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Burnside rings of fusion systems

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Abstract

In this thesis, we study the relations between a saturated fusion system $\mathcal{F}$ on S and the single and double Burnside rings, $A(S)$ and $A(S,S)$, of the p-group S. In particular, we generalize the Burnside rings of groups to Burnside rings of fusion systems. The Burnside ring of $\mathcal{F}$ is defined as the subring $A(\mathcal{F})$ of $A(S)$ consisting of the $\mathcal{F}$-stable elements; and similarly for the double Burnside rings.

We describe the algebraic properties of $A(\mathcal{F})$, which is a free abelian group with one generator per $\mathcal{F}$-conjugacy class of subgroups. For a realizable fusion system $\mathcal{F}_S(G)$, we also show how $A(\mathcal{F}_S(G))$ relates to the Burnside ring $A(G)$ of the group G.

Certain "$\mathcal{F}$-characteristic" elements of the double Burnside ring $A(S,S)$, play a special role in relation to the saturated fusion system $\mathcal{F}$. We prove their existence (and in some cases uniqueness), and we review how to reconstruct a fusion system from a characteristic element. Finally, we describe a one-to-one correspondence between saturated fusion systems on S and certain idempotents of $A(S,S)_{(p)}$.

Resumé

I dette speciale undersøger vi relationerne mellem et mættet fusionssystem $\mathcal{F}$ på S og de enkelte og dobbelte Burnside-ringe, $A(S)$ og $A(S,S)$, for p-gruppen S. Specielt, vil vi generalisere Burnside-ringene for grupper til Burnside-ringe for fusionssystemer. Burnside-ringen for $\mathcal{F}$ er defineret som delringen $A(\mathcal{F})$ af $A(S)$ der består af de $\mathcal{F}$-stabile elementer; og tilsvarende for den dobbelte Burnside-ring.

Vi beskriver de algebraiske egenskaber ved $A(\mathcal{F})$, der er en fri abelsk gruppe med én frembringer per $\mathcal{F}$-konjugeretklasse af undergrupper. For et realiserbart fusionssystem $\mathcal{F}_S(G)$, viser vi også hvordan $A(\mathcal{F}_S(G))$ relaterer til Burnside-ringen $A(G)$ for gruppen G.

I den dobbelte Burnside ring $A(S,S)$ findes "$\mathcal{F}$-karakteristiske elementer" der spiller en særlig rolle i forhold til det mættede fusionssystem $\mathcal{F}$. Vi beviser deres eksistens (og i visse tilfælde entydighed), og vi gennemgår hvordan man kan rekonstruere et fusionssystem ud fra et karakteristisk element. Til afslutning beskriver vi en én-til-én-korrespondance mellem mættede fusionssystemer på S og visse idempotenter i $A(S,S)_{(p)}$.

Contents

Introduction

This thesis studies relations between a fusion system on a p-group S and the single/double Burnside ring of S. In particular, we attempt to give a sensible generalization of these Burnside rings to the fusion system – extending the work done by Diaz-Libman in [DL09]. The thesis naturally falls into two parts, the first part concerning the single Burnside ring, and the second part concerning the double Burnside ring.

Burnside rings are completely algebraic structures, of algebraic interest, but arise frequently in equivariant algebraic topology. The single Burnside ring $A(G)$ of a finite group G show up as the ring of equivariant pointed self-maps $[S^V, S^V]_G^*$ of the representation sphere S^V – for a sufficiently large representation V. The Segal conjecture also states that $A(G)_I^\wedge \cong \pi_S^0(BG)$, relating a suitable completion of the Burnside ring to the stable cohomotopy of the classifying space. Similarly, the Burnside modules and double Burnside rings relate to stable maps of the classifying spaces, $A(G,H)_I^\wedge \cong \{BG_+, BH_+\}$.

Overview

Part 1 begins with a review of properties of the Burnside ring $A(G)$ of a finite group G, and recall structural results concerning $A(G)$ in relation to a prime p. Guided by this insight, we make a generalization of Burnside rings to the context of saturated fusion systems.

We define the Burnside ring $A(\mathcal{F})$ of a saturated fusion system F on S, as the subring consisting of the $\mathcal{F}$-stable elements in $A(S)$. We prove that $A(\mathcal{F})$ is a free abelian group with one generator per $\mathcal{F}$-conjugation class of subgroups; and we prove that $A(\mathcal{F})$ has several other properties similar to the Burnside ring of a group. We also show that for a realizable fusion system $\mathcal{F}_S(G)$, the p-localized Burnside ring $A(F_S(G))_{(p)}$ is isomorphic to an appropriate subring of $A(G)_{(p)}$.

Although this definition of a Burnside ring of a fusion system is quite natural, it has not to our knowledge been studied systematically previously in the literature, although the definition is mentioned in passing, e.g. in [Gel10].

At the end of part 1, we prove that the definition of the Burnside ring $A(\mathcal{F})$ given here, agrees with the Burnside ring defined by Antonio Diaz and Assaf Libman in [DL09], when we restrict our view to the $\mathcal{F}$-centric subgroups of S.

Part 2 of this thesis concerns the double Burnside rings and Burnside modules. We follow the general approach of [RS09], by Kári Ragnarsson and Radu Stancu, recalling the double Burnside ring $A(S, S)$ of a p-group S and how certain elements interact with a fusion system on S.

We review the proof of Broto-Levi-Oliver, in [BLO03], that there exist certain "left/right/fully characteristic elements" in $A(S, S)$ for a saturated fusion system $\mathcal{F}$. Using similar methods, and the theory developed in part 1, we then give a new proof that $A(S, S)_{(p)}$ contains a fully $\mathcal{F}$-characteristic idempotent – a result previously proven in [Rag06] using the p-completion $A(S, S)^{\wedge}_{p}$.

We also provide a counterexample to the claim in [Rag06] that $A(S, S)_{(p)}$ contains at most one left characteristic idempotent for $\mathcal{F}$ – the uniqueness of fully characteristic idempotents still holds however.

At the end, we review the constructions, from [RS09], giving a way to reconstruct a saturated fusion system from a characteristic element. This leads us to the result by Ragnarsson-Stancu that the saturated fusion systems on a p-group S are in one-to-one correspondence with the fully characteristic idempotents in $A(S, S)_{(p)}$.

On the road to this final correspondence, we define the double Burnside ring $A(\mathcal{F}, \mathcal{F})$ in a way mimicking the definition of $A(\mathcal{F})$. Using the characteristic idempotent $\omega_{\mathcal{F}}$ for $\mathcal{F}$, we then describe a $\mathbb{Z}_{(p)}$-basis for $A(\mathcal{F}, \mathcal{F})_{(p)}$. In [Rag06], Ragnarsson proves a relation between the Burnside modules $A(\mathcal{F}_1, \mathcal{F}_2)$ defined in this thesis, and the homotopy classes of maps $[\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2]$ between the classifying spectra of the saturated fusion systems – a result similar to the one for Burnside modules of finite groups.

Conventions

Throughout this thesis, p denotes some prime number. We let S denote a finite p group, and $\mathcal{F}$ is a fusion system on S. Many of the results require that $\mathcal{F}$ is saturated – and either the introduction to a section or the individual results specify when this is the case.

We let G be a finite group with S as a p-subgroup. Unless otherwise mentioned, we require that S is in fact a Sylow-p-subgroup of G. We let $N_G(H,K)$ denote the transporter in G from H to K

$$N_G(H,K) := \{g \in G \mid c_g(H) \leq K\};$$

where c_g is the conjugation map $c_g(x) := gxg^{-1}$ for $g, x \in G$. Conjugating g on a subgroup H, we get either ${}^gH := c_g(H) = gHg^{-1}$ or $H^g := (c_g)^{-1}(H) = g^{-1}Hg$.

$\mathrm{Aut}_G(H)$ is the group of automorphisms of $H \leq G$ induced by G-conjugation, so $\mathrm{Aut}_G(H) := N_G(H)/C_G(H)$ is just the quotient of the normalizer by the centralizer. We let $\mathrm{Inn}(G) := \mathrm{Aut}_G(G) = G/Z(G)$ be the group of inner automorphisms; and $\mathrm{Out}_G(H) := \mathrm{Aut}_G(H)/\mathrm{Inn}(H)$ is the group of outer automorphisms induced by G on $H \leq G$.

We use the symbol $\leq$ to denote subgroups, submodules, subrings and so on, and $<$ then denotes a proper inclusion. We denote G-conjugation by $H \sim_G K$ for subgroups $H, K \leq G$; and if H is subconjugate to K, we write $H \precsim_G K$ (or $H \prec_G K$ if $|H| < |K|$).

The rings considered in this thesis are not necessarily unital, i.e. they do not need to have a 1-element/neutral element for the multiplication. Similarly a subring is just an additive subgroup closed under multiplication. If for instance $\omega \in R$ is an idempotent, then $\omega R \omega$ is a subring of R which is unital with ω as the 1-element.

0 Fusion systems

The next two pages contain a incredibly short introduction to fusion systems without proofs. The aim is to introduce the concepts from the theory of fusion systems that will be used in the main parts of the thesis, and to establish the relevant notation concerning fusion systems. For a proper introduction to fusion systems see for instance Part I of "Fusion Systems in Algebra and Topology" by Aschbacher, Kessar and Oliver, [AKO10].

Definition 0.1. A *fusion system* $\mathcal{F}$ on a p-group S, is a category where the objects are the subgroups of S, and for all $P, Q \leq S$ the morphisms must satisfy:

(i) Every morphism $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$ is an injective group homomorphism, and the composition of morphisms in $\mathcal{F}$ is just composition of group homomorphisms.

(ii) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P, Q)$, where

$$\mathrm{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}$$

is the set of group homomorphisms $P \to Q$ induced by S-conjugation.

(iii) For every morphism $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$, the group isomorphisms $\varphi\colon P \to \varphi P$ and $\varphi^{-1}\colon \varphi P \to P$ are elements of $\mathrm{Mor}_{\mathcal{F}}(P, \varphi P)$ and $\mathrm{Mor}_{\mathcal{F}}(\varphi P, P)$ respectively.

We also write $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ or just $\mathcal{F}(P, Q)$ for the morphism set $\mathrm{Mor}_{\mathcal{F}}(P, Q)$. Furthermore, the group $\mathcal{F}(P, P)$ of automorphisms is denoted by $\mathrm{Aut}_{\mathcal{F}}(P)$; and we define the outer $\mathcal{F}$-automorphisms as $\mathrm{Out}_{\mathcal{F}}(P) := \mathrm{Aut}_{\mathcal{F}}(P)/\,\mathrm{Inn}(P)$.

Example 0.2. Let S be a p-subgroup of a finite group G. The fusion system of G over S, denoted $\mathcal{F}_S(G)$, is the fusion system on S where the morphisms from $P \leq S$ to $Q \leq S$ are the homomorphisms induced by G-conjugation:

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) := \mathrm{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\}.$$

Unless otherwise mentioned, we only consider $\mathcal{F}_S(G)$ in the case where S is a Sylow-p-subgroup of G.

The smallest fusion system on S, is the fusion system $\mathcal{F}_S(S)$ consisting only of the homomorphisms induced by S-conjugation. We write $\mathcal{F}_S$ as shorthand notation for $\mathcal{F}_S(S)$.

Definition 0.3. Two subgroup $P, Q \leq S$ are said to be *$\mathcal{F}$-conjugate*, written $P \sim_{\mathcal{F}} Q$, if there exists an $\mathcal{F}$-isomorphism $\varphi\colon P \to Q$, i.e. a map $\varphi \in \mathcal{F}(P, Q)$ which is a group isomorphism.

$\mathcal{F}$-conjugation is an equivalence relation, and the set of $\mathcal{F}$-conjugates to P is denoted by $[P]_{\mathcal{F}}$ or just $[P]$. The set of $\mathcal{F}$-conjugacy classes of subgroups in S is denoted by $C(\mathcal{F})$.

Since all S-conjugation maps are in $\mathcal{F}$, a conjugacy class $[P]_{\mathcal{F}}$ can be partitioned into the disjoint S-conjugacy classes of subgroups $Q \in [P]_{\mathcal{F}}$. We write $P \sim_S Q$ if P and Q are S-conjugate, the S-conjugacy class of P is written $[P]_S$ or $[P]$, and we write $C(S)$ for the set of S-conjugacy classes of subgroups in S.

We say that Q is $\mathcal{F}$- or *S-subconjugate* to P if Q is respectively $\mathcal{F}$- or S-conjugate to a subgroup of P, and we denote this by $Q \precsim_{\mathcal{F}} P$ or $Q \precsim_S P$ respectively.

Remark 0.4. Suppose $\mathcal{F} = \mathcal{F}_S(G)$. We then have $Q \precsim_{\mathcal{F}} P$ if and only if Q is G-conjugate to a subgroup of P; and the $\mathcal{F}$-conjugates of P, are just those G-conjugates of P which are contained in S.

Definition 0.5. A subgroup $P \leq S$ is *fully $\mathcal{F}$-normalized* if $|N_S P| \geq |N_S Q|$ for all $Q \in [P]_{\mathcal{F}}$; and P is *fully $\mathcal{F}$-centralized* if $|C_S P| \geq |C_S Q|$ for all $Q \in [P]_{\mathcal{F}}$.

Definition 0.6. A fusion system $\mathcal{F}$ on S is called *saturated* if the following properties are satisfied for all $P \leq S$:

(i) If P is fully $\mathcal{F}$-normalized, then P is fully $\mathcal{F}$-centralized, and $\mathrm{Aut}_S(P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$).

(ii) Every homomorphism $\varphi \in \mathcal{F}(P, S)$ where $\varphi(P)$ is fully $calF$-centralized, extends to a homomorphism $\varphi \in \mathcal{F}(N_\varphi, S)$ where

$$N_\varphi := \{x \in N_S(P) \mid \exists y \in S\colon \varphi \circ c_x = c_y \circ \varphi\}.$$

Proposition 0.7 ([AKO10, Theorem 2.3])**.** *If $S \in Syl_p(G)$, then $\mathcal{F}_S(G)$ is saturated. In particular, $\mathcal{F}_S$ is always saturated.*

Saturated fusion systems on the form $\mathcal{F}_S(G)$ are called realizable*, and other saturated fusion systems are* exotic*.*

Lemma 0.8 ([AKO10, Lemma 2.6(c)])**.** *Let $\mathcal{F}$ be saturated. Suppose that $P \leq S$ is fully normalized, then for each $Q \in [P]_{\mathcal{F}}$ there exists a $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$.*

1 (Single) Burnside rings

1.1 The Burnside ring of a group

This section contains a short introduction to the Burnside of a group and how the Burnside ring embeds as a subring in a suitable product ring $\prod \mathbb{Z}$.

Definition 1.1.1. The isomorphism classes of G-sets form a free commutative monoid with disjoint union as the addition, and the transitive G-sets as generators. Let $A(G)$ be the Grothendieck group of this monoid.

We define a product in $A(G)$ by putting $X \cdot Y := X \times Y$ for G-sets X and Y and then extending bilinearly. This works since the Cartesian product distributes over disjoint union. The resulting ring $A(G)$ is called the *Burnside ring of* G. If confusion with the "double Burnside ring" introduced in part 2, is possible, we might use the term "single Burnside ring" to emphasize the difference between the two types of Burnside rings.

Remark 1.1.2. Let $C(G)$ be the set of conjugacy classes of subgroups in G.

The Burnside ring $A(G)$ is a free $\mathbb{Z}$-module with one basis element $[H]$ for each conjugacy class $[H] \in C(G)$. The basis element, $[H]$ or $[H]_G$, is simply the isomorphism class of the transitive G-set G/H.

The multiplication in $A(G)$ can be described for the basis elements via the following the double coset formula

$$[H] \cdot [K] = \sum_{\overline{x} \in H\backslash G/K} [H \cap {}^xK]. \tag{1.1}$$

The Burnside ring $A(G)$ of a group G is unital, with $[G]$ a the 1-element.

Definition 1.1.3. For each $[H] \in C(G)$, we let $c_{[H]} \colon A(G) \to \mathbb{Z}$ be the homomorphism sending $X \in A(G)$ to the $[H]$-coefficient of X (when written in the standard basis of $A(G)$). Hence we have

$$X = \sum_{[H] \in C(G)} c_{[H]}(X) \cdot [H]$$

for all $X \in A(G)$. For a G-set X, $c_{[H]}(X)$ is the number of orbits in X whose stabilizer subgroups are conjugate to H.

Definition 1.1.4. For every G-set X and $H \leq G$, we let $\Phi_{[H]}(X) := \left|X^H\right|$ be the number of H-fixed-points (which depends only on the conjugacy class of H). Since we have

$$\left|(X \sqcup Y)^H\right| = \left|X^H\right| + \left|Y^H\right|, \qquad \text{and} \qquad \left|(X \times Y)^H\right| = \left|X^H\right| \cdot \left|Y^H\right|$$

for all G-sets X and Y, the map $\Phi_{[H]}$ extends to a ring homomorphism $\Phi_{[H]}\colon A(G) \to \mathbb{Z}$.

Lemma 1.1.5. *Let $H, K \leq G$. Then*

$$\Phi_{[K]}([H]) = \frac{|N_G(K,H)|}{|H|}.$$

In particular $\Phi_{[K]}([H]) \neq 0$ if and only if $K \precsim_G H$.

Proof. We count the $x \in G$ such that $\overline{x} \in (G/H)^K$. On one hand there are $|H| \cdot \Phi_{[K]}([H])$ such elements, since every coset of G/H has $|H|$ elements. On the other hand, we have $xH \in (G/H)^K$ if and only if $kxH = xH$, i.e. $x^{-1}kx \in H$, for all $k \in K$; so the elements we are counting are precisely all x where $x^{-1} \in N_G(K,H)$. □

Definition 1.1.6. Let $\Phi = \Phi^G\colon A(G) \xrightarrow{\prod_{[H]} \Phi_{[H]}} \prod_{[H] \in C(G)} \mathbb{Z}$ be the ring-homomorphism with $\Phi_{[H]}$ as the $[H]$-coordinate. We call Φ the *homomorphism of marks for* $A(G)$.

For an element $f \in \prod_{[H]\in C(G)} \mathbb{Z}$, we denote the $[H]$-coordinate of f by $f_{[H]}$.

Proposition 1.1.7. *The homomorphism Φ is injective and hence embeds $A(G)$ as a subring of $\widetilde{\Omega}(G) := \prod_{[H]\in C(G)} \mathbb{Z}$. Furthermore, we have the following isomorphism of $\mathbb{Z}$-modules:*

$$\operatorname{coker} \Phi \cong \prod_{[H]\in C(G)} (\mathbb{Z}/\left|W_G H\right| \mathbb{Z}),$$

where $W_G H$ is the quotient $W_G H := N_G H / H$.

Proof. We choose some total order of the conjugacy classes $[H] \in C(G)$ such that $|H| > |K|$ implies $[H] < [K]$. In then holds in particular that $K \precsim_G H$ implies $[H] \leq [K]$.

We then consider the matrix M describing Φ in terms of the ordered bases of $A(G)$ and $\widetilde{\Omega}(G)$. Since $M_{[K],[H]} := \Phi_{[K]}([H])$ is zero unless $H \sim K$ or $|H| > |K|$, we conclude that M is a lower triangular matrix. The diagonal elements of M are

$$M_{[H],[H]} = \Phi_{[H]}([H]) = \frac{|N_G(H)|}{|H|} = \left|W_G H\right|.$$

All the diagonal entries are non-zero, so Φ is injective. Furthermore the cokernel satisfies

$$\operatorname{coker} \Phi \cong \prod_{[H]\in C(G)} (\mathbb{Z}/M_{[H],[H]}\mathbb{Z}) = \prod_{[H]\in C(G)} (\mathbb{Z}/\left|W_G H\right| \mathbb{Z}).$$

□

Definition 1.1.8. We define the *obstruction group* $Obs(G) := \prod_{[H]\in C(G)}(\mathbb{Z}/|W_GH|\mathbb{Z})$. The following theorem then gives an explicit description of $Obs(G)$ as the cokernel of Φ.

Theorem 1.1.9 ([Yos90, Proposition 2.9])**.** *Let $\Psi\colon \widetilde{\Omega}(G) \to Obs(G)$ be given by the $[H]$-coordinate functions*

$$\Psi_{[H]}(f) := \sum_{\overline{g}\in W_GH} f_{[\langle g\rangle H]} \pmod{|W_GH|}.$$

The following sequence of $\mathbb{Z}$-modules is then exact:

$$0 \to A(G) \xrightarrow{\Phi} \widetilde{\Omega}(G) \xrightarrow{\Psi} Obs(G) \to 0.$$

Proof. From proposition 1.1.7 we already know that Φ is injective , and that $|\operatorname{coker}\Phi| = \prod_{[H]_G}|W_GH|$.

We factor Ψ as

$$\prod_{[H]}\mathbb{Z} \xrightarrow{\widetilde{\Psi}} \prod_{[H]}\mathbb{Z} \xrightarrow{\pi} \prod_{[H]\in C(G)}(\mathbb{Z}/|W_GH|\mathbb{Z}),$$

where π is the canonical surjection and $\widetilde{\Psi}$ is given by

$$\widetilde{\Psi}_{[H]}(f) := \sum_{\overline{g}\in W_GH} f_{[\langle g\rangle H]}.$$

Let M be the matrix corresponding to $\widetilde{\Psi}$ – where we order the conjugacy classes $[H] \in C(G)$ as in the proof of 1.1.7. Then M is a lower triangular matrix with only 1's in the diagonal, so $\widetilde{\Psi}$ is surjective; hence we conclude that Ψ is surjective as well.

Both $\Phi(A(G))$ and $\ker\Psi$ have index in $\widetilde{\Omega}(G)$ equal to $\prod_{[H]_G}|W_GH|$; so to show that $\Phi(A(G)) = \ker\Psi$ it is enough to show $\Psi\Phi = 0$. For every G-set $X \in A(G)$ we have

$$\begin{aligned}\Psi(\Phi(X))_{[H]} &= \sum_{\overline{g}\in W_GH} \Phi_{[\langle g\rangle H]}(X)\\ &= \sum_{\overline{g}\in W_GH} \left|X^{\langle g\rangle H}\right|\\ &= \sum_{\overline{g}\in W_GH} \left|(X^H)^g\right|\\ &= |W_GH|\cdot\left|W_GH\backslash X^H\right|\\ &\equiv 0 \pmod{|W_GH|}\end{aligned}$$

by the "orbit-counting formula"[1] when W_GH acts on X^H. This shows that $\Psi(\Phi(X)) = 0$ for all G-sets $X \in A(G)$, and thus also for the rest of the elements in $A(G)$. □

[1] Also known as: "Burnside's lemma", "the Cauchy-Frobenius lemma", "the lemma that is not Burnside's" and more. . .

Corollary 1.1.10. *Since $\mathbb{Z}_{(p)} \otimes -$ is exact, we still have a short-exact sequence when we p-localize:*

$$0 \to A(G)_{(p)} \xrightarrow{\Phi_{(p)}} \widetilde{\Omega}(G)_{(p)} \xrightarrow{\Psi_{(p)}} Obs(G)_{(p)} \to 0.$$

As $\mathbb{Z}_{(p)}$-modules, we have $A(G)_{(p)} \cong \oplus_{[H]_G} \mathbb{Z}_{(p)}$ and $Obs(G)_{(p)} = \prod_{[H]_G} \mathbb{Z}_{(p)} / |W_G H|_p \mathbb{Z}_{(p)}$ where $|W_G H|_p$ is the order of a Sylow-p-subgroup of $W_G H$; and as rings we have $\widetilde{\Omega}(G)_{(p)} = \prod_{[H]_G} \mathbb{Z}_{(p)}$.

1.2 The p-subgroup Burnside ring

The fusion system of a group G models the structure that G induces on a Sylow-p-subgroup. If we hope to generalize the concept of Burnside rings to fusion systems; it is therefore natural to first restrict our view to the p-subgroups of a group G. This leads to the subring $A(G,p)$ of G-sets where all stabilizer subgroups a p-groups.

Definition 1.2.1. Let $S_p(G)$ be the collection of p-subgroups of G. We define $A(G;p)$ to be the submodule of $A(G)$ generated by the basis elements $[P]$ where $P \in S_p(G)$. Alternatively, $A(G;p)$ is the submodule generated by the G-sets where all stabilizer subgroups are p-groups.

Since $S_p(G)$ is closed under taking subgroups, the double coset formula (1.1) gives that $A(G;p)$ is closed under the multiplication; so $A(G;p)$ is a subring of $A(G)$, and we call it the *p-subgroup Burnside ring of G*.

We also define $C(G;p)$ as the set of G-conjugacy classes of $S_p(G)$; and we define the product ring $\widetilde{\Omega}(G;p) := \prod_{[Q]\in C(G;p)} \mathbb{Z}$.

Remark 1.2.2. The p-subgroup Burnside ring $A(G;p)$ is not necessarily unital – the 1-element of $A(G)$, $[G]_G$, is only in $A(G;p)$ if G is a p-group. However, we will later show that the p-localization $A(G;p)_{(p)}$ has a 1-element; and $A(G;p)_{(p)}$ also turns out to depend only on the fusion system $\mathcal{F}_S(G)$ where $S \in Syl_p(G)$ (see 1.4.7).

Proposition 1.2.3. *Let $\Phi = \Phi^{G;p} \colon A(G;p) \to \widetilde{\Omega}(G;p)$ be given by*

$$\Phi([P])_{[Q]} := \Phi_{[Q]}([P]) = \left|(G/P)^Q\right| = \frac{|N_G(Q,P)|}{|P|}$$

on the basis elements of $A(G;p)$. We call Φ the homomorphism of marks *for $A(G;p)$, and Φ is an injective ring-homomorphism with*

$$\operatorname{coker} \Phi \cong \prod_{[Q]\in C(G;p)} \mathbb{Z}/ |W_G Q| \mathbb{Z}.$$

Proof. Let $\Phi^G \colon A(G) \to \widetilde{\Omega}(G)$ be the ring-homomorphism from proposition 1.1.7; then Φ^G restricts to a ring-homomorphism $\Phi^G \colon A(G;p) \to \widetilde{\Omega}(G)$.

Let $[P] \in C(G;p)$. Whenever H is not a p-subgroup, we have $H \not\lesssim_G P$, and consequently $\Phi^G_{[H]}([P]) = 0$ per lemma 1.1.5. It then follows that $\Phi^G(A(G;p))$ is actually contained in the subring $\prod_{[Q]\in C(G;p)} \mathbb{Z} \leq \widetilde{\Omega}(G)$. So Φ^G simply induces the injective homomorphism $\Phi\colon A(G;p) \to \widetilde{\Omega}(G;p)$ by setting

$$\Phi_{[Q]}([P]) := \Phi^G_{[Q]}([P]) = \left|(G/P)^Q\right|,$$

for all p-subgroups $P, Q \leq G$.

We order the conjugacy classes $[P] \in C(G;p)$ by decreasing order of P (as in the proof of proposition 1.1.7). The matrix M associated to Φ is then a lower triangular matrix with diagonal entries $M_{[P],[P]} = |W_G P|$ which are non-zero. It follows that the cokernel is

$$\operatorname{coker}\Phi \cong \prod_{[Q]\in C(G;p)} \mathbb{Z}/\left|W_G Q\right|\mathbb{Z}. \qquad \square$$

Definition 1.2.4. We define the (p-localized) obstruction group

$$Obs(G;p)_{(p)} := \prod_{[Q]\in C(G;p)} \mathbb{Z}_{(p)}/\left|W_G Q\right|_p \mathbb{Z}_{(p)}.$$

We also define the homomorphism $\Psi_{(p)} = \Psi^{G;p}_{(p)}\colon \widetilde{\Omega}(G;p)_{(p)} \to Obs(G;p)_{(p)}$ given by

$$\Psi_{(p)}(f)_{[Q]} := \sum_{\overline{g}\in (W_G Q)_p} f_{[\langle g\rangle Q]} \pmod{\left|W_G Q\right|_p},$$

where $(W_G Q)_p$ is some Sylow-p-subgroup of $W_G Q$ and $\left|W_G Q\right|_p$ is its order.

Let $\widetilde{Q} \sim_G Q$. Any Sylow-p-subgroup of $W_G\widetilde{Q}$ then corresponds to a Sylow-p-subgroup $\widetilde{S}$ of the normalizer $N_G\widetilde{Q}$, because $\widetilde{Q}$ is a p-group. Since $N_G\widetilde{Q} \sim_G N_G Q$, we get that $\widetilde{S}$ is G-conjugate to all $S \in Syl_p(N_G Q)$. It follows that the definition of $\Psi_{(p)}(f)_{[\widetilde{Q}]_G}$ doesn't depend on the choice of $\widetilde{Q} \in [Q]_G$ or the choice of Sylow-p-subgroup of $W_S\widetilde{Q}$.

Theorem 1.2.5 ([Yos90, Theorem 3.10])**.** *We have a short-exact sequence*

$$0 \to A(G;p)_{(p)} \xrightarrow{\Phi_{(p)}} \widetilde{\Omega}(G;p)_{(p)} \xrightarrow{\Psi_{(p)}} Obs(G;p)_{(p)} \to 0.$$

Proof. The homomorphism $\Phi_{(p)}$ is injective since Φ is, by proposition 1.2.3. As in the proof of theorem 1.1.9, $\Psi_{(p)}$ is surjective since it is represented by a triangular matrix with all 1's in the diagonal.

When localizing Φ we get that

$$\operatorname{coker}(\Phi_{(p)}) = \operatorname{coker}(\Phi)_{(p)} \cong \prod_{[Q]\in C(G;p)} \mathbb{Z}_{(p)}/\left|W_G Q\right|_p \mathbb{Z}_{(p)}.$$

We conclude that $\Phi_{(p)}(A(G;p)_{(p)})$ and $\ker \Psi_{(p)}$ both have index in $\widetilde{\Omega}(G;p)_{(p)}$ equal to $\prod_{[Q]} |W_G Q|_p$; so it suffices to show $\Psi_{(p)}\Phi_{(p)} = 0$. For every G-set $X \in A(G;p)_{(p)}$ we get

$$\begin{aligned} \Psi_{(p)}(\Phi_{(p)}(X))_{[Q]} &= \sum_{\overline{g} \in (W_G Q)_p} \left| X^{\langle g \rangle Q} \right| \\ &= |(W_G Q)_p| \cdot \left| (W_G Q)_p \backslash X^Q \right| \equiv 0 \pmod{|W_G Q|_p} \end{aligned}$$

by the "orbit-counting formula" as in the proof of 1.1.9. It follows that $\Psi_{(p)}\Phi_{(p)} = 0$, so the sequence is exact. □

1.3 Two quick lemmas

Here follows two quick homological algebra lemmas needed in the following sections.

Lemma 1.3.1. *Assume that we have the following commutative diagram of modules over a commutative ring R:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{\Phi}{\hookrightarrow} & B & \overset{\Psi}{\longrightarrow} & C & \longrightarrow & 0 \\ & & \alpha \uparrow & & \beta \uparrow & & \rho \uparrow & & \\ 0 & \longrightarrow & A' & \overset{\Phi'}{\hookrightarrow} & B' & \overset{\Psi'}{\longrightarrow} & C' & \longrightarrow & 0 \end{array}$$

where the rows are short-exact sequences.

Then ρ is injective if and only if $A' = A \cap B'$ as submodules of B.

Proof. Let $\gamma := \beta\Phi' = \Phi\alpha$. We always have $\gamma A' \leq \Phi A \cap \beta B'$, i.e. $A' \leq A \cap B$ as submodules of B; so we have to prove that ρ is injective if and only if $\Phi A \cap \beta B' \leq \gamma A'$.

We first assume that ρ is injective, and let $b \in \Phi A \cap \beta B'$ be arbitrary. Then there are $a \in A$ and $b' \in B'$ with $\Phi a = \beta b' = b$. By commutativity and exactness, we have

$$\rho \Psi' b' = \Psi \beta b' = \Psi \Phi a = 0,$$

so $\Psi' b' = 0$. Hence there exist $a' \in A'$ such that $\Phi' a' = b'$, from which it follows that $b = \gamma a'$.

Assume now that $\Phi A \cap \beta B' \leq \gamma A'$, and let $c' \in \ker \rho$. There exist $b' \in B'$ with $\Psi' b' = c'$; and by commutativity we get

$$\Psi \beta b' = \rho \Psi' b' = \rho c' = 0.$$

Exactness then gives an $a \in A$ with $\Phi a = \beta b'$. It follows that $\beta b' \in \Phi A \cap \beta B' \leq \gamma A'$, so there is an $a' \in A$ with $\gamma a' = \beta b'$. By injectivity of β we have $b' = \Phi' a'$, hence $c' = \Psi' \Phi' a' = 0$. □

Corollary 1.3.2. *Let $A, B \leq M$ be R-modules; and let $S \subseteq R$ be a multiplicative system. Then $S^{-1}(A \cap B) = S^{-1}A \cap S^{-1}B$.*

In particular, this holds for $\mathbb{Z}$-modules and p-localization.

Proof. Consider the following commutative diagram with short-exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & M & \longrightarrow & C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & A \cap B & \hookrightarrow & B & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Lemma 1.3.1 then says that ρ is injective. Since the $S^{-1}(-)$-functor is exact, the result follows if we apply $S^{-1}(-)$ to the diagram and then use lemma 1.3.1 in reverse. □

Lemma 1.3.3 (Generalization of idea in [DL09])**.** *Let B be a product ring $\prod_{\lambda \in \Lambda} \mathbb{Z}_{(p)}$, and let $A \leq B$ be a subring. Assume that we have a short-exact sequence*

$$0 \to A \to B \to C \to 0$$

of $\mathbb{Z}_{(p)}$-modules, where C is finite (hence C is a finite abelian p-group).

In that case, any element $\alpha \in A$ which is invertible in B, is invertible in A as well.

Proof. Multiplikation with α gives a map of the short-exact sequence to itself:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot\alpha} & & \downarrow{\scriptstyle \cdot\alpha} & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Since α is invertible in B, the map $B \xrightarrow{\cdot\alpha} B$ is an isomorphism. Using the surjectivity part of the five lemma, it follows that ρ is surjective. Since C is finite, we conclude that ρ is in fact an isomorphism. With another application of the five lemme we see that $A \xrightarrow{\cdot\alpha} A$ is an isomorphism as well, hence α must be invertible in A. □

1.4 The Burnside ring of a fusion system

Whenever we have a homomorphism $\varphi \colon G \to H$, we can make any H-set X into a G-set by letting G act through φ, i.e. by defining $g \cdot x := \varphi(g)x$. However, when we consider a conjugation map $c_{g_0} \colon G \to G$ and a G-set X, it doesn't matter whether we let G act by the action given on X, $g \cdot x := gx$, or we let G act through the conjugation map, $g \cdot x := c_{g_0}(g)x$. The orbits of the two actions are the same, and we just conjugate the stabilizer subgroups into each other.

This turns out to be the key of describing how $A(G,p)_{(p)}$ embeds into $A(S)_{(p)}$, because any G-set in $A(G,p)_{(p)}$ is unchanged whether we act through some G-induced map or not.

Definition 1.4.1. Assume that $S \in Syl_p(G)$. Every G-set is then also an S-set by restricting the action. This induces a ring-homomorphism $A(G) \to A(S)$, and in particular we get the ring-homomorphism $r\colon A(G;p) \to A(S)$.

If $P, Q \leq S$ are conjugate in S, then they are conjugate in G. We therefore have a well-defined map $i\colon C(S) \to C(G;p)$. Since every p-subgroup of G is conjugate to a subgroup of S, we conclude that i is in fact surjective. The map $i\colon C(S) \to C(G;p)$ then induces a ring-homomorphism

$$i^*\colon \prod_{[Q]\in C(G;p)} \mathbb{Z} \to \prod_{[Q]\in C(S)} \mathbb{Z}$$

which is injective since i is surjective.

Theorem 1.4.2. *If $S \in Syl_p(G)$, then the short-exact sequences from corollary 1.1.10 and theorem 1.2.5 fit together in a commutative diagram of $\mathbb{Z}_{(p)}$-modules:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(S)_{(p)} & \xrightarrow{\Phi^S_{(p)}} & \widetilde{\Omega}(S)_{(p)} & \xrightarrow{\Psi^S_{(p)}} & Obs(S)_{(p)} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle r_{(p)}} & & \uparrow{\scriptstyle i^*_{(p)}} & & \uparrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & A(G;p)_{(p)} & \xrightarrow{\Phi^{G;p}_{(p)}} & \widetilde{\Omega}(G;p)_{(p)} & \xrightarrow{\Psi^{G;p}_{(p)}} & Obs(G;p)_{(p)} & \longrightarrow & 0
\end{array}$$

The induced homomorphism ρ is injective, and $A(G;p)_{(p)} = \widetilde{\Omega}(G;p)_{(p)} \cap A(S)_{(p)}$ when considered as subrings of $\widetilde{\Omega}(S)_{(p)}$.

Proof. Let $[X]_G \in A(G;p)$ be the isomorphism class of a G-set X; and let $[X]_S$ be the corresponding isomorphism class of X as an S-set. Then

$$\Phi^S(r([X]_G))_{[Q]_S} = \Phi^S([X]_S)_{[Q]_S} = \left|X^Q\right| = \Phi^{G;p}([X]_G)_{[Q]_G} = i^*(\Phi^{G;p}([X]_G))_{[Q]_S}$$

for all subgroups $Q \leq S$, and hence $\Phi^S(r([X]_G)) = i^*(\Phi^{G;p}([X]_G))$. It follows that $\Phi^S \circ r = i^* \circ \Phi^{G;p}$ which also holds when p-localizing; and thus the left square of the diagram commutes. Since $\Phi^{G;p}$ and i^* are injective, we also conclude that $r\colon A(G;p) \to A(S)$ is injective.

The induced map $\rho\colon Obs(G;p)_{(p)} \to Obs(S)_{(p)}$ given by

$$\rho(\Psi^{G;p}_{(p)}(f)) := \Psi^S_{(p)}(i^*_{(p)}(f))$$

is obviously well-defined by short-exactness and the fact that the diagram's left square commutes.

Assume that $f \in \widetilde{\Omega}(G;p)_{(p)}$ satisfies $\rho(\Psi^{G;p}_{(p)}(f)) = 0$ and let $[Q] \in C(G;p)$ be arbitrary. Let $N \in Syl_p(N_G Q)$, then N is conjugate to some subgroup $\widetilde{N} = {}^g N$ contained in S since $S \in Syl_p(G)$. Furthermore, $\widetilde{N}$ is a Sylow-p-subgroup of $N_G({}^g Q)$. We let $\widetilde{Q} := {}^g Q$. and thus have $\widetilde{Q} \leq \widetilde{N} \leq S$. We also get $\widetilde{N} = N_S(\widetilde{Q})$, since $\widetilde{N} \in Syl_p(N_G \widetilde{Q})$.

By assumption $\Psi^S_{(p)}(i^*_{(p)}(f)) = 0$, which implies

$$0 = \Psi^S_{(p)}(i^*_{(p)}(f))_{[\widetilde{Q}]_S} = \sum_{\overline{s} \in \widetilde{N}/\widetilde{Q}} f_{[\langle s \rangle Q]} \pmod{\left|\widetilde{N}/\widetilde{Q}\right|}$$

since $W_S \widetilde{Q} = \widetilde{N}/\widetilde{Q}$.

Because $\widetilde{Q}$ is a p-group and $\widetilde{N} \in Syl_p(N_G \widetilde{Q})$, it follows that $\widetilde{N}/\widetilde{Q}$ is a Sylow-p-subgroup of $Syl_p(W_G \widetilde{Q})$. The value of $\Psi^{G;p}_{(p)}(f)_{[Q]_G}$ doesn't depend on the choice of $\widetilde{Q} \in [Q]_G$ or the choice of Sylow-p-subgroup of $N_G \widetilde{Q}$ (see definition 1.2.4); so we get that

$$\Psi^{G;p}_{(p)}(f)_{[Q]_G} = \sum_{\overline{g} \in \widetilde{N}/\widetilde{Q}} f_{[\langle g \rangle Q]} \equiv 0 \pmod{\left|\widetilde{N}/\widetilde{Q}\right|}.$$

Since $[Q] \in C(G;p)$ was arbitrary, we thus get $\Psi^{G;p}_{(p)}(f) = 0$, so ρ is injective.

Application of lemma 1.3.1 then gives $A(G;p)_{(p)} = \widetilde{\Omega}(G;p)_{(p)} \cap A(S)_{(p)}$ as subrings of $\widetilde{\Omega}(S)_{(p)}$ □

Lemma 1.4.3 ([Gel10, Proposition 3.2.3]). *Let $X \in A(S)$ be an S-set. For any $\varphi \in \mathcal{F}(P,S)$ we let ${}^{\varphi}_{P}X$ denote X considered as a P-set with action $p \cdot x := \varphi(p)x$. By linear extension, this gives a homomorphism $r_\varphi \colon A(S) \to A(P)$, and we let ${}^{\varphi}_{P}X := r_\varphi(X)$ for $X \in A(S)$ in general. For the inclusion* incl: $P \hookrightarrow S$ *we use the shorthand notation ${}_PX := {}^{\mathrm{incl}}_{P}X$.*

The following are equivalent for any S-set X:

(i) $\left|X^P\right| = \left|X^Q\right|$ *for all pairs $P, Q \leq S$ with $P \sim_{\mathcal{F}} Q$.*

(ii) $\left|X^P\right| = \left|X^{\varphi P}\right|$ *for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$.*

(iii) $[{}_PX] = [{}^{\varphi}_{P}X]$ *in $A(P)$ for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$.*

(iv) The P-sets ${}^{\varphi}_{P}X$ and ${}_PX$ are isomorphic for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$.

For a general element $X \in A(S)$, the conditions translate to:

(i) $\Phi_{[P]}(X) = \Phi_{[Q]}(X)$ *for all pairs $P, Q \leq S$ with $P \sim_{\mathcal{F}} Q$.*

(ii) $\Phi_{[P]}(X) = \Phi_{[\varphi P]}(X)$ *for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$.*

(iii) ${}_PX = {}^{\varphi}_{P}X$ *in $A(P)$ for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$.*

*$X \in A(S)$ is called $\mathcal{F}$-*stable *if it satisfies these properties.*

Proof. For an S-set X, (iii)⇔(iv) follows directly from the definition of the Burnside ring $A(P)$.

Let $\Phi^P\colon A(P) \to \widetilde{\Omega}(P)$ be the mark homomorphism for $A(P)$, then $\Phi^P_{[R]}({}_PX) = \Phi_{[R]}(X)$ for all $R \leq P$.

By the definition of the P-action on ${}^{\varphi}_{P}X$, we have $({}^{\varphi}_{P}X)^P = X^{\varphi P}$ for any S-set X. This generalizes to

$$\Phi^P_{[P]}({}^{\varphi}_{P}X) = \Phi_{[\varphi P]}(X)$$

for general $X \in A(S)$.

Assume (iii). Then we immediately get

$$\Phi_{[P]}(X) = \Phi^P_{[P]}({}_PX) = \Phi^P_{[P]}({}^{\varphi}_{P}X) = \Phi_{[\varphi P]}(X)$$

for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$; which proves (iii)⇒(ii).

Assume (ii). Let $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$. By assumption, we have $\Phi_{[\varphi R]}(X) = \Phi_{[R]}(X)$ for all $R \leq P$, hence

$$\Phi^P_{[R]}({}^{\varphi}_{P}X) = \Phi^R_{[R]}({}^{\varphi}_{R}X) = \Phi_{[\varphi R]}(X) = \Phi_{[R]}(X) = \Phi^P_{[R]}({}_PX).$$

Since Φ^P is injective, we get ${}^{\varphi}_{P}X = {}_PX$; so (ii)⇒(iii).

Finally, we have (ii)⇔(i) for all $X \in A(S)$, because the $Q \leq S$ with $Q \sim_{\mathcal{F}} P$ are precisely the images of maps $\varphi \in \mathcal{F}(P,S)$. □

Remark 1.4.4. The product ring $\widetilde{\Omega}(G;p)_{(p)}$ is the subring of $\widetilde{\Omega}(S)_{(p)}$ consisting of those f where $f_{[P]} = f_{[Q]}$ whenever P and Q are G-conjugate.

The last part of theorem 1.4.2 can then be rephrased as: An element $f \in \widetilde{\Omega}(S)_{(p)}$ is in the p-localized Burnside ring $A(G;p)_{(p)}$ if and only if $f \in A(S)_{(p)}$ and $f_{[P]} = f_{[Q]}$ for all pairs $P, Q \leq S$ that are G-conjugate. I.e. $A(G;p)_{(p)}$ consists of the $\mathcal{F}_S(G)$-stable elements of $A(S)_{(p)}$.

The ring $A(G;p)_{(p)}$ therefore only depends on the G-fusion in S; and this gives rise to the following definition of the Burnside ring of a fusion system $\mathcal{F}$.

Definition 1.4.5. Let $\widetilde{\Omega}(\mathcal{F}) := \prod_{[Q]\in C(\mathcal{F})} \mathbb{Z}$; and the map $i\colon C(S) \to C(\mathcal{F})$ sending $[Q]_S \mapsto [Q]_{\mathcal{F}}$ is well-defined and surjective, so it induces an injective ring-homomorphism $i^*\colon \widetilde{\Omega}(\mathcal{F}) \to \widetilde{\Omega}(S)$. The ring $\widetilde{\Omega}(\mathcal{F})$ is simply the subring of $\widetilde{\Omega}(S)$ consisting of the $f \in \widetilde{\Omega}(S)$ where $f_{[P]} = f_{[Q]}$ whenever $P \sim_{\mathcal{F}} Q$. In the case $\mathcal{F} = \mathcal{F}_S(G)$, these definitions are identical to $\widetilde{\Omega}(G;p)$ and i^* from definition 1.4.1.

We define the *Burnside ring of* $\mathcal{F}$, written $A(\mathcal{F})$, to be the intersection $A(S) \cap \widetilde{\Omega}(\mathcal{F})$ in $\widetilde{\Omega}(S)$. In other words, $A(\mathcal{F})$ consists of the $\mathcal{F}$-stable elements of $A(S)$, i.e. the $f \in \widetilde{\Omega}(S)$ such that $f \in A(S)$ and $f_{[P]} = f_{[Q]}$ for all pairs $P \sim_{\mathcal{F}} Q$. The Burnside ring $A(\mathcal{F})$ is unital, since the 1-element $[S]$ of $A(S)$ is $\mathcal{F}$-stable.

Definition 1.4.6. Let $\Phi^{\mathcal{F}}\colon A(\mathcal{F}) \to \widetilde{\Omega}(\mathcal{F})$ and $r\colon A(\mathcal{F}) \to A(S)$ be the inclusions. We then get a commutative diagram with short-exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(S) & \xrightarrow{\Phi^S} & \widetilde{\Omega}(S) & \xrightarrow{\Psi^S} & Obs(S) & \longrightarrow & 0 \\ & & {\scriptstyle r}\big\uparrow & & {\scriptstyle i^*}\big\uparrow & & {\scriptstyle \rho}\,\vdots\uparrow & & \\ 0 & \longrightarrow & A(\mathcal{F}) & \xrightarrow{\Phi^{\mathcal{F}}} & \widetilde{\Omega}(\mathcal{F}) & \longrightarrow & \operatorname{coker}\Phi^{\mathcal{F}} & \longrightarrow & 0 \end{array}$$

The induced map ρ is injective because $A(\mathcal{F}) = \widetilde{\Omega}(\mathcal{F}) \cap A(S)$.

Corollary 1.4.7. *Assume $\mathcal{F} = \mathcal{F}_S(G)$. Then $A(G;p) \leq A(\mathcal{F})$, and when p-localizing: $A(\mathcal{F})_{(p)} = A(G;p)_{(p)}$ as subrings of $\widetilde{\Omega}(S)_{(p)}$.*

Proof. We know that $\Phi^G\colon A(G;p) \to \widetilde{\Omega}(G;p)$ is injective, and $\widetilde{\Omega}(G;p) = \widetilde{\Omega}(\mathcal{F})$. Furthermore $r\colon A(G;p) \to A(S)$ is injective by the proof of theorem 1.4.2; so $A(G;p) \leq \widetilde{\Omega}(G;p) \cap A(S) = A(\mathcal{F})$ as subrings of $\widetilde{\Omega}(S)$.

Corollary 1.3.2 gives us that

$$A(\mathcal{F})_{(p)} = \widetilde{\Omega}(\mathcal{F})_{(p)} \cap A(S)_{(p)};$$

and since $\widetilde{\Omega}(\mathcal{F})_{(p)} = \widetilde{\Omega}(G;p)_{(p)}$, the result then follows from theorem 1.4.2. □

1.5 The structure of the Burnside ring $A(\mathcal{F})$

By definition, the Burnside ring $A(\mathcal{F})$ is a subring of the product ring $\widetilde{\Omega}(\mathcal{F})$. In this section we investigate this embedding, and we arrive at a result, similar to theorems 1.1.9 and 1.2.5, describing the image of $A(\mathcal{F})$ as it embeds in $\widetilde{\Omega}(\mathcal{F})$.

These result however all require that the fusion system $\mathcal{F}$ is saturated – except for the very first remark. So we assume that $\mathcal{F}$ is saturated throughout the section.

Remark 1.5.1. The Burnside ring $A(\mathcal{F})$ is a subring of $\widetilde{\Omega}(\mathcal{F}) = \prod_{[Q]\in C(\mathcal{F})} \mathbb{Z}$, so in particular $A(\mathcal{F})$ is a free $\mathbb{Z}$-module of rank at most $|C(\mathcal{F})|$, which is the number of $\mathcal{F}$-conjugacy classes of subgroups in S.

Lemma 1.5.2 ([BLO03, Lemma 5.4])**.** *Let $\mathcal{H}$ be a collection of subgroups of S such that $\mathcal{H}$ is closed under taking $\mathcal{F}$-subconjugates, i.e. if $P \in \mathcal{H}$, then $Q \in \mathcal{H}$ for all $Q \precsim_{\mathcal{F}} P$. Assume that $X \in A(S)$ has the property that $\Phi_{[P]}(X) = \Phi_{[P']}(X)$ for all pairs $P \sim_{\mathcal{F}} P'$, with $P, P' \notin \mathcal{H}$.*

Then there exists an element $X' \in A(\mathcal{F}) \leq A(S)$ satisfying $\Phi_{[P]}(X') = \Phi_{[P]}(X)$ and $c_{[P]}(X') = c_{[P]}(X)$ for all $P \notin \mathcal{H}$, $P \leq S$.

Proof. We proceed by induction on the size of $\mathcal{H}$.

If $\mathcal{H} = \emptyset$, then we have $X \in A(S) \cap \widetilde{\Omega}(\mathcal{F}) = A(\mathcal{F})$ be assumption.

Assume that $\mathcal{H} \neq \emptyset$, and let $P \in \mathcal{H}$ be maximal under $\mathcal{F}$-subconjugation as well as fully normalized. We define $\mathcal{H}' := \mathcal{H} \setminus [P]_{\mathcal{F}}$; then $\mathcal{H}'$ again contains all $\mathcal{F}$-subconjugates of any $H \in \mathcal{H}'$.

Let $P' \sim_{\mathcal{F}} P$. Then there is a homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S P', N_S P)$ with $\varphi(P') = P$ by lemma 0.8 since $\mathcal{F}$ is saturated. The restriction of S-actions to $\varphi(N_S P')$ gives a ring homomorphism $A(S) \to A(\varphi(N_S P'))$ that preserves the fixed-point homomorphisms $\Phi_{[Q]}$ for $Q \precsim_S \varphi(N_S P')$. For $X \in A(S)$, we then have $\Psi^{\varphi(N_S P')}(\Phi(X)) = 0$. In particular, $\Psi^{\varphi(N_S P')}_{[P]}(\Phi(X)) = 0$, so

$$\sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{[\langle s \rangle P]}(X) \equiv 0 \pmod{\left|\varphi(N_S P')/P\right|}.$$

From the assumption, we have $\Phi_{[Q]}(X) = \Phi_{[Q']}(X)$ for all $Q \sim_{\mathcal{F}} Q'$ with $Q \succ_{\mathcal{F}} P$. Specifically, we have

$$\Phi_{[\varphi(\langle s \rangle P')]}(X) = \Phi_{[\langle s \rangle P']}(X)$$

for all $s \in N_S P'$ with $s \notin P'$. It then follows that

$$\begin{aligned} \Phi_{[P]}(X) - \Phi_{[P']}(X) &= \sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{[\langle s \rangle P]}(X) - \sum_{\bar{s} \in N_S P'/P'} \Phi_{[\langle s \rangle P']}(X) \\ &\equiv 0 - 0 \pmod{\left|W_S P'\right|}. \end{aligned}$$

We can therefore define $\lambda_{[P']} := (\Phi_{[P]}(X) - \Phi_{[P']}(X)) / |W_S P'| \in \mathbb{Z}$.

Following that, we set

$$\widetilde{X} := X + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{[P']} \cdot [P'] \in A(S).$$

We obviously have $c_{[Q]}(\widetilde{X}) = c_{[Q]}(X)$ for all $Q \notin \mathcal{H}$, since $P' \in \mathcal{H}$ for all $P' \sim_{\mathcal{F}} P$.

Because $\Phi_{[Q]}([P']) = 0$ unless $Q \precsim_S P'$, we see that $\Phi_{[Q]}(\widetilde{X}) = \Phi_{[Q]}(X)$ for every $Q \notin \mathcal{H}$. Secondly, we calculate $\Phi_{[P']}(\widetilde{X})$ for each $P' \sim_{\mathcal{F}} P$:

$$\begin{aligned} \Phi_{[P']}(\widetilde{X}) &= \Phi_{[P']}(X) + \sum_{[\widetilde{P}]_S \subseteq [P]_{\mathcal{F}}} \lambda_{[\widetilde{P}]} \cdot \Phi_{[P']}([\widetilde{P}]) \\ &= \Phi_{[P']}(X) + \lambda_{[P']} \cdot \Phi_{[P']}([P']) = \Phi_{[P']}(X) + \lambda_{[P']} \left|W_S P'\right| \\ &= \Phi_{[P]}(X); \end{aligned}$$

which is independent on the choice of $P' \in [P]_{\mathcal{F}}$.

By induction we can then apply the lemma to $\widetilde{X}$ and the smaller collection $\mathcal{H}'$. We get an $X' \in A(\mathcal{F})$ with $\Phi_{[Q]}(X') = \Phi_{[Q]}(\widetilde{X})$ and $c_{[Q]}(X') = c_{[Q]}(\widetilde{X})$ for all $Q \notin \mathcal{H}'$. In particular, we have $\Phi_{[Q]}(X') = \Phi_{[Q]}(\widetilde{X}) = \Phi_{[Q]}(X)$ and $c_{[Q]}(X') = c_{[Q]}(\widetilde{X}) = c_{[Q]}(X)$ for all $Q \notin \mathcal{H}$. □

Proposition 1.5.3. *For each $[P] \in C(\mathcal{F})$, there is an element $\alpha_{[P]} \in A(\mathcal{F})$ such that*

(i) $\Phi_{[Q]}(\alpha_{[P]}) = 0$ *unless* $Q \precsim_{\mathcal{F}} P$.

(ii) $\Phi_{[P]}(\alpha_{[P]}) = |W_S P|$ *when P is a fully normalized representative of $[P]_{\mathcal{F}}$.*

The $\alpha_{[P]}$'s are linearly independent, and $\operatorname{Span}\{\alpha_{[P]}\}_{[P]\in C(\mathcal{F})}$ *has index* $\prod_{[P]\in C(\mathcal{F})} |W_S P|$ *in $\widetilde{\Omega}(\mathcal{F})$ where each chosen representative P of $[P]_{\mathcal{F}}$ is fully normalized.*

Proof. Let $P \le S$ be fully $\mathcal{F}$-normalized. We let $X \in A(S)$ be the S-set

$$X := \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} \cdot [P'] \in A(S).$$

X then satisfies that $\Phi_{[Q]}(X) = 0$ unless $Q \precsim_S P'$ for some $P' \sim_{\mathcal{F}} P$, in which case we have $Q \precsim_{\mathcal{F}} P$. For all $P', \widetilde{P} \in [P]_{\mathcal{F}}$ we have $\Phi_{[\widetilde{P}]}([P']) = 0$ unless $P' \sim_S \widetilde{P}$; and consequently

$$\Phi_{[P']}(X) = \frac{|N_S P|}{|N_S P'|} \cdot \Phi_{[P']}([P']) = \frac{|N_S P|}{|N_S P'|} \cdot \left|W_S P'\right| = |W_S P|\,.$$

Let $\mathcal{H} := \{Q \le S \mid Q \prec_{\mathcal{F}} P\}$, then $\Phi_{[Q]}(X) = \Phi_{[Q']}(X)$ for all pairs $Q \sim_{\mathcal{F}} Q'$ not in $\mathcal{H}$. Using lemma 1.5.2 we get some $\alpha_{[P]} \in A(\mathcal{F})$ with the required properties.

Consider the map $\Phi\colon \operatorname{Span}\{\alpha_{[P]}\}_{[P]\in C(\mathcal{F})} \to \widetilde{\Omega}(\mathcal{F})$ and order the conjugacy classes $[P]_{\mathcal{F}}$ by decreasing order of P. Then Φ is represented by a lower triangular matrix M with non-zero diagonal entries $M_{[P],[P]} = |W_S P|$ (where P is fully normalized). □

Corollary 1.5.4. *We have* $\operatorname{rank} A(\mathcal{F}) = |C(\mathcal{F})|$, *and*

$$\left|\operatorname{coker} \Phi^{\mathcal{F}}\right| \le \prod_{[P]\in C(\mathcal{F})} |W_S P|$$

where each chosen P is fully normalized.

Proof. This follows from proposition 1.5.3 since $\operatorname{Span}\{\alpha_{[P]}\}_{[P]\in C(\mathcal{F})} \subseteq A(\mathcal{F})$, so the latter has smaller index in $\widetilde{\Omega}(\mathcal{F})$. In particular the index of $A(\mathcal{F})$ in $\widetilde{\Omega}(\mathcal{F})$ is finite, so $\operatorname{rank} A(\mathcal{F}) = \operatorname{rank} \widetilde{\Omega}(\mathcal{F})$. □

Definition 1.5.5. We define the obstruction group

$$Obs(\mathcal{F}) := \prod_{\substack{[P]\in C(\mathcal{F})\\ P \text{ f.n.}}} (\mathbb{Z}/\left|W_S P\right|\mathbb{Z}),$$

where 'f.n.' is short for 'fully normalized'.

When $P' \sim_{\mathcal{F}} P$ where P, P' are both fully normalized, then $N_S P' \sim_{\mathcal{F}} N_S P$ by lemma 0.8. For all $f \in \widetilde{\Omega}(\mathcal{F})$ we therefore have

$$\Psi^S_{[P']}(f) = \sum_{\overline{s}\in W_S P'} f_{[\langle s\rangle P']} = \sum_{\overline{s}\in W_S P} f_{[\langle s\rangle P]} = \Psi^S_{[P]}(f).$$

We now choose a specific fully normalized representative P for each $\mathcal{F}$-conjugacy class $[P] \in C(\mathcal{F})$. Let $\pi\colon Obs(S) \twoheadrightarrow Obs(\mathcal{F})$ be the canonical projection discarding all coordinates except for the coordinates corresponding to the chosen representatives. We then define

$$\Psi^{\mathcal{F}} : \widetilde{\Omega}(\mathcal{F}) \xrightarrow{i^*} \widetilde{\Omega}(S) \xrightarrow{\Psi^S} Obs(S) \xrightarrow{\pi} Obs(\mathcal{F}).$$

For all $f \in \widetilde{\Omega}(\mathcal{F})$ the $[P]$-coordinate map is given by

$$\Psi^{\mathcal{F}}_{[P]}(f) := \Psi^{S}_{[P]}(f) = \sum_{\overline{s} \in W_S P} f_{[\langle s \rangle P]} \pmod{|W_S P|},$$

when P is the chosen representative. However, as described earlier, $\Psi^S_{[P]}(f)$ doesn't depend on the choice of fully normalized $P \in [P]_{\mathcal{F}}$ when $f \in \widetilde{\Omega}(\mathcal{F})$. Hence the homomorphism $\Psi^{\mathcal{F}}$ doesn't depend on the chosen representatives (as long as they are fully normalized).

As in the proof of theorem 1.1.9, $\Psi^{\mathcal{F}}$ is represented by a lower triangular matrix with only 1's in the diagonal; hence $\Psi^{\mathcal{F}}$ is surjective. We thus get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(S) & \xrightarrow{\Phi^S} & \widetilde{\Omega}(S) & \xrightarrow{\Psi^S} & Obs(S) & \longrightarrow & 0 \\
 & & \uparrow r & & \uparrow i^* & & \downarrow \pi & & \\
0 & \longrightarrow & A(\mathcal{F}) & \xrightarrow{\Phi^{\mathcal{F}}} & \widetilde{\Omega}(\mathcal{F}) & \xrightarrow{\Psi^{\mathcal{F}}} & Obs(\mathcal{F}) & \longrightarrow & 0
\end{array}$$

The bottom row isn't a priori exact at $\widetilde{\Omega}(\mathcal{F})$, but the following theorem tells us that this is the case.

Theorem 1.5.6. *The following sequence of $\mathbb{Z}$-modules is short-exact:*

$$0 \to A(\mathcal{F}) \xrightarrow{\Phi^{\mathcal{F}}} \widetilde{\Omega}(\mathcal{F}) \xrightarrow{\Psi^{\mathcal{F}}} Obs(\mathcal{F}) \to 0.$$

Proof. The only place where we don't know exactness already, is at $\widetilde{\Omega}(\mathcal{F})$.

If $\pi\colon Obs(S) \twoheadrightarrow Obs(\mathcal{F})$ is the projection chosen earlier, then

$$\Psi^{\mathcal{F}}\Phi^{\mathcal{F}} = \pi\Psi^S i^* \Phi^{\mathcal{F}} = \pi\Psi^S\Phi^S r = 0$$

by exactness in theorem 1.1.9.

From $\Phi^{\mathcal{F}}(A(\mathcal{F})) \leq \ker \Psi^{\mathcal{F}}$ as well as corollary 1.5.4, we then get

$$\begin{aligned}
\prod_{\substack{[P]\in C(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P| &= |Obs(\mathcal{F})| = \left|\widetilde{\Omega}(\mathcal{F}) : \ker \Psi^{\mathcal{F}}\right| \\
&\leq \left|\widetilde{\Omega}(\mathcal{F}) : \Phi^{\mathcal{F}}(A(\mathcal{F}))\right| = \left|\operatorname{coker} \Phi^{\mathcal{F}}\right| \leq \prod_{\substack{[P]\in C(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P| .
\end{aligned}$$

This is only possible if $\ker \Psi^{\mathcal{F}}$ and $\Phi^{\mathcal{F}}(A(\mathcal{F}))$ have the same index in $\widetilde{\Omega}(\mathcal{F})$; and thus we conclude $\ker \Psi^{\mathcal{F}} = \Phi^{\mathcal{F}}(A(\mathcal{F}))$. $\square$

Corollary 1.5.7. *We have the following commutative diagram of $\mathbb{Z}$-modules:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(S) & \xrightarrow{\Phi^S} & \widetilde{\Omega}(S) & \xrightarrow{\Psi^S} & Obs(S) & \longrightarrow & 0 \\ & & {\scriptstyle r}\uparrow & & {\scriptstyle i^*}\uparrow & & {\scriptstyle \rho}\uparrow & & \\ 0 & \longrightarrow & A(\mathcal{F}) & \xrightarrow{\Phi^{\mathcal{F}}} & \widetilde{\Omega}(\mathcal{F}) & \xrightarrow{\Psi^{\mathcal{F}}} & Obs(\mathcal{F}) & \longrightarrow & 0 \end{array}$$

The rows are short-exact, and the induced map ρ is injective.

Proof. The short-exact sequences come from theorems 1.1.9 and 1.5.6. The homomorphism ρ is injective by lemma 1.3.1 since $A(\mathcal{F}) = A(S) \cap \widetilde{\Omega}(\mathcal{F})$ by definition. □

1.6 The p-localized Burnside ring $A(\mathcal{F})_{(p)}$

We now consider the p-localization of the Burnside ring, $A(\mathcal{F})_{(p)}$. This enables us to give more detailed variations of results from the previous section; and we construct a basis for $A(\mathcal{F})_{(p)}$ which corresponds to the basis of $A(G;p)_{(p)}$ in the realizable case.

As in the previous section, all results require that $\mathcal{F}$ is saturated. We also let Φ and Ψ denote the p-localizations $\Phi_{(p)}$ and $\Psi_{(p)}$.

Observation 1.6.1. Assume that $\mathcal{F} = \mathcal{F}_S(G)$, and consider the basis element $[G/S] \in A(G;p)_{(p)} = A(\mathcal{F})_{(p)}$. Since S is a Sylow-p-subgroup of G, $|G/S|$ is coprime to p. For every $P \leq S$, we then have

$$\Phi_{[P]}([G/S]) = \left|(G/S)^P\right| \equiv |G/S| \not\equiv 0 \pmod p$$

since P is a p-group.

The element $[G/S] \in A(\mathcal{F})_{(p)}$ is thus invertible in $\widetilde{\Omega}(\mathcal{F})_{(p)}$, hence it is also invertible in $A(\mathcal{F})_{(p)}$ according to lemma 1.3.3. For each $[P] \in C(\mathcal{F})$, we set $\beta_{[P]} := \frac{[G/P]}{[G/S]} \in A(\mathcal{F})_{(p)}$. Because the $[G/P]$'s are basis of $A(G;p)_{(p)} = A(\mathcal{F})_{(p)}$, and $[G/S]$ is invertible, it follows that the $\beta_{[P]}$'s are another basis of $A(\mathcal{F})_{(p)}$.

For all $Q, P \leq S$, the centralizer $C_G(Q)$ acts freely on $N_G(Q,P)$ from the right. The orbits just correspond to the morphisms of $\mathcal{F}(Q,P)$, so we have $|N_G(Q,P)| \,/\, |C_G(Q)| = |N_G(Q,P)/C_G(Q)| = |\mathcal{F}(Q,P)|$. The $[Q]$-coordinate of the basis element $\beta_{[P]}$ is then equal to

$$\Phi_{[Q]}(\beta_{[P]}) = \frac{\Phi_{[Q]}([G/P])}{\Phi_{[Q]}([G/S])} = \frac{|N_G(Q,P)| \cdot |S|}{|P| \cdot |N_G(Q,S)|} = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|},$$

which only depends on the fusion system $\mathcal{F}$ and not on G.

The $\beta_{[P]}$'s defined by $\Phi_{[Q]}(\beta_{[P]}) := \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|}$ are thus a basis of $A(\mathcal{F})_{(p)}$ depending only on the fusion system itself. We have proven this for realizable fusion systems, so we might hope that a similar definition works for saturated fusion systems in general.

Lemma 1.6.2. *Let $P \le S$ be fully $\mathcal{F}$-normalized. Then $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$, where $p \nmid k$.*

Equivalently, $|\mathcal{F}(P,S)| = \frac{|S|}{|C_S P|} \cdot k'$, with $p \nmid k'$, when $P \le S$ is fully $\mathcal{F}$-normalized.

Proof. We have $|\mathcal{F}(P,S)| = |\mathrm{Aut}_{\mathcal{F}}(P)| \cdot |[P]_{\mathcal{F}}|$ for all $P \le S$. When P is fully $\mathcal{F}$-normalized, we furthermore have

$$|\mathrm{Aut}_{\mathcal{F}}(P)| = |\mathrm{Aut}_S(P)| \cdot k'' = \tfrac{|N_S P|}{|C_S P|} \cdot k''$$

where $p \nmid k''$ since $\mathcal{F}$ is saturated. It follows that the two statements in the lemma are equivalent for $P \le S$ fully normalized.

We proceed by induction on the index $|S : P|$. If $P = S$, then $|[S]_{\mathcal{F}}| = 1 = \frac{|S|}{|N_S S|} \cdot 1$.

Assume $P < S$ fully normalized; since $P \ne S$, we then have $P < N_S P$. The $\mathcal{F}$-conjugacy class $[P]_{\mathcal{F}}$ is a disjoint union of the S-conjugacy classes $[Q]_S$ where $Q \sim_{\mathcal{F}} P$. The S-conjugacy class $[Q]_S$ has $|S| \,/\, |N_S Q|$ elements; and $\frac{|S|}{|N_S Q|}$ is divisible by $\frac{|S|}{|N_S P|}$ since P is fully normalized. In particular, $\frac{|S|}{|N_S P|}$ divides $|[P]_{\mathcal{F}}|$.

Furthermore, we have $|[Q]_S| \,\frac{|N_S P|}{|S|} \equiv 0 \pmod p$ whenever $Q \sim_{\mathcal{F}} P$ isn't fully normalized. It follows that

$$\begin{aligned}
|[P]_{\mathcal{F}}| \,\frac{|N_S P|}{|S|} &= \sum_{[Q]_S \subseteq [P]_{\mathcal{F}}} |[Q]_S| \,\frac{|N_S P|}{|S|} \\
&\equiv \sum_{\substack{[Q]_S \subseteq [P]_{\mathcal{F}} \\ Q \text{ f.n.}}} |[Q]_S| \,\frac{|N_S P|}{|S|} = \left|[P]_{\mathcal{F}}^{\text{f.n.}}\right| \frac{|N_S P|}{|S|} \pmod p,
\end{aligned}$$

where $[P]_{\mathcal{F}}^{\text{f.n.}}$ is the set of $Q \sim_{\mathcal{F}} P$ which are fully normalized. We conclude that $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$, with $p \nmid k$, if and only if $\left|[P]_{\mathcal{F}}^{\text{f.n.}}\right| = \frac{|S|}{|N_S P|} \cdot k'$, with $p \nmid k'$.

Let $Q \sim_{\mathcal{F}} P$ be fully normalized. Since P is fully normalized, we have a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi Q = P$ by lemma 0.8; and since Q is fully normalized, φ is an isomorphism. It follows that every $Q \in [P]_{\mathcal{F}}^{\text{f.n.}}$ is a normal subgroup of exactly one element of $[N_S P]_{\mathcal{F}}$, namely $N_S Q \in [N_S P]_{\mathcal{F}}$.

Let $K \sim_{\mathcal{F}} N_S P$. We let $[P]_{\mathcal{F}}^{\lhd K}$ denote the set of $Q \sim_{\mathcal{F}} P$ such that $Q \lhd K$. Any $Q \lhd K$ is in particular fully normalized since $|K| = |N_S P|$. Any $\mathcal{F}$-isomorphism $N_S P \xrightarrow{\sim} K$ gives a bijection $[P]_{\mathcal{F}}^{\lhd N_S P} \xrightarrow{\sim} [P]_{\mathcal{F}}^{\lhd K}$.

The set $[P]_{\mathcal{F}}^{\text{f.n.}}$ is thus seen to be the disjoint union of the sets $[P]_{\mathcal{F}}^{\lhd K}$ where $K \sim_{\mathcal{F}} N_S P$, and these sets all have the same number of elements as $[P]_{\mathcal{F}}^{\lhd N_S P}$.

Let $K \sim_{\mathcal{F}} N_S P$ be fully normalized, then there is some $Q \in [P]_{\mathcal{F}}^{\lhd K}$. We have $Q \sim_{\mathcal{F}} P$, and Q is fully normalized with $N_S Q = K$ which is itself fully normalized. By letting Q take the place of P, we can therefore assume that $N_S P$ is fully normalized.

Any two elements $Q, R \in [P]_{\mathcal{F}}^{\lhd N_S P}$ are mapped $Q \xrightarrow{\sim} R$ by some $\mathcal{F}$-automorphism of $N_S P$ (since $N_S P$ is the normalizer of both Q and R); hence $\mathrm{Aut}_{\mathcal{F}}(N_S P)$ acts transitively on $[P]_{\mathcal{F}}^{\lhd N_S P}$. Let $X \le \mathrm{Aut}_{\mathcal{F}}(N_S P)$ be the subgroup stabilizing P under this action; so

$$\left|[P]_{\mathcal{F}}^{\lhd N_S P}\right| = |\mathrm{Aut}_{\mathcal{F}}(N_S P) : X|\,.$$

The number of elements in $[P]_{\mathcal{F}}^{\text{f.n.}}$ is then equal to

$$\left|[P]_{\mathcal{F}}^{\text{f.n.}}\right| = |[N_S P]_{\mathcal{F}}| \cdot |\mathrm{Aut}_{\mathcal{F}}(N_S P) : X| \,.$$

We know that $\frac{|S|}{|N_S P|}$ divides $\left|[P]_{\mathcal{F}}^{\text{f.n.}}\right|$; and by the induction assumption we have $|[N_S P]_{\mathcal{F}}| = \frac{|S|}{|N_S(N_S P)|} \cdot k$, where $p \nmid k$, since $N_S P$ is fully normalized. We can therefore conclude that $\frac{|N_S(N_S P)|}{|N_S P|}$ divides $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : X|$.

We now consider the following diagram of subgroups of $\mathrm{Aut}_{\mathcal{F}}(N_S P)$:

$$\begin{array}{ccc} & \mathrm{Aut}_{\mathcal{F}}(N_S P) & \\ \diagup & & \diagdown \\ X & & \mathrm{Aut}_S(N_S P) \\ \diagdown & & \diagup \\ & X \cap \mathrm{Aut}_S(N_S P) & \end{array}$$

The index $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : \mathrm{Aut}_S(N_S P)|$ is coprime to p since $\mathrm{Aut}_S(N_S P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(N_S P)$ by saturation of $\mathcal{F}$. We have $C_S(N_S(P)) \leq C_S(P) \leq N_S P$, which tells us that $C_S(N_S P) = Z(N_S P)$; and consequently

$$\mathrm{Aut}_S(N_S P) \cong N_S(N_S P)/Z(N_S P).$$

From the definition of X, we get that

$$\begin{aligned} X \cap \mathrm{Aut}_S(N_S P) &= \{\varphi \in \mathrm{Aut}_{\mathcal{F}}(N_S P) \mid \varphi P = P\} \cap \{c_s \in \mathrm{Aut}_{\mathcal{F}}(N_S P) \mid s \in N_S(N_S P)\} \\ &= \{c_s \in \mathrm{Aut}_{\mathcal{F}}(N_S P) \mid s \in N_S P\} = Inn(N_S P) \cong N_S P/Z(N_S P). \end{aligned}$$

The index $|\mathrm{Aut}_S(N_S P) : X \cap \mathrm{Aut}_S(N_S P)|$ is therefore equal to $\frac{|N_S(N_S P)|}{|N_S P|}$.

The right side of the subgroup diagram shows that the highest power of p dividing $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : X \cap \mathrm{Aut}_S(N_S P)|$ is $\frac{|N_S(N_S P)|}{|N_S P|}$. The highest power of p dividing $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : X|$ is thus at most $\frac{|N_S(N_S P)|}{|N_S P|}$ – and we already know that this power of p divides $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : X|$. We conclude that $|\mathrm{Aut}_{\mathcal{F}}(N_S P) : X| = \frac{|N_S(N_S P)|}{|N_S P|} \cdot k'$ for some k' coprime to p; and we finally have

$$\begin{aligned} \left|[P]_{\mathcal{F}}^{\text{f.n.}}\right| &= |[N_S P]_{\mathcal{F}}| \cdot |\mathrm{Aut}_{\mathcal{F}}(N_S P) : X| \\ &= \frac{|S|}{|N_S(N_S P)|} \cdot \frac{|N_S(N_S P)|}{|N_S P|} \cdot kk' \\ &= \frac{|S|}{|N_S P|} \cdot kk'; \end{aligned}$$

and $p \nmid kk'$. □

Lemma 1.6.3. *Let $P, Q \le S$, then $|[Q]_\mathcal{F}|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$ for all $Q' \sim_\mathcal{F} Q$; and furthermore*

$$\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \left|(S/P)^{Q'}\right| = \frac{\left|[Q]_\mathcal{F}^{\le P}\right| \cdot |S|}{|P| \cdot |[Q]_\mathcal{F}|} = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|} \in \mathbb{Z}_{(p)},$$

where $[Q]_\mathcal{F}^{\le P}$ is the set of $Q' \sim_\mathcal{F} Q$ with $Q' \le P$.

Proof. From lemma 1.6.2 we know that $|[Q]_\mathcal{F}| = \frac{|S|}{|N_S Q_0|} \cdot k$, with $p \nmid k$, where $Q_0 \sim_\mathcal{F} Q$ is fully normalized. At the same time, we have $|[Q']_S| = \frac{|S|}{|N_S Q'|}$; hence $|[Q]_\mathcal{F}|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$.

We try to simplify the sum in the lemma:

$$\begin{aligned}
\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \left|(S/P)^{Q'}\right| &= \frac{1}{|[Q]_\mathcal{F}|} \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|S|}{|N_S(Q')|} \cdot \frac{|N_S(Q',P)|}{|P|} \\
&= \frac{|S|}{|P| \cdot |[Q]_\mathcal{F}|} \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|N_S(Q',P)|}{|N_S(Q')|} \\
&= \frac{|S|}{|P| \cdot |[Q]_\mathcal{F}|} \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \left|\{R \in [Q']_S \mid R \le P\}\right| \\
&= \frac{|S|}{|P| \cdot |[Q]_\mathcal{F}|} |\{R \in [Q]_\mathcal{F} \mid R \le P\}| \\
&= \frac{\left|[Q]_\mathcal{F}^{\le P}\right| \cdot |S|}{|P| \cdot |[Q]_\mathcal{F}|} = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|}.
\end{aligned}$$

The last equality is just multiplying with $|\mathrm{Aut}_\mathcal{F}(Q)|$ in both the numerator and the denominator. ☐

Lemma 1.6.4. *Let $P \le S$, and assume that $X \in A(S)_{(p)} \le \widetilde{\Omega}(S)_{(p)}$ satisfies $\Phi_{[Q]}(X) = \Phi_{[Q']}(X)$ for all $Q \sim_\mathcal{F} Q'$ with $Q \succ_\mathcal{F} P$.*

Then there exists $X' \in A(S)_{(p)}$ with

(i) $\Phi_{[Q]}(X') = \Phi_{[Q']}(X')$ *for all* $Q \sim_\mathcal{F} Q'$ *with* $Q \succsim_\mathcal{F} P$.

(ii) $\Phi_{[Q]}(X') = \Phi_{[Q]}(X)$ *for all* $Q \not\precsim_\mathcal{F} P$.

(iii) $c_{[Q]}(X') = c_{[Q]}(X)$ *for all* $Q \not\sim_\mathcal{F} P$, *and*

$$\sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X') = \sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X).$$

(iv) For every $Q \le S$:

$$\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X') = \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X).$$

Proof. Assume without loss of generality that P is fully $\mathcal{F}$-normalized, and let $P' \sim_{\mathcal{F}} P$. Then there is a homomorphism $\varphi \in \operatorname{Hom}_{\mathcal{F}}(N_S P', N_S P)$ with $\varphi(P') = P$ by lemma 0.8.

As in the proof of lemma 1.5.2, we can then use the assumption $\Phi_{[Q]}(X) = \Phi_{[Q']}(X)$ for all $Q \sim_{\mathcal{F}} Q'$ with $Q, Q' \succ_{\mathcal{F}} P$, as well as the fact $X \in A(S)_{(p)}$, to prove that

$$\Phi_{[P]}(X) - \Phi_{[P']}(X) \equiv 0 \pmod{\left|W_S P'\right|}.$$

We can therefore define $\lambda_{[P']} := (\Phi_{[P]}(X) - \Phi_{[P']}(X)) / \left|W_S P'\right| \in \mathbb{Z}_{(p)}$.

We recall from lemma 1.6.2 that $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$ where $p \nmid k$. Since k is invertible in $\mathbb{Z}_{(p)}$, we can define

$$c := \left(\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{[P']} \right) \Big/ k \quad \in \mathbb{Z}_{(p)},$$

as well as $\mu_{[P']} := \lambda_{[P']} - \frac{|W_S P|}{|W_S P'|} c \in \mathbb{Z}_{(p)}$.

We claim that

$$X' := X + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \mu_{[P']} \cdot [P'] \in A(S)_{(p)}$$

satisfies the properties in the lemma. We at least have $c_{[Q]}(X') = c_{[Q]}(X)$ for all $Q \not\sim_{\mathcal{F}} P$, which is the first part of (iii).

Because $\Phi_{[Q]}([P']) = 0$ unless $Q \precsim_S P'$, we see that $\Phi_{[Q]}(X') = \Phi_{[Q]}(X)$ for every $Q \not\precsim_{\mathcal{F}} P$, hence (ii) holds. Thirdly, we calculate $\Phi_{[P']}(X')$ for each $P' \sim_{\mathcal{F}} P$:

$$\begin{aligned}
\Phi_{[P']}(X') &= \Phi_{[P']}(X) + \sum_{[\widetilde{P}]_S \subseteq [P]_{\mathcal{F}}} \mu_{[\widetilde{P}]} \cdot \Phi_{[P']}([\widetilde{P}]) \\
&= \Phi_{[P']}(X) + \mu_{[P']} \cdot \Phi_{[P']}([P']) = \Phi_{[P']}(X) + \mu_{[P']} \left|W_S P'\right| \\
&= \Phi_{[P']}(X) + \lambda_{[P']} \left|W_S P'\right| - \frac{|W_S P|}{\left|W_S P'\right|} c \cdot \left|W_S P'\right| \\
&= \Phi_{[P]}(X) - |W_S P|\, c;
\end{aligned}$$

which is independent on the choice of $P' \in [P]_{\mathcal{F}}$. This proves (i) in the case $Q, Q' \sim_{\mathcal{F}} P$; the case $Q \sim_{\mathcal{F}} Q'$ with $Q, Q' \succ_{\mathcal{F}} P$ follows from the assumption and (ii).

The definition of c ensures that

$$\begin{aligned}
\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|W_S P|}{\left|W_S P'\right|} c &= c \cdot \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{\left|N_S P'\right|} = c \cdot \frac{|N_S P|}{|S|} \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \left|[P']_S\right| \\
&= c \cdot \frac{|N_S P|}{|S|} \left|[P]_{\mathcal{F}}\right| = ck = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{[P']};
\end{aligned}$$

which in turn gives us

$$\sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X') - \sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X) = \sum_{[P']_S \subseteq [P]_\mathcal{F}} \mu_{[P']}$$
$$= \sum_{[P']_S \subseteq [P]_\mathcal{F}} \lambda_{[P']} - \sum_{[P']_S \subseteq [P]_\mathcal{F}} \frac{|W_S P|}{|W_S P'|} c = 0.$$

Hence the rest of (iii) holds.

By lemma 1.6.3, every $P' \sim_\mathcal{F} P$ satisfy

$$\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}([P']) = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|}$$

for all $Q \le S$.

We can then prove the final property of X', where we also use $\sum_{[P']_S \subseteq [P]_\mathcal{F}} \mu_{[P']} = 0$:

$$\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X')$$
$$= \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X) + \left(\sum_{[P']_S \subseteq [P]_\mathcal{F}} \mu_{[P']} \left(\sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}([P']) \right) \right)$$
$$= \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X) + \left(\sum_{[P']_S \subseteq [P]_\mathcal{F}} \mu_{[P']} \cdot \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|} \right)$$
$$= \sum_{[Q']_S \subseteq [Q]_\mathcal{F}} \frac{|[Q']_S|}{|[Q]_\mathcal{F}|} \Phi_{[Q']}(X) + 0. \qquad \square$$

Lemma 1.6.5 (p-localized variation of lemma 1.5.2)**.** *Let $\mathcal{H}$ be a collection of subgroups of S such that $\mathcal{H}$ is closed under taking $\mathcal{F}$-subconjugates. Assume that $X \in A(S)_{(p)}$ has the property that $\Phi_{[P]}(X) = \Phi_{[P']}(X)$ for all pairs $P \sim_\mathcal{F} P'$, with $P, P' \notin \mathcal{H}$.*

Then there exists an element $X' \in A(\mathcal{F})_{(p)} \le A(S)_{(p)}$ satisfying

(i) $\Phi_{[P]}(X') = \Phi_{[P]}(X)$ *for all* $P \notin \mathcal{H}$, $P \le S$.

(ii) $c_{[P]}(X') = c_{[P]}(X)$ *for all* $P \notin \mathcal{H}$, *and for all* $P \in \mathcal{H}$ *we have*

$$\sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X') = \sum_{[P']_S \subseteq [P]_\mathcal{F}} c_{[P']}(X).$$

(iii) For every $P \le S$:

$$\Phi_{[P]}(X') = \sum_{[P']_S \subseteq [P]_\mathcal{F}} \frac{|[P']_S|}{|[P]_\mathcal{F}|} \Phi_{[P']}(X).$$

Proof. We proceed by induction on the size of $\mathcal{H}$. If $\mathcal{H}=\emptyset$, then $X \in A(S)_{(p)} \cap \widetilde{\Omega}(\mathcal{F})_{(p)} = A(\mathcal{F})_{(p)}$ by assumption. The first two properties are also vacuously true when $X'=X$. Finally, since $X \in A(\mathcal{F})_{(p)}$, we have $\Phi_{[P]}(X) = \Phi_{[P']}(X)$ for all pairs $P \sim_{\mathcal{F}} P'$, and therefore

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|[P']_S|}{|[P]_{\mathcal{F}}|} \Phi_{[P']}(X) = \Phi_{[P]}(X) \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|[P']_S|}{|[P]_{\mathcal{F}}|} = \Phi_{[P]}(X)\cdot 1.$$

Assume that $\mathcal{H} \neq \emptyset$, and let $P \in \mathcal{H}$ be maximal under $\mathcal{F}$-subconjugation. We define $\mathcal{H}' := \mathcal{H} \setminus [P]_{\mathcal{F}}$, then $\mathcal{H}'$ again contains all $\mathcal{F}$-subconjugates of any $H \in \mathcal{H}'$.

Since P is maximal in $\mathcal{H}$, we see that P and X satisfies the assumptions in lemma 1.6.4; hence we get $\widetilde{X} \in A(S)_{(p)}$ satisfying

(i') $\Phi_{[Q]}(\widetilde{X}) = \Phi_{[Q']}(\widetilde{X})$ for all $Q \sim_{\mathcal{F}} Q'$ with $Q \gtrsim_{\mathcal{F}} P$.

(ii') $\Phi_{[Q]}(\widetilde{X}) = \Phi_{[Q]}(X)$ for all $Q \not\gtrsim_{\mathcal{F}} P$.

(iii') $c_{[Q]}(\widetilde{X}) = c_{[Q]}(X)$ for all $Q \not\sim_{\mathcal{F}} P$, and

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{[P']}(\widetilde{X}) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{[P']}(X).$$

(iv') For every $Q \leq S$:

$$\sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(\widetilde{X}) = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(X).$$

Properties (i') and (ii') together with the assumption about X, give that $\Phi_{[Q]}(\widetilde{X}) = \Phi_{[Q']}(\widetilde{X})$ for all pairs $Q \sim_{\mathcal{F}} Q'$ with $Q \notin \mathcal{H}'$. By induction we therefore get an element $X' \in A(\mathcal{F})_{(p)}$ such that

(i") $\Phi_{[Q]}(X') = \Phi_{[Q]}(\widetilde{X})$ for all $Q \notin \mathcal{H}'$, $Q \leq S$.

(ii") $c_{[Q]}(X') = c_{[Q]}(\widetilde{X})$ for all $Q \notin \mathcal{H}'$, and for all $Q \in \mathcal{H}'$ we have

$$\sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{[Q']}(X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{[Q']}(\widetilde{X}).$$

(iii") For every $Q \leq S$:

$$\Phi_{[Q]}(X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(\widetilde{X}).$$

From (ii') and (i") as well as (iii') and (ii") we get $\Phi_{[Q]}(X') = \Phi_{[Q]}(X)$ and $c_{[Q]}(X') = c_{[Q]}(X)$ for all $Q \notin \mathcal{H}$. This proves (i) and the first part of (ii).

When $Q \in \mathcal{H}'$, we have $c_{[Q']}(\widetilde{X}) = c_{[Q']}(X)$ for $Q' \sim_{\mathcal{F}} Q$, and consequently (ii”) gives

$$\sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{[Q']}(X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{[Q']}(\widetilde{X}) = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{[Q']}(X).$$

Since $P \notin \mathcal{H}'$, we have $c_{[P']}(X') = c_{[P']}(\widetilde{X})$ for $P' \sim_{\mathcal{F}} P$, so we also get

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{[P']}(X') = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{[P']}(\widetilde{X}) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{[P']}(X);$$

this time thanks to (iii’). This proves the rest of (ii).

Finally, (iii”) combined with (iv’) say that

$$\Phi_{[Q]}(X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(\widetilde{X}) = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(X)$$

for all $Q \leq S$. □

Proposition 1.6.6. *Let $X \in A(S)_{(p)}$, and define $X' \in \widetilde{\Omega}(S)_{(p)}$ by*

$$\Phi_{[Q]}(X') := \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}(X).$$

Then $X' \in A(\mathcal{F})_{(p)}$.

Proof. This is just lemma 1.6.5 applied to $X \in A(S)_{(p)}$ with $\mathcal{H}$ being the collection of all subgroups of S. □

Remark 1.6.7. The element X' is constructed by taking a weighted mean, for each conjugacy class $[Q]_{\mathcal{F}}$, of the corresponding coordinates of X. The definition of X' thus preserves addition and scalar multiplication; and we therefore have a projection of $\mathbb{Z}_{(p)}$-modules $\pi\colon A(S)_{(p)} \to A(\mathcal{F})_{(p)}$ given by $\pi(X) := X'$.

If $X \in A(\mathcal{F})_{(p)}$ already, then $\Phi_{[Q']}(X) = \Phi_{[Q]}(X)$ for all $Q' \sim_{\mathcal{F}} Q$; hence $\Phi_{[Q]}(X') = \Phi_{[Q]}(X)$ for all $Q \leq S$, i.e. $\pi X = X$.

Lemma 1.6.8. *Let $P \leq S$, and define $\beta_{[P]} \in \widetilde{\Omega}(\mathcal{F})_{(p)}$ by*

$$\Phi_{[Q]}(\beta_{[P]}) := \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|}.$$

Then $\beta_{[P]} \in A(\mathcal{F})_{(p)}$.

Proof. Applying proposition 1.6.6 to the basis element $[P] \in A(S)_{(p)}$, we get $\beta_{[P]} := \pi([P]) \in A(\mathcal{F})_{(p)}$ with

$$\Phi_{[Q]}(\beta_{[P]}) := \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{[Q']}([P]) = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|} \in \mathbb{Z}_{(p)}$$

thanks to lemma 1.6.3. □

Theorem 1.6.9. *The elements $\beta_{[P]} \in A(\mathcal{F})_{(p)}$ defined by*

$$\Phi_{[Q]}(\beta_{[P]}) := \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,S)|} \in \mathbb{Z}_{(p)}$$

are a basis for the p-localized Burnside ring $A(\mathcal{F})_{(p)}$; with one basis element per conjugacy class $[P] \in C(\mathcal{F})$.

Proof. We order the $\mathcal{F}$-conjugacy classes $[P]$ according to decreasing order of P. The homomorphism $\Phi\colon \operatorname{Span}\{\beta_{[P]}\}_{[P]\in C(\mathcal{F})} \to \widetilde{\Omega}(\mathcal{F})_{(p)}$ is then represented a matrix M which is lower triangular with diagonal entries

$$M_{[P],[P]} = \frac{|\mathcal{F}(P,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(P,S)|} = \frac{|S|}{|P| \cdot |[P]_{\mathcal{F}}|}$$

When P is fully normalized, we have $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$, where $p \nmid k$, thanks to lemma 1.6.2. Using this, we get

$$M_{[P],[P]} = \frac{|N_S P|}{|P| \cdot k} = \frac{|W_S P|}{k}.$$

We conclude that the $\beta_{[P]}$'s are linearly independent, and furthermore that the index of $\operatorname{Span}\{\beta_{[P]}\}_{[P]\in C(\mathcal{F})}$ in $\widetilde{\Omega}(\mathcal{F})_{(p)}$ is equal to

$$\prod_{\substack{[P]\in C(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P| = \left|Obs(\mathcal{F})_{(p)}\right|.$$

It follows that the $\beta_{[P]}$'s span all of $A(\mathcal{F})_{(p)}$. □

1.7 Centric Burnside rings

For a group G, the p-centric Burnside ring $A^{p\text{-}cent}(G)$ is defined as the quotient $A(G;p)/A^{\neg p\text{-}cent}(G)$ where we mod out the ideal in $A(G;p)$ generated by the non-p-centric subgroups. In this section, we consider the similar construction for the Burnside ring of a fusion system; and we show how this again relates to a suitable quotient of $A(S)$.

Since the arguments are based on the previous sections, we still require that $\mathcal{F}$ is a saturated fusion system.

Definition 1.7.1. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if every $P' \sim_{\mathcal{F}} P$ satisfies that $C_S(P') \leq P'$. In that case, $C_S(P') = Z(P')$ for all $P' \sim_{\mathcal{F}} P$; and every $P' \sim_{\mathcal{F}} P$ is also $\mathcal{F}$-centric. If P is $\mathcal{F}$-centric, we say that $[P]_{\mathcal{F}}$ is *$\mathcal{F}$-centric* as well.

Let $C^{cent}(\mathcal{F}) \subseteq C(\mathcal{F})$ and $C^{\mathcal{F}\text{-}cent}(S) \subseteq C(S)$ be the sets of $\mathcal{F}$- and S-conjugacy classes of $\mathcal{F}$-centric subgroups respectively. Similarly, we let $C^{\neg cent}(\mathcal{F}) \subseteq C(\mathcal{F})$ and $C^{\neg \mathcal{F}\text{-}cent}(S) \subseteq C(S)$ be the sets of conjugacy classes of non-$\mathcal{F}$-centric subgroups.

Remark 1.7.2. If $\mathcal{F} = \mathcal{F}_S(G)$, then $P \leq S$ is $\mathcal{F}$-centric if and only if P is a p-centric subgroup of G, i.e. $Z(P) \in Syl_p(C_G(P))$.

Remark 1.7.3. Assume $P \leq Q$ and P is $\mathcal{F}$-centric. Any $Q' \sim_{\mathcal{F}} Q$ then contains $P' \leq Q'$ such that $P' \sim_{\mathcal{F}} P$; and consequently $C_S(Q') \leq C_S(P') \leq P' \leq Q'$. Hence Q is $\mathcal{F}$-centric as well.

Definition 1.7.4. Let $\widetilde{\Omega}^{\neg cent}(\mathcal{F})$ be the ideal of $\widetilde{\Omega}(\mathcal{F})$ consisting of $f \in \widetilde{\Omega}(\mathcal{F})$ where $f_{[Q]} = 0$ for all $\mathcal{F}$-centric Q's. We denote the quotient ring by

$$\widetilde{\Omega}^{cent}(\mathcal{F}) := \widetilde{\Omega}(\mathcal{F})/\widetilde{\Omega}^{\neg cent}(\mathcal{F}) \cong \prod_{[Q]\in C^{cent}(\mathcal{F})} \mathbb{Z}.$$

We let $A^{\neg cent}(\mathcal{F})$ be the intersection

$$A^{\neg cent}(\mathcal{F}) := A(\mathcal{F}) \cap \widetilde{\Omega}^{\neg cent}(\mathcal{F}).$$

Since $\widetilde{\Omega}^{\neg cent}(\mathcal{F})$ is an ideal of $\widetilde{\Omega}(\mathcal{F})$, and $A(\mathcal{F})$ is a subring of $\widetilde{\Omega}(\mathcal{F})$, it follows that $A^{\neg cent}(\mathcal{F})$ is an ideal of $A(\mathcal{F})$.

We define the *centric Burnside ring of* $\mathcal{F}$ as the quotient ring

$$A^{cent}(\mathcal{F}) := A(\mathcal{F})/A^{\neg cent}(\mathcal{F}).$$

Proposition 1.7.5. *Let $\alpha_{[P]}$, $[P] \in C(\mathcal{F})$, be a basis for $A(\mathcal{F})$ satisfying the properties in proposition 1.5.3. Then*

$$A^{\neg cent}(\mathcal{F}) = \mathrm{Span}\{\alpha_{[P]} \mid [P]_{\mathcal{F}} \textit{ non-centric}\},$$

and $A^{cent}(\mathcal{F})$ has a basis consisting of $\overline{\alpha_{[P]}}$ where $[P]_{\mathcal{F}}$ is centric.

Recall the basis $\beta_{[P]}$ of $A(\mathcal{F})_{(p)}$ from theorem 1.6.9. The elements $\beta_{[P]}$ where $[P]_{\mathcal{F}}$ is non-centric, gives a basis for $A^{\neg cent}(\mathcal{F})_{(p)}$; and $\overline{\beta_{[P]}}$, $[P]_{\mathcal{F}}$ centric, gives a basis for $A^{cent}(\mathcal{F})_{(p)}$.

Proof. We order the conjugacy classes of $\mathcal{F}$ the following way: The $\mathcal{F}$-centric subgroups are all ordered before the non-$\mathcal{F}$-centric subgroups; and these two subgroup-families are ordered internally according to decreasing order of the subgroups.

The $\mathcal{F}$-centric subgroups are closed under $\mathcal{F}$-conjugation and taking supergroups, i.e. if P is $\mathcal{F}$-centric, then every $Q \succsim_{\mathcal{F}} P$ is also $\mathcal{F}$-centric. We therefore have $\Phi_{[Q]}([P]) = 0$ whenever Q is $\mathcal{F}$-centric and P is non-$\mathcal{F}$-centric (see lemma 1.1.5).

With the described ordering of $C(\mathcal{F})$, the matrix corresponding to $\Phi\colon A(\mathcal{F}) \to \widetilde{\Omega}(\mathcal{F})$ becomes a block matrix

$$\begin{pmatrix} M_{cent} & 0 \\ * & M_{\neg cent} \end{pmatrix}$$

where M_{cent} and $M_{\neg cent}$ both are lower triangular matrices with non-zero diagonal entries.

The submatrix $M_{\neg cent}$ represents the restriction $\Phi\colon A^{\neg cent}(\mathcal{F}) \to \widetilde{\Omega}^{\neg cent}(\mathcal{F})$, and M_{cent} represents the induced map $\overline{\Phi}\colon A^{cent}(\mathcal{F}) \to \widetilde{\Omega}^{cent}(\mathcal{F})$ between the quotient rings. Since M_{cent} has non-zero determinant, the $\overline{\alpha_{[P]}}$'s with $[P]_{\mathcal{F}}$ centric gives a basis for $A^{cent}(\mathcal{F})$. In particular, they are linearly independent in $A^{cent}(\mathcal{F}) = A(\mathcal{F})/A^{\neg cent}(\mathcal{F})$, hence $A^{\neg cent}(\mathcal{F})$ is spanned by the $\alpha_{[P]}$'s with $[P]_{\mathcal{F}}$ non-centric.

A completely similar argument gives the stated results about the p-localizations $A^{\neg cent}(\mathcal{F})_{(p)}$ and $A^{cent}(\mathcal{F})_{(p)}$. □

Remark 1.7.6. Assume $\mathcal{F} = \mathcal{F}_S(G)$. We let

$$\widetilde{\Omega}^{\neg p\text{-}cent}(G;p) = \prod_{\substack{[P]\in C(\mathcal{F}) \\ P \text{ is non-}p\text{-centric in } G}} \mathbb{Z} = \widetilde{\Omega}^{\neg cent}(\mathcal{F}).$$

Furthermore, we let $A^{\neg p\text{-}cent}(G) \leq A(G;p)$ be the ideal generated by $[P] \in A(G;p)$ where P is a non-p-centric p-subgroup of G; then $A^{\neg p\text{-}cent}(G) = A(G;p) \cap \widetilde{\Omega}^{\neg p\text{-}cent}(G;p)$. The p-centric Burnside ring of G is defined as the quotient $A^{p\text{-}cent}(G) := A(G;p)/A^{\neg p\text{-}cent}(G)$.

Using corollary 1.4.7 we then see that $A^{\neg cent}\mathcal{F}_{(p)} = A^{\neg p\text{-}cent}(G)_{(p)}$ and $A^{cent}(\mathcal{F})_{(p)} = A^{p\text{-}cent}(G)_{(p)}$.

Definition 1.7.7. We consider the map $\Psi\colon \widetilde{\Omega}(\mathcal{F}) \to Obs(\mathcal{F})$. Using the same basis-ordering as in the proof of proposition 1.7.5, Ψ is represented by a block matrix

$$\begin{pmatrix} N_{cent} & 0 \\ * & N_{\neg cent} \end{pmatrix}$$

since $\Psi_{[Q]}(f) = 0$ whenever $f \in \widetilde{\Omega}^{\neg cent}(\mathcal{F})$ and Q is $\mathcal{F}$-centric. Furthermore N_{cent} and $N_{\neg cent}$ both have only 1's in the diagonal. It follows that the restriction

$$\Psi\colon \widetilde{\Omega}^{\neg cent}(\mathcal{F}) \to \prod_{\substack{[Q]\in C^{\neg cent}(\mathcal{F}) \\ Q \text{ f.n.}}} \mathbb{Z}/\left|W_S Q\right|\mathbb{Z}$$

is surjective.

We define

$$Obs^{\neg cent}(\mathcal{F}) := \prod_{\substack{[Q]\in C^{\neg cent}(\mathcal{F})\\ Q \text{ f.n.}}} \mathbb{Z}/\left|W_S Q\right|\mathbb{Z}$$

which thus is the cokernel of

$$\Phi\colon A^{\neg cent}(\mathcal{F}) \to \widetilde{\Omega}^{\neg cent}(\mathcal{F}).$$

We also define $Obs^{cent}(\mathcal{F})$ as the quotient module

$$Obs^{cent}(\mathcal{F}) := Obs(\mathcal{F})/Obs^{\neg cent}(\mathcal{F}) \cong \prod_{\substack{[Q]\in C^{cent}(\mathcal{F})\\ Q \text{ f.n.}}} \mathbb{Z}/\left|W_S Q\right|\mathbb{Z}.$$

The matrix N_{cent} then represents the induced homomorphism $\overline{\Psi}\colon \widetilde{\Omega}^{cent}(\mathcal{F}) \to Obs^{cent}(\mathcal{F})$; hence $\overline{\Psi}$ is given by

$$\overline{\Psi}_{[Q]}(f) = \sum_{\overline{s}\in W_S Q} f_{[\langle s\rangle Q]} \pmod{|W_S Q|}$$

when $Q \leq S$ is a fully normalized, $\mathcal{F}$-centric subgroup.

Remark 1.7.8. All these $\mathbb{Z}$-modules and homomorphisms fit together in a square of short-exact sequences (where we have omitted the 0 at each end):

$$\begin{array}{ccccc}
A^{cent}(\mathcal{F}) & \xrightarrow{\overline{\Phi}} & \widetilde{\Omega}^{cent}(\mathcal{F}) & \xrightarrow{\overline{\Psi}} & Obs^{cent}(\mathcal{F}) \\
\uparrow & & \uparrow & & \uparrow \\
A(\mathcal{F}) & \xrightarrow{\Phi} & \widetilde{\Omega}(\mathcal{F}) & \xrightarrow{\Psi} & Obs(\mathcal{F}) \\
\uparrow & & \uparrow & & \uparrow \\
A^{\neg cent}(\mathcal{F}) & \xrightarrow{\Phi} & \widetilde{\Omega}^{\neg cent}(\mathcal{F}) & \xrightarrow{\Psi} & Obs^{\neg cent}(\mathcal{F})
\end{array}$$

The top row is exact since all the other rows and columns are exact.

Definition 1.7.9 (The analogous constructions for $A(S)$ and $\widetilde{\Omega}(S)$)**.** The stated results about the following constructions are all proved completely analogously to 1.7.4–1.7.8.

Let $\widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S)$ be the ideal in $\widetilde{\Omega}(S)$ consisting of the $f \in \widetilde{\Omega}(S)$ with $f_{[Q]} = 0$ for all Q that are $\mathcal{F}$-centric. We denote the quotient ring by

$$\widetilde{\Omega}^{\mathcal{F}\text{-}cent}(S) := \widetilde{\Omega}(S)/\widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S) \cong \prod_{[Q]\in C^{\mathcal{F}\text{-}cent}(S)} \mathbb{Z}.$$

Let $A^{\neg\mathcal{F}\text{-}cent}(S) := A(S) \cap \widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S)$ which is an ideal of $A(S)$. We denote the quotient ring by $A^{\mathcal{F}\text{-}cent}(S)$.

The ideal $A^{\neg\mathcal{F}\text{-}cent}(S)$ is generated by the basis elements $[P] \in A(S)$ where P is non-$\mathcal{F}$-centric; i.e. an S-set X is in $A^{\neg\mathcal{F}\text{-}cent}(S)$ if and only if all stabilizer subgroups of $x \in X$ are non-$\mathcal{F}$-centric. The quotient ring $A^{\mathcal{F}\text{-}cent}(S)$ has a basis consisting of $\overline{[P]}$ where P is $\mathcal{F}$-centric.

We define the obstruction groups

$$Obs^{\neg\mathcal{F}\text{-}cent}(S) := \prod_{[Q]\in C^{\neg\mathcal{F}\text{-}cent}(S)} \mathbb{Z}/\left|W_S Q\right|\mathbb{Z}$$

and

$$Obs^{\mathcal{F}\text{-}cent}(S) := Obs(S)/Obs^{\neg\mathcal{F}\text{-}cent}(S) \cong \prod_{[Q]\in C^{\mathcal{F}\text{-}cent}(S)} \mathbb{Z}/\left|W_S Q\right|\mathbb{Z}.$$

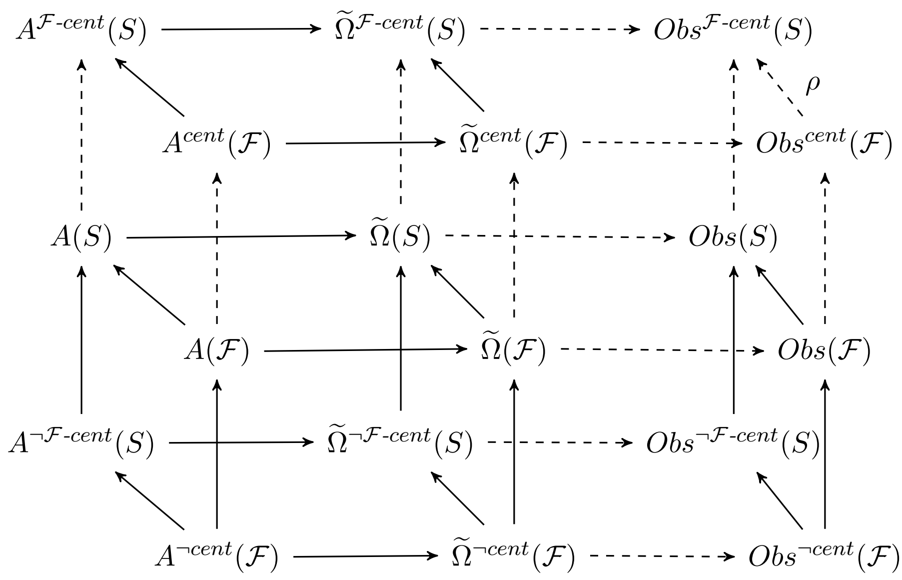
Then $\Psi\colon \widetilde{\Omega}(S) \to Obs(S)$ restricts to $\Psi\colon \widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S) \to Obs^{\neg\mathcal{F}\text{-}cent}(S)$ as well as induces $\overline{\Psi}\colon \widetilde{\Omega}^{\mathcal{F}\text{-}cent}(S) \to Obs^{\mathcal{F}\text{-}cent}(S)$. The induced homomorphism $\overline{\Psi}$ is given by

$$\overline{\Psi}_{[Q]}(f) = \sum_{\overline{s}\in W_S Q} f_{[\langle s\rangle Q]} \pmod{|W_S Q|}$$

when $Q \leq S$ is $\mathcal{F}$-centric.

All these maps fit together in a commutative square of short-exact sequences similar to remark 1.7.8.

Lemma 1.7.10. *We have the following commutative diagram of $\mathbb{Z}$-modules. Every row and column is short-exact (where we have omitted the 0 at each end); and all the non-dashed arrows denote injective homomorphisms.*



Proof. We have $\widetilde{\Omega}^{\neg cent}(\mathcal{F}) = \widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S) \cap \widetilde{\Omega}(\mathcal{F})$ and $A^{\neg cent}(\mathcal{F}) = A(S) \cap \widetilde{\Omega}^{\neg\mathcal{F}\text{-}cent}(S) \cap \widetilde{\Omega}(\mathcal{F})$. The diagram then follows by repeated application of lemma 1.3.1. □

Proposition 1.7.11. *The induced homomorphism $\rho\colon Obs^{cent}(\mathcal{F}) \to Obs^{\mathcal{F}\text{-}cent}(S)$ is injective; and $A^{cent}(\mathcal{F}) = A^{\mathcal{F}\text{-}cent}(S) \cap \widetilde{\Omega}^{cent}(\mathcal{F})$ as subrings of $\widetilde{\Omega}^{\mathcal{F}\text{-}cent}(S)$.*

Proof. Consider the top layer of the diagram in lemma 1.7.10:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\mathcal{F}\text{-}cent}(S) & \xrightarrow{\overline{\Phi}^S} & \widetilde{\Omega}^{\mathcal{F}\text{-}cent}(S) & \xrightarrow{\overline{\Psi}^S} & Obs^{\mathcal{F}\text{-}cent}(S) & \longrightarrow & 0 \\ & & \overline{r}\uparrow & & \overline{i^*}\uparrow & & \rho\uparrow & & \\ 0 & \longrightarrow & A^{cent}(\mathcal{F}) & \xrightarrow{\overline{\Phi}^{\mathcal{F}}} & \widetilde{\Omega}^{cent}(\mathcal{F}) & \xrightarrow{\overline{\Psi}^{\mathcal{F}}} & Obs^{cent}(\mathcal{F}) & \longrightarrow & 0 \end{array}$$

Let $\overline{\Psi}^{\mathcal{F}}(f) \in Obs^{cent}(\mathcal{F})$ be an arbitrary element of $\ker\rho$, i.e. $\rho(\overline{\Psi}^{\mathcal{F}}(f)) = 0$. For every fully normalized, $\mathcal{F}$-centric $Q \leq S$ we then have

$$\overline{\Psi}^{\mathcal{F}}(f)_{[Q]_{\mathcal{F}}} = \sum_{\overline{s}\in W_S Q} f_{[\langle s\rangle Q]} \pmod{|W_S Q|} = \overline{\Psi}^{S}(\overline{i^*}(f))_{[Q]_S} = \rho(\overline{\Psi}^{\mathcal{F}}(f))_{[Q]_S} = 0.$$

Because $\overline{\Psi}^{\mathcal{F}}(f)_{[Q]_{\mathcal{F}}} = 0$ for all fully normalized, $\mathcal{F}$-centric $Q \leq S$, we conclude that $\overline{\Psi}^{\mathcal{F}}(f) = 0$; so $\ker\rho = 0$.

Lemma 1.3.1 then tells us that $A^{cent}(\mathcal{F}) = A^{\mathcal{F}\text{-}cent}(S) \cap \widetilde{\Omega}^{cent}(\mathcal{F})$. □

The Diaz-Libman centric Burnside ring

What follows is a short introduction to the centric Burnside ring $\widetilde{A}^{cent}(\mathcal{F})$ as defined by Antonio Diaz and Assaf Libman in [DL09], as well as a proof that $\widetilde{A}^{cent}(\mathcal{F})_{(p)} \cong A^{cent}(\mathcal{F})_{(p)}$.

Definition 1.7.12. For $P, Q \in S$, the group of inner automorphisms $\mathrm{Inn}(P)$ acts on $\mathcal{F}(Q, P)$ by post-composition. We define the *centric orbit category of $\mathcal{F}$* as the category $\mathcal{C}$ with the $\mathcal{F}$-centric subgroups as objects and morphism sets $\mathcal{C}(Q, P) := Inn(P)\backslash\mathcal{C}(Q, P)$.

Proposition 1.7.13 ([DL09, Proposition 2.7])**.** *Let $Q, P \leq S$ be $\mathcal{F}$-centric, and let $s \in N_S Q$. Then there is a bijection $\mathcal{C}(\langle s\rangle Q, P) \simeq \mathcal{C}(Q, P)^s$, where $C(Q, P)^s$ is the fixed-point set when $s \in N_S Q$ acts by pre-conjugation.*

Proof. [DL09]. □

Definition 1.7.14. The Burnside ring $\widetilde{A}^{cent}(\mathcal{F})$ as defined by Diaz and Libman has a basis element $[P] \in \widetilde{A}^{cent}(\mathcal{F})$ for each $[P] \in C^{cent}(\mathcal{F})$; and the ring is characterized by a mark homomorphism $\widetilde{\Phi}\colon \widetilde{A}^{cent}(\mathcal{F}) \hookrightarrow \widetilde{\Omega}^{cent}(\mathcal{F})$ defined by

$$\widetilde{\Phi}_{[Q]}([P]) := |\mathcal{C}(Q, P)|$$

for all $\mathcal{F}$-centric $P, Q \leq S$.

Proposition 1.7.15. *We have* $\widetilde{A}^{cent}(\mathcal{F}) \leq A^{cent}(\mathcal{F})$*, and in the p-localized case* $\widetilde{A}^{cent}(\mathcal{F})_{(p)} = A^{cent}(\mathcal{F})_{(p)}$ *as subrings of* $\widetilde{\Omega}^{cent}(\mathcal{F})_{(p)}$.

Proof. Let $[P]_\mathcal{F} \in C^{cent}(\mathcal{F})$ and consider $\widetilde{\Phi}([P]) \in \widetilde{\Omega}^{cent}(\mathcal{F})$. If we show that the homomorphism $\overline{\Psi}\colon \widetilde{\Omega}^{cent}(\mathcal{F}) \to Obs^{cent}(\mathcal{F})$ sends $\widetilde{\Phi}([P])$ to 0, then $\widetilde{\Phi}([P])$ corresponds to an element $\delta_{[P]}$ of $A^{cent}(\mathcal{F})$.

For every Q that is fully normalized and $\mathcal{F}$-centric, $N_S Q$ acts on $\mathcal{C}(Q,P)$ by pre-conjugation. Every $q \in Q$ acts trivially on $\mathcal{C}(Q,P)$ since for all $\varphi \in \mathcal{C}(Q,P)$ we have

$$[\varphi \circ c_q] = [c_{\varphi q} \circ \varphi] = [\varphi]$$

in $\mathcal{C}(Q,P)$ since $c_{\varphi q} \in Inn(P)$. We thus have an induced action of $W_S Q$ on $\mathcal{C}(Q,P)$ by pre-conjugation, which in turn gives us

$$\begin{aligned}
\overline{\Psi}_{[Q]}(\widetilde{\Phi}([P])) &= \sum_{\overline{s}\in W_S Q} \widetilde{\Phi}_{[\langle s\rangle Q]}([P]) \\
&= \sum_{\overline{s}\in W_S Q} |\mathcal{C}(\langle s\rangle\, Q, P)| \\
&= \sum_{\overline{s}\in W_S Q} |\mathcal{C}(Q,P)^s| \\
&= |W_S Q| \cdot |\mathcal{C}(Q,P)/W_S Q| \equiv 0 \pmod{|W_S Q|}.
\end{aligned}$$

This shows that $\overline{\Psi}(\widetilde{\Phi}([P])) = 0$, so there is a $\delta_{[P]} \in A^{cent}(\mathcal{F})$ with $\overline{\Phi}(\delta_{[P]}) = \widetilde{\Phi}([P])$. We conclude that $\widetilde{A}^{cent}(\mathcal{F}) \leq A^{cent}(\mathcal{F})$ as subrings of $\widetilde{\Omega}^{cent}(\mathcal{F})$.

For all $\mathcal{F}$-centric $P, Q \leq S$ we have

$$|\mathcal{C}(Q,P)| = |\mathcal{F}(Q,P)|\, \frac{|Z(Q)|}{|P|},$$

since $C_P(\varphi Q) = Z(\varphi Q)$ for all $\varphi \in \mathcal{F}(Q,P)$ because Q is $\mathcal{F}$-centric.

For every fully normalized, $\mathcal{F}$-centric $Q \leq S$, we then have

$$|\mathcal{C}(Q,S)| = |\mathcal{F}(Q,S)|\, \frac{|Z(Q)|}{|S|} = \frac{|S| \cdot |Z(Q)|}{|C_S Q| \cdot |S|} \cdot k = k$$

for some k coprime to p satisfying $|\mathcal{F}(Q,S)| = \frac{|S|}{|C_S Q|} \cdot k$ (see lemma 1.6.2).

We thus get that $\widetilde{\Phi}_{(p)}([S]) = \overline{\Phi}_{(p)}(\delta_{[S]})$ is invertible in $\widetilde{\Omega}^{cent}(\mathcal{F})_{(p)}$, so by lemma 1.3.3 $\delta_{[S]}$ is invertible in $A^{cent}(\mathcal{F})_{(p)}$.

Consider now the element $\delta_{[P]}/\delta_{[S]} \in A^{cent}(\mathcal{F})_{(p)}$ for each $\mathcal{F}$-centric $P \leq S$. This element satisfies

$$\overline{\Phi}_{(p)}\left(\frac{\delta_{[P]}}{\delta_{[S]}}\right)_{[Q]} = \frac{\widetilde{\Phi}_{(p)}([P])_{[Q]}}{\widetilde{\Phi}_{(p)}([S])_{[Q]}} = \frac{|\mathcal{C}(Q,P)|}{|\mathcal{C}(Q,S)|} = \frac{|\mathcal{F}(Q,P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q,P)|},$$

for all $\mathcal{F}$-centric $Q \leq S$. The element $\delta_{[P]}/\delta_{[S]}$ is then simply the basis element $\overline{\beta_{[P]}}$ of $A^{cent}(\mathcal{F})_{(p)}$; and we conclude that the $\delta_{[P]}$'s give a basis of $A^{cent}(\mathcal{F})_{(p)}$ as well. □

2 Double Burnside rings

The second part of this thesis is mostly based on sections 3-6 of the article [RS09], supplemented by some proofs and additional results of my own.

2.1 Burnside modules

The Burnside modules arise when we instead of sets with only one group action, consider "bisets" that have both a left and a right action. We define a category $\mathcal{A}$ with the bisets as morphisms; and the double Burnside ring of a group G is then the set of endomorphism $\mathcal{A}(G,G)$.

In the following, we let G, H and K denote finite groups.

Definition 2.1.1. A (G,H)*-biset* (or just (G,H)*-set*) is a set equipped with a right G-action and a left H-action such that the two actions commute. A biset is *left-free* if the left action (i.e. the H-action) is free, and it is *right-free* if the right action is free. A biset which is both left- and right-free, is called *bifree*.

Remark 2.1.2. Given a (G,H)-biset X, we obtain a $(H\times G)$-set $\widehat{X}$ with the same underlying set, and the $(H\times G)$-action given by $(h,g)x := hxg^{-1}$. Conversely, any $(H\times G)$-set corresponds to a (G,H)-biset, and we shall use this bijective correspondence heavily in the rest of the thesis.

We say that a (G,H)-biset X is (G,H)*-transitive* if X is transitive when considered as a $(H\times G)$-set.

Definition 2.1.3. The isomorphism classes of finite, left-free (G,H)-sets form a free commutative monoid with disjoint union as the addition. We define the *Burnside module of G and H*, $A(G,H)$, to be the Grothendieck group of this monoid.

Remark 2.1.4. The Burnside module $A(G,H)$ is a free $\mathbb{Z}$-module with a basis consisting of the isomorphism classes of left-free (G,H)-sets that cannot be decomposed as disjoint unions of strictly smaller (G,H)-sets. These are the left-free (G,H)-bisets that are (G,H)-transitive, i.e. transitive when considered as $(H\times G)$-sets. Proposition 2.1.8 gives a more explicit description of this basis.

Definition 2.1.5. A (G,H)*-pair* is a pair (K,φ), where $K\le G$ and $\varphi\colon K\to H$ is a group homomorphism.

From every (G,H)-pair (K,φ), we obtain a left-free (G,H)-set

$$H \times_{(K,\varphi)} G := (H \times G)/\sim$$

where $\sim$ is the relation

$$(h,kg) \sim (h\varphi(k),g), \text{ for all } g \in G,\ h \in H \text{ and } k \in K.$$

The isomorphism class of $H \times_{(K,\varphi)} G$ is an element of $A(G,H)$ which we denote by $[K,\varphi]_G^H$ or just $[K,\varphi]$.

The $(H \times G)$-set corresponding to $H \times_{(K,\varphi)} G$, is isomorphic to

$$(H \times G)/\Delta(K,\varphi),$$

where $\Delta(K,\varphi) \leq H \times G$ is the "graph" of (K,φ), i.e.

$$\Delta(K,\varphi) := \{(\varphi(k),k) \mid k \in K\}.$$

The isomorphism of $(H\times G)$-sets $H \times_{(K,\varphi)} G \xrightarrow{\sim} (H \times G)/\Delta(K,\varphi)$ is simply given by $(h,g) \mapsto (h,g^{-1})$.

Definition 2.1.6. We say that two (G,H)-pairs (K,φ) and (L,ψ) are (G,H)-*conjugate*, or just *conjugate*, written $(K,\varphi) \underset{(G,H)}{\sim} (L,\psi)$, if the graphs $\Delta(K,\varphi)$ and $\Delta(L,\psi)$ are conjugate in $H \times G$. We let $[K,\varphi]_{(G,H)}$ or $[K,\varphi]$ denote the (G,H)-conjugacy class of (K,φ); and we let $C(G,H)$ denote the set of conjugacy classes.

The element $[K,\varphi]_G^H \in A(G,H)$ depends only on the conjugacy class of $\Delta(K,\varphi)$ in $H \times G$, hence only on the class $[K,\varphi]_{(G,H)} \in C(G,H)$.

We say that (K,φ) is *subconjugate* to (L,ψ), written $(K,\varphi) \underset{(G,H)}{\precsim} (L,\psi)$, if $\Delta(K,\varphi)$ subconjugate to $\Delta(L,\psi)$ in $H \times G$. In that case, every element of $[K,\varphi]_{(G,H)}$ is subconjugate to every element of $[L,\psi]_{(G,H)}$; and we say that $[K,\varphi]_{(G,H)}$ is *subconjugate* to $[L,\psi]_{(G,H)}$ as well (written $[K,\varphi] \underset{(G,H)}{\precsim} [L,\psi]$).

Lemma 2.1.7. *Let (K,φ) and (L,ψ) be (G,H)-pairs. Then $(K,\varphi) \underset{(G,H)}{\precsim} (L,\psi)$ if and only if there exist $x \in N_G(K,L)$ and $y \in H$ such that $c_y \circ \varphi = \psi \circ c_x$.*

We have $(K,\varphi) \underset{(G,H)}{\sim} (L,\psi)$ if and only if the additional condition $|K| = |L|$ is satisfied.

Proof. Assume $(K,\varphi) \underset{(G,H)}{\precsim} (L,\psi)$. Then by definition we have $\Delta(K,\varphi)$ subconjugate to $\Delta(L,\psi)$ in $H\times G$, so there exist $x \in G$, $y \in H$ such that ${}^{(y,x)}\Delta(K,\varphi) \leq \Delta(L,\psi)$. The elements of ${}^{(y,x)}\Delta(K,\varphi)$ are $(c_y(\varphi(k)), c_x(k))$, with $k \in K$.

The inclusion ${}^{(y,x)}\Delta(K,\varphi) \leq \Delta(L,\psi)$ then simply means that $c_x(k) \in L$ and $\psi(c_x(k)) = c_y(\varphi(k))$ for all $k \in K$; hence $x \in N_G(K,L)$ and $\psi c_x = c_y \varphi$.

Conversely, if $\psi c_x = c_y\varphi$, then we get ${}^{(y,x)}\Delta(K,\varphi) \le \Delta(L,\psi)$.

The graphs $\Delta(K,\varphi)$ and $\Delta(L,\psi)$ are conjugate if and only if $\Delta(K,\varphi) \precsim_{H\times G} \Delta(L,\psi)$ and the two sets have the same size. The second part of the lemma then follows by remarking that $|\Delta(K,\varphi)| = |K|$ and $|\Delta(L,\psi)| = |L|$. □

Proposition 2.1.8 ([RS09, Lemma 3.6])**.** *The Burnside module $A(G,H)$ is a free $\mathbb{Z}$-module with one basis element $[K,\varphi]_G^H$ for each conjugacy class $[K,\varphi]$ of (G,H)-pairs. We call this the* standard basis *of $A(G,H)$.*

Proof. Let (K,φ) and (L,ψ) be two (G,H)-pairs. The $(H\times G)$-sets $(H\times G)/\Delta(K,\varphi)$ and $(H\times G)/\Delta(L,\psi)$ are isomorphic if and only if $(K,\varphi) \underset{(G,H)}{\sim} (L,\psi)$. Thus the different conjugacy classes of (G,H)-pairs give rise to non-isomorphic, transitive $(H\times G)$-sets. To complete the proof, we therefore only need to prove that every left-free, (G,H)-transitive (G,H)-biset is of the form $H\times_{(K,\varphi)} G$ for some pair (K,φ); or equivalently that every transitive $(H\times G)$-set with free H-action is of the form $(H\times G)/\Delta(K,\varphi)$.

Let $(H\times G)/X$ be transitive with $X\le H\times G$; and assume that the H-action is free. Let $(h,g)\in X$; then since the H-action is free, all the elements $\overline{(yh,g)}\in (H\times G)/X$, with $y\in H$, are different. In particular, $(yh,g)\in X$ only holds for $y=1$. For every $g\in G$ there is thus at most one element of X with g as the second coordinate.

We put $K := \pi_2(X)\le G$. Then every $k\in K$, satisfies that there exist a unique $\varphi(k)\in H$ such that $(\varphi k,k)\in X$. Because X is a subgroup of $H\times G$, we then get for all $k,k'\in K$ that

$$(\varphi(kk'),kk') = (\varphi k,k)\cdot(\varphi k',k').$$

Hence $\varphi(kk')=\varphi(k)\varphi(k')$, so φ is a homomorphism and $X=\Delta(K,\varphi)$. □

Definition 2.1.9. For each $[K,\varphi]\in C(G,H)$, we let $c_{[K,\varphi]}\colon A(G,H)\to\mathbb{Z}$ be the homomorphism sending $X\in A(G,H)$ to the $[K,\varphi]$-coefficient of X (when written in the standard basis of $A(G,H)$). Hence we have

$$X = \sum_{[K,\varphi]\in C(G,H)} c_{[K,\varphi]}(X)\cdot[K,\varphi]$$

for all $X\in A(G,H)$.

Definition 2.1.10. Let X be a (G,H)-biset, and let (K,φ) be a (G,H)-pair. We then define the set

$$X^{(K,\varphi)} := \{x\in X \mid \forall k\in K : \varphi(k)x = xk\} = \{x\in X \mid \forall k\in K : \varphi(k)xk^{-1} = x\}.$$

This is just the set of fixed-points $X^{\Delta(K,\varphi)}$ when we consider X as a $(H\times G)$-set. Since $\left|X^{\Delta(K,\varphi)}\right|$ only depends on the conjugacy class of $\Delta(K,\varphi)$, then $\left|X^{(K,\varphi)}\right|$ depends only on $[K,\varphi]_{(G,H)}$ as well.

Definition 2.1.11. The Burnside module $A(G,H)$ can be considered as the submodule of $A(H\times G)$ generated by the basis elements $[\Delta(K,\varphi)]\in A(H\times G)$, corresponding to the basis elements $[K,\varphi]\in A(G,H)$. The $[\Delta(K,\varphi)]$'s are basis elements of $A(H\times G)$ since they are transitive $(H\times G)$-sets. The module $A(G,H)$ then inherits a *(module) homomorphism of marks* $\Phi\colon A(G,H)\hookrightarrow \prod_{X\in C(H\times G)}\mathbb{Z}$ (see proposition 1.1.7).

Any subgroup $X\le H\times G$ that is subconjugate in $H\times G$ to a graph $\Delta(K,\varphi)$, is itself a graph $X=\Delta(L,\psi)$ for some $L\le G$ and $\psi\colon L\to H$. As a consequence, $\Phi(A(G,H))$ is contained in the subring $\prod_{[K,\varphi]}\mathbb{Z}$; and Φ becomes an injective module homomorphism

$$\Phi\colon A(G,H)\xrightarrow{\prod_{[K,\varphi]}\Phi_{[K,\varphi]}}\prod_{[K,\varphi]\in C(G,H)}\mathbb{Z}.$$

Here $\Phi_{[K,\varphi]}\colon A(G,H)\to\mathbb{Z}$ denotes the $[K,\varphi]$-coordinate of Φ; which is given by

$$\Phi_{[K,\varphi]}(X)=\left|X^{\Delta(K,\varphi)}\right|=\left|X^{(K,\varphi)}\right|$$

for any left-free (G,H)-biset X – and then extended linearly.

Lemma 2.1.12. *Let $[K,\varphi],[L,\psi]\in C(G,H)$. Then*

$$\Phi_{[K,\varphi]}([L,\psi]_G^H)=\frac{|N_{\varphi,\psi}|}{|L|}\cdot|C_H(\varphi K)|\,.$$

where

$$N_{\varphi,\psi}=\{x\in N_G(K,L)\mid \exists y\in H\colon c_y\varphi=\psi c_x\}.$$

In particular, $\Phi_{[K,\varphi]}([L,\psi]_G^H)\neq 0$ if and only if $[K,\varphi]\underset{(G,H)}{\precsim}[L,\psi]$.

Proof. From lemma 1.1.5, we know that

$$\begin{aligned}\Phi_{[K,\varphi]}([L,\psi])&=\frac{|N_{H\times G}(\Delta(K,\varphi),\Delta(L,\psi))|}{|\Delta(L,\psi)|}\\&=\frac{|\{(y,x)\in H\times G\mid c_y\varphi=\psi c_x\}|}{|L|}\\&=\frac{|N_{\varphi,\psi}|\cdot|C_H(\varphi K)|}{|L|}\end{aligned}$$

The equality $N_{H\times G}(\Delta(K,\varphi),\Delta(L,\psi))=\{(y,x)\in H\times G\mid c_y\varphi=\psi c_x\}$ follows from arguments similar to the proof of lemma 2.1.7. ☐

Remark 2.1.13. Let $\varphi\colon K\to H$ and $\psi\colon L\to H$ where $K,L\le G$. The set $N_{\varphi,\psi}$ defined in the previous lemma, is then a bifree (N_φ,N_ψ)-biset:

Let $g\in N_\varphi$, $u\in N_{\varphi,\psi}$ and $h\in N_\psi$. Then there exist $x,n,y\in H$ such that $c_x\varphi=\varphi c_g$, $c_n\varphi=\psi c_u$ and $c_y\psi=\psi c_h$; which combines to

$$c_{ynx}\varphi=c_yc_nc_x\varphi=c_yc_n\varphi c_g=c_y\psi c_uc_g=\psi c_hc_uc_g=\psi c_{hug}.$$

We therefore have $hug \in N_{\varphi,\psi}$; and hence N_φ and N_ψ act by right- and left-multiplication respectively. Since the actions are just the translation-actions in G, both actions are free.

In particular, since $L \leq N_\psi$, L acts freely on $N_{\varphi,\psi}$ so the fraction $\frac{|N_{\varphi,\psi}|}{|L|}$ in lemma 2.1.12 is an integer.

2.2 The Burnside category

Definition 2.2.1. There is a composition $\circ\colon A(H,K) \times A(G,H) \to A(G,K)$ given by

$$[Y] \circ [X] := [Y \times_H X]$$

for bisets $Y \in A(H,K)$ and $X \in A(G,H)$, and then extending bilinearly.

The biset $Y \times_H X$ is defined as $Y \times X/\sim$ where the relation $\sim$ is $(y, hx) \sim (yh, x)$ for all $x \in X$, $y \in Y$ and $h \in H$. Furthermore, $Y \times_H X$ inherits left-freeness: Since X is left-free, an element (y,x) is only $\sim$-equivalent to one element with second-coordinate x - namely itself. So if $\overline{(ky,x)} = \overline{(k'y,x)}$ in $Y \times_H X$, then it follows that $ky = k'y$; and since Y is left-free as well, we must have $k = k'$.

Lemma 2.2.2. *Let $A \leq G$ and $B \leq H$ be subgroups, and let $X \in A(G,H)$ be a biset. We can then consider X as a (A,B)-biset X_A^B by restricting the actions; and we have*

$$X_A^B = [B, id_B]_H^B \circ X_G^H \circ [A, \mathrm{incl}]_A^G.$$

Proof. The element $[B, id_B]_H^B$ is just represented by H considered as a (H,B)-set, and $[A, \mathrm{incl}]_A^G$ is just G considered as (A,G)-set. We then clearly have

$$H \times_H X \times_G G \cong X$$

as (A,B)-sets. □

Definition 2.2.3. The *Burnside category* $\mathcal{A}$ is the category with the finite groups as objects and with morphism sets $\mathrm{Mor}_{\mathcal{A}}(G,H) := A(G,H)$. The composition in $\mathcal{A}$ is the composition $\circ$ defined in 2.2.1.

Each morphism set is a finitely generated free $\mathbb{Z}$-module, and the composition is bilinear. The identity morphism of a group G, is $[G, id]_G^G \in A(G,G)$, which is just G considered as a (G,G)-biset.

Definition 2.2.4. The *double Burnside ring* of a group G, is simply defined to be the ring $A(G,G) = \mathrm{Mor}_{\mathcal{A}}(G,G)$ of endomorphisms in the Burnside category. The ring $A(G,G)$ is unital since $[G, id]_G^G$ is a neutral element for the composition $\circ$.

Proposition 2.2.5. *The composition of basis elements can be described via the following double coset formula:*

$$[B,\psi]_H^K \circ [A,\varphi]_G^H = \sum_{\overline{y}\in B\backslash H/\varphi(A)} [A\cap\varphi^{-1}(B^y),\psi c_y \varphi]_G^K. \tag{2.1}$$

In particular if $B = H$, the formula simplifies to

$$[H,\psi]_H^K \circ [A,\varphi]_G^H = [A,\psi\varphi]_G^K;$$

similarly if $\varphi(A) = H$, then

$$[B,\psi]_H^K \circ [A,\varphi]_G^H = [\varphi^{-1}(B),\psi\varphi]_G^K.$$

Proof. The (G,K)-set that results from the composition, can be described as

$$[B,\psi]_H^K \circ [A,\varphi]_G^H \cong K\times H\times G/\sim,$$

where $(z,y,ax)\sim(z,y\varphi(a),x)$ and $(z,by,x)\sim(z\psi(b),y,x)$ for all $x\in G$, $y\in H$, $z\in K$, $a\in A$ and $b\in B$. We will describe how this biset $K\times H\times G/\sim$ decomposes into (G,K)-orbits.

Since $z^{-1}(z,y,x)x^{-1} = (1,y,1)$, every (G,K)-orbit contains at least one element on the form $(1,y,1)$. For any element $(1,y,1)$ we then try to determine the stabiliser subgroup in $K\times G$.

Assume that $k\in K$, $g\in G$ satisfy $k(1,y,1)g^{-1}\sim(1,y,1)$ or equivalently that $k^{-1}(1,y,1)g\sim(1,y,1)$. Then we must have $g\in A$, since otherwise $(*,*,g)\sim(*,*,1)$ is impossible. This gives us that

$$k^{-1}(1,y,1)g = (k^{-1},y,g)\sim(k^{-1},y\varphi(g),1) = (k^{-1},c_y(\varphi(g))\cdot y,1).$$

Since $(k^{-1},c_y(\varphi(g))\cdot y,1)\sim(1,y,1)$, we conclude that $c_y(\varphi(g))\in B$, i.e. $g\in\varphi^{-1}(B^y)$; and we get

$$(k^{-1},c_y(\varphi(g))\cdot y,1)\sim(k^{-1}\cdot\psi(c_y(\varphi(g))),y,1).$$

Finally we conclude that we must have $k=(\psi c_y\varphi)(g)$. Conversely, if $g\in A\cap\varphi^{-1}(B^y)$ and $k=(\psi c_y\varphi)(g)$, then the above calculations show that $k^{-1}(1,y,1)g\sim(1,y,1)$. The stabiliser subgroup of $(1,y,1)$ is therefore $\Delta(A\cap\varphi^{-1}(B^y),\psi c_y\varphi)$; hence the orbit of $(1,y,1)$ is isomorphic to

$$(K\times G)/\Delta(A\cap\varphi^{-1}(B^y),\psi c_y\varphi)$$

and thus an element of the isomorphism class $[A\cap\varphi^{-1}(B^y),\psi c_y\varphi]$.

When are two elements $(1, y, 1)$ and $(1, y', 1)$ then in the same (G, K)-orbit? If they are in the same orbit, then there exist $k \in K$, $g \in G$ such that $(1, y', 1) \sim k(1, y, 1)g = (k, y, g)$. Then there must also exist $a \in A$ and $b \in B$ such that $(1, y', 1) = (k\psi(b), b^{-1}y\varphi(a), a^{-1}g)$; from which we then conclude $y' \in By\varphi(A)$.

Conversely, if $y' \in By\varphi(A)$, we have $y' = by\varphi(a)$ for some $a \in A$, $b \in B$. It then follows that

$$(1, y', 1) = (1, by\varphi(a), 1) \sim (\psi(b), y, a) = \psi(b)(1, y, 1)a$$

so $(1, y, 1)$ and $(1, y', 1)$ are in the same (G, K)-orbit.

In total, $K \times H \times G/\sim$ consists of one orbit for each $\overline{y} \in B\backslash H/\varphi(A)$, and that orbit is a representative of $[A \cap \varphi^{-1}(B^y), \psi c_y \varphi]$. □

Corollary 2.2.6. *If $A \leq G$ and $\varphi \in \mathrm{Hom}(A, H)$, then*

$$[A, \varphi]_G^H = [\varphi A, \mathrm{incl}]_{\varphi A}^H \circ [A, \varphi]_A^{\varphi A} \circ [A, id_A]_G^A.$$

Proof. Immediate from the two simple cases of proposition 2.2.5. □

Definition 2.2.7. The *augmentation* $\varepsilon\colon A(G, H) \to \mathbb{Z}$ is the homomorphism defined on bisets $X \in A(G, H)$ as the number of (free) orbits of the left action:

$$\varepsilon(X) := |H\backslash X| = |X| \,/\, |H|\,,$$

and then extending linearly to all of $A(G, H)$. A basis element $[K, \varphi] \in A(G, H)$ has $|G \times H| \,/ |\Delta(K, \varphi)|$ elements, so $\varepsilon([K, \varphi]) = |G| \,/\, |K|$.

For bisets $X \in A(G, H)$ and $Y \in A(H, K)$, we have $|Y \times_H X| = |Y| \cdot |X| \,/\, |H|$ since the action of H on X is free. It follows that

$$\varepsilon([Y] \circ [X]) = \tfrac{|Y| \cdot |X|}{|H|} / \,|K| = \tfrac{|Y|}{|K|} \cdot \tfrac{|X|}{|H|} = \varepsilon([Y]) \cdot \varepsilon([X]).$$

By linearity this holds for all elements in the Burnside modules, hence we conclude that ε sends composition in $\mathcal{A}$ to products in $\mathbb{Z}$. In particular, $\varepsilon\colon A(G, G) \to \mathbb{Z}$ is a ring homomorphism.

The free Burnside modules

Definition 2.2.8. Let X be a (G, H)-biset, then we get the *opposite* (H, G)-biset X^{op} by reversing the actions on X: For any $x \in X$, let x^{op} be x considered as an element of X^{op}, then $gx^{\mathrm{op}}h := h^{-1}xg^{-1}$. The $(H \times G)$-set corresponding to X, and the $(G \times H)$-set corresponding to X^{op} are equivalent in the obvious way: $(h, g)x = hxg^{-1} = gx^{\mathrm{op}}h^{-1} = (g, h)x^{\mathrm{op}}$.

The opposite of a left-free biset is right-free and vice-versa, so we don't get a map between Burnside modules in general. We do however get a correspondence between the bifree bisets.

Definition 2.2.9. The *free Burnside module* $A_{fr}(G,H)$ is the submodule of $A(G,H)$ generated by the isomorphism classes of bifree (G,H)-sets. We call all the elements $X \in A_{fr}(G,H)$ *bifree.*

We then have the *opposite homomorphism*

$$\mathrm{op}\colon A_{fr}(G,H) \to A_{fr}(H,G)$$

sending a bifree biset to its opposite.

Definition 2.2.10. An element $X \in A_{fr}(G,G)$ is called *symmetric* if $X^{\mathrm{op}} = X$.

Lemma 2.2.11. *Let $X \in A_{fr}(G,H)$ and $Y \in A_{fr}(H,K)$. Then $Y \circ X \in A_{fr}(G,K)$ and $(Y \circ X)^{\mathrm{op}} = X^{\mathrm{op}} \circ Y^{\mathrm{op}}$.*

Proof. For any (G,H)-biset X and (H,K)-biset Y, we have $(Y \times_H X)^{\mathrm{op}} = X^{\mathrm{op}} \times_H Y^{\mathrm{op}}$. As stated in definition 2.2.1, the composition of left-free bisets is left-free. When X and Y are right-free, the bisets X^{op} and Y^{op} are left-free; we conclude that $(Y \times_H X)^{\mathrm{op}}$ is left-free, i.e. $Y \times_H X$ is right-free.

This proves that the composition of bifree bisets is again bifree, and that $([Y] \circ [X])^{\mathrm{op}} = [X]^{\mathrm{op}} \circ [Y]^{\mathrm{op}}$. The result then extends linearly to all of $A_{fr}(G,H)$ and $A_{fr}(H,K)$. □

Lemma 2.2.12. *A basis element $[K,\varphi] \in A(G,H)$ is bifree if and only if φ is injective. We therefore call a (G,H)-pair (K,φ)* free *if φ.*

Furthermore, since a biset X is only bifree when all (G,H)-transitive components are bifree, we conclude that $A_{fr}(G,H)$ has a basis consisting of the $[K,\varphi]_G^H$ where φ is injective.

Proof. Consider the (G,H)-set $X = H \times_{(K,\varphi)} G$ representing $[K,\varphi]$. If $\ker\varphi = 1$, then $(h,g) \sim (h,g')$ implies $g = g'$. For any $(h,g) \in X$ and $g' \in G$ satisfying $(h,g)g' \sim (h,g)$, we then have $gg' = g$, i.e. $g' = 1$; hence the G-action is free.

Assume that φ is not injective, and choose some $1 \neq k \in \ker\varphi$. We then get $(1,1)k = (1,k) \sim (1\varphi(k),1) = (1,1)$, so the G-action isn't free. □

Lemma 2.2.13. *For a free (G,H)-pair (K,φ), we have*

$$([K,\varphi]_G^H)^{\mathrm{op}} = [\varphi K, \varphi^{-1}]_H^G.$$

Proof. $[K,\varphi]_G^H$ corresponds to the $(H \times G)$-set $H \times G/\Delta(K,\varphi)$. The $(G \times H)$-set $(H \times G/\Delta(K,\varphi))^{\mathrm{op}}$ then has the action $(g,h) \cdot \overline{(y,x)} = \overline{(hy, xg^{-1})}$. We have a $(G \times H)$-isomorphism $(H \times G/\Delta(K,\varphi))^{\mathrm{op}} \xrightarrow{\sim} G \times H/\Delta(\varphi K, \varphi^{-1})$ given by $(y,x) \mapsto (x^{-1}, y^{-1})$, which is well-defined since $(y,x) \in \Delta(K,\varphi)$ exactly when $(x^{-1}, y^{-1}) \in \Delta(\varphi K, \varphi^{-1})$. □

Lemma 2.2.14. *If $X \in A_{fr}(G,H)$ and (K,φ) is a free (G,H)-pair, then*

$$\Phi_{[K,\varphi]}(X) = \Phi_{[\varphi K,\varphi^{-1}]}(X^{\mathrm{op}}).$$

Proof. It is enough to consider bisets $X \in A_{fr}(G,H)$. The fixed-point set $(X^{\mathrm{op}})^{(\varphi K,\varphi^{-1})}$ consists of the $x \in X$ such that $(\varphi k)xk^{-1} = x$ for all $\varphi k \in \varphi K$; but this is simply the set $X^{(K,\varphi)}$, hence $\left|(X^{\mathrm{op}})^{(\varphi K,\varphi^{-1})}\right| = \left|X^{(K,\varphi)}\right|$. □

2.3 The double Burnside ring of a fusion system

We introduce a concept of $\mathcal{F}$-stable elements, similar to the definition of $\mathcal{F}$-stability in the single Burnside ring. As for the single Burnside ring, we then define the double Burnside ring of $\mathcal{F}$, as the subring of $A(S,S)$ consisting of the $\mathcal{F}$-stable elements.

We also define $\mathcal{F}$-generated elements, and give results concerning the "characteristic" elements that are both $\mathcal{F}$-stable and $\mathcal{F}$-generated.

Definition 2.3.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over p-groups S_1 and S_2 respectively. Let (P,φ), (Q,ψ) be (S_1,S_2)-pairs.

We say that (P,φ) is $(\mathcal{F}_1,\mathcal{F}_2)$-*subconjugate* to (Q,ψ), written $(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (Q,\psi)$, if there exist $\rho_1 \in \mathcal{F}_1(P,Q)$ and $\rho_2 \in \mathcal{F}_2(\varphi P,\psi Q)$ such that $\rho_2\varphi = \psi\rho_1$.

Equivalently, we have $(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (Q,\psi)$ if and only if the graph $\Delta(P,\varphi)$ is $(\mathcal{F}_2 \times \mathcal{F}_1)$-subconjugate to $\Delta(Q,\psi)$. Here $\mathcal{F}_2 \times \mathcal{F}_1$ is a fusion system on $S_2 \times S_1$; and if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both saturated, then $\mathcal{F}_2 \times \mathcal{F}_1$ is also saturated (see [AKO10, Theorem 6.6]).

We say that (P,φ) and (Q,ψ) are $(\mathcal{F}_1,\mathcal{F}_2)$-*conjugate*, written $(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} (Q,\psi)$, if we have subconjugacy both ways.

To get $(\mathcal{F}_1,\mathcal{F}_2)$-conjugacy, it is actually enough to have $(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (Q,\psi)$ and $|P| = |Q|$. In that case, $\Delta(P,\varphi) \underset{\mathcal{F}_1\times\mathcal{F}_2}{\sim} \Delta(Q,\psi)$ since we have subconjugacy between two graphs of the same order. We then conclude that there are isomorphisms $\rho_1 \in \mathcal{F}_1(P,Q)$ and $\rho_2 \in \mathcal{F}_2(\varphi P,\psi Q)$ such that $\rho_2\varphi = \psi\rho_1$.

The set of $(\mathcal{F}_1,\mathcal{F}_2)$-conjugacy classes of (S_1,S_2)-pairs is denoted by $C(\mathcal{F}_1,\mathcal{F}_2)$.

$\mathcal{F}$-generation

Remark 2.3.2. Let (P,φ) be an (S,S)-pair with $\varphi \in \mathcal{F}(P,S)$. Then every (S,S)-subconjugate $(Q,\psi) \underset{(S,S)}{\precsim} (P,\varphi)$ has $\psi \in \mathcal{F}(Q,S)$. This is because lemma 2.1.7 gives $\psi = c_y\varphi c_x$ for suitable $x \in N_S(Q,P)$, $y \in S$; and $\mathcal{F}$ is closed under restriction and S-conjugation.

Definition 2.3.3. For $P, Q \leq S$, we define $A_{\mathcal{F}}(P,Q)$ to be the submodule of $A(P,Q)$ generated by the basis elements $[T,\varphi]_P^Q$ where $\varphi \in \mathcal{F}(T,Q)$ and $T \leq P$. An element $X \in A(P,Q)$ is *$\mathcal{F}$-generated* if $X \in A_{\mathcal{F}}(P,Q)$.

Let $\mathcal{A}_{\mathcal{F}}$ be the subcategory whose objects are the subgroups of S, and whose morphism sets are $A_{\mathcal{F}}(P,Q)$. This is well-defined since the composite of two $\mathcal{F}$-generated basis elements $[B,\psi]_Q^R \circ [A,\varphi]_P^Q$ is $\mathcal{F}$-generated thanks to the double coset formula (2.1); and the identity elements $[P, id]_P^P$ are $\mathcal{F}$-generated.

Every morphism in the fusion system $\mathcal{F}$ is an injective homomorphism, so $[T,\varphi]_P^Q \in A_{fr}(P,Q)$ whenever $\varphi \in \mathcal{F}(T,Q)$; hence we have $A_{\mathcal{F}}(P,Q) \leq A_{fr}(P,Q)$ for all $P, Q \leq S$.

For every $\varphi \in \mathcal{F}(T,Q)$, with $T \leq P$, we have $\varphi^{-1} \in \mathcal{F}(\varphi P, P)$. Any basis element $[T,\varphi] \in A_{\mathcal{F}}(P,Q)$ then satisfies $[T,\varphi]^{\mathrm{op}} = [\varphi T, \varphi^{-1}] \in A_{\mathcal{F}}(Q,P)$. We conclude that $A_{\mathcal{F}}(P,Q)^{\mathrm{op}} = A_{\mathcal{F}}(Q,P)$ for all $P, Q \leq S$.

$\mathcal{F}$-stability

Definition 2.3.4. We say that an element $X \in A(S,H)$ is *right $\mathcal{F}$-stable* if we for every $\varphi \in \mathcal{F}(P,S)$, $P \leq S$, have the following equation in $A(P,H)$:

$$X \circ [P,\varphi]_P^S = X \circ [P, \mathrm{incl}]_P^S.$$

Similarly, $X \in A(H,S)$ is *left $\mathcal{F}$-stable* if for every $\varphi \in \mathcal{F}(P,S)$, $P \leq S$, the following equation holds in $A(H,P)$:

$$[\varphi P, \varphi^{-1}]_S^P \circ X = [P, id_P]_S^P \circ X.$$

We say that $X \in A(S,S)$ is *(fully) $\mathcal{F}$-stable* if it is both left and right $\mathcal{F}$-stable.

$\mathcal{F}$-stability is preserved under addition, so the left, right and fully $\mathcal{F}$-stable elements form submodules of $A(S,H)$, $A(H,S)$ and $A(S,S)$ respectively.

Remark 2.3.5. A bifree element $X \in A_{fr}(S,H)$ is right $\mathcal{F}$-stable if and only if X^{op} is left $\mathcal{F}$-stable; because $([P,\varphi]_P^S)^{\mathrm{op}} = [\varphi P, \varphi^{-1}]_S^P$ (see lemma 2.2.13).

Remark 2.3.6. Every element of $A(S,H)$ is right $\mathcal{F}_S$-stable since $[P,\varphi]_P^S = [P, \mathrm{incl}]_P^S$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}_S}(P,S)$. Similarly, every element of $A(H,S)$ is left $\mathcal{F}_S$-stable; and every element of $A(S,S)$ is $\mathcal{F}_S$-stable.

Remark 2.3.7. The definition of $\mathcal{F}$-stability resembles the definition of $\mathcal{F}$-stability in the single Burnside ring $A(S)$ (see 1.4.3). We take an S-set and restrict the action to $P \leq S$ in two ways: We restrict along the inclusion $P \hookrightarrow S$, and we restrict along a $\varphi \in \mathcal{F}(P,S)$. The stability property then requires that these restricted P-actions give rise to isomorphic P-sets.

In the context of the single Burnside ring, lemma 1.4.3 gives us an alternate characterization of $\mathcal{F}$-stability in terms of the mark homomorphism for $A(S)$. The following lemma tells us that a similar result holds for left/right $\mathcal{F}$-stability in the double Burnside ring $A(S,S)$.

Lemma 2.3.8 ([RS09, Lemma 4.8])**.** *Let $X \in A(S,S)$. We then have the following characterizations of $\mathcal{F}$-generation and left/right $\mathcal{F}$-stability in terms of the mark homomorphism:*

(i) X is $\mathcal{F}$-generated if and only if $\Phi_{[Q,\psi]}(X) = 0$ for all conjugacy classes $[Q,\psi] \in C(S,S)$ where $\psi \notin \mathcal{F}(Q,S)$.

(ii) X is right $\mathcal{F}$-stable if and only if

$$\Phi_{[Q,\psi]}(X) = \Phi_{[\varphi Q,\psi\varphi^{-1}]}(X)$$

for all $[Q,\psi] \in C(S,S)$ and $\varphi \in \mathcal{F}(Q,S)$.

(iii) X is left $\mathcal{F}$-stable if and only if

$$\Phi_{[Q,\psi]}(X) = \Phi_{[Q,\varphi\psi]}(X)$$

for all $[Q,\psi] \in C(S,S)$ and $\varphi \in \mathcal{F}(\psi Q,S)$.

Proof. (i): Assume that X is $\mathcal{F}$-generated, and let (Q,ψ) be an (S,S)-pair.

If $\Phi_{[Q,\psi]}([P,\varphi]) = 0$ for all pairs (P,φ) with $c_{[P,\varphi]}(X) \neq 0$, then $\Phi_{[Q,\psi]}(X) = 0$. If we assume $\Phi_{[Q,\psi]}(X) \neq 0$, we therefore get $\Phi_{[Q,\psi]}([P,\varphi]) \neq 0$ for some pair (P,φ) with $c_{[P,\varphi]}(X) \neq 0$; so (Q,ψ) is subconjugate to (P,φ) (by lemma 2.1.12). Since X is $\mathcal{F}$-generated, $\varphi \in \mathcal{F}(P,S)$, so remark 2.3.2 gives $\psi \in \mathcal{F}(Q,S)$.

Assume that X is not $\mathcal{F}$-generated, and choose $[Q,\psi] \in C(S,S)$ maximal among the classes with $c_{[Q,\psi]}(X) \neq 0$ and $\psi \notin \mathcal{F}(Q,S)$. Lemma 2.1.12 and remark 2.3.2 then imply that

$$\Phi_{[Q,\psi]}(X) = c_{[Q,\psi]}(X) \cdot \Phi_{[Q,\psi]}([Q,\psi])$$

which is non-zero by lemma 2.1.12.

(ii): We first consider the case where $X \in A(S,S)$ is a biset. Let $Q \leq P \leq S$, $\varphi \in \mathrm{Inj}(P,S)$ and $\psi \in \mathrm{Hom}(Q,S)$. We have

$$X \circ [P,\varphi]_P^S = [X \times_S (S \times_{(P,\varphi)} P)]_P^S = [X \times P/(x,p) \sim (x\varphi(p),1)]_P^S,$$

so $X \circ [P,\varphi]_P^S$ is just the set X where P acts by $x \cdot p = x\varphi(p)$, and the left S-action is unchanged. The fixed-point set $(X \circ [P,\varphi]_P^S)^{(Q,\psi)}$ then satisfy

$$\begin{aligned}(X \circ [P,\varphi]_P^S)^{(Q,\psi)} &= \{x \in X \mid \forall q \in Q\colon \psi(q)x\varphi(q^{-1})\} \\ &= \{x \in X \mid \forall r \in \varphi Q\colon \psi(\varphi^{-1}(r))xr^{-1}\} \\ &= X^{(\varphi Q,\psi\varphi^{-1})}\end{aligned}$$

where the second equality holds because φ is injective. When $\varphi = \mathrm{incl}$, we simple get $(X \circ [P, \mathrm{incl}]_P^S)^{(Q,\psi)} = X^{(Q,\psi)}$.

Generalizing to all $X \in A(S,S)$ we get

$$\Phi_{[Q,\psi]}(X \circ [P,\varphi]_P^S) = \Phi_{[\varphi Q, \psi\varphi^{-1}]}(X), \quad \text{and} \quad \Phi_{[Q,\psi]}(X \circ [P, \mathrm{incl}]_P^S) = \Phi_{[Q,\psi]}(X). \tag{2.2}$$

If we now assume that $X \in A(S,S)$ is right $\mathcal{F}$-stable, then we immediately get

$$\Phi_{[Q,\psi]}(X) = \Phi_{[\varphi Q, \psi\varphi^{-1}]}(X)$$

for all $\psi \in \mathrm{Hom}(Q,S)$ and $\varphi \in \mathcal{F}(Q,S)$ by putting $(P,\varphi) := (Q,\varphi)$ in (2.2).

Conversely, let $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$, and assume that for all $Q \leq P$ and $\psi \in \mathrm{Hom}(Q,S)$ we have $\Phi_{[Q,\psi]}(X) = \Phi_{[\varphi Q, \psi\varphi^{-1}]}(X)$. Then (2.2) says that the mark homomorphism

$$\Phi\colon A(P,S) \xrightarrow{\prod_{[Q,\psi]} \Phi_{[Q,\psi]}} \prod_{[Q,\psi] \in C(P,S)} \mathbb{Z}$$

has the same image at $X \circ [P,\varphi]_P^S$ and $X \circ [P, \mathrm{incl}]_P^S$, hence they are equal since Φ is injective (see definition 2.1.11).

(iii): Analogous to proof of (ii). □

Definition 2.3.9. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over p-groups S_1 and S_2 respectively. We then define the Burnside module $A(\mathcal{F}_1, \mathcal{F}_2)$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ as the submodule of $A(S_1, S_2)$ consisting of the right $\mathcal{F}_1$-stable, left $\mathcal{F}_2$-stable elements.

If $X \in A(\mathcal{F}_1, \mathcal{F}_2)$ and $Y \in A(\mathcal{F}_2, \mathcal{F}_3)$, then $Y \circ X$ inherits right $\mathcal{F}_1$-stability and left $\mathcal{F}_3$-stability, so $Y \circ X \in A(\mathcal{F}_1, \mathcal{F}_3)$. We might hope that this defines a category with objects the fusion systems on p-groups, and morphism sets $A(\mathcal{F}_1, \mathcal{F}_2)$; but we generally lack identity elements $\omega_{\mathcal{F}_i} \in A(\mathcal{F}_i, \mathcal{F}_i)$.

However, if we take the p-localizations $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ as morphism sets instead, and require that the fusion systems are saturated, then we actually do get a category (see proposition 2.4.12).

Remark 2.3.10. The right $\mathcal{F}_1$-stable elements of $A(S_1, S_2)$ are simply the elements of the submodule $A(\mathcal{F}_1, \mathcal{F}_{S_2})$, since all $X \in A(S_1, S_2)$ are left $\mathcal{F}_{S_2}$-stable.

Similarly $A(\mathcal{F}_{S_1}, \mathcal{F}_2)$ consists of the left $\mathcal{F}_2$-stable elements in $A(S_1, S_2)$.

Definition 2.3.11. The *double Burnside ring* of $\mathcal{F}$ is defined as the ring $A(\mathcal{F}, \mathcal{F})$ of $\mathcal{F}$-stable elements in $A(S,S)$. The double Burnside ring is not necessarily unital, since the 1-element of $A(S,S)$, $[S, id]_S^S$, is usually not $\mathcal{F}$-stable. For a saturated fusion system $\mathcal{F}$, we will however see that the p-localization $A(\mathcal{F}, \mathcal{F})_{(p)}$ has the characteristic idempotent of $\mathcal{F}$ as 1-element (see proposition 2.4.12).

Lemma 2.3.12. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on p-groups S_1 and S_2 respectively. Assume that $X_1 \in A(S_1, S_1)$ is left $\mathcal{F}_1$-stable, and that $X_2 \in A(S_2, S_2)$ is right $\mathcal{F}_2$-stable.*

If two (S_1, S_2)-pairs (P, φ), (Q, ψ) are $(\mathcal{F}_1, \mathcal{F}_2)$-conjugate, then

$$X_2 \circ [P, \varphi] \circ X_1 = X_2 \circ [Q, \psi] \circ X_1.$$

Proof. Let $\rho_1 \in \mathcal{F}_1(P, Q)$ and $\rho_2 \in \mathcal{F}_2(\varphi P, \psi Q)$ be isomorphisms such that $\rho_2\varphi = \psi\rho_1$, i.e. $\psi = \rho_2\varphi\rho_1^{-1}$.

Using the simple cases of the double coset formula (see 2.2.5), we then get

$$\begin{aligned} X_2 \circ [Q, \psi]_{S_1}^{S_2} \circ X_1 &= X_2 \circ [Q, \rho_2\varphi\rho_1^{-1}]_{S_1}^{S_2} \circ X_1 \\ &= X_2 \circ [\varphi P, \rho_2]_{\varphi P}^{S_2} \circ [P, \varphi]_P^{\varphi P} \circ [Q, \rho_1^{-1}]_{S_1}^{P} \circ X_1 \\ &= X_2 \circ [\varphi P, \mathrm{incl}]_{\varphi P}^{S_2} \circ [P, \varphi]_P^{\varphi P} \circ [P, id_P]_{S_1}^{P} \circ X_1 \\ &= X_2 \circ [P, \varphi]_{S_1}^{S_2} \circ X_1. \end{aligned}$$

□

Characteristic elements

We now introduce the characteristic elements of a fusion system $\mathcal{F}$. Though it might not seem so at first, theorem 2.6.21 states that the existence of characteristic elements require that $\mathcal{F}$ is saturated. So though we don't explicitly require saturation in the next sections, the results actually only relate to saturated fusion systems.

Definition 2.3.13. We say that an element $X \in A(S, S)$ is a *right/left/fully characteristic element* for $\mathcal{F}$ if it has the following three properties:

(i) X is $\mathcal{F}$-generated, i.e. $X \in A_{\mathcal{F}}(S, S) \leq A_{fr}(S, S)$.

(ii) X is right/left/fully $\mathcal{F}$-stable.

(iii) $\varepsilon(X)$ is coprime to p.

Lemma 2.3.14. *An element $X \in A_{fr}(S, S)$ is right $\mathcal{F}$-characteristic if and only if X^{op} is left $\mathcal{F}$-characteristic. If X is right $\mathcal{F}$-characteristic, then $X^{\mathrm{op}} \circ X$ is symmetric and fully $\mathcal{F}$-characteristic.*

Similar results hold for X left $\mathcal{F}$-characteristic simply by applying the lemma to X^{op}: If X is left $\mathcal{F}$-characteristic, then $X \circ X^{\mathrm{op}}$ is symmetric and fully $\mathcal{F}$-characteristic.

Proof. Symmetry of $X^{\mathrm{op}} \circ X$ follows from lemma 2.2.11. Stability of X^{op} and $X^{\mathrm{op}} \circ X$ is immediate from the definition (and remark 2.3.5). The augmentation property is true since $\varepsilon(X^{\mathrm{op}}) = \varepsilon(X)$ and $\varepsilon(X^{\mathrm{op}} \circ X) = \varepsilon(X)^2$. Finally $\mathcal{F}$-generation holds since $(-)^{\mathrm{op}}$ and $\circ$ both preserve $\mathcal{F}$-generation (see definition 2.3.3). □

Proposition 2.3.15. *If $\mathcal{F} = \mathcal{F}_S(G)$, then G considered as a (S, S)-set, G_S^S, is a characteristic element for $\mathcal{F}$.*

Proof. We begin by remarking that

$$G_S^S = [S, id_S]_G^S \circ G \circ [S, \mathrm{incl}]_S^G,$$

according to lemma 2.2.2. We also remark that G_S^S is symmetric, either because it is obvious, or because $G = [G, id_G]_G^G$ is symmetric and $([S, \mathrm{incl}]_S^G)^{\mathrm{op}} = [S, id_S]_G^S$. Since G_S^S is symmetric, it is enough to prove that G_S^S is right $\mathcal{F}$-characteristic (see lemma 2.3.14).

The biset $[G]_S^S$ is $\mathcal{F}$-generated since we have

$$G_S^S = [S, id_S]_G^S \circ [S, \mathrm{incl}]_S^G = \sum_{\overline{g} \in S\backslash G/S} [S \cap S^g, c_g]_S^S$$

by the double coset formula, and $c_g \in \mathcal{F}(S \cap S^g, S)$ for all $g \in G$.

Let $P \le S$ and $\varphi \in \mathcal{F}(P, S)$. Then $\varphi = c_g$ for some $g \in G$, and $(P, c_g) \underset{(S,G)}{\sim} (P, \mathrm{incl})$ as (S, G)-pairs. We therefore have $[P, \varphi]_S^G = [P, \mathrm{incl}]_S^G$, and consequently

$$\begin{aligned} G_S^S \circ [P, \varphi]_P^S &= [S, id_S]_G^S \circ [S, \mathrm{incl}]_S^G \circ [P, \varphi]_P^S = [S, id_S]_G^S \circ [P, \varphi]_P^G \\ &= [S, id_S]_G^S \circ [P, \mathrm{incl}]_P^G = [S, id_S]_G^S \circ [S, \mathrm{incl}]_S^G \circ [P, \mathrm{incl}]_P^S \\ &= G_S^S \circ [P, \mathrm{incl}]_P^S. \end{aligned}$$

We conclude that G_S^S is right $\mathcal{F}$-stable.

Finally, G_S^S has augmentation $\varepsilon(G_S^S) = |G| / |S|$ which is coprime to p since $S \in Syl_p(G)$. □

Remark 2.3.16. It is only at the very end of the previous proof that we need $S \in Syl_p(G)$. If S is just some p-subgroup of G, we can still conclude that G_S^S is an $\mathcal{F}_S(G)$-generated, $\mathcal{F}_S(G)$-stable (S, S)-biset.

Theorem 2.3.17 ([BLO03, Proposition 5.5])**.** *Every saturated fusion system has a characteristic element.*

Proof. We shall use that when $\mathcal{F}$ is a saturated fusion system om S, then $\mathcal{F} \times \mathcal{F}$ is a saturated fusion system on $S \times S$, [AKO10, Theorem 6.6]. The strategy is then to apply lemma 1.5.2 with $\mathcal{F} \times \mathcal{F}$ as the fusion system.

We start by setting

$$X := \sum_{\overline{\varphi} \in \mathrm{Out}_{\mathcal{F}}(S)} [S, \varphi].$$

Since $\mathcal{F}$ is saturated, $\varepsilon(X) = |\mathrm{Out}_{\mathcal{F}}(S)| \not\equiv 0 \pmod p$. In $S \times S$, the $(\mathcal{F} \times \mathcal{F})$-conjugacy class of $\Delta(S, id_S)$ consists of all $\Delta(S, \varphi)$ where $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$. Let $\Phi\colon A(S \times S) \to \widetilde{\Omega}(S \times S)$ be the homomorphism of marks, we then have $\Phi_{\Delta(P,\varphi)} = \Phi_{[P,\varphi]}$ for all (S, S)-pairs (P, φ) – see definition 2.1.11.

By construction of X, we then have

$$\Phi_{\Delta(S,\varphi)}(X) = \Phi_{[S,\varphi]}(X) = \Phi_{[S,\varphi]}([S,\varphi]) = |Z(S)|$$

by lemma 2.1.12; and this value is independent of the choice $\Delta(S,\varphi) \in [\Delta(S,id)]_{\mathcal{F}\times\mathcal{F}}$.

The $(\mathcal{F}\times\mathcal{F})$-subconjugates of $\Delta(S,id)$ are precisely the subgroups $\Delta(P,\varphi) \leq S\times S$ with $\varphi \in \mathcal{F}(P,S)$. Lemma 1.1.5 then tells us that $\Phi_{[H]}(X) = 0$ unless the subgroup $H \leq S\times S$ is on the form $H = \Delta(P,\varphi)$ for some $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$. Let $\mathcal{H}$ be the collection $\mathcal{H} := \{\Delta(P,\varphi) \mid P < S, \varphi \in \mathcal{F}(P,S)\}$.

The collection $\mathcal{H}$ is closed under $\mathcal{F}\times\mathcal{F}$-subconjugacy; and for $H, K \leq S\times S$ that are $(\mathcal{F}\times\mathcal{F})$-conjugate, we have $\Phi_{[H]}(X) = \Phi_{[K]}(X)$ whenever $H, K \notin \mathcal{H}$. We can therefore apply lemma 1.5.2 and get an element $\widehat{X} \in A(\mathcal{F}\times\mathcal{F})$ satisfying

(i) $\Phi_{[H]}(\widehat{X}) = \Phi_{[H]}(X) = 0$ and $c_{[H]}(\widehat{X}) = c_{[H]}(X) = 0$ when H is not on the form $\Delta(P,\varphi)$ with $\varphi \in \mathcal{F}(P,S)$.

(ii) $\Phi_{[S,\varphi]}(\widehat{X}) = \Phi_{[S,\varphi]}(X) = \Phi_{[S,\varphi]}([S,\varphi])$ and $c_{[S,\varphi]}(\widehat{X}) = c_{[S,\varphi]}(X) = 1$.

From property (i) we get that $\widehat{X}$ is $\mathcal{F}$-generated.

Because $\varepsilon([P,\varphi]) \equiv 0 \pmod p$ when $P < S$ and $\varepsilon([S,\varphi]) = 1$, we get that

$$\varepsilon(\widehat{X}) \equiv \sum_{\overline{\varphi}\in \mathrm{Out}_{\mathcal{F}}(S)} c_{[S,\varphi]}(\widehat{X}) = |\mathrm{Out}_{\mathcal{F}}(S)| \not\equiv 0 \pmod p.$$

For all (S,S)-pairs (Q,ψ), all $\varphi \in \mathcal{F}(Q,S)$, and all $\rho \in \mathcal{F}(\psi Q,S)$, the pairs $(\varphi Q, \psi\varphi^{-1})$ and $(Q,\rho\psi)$ are both $(\mathcal{F}\times\mathcal{F})$-conjugate to (Q,ψ). Since $\widehat{X} \in A(\mathcal{F}\times\mathcal{F})$, we therefore get

$$\Phi_{[Q,\psi]}(\widehat{X}) = \Phi_{[\varphi Q,\psi\varphi^{-1}]}(\widehat{X}), \quad \text{and} \quad \Phi_{[Q,\psi]}(\widehat{X}) = \Phi_{[Q,\rho\psi]}(\widehat{X}).$$

Lemma 2.3.8 then gives $\mathcal{F}$-stability of $\widehat{X}$.

The element $\widehat{X} \in A(S,S)$ is thus a characteristic element for the saturated fusion system $\mathcal{F}$. □

Definition 2.3.18. We define the *ring of characteristic elements* for a saturated fusion system $\mathcal{F}$ as the intersection $A_{\mathcal{F}}(\mathcal{F},\mathcal{F}) := A_{\mathcal{F}}(S,S) \cap A(\mathcal{F},\mathcal{F})$ consisting of the $\mathcal{F}$-generated, $\mathcal{F}$-stable elements.

Let $X \in A(S,S)$ be a characteristic element for $\mathcal{F}$, then $X \in A_{\mathcal{F}}(\mathcal{F},\mathcal{F})$ by definition. Let $Y \in A_{\mathcal{F}}(\mathcal{F},\mathcal{F})$ be some $\mathcal{F}$-generated, $\mathcal{F}$-stable element that is not $\mathcal{F}$-characteristic. This implies that $\varepsilon(Y) \equiv 0 \pmod p$. However, we then have $\varepsilon(Y - X) = \varepsilon Y - \varepsilon X \not\equiv 0 \pmod p$; hence the element $Y - X \in A_{\mathcal{F}}(\mathcal{F},\mathcal{F})$ must be $\mathcal{F}$-characteristic.

Every element of $A_{\mathcal{F}}(\mathcal{F},\mathcal{F})$ is therefore either $\mathcal{F}$-characteristic or a sum of $\mathcal{F}$-characteristic elements; so $A_{\mathcal{F}}(\mathcal{F},\mathcal{F})$ is generated by the characteristic elements for $\mathcal{F}$, hence the name.

Proposition 2.3.19 ([Rag06, Proposition 5.2]). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on p-groups S_1 and S_2 respectively, and suppose that $\xi_1 \in A(S_1, S_1)$ and $\xi_2 \in A(S_2, S_2)$ are respectively left and right characteristic elements.*

Let $C(\mathcal{F}_1, \mathcal{F}_2)$ be the set of $(\mathcal{F}_1, \mathcal{F}_2)$-conjugacy classes of (S_1, S_2)-pairs, and pick a representative (P_i, φ_i) for each $i \in C(\mathcal{F}_1, \mathcal{F}_2)$. Then the collection

$$\{\xi_2 \circ [P_i, \varphi_i] \circ \xi_1 \mid i \in C(\mathcal{F}_1, \mathcal{F}_2)\}$$

form a $\mathbb{Z}$-basis for $\xi_2 \circ A(S_1, S_2) \circ \xi_1$.

Proof. Let $[Q, \psi]$ be some basis element of $A(S_1, S_2)$, then (Q, ψ) is $(\mathcal{F}_1, \mathcal{F}_2)$ conjugate to one the the chosen representatives. Let $(Q, \psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} (P_i, \varphi_i)$, then

$$\xi_2 \circ [Q, \psi] \circ \xi_1 = \xi_2 \circ [P_i, \varphi_i] \circ \xi_1$$

by lemma 2.3.12 since ξ_2 is right $\mathcal{F}_2$-stable, and ξ_1 is left $\mathcal{F}_1$-stable. Since $[Q, \psi] \in A(S, S)$ was an arbitrary basis element, we conclude that the $(\xi_2 \circ [P_i, \varphi_i] \circ \xi_1)$-elements generate all of $\xi_2 \circ A(S_1, S_2) \circ \xi_1$.

Consider some linear combination

$$\sum_{i \in C(\mathcal{F}_1,\mathcal{F}_2)} c_i \cdot (\xi_2 \circ [P_i, \varphi_i] \circ \xi_1); \tag{2.3}$$

and assume that not all c_i are zero. We then wish to prove that the linear combination is nonzero as well. Let $N := \{i \in C(\mathcal{F}_1, \mathcal{F}_2) \mid c_i \neq 0\} \neq \emptyset$, and let $j \in N$ be maximal under $(\mathcal{F}_1, \mathcal{F}_2)$-subconjugation.

We define $M(\precsim (P_i, \varphi_i)) \leq A(S_1, S_2)$ as the submodule generated by $[Q, \psi]$ with $(Q, \psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_i, \varphi_i)$.

By the double coset formula, we have $[P_i, \varphi_i]_{S_1}^{S_2} \circ [R, \rho]_{S_1}^{S_1} \in M(\precsim (P_i, \varphi_i))$ when $\rho \in \mathcal{F}_1(R, S_1)$, because in that case we have

$$(R \cap \rho^{-1}(P_i^s), \varphi_i c_s \rho) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_i, \varphi_i)$$

for all $s \in S_1$, since $c_s\rho \in \mathcal{F}_1(R \cap \rho^{-1}(P_i^s), P_i)$.

Similarly, we have $[R, \rho]_{S_2}^{S_2} \circ [P_i, \varphi_i]_{S_1}^{S_2} \in M(\precsim (P_i, \varphi_i))$ when $\rho \in \mathcal{F}_2(R, S_2)$. The characteristic elements ξ_1 and ξ_2 are $\mathcal{F}_1$- and $\mathcal{F}_2$-generated respectively, so we conclude that $\xi_2 \circ [P_i, \varphi_i] \circ \xi_1 \in M(\precsim (P_i, \varphi_i))$ for all i.

We then write $\xi_2 \circ [P_j, \varphi_j] \circ \xi_1 \in M(\precsim (P_j, \varphi_j))$ as a linear combination

$$\xi_2 \circ [P_j, \varphi_j] \circ \xi_1 = \sum_{\substack{[Q,\psi] \in C(S_1,S_2) \\ (Q,\psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_j,\varphi_j)}} c_{[Q,\psi]} \cdot [Q, \psi],$$

where $c_{[Q,\psi]} := c_{[Q,\psi]}(\xi_2 \circ [P_j, \varphi_j] \circ \xi_1)$.

The elements ξ_1 and ξ_2 have $\varepsilon(\xi_1), \varepsilon(\xi_2) \not\equiv 0 \pmod p$, since they are characteristic. The augmentation preserves products, so we get

$$\begin{aligned}\varepsilon(\xi_2) \cdot \frac{|S_1|}{|P_j|} \cdot \varepsilon(\xi_1) &= \varepsilon(\xi_2 \circ [P_j, \varphi_j] \circ \xi_1) \\ &= \sum_{\substack{[Q,\psi] \in C(S_1,S_2) \\ (Q,\psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_j,\varphi_j)}} c_{[Q,\psi]} \cdot \varepsilon([Q,\psi]) \\ &= \sum_{\substack{[Q,\psi] \in C(S_1,S_2) \\ (Q,\psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_j,\varphi_j)}} c_{[Q,\psi]} \cdot \frac{|S_1|}{|Q|}.\end{aligned}$$

The highest power of p dividing the left side, is $|S_1| / |P_j|$. We can therefore conclude that $c_{[Q,\psi]} \neq 0$ for some $(Q,\psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\precsim} (P_j, \varphi_j)$ with $|Q| = |P_j|$, since otherwise $p \cdot \frac{|S|}{|P_j|}$ would divide the right side of the equation. Then $|Q| = |P_j|$ implies that we in fact have $(Q,\psi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} (P_j, \varphi_j)$.

By maximality of $j \in N$, it follows that $c_{[Q,\psi]}\big(M(\precsim (P_i, \varphi_i))\big) = 0$ for all $i \in N$ different from j. Applying $c_{[Q,\psi]}$ to the linear combination (2.3) we then get

$$\begin{aligned} & c_{[Q,\psi]}\left(\sum_{i \in C(\mathcal{F}_1,\mathcal{F}_2)} c_i \cdot (\xi_2 \circ [P_i, \varphi_i] \circ \xi_1)\right) \\ = & \sum_{i \in C(\mathcal{F}_1,\mathcal{F}_2)} c_i \cdot c_{[Q,\psi]}(\xi_2 \circ [P_i, \varphi_i] \circ \xi_1) \\ = & \sum_{i \notin N} \underbrace{c_i}_{=0} \cdot c_{[Q,\psi]}(\xi_2 \circ [P_i, \varphi_i] \circ \xi_1) \\ & + \sum_{i \in N \setminus \{j\}} c_i \cdot \underbrace{c_{[Q,\psi]}(\xi_2 \circ [P_i, \varphi_i] \circ \xi_1)}_{=0} \\ & + \underbrace{c_j}_{\neq 0} \cdot \underbrace{c_{[Q,\psi]}(\xi_2 \circ [P_j, \varphi_j] \circ \xi_1)}_{\neq 0} \\ \neq & \, 0; \end{aligned}$$

hence the linear combination (2.3) must be nonzero. □

Corollary 2.3.20. *If $\xi_R, \xi_L \in A(S,S)$ are respectively right and left characteristic elements for $\mathcal{F}$, then the collection*

$$\{\xi_R \circ [P, \mathrm{incl}] \circ \xi_L \mid [P]_{\mathcal{F}} \in C(\mathcal{F})\}$$

is a $\mathbb{Z}$-basis for $\xi_R \circ A_{\mathcal{F}}(S,S) \circ \xi_L$.

Proof. If two $\mathcal{F}$-generated basis elements $[P,\varphi],[Q,\psi] \in A_{\mathcal{F}}(S,S)$ are $(\mathcal{F},\mathcal{F})$-conjugate then clearly we have $P \sim_{\mathcal{F}} Q$. On the other hand, if $\rho\colon P \xrightarrow{\sim} Q$ an $\mathcal{F}$-isomorphism, then $(\psi\rho\varphi^{-1})\varphi = \psi\rho$ with $\psi\rho\varphi^{-1} \in \mathcal{F}(\varphi P, \psi Q)$, shows that $(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} (Q,\psi)$.

This shows that $\xi_R \circ A_{\mathcal{F}}(S,S) \circ \xi_L$ is generated by $\xi_R \circ [P, \mathrm{incl}] \circ \xi_L$, with one basis element for each $[P]_{\mathcal{F}} \in C(\mathcal{F})$. That these basis elements are linearly independent follows from proposition 2.3.19. □

2.4 The characteristic idempotent of a fusion system

In sections 1.5-1.6 we first used lemma 1.5.2 to prove the existence of some basis for $A(\mathcal{F})$ with certain properties, but it didn't give a simple way to find such a basis. We then used the p-localized lemma 1.6.5 to actually find an explicit basis for $A(\mathcal{F})_{(p)}$. In theorem 2.3.17 we proved the existence of some $\mathcal{F}$-characteristic element in $A(S,S)$ by applying lemma 1.5.2 to the context of the double Burnside ring. In this section we then apply 1.6.5 to the p-localized case, and we get an even stronger result (theorem 2.4.11): The double Burnside ring $A(S,S)_{(p)}$ contains an $\mathcal{F}$-characteristic element $\omega_{\mathcal{F}}$ that is also idempotent.

In [Rag06], Ragnarsson uses p-completeness to prove the existence of an $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)^\wedge_p$. He then later concludes that this idempotent actually lies in $A(S,S)_{(p)}$. The proof of existence contained in this section doesn't rely on p-completeness to find $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$; and furthermore, we determine the value $\Phi(\omega_{\mathcal{F}})$ of the mark homomorphism.

Lemma 2.4.1. *An element $X \in A_{\mathcal{F}}(S,S)_{(p)}$ is a right characteristic idempotent for $\mathcal{F}$ if and only if X^{op} is a left characteristic idempotent for $\mathcal{F}$.*

Proof. By lemma 2.3.14, X is right characteristic if and only if X^{op} is left characteristic; and by lemma 2.2.11, X is idempotent if and only of X^{op} is idempotent. □

Lemma 2.4.2. *A left/right/fully characteristic idempotent $\xi \in A_{fr}(S,S)_{(p)}$ has augmentation $\varepsilon(\xi) = 1$.*

Proof. Since the augmentation is a ring-homomorphism, and ξ is an idempotent, we have $\varepsilon(\xi)^2 = \varepsilon(\xi)$; hence the augmentation $\varepsilon(\xi)$ is 0 or 1. The element ξ is also left/right/fully characteristic, so $\varepsilon(\xi) \not\equiv 0 \pmod p$ which leaves $\varepsilon(\xi) = 1$ as the only possibility. □

Definition 2.4.3. For each $\mathcal{F}$-generated $X \in A_{\mathcal{F}}(S,S)_{(p)}$ and $P \leq S$ we define

$$m_P(X) := \sum_{\substack{[P,\varphi]\in C(S,S)\\ \varphi\in\mathcal{F}(P,S)}} c_{[P,\varphi]}(X),$$

where we sum the coefficients of X corresponding to classes of (S,S)-pairs (P,φ) with $\varphi \in \mathcal{F}(P,S)$.

We similarly define

$$m^P(X) := \sum_{\substack{[P,\varphi]\in C(S,S)\\ \varphi\in\mathcal{F}(P,S)}} c_{[\varphi P,\varphi^{-1}]}(X),$$

where we sum the coefficients of X corresponding to classes of (S,S)-pairs (Q,φ^{-1}) with $\varphi^{-1} \in \mathcal{F}(Q,S)$ and $\varphi^{-1}Q = P$.

Every $\mathcal{F}$-generated element is bifree, so we can apply $(-)^{\mathrm{op}}$. Since $[P,\varphi]^{\mathrm{op}} = [\varphi P, \varphi^{-1}]$, we have $c_{[P,\varphi]}(X^{\mathrm{op}}) = c_{[\varphi P,\varphi^{-1}]}(X)$ for all $X \in A_{\mathcal{F}}(S,S)_{(p)}$.

It follows that $m_P(X^{\mathrm{op}}) = m^P(X)$ for all $X \in A_{\mathcal{F}}(S,S)_{(p)}$ and $P \leq S$.

Lemma 2.4.4. *Let $Z \in A_{\mathcal{F}}(S,S)_{(p)}$.*

If $X \in A(S,H)_{(p)}$ is right $\mathcal{F}$-stable, then

$$X \circ Z = \sum_{[P]_S\in C(S)} m_P(Z)\cdot (X \circ [P,\mathrm{incl}]_S^S).$$

Similarly, if $X \in A(H,S)_{(p)}$ is left $\mathcal{F}$-stable, then

$$Z \circ X = \sum_{[P]_S\in C(S)} m^P(Z)\cdot ([P,\mathrm{incl}]_S^S \circ X).$$

Proof. We only prove the case where X is left $\mathcal{F}$-stable; the other case is proven analogously.

Since X is left $\mathcal{F}$-stable, we have $[\varphi P,\varphi^{-1}]_S^P \circ X = [P, id_P]_S^P \circ X$ for all $\varphi \in \mathcal{F}(P,S)$. Composing with $[P,\mathrm{incl}]_P^S$ from the left, we just get $[\varphi P,\varphi^{-1}]_S^S \circ X = [P,\mathrm{incl}]_S^S \circ X$ by corollary 2.2.6.

Z is $\mathcal{F}$-generated, so it is a linear combination of the basis elements $[Q,\psi] \in A(S,S)_{(p)}$ with $\psi \in \mathcal{F}(Q,S)$. We split this linear combination according to the class $[\psi Q]_S$ of the image:

$$\begin{aligned}
Z \circ X &= \left(\sum_{\substack{[Q,\psi]\in C(S,S) \\ \psi\in\mathcal{F}(Q,S)}} c_{[Q,\psi]}(Z)\cdot[Q,\psi] \right) \circ X \\
&= \left(\sum_{[P]_S\in C(S)} \left(\sum_{\substack{[Q,\varphi^{-1}]\in C(S,S) \\ \varphi\in\mathcal{F}(P,Q) \\ \varphi \text{ iso}}} c_{[Q,\varphi^{-1}]}(Z)\cdot[Q,\varphi^{-1}] \right)\right) \circ X \\
&= \sum_{[P]_S\in C(S)} \left(\sum_{\substack{[P,\varphi]\in C(S,S) \\ \varphi\in\mathcal{F}(P,S)}} c_{[\varphi P,\varphi^{-1}]}(Z)\cdot\big([\varphi P,\varphi^{-1}]\circ X\big) \right) \\
&= \sum_{[P]_S\in C(S)} \left(\sum_{\substack{[P,\varphi]\in C(S,S) \\ \varphi\in\mathcal{F}(P,S)}} c_{[\varphi P,\varphi^{-1}]}(Z)\cdot\big([P,\text{incl}]\circ X\big) \right) \\
&= \sum_{[P]_S\in C(S)} \left(\sum_{\substack{[P,\varphi]\in C(S,S) \\ \varphi\in\mathcal{F}(P,S)}} c_{[\varphi P,\varphi^{-1}]}(Z) \right) ([P,\text{incl}]\circ X) \\
&= \sum_{[P]_S\in C(S)} m^P(Z)\cdot([P,\text{incl}]\circ X).
\end{aligned}$$

□

Lemma 2.4.5. *A right $\mathcal{F}$-characteristic element $\xi_R \in A_{\mathcal{F}}(S,S)_{(p)}$ is idempotent if and only if $m_S(\xi_R) = 1$ and $m_P(\xi_R) = 0$ for $P < S$.*

Similarly, a left $\mathcal{F}$-characteristic element $\xi_L \in A_{\mathcal{F}}(S,S)_{(p)}$ is idempotent if and only if $m^S(\xi_L) = 1$ and $m^P(\xi_L) = 0$ for $P < S$.

Proof. We prove the right characteristic case. The left characteristic case is proven analogously or just by applying $(-)^{\mathrm{op}}$ to the right characteristic case.

Since ξ_R is both $\mathcal{F}$-generated and right $\mathcal{F}$-stable, lemma 2.4.4 gives us that

$$\xi_R \circ \xi_R = \sum_{[P]_S\in C(S)} m_P(\xi_R)\cdot(\xi_R \circ [P,\text{incl}]_S^S).$$

Applying proposition 2.3.19 to the pair of fusion systems $(\mathcal{F}_S,\mathcal{F})$, where $[S,id]_S^S$ is a fully $\mathcal{F}_S$-characteristic element, we get that the elements $\xi_R \circ [P,\text{incl}]_S^S$ are linearly independent. The sum is therefore equal to $\xi_R = \xi_R \circ [S,\text{incl}]_S^S$ if and only if $m_S(\xi_R) = 1$ and $m_P(\xi_R) = 0$ for $P < S$. □

Proposition 2.4.6. *Let $\xi_R \in A_{\mathcal{F}}(S,S)_{(p)}$ be a right $\mathcal{F}$-characteristic idempotent, and let $X_R \in A(S,H)_{(p)}$ be right $\mathcal{F}$-stable. Then $X_R \circ \xi_R = X_R$.*

Similarly, a left characteristic idempotent acts trivially on the left stable elements by left-multiplication.

Proof. Since ξ_R is a right $\mathcal{F}$-characteristic idempotent, lemma 2.4.5 says that $m_S(\xi_R) = 1$ and $m_P(\xi_R) = 0$ for $P < S$. Since ξ_R is $\mathcal{F}$-generated and X_R is right $\mathcal{F}$-stable, applying lemma 2.4.4 gives us that

$$X_R \circ \xi_R = \sum_{[P]_S \in C(S)} m_P(\xi_R) \cdot (X_R \circ [P, \mathrm{incl}]_S^S) = X_R \circ [S, \mathrm{incl}]_S^S = X_R. \qquad \square$$

Proposition 2.4.7 ([Rag06, Proposition 5.6])**.** *A fusion system has at most one fully characteristic idempotent.*

Proof. Let $P_0, P_1, \dots, P_n$ be representatives of the S-conjugacy classes of subgroups in S; and choose the labeling such that

$$|P_0| \geq |P_1| \geq \cdots \geq |P_n|.$$

In particular, $P_0 = S$ and $P_n = 1$.

We then choose representatives $(P_i, \varphi_{i0}), (P_i, \varphi_{i1}), \dots, (P_i, \varphi_{im_i})$ of the (S,S)-conjugacy classes $[P_i, \varphi]_{(S,S)}$ with $\varphi \in \mathcal{F}(P_i, S)$, with the condition that $\varphi_{i0} = id_{P_i}$. Then (P_i, φ_{ij}), $0 \leq i \leq n$, $0 \leq j \leq m_i$, are a set of representatives for all classes $[P, \varphi]_{(S,S)}$ with $\varphi \in \mathcal{F}(P, S)$. The chosen ordering has the property that $(P_k, \varphi_{kl}) \underset{(S,S)}{\precsim} (P_i, \varphi_{ij})$ implies that either $(k,l) = (i,j)$ or $k > i$.

We wish to construct a system of linear equations (with one equation per pair (i,j), $0 \leq i \leq n$ and $0 \leq j \leq m_i$) that any characteristic idempotent must satisfy, and show that this system of equations has at most one solution. Assume that $\omega \in A_{\mathcal{F}}(S,S)_{(p)}$ is a characteristic idempotent for $\mathcal{F}$; we will then construct a system of equations determining the coefficients $c_{ij} := c_{[P_i, \varphi_{ij}]}(\omega)$ for all i, j. Since ω is $\mathcal{F}$-generated, these coefficients determine ω completely.

The element ω is in particular a right characteristic idempotent, so by lemma 2.4.5 we have $m_S(\omega) = 1$ and $m_P(\omega) = 0$ for $P < S$. We let equation $(0,0)$ be

$$\sum_{j=0}^{m_0} c_{0j} = 1;$$

and the $(i,0)$-equation, when $P_i < S$, is defined as the equation

$$\sum_{j=0}^{m_i} c_{ij} = 0.$$

We also have that ω is left $\mathcal{F}$-stable, so lemma 2.3.8 says in particular that

$$\Phi_{[P_k,\varphi_{kl}]}(\omega) = \Phi_{[P_k,id]}(\omega)$$

for all k, l. Define the numbers $\Phi_{ij}^{kl} := \Phi_{[P_k,\varphi_{kl}]}([P_i, \varphi_{ij}])$, and let λ_{ij}^{kl} be given by

$$\lambda_{ij}^{kl} := \Phi_{[P_k,\varphi_{kl}]}([P_i, \varphi_{ij}]) - \Phi_{[P_k,id]}([P_i, \varphi_{ij}]) = \Phi_{ij}^{kl} - \Phi_{ij}^{k0}.$$

Since $\Phi_{[P_k,\varphi_{kl}]}(\omega) = \Phi_{[P_k,id]}(\omega)$ we then get that

$$\sum_{i,j} \lambda_{ij}^{kl} \cdot c_{ij} = \Phi_{[P_k,\varphi_{kl}]}(\omega) - \Phi_{[P_k,id]}(\omega) = 0$$

for all k, l. When $0 \le k \le n$ and $0 < l \le m_k$, we therefore define the (k, l)-equation as

$$\sum_{i,j} \lambda_{ij}^{kl} \cdot c_{ij} = 0.$$

Let M be the matrix for this system of equations, then the (kl, ij)-entry of this matrix is given by

$$M_{k0,ij} := \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k; \end{cases}$$

and for $l \neq 0$ by

$$M_{kl,ij} := \lambda_{ij}^{kl} = \Phi_{ij}^{kl} - \Phi_{ij}^{k0}.$$

Let $c \in \bigoplus_{ij} \mathbb{Z}_{(p)}$ be the vector of the coefficients c_{ij} of ω, and let $b \in \bigoplus_{kl} \mathbb{Z}_{(p)}$ be the vector with $b_{00} = 1$ and $b_{kl} = 0$ elsewhere. The system of equations can then be rephrased as $Mc = b$. If we can show that $\det M \neq 0$, then there is a most one solution c, and thus at most one characteristic idempotent.

From lemma 2.1.12, we know that $\Phi_{ij}^{kl} = \frac{|N_{\varphi_{kl},\varphi_{ij}}|}{|P_i|} \cdot |C_S(\varphi_{kl}(P_k))|$ is non-zero if and only if $(P_k, \varphi_{kl}) \underset{(S,S)}{\precsim} (P_i, \varphi_{ij})$. As remarked earlier, $(P_k, \varphi_{kl}) \underset{(S,S)}{\precsim} (P_i, \varphi_{ij})$ implies that either $(k, l) = (i, j)$ or $k > i$; we therefore have that M is a lower triangular block matrix. The i'th block on the diagonal of M is

$$M_i := \begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ -a_{i0} & a_{i1} & 0 & \cdots & 0 & 0 & 0 \\ -a_{i0} & 0 & a_{i2} & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ -a_{i0} & 0 & 0 & & a_{i(m_i-2)} & 0 & 0 \\ -a_{i0} & 0 & 0 & \cdots & 0 & a_{i(m_i-1)} & 0 \\ -a_{i0} & 0 & 0 & & 0 & 0 & a_{im_i} \end{pmatrix}$$

where $a_{ij} := \Phi_{ij}^{ij} > 0$.

The determinant of M_i is

$$\det M_i = \sum_{j=0}^{m_i} \frac{\prod_{l=0}^{m_i} a_{il}}{a_{ij}}$$

which is positive since every a_{ij} is positive. The determinant $\det M = \prod_{i=0}^{n} \det M_i$ is then positive as well, in particular it is non-zero. □

Corollary 2.4.8. *Suppose that $\omega \in A_{fr}(S,S)_{(p)}$ is a characteristic idempotent for $\mathcal{F}$, then ω is symmetric.*

Proof. If ω is a characteristic idempotent for $\mathcal{F}$, then ω^{op} is as well by lemma 2.4.1. By uniqueness of characteristic idempotents (proposition 2.4.7), it follows that $\omega^{\mathrm{op}} = \omega$. □

Observation 2.4.9. In [Rag06, Remark 5.8], Ragnarsson states that the system of equations in the proof of proposition 2.4.7 must be satisfied for all right $\mathcal{F}$-characteristic idempotents. This is false, since the equations (k,l) where $l \neq 0$, originate from the left $\mathcal{F}$-stability of ω, not the right stability.

If we replace the $(k,0)$-equation by the equation

$$\sum_{\substack{i,j \text{ such that} \\ P_k = \varphi_{ij}(P_i)}} c_{ij} = m^{P_k}(\omega) = 0,$$

then the resulting system of equations will in fact determine all left characteristic idempotents of $\mathcal{F}$; and yes, there can be more than one of those (see section 2.5).

Proposition 2.4.10. *Let (P_i, φ_{ij}) be a set of representatives for the conjugacy classes of (S,S)-pairs (P,φ) with $\varphi \in \mathcal{F}(P,S)$, where the representatives are ordered as in the proof of proposition 2.4.7. Let M be a square matrix with one entry per pair of representatives (P_k, φ_{kl}) and (P_i, φ_{ij}), such that $M_{kl,ij}$ is given by*

$$M_{k0,ij} := \begin{cases} 1 & \text{if } P_k = \varphi_{ij}(P_i), \\ 0 & \text{otherwise;} \end{cases}$$

and for $l \neq 0$ by

$$M_{kl,ij} := \Phi_{[P_k,\varphi_{kl}]}([P_i,\varphi_{ij}]) - \Phi_{[P_k,id]}([P_i,\varphi_{ij}]).$$

Then the left characteristic idempotents of $\mathcal{F}$ are precisely the

$$\xi_L = \sum_{i,j} c_{ij} \cdot [P_i, \varphi_{ij}] \in A_{\mathcal{F}}(S,S)_{(p)}$$

such that $Mc = b$ where b is the vector with $b_{00} = 1$ and $b_{kl} = 0$ otherwise.

A similar system of equations characterizes the right characteristic idempotents; alternatively one can use the $(-)^{\mathrm{op}}$-correspondence between left and right characteristic idempotents (se lemma 2.4.1).

Proof. An element $X \in A(S,S)_{(p)}$ is $\mathcal{F}$-generated if and only if X can be written on the form

$$X = \sum_{i,j} c_{ij} \cdot [P_i, \varphi_{ij}]$$

with $c_{ij} \in \mathbb{Z}_{(p)}$.

Assuming X is $\mathcal{F}$ generated, then $\Phi_{[P,\varphi]}(X) = 0$ for all $\varphi \notin \mathcal{F}(P,S)$. The characterization of left $\mathcal{F}$-stability of X given in lemma 2.3.8, can then be simplified to: X is left $\mathcal{F}$-stable if and only if

$$\Phi_{[P,\varphi]}(X) = \Phi_{[P,\psi]}(X)$$

for all $\varphi, \psi \in \mathcal{F}(P,S)$ and $P \leq S$. This requirement can be simplified even further to

$$\Phi_{[P,\varphi]}(X) = \Phi_{[P,id]}(X)$$

for all $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$. The (k,l)-equation, with $l \neq 0$, is

$$\Phi_{[P_k,\varphi_{kl}]}(X) = \Phi_{[P_k,id]}(X);$$

and since the (P_k, φ_{kl})-pairs represent all pairs (P, φ) with $\varphi \in \mathcal{F}(P,S)$ and $P \leq S$, it follows that: X is left $\mathcal{F}$-stable if and only if X satisfies all (k,l)-equations with $l \neq 0$.

The $(k,0)$-equation is just $m^{P_k}(X) = 0$ for $k > 0$, and the $(0,0)$-equation is $m^S(X) = 1$. If X is a left $\mathcal{F}$-characteristic idempotent, then X satisfies the $(k,0)$-equations by lemma 2.4.5.

Assume conversely that X satisfies all (k,l)-equations, so X is in particular left $\mathcal{F}$-stable by the considerations above. The $(0,0)$-equation $m^S(X) = 1$ ensures that $\varepsilon(X) \equiv 1 \pmod p$ since $\varepsilon([P,\varphi]) \equiv 0 \pmod p$ for $P < S$. We can therefore conclude that X is left $\mathcal{F}$-characteristic. All the $(k,0)$-equations then imply that X is idempotent thanks to lemma 2.4.5. □

Theorem 2.4.11. *Every saturated fusion system has a characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$ – and it is unique by proposition 2.4.7.*

The characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$ corresponds to the basis element $\beta_{[\Delta(S,id)]}$ of $A(\mathcal{F} \times \mathcal{F}_S)_{(p)}$ (see theorem 1.6.9); and it satisfies

$$\Phi_{[P,\varphi]}(\omega_{\mathcal{F}}) = \frac{|S|}{|\mathcal{F}(P,S)|}$$

for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.

Proof. As in the proof of theorem 2.3.17, we consider a saturated fusion system on $S \times S$. This time though, we won't use $\mathcal{F} \times \mathcal{F}$. Let us instead consider the saturated fusion system $\mathcal{F} \times \mathcal{F}_S$ on $S \times S$, and we aim to apply lemma 1.6.5.

Let $X \in A(S \times S)_{(p)}$ be the class $X := [\Delta(S, id)]_{S\times S}$ corresponding to the biset $[S, id] \in A(S, S)_{(p)}$. The $\mathcal{F}\times\mathcal{F}_S$-subconjugates of $\Delta(S, id)$ are exactly the graphs $\Delta(P, \varphi)$ with $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$. We therefore have $\Phi_{[H]}(X) = 0$ whenever H is not on the form $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ by lemma 1.1.5.

We also remark that the $\mathcal{F} \times \mathcal{F}_S$-conjugates of a graph $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ are the other graphs $\Delta(P', \psi)$ with $P' \sim_S P$ and $\psi \in \mathcal{F}(P', S)$.

Using the collection

$$\mathcal{H} := \{\Delta(P, \varphi) \leq S \times S \mid P \leq S, \varphi \in \mathcal{F}(P, S)\} = \{H \leq S \times S \mid H \underset{\mathcal{F}\times\mathcal{F}_S}{\precsim} \Delta(S, id)\},$$

we are then able to use lemma 1.6.5 for the saturated fusion system $\mathcal{F}\times\mathcal{F}_S$, the collection $\mathcal{H}$ and the element $X \in A(S \times S)_{(p)}$. Thus there exists an element $X' \in A(\mathcal{F} \times \mathcal{F}_S)_{(p)}$ such that

(i) $\Phi_{[H]}(X') = \Phi_{[H]}(X) = 0$ for all $H \leq S \times S$ not on the form $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$.

(ii) $c_{[H]}(X') = c_{[H]}(X) = 0$ for all $H \leq S\times S$ not on the form $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$; and for all $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$, we have

$$\sum_{[\Delta(P,\psi)]_{S\times S}\subseteq[\Delta(P,\varphi)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\psi)]}(X') = \sum_{[\Delta(P,\psi)]_{S\times S}\subseteq[\Delta(P,\varphi)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\psi)]}(X).$$

(iii) For every $H \leq S \times S$:

$$\Phi_{[H]}(X') = \sum_{[H']_S\subseteq[H]_{\mathcal{F}}} \frac{|[H']_S|}{|[H]_{\mathcal{F}}|}\Phi_{[H']}(X).$$

By the proof of lemma 1.6.8 (and theorem 1.6.9), property (iii) implies that X' is in fact the basis element $\beta_{[\Delta(S,id)]} \in A(\mathcal{F} \times \mathcal{F}_S)$; hence we also have

$$\Phi_{[\Delta(P,\varphi)]}(X') = \frac{|\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P, \varphi), \Delta(S, id))| \cdot |S \times S|}{|\Delta(S, id)| \cdot |\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P, \varphi), S \times S)|} \tag{2.4}$$

for all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.

The first part of (ii) shows that $X' \in A(S \times S)_{(p)}$ is linear combination of elements $[\Delta(P, \varphi)]_{S\times S}$ with $\varphi \in \mathcal{F}(P, S)$. In particular X' is left free, so it corresponds to an element $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$; and this element $\omega_{\mathcal{F}}$ is $\mathcal{F}$-generated. We have $c_{[P,\varphi]}(\omega_{\mathcal{F}}) = c_{[\Delta(P,\varphi)]}(X')$ and $\Phi_{[P,\varphi]}(\omega_{\mathcal{F}}) = \Phi_{[\Delta(P,\varphi)]}(X')$ by the correspondence between (S, S)-sets and $(S \times S)$-sets.

As a consequence of (ii) we have

$$\begin{aligned}
\varepsilon(\omega_{\mathcal{F}}) &= \sum_{\substack{[P,\varphi]\in C(S,S)\\ \varphi\in\mathcal{F}(P,S)}} c_{[P,\varphi]}(\omega_{\mathcal{F}})\cdot\varepsilon([P,\varphi])\\
&= \sum_{\substack{[\Delta(P,\varphi)]_{S\times S}\\ \varphi\in\mathcal{F}(P,S)}} c_{[\Delta(P,\varphi)]}(X')\cdot\frac{|S|}{|P|}\\
&= \sum_{[P]_S\in C(S)}\left(\sum_{[\Delta(P,\varphi)]_{S\times S}\subseteq[\Delta(P,id)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\varphi)]}(X')\right)\cdot\frac{|S|}{|P|}\\
&= \sum_{[P]_S\in C(S)}\left(\sum_{[\Delta(P,\varphi)]_{S\times S}\subseteq[\Delta(P,id)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\varphi)]}(X)\right)\cdot\frac{|S|}{|P|}\\
&= c_{[\Delta(S,id)]}([\Delta(S,id)])\cdot\tfrac{|S|}{|S|}=1.
\end{aligned}$$

Because X' is in $A(\mathcal{F}\times\mathcal{F}_S)_{(p)}$ we get for all $[Q,\psi]\in C(S,S)$ and $\varphi\in\mathcal{F}(\psi Q,S)$ that

$$\Phi_{[Q,\psi]}(\omega_{\mathcal{F}})=\Phi_{[\Delta(Q,\psi)]}(X')=\Phi_{[\Delta(Q,\varphi\psi)]}(X')=\Phi_{[Q,\varphi\psi]}(\omega_{\mathcal{F}})$$

since $\Delta(Q,\psi)$ and $\Delta(Q,\varphi\psi)$ are $\mathcal{F}\times\mathcal{F}_S$-conjugate. By lemma 2.3.8, this proves that $\omega_{\mathcal{F}}$ is left $\mathcal{F}$-stable. We have thus proved that $\omega_{\mathcal{F}}\in A(S,S)_{(p)}$ is a left characteristic element for $\mathcal{F}$.

We now consider the value of $\Phi_{[P,\varphi]}(\omega_{\mathcal{F}})$, with $\varphi\in\mathcal{F}(P,S)$, in more detail using (2.4). First we remark that $\Phi_{[P,\varphi]}(\omega_{\mathcal{F}})=\Phi_{[P,id]}(\omega_{\mathcal{F}})$ since $\omega_{\mathcal{F}}$ is left $\mathcal{F}$-stable; and then

$$\begin{aligned}
\Phi_{[P,id]}(\omega_{\mathcal{F}}) &= \Phi_{[\Delta(P,id)]}(X')\\
&= \frac{|\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),\Delta(S,id))|\cdot|S\times S|}{|\Delta(S,id)|\cdot|\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),S\times S)|}.
\end{aligned}$$

The morphisms of $\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),S\times S)$ are the pairs (φ,c_s) where $c_s\in\mathcal{F}_S(P,S)$ and $\varphi\in\mathcal{F}(P,S)$, so

$$|\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),S\times S)|=|\mathcal{F}_S(P,S)|\cdot|\mathcal{F}(P,S)|\,.$$

The image of $\Delta(P,id)$ under a morphism $(\varphi,c_s)\in\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),S\times S)$ is

$$(\varphi,c_s)(\Delta(P,id))=\{(\varphi(p),c_s(p))\mid p\in P\}=\Delta({}^sP,\varphi\circ(c_s)^{-1}).$$

This image lies in $\Delta(S,id)$ if and only if $\varphi\circ(c_s)^{-1}=id_{{}^sP}$, i.e. if $\varphi=c_s$. The number of morphisms in $\mathrm{Hom}_{\mathcal{F}\times\mathcal{F}_S}(\Delta(P,id),\Delta(S,id))$ is therefore simply $|\mathcal{F}_S(P,S)|$.

Returning to the expression for $\Phi_{[P,id]}(\omega_\mathcal{F})$ we then have

$$\Phi_{[P,id]}(\omega_\mathcal{F}) = \frac{|\mathcal{F}_S(P,S)| \cdot |S \times S|}{|\Delta(S,id)| \cdot (|\mathcal{F}_S(P,S)| \cdot |\mathcal{F}(P,S)|)} = \frac{|S|}{|\mathcal{F}(P,S)|},$$

which only depends on the $\mathcal{F}$-conjugacy class of P. We conclude that for all (P,φ), with $\varphi \in \mathcal{F}(P,S)$, and (Q,ψ), with $\psi \in \mathcal{F}(Q,S)$, and such that $P \sim_\mathcal{F} Q$, we have

$$\Phi_{[P,\varphi]}(\omega_\mathcal{F}) = \Phi_{[Q,\psi]}(\omega_\mathcal{F}).$$

In particular,

$$\Phi_{[Q,\psi]}(\omega_\mathcal{F}) = \Phi_{[\varphi Q, \psi\varphi^{-1}]}(\omega_\mathcal{F})$$

for all $[Q,\psi]$ with $\psi, \varphi \in \mathcal{F}(Q,S)$. Recalling that $\Phi_{[Q,\psi]}(\omega_\mathcal{F}) = 0$ when $\psi \notin \mathcal{F}(Q,S)$, lemma 2.3.8 says that $\omega_\mathcal{F}$ is right $\mathcal{F}$-stable as well.

We have proven that $\omega_\mathcal{F}$ is fully $\mathcal{F}$-characteristic, so it just remains to show that $\omega_\mathcal{F}$ is actually idempotent. Since $\omega_\mathcal{F}$ is right $\mathcal{F}$-characteristic, we can apply lemma 2.4.5. We therefore calculate $m_P(\omega_\mathcal{F})$ for all $P \leq S$:

$$\begin{aligned}
m_p(\omega_\mathcal{F}) &= \sum_{\substack{[P,\varphi] \in C(S,S) \\ \varphi \in \mathcal{F}(P,S)}} c_{[P,\varphi]}(\omega_\mathcal{F}) \\
&= \sum_{[\Delta(P,\varphi)]_{S\times S} \subseteq [\Delta(P,id)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\varphi)]}(X') \\
&\overset{(ii)}{=} \sum_{[\Delta(P,\varphi)]_{S\times S} \subseteq [\Delta(P,id)]_{\mathcal{F}\times\mathcal{F}_S}} c_{[\Delta(P,\varphi)]}(X) \\
&= \begin{cases} 0 & \text{for } P < S, \\ c_{[\Delta(S,id)]}([\Delta(S,id)]) = 1 & \text{for } P = S; \end{cases}
\end{aligned}$$

so $\omega_\mathcal{F}$ is idempotent by lemma 2.4.5. □

Proposition 2.4.12. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems over p-groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1,S_1)_{(p)}$ and $\omega_2 \in A(S_2,S_2)_{(p)}$ be the characteristic idempotents. Then*

$$A(\mathcal{F}_1,\mathcal{F}_2)_{(p)} = \omega_2 \circ A(S_1,S_2)_{(p)} \circ \omega_1$$

with basis elements $\omega_2 \circ [P,\varphi] \circ \omega_1$, one per class $[P,\varphi] \in C(\mathcal{F}_1,\mathcal{F}_2)$.

The characteristic idempotent $\omega_\mathcal{F}$ is a multiplicative identity of $A(\mathcal{F},\mathcal{F})_{(p)}$. Furthermore, we have a category with the saturated fusion systems on p-groups as objects, the modules $A(\mathcal{F}_1,\mathcal{F}_2)_{(p)}$ as morphism sets, and the characteristic idempotents as identity morphisms.

Proof. Any element of $\omega_2 \circ A(S_1, S_2)_{(p)} \circ \omega_1$ is left $\mathcal{F}_2$-stable and right $\mathcal{F}_1$-stable and hence in $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$. Conversely, any element $X \in A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ satisfies

$$X = \omega_2 \circ X \circ \omega_1 \in \omega_2 \circ A(S_1, S_2)_{(p)} \circ \omega_1$$

by proposition 2.4.6.

The basis of $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ follows from proposition 2.3.19.

The defined category with morphism sets $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$, has $\circ$ as a well-defined composition (see definition 2.3.9); and $\omega_{\mathcal{F}}$ is the identity morphism of

$$A(\mathcal{F}, \mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A(S, S)_{(p)} \circ \omega_{\mathcal{F}}$$

because $\omega_{\mathcal{F}}$ is an idempotent. □

Proposition 2.4.13. *The ring of characteristic elements $A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$ (see definition 2.3.18) satisfies*

$$A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}.$$

It has a basis consisting of the elements $\omega_{\mathcal{F}} \circ [P, \mathrm{incl}] \circ \omega_{\mathcal{F}}$, one basis element per class $[P]_{\mathcal{F}} \in C(\mathcal{F})$.

The characteristic elements of $\mathcal{F}$ are the elements $X \in A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$ where the coefficient of X at the basis element $\omega_{\mathcal{F}} \circ [S, \mathrm{incl}] \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}}$ is coprime to p.

Proof. Any element in $\omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$ is $\mathcal{F}$-stable and a product of $\mathcal{F}$-generated elements (hence it is $\mathcal{F}$-generated as well), so it is contained in $A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$.

Conversely, let $X \in A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$. Because X is $\mathcal{F}$-stable, we then have

$$X = \omega_{\mathcal{F}} \circ X \circ \omega_{\mathcal{F}} \in \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$$

by proposition 2.4.6. We conclude that $A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$, and the basis then follows from corollary 2.3.20.

An element $X \in A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$ is $\mathcal{F}$-characteristic if and only if $\varepsilon(X) \not\equiv 0 \pmod p$. Any basis element of $A_{\mathcal{F}}(\mathcal{F}, \mathcal{F})_{(p)}$ other than $\omega_{\mathcal{F}}$ is on the form $\omega_{\mathcal{F}} \circ [P, \mathrm{incl}] \circ \omega_{\mathcal{F}}$, with $P < S$; and consequently

$$\varepsilon(\omega_{\mathcal{F}} \circ [P, \mathrm{incl}] \circ \omega_{\mathcal{F}}) = \varepsilon([P, \mathrm{incl}]) = \frac{|S|}{|P|} \equiv 0 \pmod p.$$

So whether $\varepsilon(X) \not\equiv 0 \pmod p$, depends only on the coefficient of X at the basis element $\omega_{\mathcal{F}}$. □

Remark 2.4.14. The characteristic idempotent for the minimal fusion system $\mathcal{F}_S$, is just $[S,id]_S^S$.

The ring $A_{\mathcal{F}_S}(\mathcal{F}_S,\mathcal{F}_S)_{(p)}$ of characteristic elements for $\mathcal{F}_S$, then has a basis consisting of the elements

$$[S,id]\circ[P,incl]\circ[S,id]=[P,incl]$$

with one basis element per conjugacy class $[P]_S$. Furthermore, the product of two basis elements is given by

$$[P,incl]\circ[Q,incl]=\sum_{\overline{y}\in P\backslash S/Q}[Q\cap P^y,c_y]=\sum_{\overline{y}\in P\backslash S/Q}[P\cap {}^yQ,incl].$$

If we compare this to the double coset formula for the single Burnside ring $A(S)_{(p)}$, we see that $A_{\mathcal{F}_S}(\mathcal{F}_S,\mathcal{F}_S)_{(p)}\cong A(S)_{(p)}$.

Because the characteristic idempotent $[S,id]_S^S$ is an element of the non-localized ring $A(S,S)$, we actually don't need the p-localizations above; and $A_{\mathcal{F}_S}(\mathcal{F}_S,\mathcal{F}_S)\cong A(S)$ with the basis element $[P,incl]\in A_{\mathcal{F}_S}(\mathcal{F}_S,\mathcal{F}_S)$ corresponding to $[P]\in A(S)$.

2.5 A fusion system with several left characteristic idempotents

We consider the symmetric group S_4 with the subgroup $\langle(1\ 2\ 3\ 4),(1\ 4)(2\ 3)\rangle$ of order 8 as a Sylow-2-subgroup. We denote the generators of the Sylow-subgroup by $D:=(1\ 2\ 3\ 4)$ of order 4 and $S:=(1\ 4)(2\ 3)$ of order 2; we then have $SD=D^{-1}S$, so $\langle(1\ 2\ 3\ 4),(1\ 4)(2\ 3)\rangle$ is the dihedral group D_8. We let $\mathcal{F}$ be the saturated fusion system $\mathcal{F}_{D_8}(S_4)$ where we identify $D_8=\langle(1\ 2\ 3\ 4),(1\ 4)(2\ 3)\rangle$ as described.

A particular subgroup of interest in D_8 is $V:=\left\langle D^2,S\right\rangle=\langle(1\ 3)(2\ 4),(1\ 4)(2\ 3)\rangle$ which is a copy of the Klein Four Group. The minimal fusion system $\mathcal{F}_{D_8}$ contains only 2 automorphisms of V: the identity, and the involution switching $S\leftrightarrow D^2S$. In $\mathcal{F}$ however, $\mathrm{Aut}_{\mathcal{F}}(V)$ contains all 6 permutations of the three double-transpositions of S_4. In particular, we have $c_{(1\ 2)}\colon V\to V$ mapping D^2 and S to each other.

We have two (D_8,D_8)-conjugacy classes of pairs (V,φ) with $\varphi\in\mathcal{F}(V,D_8)$: One class is the trivial one, $[V,id]_{(D_8,D_8)}=\{(V,id),(V,S\leftrightarrow D^2S)\}$. The other class is $[V,c_{(1\ 2)}]_{(D_8,D_8)}$ which contains the remaining four pairs: $(V,c_{(1\ 2)})=(V,D^2\leftrightarrow S)$, $(V,D^2\leftrightarrow D^2S)$, $(V,D^2\mapsto S\mapsto D^2S\mapsto D^2)$ and $(V,D^2\mapsto D^2S\mapsto S\mapsto D^2)$.

There are, in total, 11 different (D_8,D_8)-conjugacy classes of pairs (P,φ) with $P\le D_8$ and $\varphi\in\mathcal{F}(P,D_8)$:

$$[D_8,id],\quad [V,id],\quad [V,c_{(1\ 2)}],\quad [\left\langle D^2,DS\right\rangle,id],\quad [\langle D\rangle,id],$$
$$[\left\langle D^2\right\rangle,id],\quad [\left\langle D^2\right\rangle,c_{(1\ 2)}],\quad [\langle S\rangle,id],\quad [\langle S\rangle,c_{(1\ 2)}],\quad [\langle DS\rangle,id],\quad [1,id]$$

We will use this ordering of the classes in the following.

The homomorphism of marks for the $\mathcal{F}$-generated elements of $A(D_8, D_8)_{(2)}$ is

$$\Phi\colon A_{\mathcal{F}}(D_8, D_8)_{(2)} \to \prod_{\substack{[Q,\psi]\in C(D_8,D_8)\\ \psi\in\mathcal{F}(Q,D_8)}} \mathbb{Z}_{(2)},$$

which is given by the following 11×11-matrix M with entries $\Phi_{[Q,\psi]}([P,\varphi])$ for pairs of the 11 classes above. The rows and columns of M are ordered according to the list earlier.

$$M = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 16 & 0 & 16 & 16 & 32 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 4 & 8 & 4 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 4 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 8 & 16 & 16 & 16 & 16 & 32 & 32 & 32 & 32 & 32 & 64 \end{pmatrix}$$

By theorem 2.4.11, the $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}}$ is given by

$$\Phi_{[Q,\psi]}(\omega_{\mathcal{F}}) = \frac{|D_8|}{|\mathcal{F}(Q,D_8)|}.$$

If c is the 11-dimensional vector in $(\mathbb{Z}_{(2)})^{11}$ of coefficients $c_{[P,\varphi]}(\omega_{\mathcal{F}})$ for $\omega_{\mathcal{F}}$, then finding $\omega_{\mathcal{F}}$ amounts to solving the equation $Mc = b$ where

$$b = (2, \tfrac{4}{3}, \tfrac{4}{3}, 4, 4, \tfrac{8}{3}, \tfrac{8}{3}, \tfrac{8}{3}, \tfrac{8}{3}, 4, 8).$$

Luckily, this is easier than it might seem at first, and the solution is just

$$c = (1, -\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0);$$

so we get

$$\omega_{\mathcal{F}} = [D_8, id] - \tfrac{1}{3}[V, id] + \tfrac{1}{3}[V, c_{(1\ 2)}].$$

We might calculate the system of equations in the proof of $\omega_{\mathcal{F}}$'s uniqueness (proposition 2.4.7) and see that we get the same solution, but we won't do it here.

We will however calculate the matrix describing the left characteristic idempotents of $\mathcal{F}$ (see proposition 2.4.10). We get the following 11×11-matrix L:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -8 & -16 & 8 & -16 & -16 & -32 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -4 & -8 & 4 & 0 & 0 & 0 & 0 & -8 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The left $\mathcal{F}$-characteristic idempotents are then the elements where the coefficient vector c is a solution to $Lc = e$ where e is the vector

$$e = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Since $\omega_{\mathcal{F}}$ is in particular a left characteristic idempotent, we already know a solution:

$$(1, -\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0).$$

It is also easily seen that the vector

$$k = (0, 0, 0, 0, 0, 1, 2, -2, -1, 0, 0)$$

is in the kernel of L. The vector k corresponds to the element

$$X := [\langle D^2 \rangle, id] + 2[\langle D^2 \rangle, c_{(1\ 2)}] - 2[\langle S \rangle, id] - [\langle S \rangle, c_{(1\ 2)}].$$

Consequently X has the following properties: X is $\mathcal{F}$-generated, X is left $\mathcal{F}$-stable, and $m^P(X) = 0$ for all $P \leq S$ (in particular $\varepsilon(X) = 0$). Furthermore, $\omega_{\mathcal{F}} + \lambda X$ is a left characteristic idempotent for $\mathcal{F}$ for all $\lambda \in \mathbb{Z}_{(2)}$.

The saturated fusion system $\mathcal{F}$ therefore has more than one left characteristic idempotent – in fact, it has infinitely many.

2.6 Fusion systems induced by characteristic elements

In this section, we will show that any left/right/fully characteristic element $X \in A(S,S)_{(p)}$ for a fusion system $\mathcal{F}$ actually determines the fusion system uniquely. In particular, we shall give methods of reconstructing the fusion system given a characteristic element.

It is however not all fusion systems that have a characteristic element; for at the end we will see that the existence of a characteristic element for $\mathcal{F}$ actually implies that $\mathcal{F}$ is saturated.

This section follows sections 5 and 6 of [RS09] closely.

The stabilizer fusion systems

Definition 2.6.1. Let $X \in A(S,S)_{(p)}$.

(i) The *right stabilizer fusion system* of X is the fusion system $\mathrm{RSt}(X)$ on S with morphism sets

$$\mathrm{Hom}_{\mathrm{RSt}(X)}(P,Q) := \{\varphi \in \mathrm{Inj}(P,Q) \mid X \circ [P,\varphi]_P^S = X \circ [P, incl]_P^S\}.$$

(ii) The *left stabilizer fusion system* of X is the fusion system $\mathrm{LSt}(X)$ on S with morphism sets

$$\mathrm{Hom}_{\mathrm{LSt}(X)}(P,Q) := \{\varphi \in \mathrm{Inj}(P,Q) \mid [\varphi P, \varphi^{-1}]_S^P \circ X = [P, id_P]_S^P \circ X\}.$$

(iii) The *(full) stabilizer fusion system* of X is the intersection $\mathrm{St}(X) := \mathrm{LSt}(X) \cap \mathrm{RSt}(X)$.

These are actually fusion systems: Let us just consider $\mathrm{RSt}(X)$. All homomorphisms induced by S-conjugation are in $\mathrm{RSt}(X)$ since $[P, c_s]_P^S = [P, incl]_P^S$ for all $P \leq S$ and $s \in S$. For all pairs of homomorphisms $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ in $\mathrm{RSt}(X)$ we have

$$\begin{aligned} X \circ [P, \psi\varphi]_P^S &= X \circ [Q,\psi]_Q^S \circ [P,\varphi]_P^Q = X \circ [Q, incl]_Q^S \circ [P,\varphi]_P^Q \\ &= X \circ [P,\varphi]_P^S = X \circ [P, incl]_P^S \end{aligned}$$

so $\mathrm{RSt}(X)$ is closed under composition of homomorphisms. Finally, for every $\varphi \in \mathrm{Hom}_{\mathrm{RSt}(X)}(P,Q)$ we have

$$\begin{aligned} X \circ [\varphi P, \varphi^{-1}]_{\varphi P}^S &= X \circ [P, incl]_P^S \circ [\varphi P, \varphi^{-1}]_{\varphi P}^P \\ &= X \circ [P,\varphi]_P^S \circ [\varphi P, \varphi^{-1}]_{\varphi P}^P = X \circ [\varphi P, incl]_{\varphi P}^S; \end{aligned}$$

so we get $\varphi^{-1} \in \mathrm{Hom}_{\mathrm{RSt}(X)}(\varphi P, P)$.

Lemma 2.6.2. *Let $X \in A_{fr}(S,S)_{(p)}$. We then have*

$$\mathrm{RSt}(X) = \mathrm{LSt}(X^{\mathrm{op}}) \quad \textit{and} \quad \mathrm{LSt}(X) = \mathrm{RSt}(X^{\mathrm{op}}).$$

In particular, if X is symmetric, then

$$\mathrm{RSt}(X) = \mathrm{LSt}(X) = \mathrm{St}(X).$$

Proof. This follows from lemmas 2.2.13 and 2.2.11. □

Lemma 2.6.3. *Let $X \in \mathcal{A}(S,S)_{(p)}$. Then the following hold:*

(i) For all $[Q,\psi] \in C(S,S)$ and $\varphi \in \mathrm{Hom}_{\mathrm{RSt}(X)}(Q,S)$

$$\Phi_{[Q,\psi]}(X) = \Phi_{[\varphi Q, \psi\varphi^{-1}]}(X).$$

(ii) For all $[Q,\psi] \in C(S,S)$ and $\varphi \in \mathrm{Hom}_{\mathrm{LSt}(X)}(\psi Q,S)$

$$\Phi_{[Q,\psi]}(X) = \Phi_{[Q,\varphi\psi]}(X).$$

Proof. The result follows from lemma 2.3.8 since X is right $\mathrm{RSt}(X)$-stable and left $\mathrm{LSt}(X)$-stable per definition. □

The fixed-point and orbit-type (pre-)fusion systems

Definition 2.6.4. As *pre-fusion system* $\mathcal{P}$ on S is a family of sets

$$\mathcal{P}(P,Q) = \mathrm{Hom}_{\mathcal{P}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$$

of injective group homomorphisms, for all $P, Q \leq S$. The only other condition that $\mathcal{P}$ must satisfy, is that if $\varphi \in \mathcal{P}(P,Q)$ and $\varphi(P) \leq R \leq S$, then the composite $P \xrightarrow{\varphi} \varphi P \hookrightarrow R$ is in $\mathcal{P}(P,R)$; i.e. we can freely extend and restrict the target of morphisms. A pre-fusion system $\mathcal{P}$ is therefore determined by the sets $\mathcal{P}(P,S)$ for $P \leq S$.

There is however no requirement that $\mathcal{P}$ is a category, or even that it contains the identity homomorphisms.

Definition 2.6.5. Let $\mathcal{P}$ be a pre-fusion system on S. The *closure of* $\mathcal{P}$, written $\overline{\mathcal{P}}$, is the smallest fusion system on S containing $\mathcal{P}$; i.e. $\overline{\mathcal{P}}$ is the intersection all fusion systems containing $\mathcal{P}$. We say that $\mathcal{P}$ is *closed* if $\overline{\mathcal{P}} = \mathcal{P}$, i.e. if $\mathcal{P}$ is already a fusion system.

Definition 2.6.6. A pre-fusion system $\mathcal{P}$ on S is *level-wise closed* if the following holds for all $P, Q, R \leq S$:

(i) $\mathrm{Hom}_S(P,Q) \subseteq \mathcal{P}(P,Q)$.

(ii) If $\varphi \in \mathcal{P}(P, Q)$ is a group isomorphism, then $\varphi^{-1} \in \mathcal{P}(Q, P)$.

(iii) If $\varphi \in \mathcal{P}(P, Q)$ and $\psi \in \mathcal{P}(Q, R)$ are group isomorphisms, then $\psi\varphi \in \mathcal{P}(P, R)$.

If a pre-fusion system $\mathcal{P}$ is level-wise closed as well af closed under restriction of homomorphisms, then $\mathcal{P}$ is a fusion system. A level-wise closed pre-fusion system is thus "almost" a fusion systems, and we can define some the same concepts.

When $\mathcal{P}$ is level-wise closed, the morphism set $\mathcal{P}(P, P)$ is a group for all $P \leq S$, and we define $\mathrm{Aut}_{\mathcal{P}}(P) := \mathcal{P}(P, P)$. The concept of $\mathcal{P}$-conjugacy ($P \sim_{\mathcal{P}} Q$ if they are isomorphic by an isomorphism in $\mathcal{P}$) is a well-defined equivalence relation; and we extend the notions of fully normalized/centralized subgroups to this context.

We can therefore consider the saturation axioms for a level-wise closed pre-fusion system.

Definition 2.6.7. A level-wise closed pre-fusion system $\mathcal{P}$ is *saturated at* $P \leq S$ if the following holds:

(i) If $Q \in [P]_{\mathcal{P}}$ is fully $\mathcal{P}$-normalized, then Q is fully $\mathcal{P}$-centralized and $\mathrm{Aut}_S(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{P}}(Q)$.

(ii) If $\varphi \in \mathcal{P}(P, S)$ with φP fully $\mathcal{P}$-centralized, then φ extends to a homomorphism $\overline{\varphi} \in \overline{\mathcal{P}}(N_\varphi, S)$ in the closure of $\mathcal{P}$.

Definition 2.6.8. Let $X \in A_{fr}(S, S)_{(p)}$.

(i) The *orbit-type pre-fusion system* of X is the pre-fusion system $\text{Pre-Orb}(X)$ on S with morphism sets

$$\mathrm{Hom}_{\text{Pre-Orb}(X)}(P, Q) := \{\varphi \in \mathrm{Inj}(P, Q) \mid c_{[P,\varphi]}(X) \neq 0\}.$$

The *orbit-type fusion system* of X is the closure $\mathrm{Orb}(X) := \overline{\text{Pre-Orb}(X)}$.

(ii) The *fixed-point pre-fusion system* of X is the pre-fusion system $\text{Pre-Fix}(X)$ on S with morphism sets

$$\mathrm{Hom}_{\text{Pre-Fix}(X)}(P, Q) := \{\varphi \in \mathrm{Inj}(P, Q) \mid \Phi_{[P,\varphi]}(X) \neq 0\}.$$

The *fixed-point fusion system* of X is the closure $\mathrm{Fix}(X) := \overline{\text{Pre-Fix}(X)}$.

Remark 2.6.9. Both $\text{Pre-Orb}(X)$ and $\text{Pre-Fix}(X)$ are closed under pre- and post-S-conjugation, since the (S, S)-conjugacy classes are unchanged: $[P, \varphi]_{(S,S)} = [P^s, c_t\varphi c_s]_{(S,S)}$ for all $s, t \in S$ and $\varphi \in \mathrm{Inj}(P, S)$.

In particular when $\mathcal{P}$ is either of $\text{Pre-Orb}(X)$ and $\text{Pre-Fix}(X)$; S acts by post-conjugation on the morphism sets $\mathcal{P}(P, S)$, so i makes sense to define $\mathrm{Rep}_{\mathcal{P}}(P, S) := \mathrm{Inn}(S) \backslash \mathcal{P}(P, S)$.

Lemma 2.6.10. *For every* $X \in A_{fr}(S,S)_{(p)}$ *we have* $\mathrm{Orb}(X) = \mathrm{Fix}(X)$.

Proof. It is enough to prove $\text{Pre-Orb}(X) \subseteq \mathrm{Fix}(X)$ and $\text{Pre-Fix}(X) \subseteq \mathrm{Orb}(X)$ since the equality then follows by taking closure.

Suppose that $\psi \in \mathrm{Hom}_{\text{Pre-Orb}(X)}(Q,S)$ so $c_{[Q,\psi]}(X) \neq 0$. Let $[P,\varphi]$ be maximal (under (S,S)-subconjugacy) with $c_{[P,\varphi]}(X) \neq 0$ and $(Q,\psi) \underset{(S,S)}{\precsim} (P,\varphi)$. By the maximality of $[P,\varphi]$ we get from lemma 2.1.12 that

$$\Phi_{[P,\varphi]}(X) = c_{[P,\varphi]}(X) \cdot \Phi_{[P,\varphi]}([P,\varphi]) \neq 0$$

so $\varphi \in \mathrm{Hom}_{\mathrm{Fix}(X)}(P,S)$. Because $\mathrm{Fix}(X)$ is a fusion system, and $(Q,\psi) \underset{(S,S)}{\precsim} (P,\varphi)$, we have $\psi \in \mathrm{Hom}_{\mathrm{Fix}(X)}(Q,S)$ (see remark 2.3.2).

Suppose now that $\psi \in \mathrm{Hom}_{\text{Pre-Fix}(X)}(Q,S)$, so $\Phi_{[Q,\psi]}(X) \neq 0$. From lemma 2.1.12 we conclude that we must have $c_{[P,\varphi]}(X) \neq 0$ for some $(P,\varphi) \underset{(S,S)}{\succsim} (Q,\psi)$. It follows that $\varphi \in \mathrm{Hom}_{\mathrm{Orb}(X)}(P,S)$ and then $\psi \in \mathrm{Hom}_{\mathrm{Orb}(X)}(Q,S)$ as well. □

Lemma 2.6.11. *Let* $X \in A_{fr}(S,S)_{(p)}$. *For every* $P \leq S$ *and* $\varphi \in \mathrm{Inj}(P,S)$ *we then have* $\varphi \in \mathrm{Hom}_{\text{Pre-Fix}(X)}(P,S)$ *if and only if* $\varphi^{-1} \in \mathrm{Hom}_{\text{Pre-Fix}(X^{\mathrm{op}})}(\varphi P,S)$.

In particular, if $\text{Pre-Fix}(X)$ *is level-wise closed, then* $\text{Pre-Fix}(X^{\mathrm{op}}) = \text{Pre-Fix}(X)$.

Proof. The result follows immediately from lemma 2.2.14. □

Congruence relations for Pre-Fix

Lemma 2.6.12. *Let* $X \in A_{fr}(S,S)_{(p)}$ *and let* $P \leq S$. *We also put* $\mathcal{P} := \text{Pre-Fix}(X)$. *Recall that* $\text{Pre-Fix}(X)$ *is closed under post-*S*-conjugation (see remark 2.6.9).*

(i) For each $\varphi \in \mathcal{P}(P,S)$, *the number* $\Phi_{[P,\varphi]}(X)$ *is divisible by* $|C_S(\varphi P)|$; *and*

$$\sum_{\overline{\varphi} \in \mathrm{Rep}_{\mathcal{P}}(P,S)} \frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} \equiv \varepsilon(X) \pmod p.$$

(ii) Let $[P]_{\mathcal{P}}$ *be the set of* $Q \leq S$ *where* $Q = \varphi P$ *for some* $\varphi \in \mathcal{P}(P,S)$; *then* $[P]_{\mathcal{P}}$ *is closed under* S*-conjugation (remark 2.6.9).*

For every $Q \in [P]_{\mathcal{P}}$, *the number* $\sum_{\varphi \in \mathcal{P}(P,Q)} \Phi_{[P,\varphi]}(X)$ *is divisible by* $|N_S Q|$; *and*

$$\sum_{[Q]_S \subseteq [P]_{\mathcal{P}}} \frac{\sum_{\varphi \in \mathcal{P}(P,Q)} \Phi_{[P,\varphi]}(X)}{|N_S Q|} \equiv \varepsilon(X) \pmod p.$$

Be aware that taking sums over $\overline{\varphi} \in \mathrm{Rep}_{\mathcal{P}}(P,S)$ *is not the same as taking sums over* $[P,\varphi]_{(S,S)}$ *with* $\varphi \in \mathcal{P}(P,S)$. *If for instance* $\varphi_1 = \varphi_2 \circ c_s$ *for some* $s \in N_S P$, *then* $[P,\varphi_1] = [P,\varphi_2]$ *even though we might have* $\overline{\varphi_1} \neq \overline{\varphi_2}$.

Proof. It is sufficient to consider the case where X is an (S,S)-set, since the general case then follows by linearity.

To distinguish the two S-actions, we regard X as a (S_1,S_2)-biset (with $S_1 = S_2 = S$). The set of S_2-orbits $S_2\backslash X$ is then an S_1-set, and we let $\pi\colon X \to S_2\backslash X$ be the projection (which preserves the S_1-action).

Let $X_0 \subseteq X$ be the pre-image $X_0 := \pi^{-1}((S_2\backslash X)^P)$, then X_0 is a left-free (P,S_2)-set. Furthermore we have $\pi(xp) = \pi(x)$ for all $x \in X_0$ and $p \in P$, so for every $x \in X_0$ and $p \in P$ there is a $\theta_x(p) \in S_2$ such that $\theta_x(p)x = xp$; and $\theta_x(p)$ is unique since S_2 acts freely. Because $xpp' = \theta_x(p)xp' = \theta_x(p)\theta_x(p')x$, it even follows that θ_x is a group homomorphism $\theta_x\colon P \to S_2$.

We thus get a map $\theta\colon X_0 \to \mathrm{Hom}(P,S_2)$ by $x \mapsto \theta_x$. A pre-image $\theta^{-1}(\varphi)$ consists of the $x \in X_0$ with $\varphi(p)x = xp$ for all $p \in P$, which are just the $x \in (X_0)^{(P,\varphi)}$. Since every $x \in X^{(P,\varphi)}$ has $\pi(x) \in (S_2\backslash X)^P$, it follows that $\theta^{-1}(\varphi) = (X_0)^{(P,\varphi)} = X^{(P,\varphi)}$. In particular, if $\theta^{-1}(\varphi) \neq \emptyset$ then φ is injective (since X is bifree), and we then furthermore have $\varphi \in \mathcal{P}(P,S_2)$ by the definition of $\mathcal{P} = \text{Pre-Fix}(X)$. We conclude that θ is in fact a map $\theta\colon X_0 \to \mathcal{P}(P,S_2)$

The pre-images are a partition of X_0, so we have

$$|X_0| = \sum_{\varphi \in \mathcal{P}(P,S_2)} \left|\theta^{-1}(\varphi)\right| = \sum_{\varphi \in \mathcal{P}(P,S_2)} \Phi_{[P,\varphi]}(X).$$

For $s \in S_2$, $x \in X_0$ and $p \in P$ we have $(sx)\cdot p = (s\theta_x(p)s^{-1})\cdot(sx)$; hence $\theta_{sx} = c_s \circ \theta_x$. The map $\theta\colon X_0 \to \mathcal{P}(P,S_2)$ therefore preserves the S_2-action, hence θ induces a map $\overline{\theta}\colon S_2\backslash X_0 \to \mathrm{Rep}_{\mathcal{P}}(P,S_2)$.

Let $\mathrm{im}\colon \mathcal{P}(P,S_2) \to [P]_{\mathcal{P}}$ be the surjective map sending φ to its image in S_2. This induces a surjective map $\overline{\mathrm{im}}\colon \mathrm{Rep}_{\mathcal{P}}(P,S_2) \to \overline{[P]_{\mathcal{P}}}$ where $\overline{[P]_{\mathcal{P}}}$ is the set of S_2-conjugacy classes $[Q]_{S_2}$ where $Q \in [P]_{\mathcal{P}}$. These maps fit into a commutative diagram:

$$\begin{array}{ccccc} X_0 & \xrightarrow{\theta} & \mathcal{P}(P,S_2) & \xrightarrow{\mathrm{im}} & [P]_{\mathcal{P}} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ S_2\backslash X_0 & \xrightarrow{\overline{\theta}} & \mathrm{Rep}_{\mathcal{P}}(P,S_2) & \xrightarrow{\overline{\mathrm{im}}} & \overline{[P]_{\mathcal{P}}} \end{array}$$

We now separate the parts (i) and (ii) of the lemma.

(i): The orbit of $\varphi \in \mathcal{P}(P,S_2)$ under the S_2-action contains $|S_2 : C_{S_2}(\varphi P)|$ distinct homomorphisms. Each φ' with $\overline{\varphi'} = \overline{\varphi}$, is S_2-conjugate to φ, so $[P,\varphi']_{(S_1,S_2)} = [P,\varphi]_{(S_1,S_2)}$ and $\Phi_{[P,\varphi']}(X) = \Phi_{[P,\varphi]}(X)$. We therefore have

$$\left|(\pi\circ\theta)^{-1}(\overline{\varphi})\right| = \frac{|S_2|}{|C_{S_2}(\varphi P)|}\Phi_{[P,\varphi]}(X).$$

Since S_2 acts freely on X_0, we get

$$\left|\overline{\theta}^{-1}(\overline{\varphi})\right| = \frac{\left|(\pi\circ\theta)^{-1}(\overline{\varphi})\right|}{|S_2|} = \frac{\Phi_{[P,\varphi]}(X)}{|C_{S_2}(\varphi P)|}.$$

In particular $|C_{S_2}(\varphi P)|$ divides $\Phi_{[P,\varphi]}(X)$. When we add the sizes of the pre-images of $\overline{\theta}$, we obtain

$$\begin{aligned}\sum_{\overline{\varphi}\in \mathrm{Rep}_{\mathcal{P}}(P,S)} \frac{\Phi_{[P,\varphi]}(X)}{|C_{S_2}(\varphi P)|} &= \sum_{\overline{\varphi}\in \mathrm{Rep}_{\mathcal{P}}(P,S)} \overline{\theta}^{-1}(\overline{\varphi}) = |S_0\backslash X_0| \\ &= \left|(S_0\backslash X)^P\right| \equiv |S_0\backslash X| = \varepsilon(X) \pmod p.\end{aligned}$$

(ii): A conjugacy class $[Q]_{S_2}\in \overline{[P]_{\mathcal{P}}}$ contains $|S_2 : N_{S_2}Q|$ distinct subgroups. For each $Q^s\in [Q]_{S_2}$, $s\in S_2$, we have

$$\begin{aligned}\left|(\mathrm{im}\circ\theta)^{-1}(Q^s)\right| &= \sum_{\psi\in\mathcal{P}(P,Q^s)} \left|\theta^{-1}(\psi)\right| = \sum_{\psi\in\mathcal{P}(P,Q^s)} \Phi_{[P,\psi]}(X) \\ &= \sum_{c_s\psi\in\mathcal{P}(P,Q)} \Phi_{[P,\psi]}(X) = \sum_{c_s\psi\in\mathcal{P}(P,Q)} \Phi_{[P,c_s\psi]}(X) = \sum_{\varphi\in\mathcal{P}(P,Q)} \Phi_{[P,\varphi]}(X),\end{aligned}$$

which is independent of $Q^s\in[Q]_{S_2}$. It follows that

$$\left|(\pi\circ\mathrm{im}\circ\theta)^{-1}([Q]_{S_2})\right| = \frac{|S_2|}{|N_{S_2}Q|}\sum_{\varphi\in\mathcal{P}(P,Q)} \Phi_{[P,\varphi]}(X),$$

and in $S_2\backslash X_0$:

$$\left|(\overline{\mathrm{im}}\circ\overline{\theta})^{-1}([Q]_{S_2})\right| = \frac{\left|(\pi\circ\mathrm{im}\circ\theta)^{-1}([Q]_{S_2})\right|}{|S_2|} = \frac{\sum_{\varphi\in\mathcal{P}(P,Q)}\Phi_{[P,\varphi]}(X)}{|N_{S_2}Q|}.$$

In particular, $|N_{S_2}Q|$ divides $\sum_{\varphi\in\mathcal{P}(P,Q)}\Phi_{[P,\varphi]}(X)$. Furthermore, we again use that $S_2\backslash X_0$ is the disjoint union of the pre-images $(\overline{\mathrm{im}}\circ\overline{\theta})^{-1}([Q]_{S_2})$:

$$\sum_{[Q]_{S_2}\subseteq[P]_{\mathcal{P}}} \frac{\sum_{\varphi\in\mathcal{P}(P,Q)}\Phi_{[P,\varphi]}(X)}{|N_{S_2}Q|} = |S_2\backslash X_0| \equiv \varepsilon(X) \pmod p. \qquad \square$$

Lemma 2.6.13. *Let $X\in A_{fr}(S,S)_{(p)}$ with $\varepsilon(X)\not\equiv 0 \pmod p$, and let $P\le S$. We also put $\mathcal{P} := \mathrm{Pre\text{-}Fix}(X)$. Assume that for all $\varphi,\psi\in\mathcal{P}(P,S)$ we have $\Phi_{[P,\varphi]}(X)=\Phi_{[P,\psi]}(X)$. As in lemma 2.6.12(ii) we let $[P]_{\mathcal{P}}$ be the set of $Q\le S$ where $Q=\varphi P$ for some $\varphi\in\mathcal{P}(P,S)$.*

(i) For all $\varphi\in\mathcal{P}(P,S)$, it holds that $\varphi(P)$ is fully centralized among the $Q\in[P]_{\mathcal{P}}$ if and only if

$$\frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|}\not\equiv 0 \pmod p.$$

(ii) Suppose that $\mathcal{P}$ is levelwise closed. For all $\varphi \in \mathcal{P}(P,S)$ we then have

$$\sum_{\psi \in \mathcal{P}(P,\varphi P)} \Phi_{[P,\psi]}(X) = |\mathrm{Aut}_{\mathcal{P}}(P)| \cdot \Phi_{[P,\varphi]}(X),$$

and $\varphi(P)$ is fully $\mathcal{P}$-normalized is and only if

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(P)| \cdot \Phi_{[P,\varphi]}(X)}{|N_S(\varphi P)|} \not\equiv 0 \pmod p.$$

Proof. By assumption there is a $k \in \mathbb{Z}_{(p)}$ such that $\Phi_{[P,\varphi]}(X) = k$ for all $\varphi \in \mathcal{P}(P,S)$.

(i): From lemma 2.6.12(i) we then have

$$\sum_{\overline{\varphi} \in \mathrm{Rep}_{\mathcal{P}}(P,S)} \frac{k}{|C_S(\varphi P)|} \equiv \varepsilon(X) \not\equiv 0 \pmod p;$$

and it follows that $p \nmid \frac{k}{|C_S(\varphi P)|}$ for some $\varphi \in \mathcal{P}(P,S)$. Since $|C_S(\varphi P)|$ is thus the largest power of p dividing k, we conclude that φP must be fully centralized among the $Q \in [P]_{\mathcal{P}}$; and any $Q' \in [P]_{\mathcal{P}}$ is fully centralized if and only if $p \nmid \frac{k}{|C_S Q'|}$.

(ii): Since $\mathcal{P}$ is level-wise closed (see 2.6.6), $\mathrm{Aut}_{\mathcal{P}}(P)$ acts freely and transitively on $\mathcal{P}(P,\varphi P)$ when $\varphi \in \mathcal{P}(P,S)$, hence $|\mathcal{P}(P,\varphi P)| = |\mathrm{Aut}_{\mathcal{P}}(P)|$ for all $\varphi \in \mathcal{P}(P,S)$. For every $\varphi \in \mathcal{P}(P,S)$ we then get

$$\sum_{\psi \in \mathcal{P}(P,\varphi P)} \Phi_{[P,\psi]}(X) = |\mathcal{P}(P,\varphi P)| \cdot k = |\mathrm{Aut}_{\mathcal{P}}(P)| \cdot k = |\mathrm{Aut}_{\mathcal{P}}(P)| \cdot \Phi_{[P,\varphi]}(X).$$

From lemma 2.6.12(ii) we have

$$\sum_{[Q]_S \subseteq [P]_{\mathcal{P}}} \frac{|\mathrm{Aut}_{\mathcal{P}}(P)| \cdot k}{|N_S Q|} = \sum_{[Q]_S \subseteq [P]_{\mathcal{P}}} \frac{\sum_{\varphi \in \mathcal{P}(P,Q)} \Phi_{[P,\varphi]}(X)}{|N_S Q|} \equiv \varepsilon(X) \not\equiv 0 \pmod p.$$

We conclude that

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(P)| \cdot k}{|N_S Q|} \not\equiv 0 \pmod p,$$

for some $Q \in [P]_{\mathcal{P}}$. Since $|N_S Q|$ is thus the greatest power of p dividing $|\mathrm{Aut}_{\mathcal{P}}(P)| \cdot k$, we conclude that Q must be fully $\mathcal{P}$-normalized; and any $Q' \in [P]_{\mathcal{P}}$ is fully normalized if and only if

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(P)| \cdot k}{|N_S Q'|} \not\equiv 0 \pmod p. \qquad \square$$

Recovering $\mathcal{F}$ from a characteristic element

Lemma 2.6.14. *An element $X \in A_{fr}(S,S)_{(p)}$ with $\varepsilon(X) \not\equiv 0 \pmod p$, is right $\mathcal{F}$-characteristic if and only if*

$$\text{Pre-Orb}(X) \subseteq \mathcal{F} \subseteq \mathrm{RSt}(X).$$

Similar results hold for left and fully characteristic elements.

Proof. The condition Pre-Orb$(X) \subseteq \mathcal{F}$ is equivalent to X being $\mathcal{F}$-generated, and $\mathcal{F} \subseteq \mathrm{RSt}(X)$ is equivalent to X being right $\mathcal{F}$-stable. □

Proposition 2.6.15. *If $X \in A_{fr}(S,S)_{(p)}$ is a right characteristic element for $\mathcal{F}$, then* $\mathrm{RSt}(X) = \mathcal{F}$*. Analogous results hold for left and fully characteristic elements.*

Proof. Assume that $P \leq S$ and $\varphi\colon P \to S$ is not a morphism of $\mathcal{F}(P,S)$. Then $(P, incl)$ and (P,φ) are not $(\mathcal{F}_P,\mathcal{F})$-conjugate (P,S)-pairs. By proposition 2.3.19 it then follows that $X \circ [P, incl]_P^S$ and $X \circ [P,\varphi]_P^S$ are different basis elements of $X \circ A(P,S)_{(p)} \circ [P, id]_P^P$ since X is right $\mathcal{F}$-characteristic, and $[P, id]_P^P$ is left $\mathcal{F}_P$-characteristic. Since $X \circ [P, incl]_P^S$ and $X \circ [P,\varphi]_P^S$ are linearly independent (hence non-equal), we conclude that $\varphi \notin \mathrm{Hom}_{\mathrm{RSt}(X)}(P,S)$.

This proves $\mathrm{RSt}(X) \subseteq \mathcal{F}$, and we already have the other inclusion from lemma 2.6.14. □

Proposition 2.6.16. *If $X \in A_{fr}(S,S)_{(p)}$ is a right or left characteristic element for $\mathcal{F}$, then*

$$\text{Pre-Fix}(X) = \mathrm{Fix}(X) = \mathrm{Orb}(X) = \mathcal{F}.$$

Proof. We already have Pre-Fix$(X) \subseteq \mathrm{Fix}(X) = \mathrm{Orb}(X)$ by lemma 2.6.10, and $\mathrm{Orb}(X) \subseteq \mathcal{F}$ by taking the closure of lemma 2.6.14. We therefore only have to show $\mathcal{F} \subseteq$ Pre-Fix(X), i.e. that $\Phi_{[P,\varphi]}(X) \neq 0$ for all $\varphi \in \mathcal{F}(P,S)$.

Consider the left characteristic case. Since X is then in particular left $\mathcal{F}$-stable, lemma 2.3.8(iii) says that $\Phi_{[P,\varphi]}(X) = \Phi_{[P,\psi]}(X)$ for all $P \leq S$ and $\varphi,\psi \in \mathcal{F}(P,S)$.

Lemma 2.6.12(i) says that

$$\sum_{\overline{\varphi} \in \mathrm{Rep}_{\text{Pre-Fix}(X)}(P,S)} \frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} \equiv \varepsilon(X) \pmod p.$$

Because $\varepsilon(X) \not\equiv 0 \pmod p$ since X is characteristic, we conclude that $\Phi_{[P,\varphi]}(X) \neq 0$ for some $\varphi \in \mathrm{Hom}_{\text{Pre-Fix}(X)}(P,S) \subseteq \mathcal{F}(P,S)$. For all other $\psi \in \mathcal{F}(P,S)$ we then get $\Phi_{[P,\psi]}(X) = \Phi_{[P,\varphi]}(X) \neq 0$, and we therefore have $\mathcal{F}(P,S) \subseteq \mathrm{Hom}_{\text{Pre-Fix}(X)}(P,S)$. This proves $F \subseteq$ Pre-Fix(X) in the left characteristic case.

If X is right characteristic, then X^{op} is left characteristic, so Pre-Fix$(X^{\mathrm{op}}) = \mathcal{F}$. In particular, Pre-Fix$(X^{\mathrm{op}})$ is level-wise closed, hence Pre-Fix$(X) =$ Pre-Fix$(X^{\mathrm{op}}) = \mathcal{F}$ by lemma 2.6.11. □

Remark 2.6.17. Proposition 2.6.16 shows how to recover a fusion system $\mathcal{F}$ from a left/right/fully characteristic element X: The pre-fusion system Pre-Fix(X) defined from X always returns $\mathcal{F}$, not depending on whether X is right or left characteristic. In particular, an element $X \in A_{fr}(S,S)_{(p)}$ cannot be a left/right/fully characteristic element for more than one fusion system.

Remark 2.6.18. It is generally not true that Pre-Orb$(X) = \mathcal{F}$ when X is $\mathcal{F}$-characteristic. The fusion system $\mathcal{F}_S$ has $[S, id]_S^S$ as a characteristic element, and Pre-Orb$([S, id])$ is a pre-fusion system containing only one morphism – the identity on S.

Corollary 2.6.19. *Let $X \in A_{fr}(S,S)_{(p)}$ with $\varepsilon(X) \not\equiv 0 \pmod p$.*

If Pre-Fix$(X) \subseteq$ RSt(X) *or* Pre-Orb$(X) \subseteq$ RSt(X), *then X is a right characteristic element for* RSt(X), *and*

$$\text{Pre-Fix}(X) = \text{Fix}(X) = \text{Orb}(X) = \text{RSt}(X).$$

The similar results hold for left and fully characteristic elements.

Proof. The two conditions are equivalent since taking closure of either gives

$$\text{Fix}(X) = \text{Orb}(X) \subseteq \text{RSt}(X)$$

by lemma 2.6.10.

Assuming Pre-Orb$(X) \subseteq$ RSt(X), we get that X is a right RSt(X)-characteristic element from lemma 2.6.14 since RSt(X) is always a fusion system. The equalities then follow from proposition 2.6.16. □

Characteristic elements imply saturation

Proposition 2.6.20. *Let $X \in A_{fr}(S,S)_{(p)}$ with $\varepsilon(X) \not\equiv 0 \pmod p$, and such that $P :=$ Pre-Fix(X) is level-wise closed.*

If $P \leq S$ satisfies that $\Phi_{[P,\varphi]}(X) = \Phi_{[P,\psi]}(X)$ for all $\varphi, \psi \in \mathcal{P}(P,S)$, then $\mathcal{P}$ is saturated at P.

Proof. Let $\varphi \in \mathcal{P}(P,S)$ with $\varphi(P)$ fully normalized. Lemma 2.6.13(ii) gives

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi P)|}{|\mathrm{Aut}_S(\varphi P)|} \cdot \frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} = \frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi P)| \cdot \Phi_{[P,\varphi]}(X)}{|N_S(\varphi P)|} \not\equiv 0 \pmod p.$$

The fraction $\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi P)|}{|\mathrm{Aut}_S(\varphi P)|}$ is an integer since $\mathrm{Aut}_S(\varphi P)$ is a subgroup of $\mathrm{Aut}_{\mathcal{P}}(\varphi P)$; and $\frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} \in \mathbb{Z}_{(p)}$ since $|C_S(\varphi P)|$ divides $\Phi_{[P,\varphi]}(X)$ by lemma 2.6.12(i). We therefore conclude that

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi P)|}{|\mathrm{Aut}_S(\varphi P)|} \not\equiv 0 \pmod p \quad \text{and} \quad \frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} \not\equiv 0 \pmod p.$$

The first congruence shows that $\mathrm{Aut}_S(\varphi P) \in Syl_p(\mathrm{Aut}_{\mathcal{P}}(\varphi P))$, and the second congruence says that φP is fully centralized by lemma 2.6.13(i).

For the second saturation condition, we assume that $\varphi \in \mathcal{P}(P,S)$ with $\varphi(P)$ fully centralized. We have

$$\Phi_{[P,\varphi]}(X) = \sum_{[Q,\psi] \in C(S,S)} c_{[Q,\psi]}(X) \cdot \Phi_{[P,\varphi]}([Q,\psi]) = \sum_{[Q,\psi] \in C(S,S)} c_{[Q,\psi]}(X) \cdot \frac{|N_{\varphi,\psi}|}{|Q|} \cdot |C_S(\varphi P)|$$

by lemma 2.1.12; and from remark 2.1.13 we know that each $\frac{|N_{\varphi,\psi}|}{|Q|}$ is an integer. Since $\frac{\Phi_{[P,\varphi]}(X)}{|C_S(\varphi P)|} \not\equiv 0 \pmod p$ by lemma 2.6.13(i), there exists a (S,S)-pair (Q,ψ) with $c_{[Q,\psi]}(X) \neq 0$ and $\frac{|N_{\varphi,\psi}|}{|Q|} \not\equiv 0 \pmod p$.

Remark 2.1.13 describes $N_{\varphi,\psi}$ as a bifree (N_φ, N_ψ)-set, so in particular $Q\backslash N_{\varphi,\psi}$ is a right N_φ-set (since $Q \leq N_\psi$). Since N_φ is a p-group, we have

$$\left|(Q\backslash N_{\varphi,\psi})^{N_\varphi}\right| \equiv |Q\backslash N_{\varphi,\psi}| \not\equiv 0 \pmod p;$$

hence there exists a $x \in N_{\varphi,\psi}$ such that the orbit $Qx \in Q\backslash N_{\varphi,\psi}$ is fixed under the action of N_φ by right multiplication. This means that for every $g \in N_\varphi$ there exists $q \in Q$ such that $xg = qx$, i.e. ${}^x(N_\varphi) \leq Q$. By definition of $N_{\varphi,\psi}$ there is a $y \in S$ such that $c_y \circ \varphi = \psi \circ c_x$ as maps $P \to S$; and we now define

$$\widetilde{\varphi}\colon N_\varphi \xrightarrow{c_x} Q \xrightarrow{\psi} \psi Q \xrightarrow{(c_y)^{-1}} S.$$

The homomorphism restrict to P as

$$\widetilde{\varphi}(p) = (c_y)^{-1}(\psi(c_x(p))) = (c_y)^{-1}(c_y(\varphi(p))) = \varphi(p)$$

for $p \in P$, so $\widetilde{\varphi}\colon N_\varphi \to S$ is an extension of φ.

Finally $c_{[Q,\psi]}(X) \neq 0$ implies that $\psi \in \text{Pre-Orb}(Q,S)$ and we therefore have

$$\widetilde{\varphi} \in \text{Orb}(N_\varphi, S) = \text{Fix}(N_\varphi, S) = \overline{\mathcal{P}}(N_\varphi, S). \qquad \square$$

Theorem 2.6.21. *If a fusion system $\mathcal{F}$ on S has a left/right/fully characteristic element, then $\mathcal{F}$ is saturated.*

Proof. Let $X \in A_{fr}(S,S)$ be a left/right/fully characteristic element for $\mathcal{F}$. Either X or X^{op} is then a left characteristic element by lemma 2.3.14. Assume that X is left characteristic.

By proposition 2.6.16, we have $\mathcal{F} = \text{Pre-Fix}(X)$. Since X is left-stable we have $\Phi_{[P,\varphi]}(X) = \Phi_{[P,\psi]}(X)$ for all $P \leq S$ and $\varphi, \psi \in \text{Hom}_{\text{Pre-Fix}(X)}(P,S)$ from lemma 2.3.8. Proposition 2.6.20 then tells us that $\mathcal{F} = \text{Pre-Fix}(X)$ is saturated at all $P \leq S$. $\square$

Observation 2.6.22. Theorem 2.4.11, proposition 2.6.16 and theorem 2.6.21 together show that there is a one-to-one correspondence between saturated fusion systems on a p-group S and the idempotents in $A(S,S)_{(p)}$ that are fully characteristic elements for some fusion system.

In [RS09], Ragnarsson and Stancu show that an idempotent $X \in A(S,S)_{(p)}$ is a characteristic element for a fusion system if and only if $\varepsilon(X) = 1$ and X satisfies a Frobenius reciprocity relation:

$$(X \times X) \circ [S,\Delta]_S^{S\times S} = (X \times 1) \circ [S,\Delta]_S^{S\times S} \circ X \in A(S, S\times S)_{(p)}. \tag{2.5}$$

Consequently there is a one-to-one correspondence between saturated fusion systems on S and idempotents in $A(S,S)_{(p)}$ satisfying (2.5) and having augmentation 1.

Notation

$c_{[K,\varphi]}$ 35
For $X \in A(G,H)$, $c_{[K,\varphi]}(X)$ is the $[K,\varphi]$-coefficient of X when written in the standard basis of $A(G,H)$.
$P \sim_{\mathcal{F}} Q$ P is $\mathcal{F}$-conjugate to Q. 2
$P \sim_S Q$ P is S-conjugate to Q. 2
$(P,\varphi) \underset{(\mathcal{F}_1,\mathcal{F}_2)}{\sim} (Q,\psi)$ 41
The (S_1,S_2)-pairs (P,φ) and (Q,ψ) are $(\mathcal{F}_1,\mathcal{F}_2)$-conjugate.
$(K,\varphi) \underset{(G,H)}{\sim} (L,\psi)$ 34
The (G,H)-pairs (K,φ) and (L,ψ) are (G,H)-conjugate.
$(P,\varphi) \underset{(S_1,S_2)}{\sim} (Q,\psi)$ 34
The (S_1,S_2)-pairs (P,φ) and (Q,ψ) are (S_1,S_2)-conjugate.
$\Delta(K,\varphi)$ 33
The graph of the (G,H)-pair (K,φ), i.e. $\Delta(K,\varphi) := \{(\varphi k, k) \mid k \in K\}$.
$\Delta(P,\varphi)$ 33
The graph of the (S_1,S_2)-pair (P,φ), i.e. $\Delta(P,\varphi) := \{(\varphi p, p) \mid p \in P\}$.
ε The augmentation homomorphism $\varepsilon\colon A(G,H) \to \mathbb{Z}$. 39
$\mathcal{F}$ A fusion system on a p-group S. 1
$\mathcal{F}(P,Q)$ The set of (homo)morphisms in $\mathcal{F}$ from P to Q. 1
$\mathcal{F}_S$ Shorthand for $\mathcal{F}_S(S)$, the minimal fusion system on S. 1
$\mathcal{F}_S(G)$ 1
The (saturated) fusion system on S induced by G-conjugation, where $S \in Syl_p(G)$.
f.n. Short for "fully normalized". 15
$f_{[H]}$ The $[H]$-coordinate of $f \in \prod_{[H]\in C(G)} \mathbb{Z}$. 4
$[H]_G$, $[H]$ 3
The basis element of $A(G)$ corresponding to the conjugacy class $[H] \in C(G)$.
$[H]_G$, $[H]$ The set of G-conjugates of H. 3
$[K,\varphi]_G^H$, $[K,\varphi]$ 33
The basis element of $A(G,H)$ corresponding to the (G,H)-pair (K,φ). The corresponding $H \times G$-set is $(H \times G)/\Delta(K,\varphi)$.
$[K,\varphi]_{(G,H)}$, $[K,\varphi]$ 34
The conjugacy class of the (G,H)-pair (K,φ).
LSt(X) The left stabilizer fusion system of $X \in A(S,S)_{(p)}$ 64
N_φ 2
For $\varphi \in \mathcal{F}(P,S)$, $N_\varphi := \{x \in N_S(P) \mid \exists y \in S\colon \varphi \circ c_x = c_y \circ \varphi\}$.
$\circ$ 37
The composition $\circ\colon A(H,K) \times A(G,H) \to A(G,K)$ given by $Y \circ X := Y \times_H X$ on bisets.
$Obs^{cent}(\mathcal{F})$ 27
The obstruction group characterizing the image of $A^{cent}(\mathcal{F}) \hookrightarrow \widetilde{\Omega}^{cent}(\mathcal{F})$.
$Obs(\mathcal{F})$ 15
The obstruction group characterizing the image of $A(\mathcal{F}) \hookrightarrow \widetilde{\Omega}(\mathcal{F})$.
$Obs(G)$ 5
The obstruction group characterizing the image of $A(G) \hookrightarrow \widetilde{\Omega}(G)$.
$Obs(G;p)_{(p)}$ 7
The obstruction group characterizing the image of $A(G;p)_{(p)} \hookrightarrow \widetilde{\Omega}(G;p)_{(p)}$.

References

[AKO10] M. ASCHBACHER, R. KESSAR AND B. OLIVER: "Fusion Systems in Algebra and Topology". 2010. URL: http://www.maths.abdn.ac.uk/~bensondj/html/archive/aschbacher-kessar-oliver.html.

[BLO03] C. BROTO, R. LEVI AND B. OLIVER: "The homotopy theory of fusion systems". *J. Amer. Math. Soc.*, vol. 16(4), (2003), pp. 779–856 (electronic). ISSN 0894-0347. URL: http://dx.doi.org/10.1090/S0894-0347-03-00434-X.

[DL09] A. DÍAZ AND A. LIBMAN: "The Burnside ring of fusion systems". *Adv. Math.*, vol. 222(6), (2009), pp. 1943–1963. ISSN 0001-8708. URL: http://dx.doi.org/10.1016/j.aim.2009.06.023.

[Gel10] M. GELVIN: *Fusion action systems.* Ph.D. thesis, Massachusetts Institute of Technology. Jun. 2010. URL: http://www.math.ku.dk/~gelvin/Fusion%20Action%20Systems%20(Thesis).pdf.

[Rag06] K. RAGNARSSON: "Classifying spectra of saturated fusion systems". *Algebr. Geom. Topol.*, vol. 6, (2006), pp. 195–252. ISSN 1472-2747. URL: http://dx.doi.org/10.2140/agt.2006.6.195.

[RS09] K. RAGNARSSON AND R. STANCU: "Saturated fusion systems as idempotents in the double Burnside ring". 2009. URL: http://arxiv.org/abs/0911.0085v2.

[Yos90] T. YOSHIDA: "The generalized Burnside ring of a finite group". *Hokkaido Math. J.*, vol. 19(3), (1990), pp. 509–574. ISSN 0385-4035.