}$-conjugacy classes in $\mathcal{H}$. If $\mathcal{H} = \emptyset$ we have that $\Omega = \Omega_0$ satisfies the conditions. Assume that $\mathcal{H}$ is not empty and the lemma holds for any set $\mathcal{H}'$ with the number of $\mathcal{F}$-conjugacy classes less that $\mathcal{H}$. Pick a maximal subgroup $P \in \mathcal{H}$, which is fully centralized in $\mathcal{F}$. Let $\mathcal{H}'$ be the subset of $\mathcal{H}$ containing all subgroups not $\mathcal{F}$-conjugate to P. Then $\mathcal{H}'$ satisfies the conditions of the lemma and the number of $\mathcal{F}$-conjugacy classes in $\mathcal{H}'$ is less than in $\mathcal{H}$, so we can apply the induction hypothesis to $\mathcal{H}'$.

Consider a $Q \subseteq S$ and a $g \in Q$. Then for any $\omega \in \Omega_0^Q$ we have that $g^{-1}\omega \in \Omega_0^{g^{-1}Qg}$. Since the map $g^{-1} \colon \Omega_0^Q \to \Omega_0^{g^{-1}Qg}$ is bijective, we have that $|\Omega_0^Q| = |\Omega_0^{g^{-1}Qg}|$ for any $g \in S$. For the S-set S/Q we have that $sQ \in (S/Q)^{Q'}$ for some $Q' \subseteq S$ if and only if $s^{-1}Q's \subseteq Q$. So $|(S/Q)^{Q'}| \neq 0$ if and only if $s^{-1}Q's \subseteq Q$ for a $s \in S$.

If for a $Q \subseteq S$ there exists an $s \in S$, such that $s^{-1}Qs \subseteq P$, then $Q \in \mathcal{H}$. Hence for $Q \notin \mathcal{H}$ we have that $(S/P)^Q = \emptyset$. Consider a P' which is $\mathcal{F}$-conjugate to P. If $s^{-1}P's = P$ for some $s \in S$, then $|\Omega_0^P| = |\Omega_0^{P'}|$ and $|(S/P)^{P'}| = |(S/P)^P| = |N_S(P)/P|$. If P' is not conjugate to P, then $|(S/P)^{P'}| = 0$. So we may add orbits of the form S/P to Ω_0, such that $|\Omega_0^P| \geq |\Omega_0^{P'}|$ for any P' which is $\mathcal{F}$-conjugate to P and this does not change the sets Ω_0^Q for $Q \notin \mathcal{H}$, so the conditions still holds for such a modified Ω_0.

Fix a P' which is $\mathcal{F}$-conjugate to P. Then there exists a $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P', P)$. As $\mathcal{F}$ is saturated, we get by the Sylow condition, that there exists a $\psi \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that

$$\{(\psi\varphi)c_g(\psi\varphi)^{-1} \mid g \in N_S(P')\} \subseteq \mathrm{Aut}_S(P)$$

Then $\psi\varphi \in \mathrm{Hom}_{\mathcal{F}}(P', P)$ where P is fully normalized and hence fully centralized, so there exists an extension $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_{\psi\varphi}, S)$ of $\psi\varphi$. By the above $N_S(P') \subseteq N_{\psi\varphi}$, so we can consider $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(P'), N_S(P))$ with $\bar{\varphi}(P') = P$. For all $P' \subsetneq Q \subseteq N_S(P')$ we have by maximality that $Q \notin \mathcal{H}$. As $\bar{\varphi}(Q)$ and Q are $\mathcal{F}$-conjugate we have by assumption, that $|\Omega_0^Q| = |\Omega_0^{\bar{\varphi}(Q)}|$.

The group $N_S(P')/P'$ acts on the set $\Omega_0^{P'}$. Under this action an $x \in \Omega_0^{P'}$ has isotropy subgroup of the form Q/P' for some $P' \subseteq Q \subseteq N_S(P')$, hence it lies in a non-free orbit exactly when the stabilizer subgroup is of the form Q/P' for some $P' \subsetneq Q \subseteq N_S(P')$. Thus the set of elements in non-free orbits are exactly $\{x \in \Omega_0^Q \mid P' \subsetneq Q \subseteq N_S(P')\}$. Likewise $N_S(P')/P'$ acts on Ω_0^P via $\bar{\varphi}$ and the set of elements in non-free orbits are in this case $\{x \in \Omega_0^{\bar{\varphi}Q} \mid P' \subsetneq Q \subseteq N_S(P')\}$. Hence by the above, $\Omega_0^{P'}$ and Ω_0^P have the same number of elements in non-free orbits, so $|\Omega_0^{P'}|$ and $|\Omega_0^P|$ only differ by a number of free $N_S(P')/P'$-orbits. The free orbits all have $|N_S(P')/P'|$ elements hence

$$|\Omega_0^{P'}| \equiv |\Omega_0^P| \pmod{|N_S(P')/P'|}.$$

Put $n_{P'} = \frac{|\Omega_0^P| - |\Omega_0^{P'}|}{|N_S(P')/P'|}$, and note that $n_{P'} \geq 0$ by assumption and $n_P = 0$.

The set of subgroups which are $\mathcal{F}$-conjugate to P is closed under conjugation with elements from S, so we can pick a set of representatives for the conjugacy classes $\{P = P_1, \dots, P_n\}$. We now consider the S-set

$$\Omega_1 = \Omega_0 \coprod (\coprod_{i=1}^{n} n_{P_i}(S/P_i)).$$

As Ω_1 only differs from Ω_0 by orbits of the form S/P', where $P' \in \mathcal{H}$, we get that $\Omega_1^Q = \Omega_0^Q$ for any $Q \notin \mathcal{H}$. For P' which is $\mathcal{F}$-conjugate to P', we have that P' is S-conjugate to some P_i and

$$|\Omega_1^{P'}| = |\Omega_1^{P_i}| = |\Omega_0^{P_i}| + n_{P_i}|(S/P_i)^{P_i}| = |\Omega_0^{P_i}| + n_{P_i}|N_S(P_i)/P_i| = |\Omega_0^P| = |\Omega_1^P|$$

Then $|\Omega_1^Q| = |\Omega_1^P|$ for any $\mathcal{F}$-conjugate pair of subgroups of S, which are not elements of $\mathcal{H}'$, so by applying the induction hypothesis to this pair, we get a S-set Ω satisfying the conditions of the lemma.

Assume that Ω and Ω' are finite G-sets, where G is a finite group satisfying $|\Omega^P| = |\Omega'^P|$ for any subgroup P of G. Burnside's Lemma implies that Ω and Ω' have the same number of orbits. We will prove that Ω and Ω' are isomorphic G-sets by induction on the number of orbits. We see that $\Omega = \emptyset$ if and only if $|\Omega/G| = 0$, so Ω and Ω' are isomorphic G-set if they have no orbits. Assume that $|\Omega/G| = |\Omega'/G| > 0$, and for any pairs of G-sets $\tilde{\Omega}$ and $\tilde{\Omega}'$ with $|\tilde{\Omega}^P| = |\tilde{\Omega}'^P|$ for all $P \subseteq G$, where $|\tilde{\Omega}/G| < |\Omega/G|$ we have that $\tilde{\Omega}$ and $\tilde{\Omega}'$ are isomorphic. As $\Omega/G \neq \emptyset$, we have that $|\Omega^P|$ where $P \subseteq G$ are not all empty. Pick $x \in |\Omega^P|$, where P is a maximal subgroup of G, such that $|\Omega^P| \neq \emptyset$. Then the isotropy subgroup of x is exactly P. As $|\Omega^P| = |\Omega'^P|$, we can choose $x' \in \Omega'$ with isotropy subgroup P. Then we define a map $F \colon Gx \to Gx'$ by sending $gx \to gx'$. As both elements have the same isotropy subgroup, F is injective and the orbits have the same length namely $|G|/|P|$, so F is a G-isomorphism. Note that $y \in \Omega$ we have that $y \in |\Omega^Q|$ if and only if Q is a subgroup of the isotropy subgroup of y. For any $g \in G$ we have that the isotropy subgroups of gx and gx' agree, so by the above remark we have that the G-sets $\Omega \setminus Gx$ and $\Omega' \setminus Gx'$ satisfy the induction hypothesis. Thus there exists a G-isomorphism between them. By extending this isomorphism using F, we conclude that Ω and Ω' are isomorphic G-sets.

Let $P \subseteq S$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. For any $P' \subseteq P$ the P'-fix-points for Ω considered as a P-set via restriction is $\Omega^{P'}$, while the P'-fix-points for Ω considered as a P-set via α is $\Omega^{\alpha(P')}$. As P' and $\alpha(P')$ are $\mathcal{F}$-conjugate, we have that $|\Omega^{P'}| = |\Omega^{\alpha(P')}|$. By the above result, we have that the two P-sets are isomorphic. $\square$

For $P \subseteq S$ and $\alpha \in \mathrm{Inj}(P, S)$ we define $\Delta_P^\alpha = \{(x, \alpha(x)) \mid x \in P\}$, which is a subgroup of $S \times S$. Hence we may consider the $(S \times S)$-set $(S \times S)/\Delta_P^\alpha$. The $S \times S$-set of this form will be central in the following Proposition.

Proposition 8.8. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. There exists a $(S \times S)$-set Ω where the following holds*

- *Every orbit is of the form $(S \times S)/\Delta_P^\alpha$ for $P \subseteq S$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.*
- *For all $P \subseteq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ we have that Ω as a $(P \times S)$-set via restriction and via $\varphi \times \mathrm{id}_S$ are isomorphic.*
- $|\Omega|/|S| \equiv 1 \pmod{p}$

Furthermore there exists a $F_\Omega \in End(\mathrm{H}^*(BS))$, which is idempotent, $\mathrm{H}^*(\mathcal{F})$-linear and a homomorphism of modules over the Steenrod algebra $\mathcal{A}_p$. Furthermore $\operatorname{im} F_\Omega = \mathrm{H}^*(\mathcal{F})$.

Proof. As $\mathcal{F}$ is saturated we have that S is fully normalized and so $\operatorname{Aut}_S(S) = \operatorname{Inn}(S) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(S))$. Since $\operatorname{Out}_{\mathcal{F}}(S) = \operatorname{Aut}_{\mathcal{F}}(S)/\operatorname{Inn}(S)$, we conclude that $|\operatorname{Out}_{\mathcal{F}}(S)|$ is prime to p. So there exists $k > 0$ such that $k|\operatorname{Out}_{\mathcal{F}}(S)| \equiv 1$ modulo p. Let $\{\alpha_1, \dots, \alpha_{|\operatorname{Out}_{\mathcal{F}}(S)|}\}$ where $\alpha_i \in \operatorname{Aut}_{\mathcal{F}}(S)$ be a set of representatives for $\operatorname{Out}_{\mathcal{F}}(S)$. Consider the $(S \times S)$-set

$$\Omega_0 = k \coprod_{i=1}^{|\operatorname{Out}_{\mathcal{F}}(S)|} (S \times S)/\Delta_S^{\alpha_i}$$

For any $1 \leq i \leq |\operatorname{Out}_{\mathcal{F}}(S)|$ we have that $|\Delta_S^{\alpha_i}| = |S|$, so $|(S \times S)/\Delta_S^{\alpha_i}| = |S|$. Then $|\Omega_0| = k|\operatorname{Out}_{\mathcal{F}}(S)||S|$ and hence $|\Omega_0|/|S| \equiv 1 \pmod p$.

By the definition of orbit we see that

$$((S \times S)/\Delta_S^{\alpha_i})^{\Delta_S^{\alpha_i}} = N_{S \times S}(\Delta_S^{\alpha_i})/\Delta_S^{\alpha_i}.$$

A $(g,h) \in N_{S\times S}(\Delta_S^{\alpha_i})$ if $(gxg^{-1}, h\alpha_i(x)h^{-1}) \in \Delta_S^{\alpha_i}$ for all $x \in S$. This implies that $\alpha_i(g)\alpha_i(x)\alpha_i(g)^{-1} = h\alpha_i(x)h^{-1}$ for all $x \in S$, so $\alpha_i(g) = hz$ for some $z \in \mathrm{Z}(\alpha_i(S)) = \mathrm{Z}(S)$. We conclude that the normalizer $N_{S\times S}(\Delta_S^{\alpha_i})$ has order $|\Delta_S^{\alpha_i}||\mathrm{Z}(S)|$. Then $|((S \times S)/\Delta_S^{\alpha_i})^{\Delta_S^{\alpha_i}}| = |\mathrm{Z}(S)|$. Note that for an $\alpha \in \operatorname{Aut}_{\mathcal{F}}(S)$ we have that Δ_S^{α} is $S\times S$-conjugate to $\Delta_S^{\alpha_i}$ for some i if and only if α and α_i determines the same class in $\operatorname{Out}_{\mathcal{F}}(S)$. So $((S \times S)/\Delta_S^{\alpha_i})^{\Delta_S^{\alpha}} \neq \emptyset$ if and only if α and α_i determine the same class in $\operatorname{Out}_{\mathcal{F}}(S)$ and in this case $|((S \times S)/\Delta_S^{\alpha_i})^{\Delta_S^{\alpha}}| = |\mathrm{Z}(S)|$. As the α_i's are a set of representative for $\operatorname{Out}_{\mathcal{F}}(S)$, we conclude that $|\Omega_0^{\Delta_S^{\alpha}}| = k|\mathrm{Z}(S)|$.

Consider the following set of subgroups of $S \times S$:

$$\mathcal{H} = \{\Delta_P^{\alpha} \mid P \subsetneq S, \alpha \in \operatorname{Hom}_{\mathcal{F}}(P, S)\}.$$

A subgroup of $\Delta_P^{\alpha} \in \mathcal{H}$ is by considering the definition of Δ_P^{α} of the form $\Delta_Q^{\alpha|_Q}$, where Q is a subgroup of P and $\alpha|_Q$ is the restriction of α. Then $\alpha|_Q \in \operatorname{Hom}_{\mathcal{F}}(Q, S)$, hence $\Delta_Q^{\alpha|_Q} \in \mathcal{H}$. As $\Delta_P^{\alpha} \subseteq P \times \alpha(P)$, a morphism $\varphi \in \operatorname{Hom}_{\mathcal{F}\times\mathcal{F}}(\Delta_P^{\alpha}, S \times S)$ is of the form $\varphi = (\varphi_1 \times \varphi_2)|_{\Delta_P^{\alpha}}$, where $\varphi_1 \in \operatorname{Hom}_{\mathcal{F}}(P, S)$ and $\varphi_2 \in \operatorname{Hom}_{\mathcal{F}}(\alpha(P), S)$. Thus

$$\varphi(\Delta_P^{\alpha}) = \{(\varphi_1(x), \varphi_2\alpha(x)) \mid x \in P\} = \Delta_{\varphi_1(P)}^{\varphi_2\alpha\varphi_1^{-1}}.$$

Since φ_1 is injective, we have that $\varphi_1(P)$ is a proper subgroup of S, so $\varphi(\Delta_P^{\alpha}) \in \mathcal{H}$. Hence the set $\mathcal{H}$ is closed under taking subgroup and $\mathcal{F} \times \mathcal{F}$-conjugacy. Consider a subgroup $Q \subseteq S \times S$ which is not an element in $\mathcal{H}$. Assume that $Q = \Delta_S^{\alpha}$ for some $\alpha \in \operatorname{Aut}_{\mathcal{F}}(S)$. An $\mathcal{F} \times \mathcal{F}$-conjugate Q' to Q is then of the form $\Delta_S^{\alpha'}$ for some $\alpha' \in \operatorname{Aut}_{\mathcal{F}}(S)$. By the above we have that $|\Omega_0^Q| = k|\mathrm{Z}(S)| = |\Omega_0^{Q'}|$. Assume that Q is not of the form Δ_S^{α}. Then Q is not $S \times S$-conjugate to a subgroup of $\Delta_S^{\alpha_i}$ for all $1 \leq i \leq |\operatorname{Out}_{\mathcal{F}}(S)|$ and therefore $\Omega_0^Q = \emptyset$. As the same is true for any $\mathcal{F} \times \mathcal{F}$-conjugate of Q, we conclude that $|\Omega_0^Q| = |\Omega_0^{Q'}|$ for any pair of $\mathcal{F} \times \mathcal{F}$-conjugate subgroups of $S \times S$ which are not in $\mathcal{H}$. Since the fusion system $\mathcal{F} \times \mathcal{F}$ is saturated [10, Lemma 1.5], it follows from Lemma 8.7, that there exists an $(S \times S)$-set Ω containing Ω_0 such that $|\Omega^Q| = |\Omega^{Q'}|$ for any pair of $\mathcal{F} \times \mathcal{F}$-conjugate subgroups of $Q, Q' \subseteq S \times S$ and furthermore $\Omega^R = \Omega_0^R$ for all $r \subseteq S \times S$ with $R \notin \mathcal{H}$. Then we get directly that the isotropy subgroups of elements in $\Omega \setminus \Omega_0$ are elements of $\mathcal{H}$. Likewise the lemma implies that for all $P \subseteq S$ and $\alpha \in \operatorname{Hom}_{\mathcal{F}}(P, S)$ we have that Ω as a $(P \times S)$-set via restriction and via $\alpha \times \mathrm{id}_S$ are isomorphic.

Every orbit of Ω is isomorphic to $(S\times S)/\Delta_P^\alpha$ for some $P\subseteq S$ and $\alpha\in\mathrm{Hom}_{\mathcal{F}}(P,S)$. So we have that Ω is isomorphic to $\coprod_{i=1}^N (S\times S)/\Delta_{P_i}^{\alpha_i}$ for some $N>0$, $P_i\subseteq S$ and $\alpha_i\in\mathrm{Hom}_{\mathcal{F}}(P_i,S)$ for $1\le i\le N$. The orbits in $\Omega\setminus\Omega_0$ have this form where furthermore $P\subsetneq S$. As $|\Delta_P^\alpha|=|P|$ they satisfy $|(S\times S)/\Delta_P^\alpha|=|S|[S:P]$, where $p\mid[S:P]$, so we see that $|\Omega|\equiv|\Omega_0|\equiv|S| \pmod{p|S|}$. Then $|\Omega|/|S|\equiv 1 \pmod p$.

For $P\subseteq S$ and $\alpha\in\mathrm{Hom}_{\mathcal{F}}(P,S)$ we define $F_{(P,\alpha)}\in\mathrm{End}(\mathrm{H}^*(BS))$ as the composite

$$\mathrm{H}^*(BS)\xrightarrow{\alpha^*}\mathrm{H}^*(BP)\xrightarrow{\mathrm{trf}_P^S}\mathrm{H}^*(BS)$$

where trf_P^S is the transfer map. By the definition of $\mathrm{H}^*(\mathcal{F})$ we have that for any other $\alpha'\in\mathrm{Hom}_{\mathcal{F}}(P,S)$ we have $\alpha^*=\alpha'^*$ on $\mathrm{H}^*(\mathcal{F})$. In particular this holds for the inclusion $i_P\in\mathrm{Hom}_{\mathcal{F}}(P,S)$ and therefore $\alpha^*(r)=i_P^*(r)=\mathrm{res}_P(r)$ for any $r\in\mathrm{H}^*(\mathcal{F})$. Then for $r\in\mathrm{H}^*(\mathcal{F})$ we have using [16, Proposition 4.2.2.] that

$$F_{(P,\alpha)}(r)=\mathrm{trf}_P^S(\alpha^*(r))=\mathrm{trf}_P^S(\mathrm{res}_P(r))=[S:P]r=\frac{|(S\times S)/\Delta_P^\alpha|}{|S|}r.$$

Now define $F_\Omega=\sum_{i=1}^N F_{(P_i,\alpha_i)}$. As both α^* and the transfer maps are homomorphisms of modules over the Steenrod-algebra, the same is true for F_Ω.

Then for $r\in\mathrm{H}^*(\mathcal{F})$ we have that

$$F_\Omega(r)=\sum_{i=1}^N\frac{|(S\times S)/\Delta_{P_i}^{\alpha_i}|}{|S|}r=\frac{|\Omega|}{|S|}(r)$$

and since we are in characteric p and $|\Omega|/|S|\equiv 1 \pmod p$, we conclude that $F(r)=r$ for all $r\in\mathrm{H}^*(\mathcal{F})$.

Let $r\in\mathrm{H}^*(\mathcal{F})$ and $x\in\mathrm{H}^*(BS)$. Then we have that for any $P\subseteq S$ and $\alpha\in\mathrm{Hom}_{\mathcal{F}}(P,S)$

$$\begin{aligned}F_{(P,\alpha)}(rx)&=\mathrm{trf}_P^S(\alpha^*(rx))=\mathrm{trf}_P^S(\alpha^*(r)\alpha^*(x))=\mathrm{trf}_P^S(\mathrm{res}_P(r)\alpha^*(x))\\&=r\,\mathrm{trf}_P^S(\alpha^*(x))=rF_{(P,\alpha)}(x)\end{aligned}$$

by using the transfer formula [11, V 3.8], since P has finite index in S. As F_Ω is a finite sum of such maps, we conclude that F_Ω is $\mathrm{H}^*(\mathcal{F})$-linear.

Let $P\subseteq S$ and $\alpha\in\mathrm{Hom}_{\mathcal{F}}(P,S)$. We will now prove that $\alpha^*\circ F=\mathrm{res}_P^S\circ F$. Let $1\le i\le N$, and let D_i be a set for the double cosets representatives $S=\cup_{x\in D_i}PxP_i$. Then by [16, Theorem 4.6.2] we have that

$$\mathrm{res}_P^S\circ F_\Omega=\sum_{i=1}^N\mathrm{res}_P^S\circ\mathrm{trf}_{P_i}^S\circ\varphi_i^*=\sum_{i=1}^N\sum_{x\in D_i}\mathrm{trf}_{P\cap P_i^x}^P\circ\mathrm{res}_{P\cap P_i^x}^{P_i^x}\circ c_{x^{-1}}^*\circ\varphi_i^*.$$

Note that we have $D_i\times 1$ is a set of double coset representatives for $S\times S=\cup_{D_i\times 1}(P\times S)(x,1)\Delta_{P_i}^{\alpha_i}$. Consider the $(S\times S)$-set $(S\times S)/\Delta_{P_i}^{\alpha_i}$ as a $(P\times S)$-set via $i_P\times\mathrm{id}_S$. Then the above coset representation implies that $(S\times S)/\Delta_{P_i}^{\alpha_i}$ as a $(P\times S)$-set is the disjoint union over orbits $(P\times S)(x,1)\Delta_{P_i}^{\alpha_i}$, where $x\in D_i$. Thus for any $x\in D_i$ the isotropy subgroup of the element $(x,1)\Delta_{P_i}^{\alpha_i}$ with respect to the $(P\times S)$-action is

$$(P\times S)\cap(\Delta_{P_i}^{\alpha_i})^{(x,1)}=(P\times S)\cap\{(c_x(y),\alpha_i(y))\mid y\in P_i\}=\Delta_{P\cap P_i^x}^{\varphi_i\circ c_{x^{-1}}}.$$

Then $(S \times S)/\Delta_{P_i}^{\alpha_i}$ as a $P \times S$-set is isomorphic to $\coprod_{x \in D_i} (P \times S)/\Delta_{P \cap P_i^x}^{\varphi_i \circ c_{x^{-1}}}$. Comparing with the above formula we conclude that $\mathrm{res}_P^S \circ F_\Omega = F_{\Omega_{i_P, \mathrm{id}_S}}$, where $\Omega_{i_P, \mathrm{id}_S}$ is Ω as a $(P \times S)$-set via $i_P \times \mathrm{id}_S$. We have that $\Omega_{i_P, \mathrm{id}_S}$ and $\Omega_{\alpha, \mathrm{id}_S}$ are isomorphic as $(P \times S)$-sets.

So we want to investigate the $(P \times S)$-set $\Omega_{\alpha, \mathrm{id}_S}$. For $1 \leq i \leq N$ let $\tilde{D}_i$ be a set of coset representatives $S = \cup_{\tilde{x} \in \tilde{D}_i} \alpha(P) \tilde{x} P_i$. Then, as before, $\Omega_{\alpha, \mathrm{id}_S}$ is the disjoint union over orbits $(P \times S)(\tilde{x}, 1)\Delta_{P_i}^{\alpha_i}$, where $1 \leq i \leq N$ and $\tilde{x} \in \tilde{D}_i$. Similarly for any $\tilde{x} \in \tilde{D}_i$ the isotropy subgroup of the element $(\tilde{x}, 1)\Delta_{P_i}^{\alpha_i}$ with respect to the $P \times S$-action is

$$(\alpha^{-1} \times \mathrm{id}_S)((\alpha(P) \times S) \cap (\Delta_{P_i}^{\alpha_i})^{(\tilde{x},1)}) = \Delta_{\alpha^{-1}(\alpha(P) \cap P_i^{\tilde{x}})}^{\varphi_i \circ c_{\tilde{x}^{-1}} \circ \alpha}.$$

Isomorphic $P \times S$-sets have the same orbit representation, so we conclude that

$$\{\Delta_{P \cap P_i^x}^{\varphi_i \circ c_{x^{-1}}} \mid 1 \leq i \leq N, x \in D_i\} = \{\Delta_{\alpha^{-1}(\alpha(P) \cap P_i^{\tilde{x}}))}^{\varphi_i \circ c_{\tilde{x}^{-1}} \circ \alpha} \mid 1 \leq i \leq N, \tilde{x} \in \tilde{D}_i\}$$

Now using that the transfer map is independent of the choice of coset representatives, we conclude that

$$\begin{aligned}
\mathrm{res}_P^S \circ F_\Omega &= \sum_{i=1}^N \sum_{D_i} \mathrm{trf}_{P \cap P_i^x}^P \circ \mathrm{res}_{P \cap P_i^x}^{P_i^x} \circ c_{x^{-1}}^* \circ \varphi_i^* \\
&= \sum_{i=1}^N \sum_{\tilde{D}_i} \mathrm{trf}_{\alpha^{-1}(\alpha(P) \cap P_i^{\tilde{x}})}^P \circ (\alpha|_{\alpha^{-1}(\alpha(P) \cap P_i^{\tilde{x}}), \alpha(P) \cap P_i^{\tilde{x}}})^* \circ \mathrm{res}_{\alpha(P) \cap P_i^{\tilde{x}}}^{P_i^{\tilde{x}}} \circ c_{\tilde{x}^{-1}}^* \circ \varphi_i^* \\
&= \sum_{i=1}^N \sum_{\tilde{D}_i} (\alpha|_{P, \alpha(P)})^* \circ \mathrm{trf}_{\alpha(P) \cap P_i^{\tilde{x}}}^{\alpha(P)} \circ \mathrm{res}_{\alpha(P) \cap P_i^{\tilde{x}}}^{P_i^{\tilde{x}}} \circ c_{\tilde{x}^{-1}}^* \circ \varphi_i^* \\
&= (\alpha|_{P, \alpha(P)})^* \circ \mathrm{res}_{\alpha(P)}^S \circ F_\Omega = \alpha^* \circ F_\Omega
\end{aligned}$$

Then for every $x \in \mathrm{H}^*(BS)$, and $\varphi, \psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ we have that $\varphi^*(F_\Omega(x)) = \mathrm{res}_P^S(F_\Omega(x)) = \psi^*(F_\Omega(x))$. So the image of $F_\Omega \subseteq \mathrm{H}^*(\mathcal{F})$. As F_Ω is the identity on $\mathrm{H}^*(\mathcal{F})$, we conclude that $\mathrm{im}\, F_\Omega = \mathrm{H}^*(\mathcal{F})$ and F_Ω is idempotent. □

8.4. **The quotient fusion system.** For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and a central subgroups $V \subseteq S$ we will now construct a p-local finite group, that in some sense can be considered as the quotient of the original.

To prove that the construction gives in fact a saturated fusion system, the following lemma will we used. There exists equivalent conditions for a fusion system to be saturated, than the one used here, and the lemma is a part of the proof that they are is fact equivalent.

Lemma 8.9. *Let $\mathcal{F}$ be a fusion system on a p-group S satisfying axiom I and the extension axiom for fully normalized subgroups. Then $\mathcal{F}$ is saturated.*

Proof. To prove that $\mathcal{F}$ is saturated let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ be a morphism in $\mathcal{F}$, such that $Q = \mathrm{im}\, \varphi$ is fully centralized in $\mathcal{F}$. We now choose $R \subseteq S$ which is $\mathcal{F}$-conjugate to Q and fully normalized in $\mathcal{F}$. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$. Then $\psi \,\mathrm{Aut}_S(Q) \psi^{-1}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$, and as R is fully normalized in $\mathcal{F}$ there is by axiom I a $\chi \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\chi\psi \,\mathrm{Aut}_S(Q) \psi^{-1} \chi^{-1} \subseteq \mathrm{Aut}_S(R)$. This implies that $N_{\chi\psi} = N_S(Q)$. By the extension axiom for fully normalized subgroups we conclude that $\chi\psi$ extents to a morphism $\sigma \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(R))$. Similarly we have that

$N_\varphi \subseteq N_{\chi\psi\varphi}$, so $\chi\psi\varphi$ extends to a morphism $\tau \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, N_S(R))$. Consider $x \in N_\varphi$. Then $\varphi c_x \varphi^{-1} = c_y$ for some $y \in N_S(Q)$ and

$$c_{\tau(x)} = \chi\psi\varphi c_x(\chi\psi\varphi)^{-1} = \chi\psi c_y(\chi\psi)^{-1} = c_{\sigma(y)}.$$

Thus $\tau(x) = \sigma(y)z$ for some $z \in C_S(R)$. Since Q is fully centralized, we conclude that $\sigma(C_S(Q)) = C_S(R)$, and in particular $z = \sigma(v)$ for a $v \in C_S(Q) \subseteq N_S(Q)$. Then $\tau(x) \in \sigma(N_S(Q))$, so $\tau(N_\varphi) \subseteq \sigma(N_S(Q))$. Thus $\sigma^{-1}\tau \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, N_S(Q))$ and is an extension of φ. ☐

Lemma 8.10. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $V \subseteq S$ a subgroup of order p, such that $\mathcal{F} = C_{\mathcal{F}}(V)$. The induced fusion system $\mathcal{F}/V$ on S/V has as morphism set $\mathrm{Hom}_{\mathcal{F}/V}(P/V, Q/V) = \{\varphi/V \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)\}$. Let $\mathcal{L}/V$ be the category with objects $P/V \subseteq S/V$, where P is $\mathcal{F}$-centric and*

$$\mathrm{Mor}_{\mathcal{L}/V}(P/V, Q/V) = \mathrm{Mor}_{\mathcal{L}}(P,Q)/\delta_P(V).$$

Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the full subcategory of $\mathcal{L}$, on objects $P \subseteq S$, such that P/V is $(\mathcal{F}/V)$-centric, and $(\mathcal{L}/V)^c \subseteq \mathcal{L}/V$ the full subcategory of $\mathcal{L}/V$, on objects $P/V \subseteq S/V$, such that P/V is $(\mathcal{F}/V)$-centric. Then the following holds:

(a) *The fusion system $\mathcal{F}/V$ is saturated and $(\mathcal{L}/V)^c$ is a central linking system associated to $\mathcal{F}/V$.*
(b) *The sequence $BV \to |\mathcal{L}_0|^\wedge_p \to |(\mathcal{L}/V)^c|^\wedge_p$ is a fibration sequence.*
(c) *The inclusion $|\mathcal{L}_0|^\wedge_p \subseteq |\mathcal{L}|^\wedge_p$ is a homotopy equivalence.*
(d) *If $R_{(\mathcal{L}/V)^c}$ is an isomorphism, then $R_{\mathcal{L}}$ is an isomorphism.*

Proof. Let $V \subseteq S$ be a subgroup of order p such that $\mathcal{F} = C_{\mathcal{F}}(V)$. Since $C_{\mathcal{F}}(V)$ is a fusion system on $C_S(V)$ we have that V is normal in S. We note that S/V is then a p-group. Furthermore we have that any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ can be extended to a $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PV, QV)$ which is the identity on V. So for $P, Q \subseteq S$ containing V and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, the map $(\varphi/V)xV = \varphi(x)V$ is well-defined. It then follows easily that $\mathcal{F}/V$ is a fusion system over S/V.

As $c_g \in \mathrm{Aut}_{\mathcal{F}}(S)$ for any $g \in S$ this implies that $V \subseteq \mathrm{Z}(S)$. For a $\mathcal{F}$-centric subgroup P, we have that

$$V \subseteq \mathrm{Z}(S) \subseteq C_S(P) \subseteq P,$$

so by the above any morphism in $\mathcal{F}^c$ is the identity on V. Then for $P, Q \in \mathcal{L}$ and $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, we have that $\pi(f)|_V = \mathrm{id}_V$. Hence for any $v \in V$ we get that the following diagram is commutative by the property (C) for $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(v)"'] & Q \arrow[d, "\delta_Q(v)"'] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$

Then $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, $g \in \mathrm{Mor}_{\mathcal{L}}(Q,R)$ and $v, v' \in V$ will satisfy $g\delta_Q(v')f\delta_P(v) = gf\delta_P(v'v)$. So the composition of morphisms in $\mathcal{L}/V$ is well-defined, and it follows easily that it is a category.

For any $V \subseteq P \subseteq S$ we have a map $\mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}/V}(P/V)$ by $\varphi \mapsto \varphi/V$. Let Γ_P denote the kernel of this map. Let $N^0_S(P) = \{g \in N_S(P) \mid c_g \in \Gamma_P\}$. Then for any $g \in N^0_S(P)$ we have that $\mathrm{id}_{P/V} = c_g/V = c_{gV}$, so $N^0_S(P)/V \subseteq C_{S/V}(P/V)$.

Similarly any $gV \in C_{S/V}(P/V)$ will satisfy that $gPg^{-1} \subseteq PV = P$ and $c_g/V = \mathrm{id}_{P/V}$, so $C_{S/V}(P/V) \subseteq N_S^0(P)/V$ and thus $C_{S/V}(P/V) = N_S^0(P)/V$.

Consider a pair of subgroups $P, Q \subseteq S$ containing V, such that P/V is fully centralized in $\mathcal{F}/V$ and Q is fully normalized in $\mathcal{F}$. Since Γ_Q is a kernel, it is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Let $\varphi \in \Gamma_Q$ and $q \in Q$. Then $\varphi(q) = qv$ for some $v \in V$. Since $\varphi(v) = v$ and $|V| = p$ we have that $\varphi^p(q) = qv^p = q$. Thus Γ_Q is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and hence contained in every Sylow-p-subgroup. As Q is fully normalized and $\mathcal{F}$ is saturated, it follows from the axioms for saturation that $\Gamma_Q \subseteq \mathrm{Aut}_S(Q)$. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ be an isomorphism. As Q is fully normalized and hence fully centralized, we have by the extension axiom for saturated fusion systems that φ extends to a $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$. Let $g \in N_S^0(P)$. Then $g \in N_S(P)$ and $c_g/V = \mathrm{id}_{P/V}$, so

$$(\varphi \circ c_g \circ \varphi^{-1})/V = \varphi/V \circ c_g/V \circ (\varphi/V)^{-1} = \varphi/V \circ (\varphi/V)^{-1} = \mathrm{id}_{Q/V}$$

Hence $\varphi \circ c_g \circ \varphi^{-1} \in \Gamma_Q \subseteq \mathrm{Aut}_S(Q)$ and we conclude that $N_S^0(P) \subseteq N_\varphi$. Note that for any $g \in N_S^0(P)$ we have $\bar{\varphi}(g) \in N_S(\bar{\varphi}(P)) = N_S(Q)$. Thus for any $q \in Q$ we see that

$$c_{\bar{\varphi}(g)}(q) = \bar{\varphi}(g) q \bar{\varphi}(g)^{-1} = (\varphi \circ c_g \circ \varphi^{-1})(q)$$

and so $c_{\bar{\varphi}(g)}/V = \mathrm{id}_{Q/V}$. Hence $\bar{\varphi}(N_S^0(P)) \subseteq N_S^0(Q)$, and by this we can consider the restriction $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S^0(P), N_S^0(Q))$. By the previous remark $\bar{\varphi}/V \in \mathrm{Hom}_{\mathcal{F}/V}(C_{S/V}(P/V), C_{S/V}(Q/V))$. Since P/V is fully centralized in $\mathcal{F}/V$ we have that $|C_{S/V}(P/V)| \geq |C_{S/V}(Q/V)|$. As the map $\bar{\varphi}/V$ is injective, we conclude that it is in fact an isomorphism and $|C_{S/V}(P/V)| = |C_{S/V}(Q/V)|$. In particular we get that Q/V is fully centralized in $\mathcal{F}/V$. This implies that $|N_S^0(P)| = |N_S^0(Q)|$, forcing the injective map $\bar{\varphi}$ to be an isomorphism as well. Since $\Gamma_Q \subseteq \mathrm{Aut}_S(Q)$, we see that $\Gamma_Q = \{c_g \mid g \in N_S^0(Q)\}$ so

$$\begin{aligned}\Gamma_P = \varphi^{-1}\Gamma_Q\varphi = \{\varphi^{-1}c_g\varphi \mid g \in N_S^0(Q)\} &= \{c_{\bar{\varphi}^{-1}(g)} \mid g \in N_S^0(Q)\} \\ &= \{c_g \mid g \in N_S^0(P)\} \subseteq \mathrm{Aut}_S(P)\end{aligned}$$

As fully centralized and fully normalized are maximality conditions on the $\mathcal{F}$-conjugacy classes, such elements always exist, so we have proven the following implications:

$$Q \text{ fully normalized in } \mathcal{F} \Longrightarrow Q/V \text{ fully centralized in } \mathcal{F}/V \Longrightarrow \Gamma_Q \subseteq \mathrm{Aut}_S(Q)$$

We will now prove that $\mathcal{F}/V$ is saturated. Let $V \subseteq P \subseteq S$ be a subgroup, such that P/V is fully normalized in $\mathcal{F}/V$. We have that $N_{S/V}(P/V) = N_S(P)/V$. Since V is central in $\mathcal{F}$, any $\mathcal{F}$-conjugate to P also contains V. This implies that P is fully normalized in $\mathcal{F}$. By the proven implications we conclude that P/V is fully centralized in $\mathcal{F}/V$ and $\Gamma_P \subseteq \mathrm{Aut}_S(P)$. We have that $\mathrm{Aut}_{\mathcal{F}/V}(P/V) \cong \mathrm{Aut}_{\mathcal{F}}(P)/\Gamma_P$ and this restricts to $\mathrm{Aut}_{S/V}(P/V) \cong \mathrm{Aut}_S(P)/\Gamma_P$. As $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, the isomorphisms implies that $\mathrm{Aut}_{S/V}(P/V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}/V}(P/V))$. By Lemma 8.9 it is sufficient to prove the extension axiom for fully normalized subgroups of S/V. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}/V}(P/V, S/V)$ be a morphism in $\mathcal{F}/V$, such that $\mathrm{im}\,\varphi$ is fully normalized in $\mathcal{F}/V$. We choose $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\tilde{\varphi}/V = \varphi$. Set $Q = \mathrm{im}\,\tilde{\varphi}$. Then $\mathrm{im}\,\varphi = Q/V$ is fully normalized in $\mathcal{F}/V$, so by the above we conclude that Q is fully normalized in $\mathcal{F}$. By axiom I for $\mathcal{F}$, we get that Q is fully centralized in $\mathcal{F}$, so $\tilde{\varphi}$ extends to a $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_{\tilde{\varphi}}, S)$. Hence $\bar{\varphi}/V \in \mathrm{Hom}_{\mathcal{F}/V}(N_{\tilde{\varphi}}/V, S/V)$ is an extension of φ. Consider $xV \in N_\varphi$. Then $x \in N_S(P)$, and it satisfies

$$(\tilde{\varphi} c_x \tilde{\varphi}^{-1})/V = \varphi c_{xV} \varphi^{-1} \in \operatorname{Aut}_{S/V}(P/V) \cong \operatorname{Aut}_S(P)/\Gamma_P,$$

thus $\tilde{\varphi} c_x \tilde{\varphi}^{-1} \in \operatorname{Aut}_S(P)$ and hence $x \in N_{\tilde{\varphi}}$. So $N_\varphi \subseteq N_{\tilde{\varphi}}/V$ and therefore $\bar{\varphi}/V \in \operatorname{Hom}_{\mathcal{F}/V}(N_\varphi, S/V)$ is an extension of φ on the proper target. Hence the extension axiom holds for fully normalized subgroups and we conclude that $\mathcal{F}/V$ is saturated.

We now turn to the central linking system. Let $\pi/V \colon (\mathcal{L}/V)^c \to \mathcal{F}/V$ be the functor induced by $\pi \colon \mathcal{L} \to \mathcal{F}$, hence $(\pi/V)(P/V) = P/V$ and $(\pi/V)(\varphi/\delta_P(V)) = \pi(\varphi)/V$. For any $v \in V$ we have that $\pi(\delta_P(v)) = c_v|_P = \mathrm{id}_P$, so the functor π/V is well-defined on morphisms. Similarly for $P/V \in (\mathcal{L}/V)^c$ we define $\delta_{P/V} \colon P/V \to \operatorname{Aut}_{(\mathcal{L}/V)^c}(P/V)$ by $\delta_{P/V}(xV) = \delta_P(x)/\delta_P(V)$. Then $\delta_{P/V}$ is a well-defined monomorphism. By definition we have that π/V is the identity on objects and surjective on morphisms, since the same is true for π. We need to prove that π/V is in fact the orbit map for the free action of $\mathrm{Z}(P/V)$ on $\operatorname{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)$ via $\delta_{P/V}(\mathrm{Z}(P/V))$. For this let $P/V, Q/V \in (\mathcal{L}/V)^c$ and consider the diagram

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{L}}(P,Q) \arrow[r, "/\delta_P(V)"] \arrow[d, "\pi = /\delta_P(\mathrm{Z}(P))"'] & \operatorname{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V) \arrow[d, "\pi/V"'] \\ \operatorname{Mor}_{\mathcal{F}}(P,Q) \arrow[r, "/\Gamma_P"] & \operatorname{Mor}_{(\mathcal{F}/V)^c}(P/V, Q/V) \end{tikzcd}$$

where we indicate that three of the maps are in fact orbit map with respect to the indicated groups. The diagram commutes, as both sides represent the map $f \mapsto \pi(f)/V$. Now consider $\varphi/\delta_P(V), \psi/\delta_P(V) \in \operatorname{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)$ such that $(\pi/V)(\varphi/\delta_P(V)) = (\pi/V)(\psi/\delta_P(V))$, i.e. $\pi(\varphi)/V = \pi(\psi)/V$. As the lower horizontal map is the orbit map by Γ_P there exists an $\alpha \in \Gamma_P$ such that $\pi(\varphi) = \pi(\psi)\alpha$. Since P/V is $\mathcal{F}/V$-centric, i.e $C_{S/V}(P/V) \subseteq P/V$, the same holds for any $\mathcal{F}/V$-conjugate to P/V, thus the centralizer for P/V is isomorphic to the centralizer for any $\mathcal{F}/V$-conjugate and hence P/V is fully centralized. By a previous implication we have that $\Gamma_P \subseteq \operatorname{Aut}_S(P)$, so $\alpha = c_g$ for some $g \in S$ satisfying $\mathrm{id}_P = c_g/V = c_{gV}$. Hence $gV \in \mathrm{Z}(P/V)$ and thus $g \in P$. Now $\pi(\varphi) = \pi(\psi)c_g = \pi(\psi) \circ \pi(\delta_P(g)) = \pi(\psi \circ \delta_P(g))$. As π is the orbit map w.r.t. $\mathrm{Z}(P)$ we conclude that there is a $g' \in \mathrm{Z}(P)$ such that $\varphi = \psi \circ \delta_P(gg')$. Hence $\varphi/\delta_P(V) = \psi/\delta_P(V) \circ \delta_{P/V}(gg'V)$. Using the fact that $g' \in \mathrm{Z}(P)$ and $V \subseteq \mathrm{Z}(P)$ we conclude that $g'V \in \mathrm{Z}(P/V)$ and therefore $gg'V \in \mathrm{Z}(P/V)$. Hence π/V is the orbit map for the action of $\delta_{P/V}\,\mathrm{Z}(P/V)$ on $\operatorname{Mor}_{\mathcal{L}/V}(P/V, Q/V)$.

We will now prove that this action is indeed free. For this let $\varphi/\delta_P(V) \in \operatorname{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)$ and $gV \in \mathrm{Z}(P/V)$ satisfying $\varphi/\delta_P(V) = \varphi/\delta_P(V)\delta_{P/V}(gV)$. We then need to prove that $g \in V$. By the definition of orbit maps there exists a $v \in V$, such that $\varphi = \varphi\delta_P(gv)$. Then by applying π we conclude that $\pi(\varphi) = \pi(\varphi)c_{gv}$ in $\operatorname{Mor}_{\mathcal{F}}(P,Q)$. Since $\pi(\varphi)$ is injective we deduce that $c_{gv} = \mathrm{id}_P$, so $gv \in \mathrm{Z}(P)$. As the action on $\mathrm{Z}(P)$ on $\operatorname{Mor}_{\mathcal{L}}(P,Q)$ via δ_P is free, we conclude from $\varphi = \varphi\delta_P(gv)$ that $gv = 1$. Hence $g = v^{-1} \in V$, so the action of $\mathrm{Z}(P/V)$ on $\operatorname{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)$ via $\delta_{P/V}$ is free. The category $(\mathcal{L}/V)^c$ then satisfies axiom A for central linking systems.

Consider $xV \in P/V$ where $P/V \in (\mathcal{L}/V)^c$. Then we have by axiom (B) for $\mathcal{L}$ that

$$(\pi/V)(\delta_{P/V}(xV)) = (\pi/V)(\delta_P(x)/\delta_P(V)) = \pi(\delta_P(x))/V = c_x/V = c_{xV}$$

so axiom B holds for $(\mathcal{L}/V)^c$ as well. Consider $xV \in P/V$ and $\bar{f} = f/\delta_P(V) \in \mathrm{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)$. Then by axiom C for $\mathcal{F}$ we get that

$$\begin{aligned}\bar{f} \circ \delta_{Q/V}(\pi/V(\bar{f})(xV)) &= (f \circ \delta_Q(\pi(f)(x)))/\delta_p(V) = (f \circ \delta_P(x))/\delta_p(V)\\ &= \bar{f} \circ \delta_{P/V}(xV)\end{aligned}$$

so axiom C is also true in $(\mathcal{L}/V)^c$. Hence $(\mathcal{L}/V)^c$ is a central linking system for $\mathcal{F}/V$.

To prove part (b) we consider the functor $F\colon \mathcal{L}_0 \to (\mathcal{L}/V)^c$ given by $F(P) = P/V$ for $P \in \mathcal{L}_0$ and $F(\varphi) = \varphi/\delta_P(V)$ for $\varphi \in \mathrm{Mor}_{\mathcal{L}_0}(P, Q)$, and remark that F is well-defined due to the definition of $\mathcal{L}_0$. Let $P/V \in (\mathcal{L}/V)^c$ and consider the undercategory $P/V \downarrow F$. The objects are $\{(\tau, Q) \mid Q \in \mathcal{L}_0, \tau \in \mathrm{Mor}_{(\mathcal{L}/V)^c}(P/V, Q/V)\}$ and the morphisms from (τ_0, Q_0) to (τ_1, Q_1) is the set

$$\{\varphi \in \mathrm{Mor}_{\mathcal{L}_0}(Q_0, Q_1) \mid \varphi/\delta_{Q_0}(V) \circ \tau_0 = \tau_1\}.$$

Let $\mathcal{B}'(V)$ be the subcategory of $P/V \downarrow F$ with a unique object $(1_{P/V}, P)$ and morphisms $\delta_P(V)$. Since δ_P is injective, we get that $\mathcal{B}'(V)$ is equivalent to the category $\mathcal{B}(V)$. We will now construct a functor $G\colon P/V \downarrow F \to \mathcal{B}'(V)$. For this purpose we choose for any pair $Q_0, Q_1 \in \mathcal{L}_0$ a section $\sigma\colon \mathrm{Mor}_{\mathcal{L}/V}(Q_0/V, Q_1/V) \to \mathrm{Mor}_{\mathcal{L}}(Q_0, Q_1)$, satisfying that $\sigma(1_{Q/V}) = 1_Q$ for any $Q \in \mathcal{L}_0$. Let (τ_0, Q_0) and (τ_1, Q_1) be objects of $P/V \downarrow F$ and $\varphi\colon (\tau_0, Q_0) \to (\tau_1, Q_1)$. Then $\varphi \circ \sigma(\tau_0)$ and $\sigma(\tau_1)$ have the same image under F, so there exists $v \in V$ such that $\varphi \circ \sigma(\tau_0) = \sigma(\tau_1) \circ \delta_P(v)$. Assume that $v' \in V$ satisfies the same property. As the morphisms in $\mathcal{L}$ are monomorphisms in the categorical sense Lemma 3.1, this implies that $\delta_P(v) = \delta_P(v')$. Since δ_P is injective, we conclude that $v = v'$. So the particular v is unique and we denote it v_φ. Hence we can define G by $G(\tau, Q) = (1_{P/V}, P)$ and $G(\varphi) = \delta_P(v_\varphi)$. Let $(\tau, Q) \in P/V \downarrow F$. Then $1_Q\colon (\tau, Q) \to (\tau, Q)$ and it follows from the definition of σ that $\delta_P(v_{1_Q}) = 1_Q$. To see that G is in fact associative, let $\varphi_1\colon (\tau_0, Q_0) \to (\tau_1, Q_1)$ and $\varphi_2\colon (\tau_1, Q_1) \to (\tau_2, Q_2)$ and consider the diagram:

$$\begin{tikzcd}
P \arrow[r, "\sigma(\tau_0)"] \arrow[d, "1_P"'] & Q_0 \arrow[d, "\varphi_1"] \\
P \arrow[r, "\sigma(\tau_1) \circ \delta_P(v_{\varphi_1})"] \arrow[d, "1_P"'] & Q_1 \arrow[d, "\varphi_2"] \\
P \arrow[r, "\sigma(\tau_2) \circ \delta_P(v_{\varphi_2}) \circ \delta_P(v_{\varphi_1}) = \sigma(\tau_2) \circ \delta_P(v_{\varphi_2\varphi_1})"] & Q_2
\end{tikzcd}$$

As the morphisms in $\mathcal{L}$ are monomorphisms, we conclude that $\delta_P(v_{\varphi_2}) \circ \delta_P(v_{\varphi_1}) = \delta_P(v_{\varphi_2\varphi_1})$, and thus $G(\varphi_1) \circ G(\varphi_2) = G(\varphi_2 \circ \varphi_1)$. Hence G is a well-defined functor from $P/V \downarrow F \to \mathcal{B}'(V)$. As $\sigma(1_{P/V}) = 1_P$ is follows directly from the definition of G that G is the identity on the subcategory $\mathcal{B}'(V)$, hence $F \circ \mathrm{incl} = \mathrm{id}_{\mathcal{B}'(V)}$. For any $(\tau, Q) \in P/V \downarrow F$ we have that $\sigma(\tau)\colon (1_{P/V}, P) \to (\tau, Q)$, and by the definition of

G on morphisms this will induce a natural transformation from $\mathrm{incl} \circ F$ to $\mathrm{id}_{P/V\downarrow F}$. Hence $|F|$ is a deformation retract of $|P/V \downarrow F|$ to $|\mathcal{B}'(V)| = BV$.

Now consider $\beta \in \mathrm{Mor}_{(\mathcal{L}/V)^c}(P/V, P'/V)$. It induces a functor $\beta \downarrow F\colon P'/V \downarrow F \to P/V \downarrow F$ by $(\tau, Q) \mapsto (\tau \circ \beta, Q)$ and being the identity on morphisms. We will prove that $|\beta \downarrow F|$ is a weak homotopy equivalence. For this we consider the following composition of functors:

$$\mathcal{B}'(V) \xrightarrow{\mathrm{incl}} P'/V \downarrow F \xrightarrow{\beta \downarrow F} P/V \downarrow F \xrightarrow{G} \mathcal{B}'(V)$$

It sends $(1_{P'/V}, P')$ to $(1_{P/V}, P)$. To see what it does on morphisms, we have to identify $G(\delta_{P'}(v))$, where $\delta_{P'}(v)$ is an automorphism of (β, P'). By property C for $\mathcal{L}$ and the fact, that V is central in $\mathcal{F}$ we have that

$$\sigma(\beta)\delta_P(v) = \delta_{P'}(\pi(\sigma(\beta))(v))\sigma(\beta) = \delta_{P'}(v)\sigma(\beta).$$

Hence $G(\delta_{P'}(v)) = \delta_P(v)$. So the above composition of functors is a homotopy equivalence, and in particular a weak homotopy equivalence. As both incl and G are weak homotopy equivalences as well, we conclude that the same holds for $\beta \downarrow F$. By [30, Quillen's Theorem B] we conclude that BV is the homotopy fiber of the map $|F|\colon |\mathcal{L}_0| \to |(\mathcal{L}/V)^c|$. By Proposition 3.3 we have that $\pi_1(|(\mathcal{L}/V)^c|)$ is a quotient group of S/V and thus a finite p-group. As V is cyclic of order p we have that $\mathrm{H}_i(BV; \mathbb{F}_p)$ is a finite p-group for any i. Then by [7, Example II 5.2] $\pi_1(|(\mathcal{L}/V)^c|)$ acts nilpotently on $\mathrm{H}_i(BV; \mathbb{F}_p)$ for any i. The fibration satisfies the conditions of the mod-R fiber lemma [7, II 5.1], so we conclude that $BV^\wedge_p \to |\mathcal{L}_0|^\wedge_p \to |(\mathcal{L}/V)^c)|^\wedge_p$ is a fibration. Since BV is p-complete by [3, III 1.4 Proposition 1.10], we have that $BV^\wedge_p$ and BV are homotopy equivalent, so we may replace $BV^\wedge_p$ by BV in the given fibration.

If $\mathcal{L} = \mathcal{L}_0$ part (c) is trivial, so we assume that $\mathcal{L} \neq \mathcal{L}_0$ and pick a $P \subseteq S$ such that $P \in \mathcal{L} \setminus \mathcal{L}_0$. Hence P is $\mathcal{F}$-centric but P/V is not $\mathcal{F}/V$-centric. By definition of centric subgroup this implies that there exists a $P' \subseteq S$ which is $\mathcal{F}$-conjugate to P, and $C_{S/V}(P'/V) \neq \mathrm{Z}(P'/V)$. Let $gV \in C_{S/V}(P'/V) \setminus \mathrm{Z}(P'/V)$. Then $g \in S \setminus P'$ and satisfies $[g, P'] \subseteq V$. Consider $c_g \in \mathrm{Aut}_{\mathcal{F}}(P')$. If $c_g = c_x$ for some $x \in P'$, we would have that $gx^{-1} \in C_S(P') \setminus P'$. As P' is $\mathcal{F}$-centric, this set is empty, so a contradiction arises. Hence we conclude that c_g is not an inner automorphism of P'. So the class of c_g in $\mathrm{Out}_{\mathcal{F}}(P')$ is non-trivial and as $gV \in C_{S/V}(P'/V)$ the class lies in the kernel K for the map $\mathrm{Out}_{\mathcal{F}}(P') \to \mathrm{Out}_{\mathcal{F}/V}(P'/V)$. Since S is a p-group the class of c_g in $\mathrm{Out}_{\mathcal{F}}(P')$ has p-power order. So the elements in $\mathrm{Z}(K)$ of p-order is a nontrivial p-group, which is characteristic in K. As K is normal in $\mathrm{Out}_{\mathcal{F}}(P')$ we conclude that this is a nontrivial normal p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P')$. Hence P' and likewise P is not $\mathcal{F}$-radical. Thus $\mathcal{L}_0$ contains every $\mathcal{F}$-centric subgroup, which is also $\mathcal{F}$-radical. By Corollary 6.15 this implies that the inclusion $|\mathcal{L}_0| \hookrightarrow |\mathcal{L}|$ is a mod p-homotopy equivalence. By [7, Lemma I.5.5] this implies that the p-completion $|\mathcal{L}_0|^\wedge_p \hookrightarrow |\mathcal{L}|^\wedge_p$ is a homotopy equivalence.

For part (d) we choose a $S \times S$-set Ω with the properties from Proposition 8.8. Then V acts on Ω by $v\omega = (v, v)\omega$. The fix-point set for any $\omega \in \Omega$ is of the form Δ_P^α for some $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. As V is a group of order p, either $V \subseteq P$ or $P \cap V = 1$. Since $\varphi|_V = \mathrm{id}_V$ for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ with $V \subseteq P$, we conclude that the fix-point set any $\omega \in \Omega$ under the action of V is either V, in the case $V \subseteq P$, or 1 in the case $V \nsubseteq P$. Hence Ω^V consists exactly of the orbits from Ω of the form $(S \times S)/\Delta_P^\alpha$ where $V \subseteq P$ and is therefore also a $(S \times S)$-set. Then the first property from Proposition 8.8 clearly holds, while the second is a consequence of the fact, that an isomorphism of sets with a group action induces a bijection on the fix-point sets. The difference between Ω and Ω^V is orbits of the form $(S \times S)/\Delta_P^\alpha$, where $V \nsubseteq P$. They have $|S|[S : P]$ elements, and as $P \nsubseteq S$, we have that this is a multiple of $p|S|$. Hence $|\Omega^V|/|S| \equiv |\Omega|/|S| \equiv 1 \pmod p$. So the properties from Proposition 8.8 also hold for Ω^V, hence we replace Ω by Ω^V in the following.

We now set $\bar{\Omega} = \Omega/(V \times V)$, which may be seen as $(S/V \times S/V)$-set in natural way. Since the orbits of Ω are of the form $(S \times S)/\Delta_P^\alpha$ where $V \subseteq P \subseteq S$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, we see that the orbits of $\bar{\Omega}$ are of the form $(S/V \times S/V)/\Delta_{P/V}^{\alpha/V}$. The isomorphism between Ω as a $(P \times S)$-set via inclusion and $\alpha \times \mathrm{id}_S$ for any $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ respects the action by $V \times V$, since $\alpha|_V = \mathrm{id}_V$. Thus it will induce a well-defined bijection on $\bar{\Omega}$, which is in fact a isomorphism between $\bar{\Omega}$ as a $(P/V \times S/V)$-set via inclusion and $\alpha/V \times \mathrm{id}_{S/V}$ for any $\alpha/V \in \mathrm{Hom}_{\mathcal{F}/V}(P/V, S/V)$. The orbit $(S/V \times S/V)/\Delta_{P/V}^{\alpha/V}$ consists of $|S/V|[S/V : P/V] = |S|[S : P]/|V|$ elements, which multiplied by $|V|$ is the number of elements in $(S \times S)/\Delta_P^\alpha$. Hence $|\bar{\Omega}| = |\Omega||V|$, and so $|\bar{\Omega}|/|S/V| \equiv |\Omega|/|S| \equiv 1 \pmod p$. Hence the $(S/V \times S/V)$-set $\bar{\Omega}$ satisfies the properties from Proposition 8.8 and has the same number of orbits as Ω.

Let M_* be a free $\mathbb{Z}[S]$-resolution of $\mathbb{Z}$, and M'_* be a free $\mathbb{Z}[S/V]$-resolution of $\mathbb{Z}$. Choose a subgroup $P \subseteq S$ containing V. A set of coset representatives for V will constitute a basis for $\mathbb{Z}[S]$ as a module over $\mathbb{Z}[V]$, in particular it is a free $\mathbb{Z}[V]$-module. Then M_* will also be a free $\mathbb{Z}[V]$-resolution for $\mathbb{Z}$ and thus $\mathrm{H}^n(V; \mathbb{F}_p) = \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}[V]}(M_*, \mathbb{F}_p))$ for all n. Similarly let M'_* be a free $\mathbb{Z}[S/V]$-resolution for $\mathbb{Z}$, and it can be used to compute the cohomology of P/V in the same way. The Lyndon-Hochschild-Serre spectral sequence for the extension

$$1 \to V \to P \to P/V \to 1$$

converges to $\mathrm{H}^*(P; \mathbb{F}_p)$ and has the following E_2-page:

$$E_2^{p,q}(P) = \mathrm{H}^p(P/V, \mathrm{H}^q(V, \mathbb{F}_p)) = \mathrm{H}^p(\mathrm{Hom}_{\mathbb{Z}[P/V]}(M'_*, \mathrm{H}^q(\mathrm{Hom}_{\mathbb{Z}[V]}(M_*, \mathbb{F}_p)))).$$

The hom-tensor adjuction $\mathrm{Hom}(M', \mathrm{Hom}(M, \mathbb{F}_p))) \cong \mathrm{Hom}(M' \otimes M, \mathbb{F}_p)$ in connection with the fact that $\mathrm{Hom}_{\mathbb{Z}(Q)}(M, \mathbb{F}_p) = \mathrm{Hom}(M, \mathbb{F}_p)^Q$ for any group Q and $\mathbb{Z}[Q]$-module M, implies that for all i, j

$$\mathrm{Hom}_{\mathbb{Z}[P/V]}(M'_i, \mathrm{Hom}_{\mathbb{Z}[V]}(M_j, \mathbb{F}_p))) \cong \mathrm{Hom}_{\mathbb{Z}[P]}(M'_i \otimes M_j, \mathbb{F}_p).$$

The adjuction is natural in all entries so the given isomorphism commutes with the differentials in M_* and M'_*, and thus is a isomorphism of double chain complexes. Hence the spectral sequence is induced by the double complex $\mathrm{Hom}_{\mathbb{Z}[P]}(M'_* \otimes M_*, \mathbb{F}_p)$. The transfer map $\mathrm{Hom}_{\mathbb{Z}[P]}(M'_i \otimes M_j, \mathbb{F}_p) \to \mathrm{Hom}_{\mathbb{Z}[S]}(M'_i \otimes M_j, \mathbb{F}_p)$ given by $f \mapsto \sum g_i f g_i^{-1}$, where g_i's is set of coset representative for P, is a well-defined homomorphism. Since the differentials of M_* and M'_* are $\mathbb{Z}[S]$ and $\mathbb{Z}[S/V]$ morphisms receptively, we see that they commute with the transfer map. Hence the transfer map induces a homomorphism of double chain complexes $\mathrm{Hom}_{\mathbb{Z}[P]}(M'_* \otimes M_*, \mathbb{F}_p) \to \mathrm{Hom}_{\mathbb{Z}[S]}(M'_* \otimes M_*, \mathbb{F}_p)$, and therefore a homomorphism of spectral sequences $\mathrm{trf}\colon E_*(P) \to E_*(S)$. Since $V = \mathbb{Z}/p$ we remark that $\mathrm{H}^q(V; \mathbb{F}_p)$ is either $\mathbb{F}_p$ or 1.So a row of $E_2(P)$ is either $\mathrm{H}^*(P/V)$ or trivial. If it is $\mathrm{H}^*(P/V)$ the corresponding row of $E_2(S)$ is $\mathrm{H}^*(S/V)$ and by definition of the transfer map for cohomology we see that transfer map of the spectral sequences on this particular row is in fact the transfer map $\mathrm{H}^*(P/V) \to \mathrm{H}^*(S/V)$ from group cohomology, since g_iV will be a set of coset representatives for P/V. Similarly $\mathrm{trf}\colon E_\infty(P) \to E_\infty(S)$ is the map induced by the transfer map $\mathrm{H}^*(P) \to \mathrm{H}^*(S)$ on the filtration.

An $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ will induce a map from $\mathbb{Z}[P]$ to $\mathbb{Z}[S]$, and thus a map $\mathrm{Hom}_{\mathbb{Z}[S]}(M'_* \otimes M_*, \mathbb{F}_p) \to \mathrm{Hom}_{\mathbb{Z}[P]}(M'_* \otimes M_*, \mathbb{F}_p)$ of double chain complexes. Hence we get a map $\alpha^*\colon E_*(S) \to E_*(P)$. On the E_2 page this is induced by the map $(\alpha/V)^*\colon \mathrm{H}^*(S/V) \to \mathrm{H}^*(P/V)$ on the rows, while $\alpha^*\colon E_\infty(S) \to E_\infty(P)$ is the map induced by the map $\alpha^*\colon \mathrm{H}^*(S) \to \mathrm{H}^*(P)$ on the filtration.

If $\Omega = \coprod_i (S \times S)/\Delta_{P_i}^{\alpha_i}$ we define $\Omega_*\colon E_*(S) \to E_*(S)$ as the sum $\sum_i \mathrm{trf}_{P_i}\, \alpha_i^*$. By the previous we have that $\Omega_2\colon E_2(S) \to E_2(S)$ on rows of the form $\mathrm{H}^*(S)$ is the sum $\sum_i \mathrm{trf}_{P_i/V}(\alpha_i/V)^*$ and thus $F_{\bar{\Omega}}$ from Proposition 8.8. The Proposition implies that $F_{\bar{\Omega}}$ is idempotent. As Ω_3 on $E_3(S)$ is Ω_2 on a quotient of a subset, it is also idempotent. Hence by induction Ω_r is idempotent for all $r \geq 2$. Then for all p, q and $r \geq 2$ the sequence

$$0 \to \ker(\Omega_r) \to E_r^{pq}(S) \to \Omega_r(E_r^{pq}(S)) \to 0$$

is split-exact, so $E_r^{pq} \cong \ker(\Omega_r) \otimes \Omega_r(E_r^{pq}(S))$. As Ω_* is a morphism of spectral sequences, this splitting respects the differentials and thus gives a splitting of the spectral sequence $E_*(S) \cong \ker \Omega_* \otimes \operatorname{im} \Omega_*$. On $E_2(S)$ we have that $E_2^{pq}(S) \cong \mathrm{H}^p(S/V; H^q(V))$. As we are working with coefficients in $\mathbb{F}_p$ and $\mathrm{H}^q(V)$ is either $\mathbb{F}_p$ or 0, we have that $E_2^{pq}(S) \cong \mathrm{H}^p(S/V) \otimes H^q(V)$. By the Proposition 8.8 we see that

$$\Omega_2(E_2^{pq}(S)) \cong F_{\bar{\Omega}}(\mathrm{H}^p(S/V)) \otimes H^q(V) = \mathrm{H}^p(\mathcal{F}/V) \otimes \mathrm{H}^q(V)$$

On $E_\infty(S)$ we have that Ω is induced by F_Ω, so the spectral sequence $\Omega_*(E_*(S))$ converges to $F_\Omega(\mathrm{H}^*(S))$ This is $\mathrm{H}^*(\mathcal{F})$ by Proposition 8.8.

By part (b) $BV \to |\mathcal{L}_0| \to |(\mathcal{L}/V)^c|$ is a fibration. As both $\pi_1(|(\mathcal{L}/V)^c|)$ and $\mathrm{H}^i(BV; \mathbb{F}_p)$ for any i are finite p-groups, we have that $\pi_1(|(\mathcal{L}/V)^c|)$ acts nilpotently on $\mathrm{H}^i(BV; \mathbb{F}_p)$ for any i. By the definition of nilpotent action [7, II 4] it means that there exists a filtration of $\mathrm{H}^i(BV; \mathbb{F}_p)$ for any i such that the induced action on the quotients is trivial. As $\mathrm{H}^i(BV; \mathbb{F}_p)$ is either $\mathbb{Z}/p$ or 0 the filtration has length at most one. Thus the action of $\pi_1(|(\mathcal{L}/V)^c|)$ on $\mathrm{H}^i(BV; \mathbb{F}_p)$ is trivial for any i. We have that S/V is $\mathcal{F}/V$-centric and for any $P/V \subseteq S/V$, which is $\mathcal{F}/V$-centric there exists a $\varphi \in \mathrm{Mor}_{\mathcal{L}/V}(P/V, S/V)$, for example the orbit of a lift of the inclusion $P \hookrightarrow S$ to $\mathcal{L}$. Hence the category $(\mathcal{L}/V)^c$ is connected, so the geometric realization $|(\mathcal{L}/V)^c|$ is path-connected. Thus we may take the Serre spectral sequence E_* for the fibration to obtain a spectral sequence that converges to $\mathrm{H}^*(|\mathcal{L}_0|)$ with

$$E_2^{pq} = \mathrm{H}^p(|(\mathcal{L}/V)^c|; \mathrm{H}^q(V)) \cong \mathrm{H}^p(|(\mathcal{L}/V)^c|) \otimes \mathrm{H}^q(V).$$

Since $S \in \mathcal{L}_0$ we have that $\theta_S\colon \mathcal{B}S \to \mathcal{L}_0$. As $\theta_{S/V}\colon \mathcal{B}(S/V) \to (\mathcal{L}/V)^c$ is given by $o_{S/V} \mapsto S/V$ and $gV \mapsto \delta_S(g)/\delta_S(V)$ the following diagram commutes

$$\begin{tikzcd}
BV \arrow[r] \arrow[d, "\mathrm{id}"'] & BS \arrow[r] \arrow[d, "|\theta_S|"'] & B(S/V) \arrow[d, "|\theta_{S/V}|"'] \\
BV \arrow[r] & |\mathcal{L}_0| \arrow[r] & |(\mathcal{L}/V)^c|
\end{tikzcd}$$

and thus by naturality of the Serre spectral sequence [20, Chapter 1, page 18] induces a morphism $|\theta|_*\colon E_* \to E_*(S)$. The naturality also implies that

$$|\theta|_2^{pq}\colon \mathrm{H}^p(|(\mathcal{L}/V)^c|; \mathrm{H}^q(V)) \to \mathrm{H}^p(S/V; \mathrm{H}^q(V))$$

is exactly $|\theta_{S/V}|^* = R_{(\mathcal{L}/V)^c}$. In particular the image of $|\theta|_2^{pq}$ is contained in $\mathrm{H}^p(\mathcal{F}/V; \mathrm{H}^q(V)) = (\operatorname{im}\Omega_2^{pq})$. So $|\theta|_*$ is morphism of spectral sequences from E_* to $\operatorname{im}\Omega_*$. We also have that the map $|\theta|^*\colon \mathrm{H}^*(|\mathcal{L}_0|) \to \mathrm{H}^*(\mathcal{F})$ induces a map on successive quotient groups agreeing with $|\theta|_\infty$. We now assume that $R_{(\mathcal{L}/V)^c}\colon \mathrm{H}^*(|(\mathcal{L}/V)^c|) \to \mathrm{H}^*(\mathcal{F}/V)$ is an isomorphism. Then $|\theta|_2$ is an isomorphism, which implies that $|\theta|_r$ is an isomorphism for every $r \geq 2$. In the spectral sequence the differentials are eventually zero, so we conclude that for any p, q there exists N such that $E_\infty^{pq} = E_N^{pq}$. Then $|\theta|_\infty^{pq} = |\theta|_N^{pq}$ and thus is an isomorphism. As both $\mathcal{B}S$ and $\mathcal{L}_0$ are finite categories, we have that $\mathrm{H}^k(BS)$ and $\mathrm{H}^k(|\mathcal{L}_0|)$ are finite dimensional by cellular cohomology. We have that $\mathrm{H}^*(\mathcal{F})$ is a subring of $\mathrm{H}^*(BS)$, so the same holds for $\mathrm{H}^k(\mathcal{F})$, as $\mathbb{F}_p$ is Noetherian. Thus the filtrations of $\mathrm{H}^*(|\mathcal{L}|)$ and $\mathrm{H}^*(\mathcal{F})$ from the spectral sequences all have finite length. The map induced by $|\theta|^*\colon \mathrm{H}^*(|\mathcal{L}_0|) \to \mathrm{H}^*(\mathcal{F})$ on the successive quotients is an isomorphism, and as the filtration is finite, so is $|\theta|^*$.Let $\iota\colon \mathcal{L}_0 \to \mathcal{L}$ be the inclusion. Then $R_{\mathcal{L}} = |\theta|^* \circ \iota^*$, and hence is an isomorphism by the above and part (c). □

8.5. **Lannes T-functor and the cohomology ring of a fusion system.** Let E be an elementary abelian p-group. Lannes T_E-functor by [26, Corollary 2.4.5] is the left adjoint to $\mathrm{H}^*(BE) \otimes -$ and in particular for any unstable algebra X, Y over $\mathcal{A}_p$ we have that

$$\operatorname{Hom}_{\mathcal{K}}(T_E(X), Y) \cong \operatorname{Hom}_{\mathcal{K}}(X, \mathrm{H}^*(BE) \otimes Y)$$

Then for any map $f \in \operatorname{Hom}_{\mathcal{K}}(X, \mathrm{H}^*(BE))$ we have the adjoint map $ad(f)\colon T_E(X) \to \mathbb{F}_p$ In particular it induces a ring morphism $T_E^0(X) \to \mathbb{F}_p$ and we define $T_E(X; f) = T_E(X) \otimes_{T_E^0(X)} \mathbb{F}_p$, where the $T_E^0(X)$-module structure on $\mathbb{F}_p$ is induced by $ad(f)$. We call this the component of $T_E(X)$ at f. If $\operatorname{Hom}_{\mathcal{K}}(X, \mathrm{H}^*(BE))$ is finite then by [28, page 3] we have that $T_E(X) \cong \prod T_E(X; f)$ where f ranges over $\operatorname{Hom}_{\mathcal{K}}(X, \mathrm{H}^*(BE))$. So every in this case every map $T_E(X) \to Y$ induces a map $T_E(X; f) \to Y$ for $f \in \operatorname{Hom}_{\mathcal{K}}(X, \mathrm{H}^*(BE))$.

For any finite group P we have by [26, Theorem 3.1.5.2] that the map $\varphi \mapsto \varphi^*$ from $\operatorname{Rep}(E, P)$ to $\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(P), \mathrm{H}^*(E))$ is a bijection. According to [26, Chapther 1.8] we have that $T_E^0(\mathrm{H}^*(P))$ is a p-boolean algebra. We let $\mathcal{B}$ be the category of p-boolean algebras. Furthermore for any p-boolean algebra the functor $B \mapsto \mathbb{F}_p^{\operatorname{Hom}_{\mathcal{B}}(B, \mathbb{F}_p)}$ is the composition of an equivalence of categories and its inverse, so it is an isomorphism. Note that $\mathbb{F}_p^S$ is the set of continous functions from S to $\mathbb{F}_p$. By degree considerations, as noted in [26, 1.8.3], we have that

$$\operatorname{Hom}_{\mathcal{B}}(T_E^0(H^*(P)), \mathbb{F}_p) \cong \operatorname{Hom}_{\mathcal{K}}(T_E(H^*(P)), \mathbb{F}_p)$$

The adjunction implies that this set is in bijection with $\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(P), \mathrm{H}^*(E))$, so we conclude by the above that $T_E^0(\mathrm{H}^*(P)) \cong \mathbb{F}_p^{\operatorname{Rep}(E,P)}$. Note that $\mathbb{F}_p^{\operatorname{Rep}(E,P)}$ has the set of indicator-functions of the elements of $\operatorname{Rep}(E, P)$ as a basis, so $T_E^0(\mathrm{H}^*(P))$ has a $\mathbb{F}_p$-basis of elements x_ρ for $\rho \in \operatorname{Rep}(E, P)$. Let $\rho, \rho' \in \operatorname{Rep}(E, P)$ be different. Then with our identification we have that $ad(\rho^*)$ on $T_E^0(\mathrm{H}^*(P))$ satisfies that $ad(\rho^*)(x_\rho) = 1$ and $ad(\rho^*)(x_{\rho'}) = 0$. Furthermore x_ρ is the unique element with this property

. So $T_E(\mathrm{H}^*(P);\rho^*) = T_E(\mathrm{H}^*(P)) \otimes_{T_E^0(\mathrm{H}^*(P))} \mathbb{F}_p x_\rho$, which can be identified as $T_E(\mathrm{H}^*(P))x_\rho$. Then the isomorphism

$$T_E(\mathrm{H}^*(P)) \cong \prod_{\rho\in \mathrm{Rep}(E,P)} T_E(\mathrm{H}^*(P);\rho^*).$$

from [28] on $T_E(\mathrm{H}^*(P)) \to T_E(\mathrm{H}^*(P);\rho^*)$ given by $y \mapsto yx_\rho$ for any $\rho \in \mathrm{Rep}(E,P)$.

Let $\varphi \in \mathrm{Hom}(P,Q)$ be an injective group homomorphism. We want to determine how $T_E(\varphi^*)\colon T_E(\mathrm{H}^*(Q)) \to T_E(\mathrm{H}^*(P))$ behaves on the components. Note that composition with φ induces an injective map $\mathrm{Rep}_\varphi\colon \mathrm{Rep}(E,P) \to \mathrm{Rep}(E,Q)$. Thus we get a map $\mathbb{F}_p^{\mathrm{Rep}(E,Q)} \to \mathbb{F}_p^{\mathrm{Rep}(E,P)}$ by $g \in \mathrm{Map}(\mathrm{Rep}(E,Q),\mathbb{F}_p)$ is mapped to $\rho \mapsto g(\varphi\circ\rho)$. In particular the indicator function of $\rho' \in \mathrm{Rep}(E,Q)$ is mapped to the indicator function of ρ, where $\rho \in \mathrm{Rep}(E,P)$ with $\rho' = \varphi\circ\rho$, if such a ρ exists and to the zero function otherwise. Note that Rep_φ is injective, so there exists at most one ρ with this property. All maps used in the identification of $T_E^0(\mathrm{H}^*(P))$ as $\mathbb{F}_p^{\mathrm{Rep}(E,P)}$ are natural in P, so we conclude that $T_E(\varphi^*)\colon T_E^0(\mathrm{H}^*(Q)) \to T_E^0(\mathrm{H}^*(P))$ is corresponds to the map $\mathbb{F}_p^{\mathrm{Rep}(E,Q)} \to \mathbb{F}_p^{\mathrm{Rep}(E,P)}$ from above. Thus for $\rho' \in \mathrm{Rep}(E,Q)$ we have that, if there exists a $\rho \in \mathrm{Rep}_\varphi^{-1}(\rho')$ then $T_E(\varphi^*)(x_{\rho'}) = x_\rho$ and if $\mathrm{Rep}_\varphi^{-1}(\rho') = \emptyset$ then $T_E(\varphi^*)(x_{\rho'}) = 0$. Then we see that $T_E(\varphi^*)$ factors through the components of $T_E(\mathrm{H}^*(Q))$ to the components of $T_E(\mathrm{H}^*(P))$ as the maps $T_E(\varphi^*)\colon T_E(\mathrm{H}^*(Q);\varphi\circ\rho) \to T_E(\mathrm{H}^*(P);\rho)$ given by $yx_{\varphi\circ\rho} \mapsto T_E(\varphi^*)(y)x_\rho$ and the zero map on all other components of $T_E(\mathrm{H}^*(Q))$.

We want to see how this translates to $T_E(\mathrm{H}^*(\mathcal{F}))$ for a fusion system $\mathcal{F}$ over a p-group S. The first step is to identify $\mathrm{Hom}_\mathcal{K}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$.

Definition 8.11. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Define* $\mathrm{Rep}(E,\mathcal{F}) = \mathrm{colim}_{P\in\mathcal{O}(\mathcal{F})} \mathrm{Rep}(E,P)$.

Recall that we say $\rho,\rho' \in \mathrm{Rep}(E,P)$ are $\mathcal{F}$-conjugate if there exists a $\varphi \in \mathrm{Hom}_\mathcal{F}(\rho(E),\rho'(E))$ such that $\varphi\circ\rho = \rho'$ and remark that this is an equivalence relation. Let $\mathrm{Rep}(E,S)/\mathcal{F}$ be the set of equivalence classes of $\mathrm{Rep}(E,S)$ under $\mathcal{F}$-conjugation.

Lemma 8.12. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, then* $\mathrm{Rep}(E,\mathcal{F})$ *is* $\mathrm{Rep}(E,S)/\mathcal{F}$.

Proof. For any $\varphi \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q)$ and $\psi \in \mathrm{Rep}(E,P)$ we have that $\varphi\circ\psi$ and ψ considered as elements of $\mathrm{Rep}(E,S)$ are $\mathcal{F}$-conjugate. So the map sending $\psi \in \mathrm{Rep}(E,P)$ to the class of $i_P\circ\psi$ in $\mathrm{Rep}(E,S)/\mathcal{F}$ gives a co-cone of the diagram $\mathrm{Rep}(E,-)$. Furthermore for any co-cone of the diagram $\mathrm{Rep}(E,-)$ we have for $\varphi \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,S)$ and $\psi \in \mathrm{Rep}(E,P)$, that $\Psi_S(i_E\circ\psi) = \Psi_P(\psi) = \Psi_S(\varphi\circ\psi)$, so it factors trough $\mathrm{Rep}(E,S)/\mathcal{F}$. Hence $\mathrm{Rep}(E,S)/\mathcal{F} = \mathrm{Rep}(E,\mathcal{F})$. □

Proposition 8.13. *For any fusion system $\mathcal{F}$ over a p-group S we have that the map $[\rho]_\mathcal{F} \mapsto \rho^*\circ\iota$ is a bijection from* $\mathrm{Rep}(E,\mathcal{F})$ *to* $\mathrm{Hom}_\mathcal{K}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$, *where $[\rho]_\mathcal{F}$ is the class of $\rho \in \mathrm{Rep}(E,S)/\mathcal{F}$.*

Proof. By definition we have that $\mathrm{H}^*(\mathcal{F}) = \varprojlim_{\mathcal{O}(\mathcal{F})} \mathrm{H}^*(-)$, where $\mathcal{O}(\mathcal{F})$ is a finite category. Then [32, Corollary 3.8.8.] implies that the natural map

$$\underset{P\in\mathcal{O}(\mathcal{F})}{\mathrm{colim}}\ \mathrm{Hom}_\mathcal{B}(T_E^0(H^*(P)),\mathbb{F}_p) \to \mathrm{Hom}_\mathcal{B}(T_E^0(\mathrm{H}^*(\mathcal{F}));\mathbb{F}_p)$$

is an isomorphism. By the above stated results we have that the map $\rho \mapsto ad(\rho^*)$ is an isomorphism from $\operatorname{Rep}(E,P)$ to $\operatorname{Hom}_{\mathcal{B}}(T_E^0(H^*(P)),\mathbb{F}_p)$. This is natural in P with respect to maps from $\mathcal{O}(\mathcal{F})$, so we conclude that $\operatorname{colim}_{\mathcal{O}(\mathcal{F})}\operatorname{Rep}(E,P)\cong \operatorname{colim}_{P\in\mathcal{O}(\mathcal{F})}\operatorname{Hom}_{\mathcal{B}}(T_E^0(H^*(P)),\mathbb{F}_p)$ induced by $\rho\mapsto ad(\rho^*)$. For any unstable algebra K we see by degree considerations that elements in $\operatorname{Hom}_{\mathcal{K}}(K,\mathbb{F}_p)$ are trivial exepct for K^0, so $\operatorname{Hom}_{\mathcal{K}}(K,\mathbb{F}_p)=\operatorname{Hom}_{\mathcal{B}}(K^0,\mathbb{F}_p)$. Then we conclude that

$$\begin{aligned}\operatorname{Rep}(E,\mathcal{F}) &= \operatorname*{colim}_{\mathcal{O}(\mathcal{F})}\operatorname{Rep}(E,P)\cong \operatorname{Hom}_{\mathcal{B}}(T_E^0(\mathrm{H}^*(\mathcal{F})),\mathbb{F}_p)\\ &= \operatorname{Hom}_{\mathcal{K}}(T_E(\mathrm{H}^*(\mathcal{F})),\mathbb{F}_p)\cong \operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))\end{aligned}$$

and the isomorphism is given by $[\rho]_{\mathcal{F}}\in\operatorname{Rep}(E,\mathcal{F})$ it mapped to $\rho^*\circ\iota\colon \mathrm{H}^*(\mathcal{F})\to \mathrm{H}^*(P)$. □

Note that the above proposition implies that $\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$ is finite. As the previous discussion holds for any unstable algebras with $\operatorname{Hom}_{\mathcal{K}}(X,\mathrm{H}^*(E))$ finite, we have for any $f\in\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$ a unique element $x_f\in T_E^0(\mathrm{H}^*(\mathcal{F}))$ such that $T_E(ad(f))(x_f)=1$ and $T_E(ad(\tilde f))(x_f)=0$ for any other $\tilde f\in\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$. Note that by argument similar to the previous any injective $\varphi\colon E\to E'$ between elementary abelian p-groups induces a map $\operatorname{Rep}_\varphi\colon \operatorname{Rep}(E',\mathcal{F})\to\operatorname{Rep}(E,\mathcal{F})$, and the induced map $T_\varphi\colon T_E^0(\mathrm{H}^*(\mathcal{F}))\to T_{E'}^0(\mathrm{H}^*(\mathcal{F}))$ is exactly $x_f=\sum_{f'\in\operatorname{Rep}_\varphi^{-1}(f)}x_{f'}$.

Next we identify the elements $x_f\in T_E^0(\mathrm{H}^*(\mathcal{F}))$ for a $f\in\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$ and the corresponding components.

Proposition 8.14. *Let $\mathcal{F}$ be a fusion system over a p-group S. Let $f_1,\dots,f_n$ be a set of representatives for $\operatorname{Rep}(E,\mathcal{F})=\operatorname{Rep}(E,S)/\mathcal{F}$. For any $P\subseteq S$ and $1\le i\le n$ let $\mathcal{T}_P^i$ be the set of $\rho\in\operatorname{Rep}(E,P)$, such that $[i_P\circ\rho]_{\mathcal{F}}=[f_i]_{\mathcal{F}}$. Then $T_E(\iota)(x_{f_i^*\circ\iota})=\sum_{\rho\in\mathcal{T}_S^i}x_\rho$ and $T_E(\mathrm{H}^*(\mathcal{F});f_i^*\circ\iota)\cong\varprojlim_{P\in\mathcal{O}(\mathcal{F})}\prod_{\rho\in\mathcal{T}_P^i}T_E(\mathrm{H}^*(P);\rho^*)$.*

Proof. By [26, Theorem 2.4.1] the functor T_E is exact, so it commutes with equalizers over a finite set of morphisms. Furthermore [26, Theorem 2.4.3] implies that T_E commutes with finite products. A limit over a finite category may be expressed as a equalizer of a finite product over a finite set of morphisms. Hence T_E commutes with finite limits. By definition $\mathrm{H}^*(\mathcal{F})$ is a finite limit, so we conclude that $T_E(\mathrm{H}^*(\mathcal{F}))\cong\varprojlim_{\mathcal{O}(\mathcal{F})}T_E(\mathrm{H}^*(-))$. We now need to identity the image of $x_{f_i^*\circ\iota}$ under the isomorphism.

Let $\varphi\in\operatorname{Rep}_{\mathcal{F}}(P,Q)$. We now want to show that $\operatorname{Rep}_\varphi^{-1}(\mathcal{T}_Q^i)=\mathcal{T}_P^i$. Let $\rho'\in\mathcal{T}_Q^i$ and assume that $\rho'=\varphi\circ\rho$ for a $\rho\in\operatorname{Rep}(E,P)$. Then there exists a $\psi\in\operatorname{Hom}_{\mathcal{F}}(f_i(E),\rho'(E))$ such that $f_i=\psi\circ\rho$. Then $\varphi\circ\psi\in\operatorname{Hom}_{\mathcal{F}}(f_i(E),\rho(E))$ has the wanted property, so we see that $\rho\in\mathcal{T}_P^i$. Similary for a $\rho\in\mathcal{T}_P^i$ the corresponding $\psi\in\operatorname{Hom}_{\mathcal{F}}(f_i(E),\rho(E))$ composed with $\varphi^{-1}\in\operatorname{Hom}_{\mathcal{F}}(\varphi\circ\rho(E),\rho(E))$ will imply that $\operatorname{Rep}_\varphi(\rho)\in\mathcal{T}_Q^i$.

Now for $P\subseteq S$ and $1\le i\le n$ set $y_P^i=\sum_{\rho\in\mathcal{T}_P^i}x_\rho\in T_E^0(\mathrm{H}^*(P))$. Then for every $\varphi\in\operatorname{Rep}_{\mathcal{F}}(P,Q)$ we have by the previous discussion that

$$T_E(\varphi^*)(y_Q^i)=\sum_{\rho\in\operatorname{Rep}_\varphi^{-1}(\mathcal{T}_Q^i)}x_\rho=\sum_{\mathcal{T}_P^i}x_\rho=y_P.$$

Hence $(y_P)_{P\subseteq S}\in\varprojlim_{\mathcal{O}(\mathcal{F})}T_E(H^*(-))$. Furthermore $T_E(f_i^*)(y_S)=1$ while we have $T_E(f_j^*)(y_S)=0$ for any $j\neq i$. As every $f\in\operatorname{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}),\mathrm{H}^*(E))$ is of the form

$f_i^* \circ \iota$ for an i, we conclude that $T_E(\iota)(x_{f_i^*\circ\iota}) = (y_P^i)_{P\subseteq S}$. Then

$$\begin{aligned} T_E(\mathrm{H}^*(\mathcal{F}); f_i^* \circ \iota) = T_E(\mathrm{H}^*(\mathcal{F}))x_{f_i^*\circ\iota} &\cong \varprojlim_{P\in\mathcal{O}(\mathcal{F})} T_E(H^*(P))y_P \\ &\cong \varprojlim_{P\in\mathcal{O}(\mathcal{F})} \prod_{\rho\in\mathcal{T}_P^i} T_E(\mathrm{H}^*(P); \rho^*). \end{aligned}$$

□

If E is a elementary abelian subgroup of S, we may consider the group homomorphism mult: $C_S(E) \times E \to S$ given by $(x, y) \mapsto xy$. It induces a map $\mathrm{H}^*(BS) \to \mathrm{H}^*(BE) \otimes \mathrm{H}^*(C_S(E))$. Let $\Phi_E \colon T_E(\mathrm{H}^*(BS)) \to \mathrm{H}^*(C_S(E))$ be the adjoint to the map induced by mult. By [21, Corollary 0.3] the map $\Phi_E \colon T_E(\mathrm{H}^*(S); i) \to \mathrm{H}^*(C_S(E))$ is an isomorphism. The following lemma is the analog for fusion systems.

Lemma 8.15. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group and $E \subseteq S$ a elementary abelian subgroup, which is fully centralized in $\mathcal{F}$. Then the map $\Phi_E \circ T_E(\iota)$ is an isomorphism $T_E(\mathrm{H}^*(\mathcal{F}); i_E^* \circ \iota) \to \mathrm{H}^*(C_{\mathcal{F}}(E))$, where $i_E \colon E \to S$ is the inclusion.*

Proof. Set $\mathcal{T}_P$ for a $P \subseteq S$ to be the set of $\rho \in \mathrm{Rep}(E, P)$, such that $i_P \circ \rho$ is $\mathcal{F}$-conjugate to i_E. Now we define $\mathcal{O}_E(\mathcal{F})$ to be the category with objects (P, ρ), where $P \subseteq S$ and $\rho \in \mathcal{T}_P$, and morphism set

$$\mathrm{Mor}_{\mathcal{O}_E(\mathcal{F})}((P, \rho), (Q, \rho')) = \{\alpha \in \mathrm{Rep}_{\mathcal{F}}(P, Q) \mid \alpha \circ \rho = \rho'\}$$

Then by the above considerations we have a well-defined functor from $\mathcal{O}_E(\mathcal{F}) \to \mathcal{K}$ by setting $(P, \rho) \mapsto T_E(\mathrm{H}^*(P); \rho^*)$ and $\alpha \mapsto T_E(\alpha^*)$. Comparing the given definitions we see that $\lim_{\mathcal{O}_E(\mathcal{F})} T_E(-) \cong \lim_{P\in\mathcal{O}(\mathcal{F})} \bigoplus_{\rho\in\mathcal{T}_P} T_E(\mathrm{H}^*(P); \rho^*)$. Proposition 8.14 now implies that $T_E(\mathrm{H}^*(\mathcal{F}); i_E^* \circ \iota) \cong \lim_{\mathcal{O}_E(\mathcal{F})} T_E(-)$.

For any $(P, \rho) \in \mathcal{O}_E(\mathcal{F})$ we have that $\rho \colon E \to \rho(E)$ is a morphism in $\mathcal{F}$, so ρ is injective. This implies that $T_E(\mathrm{H}^*(P); \rho^*) \cong T_{\rho(E)}(\mathrm{H}^*(P), i)$ by the map T_ρ. We have that EP is a P-CW complex with only finite number of cells and finite isotropy groups. As EP is contractible, we see that it is mod-p-acylic. Then by [21, Corollary 0.3] we conclude that the map $\Phi_{\rho(E)} \colon T_{\rho(E)}(\mathrm{H}^*(P); i) \to \mathrm{H}^*(C_P(\rho(E))$ is an isomorphism.

Note that for any $\alpha \in \mathrm{Mor}_{\mathcal{O}_E(\mathcal{F})}((P, \rho), (Q, \rho'))$ we have that $\alpha(C_P(\rho(E))) \subseteq C_Q(\rho'(E))$, so we have a functor on $\mathcal{O}_E(\mathcal{F})$ by setting $(P, \rho) \mapsto \mathrm{H}^*(C_P(\rho(E)))$ and $\alpha \mapsto \alpha^*$. Let $(P, \rho), (Q, \rho') \in \mathcal{O}_E(\mathcal{F})$ and $\alpha \in \mathrm{Mor}_{\mathcal{O}_E(\mathcal{F})}((P, \rho), (Q, \rho'))$. In the diagram

$$\begin{tikzcd} T_E(\mathrm{H}^*(Q); \rho') \arrow[r, "T_{\rho'}"] \arrow[d, "T_E(\alpha^*)"'] & T_{\rho'(E)}(\mathrm{H}^*(Q); i) \arrow[r, "\Phi_{\rho'(E)}"] & H^*(C_Q(\rho'(E))) \arrow[d, "\alpha^*"'] \\ T_E(\mathrm{H}^*(P); \rho) \arrow[r, "T_\rho"] & T_{\rho(E)}(\mathrm{H}^*(P); i) \arrow[r, "\Phi_{\rho(E)}"] & H^*(C_P(\rho(E))) \end{tikzcd}$$

both compositions are adjoint to maps induced by the following group homomorphisms:

$$C_P(\rho(E)) \times E \xrightarrow{\mathrm{id} \times \rho} C_P(\rho(E)) \times \rho(E) \xrightarrow{\text{mult}} P \xrightarrow{\quad\alpha\quad} Q$$

$$C_P(\rho(E)) \times E \xrightarrow{\alpha \times \mathrm{id}_E} C_Q(\rho'(E)) \times E \xrightarrow{\mathrm{id} \times \rho'} C_Q(\rho'(E)) \times \rho'(E) \xrightarrow{\mathrm{mult}} Q$$

As $\alpha \circ \rho = \rho'$ the homomorphism agrees, so the above diagram commutes. Thus the two functors on $\mathcal{O}_E(\mathcal{F})$ are isomorphic.

Under the isomorphism $T_E(\mathrm{H}^*(\mathcal{F}); j_E) \cong \lim_{\mathcal{O}_E(\mathcal{F})} T_E(-)$ the map $\Phi_E \circ T_E(\iota)$ on the component $T_E(P;\rho)$ is exactly the composite $\Phi_{\rho(E)} \circ T_\rho$ and since they form a natural isomorphism, we get that

$$(\Phi_E \circ T_E(\iota))(T_E(\mathrm{H}^*(\mathcal{F}); j_E)) \cong \varprojlim_{\mathcal{O}_E(\mathcal{F})} \mathrm{H}^*(C_P(\rho(E)).$$

Let $(E,\rho) \in \mathcal{O}_E(\mathcal{F})$. Then as E is abelian, we have that $\rho(E)$ is abelian and thus $\rho(E) \subseteq C_P(\rho(E))$. Let $\mathcal{O}'_E(\mathcal{F})$ be the full subcategory of $\mathcal{O}_E(\mathcal{F})$ on objects (P,ρ) where $P \subseteq C_S(\rho(E)))$. We have a functor F on $\mathcal{O}_E(\mathcal{F})$ by setting $(P,\rho) \to (C_P(\rho(E)),\rho)$ and $\alpha \in \mathrm{Mor}_{\mathcal{O}_E(\mathcal{F})}((P,\rho),(Q,\rho'))$ to the restriction of α as a map from $C_P(\rho(E))$ to $C_Q(\rho'(E))$. As $C_P(\rho(E)) = C_{C_P(\rho(E))}(\rho(E))$ we have that $\varprojlim_{\mathcal{O}_E(\mathcal{F})} \mathrm{H}^*(C_P(\rho(E))) = \varprojlim_{\mathcal{O}_E(\mathcal{F})} \mathrm{H}^*(C_{F(P)}(\rho(E)))$. The image of F is in $\mathcal{O}'_E(\mathcal{F})$ and as it is the identity on $\mathcal{O}'_E(\mathcal{F})$ the limit $\varprojlim_{\mathcal{O}_E(\mathcal{F})} \mathrm{H}^*(C_P(\rho(E)))$ does not change when restricting to the subcategory $\mathcal{O}'_E(\mathcal{F})$. Note that for $(P,\rho) \in \mathcal{O}'_E(\mathcal{F})$ we have $C_P(\rho(E)) = P$. Hence

$$(\Phi_E \circ T_E(\iota))(T_E(\mathrm{H}^*(\mathcal{F}); j_E)) \cong \varprojlim_{\mathcal{O}'_E(\mathcal{F})} \mathrm{H}^*(P).$$

Let $(P,\rho) \in \mathcal{O}'_E(\mathcal{F})$. Since E is fully centralized there exists by Lemma 5.1 an $\mathcal{F}$ morphism $C_S(\rho)\colon C_S(\rho(E)) \to C_S(E)$ such that $C_S(\rho)$ on $\rho(E)$ is ρ^{-1}. Hence $C_S(\rho) \in \mathrm{Rep}_\mathcal{F}(P, C_S(\rho)(P))$ is a morphism in $\mathcal{O}'_E(\mathcal{F})$ from (P,ρ) to $(C_S(\rho)(P), i)$. As $C_S(\rho)$ is injective, this is an isomorphism in $\mathcal{O}'_E(\mathcal{F})$. Thus $\mathcal{O}'_E(\mathcal{F})$ is equivalent to the full subcategory on objects (P,i). We denote this subcategory $\mathcal{O}'(C_\mathcal{F}(E))$. Then the above limit does not change when replacing $\mathcal{O}'_E(\mathcal{F})$ by $\mathcal{O}'(C_\mathcal{F}(E))$.

The subcategory $\mathcal{O}'(C_\mathcal{F}(E))$ has the objects $P \subseteq C_S(E)$ containing E while the morphisms are elements in $\mathrm{Rep}_\mathcal{F}(P,Q)$, which are the identity on E. Thus it is a subcategory of $\mathcal{O}(C_\mathcal{F}(E))$ as well. Note that for any $P \subseteq C_S(E)$ we have that PE is an object of $\mathcal{O}'(C_\mathcal{F}(E))$ containing P as a subgroup. By definition of the centralizer fusion system we have that for any $\varphi \in \mathrm{Hom}_{C_\mathcal{F}(E)}(P,Q)$ there exists a lift $\tilde{\varphi} \in \mathrm{Hom}_{C_\mathcal{F}(E)}(PE,QE)$. Consider the map $\varprojlim_{\mathcal{O}'(C_\mathcal{F}(E))} \mathrm{H}^*(-) \to \varprojlim_{\mathcal{O}(C_\mathcal{F}(E))} \mathrm{H}^*(-)$ given by $(x_Q)_{Q \in \mathcal{O}'(C_\mathcal{F}(E))} \mapsto (i^*_{P \hookrightarrow PE} x_{PE})_{P \in \mathcal{O}(C_\mathcal{F}(E))}$. For any $\alpha \in \mathrm{Rep}_\mathcal{F}(P,P')$ we have that

$$\alpha^*(x_{P'}) = \alpha^*(i^*_{P' \hookrightarrow P'E} x_{P'E}) = i^*_{P \hookrightarrow PE} \tilde{\alpha}^* x_{PE} = i^*_{P \hookrightarrow PE} x_{PE},$$

so the map is well-defined. For an object P of $\mathcal{O}'(C_\mathcal{F}(E))$ we have that $PE = P$, so the given map has an inverse induced by the inclusion $\mathcal{O}'(C_\mathcal{F}(E)) \to \mathcal{O}(C_\mathcal{F}(E))$. Thus $\varprojlim_{\mathcal{O}'(C_\mathcal{F}(E))} \mathrm{H}^*(-) \cong \varprojlim_{\mathcal{O}(C_\mathcal{F}(E))} \mathrm{H}^*(-)$.

By combining the stated results we conclude that

$$(\Phi_E \circ T_E(\iota))(T_E(\mathrm{H}^*(\mathcal{F}); j_E)) \cong \varprojlim_{\mathcal{O}(C_\mathcal{F}(E))} \mathrm{H}^*(-) = \mathrm{H}^*(C_\mathcal{F}(E)).$$

$\square$

8.6. **Main Theorem on the cohomology ring of fusion systems.** We will now prove the main theorem about the cohomology ring of fusion systems, namely the identification with the cohomology ring of its p-complete classifying space, when it exists.

Let $\mathcal{K}$ be the category of unstable algebras over the mod p Steenrod algebra $\mathcal{A}_p$. Let $K \in \mathcal{K}$. We now define $A(K)$ to be the category with objects the pairs (E, f), where E is a nontrivial elementary abelian p-group and $f \in \mathrm{Mor}_{\mathcal{K}}(K, \mathrm{H}^*(BE))$, which makes $H^*(BE)$ a finitely generated K-module. As morphisms we consider

$$\mathrm{Mor}_{A(K)}((E,f),(E',f')) = \{\varphi\colon E \to E' \mid \varphi \text{ is a monomorphism}, \varphi^* f' = f\}.$$

The proof of the main theorem relies on the work in [14] on the reconstruction functor $\alpha\colon A(\mathrm{H}^*(\mathcal{F})) \to \mathcal{A}_p$. The category $A(\mathrm{H}^*(\mathcal{F}))$ can in a natural way be identified with $\mathcal{F}^e$, as described in the following lemma.

Lemma 8.16. *Let $(S, \mathcal{F}, \mathcal{L})$ be p-local finite group. Let $\mathcal{F}^e$ be the full subcategory of $\mathcal{F}$ on the set on non-trivial fully normalized subgroup of S. The functor $\beta\colon \mathcal{F}^e \to A(\mathrm{H}^*(\mathcal{F}))$ given by $\beta(E) = (E, i_E^* \circ \iota)$ and $\beta(\varphi) = \varphi$ is then an equivalence of categories.*

Proof. Let $E \in \mathcal{F}^e$. Both the inclusion $\iota\colon \mathrm{H}^*(\mathcal{F}) \hookrightarrow \mathrm{H}^*(BS)$ and the map

$$i_E^*\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(BE)$$

are maps of unstable algebras over $\mathcal{A}_p$. By Proposition 8.5, ι gives $\mathrm{H}^*(BS)$ the structure of a finitely generated module over $\mathrm{H}^*(\mathcal{F})$. For a homomorphism $\psi\colon E \to S$ it follows by [29, Corollary 2.4] that $\psi^*\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(BE)$ makes $\mathrm{H}^*(BE)$ into a finitely generated $\mathrm{H}^*(BS)$-module if any only if ψ is injective. As i_E is injective, this implies that i_E^* induces a finitely generated $\mathrm{H}^*(BS)$-module structure on $\mathrm{H}^*(BE)$. Thus we conclude that $(E, i_E^* \circ \iota) \in A(\mathrm{H}^*(\mathcal{F}))$. A $\varphi \in \mathrm{Mor}_{\mathcal{F}^e}(E, E')$ is by definition a monomorphism. Both $i_E, i_{E'} \circ \varphi \in \mathrm{Hom}_{\mathcal{F}}(E, S)$. So by definition of $\mathrm{H}^*(\mathcal{F})$ as $\varprojlim_{\mathcal{O}(\mathcal{F})} \mathrm{H}^*(-)$ we have for any $x \in \mathrm{H}^*(\mathcal{F})$ that $i_E^*(x) = (i_{E'} \circ \varphi)^*(x) = \varphi^* \circ i_{E'}^*(x)$, and hence $(i_E^* \circ \iota) = \varphi^* \circ (i_{E'}^* \circ \iota)$. Thus we conclude that φ is a morphism from $(E, i_E^* \circ \iota)$ to $(E', i_{E'}^* \circ \iota)$ in $A(\mathrm{H}^*(\mathcal{F})$. So β is well-defined and clearly a functor.

To prove that β is an equivalence of categories, we will first consider some other functors.

First we let $A(|\mathcal{L}|_p^\wedge)$ denote the category with objects the pairs (E, f), where E is a nontrivial elementary abelian p-group, and $f \in \mathrm{Mor}_{\mathsf{Top}}(BE, |\mathcal{L}|_p^\wedge)$, which makes $\mathrm{H}^*(E)$ a finitely generated $\mathrm{H}^*(|\mathcal{L}|_p^\wedge)$-module. As morphisms we consider

$$\mathrm{Mor}_{A(|\mathcal{L}|_p^\wedge)}((E,f),(E',f')) = \{\varphi\colon E \to E' \mid \varphi \text{ is a monomorphism}, f' \circ B(\varphi) \simeq f\}.$$

Let $\theta\colon BS \to |\mathcal{L}|$ be the inclusion induced by the canonical monomorphism $\delta_S\colon S \to \mathcal{L}$. Then θ^* is the composition

$$\mathrm{H}^*(|\mathcal{L}|) \xrightarrow{R_{\mathcal{L}}} \mathrm{H}^*(\mathcal{F}) \xrightarrow{\iota} \mathrm{H}^*(BS)$$

By Proposition 8.5 and Lemma 8.6 both these maps induce a finite module structure on the target, so the same holds for θ^*. By the above remark $Bi_E\colon BE \to BS$ satisfies that $(Bi_E)^*\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(BE)$ induces a finite module structure. As $|\mathcal{L}|$ is p-good by Proposition 3.3, we have that $\phi_{|\mathcal{L}|}^*\colon \mathrm{H}^*(|\mathcal{L}|_p^\wedge) \to \mathrm{H}^*(|\mathcal{L}|)$ is an isomorphism. Set $f_{\mathcal{L}} = \phi_{|\mathcal{L}|} \circ |\theta|$. From this we conclude that if we set $\tilde{B}(E) =$

$(E, f_{\mathcal{L}} \circ B(i_E))$ then $\tilde{B}(E) \in A(|\mathcal{L}|^{\wedge}_p)$. By Theorem 7.4 (b) we have that for $E, E' \in \mathcal{F}^e$

$$\begin{aligned}
&\mathrm{Mor}_{A(|\mathcal{L}|^{\wedge}_p)}(\tilde{B}(E), \tilde{B}(E')) \\
&\qquad =\{\varphi \in \mathrm{Inj}(E, E') \mid f_{\mathcal{L}} \circ B(i_{E'}) \simeq f_{\mathcal{L}} \circ B(i_E) \circ B\varphi\} \\
&\qquad =\{\varphi \in \mathrm{Inj}(E, E') \mid \exists \chi \in \mathrm{Hom}_{\mathcal{F}}(E, \varphi(E)), i_{E'} \circ \varphi = \chi \circ i_E\} \\
&\qquad =\{\varphi \in \mathrm{Inj}(E, E') \mid \varphi|_{E,\varphi(E)} \in \mathrm{Hom}_{\mathcal{F}}(E, \varphi(E))\} = \mathrm{Hom}_{\mathcal{F}}(E, E').
\end{aligned}$$

By the above we get a well-defined functor $\tilde{B}(-)\colon \mathcal{F}^e \to A(|\mathcal{L}|^{\wedge}_p)$ by setting $\tilde{B}(E) = (BE, f_{\mathcal{L}} \circ B(i_E))$ and $\tilde{B}(\varphi) = \varphi$, which is an isomorphism on the morphism classes. Consider a $(E, f) \in A(|\mathcal{L}|^{\wedge}_p)$. By Theorem 7.4 there exists a $\rho \in \mathrm{Hom}(E, S)$ such that $f \simeq f_{\mathcal{L}} \circ B(\rho)$. Then $f^* = (f_{\mathcal{L}} \circ B(\rho))^*\colon \mathrm{H}^*(|\mathcal{L}|^{\wedge}_p) \to \mathrm{H}^*(BE)$, so $(E, f_{\mathcal{L}} \circ B(\rho)) \in A(|\mathcal{L}|^{\wedge}_p)$. This implies that $B(\rho)\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(BE)$ induces a structure as a finitely generated module, so by the above remark we see that ρ is injective. Then $\rho(E)$ is non-trivial elementary abelian subgroup of S, so $(\rho(E), f_{\mathcal{L}} \circ B(i_{\rho(E)})) \in A(|\mathcal{L}|^{\wedge}_p)$ and $\rho|_{E,\rho(E)}\colon (E, f) \to (\rho(E), f_{\mathcal{L}} \circ B(i_{\rho(E)}))$ is an isomorphism in $A(|\mathcal{L}|^{\wedge}_p)$. Furthermore there exists an isomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(\rho(E), E')$, where E' is fully centralized in $\mathcal{F}$. Then Theorem 7.4 also implies that $f_{\mathcal{L}} \circ B(i_{\rho(E)}) \simeq f_{\mathcal{L}} \circ B(i_{E'} \circ \psi)$ Then $\psi\colon (\rho(E), f_{\mathcal{L}} \circ B(i_{\rho(E)})) \to \tilde{B}(E')$ is an isomorphism i $A(|\mathcal{L}|^{\wedge}_p)$. So we conclude that $\tilde{B}(-)$ is an equivalence of categories.

Let $(E, f) \in A(|\mathcal{L}|^{\wedge}_p)$. Then $H^*(f)\colon \mathrm{H}^*(|\mathcal{L}|^{\wedge}_p) \to \mathrm{H}^*(BE)$ induces a finitely generated module structure, so the same holds for $\mathrm{H}^*(f) \circ (\phi^*_{|\mathcal{L}|})^{-1}\colon \mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(BE)$. Hence if we set $\mathrm{H}^*(E, f) = (E, \mathrm{H}^*(f) \circ (\phi^*_{|\mathcal{L}|})^{-1})$ then $\mathrm{H}^*(E, f) \in A(\mathrm{H}^*(|\mathcal{L}|))$. Since any $\varphi \in \mathrm{Mor}_{A(|\mathcal{L}|^{\wedge}_p)}((E, f), (E', f'))$ is injective and satisfies $f' \circ B\rho \simeq f$, applying H^* gives that $\varphi^* \circ \mathrm{H}^*(f') = \mathrm{H}^*(f)$, so $\mathrm{H}^*(\varphi) := \varphi$ is also a morphism from $\mathrm{H}^*(E, f)$ to $\mathrm{H}^*(E', f')$ in $A(\mathrm{H}^*(|\mathcal{L}|))$. Hence we have a functor $\mathrm{H}^*(-)\colon A(|\mathcal{L}|^{\wedge}_p) \to A(\mathrm{H}^*(|\mathcal{L}|))$ by the above definitions. As $\mathcal{L}$ only has finitely many objects and each morphism set is finite, it follows from cellular cohomology that $\mathrm{H}^*(|\mathcal{L}|)$ is finite dimensional in each degree. By [26, Theorem 3.1.1.] the map $\mathrm{H}^*(-) \circ (\phi^*_{|\mathcal{L}|})^{-1}$ induces a bijection from $[BE, |\mathcal{L}|^{\wedge}_p]$ to $\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(|\mathcal{L}|), \mathrm{H}^*(BE))$. Then for $(E, f), (E', f') \in A(|\mathcal{L}|^{\wedge}_p)$ we have bijections

$$\begin{aligned}
&\mathrm{Mor}_{A(\mathrm{H}^*(|\mathcal{L}|))}(\mathrm{H}^*(E, f), \mathrm{H}^*(E', f')) \\
&\qquad =\{\varphi \in \mathrm{Inj}(E, E') \mid \varphi^* \circ \mathrm{H}^*(f') \circ (\phi^*_{|\mathcal{L}|})^{-1} = \mathrm{H}^*(f) \circ (\phi^*_{|\mathcal{L}|})^{-1}\} \\
&\qquad =\{\varphi \in \mathrm{Inj}(E, E') \mid f' \circ B\varphi \simeq f\} \\
&\qquad = \mathrm{Mor}_{A(|\mathcal{L}|^{\wedge}_p)}((E, f), (E', f')).
\end{aligned}$$

Consider a $(E, f) \in A(\mathrm{H}^*(|\mathcal{L}|))$. The bijection also ensures that there exists an $[\tilde{f}] \in [BE; |\mathcal{L}|^{\wedge}_p]$, such that $H^*(f) \circ (\phi^*_{|\mathcal{L}|})^{-1} = \tilde{f}$. As $\phi^*_{|\mathcal{L}|}$ is an isomorphism, the assumptions on f imply that $(E, \tilde{f}) \in A(|\mathcal{L}|^{\wedge}_p)$. Hence $(E, f) = \mathrm{H}^*(E, \tilde{f})$. We therefore conclude that $\mathrm{H}^*\colon A(|\mathcal{L}|^{\wedge}_p) \to A(\mathrm{H}^*(|\mathcal{L}|))$ is an equivalence of categories.

The map $R_{\mathcal{L}}\colon \mathrm{H}^*(|\mathcal{L}|) \to \mathrm{H}^*(\mathcal{F})$ is a morphism of unstable algebras which induces a finite module structure. So it induces a functor $A(R_{\mathcal{L}})\colon A(\mathrm{H}^*(\mathcal{F})) \to A(\mathrm{H}^*(|\mathcal{L}|))$ by $(E, f) \mapsto (E, f \circ R_{\mathcal{L}})$ and the identity on morphisms. By Lemma 8.6 $R_{\mathcal{L}}$ is an F-isomorphism. By [27, Corollary 6.5.2] precomposition with $R_{\mathcal{L}}$ induces a bijection from $\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}), \mathrm{H}^*(E))$ to $\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(|\mathcal{L}|), \mathrm{H}^*(E))$ for all elementary abelian p-groups E. By arguments similar to the case of the functor $\mathrm{H}^*(-)$, we conclude that $A(R_{\mathcal{L}})$ also is an equivalence of categories.

Now we consider the composition

$$\mathcal{F}^e \xrightarrow{\beta} A(\mathrm{H}^*(\mathcal{F})) \xrightarrow{A(R_{\mathcal{L}})} A(\mathrm{H}^*(|\mathcal{L}|))$$

which sends $E \in \mathcal{F}^e$ to $(E, i_E^* \circ \iota \circ R_{\mathcal{L}})$ and is the identity on morphisms. As $\theta^* = \iota \circ R_{\mathcal{L}}$ it follows from the above definitions of functors that this composition agrees with $\mathrm{H}^*(-) \circ \tilde{B}(-) \colon \mathcal{F}^e \to A(\mathrm{H}^*(|\mathcal{L}|))$. As the composition of β with an equivalence of categories is itself an equivalence, we conclude that the same holds for β.

$\square$

Theorem 8.17. *For any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ the homomorphism*

$$R_{\mathcal{L}} \colon \mathrm{H}^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \to \mathrm{H}^*(\mathcal{F})$$

is an isomorphism and $\mathrm{H}^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$ *is a Noetherian ring.*

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. It follows from Proposition 8.5 that $\mathrm{H}^*(\mathcal{F})$ is Noetherian, so it is enough to show that $R_{\mathcal{L}}$ is an isomorphism.

Define a relation on p-local finite groups by $(S', \mathcal{F}', \mathcal{L}') < (S, \mathcal{F}, \mathcal{L})$ if $|S'| < |S|$ or if $\mathcal{F}'$ is a proper subcategory of $\mathcal{F}$ in the case $S' = S$. This relation is irreflexive. As S is a finite group and $\mathcal{F}$ only contains finitely many morphisms any descending chain ending in $(S, \mathcal{F}, \mathcal{L})$ is finite. Hence the relation is well-founded, so the induction principle applies.

The minimal element with respect to this relation is the p-local finite group $(1, \mathcal{F}_1, \mathcal{L}_1)$, where $\mathcal{F}_1$ is the category with $\mathrm{Ob}(\mathcal{F}_1) = \{1\}$ and $\mathrm{Mor}_{\mathcal{F}_1}(1,1) = \{\mathrm{id}_1\}$, and $\mathcal{L}_1 = \mathcal{F}_1$. Then $\mathcal{O}^c(\mathcal{F}_1) = \mathcal{F}_1$, so

$$\mathrm{H}^*(\mathcal{F}_1) = \varprojlim_{\mathcal{O}^c(\mathcal{F}_1)} \mathrm{H}^*(-; \mathbb{F}_p) = \mathrm{H}^*(B1; \mathbb{F}_p).$$

As $\mathcal{L}_1 = \mathcal{B}(1)$, we have that $\mathrm{H}^*(|\mathcal{L}_1|_p^\wedge; \mathbb{F}_p) = \mathrm{H}^*(B1_p^\wedge; \mathbb{F}_p)$. The map

$$R_{\mathcal{L}_1} \colon \mathrm{H}^*(|\mathcal{L}_1|_p^\wedge; \mathbb{F}_p) \to \mathrm{H}^*(\mathcal{F}_1)$$

is in this case ϕ_1^*, where $\phi_1 \colon B1 \to B1_p^\wedge$ is the map from the natural transformation $\mathrm{id} \to (-)_p^\wedge$. Since 1 is a p-group, we have that $B1$ is p-complete by [3, III 1.4 Proposition 1.10], i.e. the map ϕ_1 is a homotopy equivalence. Then $R_{\mathcal{L}_1} = \phi_1^*$ is an isomorphism.

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, and assume that $R_{\mathcal{L}'}$ is an isomorphism for any $(S', \mathcal{F}', \mathcal{L}') < (S, \mathcal{F}, \mathcal{L})$. The proof of $R_{\mathcal{L}}$ being an isomorphism splits into two cases.

First assume that there exists $1Q \subseteq \mathrm{Z}(S)$, such that Q is central in $\mathcal{F}$ and non-trivial. Let $V \subseteq Q$ be a subgroup of order p. Then $V \subseteq Q \subseteq \mathrm{Z}(S)$, so $C_S(V) = S$. Then $C_{\mathcal{F}}(V)$ and $\mathcal{F}$ are both fusion systems on S. Let $P, P' \subseteq S$ and $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, P')$. As Q is central in $\mathcal{F}$, there exists $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{F}}(PQ, P'Q)$ such that $\bar{\varphi}|_P = \varphi$ and $\bar{\varphi}|_Q = \mathrm{id}_Q$. The restriction $\tilde{\varphi} = \bar{\varphi}|_{PV} \in \mathrm{Mor}_{\mathcal{F}}(PV, P'V)$ then satisfies $\tilde{\varphi}|_P = \varphi$ and $\tilde{\varphi}|_V = \mathrm{id}_V$, so $\varphi \in \mathrm{Mor}_{C_{\mathcal{F}}(V)}(P, P')$. Hence $\mathcal{F} = C_{\mathcal{F}}(V)$, so V is central i $\mathcal{F}$. By Lemma 8.10 (a) $(S/V, \mathcal{F}/V, (\mathcal{L}/V)^c)$ is a p-local finite group. As $|S/V| < |S|$ the induction hypothesis implies that $R_{\mathcal{F}/V}$ is an isomorphism. Thus we conclude by Lemma 8.10 (d) that $R_{\mathcal{L}}$ is an isomorphism.

Now assume that $\mathcal{F}$ contains no non-trivial central subgroup. Consider a $Q \subseteq S$ which is fully centralized and non-trivial. Then the fusion system $C_{\mathcal{F}}(Q)$ is either a fusion system on $C_S(Q) < S$ or a proper subcategory of $\mathcal{F}$. Hence we have $(C_S(Q), C_{\mathcal{F}}(Q), C_{\mathcal{L}}(Q)) < (S, \mathcal{F}, \mathcal{L})$, so the induction hypothesis applies to this fusion system.

Recall that $\mathcal{F}^e$ is the subcategory of $\mathcal{F}$ on the non-trivial elementary abelian subgroups of S which are fully centralized in $\mathcal{F}$. For $E \in \mathcal{F}^e$ we have the forgetful functor from $\bar{C}_{\mathcal{L}}(E)$ to $\mathcal{L}$ given by $(P, \alpha) \mapsto P$. According to Corollary 5.11 these induce a homotopy equivalence $\operatorname{hocolim}_{E \in (\mathcal{F}^e)^{op}} |\bar{C}_{\mathcal{L}}(E)| \to |\mathcal{L}|$. Hence by [7, XII.4.5] there exists a spectral sequence E_r^{ij} with

$$E_2^{i*} = \varprojlim_{E \in (\mathcal{F}^e)^{op}}{}^{i} H^*(|\bar{C}_{\mathcal{L}}(E)|), \quad i \in \mathbb{Z}$$

converging to $H^*(|\mathcal{L}|)$. We will now prove that the zero column is the only non-zero column on the E_2-page and that this column is isomorphic to $\mathrm{H}^*(\mathcal{F})$.

For $(E, f) \in A(\mathrm{H}^*(\mathcal{F}))$ we have $f \in \operatorname{Mor}_{\mathcal{K}}(\mathrm{H}^*(\mathcal{F}), \mathrm{H}^*(BE))$, so we can consider component $T_E(\mathrm{H}^*(\mathcal{F}); f)$. Lannes T-functor is natural in E so a homomorphism $\varphi\colon E \to E'$ induces a map $T_\varphi\colon T_E(\mathrm{H}^*(\mathcal{F})) \to T_{E'}(\mathrm{H}^*(\mathcal{F}))$. By previous remarks we have that

$$T_\varphi(x_f) = \sum_{g \in \operatorname{Rep}_\varphi^{-1}(f)} x_g = x_{f'} + \sum_{g \in \operatorname{Rep}_\varphi^{-1}(f), g \neq f'} x_g,$$

and since $x_{f'} x_g = 0$ for $g \neq f'$ and $x_{f'} x_{f'} = x_{f'}$, we have that $T_\varphi(x_f) x_{f'} = x_{f'}$. We have a projection $T_{E'}(H^*(\mathcal{F})) \to T_{E'}(H^*(\mathcal{F}); f')$ given by $y \mapsto y x_{f'}$. The composition with T_φ on $T_E(H^*(\mathcal{F}); f)$ is then $y x_f \mapsto T_\varphi(y x_f) x_{f'} = T_\varphi(y) x_{f'}$. So T_φ induces a well-defined map $T_E(H^*(\mathcal{F}); f) \to T_{E'}(H^*(\mathcal{F}); f')$ by $y x_f \mapsto T_\varphi(y) x_{f'}$. Hence there is a functor $\alpha\colon A(\mathrm{H}^*(\mathcal{F})) \to \mathcal{K}$ given by $\alpha(E, f) = T_E(\mathrm{H}^*(\mathcal{F}); f)$ and $\alpha(\varphi) = T_\varphi$.

Let $\beta\colon \mathcal{F}^e \to A(\mathrm{H}^*(\mathcal{F}))$ be the equivalence of category from Lemma 8.16. The functor $\alpha \circ \beta\colon \mathcal{F}^e \to \mathcal{K}$ is given by $E \mapsto T_E(\mathrm{H}^*(\mathcal{F}); i_E^* \circ \iota)$. Consider $E \in \mathcal{F}^e$. By Lemma 8.15 there exists an isomorphism $\Phi_E\colon T_E(\mathrm{H}^*(\mathcal{F}); i_E^* \circ \iota) \to \mathrm{H}^*(C_{\mathcal{F}}(E))$. By the induction hypothesis $R_{C_{\mathcal{L}}(E)}\colon \mathrm{H}^*(|C_{\mathcal{L}}(E)|) \to \mathrm{H}^*(C_{\mathcal{F}}(E))$ is an isomorphism. Lemma 5.8 gives an isomorphism $\mathrm{H}^*(|F|)\colon \mathrm{H}^*(|C_{\mathcal{L}}(E)|) \to \mathrm{H}^*(|\bar{C}_{\mathcal{L}}(E)|)$. So the two functors $\alpha \circ \beta$ and $\mathrm{H}^*(|\bar{C}_{\mathcal{L}}(-)|)$ from $\mathcal{F}^e$ to $\mathcal{K}$ have isomorphic images for all objects. To see that these isomorphisms induce a natural isomorphism, we consider the following diagram (A), where $\varphi \in \operatorname{Hom}_{\mathcal{F}}(E, E')$:

$$\begin{tikzcd}
T_E(\mathrm{H}^*(\mathcal{F}); i_E^* \circ \iota) \arrow[r, "\Phi_E"] \arrow[d, "T_\varphi"'] & \mathrm{H}^*(C_{\mathcal{F}}(E)) \arrow[d, "C_S(\varphi)^*"'] & \mathrm{H}^*(|C_{\mathcal{L}}(E)|) \arrow[l, "R_{C_{\mathcal{L}}(E)}"'] \arrow[d, "\mathrm{H}^*(|C_{\mathcal{L}}(\varphi)|)"'] & \mathrm{H}^*(|\bar{C}_{\mathcal{L}}(E)|) \arrow[l, "\mathrm{H}^*(|F|)"'] \arrow[d, "\mathrm{H}^*(|\bar{C}_{\mathcal{L}}(\varphi)|)"'] \\
T_{E'}(\mathrm{H}^*(\mathcal{F}); i_{E'}^* \circ \iota) \arrow[r, "\Phi_{E'}"] & \mathrm{H}^*(C_{\mathcal{F}}(E')) & \mathrm{H}^*(|C_{\mathcal{L}}(E')|) \arrow[l, "R_{C_{\mathcal{L}}(E')}"'] & \mathrm{H}^*(|\bar{C}_{\mathcal{L}}(E')|) \arrow[l, "\mathrm{H}^*(|F|)"']
\end{tikzcd}$$

Here, $C_S(\varphi)\colon C_S(E') \to C_S(E)$ is the map from Lemma 5.1. We will now prove that all squares in the diagram commute and as all the horizontal maps are isomorphisms, this implies that $\alpha \circ \beta$ and $\mathrm{H}^*(|\bar{C}_{\mathcal{L}}(-)|)$ are isomorphic functors.

To show the first square commutes we consider the diagram (B)

$$\begin{tikzcd} T_E(\mathrm{H}^*(\mathcal{F})) \arrow[r, "\Phi_E"] \arrow[d, "T_\varphi"'] & \mathrm{H}^*(C_S(E)) \arrow[d, "C_S(\varphi)^*"] \\ T_{E'}(\mathrm{H}^*(\mathcal{F})) \arrow[r, "\Phi_{E'}"] & \mathrm{H}^*(C_S(E')) \end{tikzcd}$$

As the algebra map Φ_E on $T_E(\mathrm{H}^*(\mathcal{F}); f)$ is an isomorphism and $(yx_f)x_f = yx_f$ for $y \in \mathrm{H}^*(\mathcal{F})$, we have that $\Phi_E(x_f) \in \mathrm{H}^0(C_S(E))$ is the unit. The same is true for $\Phi_{E'}(x_{f'}) \in \mathrm{H}^0(C_S(E'))$. As $C_S(\varphi)^*$ is a ring morphism, we conclude $C_S(\varphi)^* \circ \Phi_E(x_f) = \Phi_{E'}(x_{f'})$. So from the definition of T_φ on the components, we conclude that it is sufficient to show that diagram (B) is commutative.

For this we consider the adjoint maps $\mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(E) \otimes H^*(C_S(E')$. By the definition of Φ_E the adjoint map corresponding to the upper right triangle is induced by the group-homomorphism

$$E \times C_S(E') \xrightarrow{\mathrm{id} \times C_S(\varphi)} E \times C_S(E) \xrightarrow{\mathrm{mult}} S$$

and post composed with $\iota\colon \mathrm{H}^*(\mathcal{F}) \to \mathcal{F}^*(S)$. As the adjunction is natural in E the adjoint to $\Phi_{E'} \circ T_\varphi$ is the adjoint to $\Phi_{E'}$ composed with $\varphi^* \otimes \mathrm{id}$. Hence the adjoint map corresponding to the lower triangle is induced by

$$E \times C_S(E') \xrightarrow{\varphi \times \mathrm{id}} E' \times C_S(E') \xrightarrow{\mathrm{mult}} S$$

and afterward composed with $\iota\colon \mathrm{H}^*(\mathcal{F}) \to \mathcal{F}^*(S)$. If $\varphi\colon E \to E'$ is an inclusion, it follows that the same is true for $C_S(\varphi)\colon C_S(E') \to C_S(E)$. Then the two group homomorphisms agree and hence induce the same map on cohomology. In the case where φ is an isomorphism, we have that $C_S(\varphi)|_{E'} = \varphi^{-1}$, so the diagram

$$\begin{tikzcd} E \times C_S(E') \arrow[r, "\varphi \times \mathrm{id}"] \arrow[d, "\mathrm{id}"'] & E' \times C_S(E') \arrow[r, "\mathrm{mult}"] & C_S(E') \arrow[r, "i_{C_S(E')}"] \arrow[d, "C_S(\varphi)"'] & S \\ E \times C_S(E') \arrow[r, "\mathrm{id} \times C_S(\varphi)"] & E \times C_S(E) \arrow[r, "\mathrm{mult}"] & C_S(E) \arrow[r, "i_{C_S(E)}"] & S. \end{tikzcd}$$

commutes. This implies that the second square in the diagram

$$\begin{tikzcd} \mathrm{H}^*(\mathcal{F}) \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}^*(S) \arrow[r, "i^*_{C_S(E)}"] & \mathrm{H}^*(C_S(E)) \arrow[r, "(\mathrm{id} \otimes C_S(\varphi)^*) \circ \mathrm{mult}^*"] \arrow[d, "C_S(\varphi)^*"'] & \mathrm{H}^*(E) \otimes \mathrm{H}^*(C_S(E')) \arrow[d, "\mathrm{id}"'] \\ \mathrm{H}^*(\mathcal{F}) \arrow[r, "\iota"] & \mathrm{H}^*(S) \arrow[r, "i^*_{C_S(E')}"] & \mathrm{H}^*(C_S(E')) \arrow[r, "(\varphi^* \otimes \mathrm{id}) \circ \mathrm{mult}^*"] & \mathrm{H}^*(E) \otimes \mathrm{H}^*(C_S(E')) \end{tikzcd}$$

commutes. The first square commutes by definition of $\mathrm{H}^*(\mathcal{F})$ as both inclusions and the map $C_S(\varphi)$ are $\mathcal{F}$-morphisms. So the two horizontal maps agree, and they are exactly the adjoint maps corresponding to the original diagram. As a map in $\mathcal{F}^e$ is a composition of an isomorphism and an inclusion, we have that the diagram (B) commutes for any morphism, and hence the first square of (A) is commutative.

In case of the second square of (A) pick a $P \subseteq C_S(E')$. Then the maps are induced by two functors $\mathcal{B}P \to C_{\mathcal{L}}(E)$. In both cases the object of $\mathcal{B}P$ is mapped to $C_S(E)$. A $p \in P$ is mapped to $\delta_{C_S(E)}(C_S(\varphi)(p))$ respectively $\beta_{C_S(E')} \circ \delta_{C_S(E')} \circ \beta^{-1}_{C_S(E')}$. As $\pi(\beta_{C_S(E')}) = C_S(\varphi)$ it follows from property (C) for the central linking system $\mathcal{L}$ that the two morphisms are equal. So the two functors agree and hence the induced maps in (A) commute.

For the third square of (A) the maps are induced by functors from $C_{\mathcal{L}}(E') \to \bar{C}_{\mathcal{L}}(E)$. On the objects they are given by $P \mapsto (C_S(\varphi)(P), \iota_{E \hookrightarrow \mathrm{Z}(C_S(\varphi)(P))})$ and $P \mapsto (P, \varphi)$. It follows easily from the definition of the functors, that $\beta_P \in \mathrm{Mor}_{\mathcal{L}}(P, C_S(\varphi)(P))$ for $P \in C_{\mathcal{L}}(E')$ gives rise to a natural isomorphism of the two functors. Thus their geometric realizations are homotopic, which implies the commutativity of the third square.

We conclude that the functors $\alpha \circ \beta$ and $H^*(|\bar{C}_{\mathcal{L}}(-)|)$ from $\mathcal{F}^e$ to $\mathcal{K}$ are isomorphic. So the induced map $\varprojlim^i_{E \in (\mathcal{F}^e)^{op}} H^*(|\bar{C}_{\mathcal{L}}(E)|)$ to $\varprojlim^i_{E \in (\mathcal{F}^e)^{op}} \alpha \circ \beta(E)$ is an isomorphism for any $i \in \mathbb{Z}$. As $\beta\colon \mathcal{F}^e \to A(\mathrm{H}^*(\mathcal{F}))$ is an equivalence of categories by Lemma 8.16, it induces an isomorphism $\varprojlim^i_{A(\mathrm{H}^*(\mathcal{F})))^{op}} \alpha \to \varprojlim^i_{E \in (\mathcal{F}^e)^{op}} \alpha \circ \beta$ for any $i \in \mathbb{Z}$. So we have proven that the i'th column in the E_2-page for the spectral sequence for $|\mathcal{L}|$ is isomorphic to $\varprojlim^i_{A(\mathrm{H}^*(\mathcal{F})))^{op}} \alpha$.

To identify the limits of the functor α, we consider the map $\iota\colon \mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(BS)$ of $\mathcal{K}$-algebras. By Proposition 8.5 the map makes $\mathrm{H}^*(BS)$ into a finitely generated $\mathrm{H}^*(\mathcal{F})$-module. As S is a finite p-group, we have that $\mathrm{H}^*(BS)$ is a finitely generated $\mathbb{F}_p$- algebra, so by the inclusion of rings $\mathbb{F}_p \subseteq \mathrm{H}^*(\mathcal{F}) \subseteq \mathrm{H}^*(BS)$ we get that $\mathrm{H}^*(\mathcal{F})$ is a finitely generated $\mathbb{F}_p$-algebra. The map $f_\Omega\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(\mathcal{F})$ given by Proposition 8.8 is both a morphism of $\mathrm{H}^*(\mathcal{F})$-modules and a morphism of unstable modules over the Steenrod algebra $\mathcal{A}_p$. Since it is idempotent, is it a left-inverse to the inclusion ι. The finite p-group S has nontrivial center. Choose $g \in \mathrm{Z}(S)$ of order p. Then the pair $(\mathbb{Z}/p, f) \in A(\mathrm{H}^*(BS))$, where $f\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(\mathbb{Z}/p)$ is induced by the group homomorphism $\mathbb{Z}/p \to S$ given by $1 \mapsto g$. This will be a monic central map in the sense of [14, Definition 4.1] and hence the algebra $\mathrm{H}^*(BS)$ has non-trivial center. Then $\iota\colon \mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(BS)$ satisfies the conditions of [14, Theorem 1.2], so the groups $\varprojlim^i_{A(\mathrm{H}^*(\mathcal{F})))^{op}} \alpha = 0$ for $i > 0$ and there is an isomorphism $\mathrm{H}^*(\mathcal{F}) \to \varprojlim_{A(\mathrm{H}^*(\mathcal{F})))^{op}} \alpha$ induced by the maps

$$\mathrm{H}^*(\mathcal{F}) = T_0(\mathrm{H}^*(\mathcal{F})) \xrightarrow{T_{0 \hookrightarrow E}} T_E(\mathrm{H}^*(\mathcal{F})) \xrightarrow{y \mapsto yx_f} T_E(\mathrm{H}^*(\mathcal{F}); f) = \alpha(E, f)$$

for $(E, f) \in A(\mathrm{H}^*(\mathcal{F}))$.

Hence only the zeroth column on the E_2-page for the spectral sequence is nonzero, so $E_\infty = E_2$. As the spectral sequence converges to $\mathrm{H}^*(|\mathcal{L}|)$ the zero-column is isomorphic to $\mathrm{H}^*(|\mathcal{L}|)$, and the isomorphism $\mathrm{H}^*(|\mathcal{L}|) \to \varprojlim^i_{(\mathcal{F}^e)^{op}} \mathrm{H}^* |\bar{C}_{\mathcal{F}}(-)|$ is induced by the forgetful functor. So by combining the the stated results, we see that $\mathrm{H}^*(|\mathcal{L}|)$ and $\mathrm{H}^*(\mathcal{F})$ are isomorphic. To see the resulting map is in fact $R_{\mathcal{L}}$, we will look more carefully at the maps involved.

Let $E \in \mathcal{F}^e$. Then the composite

$$\mathrm{H}^*(\mathcal{F}) \xrightarrow{T_{0 \hookrightarrow E}} \alpha \circ \beta(E) = T_E(\mathrm{H}^*(\mathcal{F}), i_E^* \iota) \xrightarrow{\Phi_E} \mathrm{H}^*(C_{\mathcal{F}}(E))$$

is the lower composition in the diagram (B) in the case where φ is the inclusion of 0 into E. As the argument only depends on φ being an $\mathcal{F}$-morphism, we conclude that

$$\begin{tikzcd} \mathrm{H}^*(\mathcal{F}) \arrow[r, "\Phi_0"] \arrow[d, "T_{0\hookrightarrow E}"'] & \mathrm{H}^*(BS) \arrow[d, "(i_{C_S(E)})^*"'] \\ T_E(\mathrm{H}^*(\mathcal{F})) \arrow[r, "\Phi_E"] & \mathrm{H}^*(C_S(E)) \end{tikzcd}$$

commutes. By definition Φ_0 is the adjoint to id_S, so it is the inclusion $\iota\colon \mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(BS)$. Hence the composition $\Phi_E \circ T_{0\hookrightarrow E}$ is the restriction of $i^*_{C_S(E)}\colon \mathrm{H}^*(BS) \to \mathrm{H}^*(C_S(E))$. Thus the map

$$\mathrm{H}^*(|\mathcal{L}|) \xrightarrow{R_{\mathcal{L}}} \mathrm{H}^*(\mathcal{F}) \xrightarrow{T_{0\hookrightarrow E}} T_E(\mathrm{H}^*(\mathcal{F}), i^*_E\iota) \xrightarrow{\Phi_E} \mathrm{H}^*(C_{\mathcal{F}}(E))$$

is equal to $i^*_{C_S(E)} \circ \iota \circ R_{\mathcal{L}} = i^*_{C_S(E)} \circ \mathrm{H}^*(|\theta|)$, which is to say that it is induced by the functor $\mathcal{B}(C_S(E)) \to \mathcal{L}$ given $o_{C_S(E)} \mapsto S$ and $g \mapsto \delta_S(g)$. Similarly the composition

$$\mathrm{H}^*(|\mathcal{L}|) \xrightarrow{\textit{forget}} \mathrm{H}^*(|\bar{C}_{\mathcal{L}}(E)|) \xrightarrow{\mathrm{H}^*(|F|)} \mathrm{H}^*(|C_{\mathcal{L}}(E)|) \xrightarrow{R_{C_{\mathcal{L}}(E)}} \mathrm{H}^*(C_{\mathcal{F}}(E))$$

is induced by the functor $\mathcal{B}(C_S(E)) \to \mathcal{L}$ given by $o_{C_S(E)} \mapsto C_S(E)$ and $g \mapsto \delta_{C_S(E)}(g)$. Let $\tilde{i}_{C_S(E)} \in \mathrm{Mor}_{\mathcal{L}}(C_S(E), S)$ be a lift of the inclusion. By property (C) for the linking system $\mathcal{L}$ this is in fact a natural transformation between the two functors $\mathcal{B}(C_S(E)) \to \mathcal{L}$. So they induce the same map in cohomology. Hence the induced maps on the limits agree making the following diagram commute.

$$\begin{tikzcd} \mathrm{H}^*(|\mathcal{L}|) \arrow[r, "R_{\mathcal{L}}"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}^*(\mathcal{F}) \arrow[r] & \varprojlim_{(\mathcal{F}^e)^{op}} \alpha\circ\beta \arrow[r] & \varprojlim_{(\mathcal{F}^e)^{op}} \mathrm{H}^*(C_{\mathcal{F}}(E)) \arrow[d, "\mathrm{id}"'] \\ \mathrm{H}^*(|\mathcal{L}|) \arrow[r] & \varprojlim_{(\mathcal{F}^e)^{op}} \mathrm{H}^*(|\bar{C}_{\mathcal{L}}(E)|) \arrow[r] & \varprojlim_{(\mathcal{F}^e)^{op}} \mathrm{H}^*(|C_{\mathcal{L}}(E)|) \arrow[r] & \varprojlim_{(\mathcal{F}^e)^{op}} \mathrm{H}^*(C_{\mathcal{F}}(E)) \end{tikzcd}$$

We remark that all maps but $R_{\mathcal{L}}$ are already shown to be isomorphism, so we conclude that $R_{\mathcal{L}}$ is an isomorphism. $\square$

9. A topological characterization of p-local finite groups.

In this chapter we will show the main theorem. Any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is up to isomorphism determined by the homotopy type of $|\mathcal{L}|_p^\wedge$. The principal step is to construct a isomorphic p-local finite group that depends nicely on the homotopy type of $|\mathcal{L}|_p^\wedge$.

Definition 9.1. *Let S be a finite p-group, X a space and $f\colon BS \to X$. We define $\mathcal{F}_{S,f}(X)$ to be the category with objects the subgroups of S and*

$$\mathrm{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) = \{\varphi \in \mathrm{Inj}(P, Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}.$$

Lemma 9.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $f = \phi_{|\mathcal{L}|} \circ |\theta_S|\colon BS \to |\mathcal{L}|_p^\wedge$, where ϕ the natural transformation from p-completion and $\theta\colon \mathcal{B}S \to \mathcal{L}$ is the functor from Definition 3.2.*

Define $\xi_\mathcal{F}\colon \mathcal{F} \to \mathcal{F}_{S,f}(|\mathcal{L}|_p^\wedge)$ to be the functor $\xi_\mathcal{F}(P) = P$ and $\xi_\mathcal{F}(\varphi) = \varphi$. Then $\xi_\mathcal{F}$ is well-defined and an isomorphism of categories, hence $\mathcal{F}_{S,f}(|\mathcal{L}|_p^\wedge)$ is a saturated fusion system over S.

Proof. Note that $\xi_\mathcal{F}$ is a bijection on the objects, so we only need to consider the morphism sets. Let P, Q be subgroups of S. As $f|_{BP} = f \circ Bi_P$, where $i_P\colon P \to S$ is the inclusion, we have by Theorem 7.4 that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{F}_{S,f}(|\mathcal{L}|_p^\wedge)}(P, Q) &= \{\varphi \in \mathrm{Inj}(P, Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\} \\
&= \{\varphi \in \mathrm{Inj}(P, Q) \mid f \circ Bi_P \simeq f \circ B(i_Q \circ \varphi)\} \\
&= \{\varphi \in \mathrm{Inj}(P, Q) \mid \exists \chi \in \mathrm{Hom}_\mathcal{F}(P, \varphi(P)), \chi \circ i_P = i_Q \circ \varphi\} \\
&= \{\varphi \in \mathrm{Inj}(P, Q) \mid \exists \varphi|_{P,\varphi(Q)} \in \mathrm{Hom}_\mathcal{F}(P, \varphi(P))\} \\
&= \mathrm{Hom}_\mathcal{F}(P, Q)
\end{aligned}$$

Note the last equation follows from axiom 2 for fusion systems. Then $\xi_\mathcal{F}$ is well-defined on morphisms and an isomorphism on the set of morphisms, so we conclude that $\xi_\mathcal{F}$ is a isomorphism of categories. As $\mathcal{F}_{S,f}(|\mathcal{L}|_p^\wedge)$ has the same objects and morphisms as $\mathcal{F}$, it trivially satisfies the conditions for a saturated fusion system over S. $\square$

The corresponding definition for central linking systems is the following:

Definition 9.3. *Let S be a finite p-group, X a space and $f\colon BS \to X$ We define $\mathcal{L}_{S,f}(X)$ to be the category with objects the $\mathcal{F}_{S,f}(X)$-centric subgroups of S and $\mathrm{Mor}_{\mathcal{L}_{S,f}(X)}(P, Q)$ be to the set*

$$\{(\varphi, [H]) \mid \varphi \in \mathrm{Hom}_\mathcal{F}(P, Q), [H] \in \mathrm{Mor}_{\pi(\mathrm{Map}(BP,X))}(f|_{BP}, f|_{BQ} \circ B\varphi)\}.$$

with composition defined as

$$(\varphi, [H])(\varphi, [H']) = (\varphi' \circ \varphi, [(H' \circ B\varphi)H])$$

where we use the standard composition in $\pi(\mathrm{Map}(BP, |\mathcal{L}|_p^\wedge))$, and $H' \circ B\varphi\colon BP \times I \to |\mathcal{L}|_p^\wedge$ is the map $(x, t) \mapsto H(B\varphi(x), t)$.

Observe that follows easily from the definition that $\mathcal{L}_{S,f}(|\mathcal{L}|_p^\wedge)$ is in fact a category with identity object $(1_P, [H_P])$, where H_P is the constant path at $f|_{BP}$.

Let P, Q be $\mathcal{F}$-centric subgroups of S and $\varphi \in \mathrm{Mor}_\mathcal{L}(P, Q)$. Axiom (C) for the central linking system $\mathcal{L}$ implies that there is a natural transformation η_φ from $\theta_P\colon\ P \to \mathcal{L}$ to $\theta_Q \circ \pi(\varphi)\colon \mathcal{B}P \to \mathcal{L}$ given by $\eta_\varphi(o_P) = \varphi$. Then $|\theta_P|$ and

$|\theta_Q| \circ B\pi(\varphi)$ are homotopic by a homotopy given by η_φ [Proposition 2.1][33]. We denote this $|\eta_\varphi|$. For any $\varphi \in \operatorname{Mor}_{\mathcal{L}}(P,Q)$ and $\psi \in \operatorname{Mor}_{\mathcal{L}}(Q,R)$ we have that $\eta_{\psi\varphi} = (\eta_\psi \circ \mathcal{B}\pi(\varphi)) \circ \eta_\varphi$, so $|\eta_{\psi\varphi}| = (|\eta_\psi| \circ B\pi(\varphi))|\eta_\varphi|$ where we use the standard way of composing homotopies. Furthermore for an $\mathcal{F}$-centric P, we have that $\eta_{1_P} = \mathrm{id}_{\mathcal{B}P}$, so $|\eta_{1_P}|$ is the constant path at $|\theta_P|$.

Note that this implies that $|\theta_P| \simeq |\theta_S| \circ Bi_P$ for any $\mathcal{F}$-centric subgroup P. By choosing a fixed lift $\iota_P \in \operatorname{Mor}_{\mathcal{L}}(P,S)$ of the inclusion, we get a homotopy $|\eta_{\iota_P}|$ between between these maps, so if we set $|\tilde{\eta}_\varphi| = \phi_{|\mathcal{L}|}(|\eta_{\iota_Q}| \circ B(\pi(\varphi)))|\eta_\varphi||\eta_{\iota_P}|^{-1})$ we have a homotopy between $f|_{BP}$ and $f|_{BQ} \circ B\varphi$, where f is the map from Lemma 9.2. Since we are using a fixed homotopy for every $\mathcal{F}$-centric subgroup P we still have that $|\tilde{\eta}_{\psi\varphi}| = (|\tilde{\eta}_\psi| \circ B\pi(\varphi))|\tilde{\eta}_\varphi|$ for any $\varphi \in \operatorname{Mor}_{\mathcal{L}}(P,Q)$ and $\psi \in \operatorname{Mor}_{\mathcal{L}}(Q,R)$ and that $|\tilde{\eta}_{1_P}|$ is homotopic to the constant path at $f|_{BP}$. If we furthermore assume that $\iota_S = 1_S$, then $|\tilde{\eta}_\varphi| = |\eta_\varphi|$ for any $\varphi \in \operatorname{Mor}_{\mathcal{L}}(S,S)$.

Proposition 9.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. Set $f\colon BS \to |\mathcal{L}|^\wedge_p$ to be the map from Lemma 9.2 and let $\xi_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)$ be given by $\xi_{\mathcal{L}}(P) = (P)$ for any $\mathcal{F}$-centric subgroup P and $\xi_{\mathcal{L}}(\varphi) = (\pi(\varphi), [|\tilde{\eta}_\varphi|])$ for any $\varphi \in \operatorname{Mor}_{\mathcal{L}}(P,Q)$. Then $\xi_{\mathcal{L}}$ is an isomorphism of categories. Furthermore if we let $\pi'\colon \mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p) \to \mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p)$ be given by $\pi'(P) = P$ and $\pi'(\varphi,[H]) = \varphi$, and for any $\mathcal{F}$-centric P we let $\delta'_P\colon P \to \operatorname{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P)$ be given by $\delta'_P(g) = (c_g, [|\tilde{\eta}_{\delta_P(g)}|])$, then $\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)$ is a central linking system associated to $\mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p)$. Furthermore the triple $(\mathrm{id}_S, \xi_{\mathcal{F}}, \xi_{\mathcal{L}})$ is an isomorphism of the p-local finite groups $(S,\mathcal{F},\mathcal{L})$ and $(S, \mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p), \mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p))$.*

Proof. For any $\varphi \in \operatorname{Mor}_{\mathcal{L}}(P,Q)$ and $\psi \in \operatorname{Mor}_{\mathcal{L}}(Q,R)$ we have that

$$\xi_{\mathcal{L}}(\psi\varphi) = (\pi(\psi\varphi), [|\tilde{\eta}_{\psi\varphi}|]) = (\pi(\psi) \circ \pi(\varphi), [|\tilde{\eta}_\psi \circ B\varphi||\tilde{\eta}_\varphi|]) = \xi_{\mathcal{L}}(\psi) \circ \xi_{\mathcal{L}}(\varphi),$$

so $\xi_{\mathcal{L}}$ respects the composition. As for any centric P we have that $[|\eta_{1_P}|]$ contains the constant path at $f|_{BP}$, we also see that $\xi_L(1_P) = (\pi(1_P), [|\tilde{\eta}_{1_P}|])$ is the unit element of $\operatorname{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,P)$. Hence $\xi_{\mathcal{L}}$ is a well-defined functor. Let P,Q be $\mathcal{F}$-centric subgroups of S. With the given definitions the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Mor}_{\mathcal{L}}(P,Q) \arrow[r, "\xi_{\mathcal{L}}"] \arrow[d, "\pi"'] & \operatorname{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q) \arrow[d, "\pi'"'] \\
\operatorname{Hom}_{\mathcal{F}}(P,Q) \arrow[r, "\xi_{\mathcal{F}}"] & \operatorname{Mor}_{\mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q)
\end{tikzcd}$$

If for some $(\varphi,[H]),(\varphi',[H']) \in \operatorname{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q)$ we have that $\pi'(\varphi,[H]) = \pi(\varphi',[H'])$, then $\varphi = \varphi'$ and $[H] = [H'][\tilde{H}]$, where $[\tilde{H}] \in \pi_1(\operatorname{Map}(BP, |\mathcal{L}|^\wedge_p)_{f|_{BP}}$. By Theorem 7.4 we have that $\operatorname{Map}(BP, |\mathcal{L}|^\wedge_p)_{f|_{BP}}$ is homotopy equivalent to $B\,\mathrm{Z}(P)$. The map inducing the homotopy $B\,\mathrm{Z}(P) \times BP \to |\mathcal{L}|^\wedge_p$ sends $g \in \mathrm{Z}(P)$ considered as the 1-simplex $g \times o_P$ to $\phi_{|\mathcal{L}|}\delta_S(g)$. By property (C) for the central linking system, we have that $\delta_S(g) \circ \iota_P = \iota_P \circ \delta_P(g)$, where $\iota_P \in \operatorname{Mor}_{\mathcal{L}}(P,S)$ is the chosen lift of the inclusion. The map $I \times BP \to |\mathcal{L}|$ induced by $g \in \mathrm{Z}(P)$ is then $|\eta_{\iota_P}||\eta_{\delta_P(g)}||\eta_{\iota_P}|^{-1}$. So the induced map $I \times BP \to |\mathcal{L}|^\wedge_p$ is exactly $|\tilde{\eta}_{\delta_P(g)}|$. Hence the map of homotopy groups $\mathrm{Z}(P) \to \pi_1(\operatorname{Map}(BP, |\mathcal{L}|^\wedge_p)_{f|_{BP}})$ is then $g \mapsto |\tilde{\eta}_{\delta_P(g)}|$. As the map is a homotopy equivalence, this is an isomorphism. Hence there exists $g \in \mathrm{Z}(P)$ such that $[\tilde{H}] = [|\tilde{\eta}_{\delta_P(g)}|]$. As $g \in Z(P)$ we have that $\delta'_P(g) = (\mathrm{id}_P, [\tilde{H}])$, and we get that $(\varphi,[H]) = (\varphi',[H']) \circ \delta'_P(g)$. Note that $\pi'(\delta'_P(g)) = \mathrm{id}_P$ for any $g \in \mathrm{Z}(P)$, so as π' respects the composition, we conclude that π' is the orbit map for the $\mathrm{Z}(P)$-action on $\mathrm{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q)$ induced by δ'_P. Observe that if $(\varphi,[H]) = (\varphi,[H]) \circ \delta'_P(g)$ for some $g \in \mathrm{Z}(P)$, then the above implies that $|\tilde{\eta}_{\delta_P(g)}| = |\tilde{\eta}_{1_P}|$ in $\pi_1(\mathrm{Map}(BP,|\mathcal{L}|^\wedge_p)_{f|_{BP}})$ and thus $g = 1$. Hence the $\mathrm{Z}(P)$-action on $\mathrm{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q)$ is free. Furthermore we have that the $\mathrm{Z}(P)$-action on $\mathrm{Mor}_{\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)}(P,Q)$ is defined in terms of the $\mathrm{Z}(P)$-action on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ via $\xi_{\mathcal{L}}$, so $\xi_{\mathcal{L}}$ is a $\mathrm{Z}(P)$-map. The commutativity of the above diagram in connection with the lower horizontal map being a bijection and the vertical maps are orbit maps with respect to the $\mathrm{Z}(P)$-action, now implies that $\xi_{\mathcal{L}}$ is a bijection as well. Thus $\xi_{\mathcal{L}}$ is an isomorphism of categories.

As $\xi_{\mathcal{L}}$ is a bijection on morphisms and $\delta'_P(g) = \xi_{\mathcal{L}}(\delta_P(g))$ for any $\mathcal{F}$-centric subgroup P and $g \in P$, we have that δ_P being injective implies that δ'_P is injective as well. Likewise the property (C) for $\mathcal{L}$ implies the corresponding property for $\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)$. In the previous paragraph we have already proved that $\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)$ satisfies property (A) and (B) for central linking systems, so $\mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p)$ is a central linking system associated to $\mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p)$. Furthermore we have that

$$(\mathrm{id}_S, \xi_{\mathcal{F}}, \xi_{\mathcal{L}})\colon (S,\mathcal{F},\mathcal{L}) \to (S, \mathcal{F}_{S,f}(|\mathcal{L}|^\wedge_p), \mathcal{L}_{S,f}(|\mathcal{L}|^\wedge_p))$$

are isomorphisms that agree on the subgroups of S. The above commutative diagram implies that this triple commutes with the projections on the linking systems and by definition $\delta'_P = \xi_{\mathcal{L}} \circ \delta_P$. So we conclude that this is in fact a isomorphism of p-local finite groups. □

Theorem 9.5. *Let $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ be p-local finite groups. Then $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are isomorphic if and only if $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$.*

Proof. If $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are isomorphic as p-local finite groups, then there exists a isomorphism of categories from $\mathcal{L}$ to $\mathcal{L}$. Then [3, Corollary 2.2 (b)] implies that $|\mathcal{L}| \simeq |\mathcal{L}'|$. As a homotopy equivalence is a mod-p-equivalence, we have by [7, Lemma I.5.5] that $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$.

Conversely assume that $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$, and let $g\colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}'|^\wedge_p$ be a homotopy equivalence. Let $f_S = \phi_{|\mathcal{L}|} \circ |\theta_S|\colon BS \to |\mathcal{L}|^\wedge_p$ and $f_{S'} = \phi_{|\mathcal{L}|} \circ |\theta_{S'}|\colon BS' \to |\mathcal{L}'|^\wedge_p$, where ψ is the natural transformation from p-completion. Then $g \circ f_S\colon BS \to |\mathcal{L}'|^\wedge_p$, so by Theorem 7.4 we have that $g \circ f_S \simeq f_{S'} \circ B\rho$ for some $\rho \in \mathrm{Hom}(S,S')$. Let $g'\colon |\mathcal{L}'|^\wedge_p \to |\mathcal{L}|^\wedge_p$ be a homotopy inverse to g. Similarly there exists some $\rho' \in \mathrm{Hom}(S',S)$ such that $g' \circ f_{S'} \simeq f_S \circ B\rho'$. Now

$$f_S \simeq g' \circ g \circ f_S \simeq g' \circ f_{S'} \circ B\rho \simeq f_S \circ B(\rho'\rho).$$

Using Theorem 7.4 (b) we conclude that $\rho'\rho \in \mathrm{Hom}_{\mathcal{F}}(S, \rho\rho'(S))$. Thus we have that $\rho'\rho$ is injective, so ρ is injective. Similarly we conclude that ρ' is injective. As both S and S' are finite groups, we get that $|S| = |S'|$ and the injective map ρ is in fact an isomorphism.

We now define a map $\rho_{\mathcal{F}}\colon \mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p) \to \mathcal{F}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)$ by $\rho_{\mathcal{F}}(P) = \rho(P)$ for any subgroup $P \subseteq S$ and for $\varphi \in \mathrm{Mor}_{\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)}(P,Q)$ we set $\rho_{\mathcal{F}}(\varphi) = \rho|_Q \circ \varphi \circ \rho^{-1}|_{\rho(P)}$. For any $\varphi \in \mathrm{Hom}(P,Q)$ we have that

$$\begin{aligned} f_S|_{BP} \simeq f_S|_{BQ} \circ B\varphi &\Longrightarrow g \circ f_S|_{BP} \simeq g \circ f_S|_{BQ} \circ B\varphi \\ &\Longrightarrow f_{S'}|_{B\rho(P)} \circ B\rho|_{BP} \simeq f_{S'}|_{B\rho(Q)} \circ B\rho|_{BQ} \circ B\varphi \end{aligned}$$

We see that for $\varphi \in \mathrm{Mor}_{\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)}(P,Q)$ we have $\rho_{\mathcal{F}}(\varphi)$ is an element of the set $\mathrm{Mor}_{\mathcal{F}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)}(\rho(P),\rho(Q))$. Hence $\rho_{\mathcal{F}}$ is well-defined. It clearly is a functor, which is bijective on objects and injective on the morphism-set. The similar construction with ρ' is injective on the morphism-sets as well. Since these sets are finite, they must have the same number of elements. Hence $\rho_{\mathcal{F}}$ is a bijection on the morphism-sets and thus an isomorphism of categories.

To construct a functor from $\mathcal{L}_{S,f_S}(|\mathcal{L}|^\wedge_p)$ to $\mathcal{L}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)$ we choose a homotopy H_ρ from $g \circ f_S$ to $f_{S'} \circ B\rho$. We set $\rho_{\mathcal{L}} \colon \mathcal{L}_{S,f_S}(|\mathcal{L}|^\wedge_p) \to \mathcal{L}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)$ to be $\rho_{\mathcal{L}}(P) = \rho(P)$ for any $\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)$-centric subgroup of S, and for $(\varphi,[H]) \in \mathrm{Mor}_{\mathcal{L}_{S,f_S}(|\mathcal{L}|^\wedge_p)}(P,Q)$ we set $\rho_{\mathcal{L}}(\varphi,[H])$ to

$$(\rho|_Q \circ \varphi \circ \rho^{-1}|_{\rho(P)}, [(H_\rho \circ B(\varphi\rho^{-1})|_{B\rho(P)})(g \circ H \circ B\rho^{-1})(H_\rho^{-1} \circ B\rho^{-1}|_{\rho(P)})]).$$

As $\rho_{\mathcal{F}}$ is an isomorphism of categories and $C_{S'}(\rho(P)) = \rho(C_S(P))$ we conclude that for any $\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)$-centric subgroup $P \subseteq S$ we have that $\rho(P)$ is an $\mathcal{F}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)$-centric subgroup of S'. We conclude that $\rho_{\mathcal{L}}$ is a well-defined map and it is straightforward to see that it respects composition and sends the unit to the unit element, and thus is a functor.

We now want to consider $\rho_{\mathcal{L}}(c_h, |\tilde{\eta}_{\delta_S(h)}|)$ for any $h \in S$. Observe that $\rho \circ c_h \circ \rho^{-1} = c_{\rho(h)}$, so the first factor of $\rho_{\mathcal{L}}(c_h, |\tilde{\eta}_{\delta_S(h)}|)$ and $(c_{\rho(h)}, |\tilde{\eta}_{\delta_{S'}(\rho(h))}|)$ agrees. We will now show that in fact they are the same element. Consider the natural transformation χ_h between the functor $\mathrm{id}_{\mathcal{B}S}$ and $\mathcal{B}(c_h)$ on $\mathcal{B}S$ by setting $\chi_h(o_S) = h$. The map $|\chi_h| \colon BS \times I \to BS$ satisfies $|\chi|(x,0) = x$ and $|\chi_h|(x,1) = Bc_g(x)$. Furthermore $\theta_S \circ \chi_h = \eta_{\delta_S(h)}$, so $f_S \circ |\chi_h| = |\tilde{\eta}_{\delta_S(h)}|$. Note similarly for $h' \in S'$ we have that $f_{S'} \circ |\chi_{h'}| = |\tilde{\eta}_{\delta_{S'}(h')}|$ Consider $F \colon BS' \times I \times I \to |\mathcal{L}'|^\wedge_p$ given by $F(x,s,t) = H_\rho(|\chi_h|(B\rho^{-1}(x),t),s)$ Then we get a map with the property that

$$\begin{aligned}
&F(x,s,0) = H_\rho(B\rho^{-1}(x),s)\\
&F(x,1,t) = f_{S'} \circ B\rho(|\chi_h|(B\rho^{-1}(x),t)) = f_{S'}(|\chi_{\rho(h)}|(x,t)) = |\tilde{\eta}_{\delta_{S'}(\rho(h))}|(x,t)\\
&F(x,0,t) = g \circ f_S(|\chi_h|(B\rho^{-1}(x),t)) = g \circ |\tilde{\eta}_{\delta_S(h)}|(B\rho^{-1}(x),t)\\
&F(x,s,1) = H_\rho(B(c_h \circ \rho^{-1})(x),s)
\end{aligned}$$

By a reparametrization we get a homotopy between

$$(H_\rho \circ B(c_h \circ \rho^{-1}))(g \circ |\tilde{\eta}_{\delta_S(h)}| \circ B\rho^{-1})(H_\rho^{-1} \circ B\rho^{-1})$$

and $|\tilde{\eta}_{\delta_{S'}}(\rho(h))|$, which is constant on $BS' \times \{0,1\}$. With our definitions we conclude that $\rho_{\mathcal{L}}(c_h, |\tilde{\eta}_{\delta_S(h)}|) = (c_{\rho(h)}, |\tilde{\eta}_{\delta_{S'}(\rho(h))}|)$. For a general $\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)$-centric subgroup $P \subseteq S$ and $h \in P$ we have that

$$|\tilde{\eta}_{\delta_P(h)}| = \psi_{|\mathcal{L}|} \circ (|\eta_{\iota_P}| \circ B(c_h|_{BP}))|\eta_{\delta_P(h)}||\eta_{\iota_P}|^{-1}) = \psi_{|\mathcal{L}|} \circ |\eta_{\iota_P \circ \delta_P(h)}||\eta_{\iota_P}|^{-1}$$

By property (C) for $\mathcal{L}$, we have that $\iota_P \circ \delta_P(h) = \delta_S(h) \circ \iota_P$, so we see that $|\tilde{\eta}_{\delta_P(h)}| = (\psi_{|\mathcal{L}|} \circ |\eta_{\delta_S(h)}|)|_{BP} = (|\tilde{\eta}_{\delta_S(h)}|)|_{BP}$. Similar results hold for $(S',\mathcal{F}',\mathcal{L}')$ and thus $\rho_{\mathcal{L}}(c_h, |\tilde{\eta}_{\delta_P(h)}|) = (c_{\rho(h)}, |\tilde{\eta}_{\delta_{\rho(P)}(\rho(h))}|)$ for all $\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)$-centric subgroups $P \subseteq S$ and $h \in P$. Let $\pi_S, \pi_{S'}$ be projection map corresponding to π' in Proposition

9.4. Then the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{L}_{S,f_S}(|\mathcal{L}|^\wedge_p)}(P,Q) & \xrightarrow{\rho_{\mathcal{L}}} & \mathrm{Mor}_{\mathcal{L}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)}(\rho(P),\rho(Q)) \\ \Big\downarrow{\scriptstyle \pi_S} & & \Big\downarrow{\scriptstyle \pi_{S'}} \\ \mathrm{Hom}_{\mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p)}(P,Q) & \xrightarrow{\rho_{\mathcal{F}}} & \mathrm{Hom}_{\mathcal{F}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p)}(\rho(P),\rho(Q)) \end{array}$$

commutes for any pair of $\mathcal{F}$-centric subgroups P,Q. Furthermore π_S is the orbit map for the free $\mathrm{Z}(P)$-action induced by $\delta'_P(h) = (c_h, |\tilde{\eta}_{\delta_P(h)}|)$ while $\pi_{S'}$ is the orbit map for the free $\mathrm{Z}(\rho(P))$-action induced by $\delta'_{\rho(P)}(h) = (c_h, |\tilde{\eta}_{\delta_{\rho(P)(h)}}|)$. As ρ is an isomorphism, we have that $\mathrm{Z}(\rho(P)) = \rho(\mathrm{Z}(P))$. For $h \in \mathrm{Z}(P)$ we have that $\delta'_{\rho(P)}(\rho(h)) = \rho_{\mathcal{L}}(\delta'_P(h))$, so we may consider $\pi_{S'}$ is the orbit map for the free $\mathrm{Z}(P)$-action induced by $\rho_{\mathcal{L}}(\delta'_P(h))$. Then $\rho_{\mathcal{L}}$ is a $\mathrm{Z}(P)$-map and as $\rho_{\mathcal{F}}$ is a bijection, we conclude that $\rho_{\mathcal{L}}$ is a bijection as well. Hence $\rho_{\mathcal{L}}$ is an isomorphism of categories. As $\rho_{\mathcal{L}}$ respects the projections and $\delta'_{\rho(P)}(\rho(h)) = \rho_{\mathcal{L}}(\delta'_P(h))$, we conclude that $(\rho, \rho_{\mathcal{F}}, \rho_{\mathcal{L}})$ is a isomorphism between the p-local finite groups $(S, \mathcal{F}_{S,f_S}(|\mathcal{L}|^\wedge_p), \mathcal{L}_{S,f_S}(|\mathcal{L}|^\wedge_p))$ and $(S, \mathcal{F}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p), \mathcal{L}_{S',f_{S'}}(|\mathcal{L}'|^\wedge_p))$. Then by Proposition 9.4 we conclude that $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic. $\square$

Theorem 9.6. *For a topological space X have we have that $X \simeq |\mathcal{L}|^\wedge_p$ for some p-local finite group $(S,\mathcal{F},\mathcal{L})$ if and only if X is p-complete and there exists a p-group S and a map $BS \to X$ such that the following conditions hold.*

(a) *The $\mathcal{F}_{S,f}(X)$ is a saturated fusion system*
(b) *There exists a homotopy equivalence between X and $|\mathcal{L}_{S,f}(X)|^\wedge_p$.*
(c) *For every $\mathcal{F}_{S,f}(X)$-centric subgroup P the map*

$$f|_{BP} \circ (-)\colon \mathrm{Map}(BP,BP)_{\mathrm{id}} \to \mathrm{Map}(BP,X)_{f_{BP}}$$

is a homotopy equivalence.

When these conditions hold, we have that $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}(X))$ is a p-local finite group.

Proof. First assume that $X \simeq |\mathcal{L}|^\wedge_p$ for some p-local finite group $(S,\mathcal{F},\mathcal{L})$. Observe that the conditions (a)-(c) only depend on the homotopy type of X, we may assume that $X = |\mathcal{L}|^\wedge_p$. By Proposition 3.3 we have that $|\mathcal{L}|$ is p-good, so $|\mathcal{L}|^\wedge_p$ is p-complete. Let $f = \psi_{|\mathcal{L}|} \circ |\theta|\colon BS \to |\mathcal{L}|^\wedge_p$. By Lemma 9.2, we have that $\mathcal{F}$ and $\mathcal{F}_{S,f}(X)$ are isomorphic, and hence $\mathcal{F}_{S,f}(X)$ is a saturated fusion system. Similarly by Proposition 9.4 we have that $\mathcal{L}$ and $\mathcal{L}_{S,f}(X)$ are isomorphic categories and we see that $X = |\mathcal{L}|^\wedge_p \simeq |\mathcal{L}_{S,f}(X)|^\wedge_p$. Since F and $\mathcal{F}_{S,f}(X)$ are isomorphic, we conclude that $P \subseteq S$ is $\mathcal{F}$-centric if and only if it is $\mathcal{F}_{S,f}(X)$-centric. According to Theorem 7.4 (c) we have that the map $B\,\mathrm{Z}(P) \times BP \to |\mathcal{L}|^\wedge_p$ given by $(g,h) \mapsto f(gh)$ induces a homotopy equivalence $\Phi\colon B\,\mathrm{Z}(P) \to \mathrm{Map}(BP,X)_{f_{BP}}$. Note that the map agrees with

$$B\,\mathrm{Z}(P) \times BP \xrightarrow{\text{mult}} BP \xrightarrow{Bi_P} BS \xrightarrow{f} |\mathcal{L}|^\wedge_p$$

Thus the adjoint diagram commutes:

$$\begin{tikzcd} B\,\mathrm{Z}(P) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}"'] & \mathrm{Map}(BP, X)_{f_{BP}} \\ B\,\mathrm{Z}(P) \arrow[r] & \mathrm{Map}(BP, BP)_{\mathrm{id}} \arrow[u, "f|_{BP} \circ -"'] \end{tikzcd}$$

The upper horizontal map is as noted a homotopy equivalence, and the same is true for the lower horizontal map by [9, Proposition 2.1]. So by commutativity we conclude that $(f|_{BP} \circ -)$ is a homotopy equivalence as well.

Assume that X is a p-complete space, S a finite p-group and $f\colon BS \to X$ such that the conditions hold. Set $\mathcal{F} = \mathcal{F}_{S,f}(X)$ and $\mathcal{L} = \mathcal{L}_{S,f}(X)$. Then by (a) we have that $\mathcal{F}$ is a saturated fusion system over S, and $|\mathcal{L}|^\wedge_p \simeq X$ by (b). So we only need to prove that $\mathcal{L}$ is a central linking system associated to $\mathcal{F}$. By construction the objects of $\mathcal{L}$ are the $\mathcal{F}$-centric subgroups of S. Let $P \subseteq S$ be a $\mathcal{F}$-centric subgroup and $g \in P$. We get a natural transformation η_g between the functor $\mathrm{id}_{\mathcal{B}P}$ and $\mathcal{B}(c_g)$ on $\mathcal{B}P$ by setting $\eta_g(o_P) = g$. Thus $|\eta_g|\colon BP \times I \to BP$, and we define $H_g = f \circ |\eta_g|\colon BP \times I \to X$.We observe that $H_g(-,0) = f|_{BP}$ and $H_g(-,1) = f|_{BP} \circ B(c_g)$, so we can define $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ by $\delta_P(g) = (c_g, [H_g])$. Note that this is a morphism of groups. By condition (c) and the noted classical result we have that

$$\mathrm{Map}(BP, X)_{f|_{BP}} \simeq \mathrm{Map}(BP, BP)_{\mathrm{id}} \simeq B\,\mathrm{Z}(P)$$

induced by the map

$$B\,\mathrm{Z}(P) \times BP \xrightarrow{\text{mult}} BP \xrightarrow{Bi_P} BS \xrightarrow{f} |\mathcal{L}|^\wedge_p$$

By considering this map we conclude that the induced isomorphism of fundamental groups $\mathrm{Z}(P) \to \pi_1(\mathrm{Map}(BP, X)_{f|_{BP}})$ is exactly $g \mapsto (c_g, [H_g])$. Similar to the arguments from the proof of Proposition 9.4 we now see that the natural projection $\pi\colon \mathcal{L} \to \mathcal{F}$ satisfies condition (A). If $g \in P$ with $\delta_P(g) = \delta_P(1)$, then $c_g = \mathrm{id}_P$, so $g \in \mathrm{Z}(P)$ and by the above isomorphism, we conclude that $g = 1$. Hence δ_P is injective for all P. By construction $\pi(\delta_P(g)) = c_g$ for any $\mathcal{F}$-centric subgroup P and $g \in P$. So it suffices to prove condition (C), i.e. for all $(\varphi, [H]) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and $g \in P$ the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "{(\varphi, [H])}"] \arrow[d, "{(c_g, [H_g])}"'] & Q \arrow[d, "{(c_{\varphi(g)}, [H_{\varphi(g)}])}"] \\ P \arrow[r, "{(\varphi, [H])}"] & Q \end{tikzcd}$$

We have that $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$ and by setting $F\colon BP \times I \times I \to X$ to $F(x, s, t) = H(|\eta_g|(x, t), s)$ we get a map with the property that

$$F(x, s, 0) = H(x, s), \quad F(x, 1, t) = H_{\varphi(g)}(B\varphi(x), t)$$

$$F(x, 0, t) = H_g(x, t), \quad F(x, s, 1) = H(Bc_g(x), s).$$

So by a suitable reparametization we get a homotopy from $(H_{\varphi(g)} \circ B\varphi)H$ to $(H \circ Bc_g)H_g$ relative to the endpoints. Then the diagram commutes and we conclude that $\mathcal{L}$ is a central linking system associated to $\mathcal{F}$. $\square$

References

[1] Jonathan L Alperin. *Local representation theory: Modular representations as an introduction to the local representation theory of finite groups*, volume 11. Cambridge University Press, 1993.

[2] Emil Artin and John T TATE. A note on finite ring extensions. *Journal of the Mathematical Society of Japan*, 3(1):74–77, 1951.

[3] Michael Aschbacher, Radha Kessar, Robert Oliver, and Bob Oliver. *Fusion systems in algebra and topology*, volume 391. Cambridge University Press, 2011.

[4] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Number 361 in Addison-Wesley Series in Mathematics. Addison-Wesley, Reading, MA, 1969. MR 0242802. Zbl 0175.03601.

[5] David J Benson. Representations and cohomology. i, volume 30 of cambridge studies in advanced mathematics, 1998.

[6] David J Benson. *Representations and Cohomology: Volume 2, Cohomology of Groups and Modules*, volume 2. Cambridge University Press, 1998.

[7] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Verlag, 1972.

[8] Glen E Brendon. *Sheaf theory*. McGraw-Hill, 1967.

[9] Carles Broto, Ran Levi, and Bob Oliver. Homotopy equivalences of p-completed classifying spaces of finite groups. *Inventiones Mathematicae*, 151(3):611–664, 2003.

[10] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *Journal of the American Mathematical Society*, 16(4):779–856, 2003.

[11] Kenneth S Brown. *Cohomology of groups*, volume 87. Springer, 1982.

[12] Andy Chermak. Fusion systems and localities. *preprint, http://www.math.ku.dk/english/research/conferences/fusionsystems2011/ChermakLocalities2.pdf*, 2011.

[13] Charles W Curtis and Irving Reiner. Methods of representation theory with applications to finite groups and orders, vol. 1. *JOHN WILEY & SONS, INC., ONE WILEY DR., SOMERSET, NJ 08873, 1981, 864*, 1981.

[14] WG Dwyer and CW Wilkerson. A cohomology decomposition theorem. *Topology*, 31(2):433–443, 1992.

[15] Leonard Evens. The cohomology ring of a finite group. *Trans. Amer. Math. Soc*, 101(2):224–239, 1961.

[16] Leonard Evens. *The cohomology of groups*. Oxford University Press on Demand, 1991.

[17] P Goerss and J Jardine. Simplicial homotopy theory, modern birkhäuser classics, (2009).

[18] Jesper Grodal. Higher limits via subgroup complexes. *The Annals of Mathematics*, 155(2):405–457, 2002.

[19] Allen Hatcher. *Algebraic Topology*. Cambridge Univ. Press, Cambridge, 2002.

[20] Allen Hatcher. Spectral sequences in algebraic topology. *Unpublished book project, available on http://www. math. cornell. edu/hatcher/SSAT/SSATpage. html*, 2004.

[21] Hans-Werner Henn. Commutative algebra of unstable k-modules, lannes' t-functor and equivariant mod-p cohomology. *Journal fur die reine und angewandte Mathematik*, 478:189, 1996.

[22] Philip S Hirschhorn. *Model categories and their localizations*, volume 99. American Mathematical Soc., 2009.

[23] J. Hollender and R.M. Vogt. Modules of topological spaces, applications to homotopy limits and e_∞ structures. *Archiv der Mathematik*, 59:115–129, 1992.

[24] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of bg via g-actions. *The Annals of Mathematics*, 135(2):227–270, 1992.

[25] Irving Kaplansky. Projective modules. *Annals of Mathematics*, 68(2):pp. 372–377, 1958.

[26] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Publications Mathématiques de l'IHÉS*, 75(1):135–244, 1992.

[27] Jean Lannes and Lionel Schwartz. Sur la structure des a-modules instables injectifs. *Topology*, 28(2):153 – 169, 1989.

[28] Guido Mislin. Lannes' t-functor and the cohomology of bg. *Compositio Mathematica*, 86(2):177–187, 1993.

[29] Daniel Quillen. The spectrum of an equivariant cohomology ring: I. *The Annals of Mathematics*, 94(3):549–572, 1971.

[30] Daniel Quillen. Higher algebraic k-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.

[31] Daniel Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Advances in Mathematics*, 28(2):101 – 128, 1978.

[32] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*. University of Chicago Press, 1994.

[33] Graeme Segal. Classifying spaces and spectral sequences. *Publications Mathématiques de l'IHÉS*, 34:105–112, 1968.

[34] Jean Pierre Serre and Leonard L Scott. *Linear representations of finite groups*, volume 42. Springer-Verlag New York, 1977.

[35] P.J. Webb. A split exact sequence of mackey functors. *Commentarii mathematici Helvetici*, 66(1):34–69, 1991.