MASTER THESIS

Elmendorf's Theorem for Cofibrantly Generated Model Categories

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Abstract

Elmendorf's Theorem in equivariant homotopy theory states that for any topological group G, the model category of G-spaces is Quillen equivalent to the category of continuous diagrams of spaces indexed by the opposite of the orbit category of G with the projective model structure. For discrete G, Bert Guillou explored equivariant homotopy theory for any cofibrantly generated model category $\mathcal{C}$ and proved an analogue of Elmendorf's Theorem assuming that $\mathcal{C}$ has "cellular" fixed point functors. We will generalize Guillou's approach and study equivariant homotopy theory also for topological, cofibrantly generated model categories. Elmendorf's Theorem will be recovered for G a compact Lie group or a discrete group.

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1 Introduction

1.1 A motivation behind Elmendorf's Theorem

Working in the category of compactly generated spaces $\mathcal{U}$, the basics from the homotopy theory of spaces, such as the notion of homotopy, weak homotopy equivalence, CW-complex etc and the Whitehead Theorem, generalize to an equivariant setting in the following natural way. Recall that in equivariant algebraic topology (see e.g. [15]) one considers *G-spaces*, i.e. spaces with a continuous action by some topological group G, and *G-maps*, i.e. maps between G-spaces which commute with the group actions. Denote the category of G-spaces by $\mathcal{U}^G$. The product of two G-spaces X and Y in $\mathcal{U}$ becomes with coordinatewise G-action the product $X \times Y$ in $\mathcal{U}^G$. Now, a *G-homotopy* from a G-space X to a G-space Y is defined as a G-map $X \times I \to Y$, where I denotes the unit interval considered as a G-space with trivial action. Using the notion of a G-homotopy, one defines a *G-homotopy equivalence* in the obvious way. Recall that a CW-complex is constructed by attaching disks D^n along their boundaries S^{n-1}. To build up a *G-CW-complex* one considers for closed subgroups H of G the homogeneous spaces G/H and attaches $G/H \times D^n$ along $G/H \times S^{n-1}$ in $\mathcal{U}^G$. A *weak G-homotopy equivalence* is a G-map $f\colon X \to Y$ such that for any closed subgroup H of G, the map $f^H\colon X^H \to Y^H$ is a weak homotopy equivalence in $\mathcal{U}$, where

$$(-)^H\colon \mathcal{U}^G \to \mathcal{U}$$

denotes the *H-fixed point functor*, sending a G-space X to its H-fixed point set

$$X^H := \{x \in X; hx = x \text{ for all } h \in H\}.$$

Then the G-Whitehead Theorem holds: A weak G-homotopy equivalence between G-CW-complexes is a G-homotopy equivalence. Taking G to be the trivial group $\{e\}$ consisting only of the neutral element, the equivariant homotopy theory of $\{e\}$-spaces is the same as ordinary homotopy theory of spaces.

Let's switch to a category theoretical viewpoint and to another generalization of homotopy theory. Quillen introduced the notion of categories which model a homotopy theory, called *model categories*. Roughly speaking, a model category is a category $\mathcal{C}$ together with three distinguished classes of morphisms of $\mathcal{C}$, called *fibrations*, *cofibrations* and *weak equivalences*, which satisfies certain axioms. For instance the category of spaces $\mathcal{U}$ with the Quillen model structure is a cofibrantly generated model category, where the fibrations are the Serre fibrations and the weak equivalences are the weak homotopy equivalences. The category $\mathcal{U}^G$ of G-spaces is a model category with the fixed point model structure, where the fibrations and weak equivalences are the G-maps which by each fixed point functor are taken to fibrations and weak equivalences in $\mathcal{U}$, respectively. Given a model category

$\mathcal{C}$ and a small category $\mathcal{D}$, there are some natural candidates of model category structures for the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ with weak equivalences the levelwise weak equivalences, i.e. the natural transformations which evaluated in each object of $\mathcal{D}$ are weak equivalences in $\mathcal{C}$. For instance if $\mathcal{C}$ is cofibrantly generated, then $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ admits the projective model structure, where the fibrations and weak equivalences are the levelwise fibrations and levelwise weak equivalences, respectively (see [8, Theorem 11.6.1]). Note that the category of G-spaces can be identified with the category of continuous functors from G to $\mathcal{U}$. Hence, if G is a discrete group, then the category $\mathcal{U}^G$ is just the functor category $\mathrm{Fun}(G, \mathcal{C})$. Recall that in $\mathrm{Fun}(G, \mathcal{C})$ with the fixed point model structure, the weak equivalences are not defined as the levelwise weak equivalences. Thus from the perspective of model category theory, equivariant homotopy theory looks artificial. This issue is resolved by Elmendorf's Theorem.

In [4], Elmendorf considered the *orbit category* $\mathcal{O}_G$, that is the full subcategory of $\mathcal{U}^G$ given by the homogeneous spaces (or orbit spaces) G/H for H a closed subgroup of G. He showed that the homotopy theory of G-spaces is reflected in a homotopy theory of the category $\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}}$ of continuous diagrams of spaces indexed by the opposite of the orbit category. In model category theoretical terms, Elmendorf's Theorem is stated as follows (see e.g. [7, Remark 1.3]).

Theorem 1.1. *For any topological group G, there is a pair of Quillen equivalences*

$$\Theta \colon \mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}} \rightleftarrows \mathcal{U}^G \colon \Phi$$

between $\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}}$ with the projective model structure and $\mathcal{U}^G$ with the fixed point model structure.

In fact, Elmendorf's Theorem also holds if one considers any set $\mathcal{F}$ of closed subgroups of G, which contains the trivial subgroup $\{e\}$, by equipping $\mathcal{U}^G$ with the $\mathcal{F}$-fixed point model structure and replacing the orbit category $\mathcal{O}_G$ with the $\mathcal{F}$-orbit category $\mathcal{O}_{\mathcal{F}}$. That is, $\mathcal{U}^G$ is the model category with fibrations and weak equivalences the G-maps which by the H-fixed point functors for $H \in \mathcal{F}$ are taken to fibrations and weak equivalences in $\mathcal{U}$, respectively, and $\mathcal{O}_{\mathcal{F}}$ is the full subcategory of $\mathcal{U}^G$ given by the orbit spaces G/H for $H \in \mathcal{F}$.

1.2 The main theorem

The goal of this paper is to show the following theorem.

Theorem 1.2. *Let $\mathcal{V}$ be either the category $\mathcal{U}$ of compactly generated spaces or the category **Set** of discrete spaces and correspondingly, let G be either a topological group or a discrete group. Let $\mathcal{F}$ be a set of closed subgroups of G, which contains the trivial subgroup $\{e\}$. Let $\mathcal{C}$ be a $\mathcal{V}$-model category, which* *is cofibrantly generated. Suppose that for any $H \in \mathcal{F}$ the H-fixed point functor $(-)^H \colon \mathcal{C}^G \to \mathcal{C}$ is cellular and that for all $H, K \in \mathcal{F}$ the functor $(G/K)^H \otimes - \colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations. Then there is a pair of Quillen equivalences*

$$\Theta \colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{C}^G \colon \Phi$$

between $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ with the projective model structure and $\mathcal{C}^G$ with the $\mathcal{F}$-fixed point model structure.

In §1.3, we elaborate briefly on the new terms. The theorem relates to existing work as follows.

Suppose that in Theorem 1.2, the set $\mathcal{F}$ contains all closed subgroups of G. If $\mathcal{V}$ is $\mathcal{U}$ and $\mathcal{C}$ is $\mathcal{U}$, then any H-fixed point functor is cellular. If in addition the topological group G is a compact Lie group or if G has the discrete topology, then the condition on the functor $(G/K)^H \otimes -$ holds. Thus, in this case one recovers Elmendorf's Theorem.

The case, where $\mathcal{V}$ is **Set** and again $\mathcal{F}$ contains all closed subgroups of G, has been studied by Bert Guillou in [6]. In this setting the condition on the functor $(G/K)^H \otimes -$ is automatically satisfied since it is given by the coproduct functor $\coprod_{(G/K)^H} - \colon \mathcal{C} \to \mathcal{C}$.

1.3 Outline of this paper

The term $\mathcal{V}$-model category mentioned in Theorem 1.2 will be defined in §2.1. It will allow a unified treatment of topological model categories, i.e. $\mathcal{U}$-model categories, and ordinary model categories which will be the same as **Set**-model categories. Roughly speaking, a $\mathcal{V}$-model category is a model category which as a category is enriched in $\mathcal{V}$, i.e. is a $\mathcal{V}$-category, such that the enrichment is compatible with the model structure of $\mathcal{V}$. By definition, every $\mathcal{V}$-model category $\mathcal{C}$ will come equipped with a tensor functor $\otimes \colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$. This explains the notion of the functor $(G/K)^H \otimes -$ in Theorem 1.2.

To equip the category $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ of $\mathcal{V}$-functors (see B.8) from $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ to $\mathcal{C}$ in Theorem 1.2 with the desired model category structure, a general existence theorem for the projective model structure will be proved in §2.2.

In §3, we define the H-fixed point functors $(-)^H \colon \mathcal{C}^G \to \mathcal{C}$ and study the $\mathcal{F}$-fixed point model structure for the category $\mathcal{C}^G$ of $\mathcal{V}$-functors from G to $\mathcal{C}$ considered in Theorem 1.2. The fibrations and weak equivalences will be the morphisms which by each H-fixed point functor with $H \in \mathcal{F}$ are taken to fibrations and weak equivalences in $\mathcal{C}$, respectively. Under a cellularity assumption on the fixed point functors and the same assumption on the functor $(G/K)^H \otimes -$ as in Theorem 1.2, an existence theorem for the $\mathcal{F}$-model structure will be proved.

In §4, we construct the mentioned pair of Quillen equivalences

$$\Theta \colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{C}^G \colon \Phi$$

and thus prove Theorem 1.2.

As an example, it will be shown in §5, that for any small $\mathcal{U}$-category $\mathcal{D}$ and any topological group G, the category $\mathcal{U}^{\mathcal{D}}$ of $\mathcal{U}$-functors from $\mathcal{D}$ to $\mathcal{U}$ with the projective model structure is a cofibrantly generated $\mathcal{U}$-model category with cellular fixed point functors. Moreover, if G is a compact Lie group or has the discrete topology then Theorem 1.2 applies, since the condition on the functor $(G/K)^H \otimes -$ holds.

To prove each of the mentioned existence theorems of model category structures, we use the generalized transport Theorem D.20 shown in §D.2 of the appendix. It enables one to lift a cofibrantly generated model structure from a category $\mathcal{C}$ to a category $\mathcal{D}$ not only along one left adjoint functor but along a set of left adjoints from $\mathcal{C}$ to $\mathcal{D}$.

The reader is assumed to have some basic knowledge of algebraic topology and category theory such as treated for instance in [19] and Mac Lane's book [14], respectively. Properties of compactly generated spaces are listed in §A. The for this paper necessary theory of categories enriched in $\mathcal{U}$ or **Set** is quickly developed in §B. The basics of model categories and in particular of compactly generated model categories are summarized in §C and §D, respectively.

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2 Topological and discrete model categories

Roughly speaking, a topological model category or $\mathcal{U}$-model category is a category enriched over the category of compactly generated spaces $\mathcal{U}$ together with a model category structure, which is assumed to be compatible with the Quillen model structure of $\mathcal{U}$ (see D.29). A discrete model category or **Set**-model category is just an ordinary model category, but which for a unified treatment is considered as enriched over the full subcategory **Set** of the category of topological spaces **Top** given by the discrete spaces.

In this section, assuming that $\mathcal{V}$ is either the category $\mathcal{U}$ or the category **Set**, we will make the notion of a $\mathcal{V}$-model category precise. Thereafter, we consider the category $\mathcal{C}^{\mathcal{D}}$ (see B.8) of $\mathcal{V}$-functors from some small $\mathcal{V}$-category $\mathcal{D}$ to a $\mathcal{V}$-model category $\mathcal{C}$ and study the projective model category structure, where the weak equivalences and fibrations are defined to be the levelwise weak equivalences and fibrations, respectively. It is well-known, that if $\mathcal{V}$ is the category **Set** and $\mathcal{C}$ is cofibrantly generated, then $\mathcal{C}^{\mathcal{D}}$ admits the projective model category structure (see e.g. [8, Theorem 11.6.1]). This fact will be recoverd from Theorem 2.7, the main result of this section. Moreover, in the case where $\mathcal{V}$ is the category $\mathcal{U}$ it will follow that $\mathcal{C}^{\mathcal{D}}$ admits the projective model category structure provided that $\mathcal{C}$ is cofibrantly generated and for any two objects d, d' of $\mathcal{D}$, the hom-space $\mathcal{D}(d, d')$ (see B.1) is cofibrant in $\mathcal{U}$.

2.1 Definition and elementary properties of $\mathcal{V}$-model categories

Let $\mathcal{V}$ be either the category $\mathcal{U}$ with the Quillen model structure or the category **Set** with the trivial model structure (see C.8), where the cofibrations are the isomorphisms and every map is both a fibration and a weak equivalence.

Recall (from B.1) that a $\mathcal{V}$-category $\mathcal{C}$ is a category, where every hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of two objects A and B of $\mathcal{C}$ is topologized as a space $\mathcal{C}(A, B)$ in $\mathcal{V}$ such that composition is continuous. Note that if the $\mathcal{V}$-category $\mathcal{C}$ is tensored and cotensored, then the tensors induce a $\mathcal{V}$-functor $\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ and the cotensors induce a $\mathcal{V}$-functor $[-,-]\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ such that there are isomorphisms

$$\mathcal{C}(X \otimes A, B) \cong \mathcal{V}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, [X, B]),$$

which are natural in the objects A, B of $\mathcal{C}$ and X of $\mathcal{V}$ (see B.18).

Definition 2.1. A *$\mathcal{V}$-model category* $\mathcal{C}$ is a tensored and cotensored $\mathcal{V}$-category $\mathcal{C}$ together with a model category structure on the underlying category such that the tensor functor $\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ satisfies the following condition: (product-pushout axiom)

For any cofibration $f\colon X \to Y$ in $\mathcal{V}$ and any cofibration $i\colon A \to B$ in $\mathcal{C}$, the induced map from the pushout $Y \otimes A \cup_{X \otimes A} X \otimes B$ to $Y \otimes B$ is a cofibration in $\mathcal{C}$, which is acyclic if either f or i is acyclic.

Example 2.2. The category $\mathcal{U}$ with the Quillen model structure is a $\mathcal{U}$-model category. Indeed, $\mathcal{U}$ is a tensored and cotensored $\mathcal{U}$-category with tensor functor the product functor $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$, since $\mathcal{U}$ is cartesian closed. For a proof that the product-pushout axiom holds, the reader is referred to [9, Proposition 4.2.11].

By the following proposition, the product-pushout axiom is the analogue of Quillen's axiom **SM7** in the definition of a closed simplicial model category (see [16, p. 2.2]).

Proposition 2.3. *Let $\mathcal{C}$ be a tensored and cotensored $\mathcal{V}$-category equipped with a model category structure on the underlying category. Then the following three conditions are equivalent:*

i) The product-pushout axiom for the tensor functor holds.

ii) For any cofibration $f\colon X \to Y$ in $\mathcal{V}$ and any fibration $p\colon C \to D$ in $\mathcal{C}$, the induced map from the cotensor $[Y, C]$ of Y and C to the pullback $[Y, D] \times_{[X,D]} [X, C]$ is a fibration in $\mathcal{C}$, which is acyclic if either f or p is acyclic.

iii) For any cofibration $i\colon A \to B$ in $\mathcal{C}$ and any fibration $p\colon C \to D$ in $\mathcal{C}$, the induced map from the hom-space $\mathcal{C}(B, C)$ to the pullback $\mathcal{C}(B, D) \times_{\mathcal{C}(A,D)} \mathcal{C}(A, C)$ is a fibration in $\mathcal{V}$, which is acyclic if either i or p is acyclic.

Proof. Using the natural isomorphisms from Corollary B.18c), one shows that the three lifting problems

$$\begin{tikzcd} Y \otimes A \cup_{X\otimes A} X \otimes B \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ Y \otimes B \arrow[r] & D \end{tikzcd} , \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & {[Y, C]} \arrow[d] \\ B \arrow[r] & {[Y, D] \times_{[X,D]} [X, C]} \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{C}(B, C) \arrow[d] \\ Y \arrow[r] & \mathcal{C}(B, D) \times_{\mathcal{C}(A,D)} \mathcal{C}(A, C) \end{tikzcd}$$

are equivalent. Thus the claim follows by the characterization of the (acyclic) fibrations and (acyclic) cofibrations via right and left lifting properties (see C.9). □

Example 2.4 (**Set**-model categories)**.** Let $\mathcal{V}$ be the category **Set**. Let $\mathcal{C}$ be a model category. Since by definition any model category is complete and cocomplete, the category $\mathcal{C}$ is a tensored and cotensored **Set**-category (see B.15) with tensor $\coprod_X A$ for an object A of $\mathcal{C}$ and an object X of **Set**. Recall that **Set** is equipped with the trivial model category structure, where the cofibrations are the isomorphisms and every morphism is a weak equivalence and a fibration. In particular, every map in **Set** is also an acyclic fibration. Hence $\mathcal{C}$ satisfies condition iii) of Proposition 2.3 and therefore $\mathcal{C}$ is a **Set**-model category.

Proposition 2.5. *Let $\mathcal{C}$ be a $\mathcal{V}$-model category. Then for any cofibrant object X of $\mathcal{V}$, the functor $X \otimes - \colon \mathcal{C} \to \mathcal{C}$ is a left Quillen functor.*

Proof. The functor $X \otimes -$ is left adjoint to $[X, -]$, where $[-,-]\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ denotes the cotensor functor. We show that $X \otimes -$ preserves cofibrations and acyclic cofibrations, assuming that X is cofibrant, i.e. that the unique morphism from the initial object $\emptyset$ to X is a cofibration. Let $i\colon A \to B$ be a cofibration in $\mathcal{C}$. Since the functors $- \otimes A$ and $- \otimes B$ are left adjoints (see B.18), they preserve initial objects. Thus

$$\begin{tikzcd}
\emptyset \otimes A \arrow[r] \arrow[d] & X \otimes A \arrow[d] \\
\emptyset \otimes B \arrow[r] & X \otimes A
\end{tikzcd}$$

is a pushout diagram and $X \otimes i$ is the induced map $X \otimes A \to X \otimes B$. By the product-pushout axiom, it follows that $X \otimes i$ is a cofibration, which is acyclic if i is acyclic. □

2.2 The projective model structure

Let $\mathcal{V}$ be as in §2.1.

Definition 2.6. Let $\mathcal{D}$ be a small $\mathcal{V}$-category and $\mathcal{C}$ a $\mathcal{V}$-model category. Recall that a morphism η in the category $\mathcal{C}^{\mathcal{D}}$ of $\mathcal{V}$-functors from $\mathcal{D}$ to $\mathcal{C}$ is just an ordinary natural transformation.

a) The morphism η is a *levelwise weak equivalence* if evaluated in any object of $\mathcal{D}$ it is a weak equivalence in $\mathcal{C}$.

b) The morphism η is a *levelwise fibration* if evaluated in any object of $\mathcal{D}$ it is a fibration in $\mathcal{C}$.

c) The category $\mathcal{C}^{\mathcal{D}}$ is said to admit the *projective model structure* if together with the levelwise weak equivalences as weak equivalences and the levelwise fibrations as fibrations it is a model category.

If $\mathcal{V}$ is the category **Set** and $\mathcal{C}$ is a cofibrantly generated model category, then for any small category $\mathcal{D}$, the category $\mathcal{C}^{\mathcal{D}}$ admits the projective model structure by [8, Theorem 11.6.1]. In the proof, denoting by $\mathrm{Ob}(D)$ the subcategory of $\mathcal{D}$ which contains only the identity morphisms, the functor category $\mathcal{C}^{\mathrm{Ob}(D)}$ is equipped with a cofibrantly generated model category structure. Afterwards, this structure is transported to $\mathcal{C}^{\mathcal{D}}$ via a left adjoint functor. In a similar way, an existence theorem of the projective model category structure for categories enriched in a monoidal model category is proven in [17, Theorem 24.4]. Working enriched over $\mathcal{V}$, we will provide an existence theorem, where instead of the smallness assumption of [17, Theorem 24.4], we assume the functor $\mathcal{D}(d,d') \otimes - \colon \mathcal{C} \to \mathcal{C}$ to be a left Quillen functor for any objects d, d' of $\mathcal{D}$. In the proof, we will lift the model category structure of $\mathcal{C}$ directly to $\mathcal{C}^{\mathcal{D}}$ using the generalized transport Theorem D.20.

Theorem 2.7. *Let $\mathcal{C}$ be $\mathcal{V}$-model category which is cofibrantly generated. Let $\mathcal{D}$ be a small $\mathcal{V}$-category. Suppose that for any two objects d, d' of $\mathcal{D}$, the functor $\mathcal{D}(d,d') \otimes - \colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations. Then $\mathcal{C}^{\mathcal{D}}$ admits the projective model structure. Moreover, $\mathcal{C}^{\mathcal{D}}$ is cofibrantly generated and a $\mathcal{V}$-model category.*

Proof. Since $\mathcal{C}$ is cotensored and cocomplete, so is the category $\mathcal{C}^{\mathcal{D}}$ (see B.23). Similarly, $\mathcal{C}^{\mathcal{D}}$ is tensored and complete. Let I and J be sets of generating cofibrations and generating acyclic cofibrations of $\mathcal{C}$, respectively. Recall (from B.20) that for any object d of $\mathcal{D}$ the evaluation functor $\mathrm{ev}_d \colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ has a left adjoint $F^d \colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$, which sends an object A of $\mathcal{C}$ to the composite

$$\mathcal{D} \xrightarrow{\mathcal{D}(d,-)} \mathcal{V} \xrightarrow{-\otimes A} \mathcal{C}.$$

We apply Theorem D.20 with the set of adjunctions

$$\left\{ F^d \colon \mathcal{C} \rightleftarrows \mathcal{C}^{\mathcal{D}} \colon \mathrm{ev}_d \right\}_d .$$

Set $FI := \bigcup_d \{\mathcal{D}(d,-) \otimes f; f \in I\}$ and $FJ := \bigcup_d \{\mathcal{D}(d,-) \otimes f; f \in J\}$. We show that the conditions i)-iv) of Remark D.21 are satisfied. For any object d of $\mathcal{D}$, the functor ev_d preserves colimits, since colimits in $\mathcal{C}^{\mathcal{D}}$ are calculated objectwise. Thus conditions iii) and iv) hold. For condition i), consider a relative FI-cell complex, i.e. the composition $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ of a λ-sequence $X \colon \lambda \to \mathcal{C}^{\mathcal{D}}$ of pushouts of maps in FI, where λ is some non-zero ordinal. Let d be an object of $\mathcal{D}$. Since ev_d preserves colimits, the composite $\mathrm{ev}_d \circ X \colon \lambda \to \mathcal{C}$ is a λ-sequence of pushouts of maps in $\bigcup_{d'} \{\mathcal{D}(d',d) \otimes f; f \in I\}$. Since by assumption $\mathcal{D}(d',d) \otimes -$ preserves cofibrations for any object d' of $\mathcal{D}$, it follows from Lemma D.11 that the composition $\mathrm{ev}_d(X_0) \to \mathrm{colim}_{\beta<\lambda}(\mathrm{ev}_d(X_\beta))$ is a cofibration in $\mathcal{C}$. Thus, using that ev_d preserves colimits, one deduces that ev_d takes the relative FI-cell complex $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ to a cofibration. Condition ii) is shown similarly. Hence, $\mathcal{C}^{\mathcal{D}}$ admits the projective model structure and is cofibrantly generated.

It follows that $\mathcal{C}^{\mathcal{D}}$ is a $\mathcal{V}$-model category since condition ii) of Proposition 2.3 holds. Indeed, let $f \colon X \to Y$ be a cofibration in $\mathcal{V}$ and let $p \colon C \to D$ be a fibration in $\mathcal{C}^{\mathcal{D}}$. For any object d of $\mathcal{D}$ the evaluation functor ev_d takes the diagram

$$\begin{tikzcd}
[Y,C] \arrow[r, "{[f,C]}"] \arrow[d, "{[Y,p]}"'] & [X,C] \arrow[d, "{[X,p]}"] \\
[Y,D] \arrow[r, "{[f,D]}"'] & [X,D]
\end{tikzcd}$$

to the diagram

$$\begin{tikzcd}
[Y,Cd] \arrow[r, "{[f,Cd]}"] \arrow[d, "{[Y,p_d]}"'] & [X,Cd] \arrow[d, "{[X,p_d]}"] \\
[Y,Dd] \arrow[r, "{[f,Dd]}"'] & [X,Dd]
\end{tikzcd}$$

by the definition of the cotensor functor of $\mathcal{C}^{\mathcal{D}}$ (see B.19). Condition ii) of Proposition 2.3 holds in $\mathcal{C}$ by assumption. One concludes that it also holds in $\mathcal{C}^{\mathcal{D}}$ using that for any object d of $\mathcal{D}$ the functor ev_d preserves limits and that the (acyclic) fibrations of $\mathcal{C}^{\mathcal{D}}$ are the levelwise (acyclic) fibrations. □

Corollary 2.8. *Let $\mathcal{C}$ be a $\mathcal{V}$-model category which is cofibrantly generated. Let $\mathcal{D}$ be a small $\mathcal{V}$-category.*

a) If $\mathcal{V}$ is ***Set****, then $\mathcal{C}^{\mathcal{D}}$ with the projective model structure is a cofibrantly generated model category.*

b) If $\mathcal{V}$ is $\mathcal{U}$ and for all objects d, d' of $\mathcal{D}$ the hom-space $\mathcal{D}(d, d')$ is cofibrant in $\mathcal{U}$, then $\mathcal{C}^{\mathcal{D}}$ with the projective model structure is a $\mathcal{U}$-model category, which is cofibrantly generated.

Proof. Part a) holds, since cofibrations and acyclic cofibrations are closed under coproducts.

Part b) follows from Proposition 2.5. □

3 The fixed point model structure

Recall that for any topological group G, the category of G-spaces $\mathcal{U}^G$ admits a model category structure, where the weak equivalences and fibrations are the maps which by each H-fixed point functor $(-)^H$ are taken to weak equivalences and fibrations in $\mathcal{U}$, respectively. In [6], considering a discrete group G, Guillou replaced the category of compactly generated spaces $\mathcal{U}$ by an arbitrary cofibrantly generated model category $\mathcal{C}$ and studied G-objects in $\mathcal{C}$, i.e. functors from G to $\mathcal{C}$. For any subgroup H of G, he defined a functor $(-)^H$ from the category $\mathrm{Fun}(G, C)$ of G-objects in $\mathcal{C}$ to the category $\mathcal{C}$, which again is called H-fixed point functor and which agrees with the ordinary H-fixed point functor if $\mathcal{C}$ is $\mathcal{U}$. Provided that the fixed point functors are what is called cellular in [6], Guillou showed that the category

$\mathrm{Fun}(G, C)$ admits a model category structure, where the weak equivalences and fibrations are the maps which by each H-fixed point functor are taken to weak equivalences and fibrations in $\mathcal{C}$, respectively. If $\mathcal{C}$ is $\mathcal{U}$, then each H-fixed point functor is cellular. Thus for discrete G, the model category structure of G-spaces is recovered.

Here, we will generalize the approach in [6] such that also topological model categories and topological groups can be considered. Assuming that $\mathcal{V}$ is either the category $\mathcal{U}$ or the category **Set**, we introduce the notion of a $\mathcal{V}$-group G and consider a $\mathcal{V}$-model category $\mathcal{C}$. For any closed subgroup H of G, we introduce a $\mathcal{V}$-functor $(-)^H \colon \mathcal{C}^G \to \mathcal{C}$, called H-fixed point functor again. Then we will study the fixed point model structure on the category $\mathcal{C}^G$ of $\mathcal{V}$-functors from G to $\mathcal{C}$, where the weak equivalences and fibrations are the maps which by each H-fixed point functor are taken to weak equivalences and fibrations in $\mathcal{C}$, respectively. It will turn out that the category $\mathcal{C}^G$ admits the fixed point model structure provided that $\mathcal{C}$ is cofibranty generated, has cellular fixed point functors and for all closed subgroups H, K of G, the functor $(G/K)^H \otimes - \colon \mathcal{C} \to \mathcal{C}$ is a left Quillen functor. If $\mathcal{V}$ is **Set**, then Guillou's development is recovered. If $\mathcal{V}$ is $\mathcal{U}$ and the $\mathcal{V}$-model category $\mathcal{C}$ is $\mathcal{U}$, then one recovers the model category structure of G-spaces also for a compact Lie group G.

Besides the described fixed point model structure on $\mathcal{C}^G$, one might be interested in model category structures where weak equivalences and fibrations are maps, which not by all but by some of the fixed point functors are taken to weak equivalences and fibrations, respectively. Correspondingly, we will in fact study $\mathcal{F}$-fixed point model structures, where $\mathcal{F}$ can be any set of closed subgroups of G containing the trivial subgroup $\{e\}$.

Let $\mathcal{V}$ be as in §2.1.

Definition 3.1. A $\mathcal{V}$-category G with one object $*$ and where each morphism g of G is an isomorphism, is called a *$\mathcal{V}$-group* if the map $G \to G$, $g \mapsto g^{-1}$, sending a group element to its inverse, is continuous.

Example 3.2. If $\mathcal{V}$ is **Set**, then a $\mathcal{V}$-group is the same as a discrete group. If $\mathcal{V}$ is $\mathcal{U}$, then a $\mathcal{V}$-group is the same as a topological group. Note that since $\mathcal{U}$ contains the discrete spaces, every discrete group can also be regarded as a $\mathcal{U}$-group.

Definition 3.3. Let G be a $\mathcal{V}$-group and let $\mathcal{C}$ be a $\mathcal{V}$-category.

a) A *G-object in $\mathcal{C}$* is a $\mathcal{V}$-functor from G to $\mathcal{C}$, i.e. an object of $\mathcal{C}^G$.

b) Let H be a closed subgroup of G. The G-object X in $\mathcal{V}$, where $X(*)$ is the set G/H of cosets gH for $g \in G$ topologized as a quotient space of G and where the map $X(g) \colon G/H \to G/H$ in $\mathcal{V}$ is defined by $g'H \mapsto gg'H$ for $g \in G$, is called an *orbit space* and denoted by G/H.

Definition 3.4. Let G be a $\mathcal{V}$-group and let $\mathcal{C}$ be a $\mathcal{V}$-model category. Let H be a closed subgroup of G. Thus H is again a $\mathcal{V}$-group. The *H-fixed point functor* $(-)^H : \mathcal{C}^G \to \mathcal{C}$ is the composite $\mathcal{V}$-functor

$$\mathcal{C}^G \longrightarrow \mathcal{C}^H \xrightarrow{\lim} \mathcal{C},$$

where $\mathcal{C}^G \to \mathcal{C}^H$ is given by restriction and lim is induced by the limit functor $\lim : \mathrm{Fun}(H, \mathcal{C}) \to \mathcal{C}$ (see B.22).

Example 3.5. Let H be a closed subgroup of a $\mathcal{V}$-group G. Let $\mathcal{C}$ be a $\mathcal{V}$-model category.

a) If H is the trivial subgroup $\{e\}$ consisting of the neutral element, then $(-)^H : \mathcal{C}^G \to \mathcal{C}$ is the functor which evaluates in the object $*$ of G.

b) If $\mathcal{C}$ is the $\mathcal{V}$-model category $\mathcal{V}$, then $(-)^H : \mathcal{V}^G \to \mathcal{V}$ takes a G-object X in $\mathcal{V}$ to its fixed point set $X^H = \{x \in X(*); X(h)x = x \text{ for all } h \in H\}$, which is a closed subset of $X(*)$.

If $\mathcal{C}$ is the $\mathcal{V}$-model category $\mathcal{V}$, then the fixed point functors are "$\mathcal{V}$-representable".

Lemma 3.6. *Let G be a $\mathcal{V}$-group and let H be a closed subgroup of G. Then there is an isomorphism*

$$\mathcal{V}^G(G/H, X) \cong X^H$$

in $\mathcal{V}$, which is natural in the G-object X in $\mathcal{V}$.

Proof. Let $f : G/H \to X$ be a map in $\mathcal{V}^G$. The morphism f evaluated in the object $*$ of G takes the coset H to a point $f_*(H)$ in X^H. Sending f to $f_*(H)$ defines the desired isomorphism. □

Remark 3.7. Let H be a closed subgroup of a $\mathcal{V}$-group G and let $\mathcal{C}$ be a $\mathcal{V}$-model category. In this context, we can consider two H-fixed point functors $(-)^H$, one from $\mathcal{C}^G$ to $\mathcal{C}$ and one from $\mathcal{V}^G$ to $\mathcal{V}$. We won't distinguish between them semantically.

To establish the fixed point model structure, we will apply the transport Theorem D.20 with the fixed point functors as right adjoints. Using the orbit spaces, one constructs the left adjoints.

Proposition 3.8. *Let G be a $\mathcal{V}$-group and let $\mathcal{C}$ be a $\mathcal{V}$-model category. Consider a closed subgroup H of G. For any object A of $\mathcal{C}$ denote by $G/H \otimes A$ the composite $\mathcal{V}$-functor*

$$G \xrightarrow{G/H} \mathcal{V} \xrightarrow{-\otimes A} \mathcal{C},$$

where G/H is an orbit space and $\otimes : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ is the tensor functor. Then $G/H \otimes - : \mathcal{C} \to \mathcal{C}^G$ defines the $\mathcal{V}$-left adjoint of the H-fixed point functor $(-)^H$.

Proof. We have to find an isomorphism

$$\mathcal{C}^G(G/H \otimes A, X) \cong \mathcal{C}(A, X^H)$$

in $\mathcal{V}$, which is natural in the object A of $\mathcal{C}$ and in the G-object X in $\mathcal{C}$. It is given by the composite

$$\begin{aligned}\mathcal{C}^G(G/H \otimes A, X) &\cong \mathcal{V}^G(G/H, \mathcal{C}(A, X))\\ &\cong \mathcal{C}(A, X)^H\\ &\cong \mathcal{C}(A, X^H),\end{aligned}$$

where the first isomorphism is induced by the isomorphism which expresses that $\mathcal{C}$ is tensored, the second isomorphism is given by Lemma 3.6 and the third one comes from the fact that $\mathcal{C}(A, -)$ preserves limits (see B.21). □

Definition 3.9. Let H be a closed subgroup of a $\mathcal{V}$-group G and let $\mathcal{C}$ be a $\mathcal{V}$-model category. The H-fixed point functor $(-)^H\colon \mathcal{C}^G \to \mathcal{C}$ is called *cellular* if

i) it preserves directed colimits of diagrams where each arrow evaluated in the object $*$ of G is a cofibration in $\mathcal{C}$,

ii) it preserves pushouts of diagrams, where one leg is given by

$$G/K \otimes f\colon G/K \otimes A \to G/K \otimes B$$

for some closed subgroup K of G and a cofibration $f\colon A \to B$ in $\mathcal{C}$, and

iii) for any closed subgroup K of G and any object A of $\mathcal{C}$ the induced map $(G/K)^H \otimes A \to (G/K \otimes A)^H$ is an isomorphism in $\mathcal{C}$.

Example 3.10. Let $\mathcal{V}$ be the category $\mathcal{U}$ and $\mathcal{C}$ the $\mathcal{V}$-model category $\mathcal{U}$. Let H be a closed subgroup of a $\mathcal{U}$-group G. Then the H-fixed point functor $(-)^H\colon \mathcal{U}^G \to \mathcal{U}$ is cellular. Indeed, recall that any cofibration in $\mathcal{U}$ is also a closed embedding (see D.26). The cellularity condition i) holds, since $(-)^H$ preserves directed colimits of diagrams where each arrow is injective as a map in $\mathcal{U}$. Condition ii) holds, since $(-)^H$ preserves pushouts of diagrams where one leg is a closed embedding as a map in $\mathcal{U}$. The map considered in condition iii) is given by

$$(G/K)^H \times A \to (G/K \times A)^H, \quad (gK, x) \mapsto (gK, x),$$

which is an isomorphism in $\mathcal{U}$.

Definition 3.11. Let $\mathcal{C}$ be a $\mathcal{V}$-model category and let G be a $\mathcal{V}$-group. Consider a set $\mathcal{F}$ of closed subgroups of G, which contains the trivial subgroup $\{e\}$. The category $\mathcal{C}^G$ is said to admit the *$\mathcal{F}$-fixed point model structure*, if it is a model category with weak equivalences and fibrations the maps which by each H-fixed point functor with $H \in \mathcal{F}$ are taken to weak equivalences and fibrations in $\mathcal{C}$, respectively.

Remark 3.12. In equivariant homotopy theory of spaces, one usually considers families of closed subgroups, which are assumed to be closed under conjugation and under taking subgroups. Considering $\mathcal{F}$-fixed point model structures, we don't make any of these two assumptions on $\mathcal{F}$. Nevertheless, note that for any $\mathcal{V}$-group G and $\mathcal{V}$-model category $\mathcal{C}$, an H-fixed point functor takes a map f of $\mathcal{C}^G$ to a weak equivalence (resp. fibration) in $\mathcal{C}$ if and only if for any conjugate subgroup H' of H, the H'-fixed point functor takes f to a weak equivalence (resp. fibration). Indeed, the functors $(-)^H$ and $(-)^{H'}$ are naturally isomorphic. If $H' = gHg^{-1}$, then the composite

$$X^H \longrightarrow X(*) \xrightarrow{X(g)} X(*)$$

induces an isomorphism $X^H \to X^{H'}$, which is natural in the G-object X in $\mathcal{C}$.

Theorem 3.13. *Let $\mathcal{C}$ be a $\mathcal{V}$-model category, which is cofibrantly generated. Let $\mathcal{F}$ be a set of closed subgroups of a $\mathcal{V}$-group G, which contains the trivial subgroup $\{e\}$. Suppose that for any $H \in \mathcal{F}$ the H-fixed point functor is cellular and that for all $H, K \in \mathcal{F}$ the functor $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations. Then $\mathcal{C}^G$ admits the $\mathcal{F}$-fixed point model structure. Moreover, $\mathcal{C}^G$ is cofibrantly generated and a $\mathcal{V}$-model category.*

Proof. To show that $\mathcal{C}^G$ with the $\mathcal{F}$-fixed point model structure is a cofibrantly generated model category, we apply the transport Theorem D.20 with the set of adjunctions

$$\left\{G/H \otimes -\colon \mathcal{C} \rightleftarrows \mathcal{C}^G\colon (-)^H\right\}_{H\in\mathcal{F}}.$$

Since $\mathcal{C}$ is cotensored and cocomplete, so is the category $\mathcal{C}^G$ (see B.23). Similarly, $\mathcal{C}^G$ is tensored and complete. Let I and J be sets of generating cofibrations and of generating acyclic cofibrations of $\mathcal{C}$, respectively. Set $FI := \bigcup_{H\in\mathcal{F}} \{G/H \otimes f; f \in I\}$ and $FJ := \bigcup_{H\in\mathcal{F}} \{G/H \otimes f; f \in J\}$. We show that the conditions i)-iv) of Remark D.21 are satisfied.

Considering condition i), let λ be a non-zero ordinal and $X\colon \lambda \to \mathcal{C}^G$ a λ-sequence of pushouts of maps in FI. We have to show that for any $H \in \mathcal{F}$, the fixed point functor $(-)^H$ takes the relative FI-cell complex

$X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ to a cofibration in $\mathcal{C}$. For any successor ordinal $\beta+1 < \lambda$, there is by construction a pushout square

$$\begin{tikzcd} G/K \otimes A \arrow[r] \arrow[d, "G/K\otimes f"'] & X_\beta \arrow[d] \\ G/K \otimes B \arrow[r] & X_{\beta+1} \end{tikzcd}$$

in $\mathcal{C}^G$ for some $K \in \mathcal{F}$ and some map $f\colon A \to B$ in $\mathcal{C}$. By the cellularity properties ii) and iii) of $(-)^H$, one obtains a pushout square

$$\begin{tikzcd} (G/K)^H \otimes A \arrow[r] \arrow[d, "(G/K)^H\otimes f"'] & (X_\beta)^H \arrow[d] \\ (G/K)^H \otimes B \arrow[r] & (X_{\beta+1})^H \end{tikzcd} .$$

Note that the functor $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations by assumption. Thus $(X_\beta)^H \to (X_{\beta+1})^H$ is a cofibration in $\mathcal{C}$. By Lemma D.11, it's enough to show that $(-)^H \circ X\colon \lambda \to \mathcal{C}$ is a λ-sequence with colimit $\operatorname{colim}_{\beta<\lambda}(X_\beta)^H$. But this holds, since we can apply the cellularity property i) of $(-)^H$. Indeed, the morphism $X_\beta \to X_{\beta+1}$ evaluated in the object $*$ of G is a cofibration in $\mathcal{C}$, since it is given by $(X_\beta)^{\{e\}} \to (X_{\beta+1})^{\{e\}}$ and since we have already shown that $(X_\beta)^{H'} \to (X_{\beta+1})^{H'}$ is a cofibration for any $H' \in \mathcal{F}$. Condition ii) is shown similarly.

Concerning condition iii), note that evaluating a relative FI-cell complex in the object $*$ of G, i.e. applying the $\{e\}$-fixed point functor, yields a cofibration in $\mathcal{C}$ as already shown. Thus iii) follows from the cellularity property i). Condition iv) is shown similarly, using that an acyclic cofibration is in particular a cofibration.

To show that $\mathcal{C}^G$ is a $\mathcal{V}$-model category, one checks analogously to the proof of Theorem 2.7 that condition ii) of Proposition 2.3 holds. □

Remark 3.14. Let G be a $\mathcal{V}$-group and H, K closed subgroups of G. Consider a $\mathcal{V}$-model category $\mathcal{C}$.

a) If $\mathcal{V}$ is the category **Set**, then $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations, since it is given by

$$\coprod_{(G/K)^H} (-)\colon \mathcal{C} \to \mathcal{C}.$$

b) If $\mathcal{V}$ is $\mathcal{U}$ and G is a compact Lie group, then $(G/K)^H$ is cofibrant in $\mathcal{U}$ and thus $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations by Proposition 2.5. Indeed, the homogeneous space G/K has a unique smooth structure such that the projection $G \to G/K$ is smooth (see e.g. [11, Theorem 3.37]). With this smooth structure, the action $G \times G/K \to G/K$ is smooth. Now, since H as a closed subgroup of G is a compact Lie group, the H-fixed point set $(G/K)^H$ is a smooth manifold (see e.g. [11, Theorem 4.14]). Therefore, $(G/K)^H$ is triangulable (see e.g. [20, p. 124]). In particular, $(G/K)^H$ has the structure of a CW-complex and hence it is cofibrant in $\mathcal{U}$.

c) Suppose $\mathcal{V}$ is $\mathcal{U}$ and $\mathcal{C}$ is $\mathcal{U}$. If G has the discrete topology, then $(G/K)^H \otimes - \colon \mathcal{U} \to \mathcal{U}$ preserves cofibrations and acyclic cofibrations, since a map f in $\mathcal{U}$ is mapped to $(G/K)^H \times f$ which is the coproduct $\coprod_{(G/K)^H} f$.

Corollary 3.15. *Let $\mathcal{V}$ be the category $\mathcal{U}$ and $\mathcal{C}$ the $\mathcal{V}$-model category $\mathcal{U}$. Let G be a compact Lie group and $\mathcal{F}$ a set of closed subgroups of G, which contains the trivial subgroup $\{e\}$. Then $\mathcal{U}^G$ with the $\mathcal{F}$-fixed point model structure is a cofibrantly generated $\mathcal{U}$-model category.*

Proof. The assumptions of Theorem 3.13 are satisfied by Example 3.10 and part b) of the remark above. □

4 Elmendorf's Theorem

Recall that Elmendorf's Theorem states that for any topological group G, there is a right Quillen equivalence from the category of G-spaces $\mathcal{U}^G$ with the fixed point model structure to the category $\mathcal{U}^{\mathcal{O}_G^{\mathrm{op}}}$ of continuous contravariant functors from the orbit category $\mathcal{O}_G$ to $\mathcal{U}$ with the projective model structure. In [6], considering a discrete group G and G-objects in a cofibrantly generated, discrete model category $\mathcal{C}$, Guillou showed that Elmendorf's Theorem also holds if $\mathcal{U}$ is replaced by $\mathcal{C}$ provided that $\mathcal{C}$ has cellular fixed point functors. In particular, Elmendorf's Theorem for discrete G is recovered by taking $\mathcal{C}$ to be the category of compactly generated spaces $\mathcal{U}$.

Here working enriched over $\mathcal{V}$ for $\mathcal{V} = \mathcal{U}$ or $\mathcal{V} = \mathbf{Set}$, we will generalize Guillou's result such that also topological model categories and topological groups can be considered. Elmendorf's Theorem will be recovered also for a compact Lie group G.

In fact, as in the previous section §3, we consider $\mathcal{F}$-fixed point model structures for a set of closed subgroups of a given $\mathcal{V}$-group G, which contains the trivial subgroup $\{e\}$. Accordingly, the notion of the $\mathcal{F}$-orbit category $\mathcal{O}_{\mathcal{F}}$ will be introduced before stating and proving the main Theorem 4.2 of this section.

Let $\mathcal{V}$ be as in §2.1.

Definition 4.1. Let G be a $\mathcal{V}$-group and $\mathcal{F}$ a set of closed subgroups of G, which contains the trivial subgroup $\{e\}$. The *$\mathcal{F}$-orbit category $\mathcal{O}_{\mathcal{F}}$* is the

$\mathcal{V}$-category defined by the full subcategory of $\mathcal{V}^G$ given by the orbit spaces G/H (see 3.3) for $H \in \mathcal{F}$.

Theorem 4.2. *Let $\mathcal{C}$ be a $\mathcal{V}$-model category, which is cofibrantly generated. Let $\mathcal{F}$ be a set of closed subgroups of a $\mathcal{V}$-group G, which contains the trivial subgroup $\{e\}$. Suppose that for any $H \in \mathcal{F}$ the H-fixed point functor is cellular and that for all $H, K \in \mathcal{F}$ the functor $(G/K)^H \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations. Then there is a pair of Quillen equivalences*

$$\Theta\colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{C}^G \colon \Phi$$

between $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ with the projective model structure and $\mathcal{C}^G$ with the $\mathcal{F}$-fixed point model structure. Moreover, the pair (Θ, Φ) induces a $\mathcal{V}$-adjunction and Φ is full and faithful.

Proof. The proof is structured as follows. First, we show that the category $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ of $\mathcal{V}$-functors from the opposite of the $\mathcal{F}$-orbit category to $\mathcal{C}$ admits the projective model structure and that the category $\mathcal{C}^G$ of G-objects in $\mathcal{C}$ admits the $\mathcal{F}$-fixed point model structure. Secondly, $\mathcal{V}$-functors $\Phi\colon \mathcal{C}^G \to \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and $\Theta\colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \to \mathcal{C}^G$ will be defined. Thirdly, we show that the underlying pair (Θ, Φ) is an adjunction by describing the counit $\varepsilon\colon \Theta\Phi \to \mathrm{id}$ and the unit $\eta\colon \mathrm{id} \to \Phi\Theta$. Thus the pair of $\mathcal{V}$-functors (Θ, Φ) is a $\mathcal{V}$-adjunction by Proposition B.11. The counit ε turns out to be a natural isomorphism. Therefore, the right adjoint Φ will be full and faithful. In the fourth step, using that for any $H \in \mathcal{F}$ the H-fixed point functor is cellular, we will show that the unit η is an isomorphism in cofibrant objects of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$. From this we will finally deduce that (Θ, Φ) is a pair of Quillen equivalences.

Central for the mentioned steps of the proof is the following characterization of the morphisms in the opposite of the $\mathcal{F}$-orbit category. For any $H \in \mathcal{F}$, there is by Lemma 3.6 an isomorphism

$$\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K, G/H) \cong (G/K)^H, \tag{4.1}$$

which is natural in the object G/K of $\mathcal{O}_{\mathcal{F}}$. Given $a \in G$ such that the coset aK is in $(G/K)^H$, i.e. such that $a^{-1}Ha \subset K$, we denote by

$$R_a\colon G/K \to G/H \tag{4.2}$$

the corresponding morphism of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$.

To show that $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ admits the projective model structure, apply Theorem 2.7. The condition that for any objects $G/K, G/H$ of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, the functor $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K, G/H) \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations, is satisfied by the isomorphism (4.1) above and the assumption on the functor $(G/K)^H \otimes -$. We have already shown that $\mathcal{C}^G$ admits the $\mathcal{F}$-fixed point model structure in Theorem 3.13.

Let's define $\Phi\colon \mathcal{C}^G \to \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$. Given a G-object X in $\mathcal{C}$, determine the $\mathcal{V}$-functor $X_*\colon \mathcal{O}_{\mathcal{F}}^{\mathrm{op}} \to \mathcal{C}$ as follows. It sends an object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ to

X^H and the morphism $R_a\colon G/K \to G/H$ of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ (see (4.2)) to the map $X_*(R_a)\colon X^K \to X^H$ in $\mathcal{C}$ induced by the the composite

$$X^K \longrightarrow X(*) \xrightarrow{X(a)} X(*).$$

Let

$$\Phi\colon \mathcal{C}^G \to \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$$

be the $\mathcal{V}$-functor which sends a G-object X in $\mathcal{C}$ to X_* and a morphism f in $\mathcal{C}^G$ to the natural transformation $\Phi(f)$ given in the object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ by $\Phi(f)_{G/H} = f^H$.

To define a $\mathcal{V}$-functor

$$\Theta\colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \to \mathcal{C}^G,$$

recall that $\mathcal{F}$ contains the trivial subgroup $\{e\}$. Evaluating an object T of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ in $G/\{e\}$ gives an object $T(G/\{e\})$ in $\mathcal{C}$. For any $g \in G$, there is a morphism $R_g\colon G/\{e\} \to G/\{e\}$ in $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ (see (4.2)). Define $\Theta(T)$ to be the G-object in $\mathcal{C}$ with $\Theta(T)(*) = T(G/\{e\})$ and action $\Theta(T)(g) = T(R_g)$. On morphisms, define Θ by evaluation in $G/\{e\}$, i.e. a map f in $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ is sent to $\Theta(f)$ in $\mathcal{C}^G$ given by $\Theta(f)_* = f_{G/\{e\}}$ in the object $*$ of G.

We construct a natural isomorphism

$$\varepsilon\colon \Theta\Phi \to \mathrm{id}_{\mathcal{C}^G}.$$

Given a G-object X in $\mathcal{C}$, the G-object $\Theta\Phi(X)$ evaluated in the object $*$ of G is $X^{\{e\}}$ by definition of Φ and Θ. Let $\varepsilon_X\colon \Theta\Phi(X) \to X$ in the object $*$ of G be defined as the natural map $X^{\{e\}} \to X(*)$. One checks, that this defines a natural isomorphism ε.

Let's define a natural transformation

$$\eta\colon \mathrm{id}_{\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}} \to \Phi\Theta.$$

Let T be an object of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$. For any object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, we have to define a map $(\eta_T)_{G/H}$ from $T(G/H)$ to $(\Phi\Theta(T))(G/H) = \Theta(T)^H$ in $\mathcal{C}$, which is natural in the object G/H of $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$. Applying T to the map $R_e\colon G/H \to G/\{e\}$ in $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$, gives a map $T(G/H) \to \Theta(T)(*)$. It induces a map $T(G/H) \to \Theta(T)^H$, which defines the natural map $(\eta_T)_{G/H}$. One checks that $(\eta_T)_T$ is a natural transformation η.

By construction, one verifies that ε and η satisfy the necessary equations to determine the adjunction (Θ, Φ) as counit and unit, respectively. Thus, (Θ, Φ) is a $\mathcal{V}$-adjunction by Proposition B.11 and Φ is full and faithful since the counit ε is a natural isomorphism.

The next step is to show that in any cofibrant object T of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$, the unit $\eta_T\colon T \to \Phi\Theta(T)$ is an isomorphism. Denoting by I the set of generating cofibrations of $\mathcal{C}$, recall that $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ is cofibrantly generated with generating cofibrations $FI := \bigcup_{G/K} \left\{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K, -) \otimes f; f \in I\right\}$. Since any cofibrant object of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ is a retract of a FI-cell complex (see D.17), we can assume that T is a FI-cell complex. Thus, there is an ordinal $\lambda > 0$ and a λ-sequence $S\colon \lambda \to \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ such that $T = \operatorname{colim}_{\beta<\lambda} S_\beta$ and such that S_0 is the initial object $\emptyset$ and for any $\beta + 1 < \lambda$ there is a pushout square

$$\begin{tikzcd} \mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-) \otimes A \arrow[r] \arrow[d, "\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-)\otimes f"'] & S_\beta \arrow[d] \\ \mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-) \otimes B \arrow[r] & S_{\beta+1} \end{tikzcd} \tag{4.3}$$

for some object G/K in $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}$ and some map $f\colon A \to B$ in I. Set $S_\lambda = T$. We will deduce by transfinite induction that η_{S_β} is an isomorphism for $\beta \leq \lambda$. To do so, we will show that the composite $\Phi\Theta$ preserves certain colimits using that as a left adjoint, the functor Θ preserves all colimits, and using the cellularity properties of the fixed point functors. When applying the cellularity properties of the H-fixed point functors for $H \in \mathcal{F}$ later on, we will silently use the following observation. Note that for any $K \in \mathcal{F}$ and morphism f of $\mathcal{C}$, the map $\Theta(\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-) \otimes f)$ is isomorphic to $G/K \otimes f$ in the category $\mathrm{Mor}(\mathcal{C}^G)$ of morphisms of $\mathcal{C}^G$. This follows, since the isomorphism in (4.1) for $H = \{e\}$ induces for any object C of $\mathcal{C}$ an isomorphism

$$\Theta(\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-) \otimes C) \cong G/K \otimes C$$

in $\mathcal{C}^G$. We are ready for the transfinite induction. For the ordinal 0, the morphism η_{S_0} is an isomorphism since Φ preserves the initial object. Consider a successor ordinal $\beta + 1 < \lambda$ and assume that η_{S_α} is an isomorphism for all $\alpha < \beta + 1$. Note that for any object C of $\mathcal{C}$, the unit η is an isomorphism in the object $\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}(G/K,-) \otimes C$ of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ by the cellularity property iii) of the fixed point functors. It follows that $\eta_{S_{\beta+1}}$ is an isomorphism since $\Phi\Theta$ preserves the pushout square as in (4.3) by the cellularity condition ii) of the fixed point functors. For the case of a non-zero limit ordinal one applies the cellularity property i), whose assumption holds by the cellularity conditions ii) and iii) for $H = \{e\}$ and since for any $K \in \mathcal{F}$, the functor $(G/K)^{\{e\}} \otimes -$ preserves cofibrations. Thus we have shown that $\eta_T = \eta_{S_\lambda}$ is an isomorphism.

Now, it's straightforward to show that (Θ, Φ) is a pair of Quillen equivalences. By construction of Φ and by definition of the model category structures on $\mathcal{C}^G$ and $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$, it follows that Φ preserves fibrations and acyclic fibrations. It's left to show that for any cofibrant object T of $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ and any fibrant G-object X in $\mathcal{C}$, a morphism $f\colon \Theta(T) \to X$ is a weak equivalence in $\mathcal{C}^G$ if and only if it's adjoint $\Phi(f)\eta_T\colon T \to \Phi(X)$ is a weak equivalence in $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$. Since T is cofibrant, we know that η_T is an isomorphism. Hence, noting that f is a weak equivalence if and only if $\Phi(f)$ is a weak equivalence, one concludes the proof.

$\square$

Corollary 4.3. *Let $\mathcal{V}$ be the category $\mathcal{U}$ and $\mathcal{C}$ the $\mathcal{V}$-model category $\mathcal{U}$. Let G be a compact Lie group or a discrete group and $\mathcal{F}$ a set of closed subgroups of G, which contains the trivial subgroup $\{e\}$. Then there is a pair of Quillen equivalences*

$$\Theta\colon \mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{U}^G \colon \Phi$$

between $\mathcal{U}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ with the projective model structure and $\mathcal{U}^G$ with the $\mathcal{F}$-fixed point model structure.

Proof. The assumptions of Theorem 4.2 are satisfied by Example 3.10 and Remark 3.14b) and c). □

5 Example: Diagrams of spaces

In this section we work enriched over $\mathcal{V} = \mathcal{U}$. In Example 3.10, we saw that the $\mathcal{U}$-model category $\mathcal{U}$ has cellular fixed point functors. Here, we consider the category $\mathcal{U}^{\mathcal{D}}$ of $\mathcal{U}$-functors from a small $\mathcal{U}$-category $\mathcal{D}$ to $\mathcal{U}$. We show that $\mathcal{U}^{\mathcal{D}}$ is a cofibrantly generated $\mathcal{U}$-model category with the projective model structure. Thereafter, we deduce that $\mathcal{U}^{\mathcal{D}}$ has cellular fixed point functors. Thus, we will be able to apply Theorem 4.2 to $\mathcal{C} = \mathcal{U}^{\mathcal{D}}$ provided that the $\mathcal{U}$-group G is a compact Lie group or has the discrete topology.

Theorem 5.1. *Consider the $\mathcal{U}$-model category $\mathcal{U}$ with the Quillen model structure. For any small $\mathcal{U}$-category $\mathcal{D}$, the category $\mathcal{U}^{\mathcal{D}}$ admits the projective model structure. Moreover, $\mathcal{U}^{\mathcal{D}}$ is cofibrantly generated and a $\mathcal{U}$-model category.*

Note that we don't assume the tensor functor $\mathcal{D}(d, d') \times - \colon \mathcal{U} \to \mathcal{U}$ to preserve cofibrations and acyclic cofibrations for objects d, d' of $\mathcal{D}$. Thus, Theorem 2.7 can not be applied. Nevertheless, the proof will be similar to the one of Theorem 2.7. Again, we will use the transport Theorem D.20. But to verify the assumptions, we will make use of the properties of **Top** described in Lemma D.26 and silently, of the fact that certain colimits in $\mathcal{U}$ can be calculated in **Top** (see A.7).

Proof of Theorem 5.1. Since $\mathcal{U}$ is cotensored, cocomplete, tensored and complete, so is the category $\mathcal{U}^{\mathcal{D}}$. Recall that $\mathcal{U}$ is cofibrantly generated with generating cofibrations I the inclusions of spheres into disks and generating acyclic cofibrations J the inclusions of disks into cylinders. We apply Theorem D.20 with the adjunctions

$$\{F^d \colon \mathcal{U} \rightleftarrows \mathcal{U}^{\mathcal{D}} \colon \mathrm{ev}_d\}_d,$$

described in Proposition B.20. Set $FI := \bigcup_d \{\mathcal{D}(d, -) \times f; f \in I\}$ and $FJ := \bigcup_d \{\mathcal{D}(d, -) \times f; f \in J\}$. We have to verify that conditions i) and ii) of Theorem D.20 hold. Let's show ii), i.e. that for any object d of

$\mathcal{D}$, the evaluation functor $\mathrm{ev}_d\colon \mathcal{U}^{\mathcal{D}} \to \mathcal{U}$ takes relative FJ-cell complexes to weak equivalences in $\mathcal{U}$. Recall that ev_d preserves colimits, thus ev_d takes any relative FJ-cell complex to a relative $\mathrm{ev}_d(FJ)$-cell complex, where $\mathrm{ev}_d(FJ) = \bigcup_{d'} \{\mathcal{D}(d', d) \times f; f \in J\}$. Note that for any object d' of $\mathcal{D}$ and any morphism f of J, the map $\mathcal{D}(d', d) \times f$ is both a closed T_1 embedding (see D.26) and an inclusion of a deformation retract. By Lemma D.26b) and d), any pushout along $\mathcal{D}(d', d) \times f$ is a closed T_1 embedding and an inclusion of a deformation retract, thus in particular also a weak equivalence in $\mathcal{U}$. It follows from Lemma D.26e) that any relative $\mathrm{ev}_d(FJ)$-cell complex is a weak equivalence. Hence condition ii) holds.

Considering condition i), we show that FJ permits the small object argument, i.e. that the domain of any relative FJ-cell complex is small relative to FJ-cell. Such a domain is of the form $\mathcal{D}(d, -) \times D^n$, where D^n denotes the n-disk. By an adjointness argument as in the proof of Remark D.21 and since ev_d preserves colimits, it's enough to show that D^n is small relative to the class of relative FJ-cell complexes evaluated in d. But this holds by Lemma D.26a), since ev_d takes a relative FJ-cell complex to a relative $\mathrm{ev}_d(FJ)$-cell complex, which is a closed T_1 embedding by Lemma D.26b).

Similarly, one shows that FI permits the small object argument. Thus $\mathcal{U}^{\mathcal{D}}$ with the projective model structure is a cofibrantly generated model category by the transport Theorem D.20.

As in the proof of Theorem 2.7, one checks that condition ii) of Proposition 2.3 holds and concludes that $\mathcal{U}^{\mathcal{D}}$ is a $\mathcal{U}$-model category. □

In order to show that $\mathcal{U}^{\mathcal{D}}$ has cellular fixed point functors, we first study its cofibrations.

Lemma 5.2. *Let $\mathcal{D}$ be a small $\mathcal{U}$-category and f a cofibration in the category $\mathcal{U}^{\mathcal{D}}$ with the projective model structure. Then for any object d of $\mathcal{D}$, the evaluation functor $\mathrm{ev}_d\colon \mathcal{U}^{\mathcal{D}} \to \mathcal{U}$ takes f to a closed embedding f_d.*

Proof. Let FI be the set of generating cofibrations of the cofibrantly generated model category $\mathcal{C}^{\mathcal{D}}$ given in the proof of Theorem 5.1. Since any cofibration in $\mathcal{C}^{\mathcal{D}}$ is a retract of a relative FI-cell complex (see D.17) and closed embeddings are closed under retraction, we can assume that f is a relative FI-cell complex. Since the evaluation functor ev_d preserves colimits, it follows that f_d is a closed embedding by Lemma D.26b). □

Proposition 5.3. *Let $\mathcal{D}$ be a small $\mathcal{U}$-category and G a $\mathcal{U}$-group. Denote the $\mathcal{U}$-model category $\mathcal{U}^{\mathcal{D}}$ with the projective model structure by $\mathcal{C}$. Then for any closed subgroup H of G, the fixed point functor $(-)^H\colon \mathcal{C}^G \to \mathcal{C}$ is cellular.*

Proof. The proof will reduce to the arguments in Example 3.10, which showed that $(-)^H\colon \mathcal{U}^G \to \mathcal{U}$ is cellular. We first study how the fixed point functors and colimits in $\mathcal{C}^G$ behave with respect to the evaluation functor $\mathrm{ev}_d\colon \mathcal{C} \to \mathcal{U}$ for any object d of $\mathcal{D}$. Precomposing with ev_d defines a functor

$$(\mathrm{ev}_d)_*\colon \mathcal{C}^G \to \mathcal{U}^G,$$

which makes the diagram

$$\begin{tikzcd} \mathcal{C}^G \arrow[r, "\mathrm{ev}_*"] \arrow[d, "(\mathrm{ev}_d)_*"'] & \mathcal{C} \arrow[d, "\mathrm{ev}_d"] \\ \mathcal{U}^G \arrow[r, "\mathrm{ev}_*"'] & \mathcal{U} \end{tikzcd}$$

commute, where $*$ in the top and bottom row denotes the object of G. Using that colimits in $\mathcal{C}^G$ and $\mathcal{U}^G$ are calculated in the object $*$ of G and that ev_d preserve colimits, one deduces that $(\mathrm{ev}_d)_*$ preserves colimits.

Since limits in $\mathcal{U}^\mathcal{D}$ are calculated objectwise, it follows that $(-)^H$ and ev_d commute in the following sense. For any map $f\colon X \to Y$ in $\mathcal{C}^G$, the map $f^H\colon X^H \to Y^H$ evaluated in d is given by

$$(\mathrm{ev}_d \circ X)^H \overset{((\mathrm{ev}_d)_*(f))^H}{\longrightarrow} (\mathrm{ev}_d \circ Y)^H,$$

where $(-)^H$ denotes the H-fixed point functor $\mathcal{U}^G \to \mathcal{U}$.

We show that $(-)^H\colon \mathcal{C}^G \to \mathcal{C}$ satisfies the cellularity property i). Let P be a directed set and $X\colon P \to \mathcal{C}^G$ a functor such that for all $\alpha \leq \beta$ in P, the map $X_\alpha(*) \to X_\beta(*)$ is a cofibration in $\mathcal{C}$. To prove that $(-)^H$ preserves the colimit of X, i.e. that the induced map

$$\mathrm{colim}_\alpha (X_\alpha)^H \longrightarrow (\mathrm{colim}\, X)^H$$

is an isomorphism in $\mathcal{C} = \mathcal{U}^\mathcal{D}$, we show that evaluated in any object d of $\mathcal{D}$ it is an isomorphism in $\mathcal{U}$. Evaluated in d, it is given by the induced map

$$\mathrm{colim}_\alpha (\mathrm{ev}_d \circ X_\alpha)^H \longrightarrow (\mathrm{colim}_\alpha\, \mathrm{ev}_d \circ X_\alpha)^H$$

since $(-)^H$ and ev_d commute and since the functors $(\mathrm{ev}_d)_*$ and ev_d preserve colimits. From Lemma 5.2 it follows that this map is an isomorphism since $(-)^H\colon \mathcal{U}^G \to \mathcal{U}$ preserves directed colimits of diagrams, where each map is injective in $\mathcal{U}$.

Similarly, one shows that the cellularity property ii) holds, by reducing it to the fact that $(-)^H\colon \mathcal{U}^G \to \mathcal{U}$ preserves pushouts of diagrams where one leg is a closed embedding as a map in $\mathcal{U}$.

The cellularity property iii) for $(-)^H\colon \mathcal{C}^G \to \mathcal{C}$ follows from the same property of $(-)^H\colon \mathcal{U}^G \to \mathcal{U}$. □

Corollary 5.4. *Let $\mathcal{D}$ be a small $\mathcal{U}$-category and G a compact Lie group or a discrete group. Let $\mathcal{F}$ be a set of closed subgroups of G, which contains* *the trivial subgroup $\{e\}$. Denote the cofibrantly generated $\mathcal{U}$-model category $\mathcal{U}^{\mathcal{D}}$ with the projective model structure by $\mathcal{C}$. Then there is a pair of Quillen equivalences*

$$\Theta\colon \mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}} \rightleftarrows \mathcal{C}^G \colon \Phi$$

between $\mathcal{C}^{\mathcal{O}_{\mathcal{F}}^{\mathrm{op}}}$ with the projective model structure and $\mathcal{C}^G$ with the $\mathcal{F}$-fixed point model structure.

Proof. Check that the assumptions in Theorem 4.2 are satisfied. The cellularity property holds by Proposition 5.3. Consider $H, K \in \mathcal{F}$. If G is a compact Lie group, then the condition on the functor $(G/K)^H \otimes - \colon \mathcal{U}^{\mathcal{D}} \to \mathcal{U}^{\mathcal{D}}$ is satisfied by Remark 3.14b). If G is discrete, then for any map f in $\mathcal{U}^{\mathcal{D}}$, the natural transformation $(G/K)^H \otimes f$ evaluated in an object d of $\mathcal{D}$ is given by $(G/K)^H \times f_d$ which is the coproduct $\coprod_{(G/K)^H} f_d$. Since colimits in $\mathcal{U}^{\mathcal{D}}$ are calculated objectwise, it follows that $(G/K)^H \otimes -$ is the coproduct functor $\coprod_{(G/K)^H} -$ and thus preserves cofibrations and acyclic cofibrations. $\square$

A k-spaces and compactly generated spaces

From a category theoretical viewpoint, the category **Top** of topological spaces and continuous maps has some drawbacks. For instance it is not cartesian closed. Therefore, in algebraic topology one often works in the category $\mathcal{U}$ of compactly generated spaces, which is a full subcategory of **Top** and which is better behaved. Compactly generated spaces are defined as weak Hausdorff k-spaces. They form a full subcategory of another well-behaved subcategory of **Top**, the category $\mathcal{K}$ of k-spaces. But they have the advantage of satisfying a seperation axiom, the weak Hausdorff condition, which is stronger than T_1 but weaker then the Hausdorff property T_2.

Here, we summarize the for the working knowledge necessary definitions and facts about these two subcategories. The reader is referred to Appendix A of [13] for the proofs.

Definition A.1. Let X be a topological space.

a) The space X is called *weak Hausdorff* if for all maps $g\colon K \to X$ in **Top** with K compact Hausdorff, the image $g(K)$ is closed in X.

b) A subset $A \subset X$ is called *compactly closed* if for all maps $g\colon K \to X$ in **Top** with K compact Hausdorff, the preimage $g^{-1}(A)$ is closed.

c) The space X is a *k-space* if every compactly closed subset is closed. Let $\mathcal{K}$ denote the full subcategory of **Top** consisting of the k-spaces.

d) The space X is called *compactly generated* if it is both weak Hausdorff and a k-space. Let $\mathcal{U}$ denote the full subcategory of $\mathcal{K}$ consisting of the compactly generated spaces.

e) The *k-space topology* on X is the topology, where the closed sets are precisely the compactly closed sets.

Example A.2. Let $\mathcal{C}$ be either the category of k-spaces $\mathcal{K}$ or the category of compactly generated spaces $\mathcal{U}$. Then the category $\mathcal{C}$ contains

a) locally compact Hausdorff spaces,

b) metric spaces,

c) closed and open subsets of spaces in $\mathcal{C}$.

Proposition A.3 (Limits, colimits, the functors k and wH)**.** *a) The inclusion functor $\mathcal{K} \to$ **Top** has a right adjoint and left inverse $k\colon$ **Top** $\to \mathcal{K}$, which takes a topological space to its underlying set equipped with the k-space topology.*

*b) The category $\mathcal{K}$ has all small limits and colimits. Colimits are created by the inclusion functor and limits are obtained by applying k to the limit in **Top**.*

*c) If X and Y are k-spaces and Y is locally compact Hausdorff, then the product $X \times Y$ in **Top** is a k-space and hence, it is also the product in $\mathcal{K}$.*

*d) Let $p : X \to Y$ be a quotient map in **Top**, where X is a k-space. Then Y is a k-space. It is weak Hausdorff and thus in $\mathcal{U}$ if and only if the preimage $(p \times p)^{-1}(\Delta)$ of the diagonal $\Delta \subset Y \times Y$ is closed in the product $X \times X$ in $\mathcal{K}$. In particular, X is in $\mathcal{U}$ if and only if the diagonal in the product $X \times X$ in $\mathcal{K}$ is closed.*

e) The inclusion functor $\mathcal{U} \to \mathcal{K}$ has a left adjoint and left inverse

$$wH : \mathcal{K} \to \mathcal{U},$$

which takes a k-space X to the quotient X/R, where $R \subset X \times X$ is the smallest closed equivalence relation.

f) The category $\mathcal{U}$ has all small limits and colimits. Limits are created by the inclusion functor and colimits are obtained by applying wH to the colimit in $\mathcal{K}$.

Proposition A.4 (Mapping space)**.** *Let X, Y be in $\mathcal{K}$. For any map $h\colon K \to X$ from a compact Hausdorff space K to X and any open subset U of Y, denote the set of maps $f\colon X \to Y$ in $\mathcal{K}$ with $f(h(K)) \subset U$ by $N(h, U)$. Let $C(X, Y)$ be the set $Hom_{\mathcal{K}}(X, Y)$ with the topology generated by the subbasis $\{N(h, U)\}$. Let Y^X be the k-space $kC(X, Y)$.*

a) *For all X, Y, Z in $\mathcal{K}$, there is a natural isomorphism*

$$Hom_{\mathcal{K}}(X \times Y, Z) \to Hom_{\mathcal{K}}(X, Z^Y),$$

which sends a map f to $\tilde{f}$ given by $\tilde{f}(x)(y) = f(x, y)$. In particular, $\mathcal{K}$ is cartesian closed.

b) *For all X, Y in $\mathcal{K}$, evaluation $Y^X \times X \to Y$ is the counit of the adjunction above.*

c) *If X is in $\mathcal{K}$ and Y is in $\mathcal{U}$, then Y^X is weak Hausdorff. Hence, for all X, Y, Z in $\mathcal{U}$, there is a natural isomorphism*

$$Hom_{\mathcal{U}}(X \times Y, Z) \to Hom_{\mathcal{U}}(X, Z^Y),$$

which sends a map f to $\tilde{f}$ given by $\tilde{f}(x)(y) = f(x, y)$. In particular, $\mathcal{U}$ is cartesian closed.

Remark A.5 (Subspace)**.** Let $\mathcal{C}$ be either the category of k-spaces $\mathcal{K}$ or the category of compactly generated spaces $\mathcal{U}$. Let Y be a space in $\mathcal{C}$ and $A \subset Y$ a subset of Y. We call A equipped with the topology obtained by applying k to the subspace $A \subset Y$ in **Top** a *subspace* of Y. Note that if X is space in $\mathcal{C}$ and $f \colon X \to Y$ is a set function with $f(X) \subset A$, then $f \colon X \to Y$ is in $\mathcal{C}$ if and only if $f \colon X \to A$ is in $\mathcal{C}$.

Proposition A.6 (Closed embeddings, quotient maps)**.** *Let $\mathcal{C}$ be either the category of k-spaces $\mathcal{K}$ or the category of compactly generated spaces $\mathcal{U}$. We call a map f in $\mathcal{C}$ a closed embedding, if it is a closed embedding in **Top**. We call f a quotient map, if it is a qoutient map in **Top**.*

a) *An arbitrary product of closed embeddings in $\mathcal{C}$ is a closed embedding.*

b) *An arbitrary coprodut of closed embeddings in $\mathcal{C}$ is a closed embedding.*

c) *Let X, X', Y, Y' be spaces in $\mathcal{C}$ and $p \colon X \to Y$, $q \colon X' \to Y'$ quotient maps. Then $p \times q \colon X \times X' \to Y \times Y'$ in $\mathcal{C}$ is a quotient map.*

Proposition A.7 (Colimits in $\mathcal{U}$, which are created in $\mathcal{K}$)**.** *The inclusion functor $\mathcal{U} \to \mathcal{K}$ creates*

a) *arbitrary coproducts,*

b) *pushouts of diagrams $X \leftarrow A \to Y$, where one leg is a closed embedding, and*

c) *directed colimits of diagrams where each arrow is injective.*

B Topological categories and discrete categories

We work both with topological categories, that is categories enriched over the category of compactly generated spaces $\mathcal{U}$ (see A.1), and with discrete categories. Discrete categories are just ordinary categories, but which for a unified treatment are viewed as categories enriched over the full subcategory **Set** of **Top** given by the discrete spaces. The standard reference for enriched category theory in general is Kelly's book [12], an introduction is given by Borceux in [2, Ch. 6].

Let $\mathcal{V}$ be either the category of compactly generated spaces $\mathcal{U}$ or the category of discrete spaces **Set**. Recall that $\mathcal{V}$ is cartesian closed, complete and cocomplete and has the one-point space $*$ as terminal object. We develop the theory of $\mathcal{V}$-categories, that is categories enriched over $\mathcal{V}$.

B.1 Definition and elementary properties of $\mathcal{V}$-categories

Definition B.1. A $\mathcal{V}$*-category* is a category $\mathcal{C}$, where for all objects A, B of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is topologized as a space in $\mathcal{V}$, called *hom-space* and denoted by $\mathcal{C}(A, B)$, such that composition $\circ$ is continuous. That is, for all objects A, B and C of $\mathcal{C}$, composition $\circ\colon \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ is a morphism in $\mathcal{V}$.

Example B.2. The category $\mathcal{V}$ itself is a $\mathcal{V}$ category, where for objects X, Y in $\mathcal{V}$ the hom-space $\mathcal{V}(X, Y)$ is the mapping space Y^X (see A.4) if $\mathcal{V} = \mathcal{U}$ and the set $\mathrm{Hom}_{\mathcal{V}}(X, Y)$ equipped with the discrete topology if $\mathcal{V} = \mathbf{Set}$.

Example B.3. If $\mathcal{V}$ is the category **Set**, then any category $\mathcal{C}$ becomes a $\mathcal{V}$-category by equipping the hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ with the discrete topology for any objects A, B of $\mathcal{C}$.

Remark B.4. Let $\mathcal{C}$ be $\mathcal{V}$-category.

a) The opposite category $\mathcal{C}^{\mathrm{op}}$ of the category $\mathcal{C}$ is a $\mathcal{V}$-category with hom-space $\mathcal{C}^{\mathrm{op}}(B, A) = \mathcal{C}(A, B)$ for any objects B and A of $\mathcal{C}^{\mathrm{op}}$.

b) If $\mathcal{D}$ is another $\mathcal{V}$-category, then the product category $\mathcal{C} \times \mathcal{D}$ of the categories $\mathcal{C}$ and $\mathcal{D}$ is a $\mathcal{V}$-category with hom-space

$$(\mathcal{C} \times \mathcal{D})(A \times A', B \times B') = \mathcal{C}(A, B) \times \mathcal{D}(A', B')$$

for any objects $A \times A'$ and $B \times B'$ of $\mathcal{C} \times \mathcal{D}$.

Definition B.5. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories.

a) A functor $F\colon \mathcal{C} \to \mathcal{D}$ between the categories $\mathcal{C}$ and $\mathcal{D}$ is called a $\mathcal{V}$-*functor*, if it is continuous, i.e. if $F\colon \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ is a morphism in $\mathcal{V}$ or all objects A, B of $\mathcal{C}$.

b) Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-functors. A *$\mathcal{V}$-natural transformation* from F to G is an ordinary natural transformation between the functors F and G. The class of $\mathcal{V}$-natural transformations from F to G is denoted by $\mathcal{V}\text{-Nat}(F, G)$.

Example B.6. Let $\mathcal{C}$ be a $\mathcal{V}$-category. The functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ induces a $\mathcal{V}$-functor $\mathcal{C}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$.

Remark B.7. Semantically, we don't distinguish between a $\mathcal{V}$-category and its underlying category, where one forgets the topology of the hom-spaces, and similarly for a $\mathcal{V}$-functor ($\mathcal{V}$-natural transformation) and its underlying functor (natural transformation). For instance a $\mathcal{V}$-category is said to be small, if it is small as a category.

Recall that a subspace X (see A.5) of a space Y in $\mathcal{U}$ is a subset X of the set Y equipped with the topology obtained by first taking the subspace topology in **Top** and then applying the k-ification functor. By a subspace X of a discrete space Y in **Set**, we mean a subset X of the set Y equipped with the discrete topology.

Definition B.8. Let $\mathcal{D}, \mathcal{C}$ be $\mathcal{V}$-categories. Suppose that $\mathcal{D}$ is small. The *category of $\mathcal{V}$-functors from $\mathcal{D}$ to $\mathcal{C}$*, denoted by $\mathcal{C}^{\mathcal{D}}$, is the $\mathcal{V}$-category with underlying category the full subcategory of the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ given by the $\mathcal{V}$-functors, where for any objects F, G of $\mathcal{C}^{\mathcal{D}}$ the hom-space $\mathcal{C}^{\mathcal{D}}(F, G)$ is a subspace of $\prod_{d \in D} \mathcal{C}(F(d), G(d))$ in $\mathcal{V}$.

Theorem B.9 ($\mathcal{V}$-Yoneda Lemma)**.** *Let $\mathcal{C}$ be a $\mathcal{V}$-category. Denote by U the forgetful functor $\mathcal{V} \to$ **Set**, which takes a space to its underlying set.*

a) For every $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{V}$ and every object A of $\mathcal{C}$, there is a bijection

$$\mathcal{V}\text{-Nat}(\mathcal{C}(A, -), F) \cong UF(A) \text{ defined by } \eta \mapsto \eta_A(\mathrm{id}_A),$$

which is natural in F and A. Furthermore, if $\mathcal{C}$ is small, then this bijection induces a natural isomorphism

$$\mathcal{V}^{\mathcal{C}}(\mathcal{C}(A, -), F) \cong F(A)$$

in $\mathcal{V}$.

b) For every $\mathcal{V}$-functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and every object A of $\mathcal{C}$, there is a bijection

$$\mathcal{V}\text{-Nat}(\mathcal{C}(-, A), F) \cong UF(A) \text{ defined by } \eta \mapsto \eta_A(\mathrm{id}_A),$$

which is natural in F and A. Furthermore, if $\mathcal{C}$ is small, then this bijection induces a natural isomorphism

$$\mathcal{V}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-, A), F) \cong F(A)$$

in $\mathcal{V}$.

Proof. Concerning the first statement of part a), note that applying U to any natural transformation $: \mathcal{C}(A,-) \to F$ gives a natural transformation $\mathrm{Hom}_{\mathcal{C}}(A,-) \to UF$. Hence by the ordinary Yoneda Lemma, it's enough to show that any natural transformation $\eta\colon \mathrm{Hom}_{\mathcal{C}}(A,-) \to UF$ induces a natural transformation $\mathcal{C}(A,-) \to F$, i.e. $\eta_B\colon \mathcal{C}(A,B) \to F(B)$ is continuous for every object B of $\mathcal{C}$. Set $a = \eta_A(\mathrm{id}_A) \in F(A)$ and write $\mathrm{eval}_a\colon \mathcal{V}(F(A),F(B)) \to F(B)$ for the map in $\mathcal{V}$ which evaluates in a. Then η_B is the composite

$$\mathcal{C}(A,B) \xrightarrow{F} \mathcal{V}(F(A),F(B)) \xrightarrow{\mathrm{eval}_a} F(B)$$

and thus is indeed continuous. To show the second statement of part a), use that for any two spaces X,Y of $\mathcal{V}$, the evaluation map $\mathcal{V}(X,Y) \times X \to Y$ is continuous.

Part b) is similar. □

Definition B.10. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a pair (F,U) of $\mathcal{V}$-functors $F\colon C \to \mathcal{D}$ and $U\colon \mathcal{D} \to \mathcal{C}$ together with an isomorphism

$$\mathcal{D}(F(A),B) \cong \mathcal{C}(A,U(B)) \tag{B.1}$$

in $\mathcal{V}$, which is natural in the objects A of $\mathcal{C}$ and B of $\mathcal{D}$. The $\mathcal{V}$-functor F and the $\mathcal{V}$-functor U of a $\mathcal{V}$-adjunction (F,U) is said to be *$\mathcal{V}$-left adjoint* to U and *$\mathcal{V}$-right adjoint* to F, respectively.

Working enriched over $\mathcal{U}$ or **Set**, enriched adjunctions between enriched functors are the same as ordinary adjunctions between the underlying functors.

Proposition B.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $U\colon \mathcal{D} \to \mathcal{C}$ are $\mathcal{V}$-functors. Then F is $\mathcal{V}$-left adjoint to $\mathcal{V}$ if and only if the underlying functor F is left adjoint to the underlying functor U.*

Proof. If (F,U) is a $\mathcal{V}$-adjunction, then applying the forgetful functor $\mathcal{V} \to$ **Set** to the isomorphism as in (B.1) shows that (F,U) is an adjunction. Conversely, if (F,U) is an adjunction, then the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(A),B) \cong \mathrm{Hom}_{\mathcal{C}}(A,U(B)),$$

natural in the objects A of $\mathcal{C}$ and B of $\mathcal{D}$, between the hom-sets is continuous as a map between the corresponding hom-spaces as already shown in the proof of the $\mathcal{V}$-Yoneda Lemma (see B.9). □

B.2 Tensors and cotensors

The enriched notion of limit and colimit is weighted limit and weighted colimit. The most important examples of these are cotensors and tensors, respectively.

Definition B.12. Let $\mathcal{C}$ be a $\mathcal{V}$-category, A an object of $\mathcal{C}$ and X an object of $\mathcal{V}$.

a) The *cotensor* of X and A is an object $[X, A]$ of $\mathcal{C}$ together with a natural isomorphism

$$\mathcal{C}(-, [X, A]) \cong \mathcal{V}(X, \mathcal{C}(-, A)).$$

b) The $\mathcal{V}$-category $\mathcal{C}$ is called cotensored, if it admits all cotensors.

c) The tensor of X and A is an object $X \otimes A$ of $\mathcal{C}$ together with a natural isomorphism

$$\mathcal{C}(X \otimes A, -) \cong \mathcal{V}(X, \mathcal{C}(A, -)).$$

d) The $\mathcal{V}$-category $\mathcal{C}$ is called tensored, if it admits all tensors.

Remark B.13. Let $\mathcal{C}$ be a $\mathcal{V}$-category. Note that for any object A of $\mathcal{C}$ and object X of $\mathcal{V}$, the tensor $X \otimes A$ and the cotensor $[X, A]$ is unique up to canonical isomorphism by the Yoneda Lemma.

Example B.14. The $\mathcal{V}$-category $\mathcal{V}$ (see B.2) is tensored and cotensored with tensor $X \otimes A = X \times A$ and cotensor $[X, A] = \mathcal{V}(X, A)$ for objects A, X of $\mathcal{V}$, since $\mathcal{V}$ is cartesian closed.

Example B.15. Let $\mathcal{V}$ be the category **Set** and let $\mathcal{C}$ be a category. Thus $\mathcal{C}$ is also a $\mathcal{V}$-category by Example B.3. If $\mathcal{C}$ is cocomplete, then $\mathcal{C}$ is tensored with tensor $X \otimes A = \coprod_X A$ for an object A of $\mathcal{C}$ and an object X of **Set**. If $\mathcal{C}$ is complete, then $\mathcal{C}$ is cotensored with cotensor $[X, A] = \prod_X A$ for an object A of $\mathcal{C}$ and an object X of **Set**.

We will not use and not define other weighted limits and weighted colimits than cotensors and tensors. To illustrate their significant role, we recall the following result.

Theorem B.16 ([2, Corollary 6.6.16])**.** *Suppose a $\mathcal{V}$-category $\mathcal{C}$ is tensored and cotensored. Then $\mathcal{C}$ admits all weighted limits if and only if $\mathcal{C}$ is complete and $\mathcal{C}$ admits all weighted colimits if and only if $\mathcal{C}$ is cocomplete.*

From the proposition below it will follow that in any tensored and cotensored $\mathcal{V}$-category, the tensors and cotensors induce $\mathcal{V}$-functors.

Proposition B.17. *Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-categories and $F \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ a $\mathcal{V}$-functor.*

a) Given for all objects A of $\mathcal{D}$, an object $X(A)$ of $\mathcal{C}$ together with a natural isomorphism $\mathcal{C}(X(A), -) \to F(A, -)$, there exists for every *morphism $f\colon A \to A'$ in $\mathcal{D}$ a unique morphism $X(f)\colon X(A) \to X(A')$ in $\mathcal{C}$ making the diagram*

$$\begin{tikzcd} \mathcal{C}(X(A),B) \arrow[r, "\cong"] & F(A,B) \\ \mathcal{C}(X(A'),B) \arrow[r, "\cong"] \arrow[u, "\mathcal{C}(X(f),B)"] & F(A',B) \arrow[u, "F(f,B)"'] \end{tikzcd} \tag{B.2}$$

commute for all objects B of $\mathcal{C}$. Furthermore, the assignment $f \mapsto X(f)$ defines a $\mathcal{V}$-functor $X\colon \mathcal{D} \to \mathcal{C}$.

b) Given for all objects A of $\mathcal{C}$, an object $X(A)$ of $\mathcal{D}$ together with a natural isomorphism $\mathcal{D}(-, X(A)) \to F(-, A)$, there exists for every morphism $f\colon A \to A'$ in $\mathcal{C}$ a unique morphism $X(f)\colon X(A) \to X(A')$ in $\mathcal{D}$ making the diagram

$$\begin{tikzcd} \mathcal{D}(B,X(A)) \arrow[r, "\cong"] \arrow[d, "\mathcal{D}(B,X(f))"'] & F(B,A) \arrow[d, "F(B,f)"] \\ \mathcal{D}(B,X(A')) \arrow[r, "\cong"] & F(B,A') \end{tikzcd} \tag{B.3}$$

commute for all objects B of $\mathcal{D}$. Furthermore, the assignment $f \mapsto X(f)$ defines a $\mathcal{V}$-functor $X\colon \mathcal{C} \to \mathcal{D}$.

Proof. We prove part a). For every object B in $\mathcal{C}$, denote the composite

$$\mathcal{C}(X(A'),B) \to F(A',B) \overset{F(f,B)}{\to} F(A,B) \to \mathcal{C}(X(A),B)$$

by $\eta_B^f = \eta_B$. Then $\{\eta_B\}_B$ is a $\mathcal{V}$-natural transformation

$$\mathcal{C}(X(A'),-) \to \mathcal{C}(X(A),-).$$

From part a) of the $\mathcal{V}$-Yoneda Lemma (see B.9), it follows that $X(f) := \eta_{X(A')}(\mathrm{id}_{X(A')})$ is the unique morphism making diagram (B.2) commute for all objects B of $\mathcal{C}$. Using that $\mathcal{C}(-,B)$ and $F(-,B)$ are functorial for all objects B of $\mathcal{C}$, one deduces the functoriality of X. The functor $X\colon \mathcal{D} \to \mathcal{C}$ is a $\mathcal{V}$-functor, since the functor $F(-, X(A'))$ is continuous and since evaluation in the point $\mathrm{id}_{X(A')} \in \mathcal{C}(X(A'), X(A'))$ is continuous.

Similarly, one shows part b) by using part b) of the $\mathcal{V}$-Yoneda Lemma. □

Corollary B.18. *Let $\mathcal{C}$ be a $\mathcal{V}$-category.*

a) If $\mathcal{C}$ is tensored, then the tensors $X \otimes A$ for objects X of $\mathcal{V}$ and A of $\mathcal{C}$ induce a $\mathcal{V}$-functor

$$\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$$

b) If $\mathcal{C}$ is cotensored, then the cotensors $[X, A]$ for objects X of $\mathcal{V}$ and A of $\mathcal{C}$ induce a $\mathcal{V}$-functor

$$[-,-]\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

c) Suppose $\mathcal{C}$ is tensored and cotensored. Then there are isomorphisms

$$\mathcal{C}(X \otimes A, B) \cong \mathcal{V}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, [X, B]),$$

which are natural in the objects A, B of $\mathcal{C}$ and X of $\mathcal{V}$. In particular the pair $(X \otimes -, [X, -])$ is a $\mathcal{V}$-adjunction for any object X of $\mathcal{V}$.

Proof. Part a) follows from Proposition B.17a) with $F\colon (\mathcal{V} \times \mathcal{C})^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ the composite

$$\mathcal{V}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{id} \times \mathcal{C}} \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \xrightarrow{\mathcal{V}} \mathcal{V}.$$

For part b), apply Proposition B.17b) with $F\colon \mathcal{C}^{\mathrm{op}} \times (\mathcal{V}^{\mathrm{op}} \times \mathcal{C}) \to \mathcal{V}$ the composite

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{id} \times \mathcal{C}} \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \xrightarrow{\mathcal{V}} \mathcal{V}.$$

Part c) follows from the definition of the tensor functor and the cotensor functor. □

Proposition B.19. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{D}$ be a small $\mathcal{V}$-category. Consider the $\mathcal{V}$-category $\mathcal{C}^{\mathcal{D}}$ of $\mathcal{V}$-functors from $\mathcal{D}$ to $\mathcal{C}$ (see B.8).*

a) If $\mathcal{C}$ is cotensored, then so is $\mathcal{C}^{\mathcal{D}}$.

b) If $\mathcal{C}$ is tensored, then so is $\mathcal{C}^{\mathcal{D}}$.

Proof. For part a), suppose that $\mathcal{C}$ is cotensored. Thus for every object X of $\mathcal{V}$, there is a $\mathcal{V}$-functor $[X, -]\colon \mathcal{C} \to \mathcal{C}$ and an isomorphism

$$\varphi\colon \mathcal{C}(B, [X, A]) \xrightarrow{\cong} \mathcal{V}(X, \mathcal{C}(B, A)),$$

which is natural in the objects B and A of $\mathcal{C}$. For any object F of $\mathcal{C}^{\mathcal{D}}$ and any object X, denote the composite $\mathcal{V}$-functor

$$\mathcal{D} \xrightarrow{F} \mathcal{C} \xrightarrow{[X,-]} \mathcal{C}$$

by $[X, F]$. Using φ, one defines an isomorphism

$$\Phi\colon \mathcal{C}^{\mathcal{D}}(G, [X, F]) \to \mathcal{V}(X, \mathcal{C}^{\mathcal{D}}(G, F))$$

in $\mathcal{V}$, which is natural in the object G of $\mathcal{C}^{\mathcal{D}}$. Hence $[X, F]$ is the cotensor of X and F.

Part b) is shown similarly. The tensor $X \otimes F$ of an object X of $\mathcal{V}$ and an object F of $\mathcal{C}^{\mathcal{D}}$ is given by the composite

$$\mathcal{D} \xrightarrow{F} \mathcal{C} \xrightarrow{X \otimes -} \mathcal{C},$$

where $\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ is the tensor functor of $\mathcal{C}$. □

Proposition B.20. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{D}$ be a small $\mathcal{V}$-category. Suppose that $\mathcal{C}$ is tensored. Then for each object d of $\mathcal{D}$, the $\mathcal{V}$-functor* $\mathrm{ev}_d\colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ *given by evaluation in d, has a $\mathcal{V}$-left adjoint $F^d\colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$.*

Proof. Fix an object d of $\mathcal{D}$. For any object A of $\mathcal{C}$, define the $\mathcal{V}$-functor $F^d_A\colon \mathcal{D} \to \mathcal{C}$ to be the composite

$$\mathcal{D} \xrightarrow{\mathcal{D}(d,-)} \mathcal{V} \xrightarrow{-\otimes A} \mathcal{C}.$$

For any morphism $f\colon A \to B$, let $F^d_f\colon F^d_A \to F^d_B$ be the $\mathcal{V}$-natural transformation given by

$$\mathcal{D}(d,d') \otimes f\colon \mathcal{D}(d,d') \otimes A \to \mathcal{D}(d,d') \otimes B$$

in the object d' of $\mathcal{D}$. Then these definitions induce a $\mathcal{V}$ functor $F^d\colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$. To show that it is $\mathcal{V}$-left adjoint to ev_d, note that there are isomorphisms

$$\mathcal{C}^{\mathcal{D}}(F^d_A, G) \cong \mathcal{V}^{\mathcal{D}}(\mathcal{D}(d,-), \mathcal{C}(A, G-)) \cong \mathcal{C}(A, G(d)),$$

in $\mathcal{V}$, which are natural in the objects A of $\mathcal{C}$ and G of $\mathcal{C}^{\mathcal{D}}$. Indeed, the first isomorphism is induced by the isomorphism

$$\mathcal{C}(X \otimes B, C) \cong \mathcal{V}(X, \mathcal{C}(B, C)),$$

natural in the objects X of $\mathcal{V}$ and B, C of $\mathcal{C}$, which expresses that $\mathcal{C}$ is tensored. The second isomorphism is given by the $\mathcal{V}$-Yoneda Lemma. □

B.3 Limits and colimits

Recall that for any category $\mathcal{C}$ and any object A of $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}(A,-)\colon \mathcal{C} \to \mathbf{Set}$ preserves limits. Furthermore, if $\mathcal{C}$ is complete, then so is the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ for any small category $\mathcal{D}$. We will see that analogous results hold in an enriched setting, provided that $\mathcal{C}$ is an enriched category which is tensored.

Proposition B.21. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and A an object of $\mathcal{C}$.*

a) If $\mathcal{C}$ is tensored, then the functor $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathcal{V}$ preserves all small limits.

b) If $\mathcal{C}$ is cotensored, then the functor $\mathcal{C}(-,A)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ takes any small colimit to a limit.

Proof. Let $\mathcal{D}$ be a small category and $F\colon \mathcal{D} \to \mathcal{C}$ a functor.

For part a), assume that F has a limiting cone $\nu\colon \lim F \to F$. We claim that the map

$$\mathcal{C}(A, \lim F) \to \lim \mathcal{C}(A, F-)$$

induced by $\mathcal{C}(A,\nu)\colon \mathcal{C}(A,\lim F) \to \mathcal{C}(A,F-)$ is an isomorphism in $\mathcal{V}$. Indeed, for any object X of $\mathcal{V}$, the map

$$\mathrm{Hom}_{\mathcal{V}}(X,\mathcal{C}(A,\lim F)) \to \mathrm{Hom}_{\mathcal{V}}(X,\lim \mathcal{C}(A,F-))$$

equals the composite

$$\begin{aligned}\mathrm{Hom}_{\mathcal{V}}(X,\mathcal{C}(A,\lim F)) &\cong \mathrm{Hom}_{\mathcal{C}}(X\otimes A,\lim F) \cong \lim \mathrm{Hom}_{\mathcal{C}}(X\otimes A,F)\\ &\cong \lim \mathrm{Hom}_{\mathcal{V}}(X,\mathcal{C}(A,F-)) \cong \mathrm{Hom}_{\mathcal{V}}(X,\lim \mathcal{C}(A,F-)).\end{aligned}$$

Thus the claim follows from the Yoneda Lemma.

Similarly, for part b) one assumes that F has a colimiting cone $F \to \operatorname{colim} F$ and shows that the map

$$\mathcal{C}(\operatorname{colim} F, A) \to \lim \mathcal{C}(F-,A)$$

induced by $\mathcal{C}(\operatorname{colim} F, A) \to \mathcal{C}(F-,A)$ is an isomorphism in $\mathcal{V}$. □

Corollary B.22. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{D}$ be a small $\mathcal{V}$-category.*

a) If $\mathcal{C}$ is tensored and complete, then the limit functor $\mathrm{Fun}(\mathcal{D},\mathcal{C}) \xrightarrow{\lim} \mathcal{C}$ induces a $\mathcal{V}$-functor $\lim\colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, i.e. the composite

$$\mathcal{C}^{\mathcal{D}} \longrightarrow \mathrm{Fun}(\mathcal{D},\mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

is a continuous functor.

b) If $\mathcal{C}$ is cotensored and cocomplete, then the colimit functor $\mathrm{Fun}(\mathcal{D},\mathcal{C}) \xrightarrow{\operatorname{colim}} \mathcal{C}$ induces a $\mathcal{V}$-functor $\operatorname{colim}\colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$.

Proof. For part a), let two $\mathcal{V}$-functors $F,G\colon \mathcal{D} \to \mathcal{C}$ be given. We show that $\lim\colon \mathcal{C}^{\mathcal{D}}(F,G) \to \mathcal{C}(\lim F,\lim G)$ is continuous. Since $\mathcal{C}$ is tensored, there is a bijection

$$\mathrm{Hom}_{\mathcal{V}}(\mathcal{C}^{\mathcal{D}}(F,G),\mathcal{C}(\lim F,\lim G)) \cong \mathrm{Hom}_{\mathcal{V}}(\mathcal{C}^{\mathcal{D}}(F,G),\lim \mathcal{C}(\lim F,G-))$$

by Proposition B.21a). We define a morphism

$$g\colon \mathcal{C}^{\mathcal{D}}(F,G) \to \lim \mathcal{C}(\lim F,G-),$$

which under the above bijection is mapped to $\lim\colon \mathcal{C}^{\mathcal{D}}(F,G) \to \mathcal{C}(\lim F,\lim G)$. Let $v\colon \lim F \to F$ denote the limiting cone. For any object d of $\mathcal{D}$, define the morphism

$$g_d\colon \mathcal{C}^{\mathcal{D}}(F,G) \to \mathcal{C}(\lim F,G(d)), \quad \eta \mapsto \eta_d v_d$$

in $\mathcal{V}$. Then $(g_d)_d$ induces the desired map g.

Part b) is similar. □

Proposition B.23. *Let $\mathcal{C}$ be a $\mathcal{V}$-category, let $\mathcal{D}$ be a small $\mathcal{V}$-category and recall that the category of $\mathcal{V}$-functors from $\mathcal{D}$ to $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathcal{D}}$.*

a) If $\mathcal{C}$ is cotensored and cocomplete, then so is $\mathcal{C}^{\mathcal{D}}$.

b) If $\mathcal{C}$ is tensored and complete, then so is $\mathcal{C}^{\mathcal{D}}$.

Proof. For part a), assume that $\mathcal{C}$ is cotensored and cocomplete. Hence $\mathcal{C}^{\mathcal{D}}$ is also cotensored by Proposition B.19a). To show that $\mathcal{C}^{\mathcal{D}}$ is cocomplete, consider a small category $\mathcal{J}$ and a functor $F\colon \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$. Since $\mathcal{C}$ is cocomplete, the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ has all small colimits and they can be calculated objectwise. Thus the composite

$$\mathcal{J} \xrightarrow{F} \mathcal{C}^{\mathcal{D}} \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$$

has a colimit $X \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$, which in the object d of $\mathcal{D}$ is given by $X_d = \mathrm{colim}(\mathrm{ev}_d F)$, where ev_d denotes the functor which evaluates in d. Write $t_d\colon \mathrm{ev}_d F \to X_d$ for the colimiting cone. We show that X is a $\mathcal{V}$-functor, i.e. that $X\colon \mathcal{D}(d, d') \to \mathcal{C}(X_d, X_{d'})$ is continuous for all objects d, d' of $\mathcal{D}$, and hence that X is the colimit of F. Since $\mathcal{C}$ is cotensored, there is a bijection

$$\mathrm{Hom}_{\mathcal{V}}(\mathcal{D}(d, d'), \mathcal{C}(X_d, X_{d'})) \cong \mathrm{Hom}_{V}(\mathcal{D}(d, d'), \lim \mathcal{C}(\mathrm{ev}_d F-, X_{d'}))$$

by Proposition B.21b). To show that X is continuous, we define a morphism

$$g\colon \mathcal{D}(d, d') \to \lim \mathcal{C}(\mathrm{ev}_d F-, X_{d'})$$

in $\mathcal{V}$, which under the above bijection is mapped to $X\colon \mathcal{D}(d, d') \to \mathcal{C}(X_d, X_{d'})$. For any object j of $\mathcal{J}$, define the map

$$g_j\colon \mathcal{D}(d, d') \to \mathcal{C}(\mathrm{ev}_d F(j), X_d), \quad f \mapsto (t_{d'})_j \circ F(j)(f)$$

in $\mathcal{V}$. Then the maps $(g_j)_j$ induce the desired map g.

Part b) is shown similarly. □

C Model category theory

Model categories have been introduced by Quillen. Model categories as treated here correspond to what was called closed model categories in [16], but we require a stronger completeness axiom. A model category is a category $\mathcal{C}$ together with three distinguished classes of morphisms of $\mathcal{C}$, called fibrations, cofibrations and weak equivalences, which satisfies certain axioms. These axioms ensure that the localization (see C.11) of $\mathcal{C}$ with respect to the weak equivalences, the so-called homotopy category of $\mathcal{C}$, exists and provide an abstract framework to do homotopy theory, in a similar way as for topological spaces. A very readable introduction to model category theory is the paper by Dwyer and Spalinski [3]. For further investigation, the books [9] by Hovey and [8] by Hirschhorn are suggested. Another, more historical point of view to present model categories is chosen by Goerss and Schemmerhorn in the monograph [5]. They treat model categories in a way to build resolutions in a non-abelian setting and thus to do homotopical algebra.

Here, omitting most proofs, we summarize the basics of model category theory following [3].

C.1 Definition and elementary properties of a model category

The following terms will be used in the definition of a model category.

Definition C.1. Let $j\colon A \to B$, $p\colon X \to Y$ be morphisms in a category $\mathcal{C}$. We say that j has the *left lifting property* (LLP) with respect to p and that p has the *right lifting property* (RLP) with respect to j, if in any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r, "t"] & Y \end{tikzcd}, \tag{C.1}$$

there exists a lift $h\colon B \to X$, i.e. a map h such that $hj = s$ and $ph = t$.

Definition C.2. A morphism $f\colon X \to X'$ in a category $\mathcal{C}$ is called a *retract* of a morphism $g\colon Y \to Y'$ of $\mathcal{C}$, if f is a retract of g as objects in the category of morphisms $\mathrm{Mor}(\mathcal{C})$ of $\mathcal{C}$, i.e. if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & Y \arrow[r, "r"'] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{id}_{X'}"'] & Y' \arrow[r, "r'"] & X' \end{tikzcd}. \tag{C.2}$$

The following two lemmas show that isomorphisms are closed under retraction and that the left and right lifting properties of a map are preserved under retraction.

Lemma C.3. *Let g be an isomorphism in a category $\mathcal{C}$ and f a retract of g. Then f is also an isomorphism.*

Proof. By definition, there exists a commutative diagram as in (C.2) with $ri = \mathrm{id}_X$ and $r'i' = \mathrm{id}_{X'}$. By assumption, g has an inverse g^{-1}. One checks that $rg^{-1}i'$ is the inverse of f. □

Lemma C.4. *Let $\mathcal{C}$ be a category and f a retract of a morphism g in $\mathcal{C}$. If g has the RLP (resp. LLP) with respect to a morphism j (resp. p), then so does f.*

Proof. Consider a commutative diagram as in (C.2), expressing f as a retract of g. We assume that g has the RLP with respect to a morphism $j\colon A \to B$ and want to find a lift in any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & X' \end{tikzcd} .$$

By assumption, there exists a lift $h\colon B \to Y$ in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[r, "i"] & Y \arrow[d, "g"] \\ B \arrow[r] & X' \arrow[r, "i'"] & Y' \end{tikzcd} .$$

The desired lift is rh.

The case, where g has the LLP with respect to a morphism p is dual. $\square$

Definition C.5. A *model category* is a category $\mathcal{C}$ together with three classes of morphisms of $\mathcal{C}$, the class of *weak equivalences*, of *fibrations* and of *cofibrations*, each of which is closed under composition and contains all identity morphisms of $\mathcal{C}$, such that the five axioms **MC1-MC5** below hold. A morphism of $\mathcal{C}$ is called an *acyclic fibration* if it is both a weak equivalence and a fibration, it is called an *acyclic cofibration* if it is both a weak equivalence and a cofibration.

MC1: (completeness) Every functor from a small category to $\mathcal{C}$ has a limit and a colimit.

MC2: (two-out-of-three) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and if two out of the three morphisms f, g and gf are weak equivalences, then so is the third.

MC3: (retract) If f is a retract of a morphism g of $\mathcal{C}$ and g is a weak equivalence, a fibration or a cofibration, then so is f.

MC4: (lifting)

i) Every cofibration has the LLP with respect to all acyclic fibrations.

ii) Every fibration has the RLP with respect to all acyclic cofibrations.

MC5: (factorization) Any morphism f of $\mathcal{C}$ can be factored as

i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and as

ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

In [16], Quillen requires a closed model category to have only all finite limits and finite colimits. On the other hand, the axioms in Hovey's book [9] are slightly stronger. There the factorizations in **MC5** are part of the structure and assumed to be functorial.

Definition C.6. Let $\mathcal{C}$ be a category. For any non-negative integer n, denote by $[n]$ the category given by the ordinal $n+1 = \{0, 1, \cdots, n\}$ and let $\mathcal{C}^{[n]}$ be the category of functors from $[n]$ to $\mathcal{C}$. Let $d^1\colon [1] \to [2]$ be the functor given by the injective function which omits 1. Precomposing with d^1 defines a functor $(d^1)^*\colon \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}$. A *functorial factorization* in $\mathcal{C}$ is a functor $T\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ which is right inverse to $(d^1)^*$, i.e. such that $(d^1)^* \circ T$ is the identity.

Remark C.7. Since any isomorphism in a category is a retract of an identity morphism, it follows that in a model category, every isomorphism is a weak equivalence, a fibration and a cofibration.

Non-trivial examples of model categories are given in section D.3. The following example ensures the existence of model categories.

Example C.8. Any complete and cocomplete category $\mathcal{C}$ can be equipped with three model category structures by determing one of the classes of weak equivalences, fibrations and cofibrations to be the class of isomorphisms and the other two classes to consist of all morphisms of $\mathcal{C}$.

Any two of the three classes of maps in a model category structure determine the third one.

Proposition C.9. *Let $\mathcal{C}$ be a model category.*

i) The cofibrations in $\mathcal{C}$ are exactly the morphisms that have the LLP with respect to all acyclic fibrations.

ii) The fibrations in $\mathcal{C}$ are exactly the morphisms that have the RLP with respect to all acyclic cofibrations.

iii) The acyclic cofibrations in $\mathcal{C}$ are exactly the morphisms that have the LLP with respect to all fibrations.

iv) The acyclic fibrations in $\mathcal{C}$ are exactly the morphisms that have the RLP with respect to all cofibrations.

By axiom **MC1**, every model category has an initial object $\emptyset$ and a terminal object $*$.

Definition C.10. An object X of a model category $\mathcal{C}$ is called *fibrant*, if the unique morphism from X to the terminal object is a fibration. It is called *cofibrant*, if the unique morphism from the initial object to X is a cofibration.

The cofibrant and fibrant objects of a model category $\mathcal{C}$ will be used in the next section to construct the homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$.

C.2 The homotopy category of a model category

Definition C.11. Let W be a class of maps of a category $\mathcal{C}$. A category $\mathcal{C}\left[W^{-1}\right]$ together with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}\left[W^{-1}\right]$ is called the *localization of C with respect to W*, if γ maps morphisms of W to isomorphisms and satisfies the following universal property: Given any functor G from $\mathcal{C}$ to a category $\mathcal{D}$, which maps morphisms of W to isomorphisms, there exists a unique functor $F\colon \mathcal{C}\left[W^{-1}\right] \to \mathcal{D}$ such that $F\gamma = G$.

If the class W of maps of a category $\mathcal{C}$ is a set, then the localization of $\mathcal{C}$ with respect to W exists and can be obtained by formally inverting the maps of W as described in [1, Proposition 5.2.2]. But if W is a proper class, then in general the construction of $\mathcal{C}\left[W^{-1}\right]$ fails to be a category since the morphisms from an object to another one may form a proper class instead of a set.

We will see that if $\mathcal{C}$ is a model category, then the localization of $\mathcal{C}$ with respect to the class of weak equivalences W exists, and thus define the homotopy category $\mathrm{Ho}(\mathcal{C})$ to be the category $\mathcal{C}\left[W^{-1}\right]$. It will have the same objects as $\mathcal{C}$ and for any cofibrant-fibrant objects X, Y of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y)$ will be given by the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ modulo some equivalence relation.

C.2.1 Homotopy relations on morphisms

Definition C.12. Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$.

a) A *cylinder object* for X is a commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r] \arrow[rr, bend right, "\mathrm{id}_X + \mathrm{id}_X"'] & C(X) \arrow[r, "p"] & X \end{tikzcd},$$

where p is a weak equivalence.

b) A *path object* for X is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "(\mathrm{id}_X, \mathrm{id}_X)"'] & P(X) \arrow[r] & X \times X \end{tikzcd},$$

where i is a weak equivalence.

Definition C.13. Two morphisms $f, g\colon A \to X$ in a model category $\mathcal{C}$ are called ...

a) ... *left homotopic* (denoted $f \overset{l}{\sim} g$), if there exists a cylinder object $C(A)$ and a commutative diagram

$$\begin{tikzcd} A\coprod A \arrow[r] \arrow[rr, bend right, "f+g"'] & C(A) \arrow[r, "H"] & X \end{tikzcd}.$$

b) ... *right homotopic* (denoted $f \overset{r}{\sim} g$), if there exists a path object $P(X)$ and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[rr, bend right, "(f{,}g)"'] & P(X) \arrow[r] & X \times X \end{tikzcd}.$$

Proposition C.14. *Let $f, g\colon A \to X$ be two morphisms in a model category $\mathcal{C}$.*

a) If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $Hom_{\mathcal{C}}(A, X)$.

b) If X is fibrant and $f \overset{l}{\sim} g$, then $fh \overset{l}{\sim} gh$ for any morphism h with codomain A.

c) If X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on $Hom_{\mathcal{C}}(A, X)$.

d) If A is cofibrant and $f \overset{r}{\sim} g$, then $hf \overset{r}{\sim} hg$ for any morphism h with domain X.

e) If A is cofibrant and X is fibrant, then the equivalence relations $\overset{l}{\sim}$ and $\overset{r}{\sim}$ on $Hom_{\mathcal{C}}(A, X)$ agree.

If A is a cofibrant object and X a fibrant object of a model category $\mathcal{C}$, then we denote the identical equivalence relations $\overset{l}{\sim}$ and $\overset{r}{\sim}$ by $\sim$ and write $\pi(A, X)$ for the set of equivalence classes $\mathrm{Hom}_{\mathcal{C}}(A, X)/\sim$.

C.2.2 Construction of the homotopy category

Let $\mathcal{C}$ be a model category. Using a suitable Axiom of Choice and axiom **MC5**i), choose for any object X of $\mathcal{C}$ a factorization

$$\emptyset \longrightarrow QX \xrightarrow{p_X} X$$

of the unique map from the initial object $\emptyset$ of $\mathcal{C}$ to X, such that QX is cofibrant and p_X is an acyclic fibration. If X is already cofibrant, let $QX = X$. Dually, choose for any object X of $\mathcal{C}$ an acyclic cofibration $i_X\colon X \to RX$ with RX fibrant such that $RX = X$ if X is fibrant.

Definition C.15. The homotopy category $\mathrm{Ho}(\mathcal{C})$ is the category with the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \pi(RQX, RQY)$ for objects X, Y.

Composition in $\mathrm{Ho}(\mathcal{C})$ is well-defined by Proposition C.14.

We construct a functor $\gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, which will turn out to be the localization of $\mathcal{C}$ with respect to the weak equivalences. For any morphism $f\colon X \to Y$ in $\mathcal{C}$, there exists a lift $Qf\colon QX \to QY$ in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "f \circ p_X"] & Y \end{tikzcd} ,$$

by axiom **MC4**i). Dually, choose for any morphism $f\colon X \to Y$ a morphism $Rf\colon RX \to RY$ such that $Rf \circ i_X = i_Y \circ f$.

Proposition C.16. *For any morphism $f\colon X \to Y$ in $\mathcal{C}$, the class $[R(Qf)]$ in $\pi(RQX, RQY)$ does not depend on the chosen lifts.*

Now, let $\gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the functor which is the identity on objects and sends a morphism $f\colon X \to Y$ in $\mathcal{C}$ to the class $[R(Qf)]$ in $\pi(RQX, RQY)$.

Theorem C.17. *The functor $\gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is the localization of the model category $\mathcal{C}$ with respect to the weak equivalences.*

C.2.3 Derived functors and Quillen pairs

To compare model categories, we will introduce Quillen pairs. These are adjoint functors between model categories, which satisfy a condition to ensure that they induce functors between the corresponding homotopy categories. The induced functors will be a special case of derived functors.

Definition C.18. Let $\mathcal{C}$ be a model category with homotopy category $(\gamma, \mathrm{Ho}(\mathcal{C}))$ and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor to a category $\mathcal{D}$.

a) A *left derived functor* LF for F is a right Kan extension of F along γ.

b) A *right derived functor* RF for F is a left Kan extension of F along γ.

Definition C.19. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories.

a) A *total left derived functor* $\mathbf{L}F$ for F is a left derived functor for the composite $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$.

b) A *total left derived functor* $\mathbf{R}F$ for F is a right derived funtor for the composite $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$.

c) If F is a left adjoint functor, which preserves cofibrations (i.e. takes cofibrations to cofibrations) and which preserves acyclic cofibrations, then F is called a *left Quillen functor*.

d) If $U\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint functor, which preserves fibrations and acyclic fibrations, then U is called a *right Quillen functor*.

e) If (F, U) is a pair of adjoint functors such that F preserves cofibrations and U preserves fibrations, then (F, U) is called a *Quillen pair*.

Remark C.20. Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U$ be an adjunction between model categories. Then the following statements are equivalent:

i) The functor F is a left Quillen functor.

ii) The functor U is a right Quillen functor.

iii) The pair (F, U) is a Quillen pair.

Definition C.21. Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U$ is a Quillen pair such that for each cofibrant object A of $\mathcal{C}$ and each fibrant object X of $\mathcal{D}$, a map $f\colon A \to U(X)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $F(A) \to X$ is a weak equivalence in $\mathcal{D}$. Then (F, U) is called a *pair of Quillen equivalences*, the functor F is called a *left Quillen equivalence* and U is called a *right Quillen equivalence*.

Proposition C.22. *If $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U$ is a Quillen pair, then the total left derived functor $\boldsymbol{L}F$ of F and the total right derived functor $\boldsymbol{R}U$ of U exist and they form an adjunction $\boldsymbol{L}F\colon \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D})\colon \boldsymbol{R}U$. If furthermore (F, U) is a pair of Quillen equivalences, then $\boldsymbol{L}F$ and $\boldsymbol{R}U$ are inverse equivalences of categories.*

D Cofibrantly generated model categories

Roughly speaking, a cofibrantly generated model category (see D.16) is a model category $\mathcal{C}$ such that there exists a set I of cofibrations and a set J of acyclic cofibrations, which generates the class of all cofibrations and the class of all acyclic cofibrations, respectively. Furthermore, the sets I and J are assumed to permit the small object argument (see D.13), a construction which provides for every morphism of $\mathcal{C}$ a factorization into maps with lifting properties. This factorization will be functorial (see C.6). We point out two more advantages of cofibrantly generated model categories. It's easier to prove that an adjunction is a Quillen pair and under certain assumptions, one can transport the model category structure along left adjoints. In practice, most model categories are cofibrantly generated.

Here, we summarize the basics about cofibrantly generated model categories following mainly Hovey's exposition [9], where also the omitted proofs can be found. In order to define cofibrantly generated model categories in §D.2, we introduce the notions of small objects and relative I-cell complexes and prove the small object argument in §D.1. The main result of §D.2 is the generalized transport Theorem D.20, which enables one to lift a cofibrantly generated model category structure from a category $\mathcal{C}$ to a category $\mathcal{D}$ not only along one left adjoint functor, but along a set of left adjoints from $\mathcal{C}$ to $\mathcal{D}$. In §D.3, the categories of spaces **Top**, $\mathcal{K}$ and $\mathcal{U}$ will be equipped with the Quillen model structure and provide examples of cofibrantly generated model categories.

Both, in the statement of the small object argument and in the definition of a cofibrantly generated model category, the following terminology concerning lifting properties will be used.

Definition D.1. Let I be a class of maps of a category $\mathcal{C}$

a) A morphism of $\mathcal{C}$ is called *I-injective*, if it has the RLP with respect to every map in I. The class of I-injective maps is denoted by I-inj.

b) A morphism of $\mathcal{C}$ is called *I-projective*, if it has the LLP with respect to every map in I. The class of I-projective maps is denoted by I-proj.

c) The class (I-inj)-proj is denoted by I-cof and its elements are called *I-cofibrations*.

Remark D.2. Given a class I of maps of a category $\mathcal{C}$, one concludes directly from the definitions, that $I \subset I$-cof. Furthermore, if J is a class of maps of $\mathcal{C}$ with $I \subset J$, then I-inj $\supset J$-inj and I-proj $\supset J$-proj, hence I-cof $\subset J$-cof.

D.1 Small objects and relative I-cell complexes

For a set I of maps in a cocomplete category $\mathcal{C}$, where the domains of the maps of I satisfy some smallness condition, the small object argument will provide a functorial factorization of any map in $\mathcal{C}$ into a relative I-cell complex (see D.10) followed by a map in I-inj. Roughly speaking, a relative I-cell complex is an object which is obtained by attaching maps in I. Thus, it will follow that any relative I-cell complex is a I-cofibration.

In order to state the rather technical definition of small objects, we first recall some terms from set theory. For further background, the reader is referred to [10]. A set T is *transitive*, if every element of T is a subset of T. A linear ordering of a set P is a *well-ordering* if every nonempty subset of P has a least element. Now, an *ordinal* is a set which is transitive and well-ordered by the binary relation $\in$. On the class of all ordinals, the binary relation $\in$ is a well-ordering, which we denote by $<$. Every ordinal β has a *successor ordinal* $\beta + 1$ and if an ordinal α is not a successor ordinal $\beta + 1$, then α is called a *limit ordinal*. In particular, an ordinal λ is a poset and thus a category with objects the elements $\beta \in \lambda$ and with exactly one morphism from β to β' if $\beta \leq \beta'$ and with $\mathrm{Hom}_\lambda(\beta, \beta') = \emptyset$ if $\beta > \beta'$.

Definition D.3. Let $\mathcal{C}$ be a cocomplete category, let D be a class of morphisms in $\mathcal{C}$ and let λ be an ordinal.

a) A *λ-sequence* in $\mathcal{C}$ is a functor $X\colon \lambda \to \mathcal{C}$, i.e. a diagram

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda),$$

such that for each non-zero limit ordinal $\gamma \in \lambda$, the induced map $\mathrm{colim}_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism.

b) If $\lambda > 0$ and X is a λ-sequence, then the natural map $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ is called the *composition* of X.

c) A *λ-sequence of maps in D* is a λ-sequence X such that for every $\beta + 1 < \lambda$, the map $X_\beta \to X_{\beta+1}$ is in D.

d) A map f in $\mathcal{C}$ is called a *transfinite composition of maps in D*, if f is the composition of a λ'-sequence of maps in D for some ordinal $\lambda' > 0$.

Remark D.4. Let $\mathcal{C}$ be a cocomplete category and let λ be an ordinal. If X is a λ-sequence in $\mathcal{C}$, then X preserves all small, non-empty colimits.

Recall that the *cardinality* $|T|$ of a set T is the least ordinal such that there exists a bijection $|T| \to T$. A *cardinal* is an ordinal κ with $\kappa = |\kappa|$.

Definition D.5. Let κ be a cardinal. A limit ordinal α is said to be *κ-filtered*, if $\sup T < \alpha$ for all subsets $T \subset \alpha$ with $|T| \leq \kappa$.

Remark D.6. Let κ be a cardinal. If κ is finite, then every non-zero limit ordinal α is κ-filtered. If κ is infinite, then the least cardinal greater than κ is κ-filtered.

Definition D.7. Let $\mathcal{C}$ be a cocomplete category, let D be a class of morphisms of $\mathcal{C}$ and let κ be a cardinal.

a) An object A of $\mathcal{C}$ is called *κ-small relative to D*, if for all κ-filtered ordinals λ and all λ-sequences X of maps in D, the induced map

$$\mathrm{colim}_{\beta<\lambda} \mathrm{Hom}_{\mathcal{C}}(A, X_\beta) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{colim}_{\beta<\lambda} X_\beta) \qquad \text{(D.1)}$$

is a bijection.

b) An object A of $\mathcal{C}$ is called *small relative to D*, if A is κ'-small relative to D for some cardinal κ'.

c) An object A of $\mathcal{C}$ is called *finite relative to D*, if A is κ'-small relative to D for some finite cardinal κ'.

Example D.8. If $\mathcal{C}$ is the category of sets and D is the class of all morphisms of $\mathcal{C}$. Then any set A is $|A|$-small relative to D. Furthermore, the object A of $\mathcal{C}$ is finite relative to D if and only if it is a finite set.

The following remark motivates the consideration of not just one or all big enough limit ordinals, but of all κ-filtered ordinals in the previous definition.

Remark D.9. Let $\mathcal{C}$ be a cocomplete category and let D be a class of morphisms of $\mathcal{C}$.

a) If κ, κ' are cardinals with $\kappa < \kappa'$, then every κ'-filtered ordinal is also κ-filtered and hence every object A which is κ-small relative to D is also κ'-small relative to D.

b) Suppose $\{A_\iota\}_\iota$ is a set of small objects relative to D, say A_ι is κ_ι-small relative to D. Then there exists a cardinal κ such that each object A_ι is κ-small relative to D. Indeed, set $\kappa = \bigcup_\iota \kappa_\iota$ and use a).

c) Suppose that D contains all identity morphisms. If κ is a cardinal and A an object of $\mathcal{C}$ such that for every limit ordinal $\lambda > \kappa$ and every λ-sequence X of maps in D the induced map as in (D.1) is a bijection, then for the least infinite ordinal ω and every ω-sequence Y of maps in D, the induced map

$$\operatorname{colim}_{\beta<\omega} \operatorname{Hom}_{\mathcal{C}}(A, Y_\beta) \to \operatorname{Hom}_{\mathcal{C}}(A, \operatorname{colim}_{\beta<\omega} Y_\beta)$$

is a bijection. Indeed, conversely if Y is a ω-sequence of maps in D, where the induced map as above is not a bijection, then consider $\lambda = \kappa + \omega$ and set $X_\beta = Y_0$ for $\beta \leq \kappa$ and $X_\beta = Y_i$ for $\beta = \kappa + i$ to obtain a λ-sequence X of maps in D, where the induced map as in (D.1) is not a bijection.

Definition D.10. Let I be a set of maps in a cocomplete category $\mathcal{C}$.

a) A *relative I-cell complex* is a transfinite composition of pushouts of maps in I. That is, a map $f \colon A \to B$ is a relative I-cell complex, if there exists an ordinal $\lambda > 0$ and a λ-sequence X in $\mathcal{C}$ such that f is the composition of X and such that for every $\beta + 1 < \lambda$, there exists a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

with $g_\beta \in I$. The class of relative I-cell complexes is denoted by I-cell.

b) An object A of $\mathcal{C}$ is called an *I-cell complex* if the map $\emptyset \to A$ is in I-cell.

Lemma D.11. *Let I be a set of maps in a cocomplete category $\mathcal{C}$. Then*

a) I-cell $\subset$ I-cof,

b) I-cell is closed under transfinite composition and

c) any pushout of coproducts of maps of I is in I-cell.

Definition D.12. Let $\mathcal{C}$ be a cocomplete category. A set I of maps of $\mathcal{C}$ is said to *permit the small object argument*, if the domain of any element of I is small relative to I-cell (see D.7 and D.10).

Theorem D.13 (The small object argument)**.** *Let $\mathcal{C}$ be a cocomplete category and I a set of maps of $\mathcal{C}$, which permits the small object argument. Then there exists a functorial factorization of any map in $\mathcal{C}$ into a relative I-cell complex followed by an I-injective map.*

We will prove the small object argument, since the actual construction of the factorization may be even more important than the existence result itself. Furthermore, the role of the smallness assumption and of relative I-cell complexes will be clarified.

Proof of Theorem D.13. Using the smallness assumption on I and Remark D.9b), choose a cardinal κ, such that the domain of any map of I is κ-small relative to I-cell. Fix a κ-filtered ordinal λ.

To define a functorial factorization $T\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ on objects, i.e. in a morphism $f\colon X \to Y$ of $\mathcal{C}$, we will construct a λ-sequence Z in $\mathcal{C}$ with $Z_0 = X$ and a natural map $p_\beta\colon Z_\beta \to Y$ factoring f as $Z_0 \to Z_\beta \xrightarrow{p_\beta} Y$, $\beta < \lambda$. For every $\beta + 1 < \lambda$, the map $Z_\beta \to Z_{\beta+1}$ will be a pushout of coproducts of maps of I and hence in I-cell by Lemma D.11c). Therefore, the composition $i\colon Z_0 \to \operatorname{colim} Z$ will also be a relative I-cell complex by Lemma D.11b). From the construction of the λ-sequence Z and the smallness assumption on I it will follow that the induced map $p := \operatorname{colim}_{\beta<\lambda} p_\beta$ is in I-inj. Thus, the composite $p \circ i$ will be the desired factorization of f.

Let's define Z and $\{p_\beta\}_{\beta<\lambda}$ by transfinite induction. Set $Z_0 := X$ and $p_0 := f$. In a limit ordinal $\alpha > 0$, set $Z_\alpha := \operatorname{colim}_{\beta<\alpha} Z_\beta$ and $p_\alpha := \operatorname{colim}_{\beta<\alpha} p_\beta$. Considering a successor ordinal $\beta + 1 < \lambda$, let $S(\beta + 1)$ be the set of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & Z_\beta \arrow[d, "p_\beta"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $g \in I$. For any square $s \in S(\beta+1)$, let $g_s\colon A_s \to B_s$ denote the map on its the left-hand side. Now, define $Z_{\beta+1}$ by the pushout square

$$\begin{tikzcd} \coprod_s A_s \arrow[r] \arrow[d, "\coprod_s g_s"'] & Z_\beta \arrow[d] \\ \coprod_s B_s \arrow[r] & Z_{\beta+1} \end{tikzcd}$$

and let $p_{\beta+1}\colon Z_{\beta+1} \to Y$ be the map induced by p_β. This concludes the construction of Z and the natural map p_β, $\beta < \lambda$. To show that the induced map $p\colon \operatorname{colim} Z \to Y$ is I-injective, consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & \operatorname{colim} Z \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $g \in I$. By the choice of κ, the domain A of g is κ-small relative to I-cell. Thus, the map $A \to \operatorname{colim} Z$ factors as $A \to Z_\beta \to \operatorname{colim} Z$ for some $\beta < \lambda$. We obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & Z_\beta \arrow[r] \arrow[d, "p_\beta"] & Z_{\beta+1} \arrow[r] \arrow[d, "p_{\beta+1}"] & \operatorname{colim} Z \arrow[d, "p"] \\ B \arrow[r] & Y \arrow[r, "="] & Y \arrow[r, "="] & Y \end{tikzcd} .$$

In particular, the square on the left-hand side is in $S(\beta+1)$. One checks that the composite $B \to \coprod_s B_s \to Z_{\beta+1} \to \operatorname{colim} Z$ gives the desired lift.

To define the functor $T\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ on morphisms, one has to associate to a given commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

a map $w\colon \operatorname{colim} Z \to \operatorname{colim} Z'$ making the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "u"'] & \operatorname{colim} Z \arrow[r, "p"] \arrow[d, "w"] & Y \arrow[d, "v"] \\ X' \arrow[r, "i'"] & \operatorname{colim} Z' \arrow[r, "p'"] & Y' \end{tikzcd}$$

commute. One defines by transfinite induction a natural map $w_\beta\colon Z_\beta \to Z'_\beta$ with $p'_\beta w_\beta = v p_\beta$ for $\beta < \lambda$ and with $w_0 = u$, such that setting $w := \operatorname{colim}_{\beta<\lambda} w_\beta$ makes T into a functor. □

Corollary D.14. *Let $\mathcal{C}$ be a cocomplete category and I a set of maps of $\mathcal{C}$, which permits the small object argument. Then for any I-cofibration $f\colon A \to B$, there exists a relative I-cell complex $g\colon A \to C$ such that f is a retract of g by a retraction of the form*

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_B"'] & C \arrow[r, "r"] & A \end{tikzcd} .$$

Proposition D.15. *Let $\mathcal{C}$ be a cocomplete category and I a set of maps of $\mathcal{C}$, which permits the small object argument. Then any object which is small relative to I-cell is also small relative to I-cof.*

D.2 Definition and properties of a cofibrantly generated model category

Definition D.16. A model category $\mathcal{C}$ is said to be *cofibrantly generated* if there exist sets I and J of maps of $\mathcal{C}$, which permit the small object argument, and such that

i) the class of acyclic fibrations is I-inj and

ii) the class of fibrations is J-inj.

Such sets I and J are called a set of *generating cofibrations* and a set of *generating acyclic cofibrations*, respectively.

Proposition D.17. *Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J.*

a) The class of cofibrations is I-cof.

b) Every cofibration is a retract of a relative I-cell complex.

c) The cofibrant objects are the retracts of I-cell complexes.

d) The domain of any map of I is small relative to the cofibrations.

e) The class of acyclic cofibrations is J-cof.

f) Every acyclic cofibration is a retract of a relative J-cell complex.

g) The domain of any map of J is small relative to the acyclic cofibrations.

For cofibrantly generated model categories it's easier to show that a pair of adjoint functors is a Quillen pair.

Lemma D.18. *Let $\mathcal{C}$ be a cofibrantly generated model category and let $\mathcal{D}$ be a model category. A left adjoint functor $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor if and only if F takes generating cofibrations to cofibrations and generating acyclic cofibrations to acyclic cofibrations.*

The following recognition theorem can be used to equip a category with a cofibrantly generated model category structure.

Theorem D.19. *Let $\mathcal{C}$ be a complete, cocomplete category and let I and J be sets of maps of $\mathcal{C}$. Let W be a class of maps of $\mathcal{C}$, which is closed under composition and contains all identity morphisms. There exists a cofibrantly generated model category structure on $\mathcal{C}$ with generating cofibrations I, generating acyclic cofibrations J and weak equivalences W if and only if the following conditions hold:*

i) W has the two-out-of-three property and is closed under retraction,

ii) I and J permit the small object argument,

iii) J-cell $\subset W \cap I$-cof,

iv) I-inj $\subset W \cap J$-inj,

v) either $W \cap I$-cof $\subset J$-cof or $W \cap J$-inj $\subset I$-inj.

The transport theorem below describes how to lift a cofibrantly generated model category structure from a category to another one along left adjoints.

Theorem D.20. *Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{D}$ be a complete, cocomplete category. Given a set of adjunctions $\{F_\iota\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U_\iota\}_\iota$, write $FI := \bigcup_\iota \{F_\iota(f); f \in I\}$ and $FJ := \bigcup_\iota \{F_\iota(f); f \in J\}$. Suppose that*

i) the sets FI and FJ permit the small object argument (see D.12) and

ii) for all ι, the functor U_ι takes relative FJ-cell complexes (see D.10) to weak equivalences.

Then there exists a cofibrantly generated model category structure on $\mathcal{D}$ with generating cofibrations FI, generating acyclic cofibrations FJ and with weak equivalences and fibrations the maps of $\mathcal{D}$ which by every U_ι are taken to weak equivalences and fibrations in $\mathcal{C}$, respectively.

This theorem in the case, where one considers only one adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ is attributed to D. M. Kan in [8, Theorem 11.3.2], but the author has not found the general statement in the literature.

Proof of Theorem D.20. Let W denote the class of maps of $\mathcal{D}$, which by every U_ι are taken to weak equivalences in $\mathcal{C}$. We apply Theorem D.19. The class W contains all identity morphisms, is closed under composition, has the two-out-of-three property and is closed under retraction by functoriality and the corresponding properties of the class of weak equivalences in $\mathcal{C}$.

The sets FI and FJ permit the small object argument by assumption.

To check that the conditions iii)-v) of Theorem D.19 are satisfied, note that for any ι, any map i in $\mathcal{C}$ and any map p in $\mathcal{D}$, the lifting problems

$$\begin{tikzcd} F_\iota(A) \arrow[r] \arrow[d, "F_\iota(i)"'] & X \arrow[d, "p"] \\ F_\iota(B) \arrow[r] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & U_\iota(X) \arrow[d, "U_\iota(p)"] \\ B \arrow[r] & U_\iota(Y) \end{tikzcd}$$

are equivalent by adjointness. One deduces that a map p of $\mathcal{D}$ is in FI-inj (resp. FJ-inj) if and only if for all ι the map $U_\iota(p)$ is in I-inj (resp. J-inj). Thus from I-inj $\subset$ J-inj follows that FI-inj $\subset$ FJ-inj and therefore that FJ-cof $\subset$ FI-cof.

Now, iii) FJ-cell $\subset W \cap FI$-cof holds by assumption ii) and since every relative FJ-cell complex is in FJ-cof by Lemma D.11a). For iv) FI-inj $\subset$ $W \cap FJ$-inj, we are left to show that every FI-injective map is in W. If a map p is in FI-inj, then for all ι the map $U_\iota(p)$ is in I-inj and thus is an acyclic fibration and in particular a weak equivalence in $\mathcal{C}$. Hence p is in W. We conclude the proof by showing v) $W \cap FJ$-inj $\subset$ FI-inj. If p is a map in $W \cap FJ$-inj, then for all ι the map $U_\iota(p)$ is in W and in J-inj. Thus the map $U_\iota(p)$ is an acyclic fibration, i.e. $U_\iota(p) \subset I$-inj. It follows that p is in FI-inj. ☐

Remark D.21. Given a set of adjunctions $\{F_\iota \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon U_\iota\}_\iota$ between a cofibrantly generated model category $\mathcal{C}$ and a complete, cocomplete category $\mathcal{D}$, use the notations FI and FJ as in Theorem D.20. Suppose that for each ι, the functor U_ι

i) takes relative FI-cell complexes to cofibrations,

ii) takes relative FJ-cell complexes to acyclic cofibrations,

iii) preserves the colimit of any λ-sequence of maps in FI-cell and

iv) preserves the colimit of any λ-sequence of maps in FJ-cell,

then the conditions i) and ii) of Theorem D.20 are satisfied.

Proof. Condition ii) of Theorem D.20 holds, since acyclic cofibrations are in particular weak equivalences.

Concerning condition i), we show that FI permits the small object argument. Let A be the domain of a map in I. We have to prove that for any ι, the object $F_\iota(A)$ is small relative to FI-cell. By Proposition D.15, there is a cardinal κ such that A is κ-small relative to I-cof. Let λ be a κ-filtered ordinal and $X\colon \lambda \to \mathcal{D}$ a λ-sequence of maps in FI-cell. Note that by adjointness, there is a commutative diagram

$$\begin{tikzcd} \operatorname{colim}_{\beta<\lambda} \operatorname{Hom}_{\mathcal{D}}(F_\iota(A), X_\beta) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}_{\mathcal{D}}(F_\iota(A), \operatorname{colim} X) \arrow[d, "\cong"] \\ \operatorname{colim}_{\beta<\lambda} \operatorname{Hom}_{\mathcal{C}}(A, U_\iota(X_\beta)) \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(A, U_\iota(\operatorname{colim} X)) \end{tikzcd} .$$

Thus it's enough to show that the map in the bottom is an isomorphism. But this holds by the choice of κ, since $U_\iota(\operatorname{colim} X) = \operatorname{colim}_{\beta<\lambda} U_\iota(X_\beta)$ by assumption iii) and since $U_\iota \circ X\colon \lambda \to \mathcal{C}$ is a λ-sequence of maps in I-cof by the assumptions iii) and i).

Similarly, one shows that FJ permits the small object argument. $\square$

D.3 Examples: Subcategories of Top

Note that each of the categories **Top** of topological spaces, $\mathcal{K}$ of k-spaces and $\mathcal{U}$ of compactly generated spaces contains the unit interval $I = [0,1]$, the n-disk D^n and its boundary S^{n-1}, $n \geq 0$. By abuse of notation we denote the set of inclusions $\left\{S^{n-1} \to D^n\right\}_{n\geq 0}$ again by I. We write J for the set of inclusions $\{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$. We will see that each category of **Top**, $\mathcal{K}$ and $\mathcal{U}$ is a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. To describe the weak equivalences and fibrations, we recall the following two definitions.

Definition D.22. A map f in **Top** is called a *Serre fibration* if it is in J-Inj.

Remark D.23. Using that for any $n \geq 0$, the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic, one deduces that a map f is a Serre fibration if and only if it has the RLP with respect to any inclusion $X \times \{0\} \cup A \times I \to X \times I$, where (X, A) is a CW-pair.

Definition D.24. A map $f\colon X \to Y$ in **Top** is called a *weak homotopy equivalence* if either both X and Y are empty or if both are non-empty and for every point $x \in X$ and every $n \geq 0$, the morphism $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ between homotopy groups is an isomorphism.

Theorem D.25. *The category **Top** admits a cofibrantly generated model category structure (called* Quillen model structure of **Top***) with generating cofibrations I, generating acyclic cofibrations J, weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations. Every object in **Top** is fibrant and the cofibrant objects are the retracts of I-cell complexes.*

Theorem D.25 is proved in [9] using the recognition Theorem D.19. One step of the proof is to show the following properties of **Top**.

Lemma D.26. *Call a closed embedding $f\colon X \to Y$ in **Top** a* closed T_1 embedding*, if every point in $Y \backslash f(X)$ is closed as a point in Y.*

a) Every topological space is small relative to the embeddings.

b) Closed T_1 embeddings are closed under pushouts and tranfinite composition.

*c) Any I-cofibration in **Top** is a closed T_1 embedding.*

d) Inclusions of deformation retracts are closed under pushouts.

*e) Let λ be a non-zero ordinal and $X\colon \lambda \to$ **Top** a λ-sequence of closed T_1 embeddings that are also weak equivalences, then the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a weak equivalence.*

We only provide a relatively short, geometric proof of the hardest step in showing Theorem D.25, which is the following lemma.

Lemma D.27. *A map $p\colon X \to Y$ in **Top**, which is both a Serre fibration and a weak homotopy equivalence, is in I-Inj.*

Proof. Consider a lifting problem

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ D^{n+1} \arrow[r, "g"] & Y \end{tikzcd} .$$

If $n = -1$, then the disk D^0 is mapped to a point $y \in Y$. We have to find a point in X which is mapped to y by p. Since p is a weak homotopy equivalence, there exists a point $x \in X$ such that $p(x)$ lies in the same path component of Y as y. Let H be a path from $p(x)$ to y. Since p is a Serre fibration, there exists a lift $\tilde{H}$ in

$$\begin{tikzcd} D^0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^0 \times I \arrow[r, "H"] & Y \end{tikzcd} .$$

The endpoint of the path $\tilde{H}$ is mapped to y by p and thus provides the solution of our lifting problem.

If $n \geq 0$, factor the inclusion of the sphere in the disk over the cylinder and the cone as $S^n \to S^n \times I \to CS^n \cong D^{n+1}$ and consider the commutative diagram

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[d] & & X \arrow[d, "p"] \\ S^n \times I \arrow[r] & D^{n+1} \arrow[r, "g"] & Y \end{tikzcd} . \tag{D.2}$$

By Remark D.23 applied to the CW-pair $(S^n, \emptyset)$ it has a lift h, since p is a Serre fibration. But since h is not necessarily constant on the top of the cylinder, it does not directly induce a solution of our lifting problem. We will deform it to a lift in (D.2) which is constant on $S^n \times \{1\}$. Under the composite $S^n \times I \to D^{n+1} \xrightarrow{g} Y$, the top $S^n \times \{1\}$ is mapped to a point $y \in Y$. Hence the image of the restricted map $h| : S^n \times \{1\} \to X$ lies in the fiber $F := p^{-1}\{y\}$. Denote by $x \in F$ the point to which $h|$ maps the basepoint of the sphere. Since p is a weak homotopy equivalence, it follows by the long exact sequence in homotopy of a Serre fibration, that $\pi_n(F, x)$ is trivial. Therefore, there exists a homotopy $H \colon S^n \times \{1\} \times I \to F$ from $h|$ to the constant map c_x. This map H and the composites $S^n \times \{0\} \times I \to S^n \xrightarrow{f} X$, $S^n \times I \times \{0\} \to S^n \times I \xrightarrow{h} X$ and $(S^n \times I) \times I \to S^n \times I \to D^{n+1} \xrightarrow{g} Y$ provide a commutative square

$$\begin{tikzcd}
(S^n \times \{1\} \cup S^n \times \{0\}) \times I \cup S^n \times I \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
S^n \times I \times I \arrow[r] & Y
\end{tikzcd}.$$

In this diagram, there exists a lift $\tilde{H}$ by Remark D.23 applied to the CW-pair $(S^n \times I, S^n \times \{1\} \cup S^n \times \{0\})$. The restricted map $\tilde{H}|_{S^n \times I \times \{1\}}$ is another lift in (D.2), but which satisfies $\tilde{H}|_{S^n \times \{1\} \times \{1\}} = c_x$. It therefore induces the solution $D^{n+1} \to X$ of the original lifting problem. $\square$

Using that the inclusion functor $\mathcal{K} \to$ **Top** preserves colimits, the recognition Theorem D.19 is applied in [9] to equip the category of k-spaces with the following model category structure.

Theorem D.28. *The category $\mathcal{K}$ of k-spaces admits a cofibrantly generated model category structure (called* Quillen model structure of $\mathcal{K}$*) with generating cofibrations I and generating acyclic cofibrations J. A map in $\mathcal{K}$ is a cofibration (fibration, weak equivalence) if and only if it is so in* ***Top****. The inclusion functor $\mathcal{K} \to$* ***Top*** *is a left Quillen equivalence.*

Since the inclusion functor $\mathcal{U} \to \mathcal{K}$ preserves certain pushouts and certain directed colimits, the model category structure of $\mathcal{K}$ can be transported along the left adjoint functor $wH \colon \mathcal{K} \to \mathcal{U}$ to the category of compactly generated spaces by applying Theorem D.20.

Theorem D.29. *The category $\mathcal{U}$ admits a cofibrantly generated model category structure (called* Quillen model structure of $\mathcal{U}$*) with generating cofibrations I and generating acyclic cofibrations J. A map in $\mathcal{U}$ is a cofibration (fibration, weak equivalence) if and only if it is so in $\mathcal{K}$. The inclusion functor $\mathcal{U} \to \mathcal{K}$ is a right Quillen equivalence.*

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