Equivalenace of The Little Disk and Cacti operads

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Abstract

The subject of this thesis is to study the Little Disk operad and the Cacti operad and show that they are equivalent as operads as presented by Kaufmann in the article [Kau05]. In doing so, we go through a preliminary study of operads, what it means for them to be equivalent and the problems involved. We introduce and use the Little Disk in the process. We furthermore introduce and show results about the recognition principle of Fiedorowicz that is used to compare operads up against the Little Cube operad via a "ziq-zaq" through B_∞ operads. We introduce and study the Cacti operad in detail while providing the means to finally apply the recognition principle. Throughout the thesis we will be elaborate on the graphical structures involved. This is both to fertilize the understanding of, but also to embrace the mathematical ideas and metaphors in, the subject.

Resumé

Emnet for nærværende speciale er at undersøge Lille Disk operaden og Kaktus operaden samt vise, at de er ækvivalente som operader, som det er præsenteret af Kaufmann i artiklen [Kau05]. I den forbindelse gennemgår vi indledende studier af operader, hvad det vil sige, at operader er ækvivalente, og de problemer, der er involveret i det. Vi introducerer og anvender Lille Disk operaden i den proces. Derudover introducerer vi og viser resultater om "genkendelsesprincipet" af Feidorowicz, som der anvendes til at måle operader om imod Lille Kube operaden via en "ziq-zaq" gennem B_∞ operader. Vi introducerer Kaktus operaden i detaljer, mens vi fremstiller de betingelser, vi har brug for, for endeligt at kunne anvende "genkendelsesprincippet". Gennem hele specialet er der fokus på den grafiske fremstilling af emnet. Det er for både at understøtte forståelsen af, men også for at hylde de matematiske ideer og metaforer iboende i emnet.

Contents

1 Introduction

1.1 Motivation

The goal for this thesis have been to show that the Little Disk operad and the Cacti operad are equivalent. The spine of the study have been Ralph M. Kaufmann's article *On several varieties of cacti and their relations* – especially the section 3.

In the process of undergoing this study I have found the need to study and get comfortable with the general notion of operads as given by May, cf. [May97]. But also a number of other mathematical preliminaries have I in the process felt the desire to study and understand well before continuing. And luckily this topic has allowed me to do so. As such one can say that a general theme throughout this thesis is to straighten out facts and understanding, visualizing presenting the mathematical ideas and metaphors that are in play. And this is exactly what I find most enjoyable and fascinating aspect of mathematics. If we could speak about the *ontology* of mathematical objects, then that is what I lust for with respect to mathematics, and it has been plentifully present in this study and will therefore also be plentiful present in this thesis.

But that "only" deals with some almost aesthetic motivations for doing this study. Natural questions to ask is "what are operads good for?" and "what can equivalence of Cacti and the Little Disk operad be used for?". They give rise to the notion of an *algebra over en operad*. In short this means that we can recover algebraic structure. For instance one of the first examples of operads is the commutative operad often called *Comm* gives us plain old commutative associative algebra with unit.

The main interest to algebraic topology, as an example, is how the Little Disk operad acts on the based double loop space, and in that way one can recover information about its structure. The Cacti operad of Voronov have been used to make a structure on the free loop space. This is done via its connection to (a flavor of) the Little Disk operad. Equivalence of operads naturally allow us to compare the objects they involve.

1.2 Structure

Roughly speaking, I have divided the work into two main sections. The first entitled *Preliminaries* and the second *The Cacti Operad*.

The first section contains all of the preliminaries to deal with the second section. One motivation for this is, as mentioned, to compare the Little Disk operad with the Cacti operad. Another is to introduce and discuss the Cacti operad while measuring it up against the Little Disk operad. In this thesis those to motives are equal. It is not to be thought of as a downgrade of the Little Disk operad.

As mentioned, we measure the Cacti operad up against the Little Disk operad via the recognition principle due to Fiedorowicz, since, as we will see in that subsection, what the recognition principle of Fiedorowicz does, is to measure operads up against the Little Cube operad.

This is one reason that the Little Disk operad is put into the preliminaries section. Another is that we also are developing the necessary language of operads. The Little Disk operad then serves as a nice first example to study and get comfortable with the notion of operads and the structure involved. So the Little Disk is both introduced and used to introduce the concept of operads. We can thus think of the preliminaries section as an introduction to operads viewed through the Little Disk operad. But only in part because:

We also introduce general concepts that we need for to show the main result. The Braid group and the Pure Braid group with their relation to the symmetric group. All do they play a significant role in the recognition principle. As well as in the introduction and initial study of the configuration space that we also present in the preliminaries section.

Finally we study the recognition principle of Fiedorowicz in the first section. It is arguable that this section is worthy of even its own entire section due to the importance in this topic, but it serves as a nice round up on the preliminary studies. It follows up on the discussion about equivalences of operads that is presented in the conclusion of the Little Disk subsection.

In the section about the Cacti operad we follow in large the structure of especially section 3 of [Kau05], but also initially present parts of section 2 of the same reference to introduce the Cacti operad before working with it. The structure there is though derived from the examples of applying the recognition principle that we have gone through in detail in the first section.

It is arguable that the subsection about the quasi-fibration since it is concept from homotopy theory in general should be put into the preliminaries section just as well as e.g. the Braid groups. But I find it natural and nice to have it right at hand and fresh in memory when it is to be used.

The section end by wrapping it all up by assembling the bits and parts to show main statement, that the Little Disk and the Cacti are equivalent as operads.

1.3 Thanks

I would like to thank the Topology Group as well as the Centre for Symmetry and Deformation for providing a welcoming and inspiring environment. It is quite amazing what have happen through the last few years at the mathematical institute here in Copenhagen. Topology has become a genre that one as a student can dive into with a safety line of people and courses. Gone is the time where a single third year course in general point-set topology acted as the nirvana.

My prime safety line and adviser for this thesis, Nathalie Wahl, I owe a great “thank you” for being so very patient and understanding when my situation were less then optimal. It has been inspiring, fun and educating to have her guidance though out the work on this thesis.

Finally I would like to thank my family, my girlfriend and our two boys for having provided invaluable support, love and motivation. I can not express in words what it means.

2 Preliminaries

In this section we will study preliminaries for the main part of the thesis that come in the following sections. In the first subsection we will look at some basic and needed definitions with respect to operads and the in the following subsection we will give and study the Little Disk operad, that is central to this thesis. In that section we will provide examples that elaborates on and explain what is going on in the definitions of the previous subsection.

In the third subsection we introduce the braid group and the Pure Braid group and give a graphical description of the element. In the forth subsection we introduce the configuration space, that for its homotopic properties are central and natural for this subject.

Finally we dive into the recognition principle and show in detail how it works.

2.1 Operads

Operads live in a setting of *symmetric monoidal categories* and we can loosely say, that an operad is a structure on some, more precisely a set of, objects from a symmetric monoidal category. In the following we let the tree-tuple of data, $(\mathcal{C}, \otimes, \kappa)$, be a symmetric monoidal category, where $\otimes$ is the product morphism and κ is the unit object that constitutes the unit morphism, $\eta\colon \kappa \to \mathcal{C}$. For further insight on monoidal categories see my 'fagprojekt' [Chr10] on $2dTQFT$ or for instance [ML98].

Definition 2.1. An operad, $\mathcal{O}$, is a set of objects from a symmetric monoidal category $\mathcal{C}$, $\mathcal{O}(n)$ for $n \in \mathbb{N}$, together with a unit morphism, $\eta\colon \kappa \to \mathcal{O}(1)$, a Σ_n action on $\mathcal{O}(n)$ for each n and (operadic) composition:

$$\omega\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \ldots n_k)$$

such that the following three properties are satisfied:

(i) Where $n := \sum_{i=1}^{k} n_i$, $j := \sum_{i=1}^{n} j_i$, $g_s := \sum_{i=1}^{s} n_i$ and $h_s := j_{g_{s-1}+1} + \cdots + j_{g_s}$ for $1 \leq s \leq k$, then the following diagram of associativity

$$\begin{tikzcd}
\mathcal{O}(k) \otimes \left(\bigotimes_{i=1}^{k} \mathcal{O}(n_i)\right) \otimes \left(\bigotimes_{r=1}^{n} \mathcal{O}(j_r)\right) \arrow[r, "\omega \otimes \mathbb{1}^j"] \arrow[dd, "shuffle"'] & \mathcal{O}(n) \otimes \left(\bigotimes_{i=1}^{n} \mathcal{O}(j_i)\right) \arrow[d, "\omega"] \\
 & \mathcal{O}(j) \\
\mathcal{O}(k) \otimes \bigotimes_{s=1}^{k} \left[\mathcal{O}(n_s) \otimes \left(\bigotimes_{q=1}^{n_s} \mathcal{O}(j_{g_{s-1}+q})\right)\right] \arrow[r, "\mathbb{1} \otimes \omega^n"] & \mathcal{O}(k) \otimes \left(\bigotimes_{s=1}^{k} \mathcal{O}(h_s)\right) \arrow[u, "\omega"']
\end{tikzcd}$$

commutes.

(ii) That these two diagrams that express unitality

$$\begin{tikzcd} \mathcal{O}(k) \otimes (\kappa)^k \arrow[r, "\cong"] \arrow[d, "\mathbb{1} \otimes \eta^k"'] & \mathcal{O}(k) \\ \mathcal{O}(k) \otimes \mathcal{O}(1)^k \arrow[ur, "\omega"'] & \end{tikzcd} \qquad \begin{tikzcd} \kappa \otimes \mathcal{O}(k) \arrow[r, "\cong"] \arrow[d, "\eta \otimes \mathbb{1}"'] & \mathcal{O}(k) \\ \mathcal{O}(1) \otimes \mathcal{O}(k) \arrow[ur, "\omega"'] & \end{tikzcd}$$

commutes.

(iii) And finally that diagrams that express Σ_n equivariance

$$\begin{tikzcd} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \arrow[r, "\sigma \otimes \sigma^{-1}"] \arrow[d, "\omega"'] & \mathcal{O}(k) \otimes \mathcal{O}(n_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(n_{\sigma(k)}) \arrow[d, "\omega"] \\ \mathcal{O}(n) \arrow[r, "\sigma(n_{\sigma(1)}, \ldots, n_{\sigma(k)})"'] & \mathcal{O}(n) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \arrow[r, "\mathbb{1} \otimes \tau_1 \otimes \cdots \otimes \tau_k"] \arrow[d, "\omega"'] & \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \arrow[d, "\omega"] \\ \mathcal{O}(n) \arrow[r, "\tau_1 \oplus \cdots \oplus \tau_k"'] & \mathcal{O}(n) \end{tikzcd}$$

also commutes where for $\sigma \in \Sigma_k, \tau_s \in \tau_{n_s}$, then $\sigma(n_1, \ldots, n_k) \in \Sigma_n$ permutes k blocks, $(1, \ldots, n_1), \ldots, (n_{k-1} + 1, \ldots, n_k)$ via σ permutes k letters and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_n$ is the block sum or direct sum of permutations.

Definition 2.2. Let $\mathcal{O}$ be an operad, then if we truncate the Σ action, we will called it a non-Σ operad.

Following a diffrent source by Kaufmann, [Kau04, Def. 4.1.3] (or [Mar08]) we define:

Definition 2.3. A *morphism* or an *operad morphism*, f, between two operads of the same symmetric monoidal category, $\mathcal{P}$ and $\mathcal{O}$, is a collection of morphisms $f_n : \mathcal{P}(n) \to \mathcal{O}(n)$ for $n \in \mathbb{N}$, that respect the operadic operation, are Σ_n equivariant and preserve the unit.

Remark 2.4. To preserve the unit is to map it to the unit of the other operad. I will explain in diagrams what it means to be Σ_n equivariant and to respect the operation. For a operad morphism, $f\colon \mathcal{O} \to \mathcal{P}$, to be Σ_n equivariant means just that it commutes with the action of Σ_n. So for $\sigma \in \Sigma_n$ that acts $\mathcal{O}(n)$ and $\mathcal{P}(n)$ we want the following diagram to commute:

$$\begin{tikzcd} \mathcal{O}(n) \arrow[r, "\sigma"] \arrow[d, "f_n"'] & \mathcal{O}(n) \arrow[d, "f_n"] \\ \mathcal{P}(n) \arrow[r, "\sigma"'] & \mathcal{P}(n). \end{tikzcd}$$

To respect the operation ω means that it also commutes with the operad morphism and that it is well-defined, so we want the following diagram to commute

$$\begin{tikzcd} \mathcal{O}(k) \otimes \bigotimes_{i=1}^{k} \mathcal{O}(n_i) \arrow[r, "f_{n_k} \otimes \bigotimes_{i_1}^{i=k} f_{n_i}"] \arrow[d, "\omega"'] & \mathcal{P}(k) \otimes \bigotimes_{i=1}^{k} \mathcal{P}(n_i) \arrow[d, "\omega"] \\ \mathcal{O}(\sum_{i=1}^{k} n_i) \arrow[r, "f_{\sum_{i=1}^{i=k} n_i}"'] & \mathcal{P}(\sum_{i=1}^{k} n_i). \end{tikzcd}$$

Definition 2.5. An operad $\mathcal{P}$ is a *suboperad* of an operad $\mathcal{O}$ if there exists an injective operad morphism from $\mathcal{P}$ to $\mathcal{O}$. [1]

Definition 2.6. An operad morphism, $f\colon \mathcal{P} \to \mathcal{O}$, is an *equivalence (of operads)* if for each n, then $f_n\colon \mathcal{P}(n) \to \mathcal{O}(n)$ is a (Σ_n equivariant) homotopy equivalence. Also, we call

$$\mathcal{P} \to \mathcal{O} \leftarrow \ldots \to \mathcal{O}'$$

a *chain of operad equivalences* if it is a chain of operad morphisms that all are operad equivalences.

Remark 2.7. Also, since homotopy equivalence is an equivalence relation we also get that operad equivalence is an equivalence relation, so if we have a chain of operads $\mathcal{P} \to \mathcal{O} \leftarrow \ldots \to \mathcal{O}'$, surely $\mathcal{P}$ and $\mathcal{O}'$ is equivalent and the directions of the arrows are indifferent. Note that this notion of equivalence makes sense because we deal with operads that take objects from the category of topological spaces.

[1] See [Kau04, Def. 4.1.3]

2.2 Little Disk

The Little Disk is a central object in this thesis, thus we will use that as an illustrating example of operads as well as casting light on its properties.

Definition 2.8. Let D^n be the standard unit disk in $\mathbb{R}^n$. For $k \in \mathbb{N}$ we call $\mathrm{Disk}_n(k) := \{f\colon \bigsqcup_{i=1}^k D_i^n \to D^n\}$ the space of embeddings of k disjoint n disks, where each f is a scaling and position-translation.

So f will take k discs, scale each of them to make them little and then position them inside a disk. As such it makes sense to think of it as an (n) disk with (k) little (n) disks disjoint inside. Sometimes we omit the type of disk and focus on the number of disks in play, in those cases, or where it is clear for which n in $\mathrm{Disk}_n(k)$ we are talking about, we will just write $\mathrm{Disk}(k)$ to refer to the operad of k disks.

With that settled we will make the following claim

Proposition 2.9. *The set of objects $\{\mathrm{Disk}_n(k)\}_{k\in\mathbb{N}}$ from the symmetric monoidal category of topological spaces, $(Top, \times, *)$, with composition*

$$\omega\colon\ \mathrm{Disk}_n(k) \times \mathrm{Disk}_n(n_1) \times \ldots \times \mathrm{Disk}_n(n_k) \to \mathrm{Disk}_n(n_1 + \cdots + n_k)$$

defined as

$$(f, f_1, \ldots f_k) \mapsto f \circ (f_{n_1} \sqcup \cdots \sqcup f_{n_k}) = \bigsqcup_{i=1}^{k} f_{D_i} \circ f_i,$$

unit morphism $\eta\colon * \mapsto \mathbb{1}_{D^n} \in Disk_n(1)$ and Σ_k action for each k defined as permutation of the label of the little disks is an operad. We will call it the little disk operad*.*

Proof. We will show that it satisfies the required properties from definition 2.1. This will also work as an opportunity to do the exercise of getting dirty hands by working with the definition and thereby comfortable.

First the associativity. We begin by presenting the diagram as it looks in this setting with the term $\mathcal{O}(j)$ from the general diagram expanded and thus better suited for doing a chase around the diagram.

$$\begin{tikzcd}
\mathrm{Disk}(k) \times \prod_{i=1}^{k} \mathrm{Disk}(n_i) \times \prod_{r=1}^{n} \mathrm{Disk}(j_r) \arrow[r, "\omega \otimes \mathbb{1}^j"] \arrow[d, "shuffle"'] & \mathrm{Disk}(n) \times \prod_{i=1}^{n} \mathrm{Disk}(j_i) \arrow[d, "\omega"] \\
\mathrm{Disk}(k) \times \prod_{s=1}^{k} \mathrm{Disk}(n_s) \times \prod_{q=1}^{n_s} \mathrm{Disk}(j_{g_{s-1}+q}) \arrow[d, "\mathbb{1} \otimes \omega^n"] & \mathrm{Disk}(\sum_{i=1}^{n} j_i) \arrow[d, equal] \\
\mathrm{Disk}(k) \times \prod_{s=1}^{k} \mathrm{Disk}(h_s) \arrow[r, "\omega"'] & \mathrm{Disk}(\sum_{s=1}^{k} h_s)
\end{tikzcd}$$

Where j_*, h_* and g_* are as in definition 2.1.
Clockwise around we get:

$$\begin{tikzcd}
(f_k, f_{n_1}, \dots, f_{n_k}, f_{j_1}, \dots, f_{j_n}) \arrow[r, mapsto, "\omega \otimes \mathbb{1}^j"] & (f_k \circ (f_{n_1} \sqcup \cdots \sqcup f_{n_k}), f_{j_1}, \dots, f_{j_n}) \arrow[d, mapsto, "\omega"] \\
 & (f_k \circ (f_{n_1} \sqcup \cdots \sqcup f_{n_k})) \circ (f_{j_1} \sqcup \cdots \sqcup f_{j_n})
\end{tikzcd}$$

and counter clockwise:

$$\begin{tikzcd}
(f_k, f_{n_1}, \dots, f_{n_k}, f_{j_1}, \dots, f_{j_n}) \arrow[d, mapsto, "shuffle"'] & \\
\begin{array}{c} (f_k, f_{n_1}, f_{j_1}, \dots, f_{j_{n_1}}, \\ f_{n_2}, f_{j_{n_1+1}}, \dots, f_{j_{n_2}}, \dots, \\ f_{n_k}, f_{j_{n_1+\cdots+n_{k-1}+1}}, \dots, f_{j_n}) \end{array} \arrow[d, mapsto, "\mathbb{1} \times \omega^k"'] & \\
\begin{array}{c} (f_k, f_{n_1} \circ (f_{j_1} \sqcup \cdots \sqcup f_{j_{n_1}}), \\ f_{n_2} \circ (f_{j_{n_1+1}} \sqcup \cdots \sqcup f_{j_{n_2}}), \dots, \\ f_{n_k} \circ (f_{j_{n_1+\cdots+n_{k-1}+1}} \sqcup \cdots \sqcup f_{j_n})) \end{array} \arrow[r, mapsto, "\omega"] & \begin{array}{c} f_k \circ \big[f_{n_1} \circ (f_{j_1} \sqcup \cdots \sqcup f_{j_{n_1}}) \sqcup \\ f_{n_2} \circ (f_{j_{n_1+1}} \sqcup \cdots \sqcup f_{j_{n_2}}) \sqcup \cdots \sqcup \\ f_{n_k} \circ (f_{j_{n_1+\cdots+n_{k-1}+1}} \sqcup \cdots \sqcup f_{j_n}) \big] \end{array}
\end{tikzcd}$$

Left is to show the equality where the two halfes of the diagram meet. Really the two diagrams above only picture the spaghetti-mess of indices in the compact formulation of the diagram in the definition. I have found it useful to at least once write it out to make it crystal clear, and that is the motivation for also presenting it here. Then the remaining equality is where the design of the operad composition is to show its worth. In short, since the embeddings of little discs are also associative, then the equality holds. We refer to example 2.10 to see the idea of how it works.

Next is the unitality. We are in the monoidal category of topological spaces, Top, with unit $*$, so the concrete diagrams we want to chase is the following

$$\begin{tikzcd} \mathrm{Disk}(k) \times *^k \arrow[r, "\cong"] \arrow[d, "\mathbb{1} \times \eta^k"'] & \mathrm{Disk}(k) \\ \mathrm{Disk}(k) \times \mathrm{Disk}(1)^k \arrow[ur, "\omega"'] & \end{tikzcd} \qquad \begin{tikzcd} * \times \mathrm{Disk}(k) \arrow[r, "\cong"] \arrow[d, "\eta \times \mathbb{1}"'] & \mathrm{Disk}(k) \\ \mathrm{Disk}(1) \times \mathrm{Disk}(k) \arrow[ur, "\omega"'] & \end{tikzcd}$$

We note first that the upper arrows follow from the category being monoidal, the natural identity morphism: $(- \times \mathbb{1}_{Top}) \cong -$ and $(\mathbb{1}_{Top} \times -) \cong -$. We will take the left diagram first.

$$\begin{tikzcd} (f, *^k) \arrow[rr, mapsto, "\cong"] \arrow[d, mapsto, "\mathbb{1} \times \eta^k"'] & & f \arrow[d, equal] \\ (f, \mathbb{1}_1, \ldots, \mathbb{1}_k) \arrow[r, mapsto, "\omega"'] & \omega(f, \mathbb{1}_1, \ldots, \mathbb{1}_k) = & f \circ (\mathbb{1}_1 \sqcup \cdots \sqcup \mathbb{1}_k) \end{tikzcd}$$

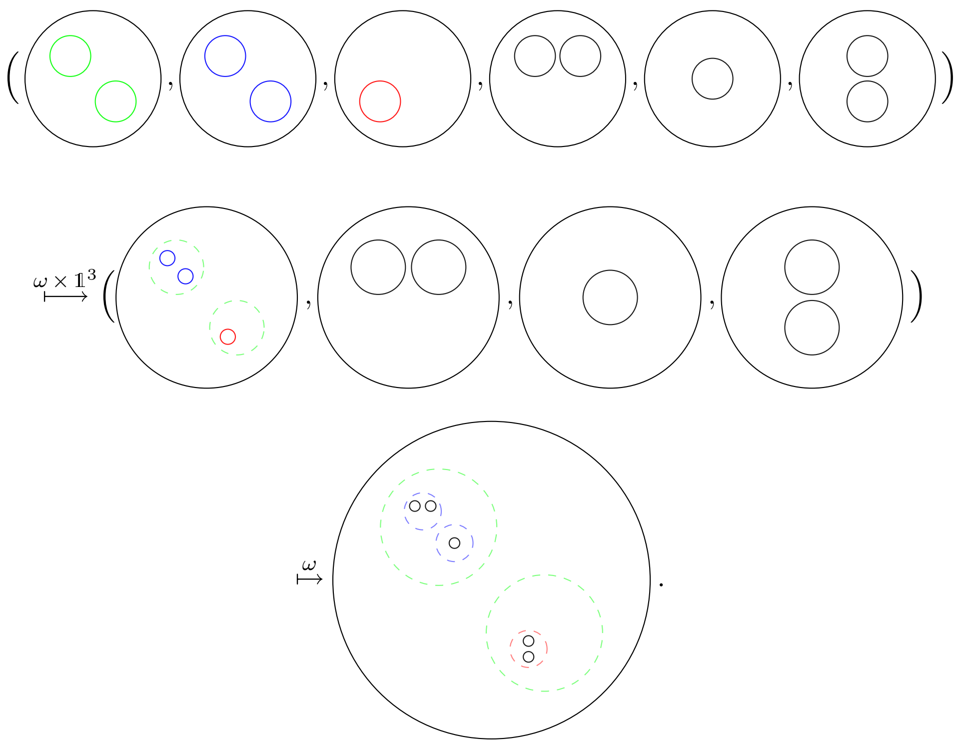
and the right

$$\begin{tikzcd} (*, f) \arrow[r, mapsto, "\cong"] \arrow[d, mapsto, "\eta \times \mathbb{1}"'] & f \arrow[d, equal] \\ (\mathbb{1}, f) \arrow[r, mapsto, "\omega"'] & \omega(\mathbb{1}, f) = \mathbb{1} \circ f. \end{tikzcd}$$

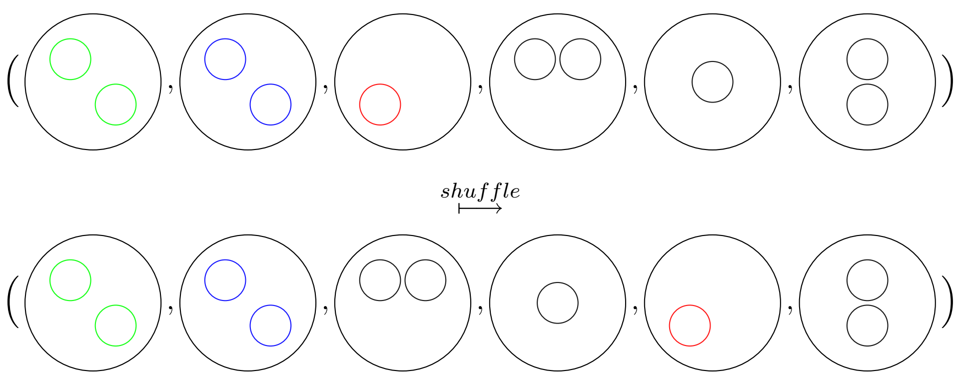
Finally, that the operad satisfies the Σ_k equivariance property is immediate as it is (just) permuting the labels, thus illustrating it in a diagram for a general setting would literally be copying the definition. Instead we will illustrate it in the concrete, but scaling, example 2.11. $\square$

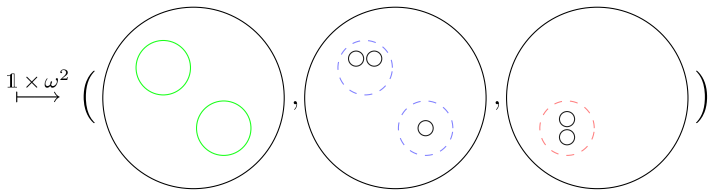
The following two examples are to illustrate in a concrete way how the operad composition of little disks satisfy the equivariance and associativity properties. Naturally included in that they also illustrate how the composition look works out pictorially. But also they exhibit how to read and understand the general definition of operads.

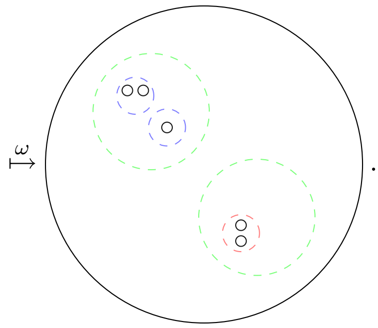
Example 2.10. We let $k = 2$, $n_1 = 2, n_2 = 1$ and $j_1 = 2, j_2 = 1, j_3 = 2$ and observe the following elements in Disk being the input of the upper left corner of the diagram. In the example the disks are unlabeled to avoid clutter, in return they are coloured and embedding history is preserved by printing the disks that illustrated the old *k-arry operation* with a dashed line. Imagine that they are ordered successively top to bottom, left to right. First we go clockwise around:



And counter clockwise:







Example 2.11. Here we will see a quite concrete example of elements of the Disk(k) operad with a concrete permutation of Σ_k satisfying the first of the equivariant diagrams. The second is analogue in fashion and thus omitted here.

Let $k = 3, n_1 = 1, n_2 = 2, n_3 = 3$ and $\sigma = (231), \sigma^{-1} = (312) \in \Sigma_3$. Then

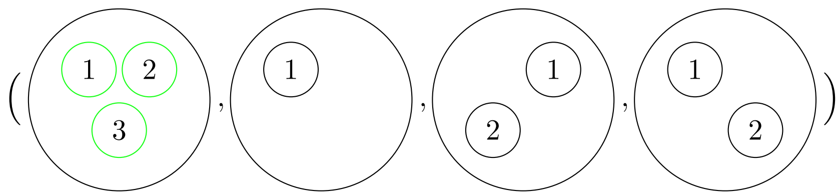
$$\omega\colon\ \mathrm{Disk}(3) \times \mathrm{Disk}(1) \times \mathrm{Disk}(2) \times \mathrm{Disk}(2) \to \mathrm{Disk}(5),$$

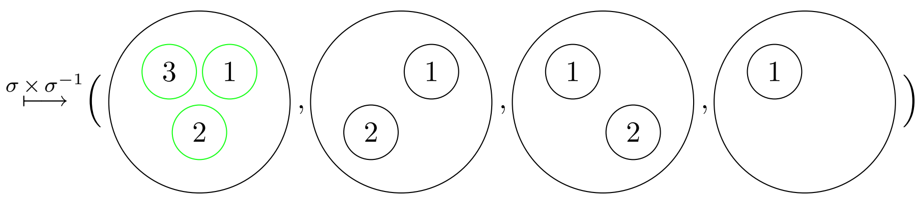
$$\sigma \times \sigma^{-1}\colon\ \mathrm{Disk}(3) \times \mathrm{Disk}(1) \times \mathrm{Disk}(2) \times \mathrm{Disk}(2) \to \mathrm{Disk}(3) \times \mathrm{Disk}(2) \times \mathrm{Disk}(2) \times \mathrm{Disk}(1) \text{ and}$$

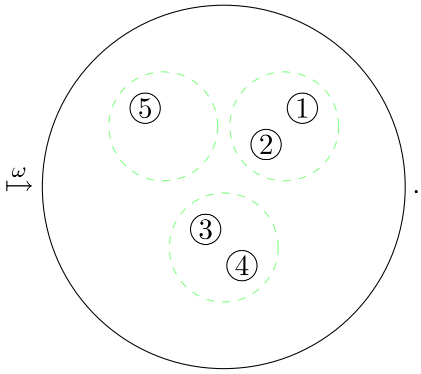
$$\sigma(2,2,1)\colon\ \mathrm{Disk}(5) \to \mathrm{Disk}(5) \in \Sigma_5 \text{ by } (23451).$$

Recall that the i'th disk, the disk labeled i, after the action of an element σ is the $\sigma(i)$'th disk before the action.

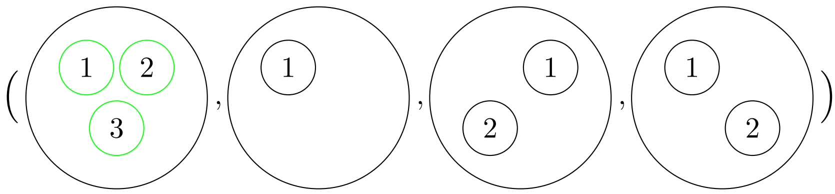
Clockwise round we get:

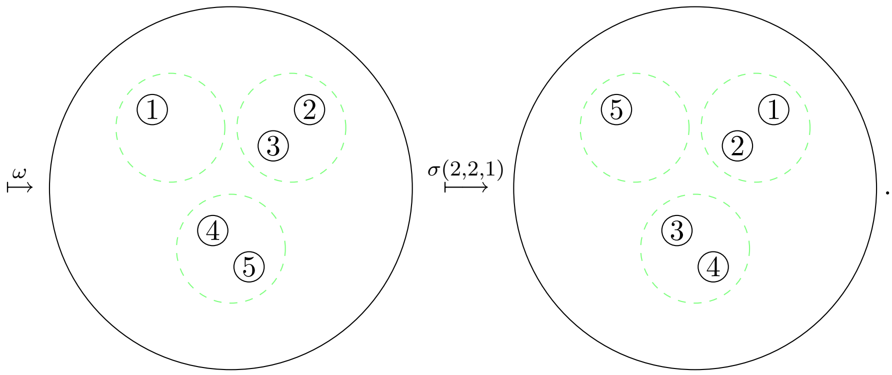






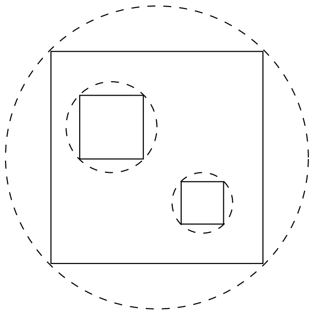
And counter clockwise:





The little cube operad, $\mathcal{C}_2$, was briefly mentioned as it is part of definition 2.17. In short, $\mathcal{C}_2$, is very similar to the little disk operad. Σ_k acts on the label the same way. On the space level we have little squares, instead of little disks. Hence we can define the operad composition completely analogous. The obvious next question would be to write down a morphism for each $\mathcal{C}_2(k)$ to $\mathrm{Disk}_2(k)$. It is not hard to think of a morphism between squares and disks, but it is harder to make it actually be a morphism of operads. We will show later indirectly, via the recognition principle, that they are equivalent as operads, but for now we will discuss some of the problems of trying to define a direct operad morphism.

Our first natural idea is to define the morphism from $\mathrm{Disk}_2 \to \mathcal{C}_2$ by putting squares inside the discs, like this:



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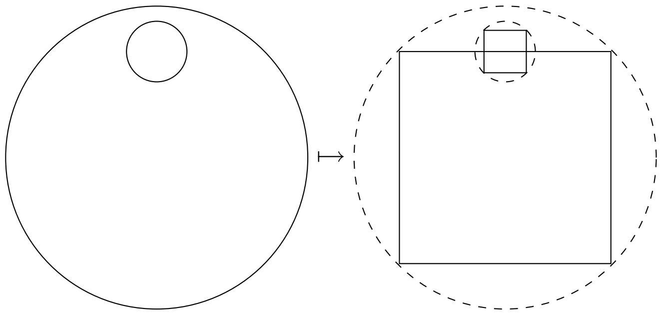
Then it will not work out as it will not be well-defined. For example.



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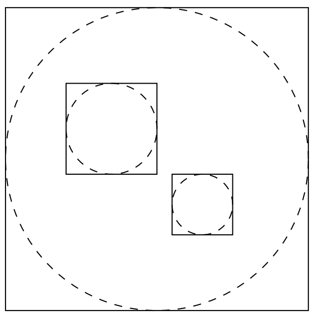
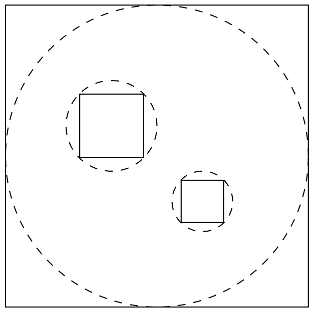
So what if we do it the other way around, and put squares outside the disks like



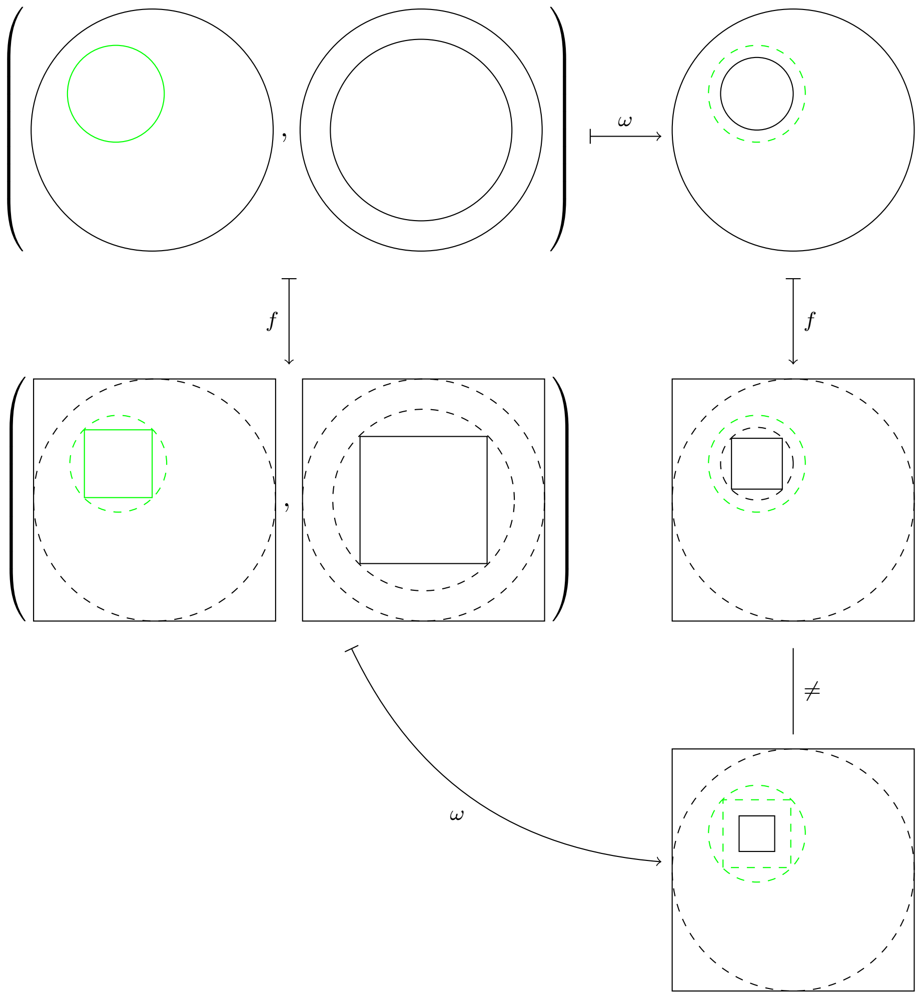
.

Then we will have analogous problems as it will not be well defined when we go back. We could well have a little square near one of the corners, that, when mapped back to Disk_2, would place the little disk out side the unit disk it should be embedded in. Or two little disks could so close that the squares that are put around them would overlap.

So what if we mix those two approaches by putting the little square inside the little disks but the big square outside the big disk. Like

 .

That will fail when composing, it will not respect the composition, so the second diagram of remark 2.4 will be non-commutative by a scaling factor. Observe the following example:



Instead one can define a bigger middle operad, where the spaces are a triples of cubes, disks and homotopies between them, that is equivalent to each of the two. Hepworth does that for framed disks and cacti with spines in [Hep10, Theorem A].

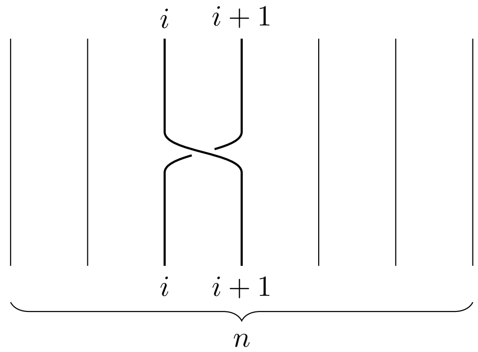
2.3 Braid groups

In this subsection we will introduce the braid groups. We will talk about the normal *braid group on n strands*, with its connection to the symmetric group. And thus also the *pure braid group*. All of them play a central role in the rest of this text.

Definition 2.12. *The braid group n stands* can be specified by a set of $n-1$ generators and some relations, namely

$$B_k := \langle \sigma_1, \ldots \sigma_{k-1} \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle$$

where for the first relation $1 \leq i \leq n-2$ and $|i-j| \geq 2$ for the second relation. The σ_i'th generator looks like this:



Remark 2.13. So what it does geometrically is, that it takes the i'th and $i+1$'th stand and braids them. It takes the i'th and moves it over the $i+1$'th strand so that it transpose the ends of the i'th and the $i+1$'th strand. If we were going under instead of over, we would express the inverse of the generator. Composing σ_i with σ_i^{-1} yields the identity element of the braid group, since it can be continuously deformed to become straight strands taking i to i.

This also explains how the braid group can be surjectively projected to the symmetric group: by forgetting how the endpoints of the strands are transposed. We can write it up explicitly by sending generator to generator but adding the relation that $\sigma_i^2 = 1$ as that expresses that we forget how the transposition happens,

$$p\colon B_k \to \Sigma_k \text{ by } \langle \sigma_i \in B_k \rangle \;\mapsto\; \langle \sigma_i \,|\, \sigma_i^2 = 1 \rangle. \tag{2.1}$$

On the other hand we can also lift a, say generator, of the symmetric group by connecting the points from the domain to the image such that they become strands. We connect point i to point $\sigma(i)$ to form a strand and also choosing if we go *over* or under *under* when two strands are to cross each other. So for the generator σ_i that transpose i and $i+1$ of Σ_k we connect the point i to the point $\sigma_i(i) = i+1$ and $i+1$ to $\sigma_i(i)$ choosing to pass the first strand over the other as they are to cross.

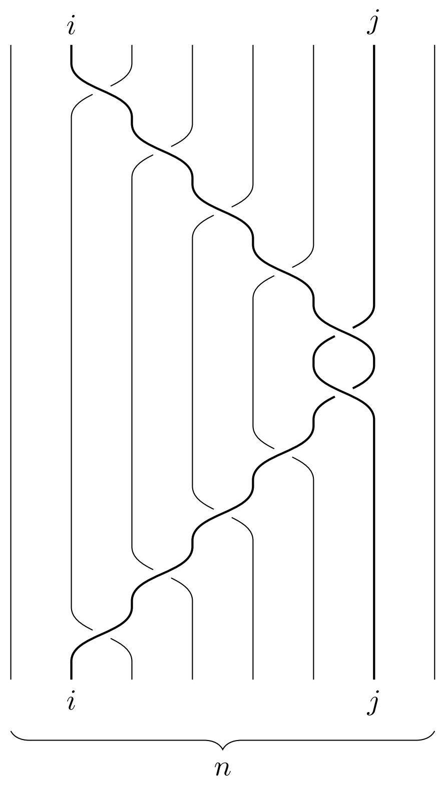
And finally the pure braid group:

Definition 2.14. *The pure braid group* is defined to be the kernel of the projection from the braid group to the symmetric group as seen in equation 2.1.

The generators $\chi_{i,j}$ – that wraps the ith strand round the j'th strand wise – are specified as

$$\chi_{j,j} = \sigma_i \sigma_{i+1} \dots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \dots \sigma_j^{-1} \tag{2.2}$$

where σ_* are the generators of the braid group, and pictured as below.



See [KT08, p. 18f] for more information on this Coexter presentation of the pure braid group.

2.4 Configuration space

Lets start off with the definition:

Definition 2.15. The space

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X \text{ s.t. } x_i \neq x_j \text{ if } i \neq j\}$$

is called the *configuration space* over X of n distinct labelled points.

What we are interested in is the configuration space over $\mathbb{R}^2$, $\mathrm{Conf}_n(\mathbb{R}^2)$. It is quite straight forward to see that on a space level $\mathrm{Conf}_k(\mathbb{R}^2)$ is homotopy equivalent to $\mathrm{Disk}_2(k)$. In one way we shrink the disks to their center to get points in the configuration space, and in the other way we blow up the points of the configuration space to obtain little disks.

Now let us have a look at the homotopy groups:

Proposition 2.16. $\mathrm{Conf}_n(\mathbb{R}^2)$ *is* $K(PB_n, 1)$.

The proof of this fact might seem superficial in this paper, but it is an good exercises for the proof of the similar statement for the spaces of $\mathrm{Cact}(n)$. As the fundamental group of the configurations space of n points in the plane is actually defining the Pure Braid group on n strands, the work here will be a lot less. On a slightly anachronistic note: Another difference is that here we can make use of a long exact sequence induced by a fibration rather then a *quasi-fibration* which is needed in the case of $\mathrm{Cact}(n)$.

Proof. We will use with out proof here, see for example [Sin06, Lemma 3.4], that the projection $p : \mathrm{Conf}_{n+1}(\mathbb{R}^2) \to \mathrm{Conf}_n(\mathbb{R}^2)$ that forgets a point by mapping $(x_1, \ldots, x_n, x_{n+1})$ to $(x_1, \ldots, x_n)$ admits a fiber bundle structure with fiber $\mathbb{R}^2$ with n points removed, which is homotopic to a wedge of n circles. This fiber bundle then induces the following long exact sequence in homotopy:

$$\pi_i(\bigvee_n S^1) \to \pi_i(\mathrm{Conf}_{n+1}(\mathbb{R}^2)) \to \pi_i(\mathrm{Conf}_n(\mathbb{R}^2)) \to \pi_{i-1}(\bigvee_n S^1) \to \\ \cdots \to \pi_0(\mathrm{Conf}_{n+1}(\mathbb{R}^2)) \to 0 \quad (2.3)$$

The universal cover of $\bigvee_n S^1$ is, a tree, contractible, hence by [Hat02, 4.2] $\pi_i(\bigvee_n S^1) = 0$ for $i \geq 2$. So from the above long exact sequence we get that

$$0 \to \pi_i \mathrm{Conf}_{n+1}(\mathbb{R}^2) \to \pi_i \mathrm{Conf}_n(\mathbb{R}^2) \to 0$$

for $i \geq 3$ implying that $\pi_i \mathrm{Conf}_{n+1}(\mathbb{R}^2) \cong \pi_i \mathrm{Conf}_n(\mathbb{R}^2)$ for $i \geq 3$. So if $\pi_3 \mathrm{Conf}_2(\mathbb{R}^2) = 0$, then also $\pi_3 \mathrm{Conf}_3(\mathbb{R}^2) = 0$, if $\pi_4 \mathrm{Conf}_2(\mathbb{R}^2) = 0$, then $\pi_4 \mathrm{Conf}_3(\mathbb{R}^2) = 0$ and so on. We observe that $\mathrm{Conf}_2(\mathbb{R}^2) \simeq S^1$. For example: fix a point, then the other point can be anywhere else in the plain, which is homotopic to the plain with a point removed, that is homotopic to S^1. So $\pi_i \operatorname{Conf}_2(\mathbb{R}^2) = 0$ for $i \geq 2$. Hence $\pi_i \operatorname{Conf}_n(\mathbb{R}^2) = 0$ for $i \geq 2$ and $n \geq 2$. $\operatorname{Conf}_1(\mathbb{R}^2) \simeq *$ so $\pi_i \operatorname{Conf}_1(\mathbb{R}^2)$ is trivial. Thus for any n the higher, greater then 2, homotopy groups vanish, so what about the case $i = 1$: $\pi_1 \operatorname{Conf}_1(\mathbb{R}^2) = 0$, $\pi_1 \operatorname{Conf}_2(\mathbb{R}^2) = \mathbb{Z}$, and in general $\pi_i \operatorname{Conf}_n(\mathbb{R}^n) = PB_n$ by definition of the pure braid group. $\square$

2.5 Recognition principle

The overall subject of this thesis is to compare two operads, one of them being the little disk. We will now look ways to recognize operads as operads that are equivalent to the little disk, the same as saying that they are E_2 as we shall see below. Finding concrete direct morphisms between operads that will be equivalences is not easy. Means to classify them, or recognize them, is useful. In [Fie98] Fiedorowicz presents the following definitions and statements:

Definition 2.17. An operad, $\mathcal{O}$, is called E_n for $n \in \{1, 2, \ldots, \infty\}$ if there exists a chain of operad equivalences from $\mathcal{O}$ to C_n. Where the operad C_n is the n-cube operad defined by Bordmann & Vogt, [BV73, p. 64].

Definition 2.18. An operad, $\mathcal{O}$, is can be called a braid operad if we in a natural way can swap the symmetric group with the braid group.

Thus for a braid operad spaces $\mathcal{O}(k)$ admits an action by the braid group, B_k, such that we get B_k-equivariance.

Before we continue let us make sure we know what a braid operad morphism is.

Definition 2.19. A braid operad morphism is an operad morphism between two braid operads with the extension that it is equivariant with respect to the braid group.

Now we will turn to the central points.

Definition 2.20. A braid operad $\mathcal{O}$ is a B_∞ operad if it is spacewise contractible and braid group B_k is free.

Theorem 2.21. *An operad $\mathcal{O}$ is E_1, if and only, if each path-component of $\mathcal{O}(k)$ is contractible and the action of Σ_k on $\pi_0(\mathcal{O}(k))$ for each k is regular.*

Theorem 2.22. *An operad $\mathcal{O}$ is E_2, if and only, if $\mathcal{O}(k)$ is connected for each k and the induced family of universal covers, $\widetilde{\mathcal{O}}(k)$, admits the structure of a B_∞ operad.*

I shall not fully prove them here, but will refer to the original source (or [SW03]) for the later statement.

Instead we will go to the example below (proposition 2.28) from [Fie]. We will use the method in this example as a template to show the main result. This is almost what Kaufmann does, but he wants to distil the template into a proposition for recognizing E_2 operads, which I will diverged slightly from. What he does, in his proposition 3.2.4, is to provide a criteria for an operad to be liftable to a B_∞ operad. This also means that we only need a weaker version of the recognition of E_2 from above, namely the direction where we end up concluding the E_2 property from having the B_∞ structure on the universal covers. Before we proceed to the promised example let us look into how to prove the weaker version of theorem 2.22.

Lemma 2.23. *A product of two B_∞ operads are again a B_∞ operad.*

Proof. Let $\mathcal{O}$ and $\mathcal{P}$ be two B_∞ operads. For the product to be an B_∞ by definition 2.20 above we need to see that it is contractible and that there is a free braid group action. So both $\mathcal{O}$ and $\mathcal{P}$ are contractible, thus the product $\mathcal{O} \times \mathcal{P}$ is also contractible. Also since for any k the action of the braid group, B_k, on each of the spaces $\mathcal{O}(k)$, $\mathcal{P}(k)$ are free then it is also free on the product. □

Lemma 2.24. *Any braid operad morphism between B_∞ operads is an equivalence of operads.*

Proof. Take a braid operad morphism $f \colon \mathcal{O} \to \mathcal{P}$ where $\mathcal{O}$ and $\mathcal{P}$ are B_∞ operads. Then since both $\mathcal{O}$ and $\mathcal{P}$ are contractible f is also a homotopy equivalence as it can be factored through a point, hence an equivalence of operads. □

Proposition 2.25. *Any two B_∞ operads are equivalent as operads.*

Proof. A direct consequence of the previous two lemmas is the following. Suppose we have two B_∞ operads $\mathcal{O}$ and $\mathcal{P}$, then if we can produce an operad morphism from the product to each from them, such that is forms a chain of operad morphisms, like,

$$\mathcal{O} \leftarrow \mathcal{O} \times \mathcal{P} \rightarrow \mathcal{P},$$

then both of these operad morphisms are operad equivalences, thus $\mathcal{O}$ and $\mathcal{P}$ are equivalent.

Left is to see that we can produce such operad morphism. We claim that the projection morphisms from the product is operad morphisms. We check; the unit is preserved. Secondly the equivariance and composition requirements are met and well-defined due to contractibility of the B_∞ operads. □

The notion of equivariance worked with in the above is with respect to action with the braid group, so it also works with a Σ action. Similar to the concept of B_∞ operads we have E_∞ operads, hence:

Remark 2.26. Any two operads that is space-vice contractible and where the Σ action is free are equivalent. E_∞ satisfies this.

Before we get back to the E_2 case, lets get the E_1 case settled. In the little cube treatment below we also see that the little interval, $\mathcal{C}_1$ meets the criteria in theorem 2.21. So if we could show that any operad that satisfies these criteria is equivalent, then we would also have shown the direction of theorem 2.21 that we use. The argument carries through as in the above except, since it is not contractible, that the product that we want to factor though is taken over the path-components. This is done to ensure that the equivariance will be well-defined.

We will now return to the E_2 case.

Proposition 2.27. *Any two operads are equivalent if their universal covers admit a B_∞ operad structure.*

Proof. Let $\mathcal{O}$ and $\mathcal{P}$ be operads such that their universal covers, $p\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ and $p^{'} : \widetilde{\mathcal{P}} \to \mathcal{P}$, are B_∞ operads. Then, by proposition 2.25 we can construct a chain of operad equivalences form the universal covers. This chain can then be extended to the following diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{O}} \arrow[d, "p"'] & \widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \widetilde{\mathcal{P}} \arrow[d, "p^{'}"] \\
\mathcal{O} & (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})/PB \arrow[l] \arrow[r] & \mathcal{P}.
\end{tikzcd}$$

The goal now is to see that arrows in the bottom row can be operad equivalences. We will now see how they can be induced from the rest of the diagram.

We will focus on only one side of the diagram. The other will be analogous.

First, the outer vertical arrow stems from the universal covering. Take universal covering $p\colon \widetilde{\mathcal{O}} \to \mathcal{O}$. We have a braid group action on $\widetilde{\mathcal{C}}$. We see that when we quotient $\widetilde{\mathcal{O}}$ with the pure braid group we get back $\mathcal{O}$ (if $\mathcal{O}$ is connected), see e.g. [Hat02, p. 71f], with a Σ action.

Now let us consider the middle vertical arrow. As PB denotes the pure braid group, the middle part of the bottom row, $(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})/PB$ is the quotient space. The map is the quotient map defined by sending points in $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}}$ to orbits in $(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})/PB$.

The map, $f\colon (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})/PB \to \mathcal{O}$, induced by taking quotient, defined by taking orbits to orbits is well-defined and homotopic. So left is only to verify that it is also an operad morphism.

First we will check that $(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})$ is an operad such that it makes sense to talk about operad morphisms. We take the operadic composition to be the one induced by the quotient map of the composition in $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}}$, $\widetilde{\omega}$. In other words defined through the following diagram:

$$\begin{tikzcd}
(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(k) \times (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(n_1) \times \cdots \times (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(n_k) \arrow[r, "\widetilde{\omega}"] \arrow[d] & (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(n) \arrow[d] \\
(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(k)/PB \times \prod_i^k (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(n_i)/PB \arrow[r, "\omega"'] & (\widetilde{\mathcal{O}} \times \widetilde{\mathcal{P}})(n)/PB
\end{tikzcd}$$

So immediately all the needed structural properties required on ω is met since they are assumed met for $\widetilde{\omega}$.

Then by analogous argument the map f carries the structure of an operad morphism.

Hence we can conclude that $\mathcal{O}$ and $\mathcal{P}$ are equivalent. □

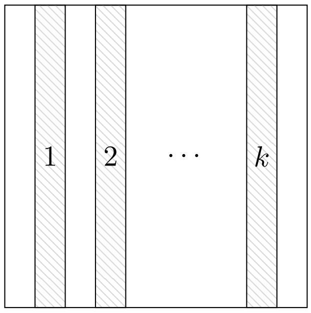
Now as a corollary, if we plug in $\mathcal{C}_2$, while assuring that it has the needed properties, which we show in proposition 2.28, then we have shown the reverse direction of theorem 2.22.

Proposition 2.28. *The little* 2 *cube operad,* $\mathcal{C}_2$*, is* E_2*. Or more in the spirit of [Fie]: The universal covers of the spaces of* $\mathcal{C}_2$ *admits a* B_∞ *structure.*

Proof. When given the $\mathcal{C}_2$ operad with an operad structure operation and Σ_k action, then we look at the projection $p\colon \widetilde{\mathcal{C}_2}(k) \to \mathcal{C}_2(k)$ where $\widetilde{\mathcal{C}_2}(k)$ is the universal cover of $\mathcal{C}_2$. Now the goal is to construct the needed structure for $\widetilde{\mathcal{C}_2}$ to be a B_∞ operad from the structure on $\mathcal{C}_2$. So per definition 2.20 $\widetilde{\mathcal{C}_2}$ need: to have an operad structure, make sure that for each k, the spaces $\widetilde{\mathcal{C}_2}(k)$ are contractible and to admit a free action by the braid group, B_k. In other words we want to lift the operad structure of $\mathcal{C}_2$ to a B_∞ structure on $\widetilde{\mathcal{C}_2}$ by lifting the composition, ω to a composition $\widetilde{\omega}$ and the Σ_k action on $\mathcal{C}$ to a B_k action on $\widetilde{\mathcal{C}_2}$.

Let $\mathcal{C}_1$ be the little 1-cube, or interval, operad. We see that each path-component is contractible: For some k, a point in $\mathcal{C}_1(k)$ consists of k small (labelled) intervals embedded in the unit interval. Two points are in the same path-component exactly when the sequence of their labels are the same. Suppose for illustrative purpose that $k = 2$, then the sequence of the labels can be either $(1, 2)$ or $(2, 1)$. A point in $\mathcal{C}_1(2)$ that has the labels orders as $(1, 2)$ can not be moved to a point with the intervals labeled $(2, 1)$ as they live in $\mathbb{R}^1$ the little intervals can not be moved past each other. Then for any path-component fix a point, then there is a path from any of the other points to this point, thus each path-component is contractible. Also for any two points, $x, y \in \pi_0(\mathcal{C}_1(k))$ there is exactly one element, $\sigma \in \Sigma_k$ such that $\sigma(x) = y$, meaning that the action of Σ_k on $\pi_0(\mathcal{C}_1(k))$ is regular. Hence per theorem 2.21 it is E_1.

We can embed $i\colon \mathcal{C}_1 \hookrightarrow \mathcal{C}_2$ as bottom line as we cross it with the unit interval we get a point in $\mathcal{C}_2$ as in the following illustration.



What we a seeking to construct is a lift of the map ω, call it $\widetilde{\omega}$, from the diagram below

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_2(k) \times \widetilde{\mathcal{C}}_2(n_1) \times \cdots \times \widetilde{\mathcal{C}}_2(n_k) \arrow[r, "\widetilde{\omega}"] \arrow[d, "p"'] & \widetilde{\mathcal{C}}_2(n_1 + \cdots + n_k) \arrow[d, "p"] \\
\mathcal{C}_2(k) \times \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_k} \arrow[r, "\omega"'] & \mathcal{C}(n_1 + \cdots + n_{n_k}).
\end{tikzcd}$$

For that we will use the one part of $\mathcal{C}_1$ being E_1 namely that each path-component is contractible.

When we take the non-Σ version of a E_1 operad we get only one (contractible) path-component. So in this case the non-Σ version of $\mathcal{C}_1$ is the unordered little interval operad. This will act as a basepoint in $\mathcal{C}_2$.

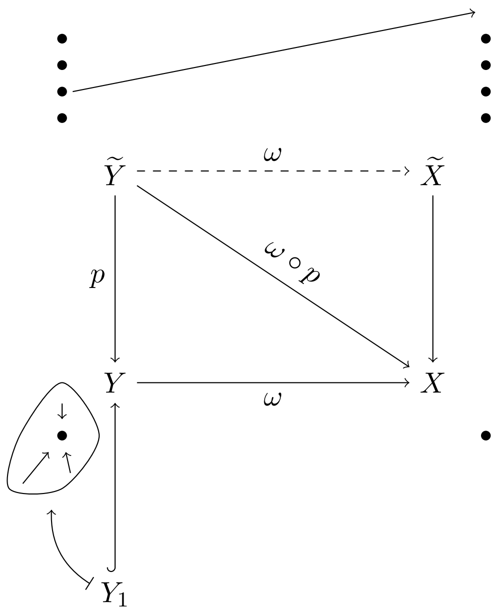
The morphism $\widetilde{\omega}$ is a lift if, by definition of lift, $\omega \circ p = p \circ \widetilde{\omega}$. By the unique lifting property we need only, since the domain of ω is connected, to specify what our lift does to the basepoint to ensure uniqueness of the lift. It should be mapped to the basepoint in $\widetilde{\mathcal{C}}_2$.

The fiber of $\mathcal{C}_1(n)$, $p^{-1}(\mathcal{C}_1(n)) \subset \widetilde{\mathcal{C}}_2$, is a disjoint union of components, since p is a covering. And hence p also maps each component homeomorphically to $\mathcal{C}_1(n)$. Now for any n we pick one of the components of the cover and call is $\widetilde{\mathcal{C}}_1$. $\widetilde{\mathcal{C}}_1$ is just a sheet in the cover, and should not be confused with the entire cover over $\mathcal{C}_1$ even though it is an abuse of (standard) notation.

So we can define the unique lift $\widetilde{\omega}$ such that

$$\widetilde{\mathcal{C}}_1(k) \times \widetilde{\mathcal{C}}_1(n_1) \times \cdots \times \widetilde{\mathcal{C}}_1(n_k) \mapsto \widetilde{\mathcal{C}}_1(n_1 + \cdots + n_k).$$

The situation is made visible in the following drawing. There have been made a slight but compatible change of names. The dots in the top part of the picture are to illustrate the fiber over the lower dot. The Y_1 is included in Y as the subspace that contracts to the basepoint; the dot next to Y.



As we need to get the operad structure on $\widetilde{\mathcal{C}}_2$ we need to construct the identity in $\widetilde{\mathcal{C}}_2(1)$. We have a unit $1 \in \mathcal{C}_1(1) \subset \mathcal{C}_2(1)$ from the operad structure on $\mathcal{C}_2$. So we pick in the fiber over the identity in $\mathcal{C}_2(1)$, $p^{-1}(1)$, an element such that it maps back to our original 1 by p. We will just call it 1, and thus the unit morphism is $\eta\colon * \mapsto 1$.

Before we continue we will show that the unitality and associativity requirements are met. I will now present the idea for the unitality diagram from definition 2.1 and the other check carries through in a similar way.

The bottom line of unitality diagram, $\widetilde{\mathcal{C}}_2(k) \times \widetilde{\mathcal{C}}_2(1) \xrightarrow{\widetilde{\omega}} \widetilde{\mathcal{C}}_2(k)$ can be naturally extended by the commutative square from the definition of $\widetilde{\omega}$.

$$\begin{tikzcd}
 & \widetilde{\mathcal{C}}_2(k) \times *^k \arrow[dl, "\mathbb{1} \times \eta^k"'] \arrow[dr, "f"] & \\
\widetilde{\mathcal{C}}_2(k) \times \widetilde{\mathcal{C}}_2(1) \arrow[rr, "\widetilde{\omega}"] \arrow[d, "p"'] & & \widetilde{\mathcal{C}}_2(k) \arrow[d, "p"] \\
\mathcal{C}_2(k) \times \mathcal{C}_2(1) \arrow[rr, "\omega"'] & & \mathcal{C}_2(k)
\end{tikzcd}$$

So we want to show that when f is an isomorphism, then the triangle commutes. That is equivalent to showing that if we assume that the triangle commute, then f is an isomorphism. We note that under that assumption the entire diagram commutes.

The projection that constitutes the universal cover over $\mathcal{C}_2$ also gives a map $p\colon \widetilde{\mathcal{C}}_2(k) \times *^k \to \mathcal{C}_2(k) \times *^k$, then by unitality in $\mathcal{C}_2$, where we denote the unit morphism $\widehat{\eta}$, we get the following big diagram of two triangles and three squares.

$$
\begin{tikzcd}
 & & \widetilde{\mathcal{C}}_2(k) \times *^k \arrow[dll, "\mathbb{1} \times \eta^k"'] \arrow[drr, "f"] \arrow[dd, "p"] & & \\
\widetilde{\mathcal{C}}_2(k) \times \widetilde{\mathcal{C}}_2(1) \arrow[rrrr, "\widetilde{\omega}"] \arrow[dd, "p"'] & & & & \widetilde{\mathcal{C}}_2(k) \arrow[dd, "p"] \\
 & & \mathcal{C}_2(k) \times *^k \arrow[dll, "\mathbb{1} \times \widehat{\eta}^k"'] \arrow[drr, "\mathbb{1}"] & & \\
\mathcal{C}_2(k) \times \mathcal{C}_2(1) \arrow[rrrr, "\omega"'] & & & & \mathcal{C}_2(k)
\end{tikzcd}
$$

Now from this diagram, we see that f is a lift of the $\mathbb{1}$ below it. We see that the left square commutes by taking an arbitrary point in $\widetilde{\mathcal{C}}_2 \times *^k$ and map it around. Since, as mentioned, the front square and the two triangle commutes (the upper by assumption) and then also the right square commutes. So it is a lift of the identity. Hence f is the identity if it is the lift that takes basepoint to basepoint. In other words we want to map $\widetilde{\mathcal{C}}_1(k)$ in $\widetilde{\mathcal{C}}_2(k) \times *^k$ to $\widetilde{\mathcal{C}}_1(k)$ in $\widetilde{\mathcal{C}}_2(k)$ via f. Now again, since the upper triangle, by assumption, commutes, checking the basepoint condition on f amounts to verify that the basepoints maps the right way around via $\widetilde{\omega} \circ \mathbb{1} \times \eta^k$. And this happens exactly due to definition of $\widetilde{\omega}$. First it will be mapped to $\widetilde{\mathcal{C}}_1(k) \times \widetilde{\mathcal{C}}_1(1)$ that is mapped to $\widetilde{\mathcal{C}}_1(k)$ by $\widetilde{\omega}$.

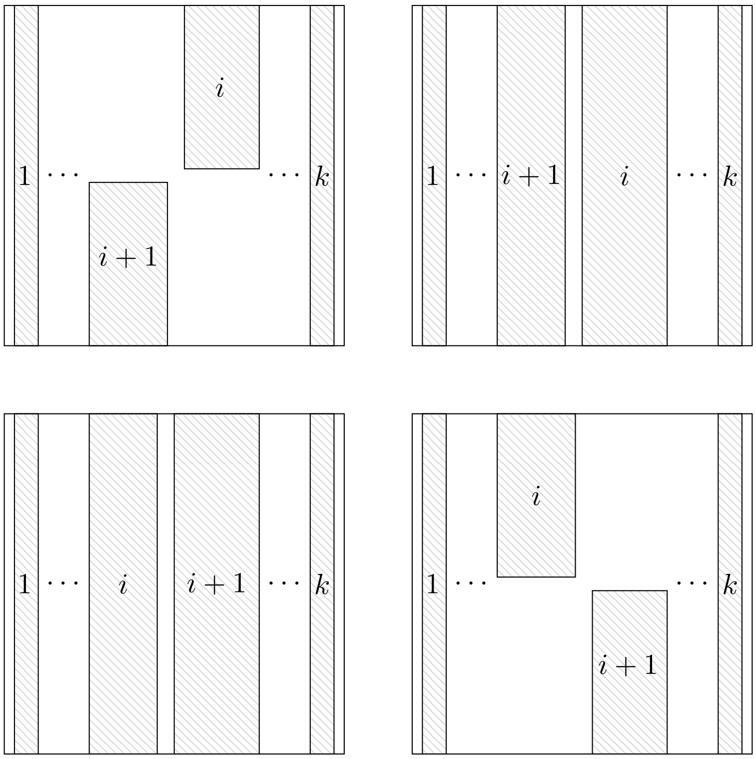
We recall that for $\widetilde{\mathcal{C}}_2$ to be a B_∞ operad we first of all need to confirm that it admits an operad structure. Then we need to show some additional properties of the structure are satisfied. Normally for operads they have a Σ_k action, but what we are setting out to give $\widetilde{\mathcal{C}}_2$ is an action of B_k, the braid group. Now the braid group can be projected to the symmetric group. Supplying an action of the braid group is in a sense giving more than we are asked for, as it can be projected to a normal Σ action. Sometimes an operad with a braid group action, instead of just a symmetric group action, is called a *braid operad*. The requirements are almost the same. The unital and associativity requirements coincide and the equivariance requirement is naturally modified to match (see [Fie, definition 3.2]). As we need to show a property of this action, that it is free, we will thus define it.

We are looking to define the action $\widetilde{\mathcal{C}}_2(k) \times B_k \to \widetilde{\mathcal{C}}_2(k)$. The braid group can be specified by a set of $k-1$ generators and some relations, namely

$$B_k := \langle \sigma_1, \dots \sigma_{k-1} \,|\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$$

as seen in definition 2.12. To construct the braid group action on the little cubes we will do exactly as discussed in remark 2.13. Metaphorically this is exactly what we will be doing in the following.

We have the Σ action on $\mathcal{C}_2$. And, again, we want to lift that action. It suffices to lift the generators to describe the action. We pick therefore a point in $\mathcal{C}_1$, c, that by i sit in $\mathcal{C}_2$. The first picture in the series below is an illustration of such point $i(c)$. The last picture illustrates $\sigma_i(i(c))$. The two pictures in between show how $i(c)$ can be moved to $\sigma_i(i(c))$ in a path $\alpha\colon I \times \mathcal{C}_2 \to \mathcal{C}_2$.



In the path we see how we have chosen to move little square labelled i over the one labeled $i + 1$ hence completely compatible with the metaphor we have in play for the braid group, mentioned above. We also note how the size of the little intervals in the bottom of the pictures end up having the same size in analogous to how the strands moved around in braid group.

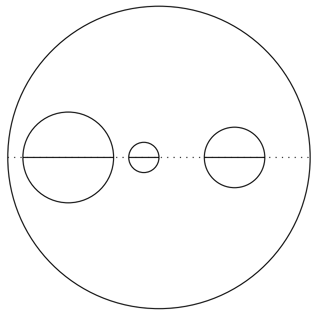
Since $\mathcal{C}_1$ is E_1 we know that the Σ action on the path-components is regular. This means that the choice made above can be arbitrarily made. When we apply the chosen σ_i we do not go to the same point (up to homotopy, as mentioned previously, due to the other property of being E_1.)

Now due to unique path lifting, we get, when we lift the constructed path α to a unique path in $\widetilde{\mathcal{C}}_2$ which describes the braid action on $\widetilde{\mathcal{C}}_2$. $\square$

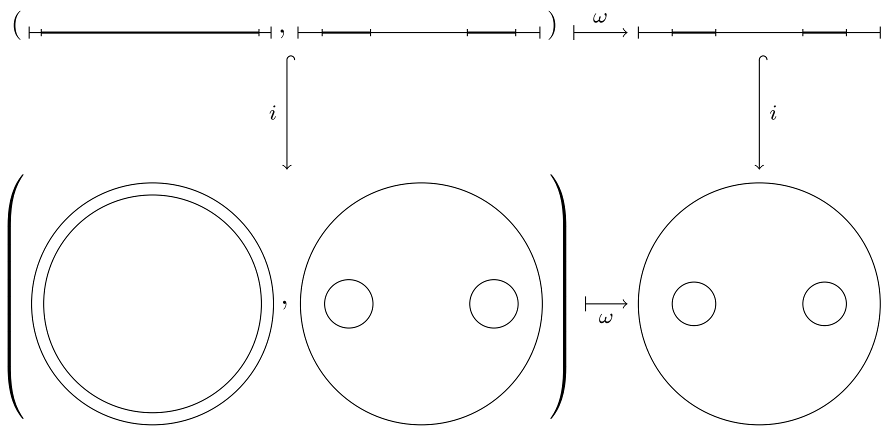
Following up we will now show that we can apply the same series of arguments to Disk_2.

Proposition 2.29. *The universal cover of* Disk_2 *admits a* B_∞ *operad structure.*

Proof. The embedding of Disk_1. The little 1-disks are really little intervals embedded in the unit interval. We can thus embed Disk_1 in Disk_2 in the following way. The unit interval that is hosting the little intervals we embed as the diameter of the unit disk that hosts the little 2-disks. Then for each little interval we pick the center and use that as center for a disk that has radius half the length of the little interval. An label-less example is pictured below.



Disk_1 is a suboperad of Disk_2. The embedding from above is an inclusion on space level, $i\colon \mathrm{Disk}_1 \to \mathrm{Disk}_2$, and it makes Disk_1 a suboperad of Disk_2. By definition it is injective, so to satisfy definition 2.5 of being a suboperad, then we lack only to verify that it is an operad morphism. Clearly the inclusion does noting to the labels, and hence the Σ action on the labels is not touched and hence still Σ-equivariant. Also the operad composition on Disk_2 commutes with the inclusion and preserves also the unit. Here is an example, that will scale to the general case, to visualize:



Disk_1 is E_1. As in proposition 2.28 we need to show this property as we need it to be contractible as non-Σ operad and have regular Σ action on the path-components. Let us start by showing that each path-component is contractible. Since on space level Disk_1 and $\mathcal{C}_1$ are clearly homotopic as they are identical, hence also Disk_1 is contractible. The $k!$ path-components of $\mathrm{Disk}_1(k)$ can be named by the order of the labels. In other words, the elements of a path-component is exactly those elements of Disk_1 where the order of little 1-disks are labelled the same. Hence, as in the cube case, the Σ_k action is regular.

With universal cover $p\colon \widetilde{\mathrm{Disk}}_2(k) \to \mathrm{Disk}_2(k)$ and ensuring that the needed properties of Disk_1 are met, we can now construct the lift of the composition in Disk_2 to an operad composition in $\widetilde{\mathrm{Disk}}_2$ in the same way as was done in the cube case. So $\widetilde{\omega}$ is the lift that makes the following diagram

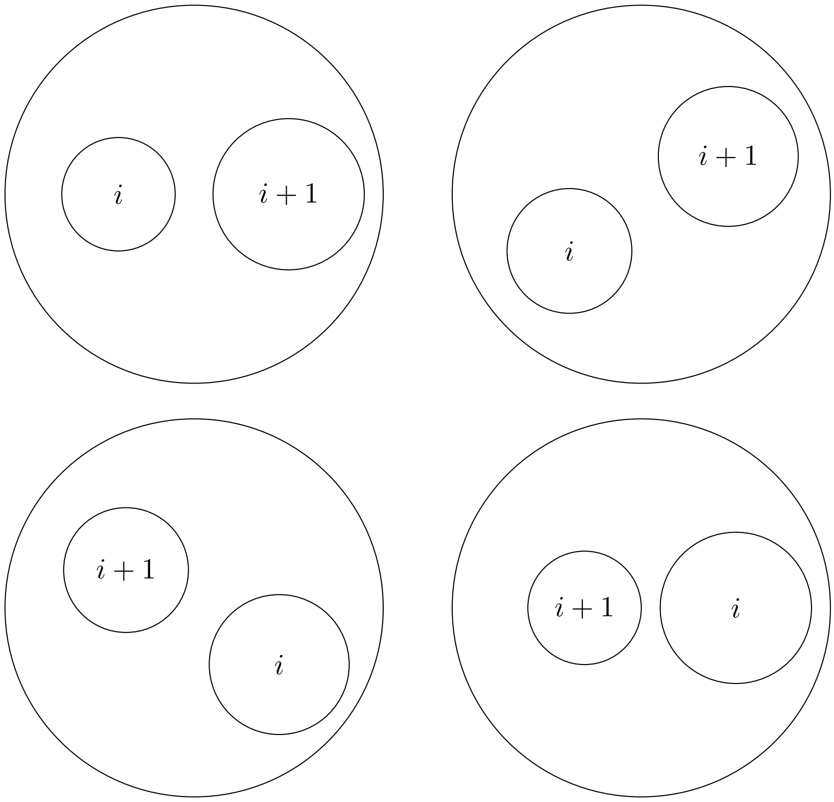
$$\begin{tikzcd}
\widetilde{\mathrm{Disk}}_2(k)\times\widetilde{\mathrm{Disk}}_2(n_1)\times\ldots\times\widetilde{\mathrm{Disk}}_2(n_k) \arrow[r, "\widetilde{\omega}"] \arrow[d] & \widetilde{\mathrm{Disk}}_2(n_1+\cdots+n_k) \arrow[d] \\
\mathrm{Disk}_2(k)\times\mathrm{Disk}_2(n_1)\times\ldots\times\mathrm{Disk}_2(n_k) \arrow[r, "\omega"'] & \mathrm{Disk}_2(n_1+\cdots+n_k)
\end{tikzcd}$$

commute. And it is unique by the following choice of basepoint

$$\widetilde{\mathrm{Disk}}_1(k)\times\widetilde{\mathrm{Disk}}_1(n_1)\times\ldots\times\widetilde{\mathrm{Disk}}_1(n_k)\mapsto\widetilde{\mathrm{Disk}}_1(n_1+\cdots+n_k)$$

that can be done due to Disk_1s ability to act as a basepoint. We recall that $\widetilde{\mathrm{Disk}}_1$ is not the cover over Disk_1 but only a sheet in the cover over Disk_1. Also we have unique unit element of $\widetilde{Disk}_2(1)$ to be 1 from the sheet $\widetilde{\mathrm{Disk}}(1)$ that will be projected down to the identity element of $\mathrm{Disk}_1 \subset \mathrm{Disk}_2$.

As for the cube case the unit and composition admit to the operad structure. For complete braid operad structure we need to define the braid action. Again we will lift it from the Σ action on Disk_2 by describing a path $I\times\mathrm{Disk}_2 \to \mathrm{Disk}_2$ that induces the braid action when that path is lifted to the cover. So to get dirty, we move the disk labelled $i+1$ over the disk labeled i and leave the other disks unmoved. The following series of pictures illustrate, as we read them left to right.



Is the braid action free? We see quite clearly only the identity element in the braid group will fix a point in the cover, so it is.

The last but not least point we have to make is to show that the cover is contractible. We recall from the recognition principle that this was a important step in constructing the actual equivalence of operads.

The first thing we note is that the fundamental group of the universal cover, $\widetilde{\mathrm{Disk}_2}$ is simply-connected. This follows from the definition of universal cover. So left is to deal with the homotopy groups for $n \geq 2$. About them we know, that the projection that constitutes the universal covering also induces isomorphism on the fundamental groups, π_n, for that range of n, see [Hat02, Prop. 4.1]. Since Disk_2 is $K(PB, 1)$, then that implies that $\pi_n(\mathrm{Disk}_2)$ is trivial for $n \geq 2$. In other words all the homotopy groups of the universal cover are trivial.

By Whitehead we know that if a map between CW complexes induces a weak homotopy equivalence then it is a homotopy equivalence. We also know, if a space is homotopic to a CW complex, then the cover of that space is homotopic to a CW complex. Disk_2 is (homotopic to) a CW complex. Hence it follows that the the universal cover is homotopic to a point.

□

Lemma 2.30. Disk_2 *is connected.*

Proof. It is homotopic to $\mathrm{Conf}(\mathbb{R}^2)$ that by proposition 2.16 is $K(PB, 1)$ hence simply-connected, thus Disk_2 is connected. □

Corollary 2.31. *The operad* Disk_2 *is* E_2

Proof. This follows directly from applying lemma 2.30 and proposition 2.29 to theorem 2.22. □

Now as a reminder, and almost too redundant, I will like to mention the following immediate corollary that follows strait from the definition of E_2, 2.17:

Corollary 2.32. *The operads* Disk_2 *and* $\mathcal{C}_2$ *are equivalent.*

It is still worth to highlight as an end of the preliminaries section as it ties it all up: We have shown that $Disk_2$ and $\mathcal{C}_2$ can be linked together via a chain of operad equivalences in a specific *ziq-zaq* way that will link them to any (connected, topological) operad which has a universal cover that admits a B_∞ structure. So when we in the next section apply the same method to the (soon to be properly defined) topological operad Cact, then we will have shown that it is E_2 by the recognition principle theorem 2.22 and hence also equivalent to Disk_2, which was the original goal.

3 The Cacti Operad

In this part of the thesis I will introduce and discuss the cacti operad, Cact, and show that it is equivalent to the little disk operad, Disk_2.

3.1 Definition and operad structure

Definition 3.1. A spineless cactus with n lobes is a planer rooted tree-like configurations of n S^1's with labels such that the zero point of each S^1 intersects with another S^1 below except for the lowest S^1 for which the zero point is called the global zero. The $i'th$ lobe is the circle labelled i. By the outside circle of a cactus we shall understand the circle with boundary of the same length as the boundary of the cactus.

That it is tree-like means that if the circles where disks, ie. solid, the cactus would be contractible. Also due to the tree structure each node in the tree, here lobes in the cactus, has a natural orientation towards the, specified, root. And this gives rise to a partial ordering of the lobes.

The orientation of the outside circle induces an orientation of the lobes. But is also a feature of being planer as a tree, i.e. a tree embedded in a plane, since then the lobes can be given a cyclic order, or simply just orientated. Each individual lobe and the outside circle also has an orientation given by for example a parametrization of the circles, over the interval 0 to 2π the unit circle can be described as the points $(\cos\nu, sin\nu)$ were ν runs through that interval. For $\nu = 0$ we have the zero point of the circle.

Definition 3.2. For each lobe in each cacti the segments of the lobe that is between each intersection point, between intersection point and the zero point, or, if there is no intersection points, from the zero back to itself, is called the *arcs*. The arc a representing a vertex v is the arc segment immediately after in the order of the outside circle.

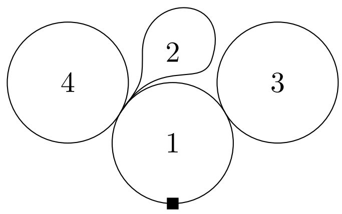
The above definition allows us to talk about the length of the arcs of a lobe. Notice that when the arc goes from zero and back to itself, then as the definition talks about arcs of specific lobes, we are not passing zero points of other lobes in the way back.

Definition 3.3. The space of all cacti with n lobes we denote $\mathrm{Cact}(n)$.

In this setting all lobes have the same length as S^1, or in other words they have unit radii. We assign a word to that:

Definition 3.4. We call the set Cact for the set of normalized cacti if all lobes have unit radii and denote it Cact^1

Example 3.5. Below is an example of an element in $\mathrm{Cact}(4)$. The lobes labelled 4 and 2 intersect the lobe labelled 1 in the same place, whereas the lobe labelled 3 intersects the lobe 1 in a different place. The lobe labelled 1 has the global zero as its zero. The lobes labelled $2, 3, 4$ have one piece of arc, while the lobe labeled 1 has tree arc pieces. One form the (global) zero to the intersection with the lobes 4 and 2, one from the intersection point of the lobe labeled 3 to the zero point and one in between the two intersection points.



This definition of cacti is in the literature often called *spineless cacti.* This is due to the fact that the original definition of cacti was with, so-called, spines and due to Voronov in [Vor05]. It echoes here as follows:

Definition 3.6. A cacti with spine is a cacti where the intersection points need not be the zero points and added a basepoint to one of the lobes.

So the literature often deals with cacti and spineless cacti where the former is assumed to be with spines. Here we focus on spineless cacti, so we will refer to then as cacti and the others as *cacti with spines* if we want to mention them. I will not go into detail about it here, but mainly mention it to avoid confusion.

We will now be more specific about how any cactus can be made into a tree of a special kind.

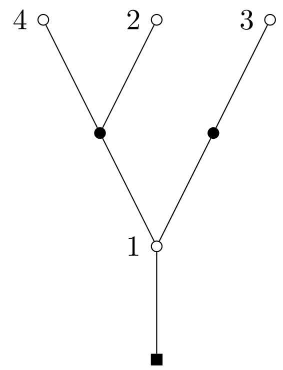
Definition 3.7. A graph can be considered coloured if the vertices are coloured. It is bi-coloured if it is coloured by two colours. By a black and white tree we shall understand a bi-coloured tree with the colours back and white.

So in the definition there is no restriction on how the vertices are to be connected. In the definition below we establish a restriction by constructing a black and white tree from a cactus.

Definition 3.8. Given a cactus we can construct the dual black and white tree of it. It is done in the following way. Each lobe is made into a white vertex. And each intersection point is made into a black vertex. The edges of the tree are then constructed in the following way. We connect each black vertex, that marks the intersection point of some lobes with the white vertices that represent the lobes that were intersecting in that point.

The word dual works fine here because we deal with normalized cacti. If the cacti were not normalized, then the word dual would be slightly off, since the graphs we deal with are not assumed to be metric. The metric information, that of the size of the lobes, will not be forgotten, but encoded differently as we will see later.

Example 3.9. The following example is the dual black and white tree of the previous (and only so far) illustrated example of a cactus. The circles with the numbers in them are the white lobes with their labels, the black square is the black vertex that is also the global zero and the round dots are the remaining black vertices.



The illustration is cheating a little bit, because the white vertices looks like small circles and hence we could measure the arc lengths between each edge. This information is not supposed to be included in the black and white tree, as it it not metric. We could have done that, but instead it is encoded differently as we will see later.

First we shall continue with more notation with respect to the black and white trees that will aid the construction of the encoding of the arc lengths.

We recall that we have a partial ordering of the vertices such that we have en orientation to the global root. Also a general notion from graph theory is that an edge, e_v is said to be *incident* to a vertex v if the one end of the edge is connected to the vertex independently of a possible orientation of the graph. In contrast when we *go* to a vertex by an edge we will be true to orientation.

Definition 3.10. For a dual black and white tree of a cacti we call the edged that go from a black vertex to a white vertex of a *white edge*, and we denote the set of withe edges E_w. The *set of white vertices* we denote V_w. For a white vertex, $v \in V_w$, we call the *number of edges going to* v for $|v|$. The set of dual black and white trees with n labelled white vertices we will simply denote $\mathcal{T}(n)$. If the dual black and white tree $\tau \in \mathcal{T}(n)$ is derived from the cactus c we will sometimes refer to it as τ_c to get the genesis straight. If the set of white vertices, V_w, is associated to a tree τ, we denote it $V_w(\tau)$. If the set of white edges, E_w, is associated to a tree τ, we denote it $E_w(\tau)$. And finally the set of edges of a tree $\tau \in \mathcal{T}$, independent of colour, is denoted $E(\tau)$.

Remark 3.11. For a white vertex v, note that $|v|$ is both the number of white edges incident to v and the total number of edges incident to v minus one.

Example 3.12. The dual black and white tree from example 3.9 is an element in $\mathcal{T}(4)$.

The moral of these constructions is expressed in the following definition about the topological type of cactus.

Definition 3.13. By the *topological type* of a normalized cactus $c \in \mathrm{Cact}^1(n)$ we mean the tree $\tau_c \in \mathcal{T}(n)$ where the labels are induced from the labeling of the lobes of c.

This means that given two cacti with the same number of lobes, then we can compare them up to the central topological features by comparing their associated dual black and white tree.

What we are missing is to deal with the length of the arcs in the cactus. We can think of that as a mere geometric feature of a cactus. So to differentiate between cacti with the same topological type but with varying arc lengths, we shall need the following lemma.

Lemma 3.14. *A normalized cactus is entirely described by its topological type and the length of the arcs.*

Proof. We are to show a bijective correspondence, so first assume that we have a cactus. Then from that cactus, c, we have constructed the tree, τ_c, that is the topological type of c. Definition 3.8 show how that is done. As mentioned the remaining data that is not encoded in the topological type is the arc length of the arcs of the cactus.

Now we need to show that we have an inverse method to get back to that exact cactus. So we take the tree. Then we blow up the white vertices such that they become unit circles. While we do that we make sure they do not intersect with each other, for instance by extending the edges suitably. We label the circles according to the labelling of the white vertices. Then contract the edges that are not white. By doing that the black vertices will meet the circles. The point where this happens will mark the root point of each circle. There is a black vertex to each white, so it is well-defined. The circle for which the special square black point meets the circle will be the global zero of the cactus. Left is to contract the white edges. While doing this deforming the circles might be necessary such that they do not intersect in new places. When this is done we have a cactus, and the (possibly deformed) circles are the lobes. To make it the same cactus as we started with we slide the intersection points around such that the arc length matches. $\square$

We would like to be more specific about how to encode the lengths of the arcs. This is the motive for lemma 3.16 below. For that lemma to make sense, we first need to extend our language with the concept of simplices.

Definition 3.15. By Δ^n we denote the n-simplex, where by $|\Delta^n|$ denotes the realization in $\mathbb{R}^{n+1}$ as the point set $\{(t_1, \ldots, t_{n+1})\}$ where $\sum_i t_i = 1$. The interior of $|\Delta^n|$ is denoted $|\mathring{\Delta}^n|$.

Lemma 3.16. *The length of the arcs of a lobe in a normalized cactus, c, that correspond to a white vertex v in the black and white tree of c is in a bijective correspondence with the open simplex* $\mathring{\Delta}^{|v|}$.

Proof. This follows almost immediately from the definitions. The points in the simplex is a $n+1$ tuple whose sum is 1. The sum of the length of the arcs of any given lobe in a normalized cactus is 1. Any lobe can be divided into $|v|+1$ arc segments, as mentioned in 3.11. Thus it follows. □

Example 3.17. As an example, take the lobe labelled 1 from the example 3.5. As discussed there, it has 3 arc segments. In example 3.9 we have the tree representing its topological type. The white vertex of that tree that is labelled 1 has 2 white edges going to it, and 3 edged incident to it.

Remark 3.18. Suppose we did not restrict to the interior of the simplices. It would then be allowed to take a point on the boundary of the simplex. While this point would be a $n+1$ tuple where the sum would be 1 as required, it would imply that one arc segment should have length 0. That would mean that the corresponding lobe would be divided into only n arc segments, which would be a mismatch.

Now for each tree we will define a product of simplices derived from the tree. This is the feature we have been seeking to encode the arc lengths of an entire cactus.

Definition 3.19. For each $\tau \in \mathcal{T}(n)$ we define the following product

$$\Delta(\tau) = \prod_{v \in V_w(\tau)} |\Delta^{|v|}|$$

of realizations of simplices. And similarly for the interior;

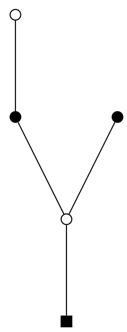
$$\mathring{\Delta}(\tau) = \prod_{v \in V_w(\tau)} |\mathring{\Delta}^{|v|}|.$$

Remark 3.20. The product has an order while the index set, the white vertices, does not in it self carry an order. But sine a $\Delta(\tau)$ is associated a τ then we get the order from the orientation of the tree.

Example 3.21. For the three, τ, from example 3.9, we get the following product

$$\Delta(\tau) = |\Delta^2| \times |\Delta^0| \times |\Delta^0| \times |\Delta^0|.$$

Example 3.22. Not any product of realizations of simplices gives rise to a tree $\tau \in \mathcal{T}$. Take as an example the product $|\Delta^2| \times |\Delta^0|$. If we try to read back a tree it would give the following



which, although being a black and white planer tree, is not an element of $\mathcal{T}$.

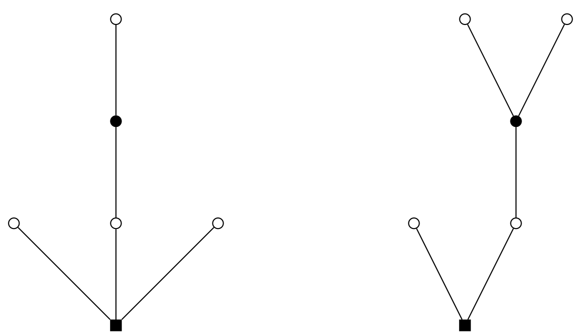
Proposition 3.23. *On space level* $\mathrm{Cact}^1(n) = \coprod_{\tau \in \mathcal{T}(n)} \mathring{\Delta}(\tau)$.

Proof. By definition 3.19, lemma 3.16 and 3.14 this follows. $\square$

The notion of topological type can be extended further to the notion that answer the indexing question when we later will equip Cact^1 with a CW complex structure.

Definition 3.24. The set of dual black and white trees with n white vertices and k white edges is denoted $\mathcal{T}(n)^k$.

Example 3.25. As an example of the above definition; the following two trees, τ_1, τ_2, are the elements from $\mathcal{T}(4)^1$. For this illustration the actual labels of the vertices are not relevant, hence not printed.



We will now talk about what we will mean by *degeneration* of a cactus, and how it is done for the tree $\tau \in \mathcal{T}$ counterpart.

Definition 3.26. Given a cactus, c, with an arc segment, a, such that $|a| < 1$, then we say that we *degenerate* it by the arc a when we homotopically contract it to a point.

Remark 3.27. We observe that for a cactus, c, as in the above definition we need in order to have an arc segment of non unit length to have at least one intersection point away from the global zero. Naturally also $c \in \mathrm{Cact}^1(n)$ for $n > 1$.

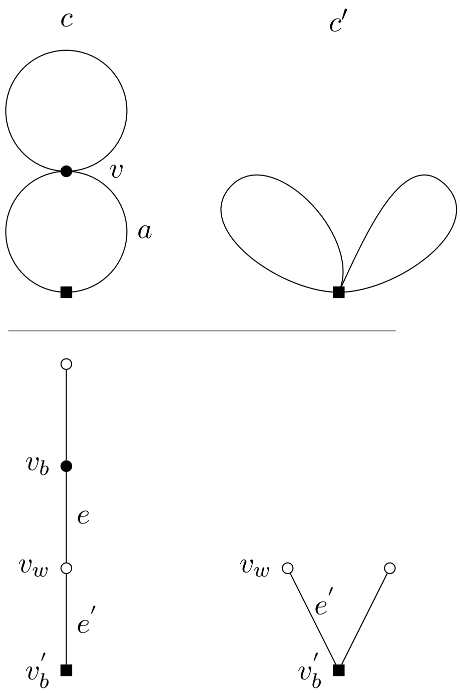
Also when we degenerate a cactus, c, as in the above definition, then we still keep the number of lobes and all the indices intact.

Now to the analogous concept for trees.

Definition 3.28. Given a tree $\tau \in \mathcal{T}(n)$. Let the edge e be incident to an white vertex v_w with $|v_w| > 0$. Let v_b be the other vertex incident to e. Then let $e^{'}$ be the edge (also) incident to v_w but preceding e when considering the orientation of the tree. Name the other vertex incident to $e^{'}$ $v_b^{'}$.

The degeneration of τ *by the edge* e is then defined to be the tree $\tau^{'}$ by contracting the edge e of τ while making the tree temporarily disconnected and then identifying the vertex v_b with the vertex $v_b^{'}$. Then the tree is again connected and belonging to $\mathcal{T}(n)$ and the ordering of the branch above $v_b^{'}$ are kept intact.

Example 3.29. The preceding two definitions are illustrated in the following. It is showing almost literally the situation of the definition of degenerating a tree.



The cactus illustration is the cactus counterpart. This also illustrates the general case as this small tree/cactus can be seen as a sub-tree/-cactus, where there would just be appended more “vegetation” in either place where it makes sense.

We will now define a CW complex and then show its relation to the set $\mathrm{Cact}(n)$. This is a milestone in this section. By that we gain a topological structure on Cact and with little more work a composition of cacti that equip Cact with an operad structure.

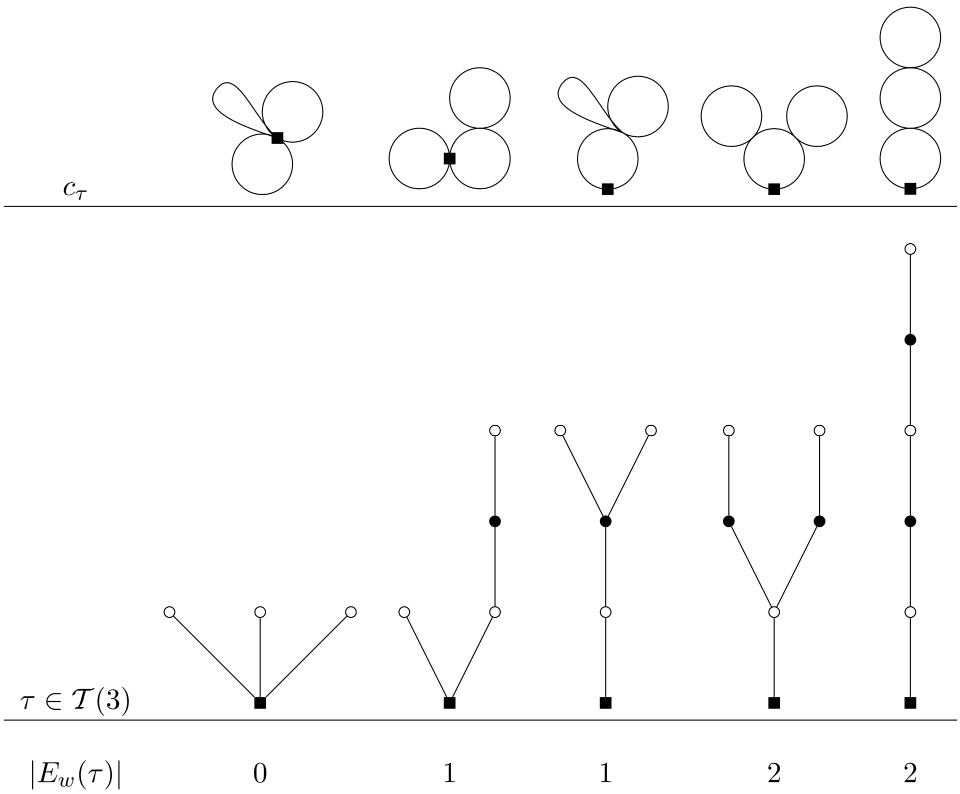
Definition 3.30. Call $\Delta(\tau)$ a *cell.* By the v'th face of $\Delta(\tau)$, where v is a white vertex of τ, we mean the subset of $\Delta(\tau)$ that has the v'th coordinate equal to zero.

Example 3.31. The subset that has the v'th coordinate equal to zero really is just the subset that goes on dimension down in the product of simplices. So if $\Delta(\tau) = |\Delta^{r_1}| \times \cdots \times |\Delta^{r_n}|$ then the v'th face lives in say Δ^{r_i}. Then when this has one of the coordinates equal to zero, it means that the v'th face of Δ^{r_i} is the set, where $\{(t_o, \ldots, t_{r_i})| \sum t_j = 1, t_l = 0\}$ that equals $\Delta^{r_i - 1}$. Hence the v'th face of $\Delta(\tau)$ is the subset $|\Delta^{r_1}| \times \cdots \times |\Delta^{r_i - 1}| \times \cdots \times |\Delta^{r_n}|$.

We need this concept such that the attaching maps in the following CW complex construction makes sense.

Definition 3.32. Let $K(n)$ be the CW complex that has as k-cells $C(\tau) := \Delta(\tau)$ indexed by $\tau \in \mathcal{T}(n)^k$. Where τ is the topological type of a cactus c with a vertex v and an arc segment a at v, then $\tau^{'}$ is the topological type of a cactus $c^{'}$ that is the degeneration of the cactus c by the arc a. We attach by identifying the v'th face of $\Delta(\tau)$ to $\Delta(\tau^{'})$. We call the the attaching map e_τ and its restriction to the interior for $\mathring{e}_\tau$.

Example 3.33. In this example we will look at the CW complex $K(3)$ in order to shed some light on the definition of the CW complex $K(n)$.



In the table above, in the first row, is listed the different "types" of normalized cacti of 3 lobes, then in the second row is their associated topological types. Note that this list is exhaustive. Note also that this list correspond to that of Cact(3), meaning that the cells of the CW complex match the elements of Cact(3). Below that is listed the number of white edges in the trees. This is the number that indicates the CW complex cell dimension.

Theorem 3.34. *The CW complex $K(n)$ and the set* $\mathrm{Cact}^1(n)$ *are in a bijective correspondence.*

Proof. Follow now directly from the proposition 3.23 and definition. □

Definition 3.35. Σ_n acts on $\mathrm{Cact}^1(n)$ by acting on the labels.

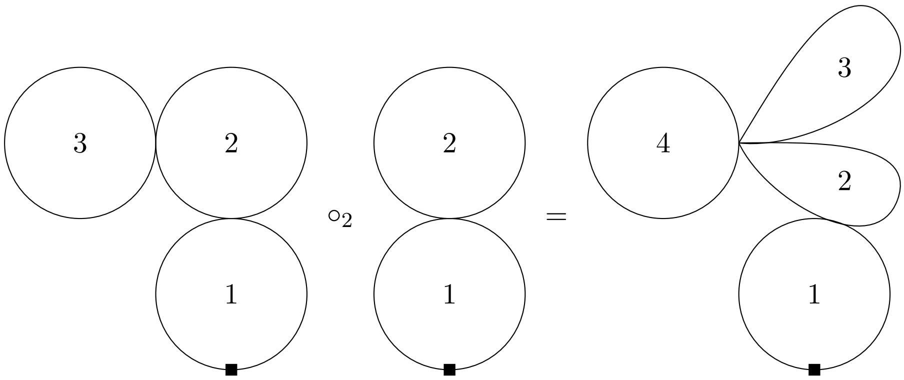
Definition 3.36. Let $\eta\colon * \to \mathrm{Cact}^1(1)$ be the unit morphism.

Definition 3.37. As composition of cacti from Cact^1 consider the following map

$$\circ_i\colon \mathrm{Cact}^1(n) \times \mathrm{Cact}^1(m) \to \mathrm{Cact}^1(n+m-1)$$

defined by the following way. It takes a cactus $c_n \in \mathrm{Cact}^1(n)$ and $c_m \in \mathrm{Cact}^1(m)$. We want to glue in the cactus c_m into the i'th lobe of c_n. So we rescale the i'th lobe of c to have the same size as the outside circle of c_m and then we can glue in c_m by identifying the outside circle of c_m to the scaled i'th lobe of c_n such that the global zero of c_m is assigned to the zero point of the scaled lobe. While gluing we shift the labels such that the labels of the glued in part will be $i, 1+i, \ldots, m-1+i$. The first $i-1$ labels of c_n will stay intact and the last part will be shifted by m, so they become $m+i, m+i+1, \ldots, m+n-1$.

Example 3.38. We take an element $c_n \in \mathrm{Cact}^1(3)$ and $c_m \in \mathrm{Cact}^1(2)$ and display the composition $c_n \circ_2 c_m$.



In the cactus c_n we see that the lobe labelled 2 has two arc segments, the first of 1/4 length and the second of 3/4 length. That means that when gluing in the cactus c_m that consists of two lobes, then we need to glue in the point that is half the way around of the lobe labelled 1 to the previous mentioned point of intersection that divide the lobe labeled 2 of c_n into two segments. In this case this turns out to be exactly the point of intersection between the lobes of c_m as they meet halfway around the lobe (of c_m) labeled 1.

While the composition $\circ_i$ is not quite what we might have hoped for in order to realize Cact^1 as an operad. But due to [Mar08, Proposition 13] it suffices with such composition given it satisfies an associativity requirement:

Proposition 3.39. *Given a composition*

$$\circ_i \colon \mathcal{O}(n) \otimes \mathcal{O}(k) \to \mathcal{O}(n+k-1)$$

that satisfies the following associativity requirements: for each $f \in \mathcal{O}(a)$, $g \in \mathcal{O}(b)$ and $h \in \mathcal{O}(c)$,

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g & \textit{for } 1 \leq i \leq j, \\ f \circ_j (g \circ_{i-j+1} h) & \textit{for } j \leq i < b+j \textit{ and} \\ (f \circ_{i-b+1} h) \circ_j g & \textit{for } j+b \leq i \leq a+b-1, \end{cases}$$

then there can be constructed a composition of operads

$$\omega\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \ldots \otimes \mathcal{O}(n_k)$$

from $\circ_i$ *in the following way,*

$$\omega\colon (f, g_1, \ldots, g_k) \mapsto (\cdots((f \circ_k g_k) \circ_{k-1} g_{k-1}) \cdots) \circ_1 g_1.$$

where $f \in \mathcal{O}(k)$ *and* $g_i \in \mathcal{O}(n_i)$ *for* $1 \leq i \leq k$*.*

So we are back on track and can state this conclusive corollary:

Corollary 3.40. *Cact*^{1} *is an (topological) operad.*

Until now we have dealt with normalized cacti (without spine). Now we will run though the construction for the general (spineless) cacti of lobes of varying radii, Cact. We already have defined it, and now claim that:

Lemma 3.41. *As spaces* $\mathrm{Cact}(n) = \mathrm{Cact}^1(n) \times \mathbb{R}^n_{>0}$.

Proof. This is fairly easily shown as we realize that we can describe the general cacti from $\mathrm{Cact}(n)$ by a normalised cacti from Cact^1 by adding the information of how much we need to rescale each lobe, which can be described by a point in $\mathbb{R}^n_{>0}$. The reason to limit $\mathbb{R}^n$ to positive coordinates is to make this well-defined. For example a zero coordinate would kill a lobe and a negative coordinate would make the parametrization run in reverse direction and hence make clutter. In the other direction; we could likewise factor out the information in a general cacti to a normalizes and the point in $\mathbb{R}^n_{>0}$. □

Remark 3.42. From lemma 3.41 we see that Cact^1 induces a topology on Cact.

Remark 3.43. The method of gluing we defined for normalized cacti works also in the setting of cacti with lobes of varying radii. We will call it by the same name,

$$\circ_i\colon\ \mathrm{Cact}(n) \times \mathrm{Cact}(m) \to \mathrm{Cact}(n + m - 1).$$

We can give a direct construction analogous to that of normalized cacti. We can also define it though lemma 3.41 by going back and forth while using the gluing defined for normalized cacti.

Thus we arrive as the following final statement:

Proposition 3.44. *The gluing,* $\circ_i$*, makes* Cact *an operad.*

3.2 Corolla cacti

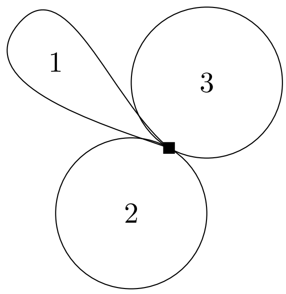
In this subsection we will define and study the *corolla cacti*. We will show that it is an E_1 sub-operad of Cact.

In the literature the corolla cacti is often referred to as the *spineless corolla cacti*. In the previous section we have chose to contract the term "spineless" because we deal mainly with spineless cacti so it would by hideous to carry that term around. For the same reason we contract the spineless corolla cacti to just corolla cacti. Should we need to, we can talk about a corolla cacti with spine.

Kaufmann defines it the following way.

Definition 3.45. A corolla cacti is a cacti that where all the intersection points are at the global zero. The set of corolla cacti with n lobes is denoted $\mathrm{CC}(n)$.

Example 3.46. We notice that this is equivalent to cacti with $|E_w| = 0$, thus we have as an example the following image of a corolla cacti from $\mathrm{CC}(3)$.



Lemma 3.47. CC *is a suboperad of* Cact.

Proof. First we observe that as spaces, for any n, $CC(n) \subset \mathrm{Cact}(n)$, which we can use as an injective map, $i\colon \mathrm{CC}(n) \to \mathrm{Cact}(n)$. Left is, by definition 2.5 to verify that this map can be seen as a morphism of operads respecting the definition 2.3.

It preserves the unit. And is also Σ_n-equivariant. But let us exercise the composition we have defined on Cact to verify that is respected by the inclusion.

We do not need to show it for ω but only for $\circ_i$ as the corresponding diagram for ω will commute accordingly if the one for $\circ_i$ commutes. So observe the following diagram:

$$\begin{array}{ccc} \mathrm{CC}(n) \times \mathrm{CC}(m) & \xrightarrow{\circ_i} & \mathrm{CC}(n+m-1) \\ \downarrow i & & \downarrow i \\ \mathrm{Cact}(n) \times \mathrm{Cact}(m) & \xrightarrow[\circ_i]{} & \mathrm{Cact}(n+m-1) \end{array}$$

When we compose to corolla cacti, c_n, c_m via $\circ_i$, for $1 \leq i \leq n$, then we get exactly a corolla cacti in $\mathrm{CC}(m+n-1)$. Thus it is clear that $\circ_i$ commutes with the inclusion. □

Proposition 3.48. *Corolla Cacti,* $\mathrm{CC}(n)$*, is* E_1*.*

Proof. By theorem 2.21 we have our check list. First we need to see that the path-components are contractible. First we notice that there is $n!$ number of path-components. For each point in $\mathrm{CC}(n)$ the lobe are labelled. And whenever any two points in $\mathrm{CC}(n)$ have a different linear ordering of the labels then they belong to two different path-components, as there is no path in $\mathrm{CC}(n)$ between them. In return any two points of the same path-components are connected by a path where the lobes are homotopically deformed from the one to the other. In other words, then path-components are all contractible.

Lastly we need to see that the action of Σ_n on $\pi_0(\mathrm{CC}(n))$ is regular. So consider two points $x, y \in \mathrm{CC}(n)$, then there exists exactly one element, $\sigma \in \Sigma_n$ that moves the one to they other, $\sigma(x) = y$. The action works on the labels, and due to contractibility of the path-components we can view $\mathrm{CC}(n)$ as Σ_n. □

3.3 Forgetful map

In this subsection we will construct a projection $p\colon \mathrm{Cact}(n+1) \to \mathrm{Cact}(n)$ called the *forgetful map*. Also we will compute the fiber and show a few properties we will exploit later on.

Definition 3.49. For a cactus $c \in \mathrm{Cact}(n)$ the following construction based on it is called *the completed chord diagram of* c. And we will denote it as $\mathrm{Chord}(c)$.

We start by the global zero of the cactus c. Then we take the outside circle of c. While we traverse the outside circle we mark all the zero points on the outside circle. We recall that all the zero points are intersections points between two or more lobes of the cactus, except that the global zero might not be an intersection point.

All the marked points on the outside circle will correspond to intersection point of n lobes there will be n marked points. If the global zero was not an intersection point it will be addition point on the outside circle as well.

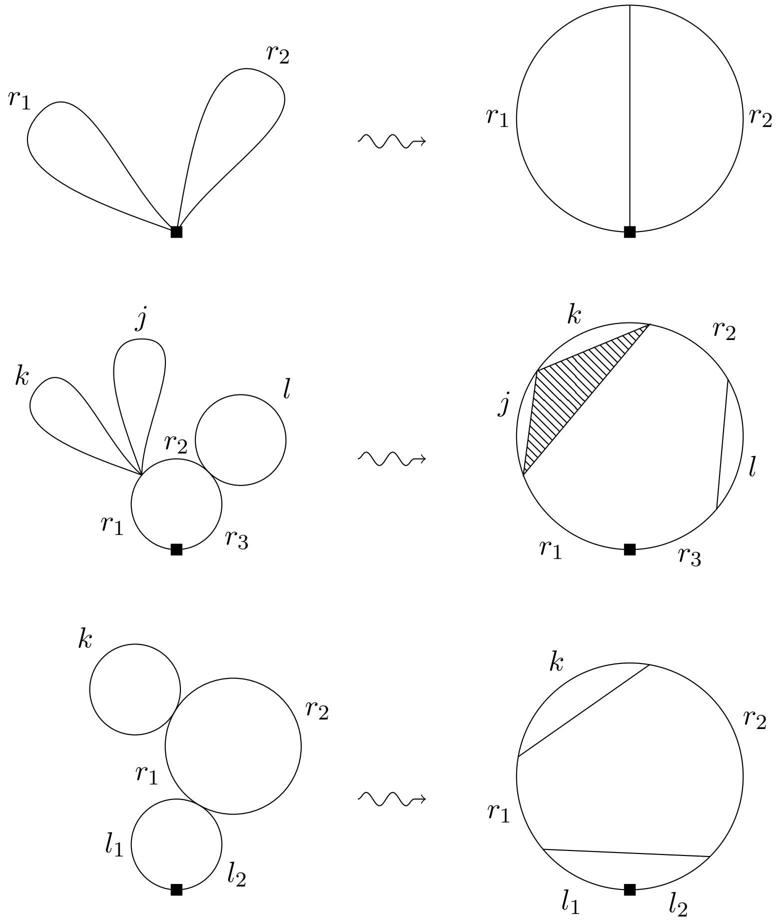
All the arc segments on the outside circle will be of length corresponding the lobe segment they originate from.

Between all the n marked points that come from the same intersection point we will insert a $n-1$ simplex such that the vertices of the simplex is identified with the marked points. What we have then is the complete chord diagram of the cactus c.

The complete chord diagram is an object in the category of topological spaces.

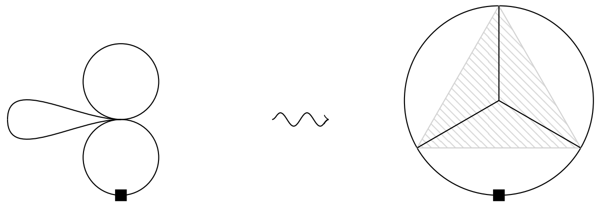
Before we continue with its sister definition we will see some examples.

Example 3.50. I will show the following three examples to illustrate the definition. By combining the idea presented in them it should be easy to construct the chord diagram of any cacti.



Definition 3.51. The subspace of a completed chord diagram of a cactus called *the spine* of the completed chord diagram is the following: For each simplex we locate the barycenter and then we connect that by a straight line to each of the vertices of the same simplex on the outside circle.

Example 3.52. In this example is illustrated a cactus and the spine of the completed chord diagram of the cactus. The completed chord diagram is drawn underneath in a light grey colour to aid the illustration. It is not part of the spine.



Lemma 3.53. *A cactus is homotopic to its completed chord diagram and to the spine of the completed chord diagram.*

Proof. First consider the completed chord diagram and its spine. By the definition of the spine we see, that this subspace is a deformation retraction of the complete chord diagram it sits in. Hence they are homotopic.

Then the spine and the cactus case. All the straight lines in the spine are contractible. By collapsing those we get exactly the original cactus back. And we can do the inverse as follows: Take a cactus, c. We then produce the completed chord diagram of c as a template. Then the barycenters of the simplices of the chord diagram are well-defined, and we can identify the vertices of c to the barycenters. With the barycenters in the right place we can blow up the vertices by a line for each lobe incident to the intersection that specifies the vertex. We do that in a way such that we do not alter the length of the arcs of the lobes and until we arrive at the outside circle. □

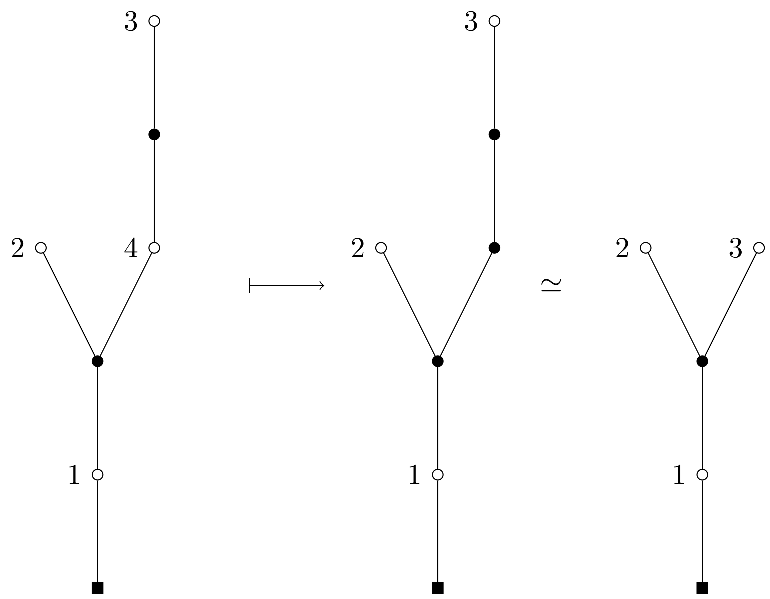
We will now continue with defining forgetful maps. We need, as mentioned, a map projecting cacti to cacti of one lobe less. That will be defined though a projection on trees.

Definition 3.54. The map

$$p^T : \mathcal{T}(n+1) \to \mathcal{T}(n)$$

that maps a tree $\tau \in \mathcal{T}(n+1)$ to the image $p^T(\tau) \in \mathcal{T}(n)$ we will call *forgetful*. The image is the tree $p^T(\tau) \in \mathcal{T}(n)$ we get from the original τ by forgetting the vertex labelled $n+1$ and corresponding edges. We do that be forgetting the label and colour the vertex black and contract the edges incident to it. The black vertex immediately above and below (the former white edge labelled $n+1$, now black unlabelled) will then be identified. If the vertex $n+1$ has only one edge incident to it, then we contract down to the first white edge.

Example 3.55. Here the forgetful map p^T is illustrated by taking a tree $\tau \in \mathcal{T}(4)$ and mapping it to $\mathcal{T}(3)$.



The map $p^T : \mathcal{T}(n+1) \to \mathcal{T}(n)$ induces a similar forgetful map on the simplex representation, $\Delta(\tau)$:

Definition 3.56. Where $\tau \in \mathcal{T}$, $e \in E(\tau)$, then let

$$p^{\Delta(\tau)} \colon \Delta(\tau) \to \Delta(p^T(\tau))$$

be given by

$$\Delta(\tau) \ni (x_e) \mapsto (x'_e) \in \Delta(p^T(\tau))$$

where the edges of the original tree τ that is not contracted are identified with the edges of $p^T(\tau)$.

We recall that we have a bijection $\mathring{e}_\tau$ between $c \in \mathrm{Cact}$ and $\Delta(\tau_c)$ following theorem 3.34. So we can go from a cactus til its point in a simplex representation and back again. The simplices are associated to trees. So by the map $p^{\Delta(\tau)} \colon \Delta(\tau) \to \Delta(p^T(\tau))$, then we can make the following definition of a forgetful projection on Cact:

Definition 3.57. Define a map

$$p' \colon \mathrm{Cact}^1(n+1) \to \mathrm{Cact}^1(n)$$

through the composition

$$\mathrm{Cact}^1(n+1) \ni c' \mapsto \mathring{e}_{p^T(\tau)} \circ p^{\Delta(\tau)} \circ \mathring{e}_\tau^{-1}(c') \in \mathrm{Cact}^1(n).$$

Let $c = (c', (r_1, \ldots, r_{n+1}))$ and define the map

$$p \colon \mathrm{Cact}(n+1) \to \mathrm{Cact}(n)$$

through p' by

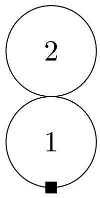
$$c \mapsto (p'(c'), (r_1, \ldots, r_n)).$$

Proposition 3.58. *The fiber of the forgetful map p over a cactus $c \in \mathrm{Cact}$ is homotopic to the completed chord diagram of c times $\mathbb{R}^n_{>0}$; $p^{-1}(c) \simeq \mathrm{Chord}(c) \times \mathbb{R}^n_{>0}$.*

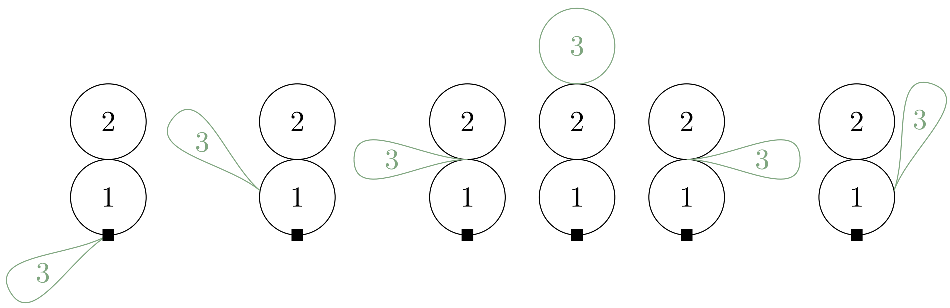
The fiber of the forgetful map p over a cactus $c' \in \mathrm{Cact}^1$ is homotopic to the completed chord diagram of c'; $p^{-1}(c') \simeq \mathrm{Chord}(c)$.

To motivate the proof of this lemma, we will look at the fiber over a specific point to illustrate its nature.

Example 3.59. Let us consider the cactus $c \in \mathrm{Cact}(2)$ pictured as:

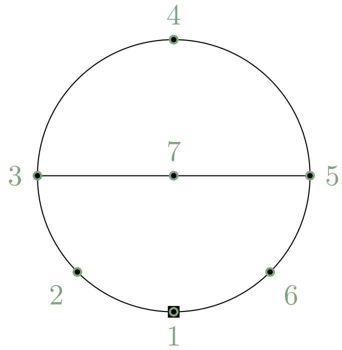


In this example we shall see some of the points in the fiber over c as an illustration and where they fit as points in $\mathrm{Chord}(c)$. In short the fiber consist of cacti of $2+1$ lobes where the extra lobe is what we will present now. Around the outside circle of c there is a continuum of places where an extra lobe could intersect. Metaphorically we can say that we add a lobe by the global zero and then slide the intersection point around the outside circle. The following illustration displays some cacti of such path.



If we took the first four of those cacti, ignored the labels and rendered the topological types of them, then those types will span the fiber. With the labels, and also metaphorically and essentially differently, we need the cacti where the extra lobe is added in between the two original lobes:

Hence the entire fiber as the complete chord diagram over c, where the points used to illustrate are pinned to, are:



Inspired by the above example here is a proof in short.

Proof. First of all we notice that between the normalized version and the non-normalized is a factor of $\mathbb{R}_{>0}$ that scales the chord diagram. So we can deal with the rest of the non-normalized case and that for normalized cacti in one go.

We consider an arbitrary cactus c in $\mathrm{Cact}^1(n)$. We want to give a description of the fiber over it by the forgetful map, and see how it is $\mathrm{Chord}(c)$. Like in example 3.59 we can attach the $n+1$'th lobe at any point around the outside circle of c, hence that is a subset of fiber over c.

Left is to consider the possible intersection points of c. For any such point of intersection of lobes in c, there is a zero point. We can take this point and blow it by a k simplex, if we assume, there is $k+1$ lobes intersecting at this point. We do that for each point of intersection until we reach the outside circle. Any point in those simplices will be in the fiber as, when attaching the $n+1$'th lobe there, would be a point in the fiber, as it would be the same as splitting c at an intersection point and glue in the extra lobe in an order preserving manner with respect to lobes of c. In short: We have described the possibilities of adding a lobe to c to make it a point in the fiber over c and seen that this actually describes $\mathrm{Chord}(c)$. $\square$

Corollary 3.60. *The fiber of the forgetful map p over $c \in \mathrm{Cact}$ is homotopic to the cactus $c \in \mathrm{Cact}(n)$ that further is homotopic to a wedge of circles, $\bigvee_n s^1$.*

Proof. Clearly the later statement holds since we can define a homotopy that slides all the lobes of the cactus by intersection points down to the global zero.

The first statement follows from the previous proposition and lemma 3.53. □

Now that we have computed the fiber of the forgetful map, p, we will as promised in the beginning of this subsection continue to show further properties with respect to it which we will use in subsection *Homotopy type of Cacti 3.5.*

Definition 3.61. Let c and c' be cacti from Cact^1 and let τ and τ' be their associated topological type. If

$$e_{\tau'}(\Delta(\tau')) \subset e_{\tau}(\Delta(\tau)) \subset \mathrm{Cact}^1 \tag{3.1}$$

where are the maps $e_{\_}$ that stem from definition 3.32 (through theorem 3.34), then we say that c' is *derived* from c and likewise that τ' is *derived* from τ.

Further; if the inclusion is strict then we call c' a *degeneration* of c, and also τ' and *degeneration* of τ.

Here is a few examples to illustrate the definition and to show that this notion of *degeneration* is compatible with the one we have defined previously for trees that represent a topological type of cacti in definition 3.28.

Example 3.62. This example aims to illustrate the simple case of equality in equation (3.1). So the setting is that we have to cacti $c, c' \in \mathrm{Cact}^1(n)$ with associated topological types τ, τ'. Suppose $\tau = \tau'$, then $\Delta(\tau) = \Delta(\tau') = \prod_i |\Delta^{r_i}|$ by definition 3.19. Now since there is equality $\Delta(\tau) = \Delta(\tau')$ then also $e_\tau(\Delta(\tau)) = e_{\tau'}(\Delta(\tau'))$.

So the cacti c and c' are cacti with the same number of lobes, same topological type, meaning that for them to be different the length of the arc sections between intersection points are not all identical.

Of course the case where the inclusion is strict is of most interest. In fact:

Remark 3.63. When the inclusion, from equation (3.1), is strict then it induces a partial order on the trees in $\mathcal{T}$: $\tau' \prec \tau$ if τ' is a degeneration of τ.

Example 3.64. Let the cacti c and c' be as in examples 3.29. We let their topological types be τ and τ'. In the example 3.29 c' was a degeneration of c. We you like to see that this still holds with the notion of definition 3.61, both to exercise the use of the definition but also, to observe how the two notions are compatible. Hence we would like to verify that

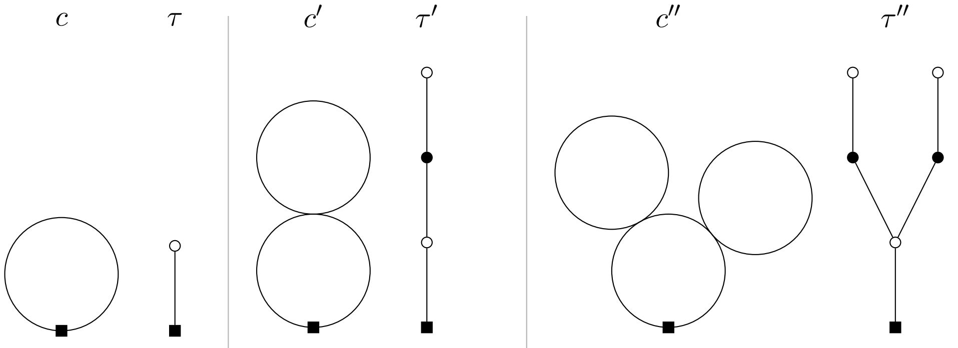
$$e_\tau(\Delta(\tau)) \subsetneq e_{\tau'}(\Delta(\tau')). \tag{3.2}$$

The equation (3.2) holds by definition of the inclusion in equation (3.1) if the inclusion of spaces $\Delta(\tau) \subsetneq \Delta(\tau')$ does. As we have that $\Delta(\tau) = |\Delta^2| \times |\Delta^0| \times |\Delta^0|$ and $\Delta(\tau) = |\Delta^1| \times |\Delta^0| \times |\Delta^0|$, we see it does hold.

This also mean that $\tau' \prec \tau$.

Definition 3.65. For a tree $\tau'' \in \mathcal{T}$ such that $c, c' \in e_{\tau''}(\Delta(\tau''))$, then we say that c and c' (and τ_c and $\tau'_{c'}$) share *the common type* τ''.

Example 3.66. The following pairs of cacti with their topological types illustrate definition 3.65 above



$\Delta(\tau) = |\Delta^0|$ | $\Delta(\tau') = |\Delta^1| \times |\Delta^0|$ | $\Delta(\tau'') = |\Delta^2| \times |\Delta^0| \times |\Delta^0|$

Definition 3.67. Where c and c' are cacti with common type τ'', then by $d_{\tau''}(c, c')$ we denote the *distance* between c and c' when lifted to the product of realization with respect to τ'', i.e. $\Delta(\tau'')$.

Definition 3.68. Let $c \in \mathrm{Cact}(n)$ and $\tau \in \mathcal{T}(n)$ be such that $c \in e_\tau(\Delta(\tau))$ and let $\varepsilon > 0$, then we define the following sets:

$$U(c, \varepsilon, \tau) = \{c' \in e_\tau(\Delta(\tau)) \text{ st. } d_\tau(c, c') < \varepsilon\}$$

and

$$U(c, \varepsilon) = \bigcup_{\tau \text{ st. } c \in e_\tau(\Delta(\tau))} U(c, \varepsilon, \tau). \tag{3.3}$$

Further if $c' \in U(c, \varepsilon)$ implies that c is a degeneration of c', then we call ε *small* for c.

Remark 3.69. Assume that ε is small for a cactus c, then that implies that any $c' \in U(c, \varepsilon)$ will be a degeneration of c.

The following lemma collects three properties regarding $U(c, \varepsilon)$.

Lemma 3.70. *1. For ε is small w.r.t. c then $U(c, \varepsilon) \simeq *$.*

2. For $c \in \mathrm{Cact}^1$ the sets $\mathcal{U} := \{U(c, \varepsilon)\}$ is an open covering of Cact^1.

3. For $U(c, \varepsilon), U(c', \varepsilon') \in \mathcal{U}$ then to each $c'' \in U(c, \varepsilon) \cap U(c', \varepsilon')$ there exist an ε'' such that $c'' \in U(c'', \varepsilon'') \subset U(c, \varepsilon) \cap U(c', \varepsilon')$.

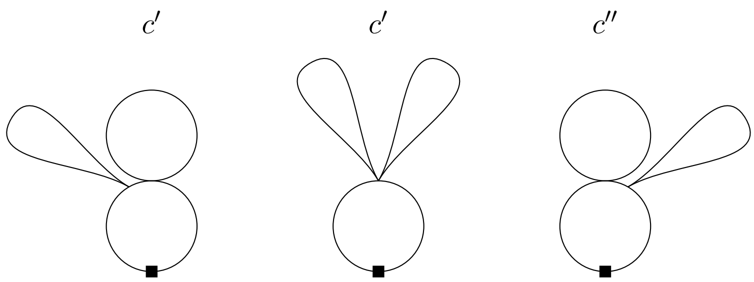
Proof. The first claim follows from a homotopy, $H \colon U(c, \varepsilon) \times I \to U(c, \varepsilon)$, given as: We take a point $c' \in U(c, \varepsilon)$ with corresponding topological type τ'. And then we think of $c'(t)$ as the lift to $\Delta(\tau')$ that is $\varepsilon - t/\varepsilon$ away from the lift of c in $\Delta(\tau')$. Now this straight lined path of $c'(c)$ contracts to c. Note how this visualize the sets $U(c, \varepsilon)$.

The second claim is trivial.

The third claim. Let τ, τ' and τ'' be the topological types of c, c' and c''. Further let $d_1 = d_{\tau''}(c, c'')$ and $d_2 = d_{\tau''}(c', c'')$. Then we choose an $\varepsilon'' < \min(\sqrt{\varepsilon^2 - d_1^2}, \varepsilon^2 - d_2^2)$. We now need to show that $U(c'', \varepsilon'') \subset U(c, \varepsilon) \cap U(c', \varepsilon')$, but that follows from the partial order between the topological types that has been induces by the standard partial order induced by inclusion of sets. Clearly $U(c'', \varepsilon'')$ is included in both $U(c, \varepsilon)$ and $U(c', \varepsilon')$. $\square$

Before we continue we will have more concrete look at the space $U(c, \varepsilon)$ in an example.

Example 3.71. We can think of the space $U(c, \varepsilon)$ as a "ball" around the cactus c by lifting to the $\Delta(\tau_c)$ level. Then we can think of balls around a point somewhere in the product of realizations of simplices. The conceptual tricky thing here, and the reason we take union over the topological types, is, that the balls can somewhat go through boarders of simplices. Consider the following illustration of three quite similar cacti:



They look close to be the same, right? The arc segment between the lobes that does not contain the global zero for c' and c'' is quite small, and collapsing that we have c as the middle between them. We would like this sense of *closeness* to be guiding. The problem is then, that the topological type and thereby the associated $\Delta(\tau)$-space is different from each other. The topological type of c' and c'' is up to labelling identical, but they differ from that of c. We have that $\Delta(\tau_c) = |\Delta^2| \times |\Delta^0| \times |\Delta^0|$ and $\Delta(\tau_{c'}) = |\Delta^1| \times |\Delta^0| \times |\Delta^0|$. For this reason we union over $\tau \in \mathcal{T}$ where the distance when lifted is small.

Lemma 3.72. *$(p'^{-1}(U(c,\varepsilon)), p'^{-1}(c))$ is homotopic to $(U(c,\varepsilon), c)$.*

We have an obvious map from $(p'^{-1}(U(c,\varepsilon)), p'^{-1}(c))$ to $(U(c,\varepsilon), c)$ that is p itself. Then a section, s, the other way could be a the map that attaches a lobe labelled $n+1$ at the global zero of c. As the global zero can be seen a canonical basepoint this is well defined. Clearly s is a section as $p \circ s = \mathbb{1}$. The goal is then to be convinced that $s \circ p \simeq \mathbb{1}$. This is how Kaufmann does it. Alternatively:

Proof. We have just found that $U(c,\varepsilon)$ is contractible, hence also the relative homotopy groups $\pi_i(U(c,\varepsilon), c)$ are trivial. In other words, if we can argue that $p'^{-1}(U(c,\varepsilon))$ retracts to $p'^{-1}(c)$ then the relative homotopy groups $\pi_i(p'^{-1}(U(c,\varepsilon)), p'^{-1}(c))$ are also trivial. This means they are weakly homotopic and as everything is CW complexes we can conclude that they are homotopic.

Think of $c \in \mathrm{Cact}^1(n)$. Then think of some $c' \in U(c,\varepsilon)$. Now by the same argument as when we define the homotopy in lemma 3.70 and idea as in example 3.71 then points in the fiber over c' can be retracted to a point in the fiber over c. □

3.4 Quasi-fibration

In this subsection we will introduce a concept like that of a fibration called *quasi-fibration*, but weaker: A fibration is a quasi-fibration. The conclusion of this subsection is, however, that the quasi-fibration induces a long exact sequence in homotopy like a fibration does.

On to the formalities. First the definition according to [DT58].

Definition 3.73. A surjective map $p\colon E \to B$, where it holds that for each $x \in B$, $y \in p^{-1}(x)$ and $i \geq 0$, then

$$p_*\colon \pi_i(E, p^{-1}(x)) \cong \pi_i(B, x),$$

is called a *quasi-fibration*.

And then the proposition that we need. An argument for it can be found in [May90, page 93].

Proposition 3.74. *A quasi-fibration, p as in the above definition, admits a long exact sequence in homotopy:*

$$\cdots \to \pi_{i+1}(B,x) \to \pi_i(p^{-1}(x), y) \to \pi_i(E,y) \to \pi_i(B,x) \to \cdots \to \pi_0(B,x).$$

Now Dold and Thom continues with the following criteria for quasi-fibration, that we, while following Kaufmann, steered to satisfy in the previous subsection and as we will exploit in the following subsection.

Definition 3.75. Let $p\colon E \to B$ be a map. Let a subset $U \subset B$ having the property that both $U \subset p(E)$ and for each $x \in U$, $y \in p^{-1}(x)$ and $i \geq 1$ then

$$p_*\colon \pi_i(p^{-1}(U), p^{-1}(x)) \cong \pi_i(U, x).$$

Then U is called *distinguished*.

We can re-phrase this as: U is called distinguished if the restriction of p by $p^{-1}(U)$, $p_U\colon p^{-1}(U) \to U$, is a quasi-fibration. Both Dold & Thom and May formulate it as such, and the word distinguished is borrowed form May as a translation of the German word *ausgezeichnet* used in [DT58, definition 2.1].

Then the criteria we were seeking, [DT58, Satz 2.2]:

Proposition 3.76. *Let $p\colon E \to B$ be map and let $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ be a family of distinguished open sets that has the following properties:*

- *The family $\mathcal{U}$ is an open cover of B and*
- *for $U_k, U_m \in \mathcal{U}$ then to each $x \in U_i \cap U_k$ there exists a $U_m \in \mathcal{U}$ such that $x \in U_m \subset U_i \cap U_k$,*

then B is distinguished, thus p is a quasi-fibration.

3.5 Homotopy type of Cacti

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Proposition 3.77. *The forgetful maps $p\colon \mathrm{Cact}(n+1) \to \mathrm{Cact}(n)$ and $p\colon \mathrm{Cact}(n+1) \to \mathrm{Cact}(n)$ are quasi-fibrations.*

Proof. Due to lemma 3.70 and 3.72 it follows from proposition 3.76 that p' is a quasi-fibration.

Take a point $c = (c', \mathbf{r}) \in \mathrm{Cact}(n)$. By the definitions of c and Cact^1 we have that

$$\pi_i(\mathrm{Cact}(n+1), p^{-1}(c)) = \pi_i(\mathrm{Cact}^1(n+1) \times \mathbb{R}^{n+1}_{>0}, p^{-1}(c', \mathbf{r})).$$

As $\mathbb{R}^{n+1}_{>0} \simeq *$ then $\mathrm{Cact}^1(n+1) \times \mathbb{R}^{n+1}_{>0} \simeq \mathrm{Cact}^1(n+1)$ also $p^{-1}(c', \mathbf{r}) \simeq p'^{-1}(c')$, so

$$\pi_i(\mathrm{Cact}^1(n+1) \times \mathbb{R}^{n+1}_{>0}, p^{-1}(c', \mathbf{r})) = \pi_i(\mathrm{Cact}^1(n+1), p'^{-1}(c').$$

As p' is a quasi-fibration, then per definition 3.73,

$$\pi_i(\mathrm{Cact}^1(n+1), p'^{-1}(c') = \pi_i(\mathrm{Cact}^1(n), c').$$

Finally we can get back to the domain of Cact: As $(\mathrm{Cact}^1(n), c')$ is homotopic to the pair $(\mathrm{Cact}(n), (c', \mathbf{r})) = (\mathrm{Cact}(n), c)$.

So all in all we have that

$$\pi_i(\mathrm{Cact}(n+1), p^{-1}(c)) = \pi_i(\mathrm{Cact}(n), (c))$$

that by definition 3.73 means that also p is a quasi-fibration. □

Proposition 3.78. *For each n the space* $\mathrm{Cact}(n)$ *is* $K(PB_n, 1)$.

Proof. By proposition 3.77 we have that $p\colon \mathrm{Cact}(n+1) \to \mathrm{Cact}(n)$ is a quasi-fibration. Then by proposition 3.74 p induces a long exact sequence in homotopy. Take $\pi_i(p^{-1}(c)) = \pi_i(\bigvee_n S^1)$ as the fiber over c, then we get:

$$\cdots \to \pi_i(\bigvee_n S^1) \to \pi_i(\mathrm{Cact}(n+1)) \xrightarrow{p_*} \pi_i(\mathrm{Cact}(n)) \to \pi_{i-1}(\bigvee_n S^1) \to \\ \cdots \to \pi_0(\mathrm{Cact}(n)). \quad (3.4)$$

The higher homotopy groups of S^1 are trivial, that is $\pi_i(\bigvee_n S^1) = 0$ for $i \geq 2$. So for $i \geq 3$ we get a series of short sequences as

$$0 \to \pi_i(\mathrm{Cact}(n+1)) \to \pi_i(\mathrm{Cact}(n)) \to 0,$$

implying that $\pi_i(\mathrm{Cact}(n+1)) \cong \pi_i(\mathrm{Cact}(n))$ for $i \geq 3$.

We have dealt with $\pi_i(\mathrm{Cact}(n)$ for all $n \geq 1$ and $i \geq 3$. Now we will look into the $i = 2$ case. Analogous to the argument in the proof of proposition 2.16 we see that $\mathrm{Cact}(1) \simeq *$ and $\mathrm{Cact}(2) \simeq S^1$, that implies that $\pi_2(\mathrm{Cact}(1)) = 0$ and $\pi_2(\mathrm{Cact}(2)) = 0$. When we plug that into the relevant part where $i = 2$ and $n = 2$ of the long exact sequence in (3.4) we get

$$\begin{tikzcd} 0 \arrow[r, "f"] & \pi_2(\mathrm{Cact}(3)) \arrow[r, "g"] & \pi_2(\mathrm{Cact}(2)) \arrow[r] \arrow[d, equal] & \pi_1(\bigvee_2 S^1) \\ & & 0 & \end{tikzcd}$$

which is exact at $\pi_2(\mathrm{Cact}(3))$ hence $0 = \mathrm{img}(f) = \ker(g) = \pi_3(\mathrm{Cact}(3))$. Then by induction over n we get trivial groups all the way, so $\pi_2(\mathrm{Cact}(n)) = 0$ for $n \geq 1$.

Summing up we have $\pi_i(\mathrm{Cact}(n)) = 0$ for $i \geq 2$ and $n \geq 1$, and hence lack to deal with the case where $i = 1$, which we will do in the following.

We aim to show that $\pi_1(\mathrm{Cact}(n)) = PB_n$ for all n. As a start we see that $\pi_1(\mathrm{Cact}(1)) = 0 = PB_1$ and $\pi_1(\mathrm{Cact}(2)) = \mathbb{Z} = PB_2$. We want to show it by induction over n. So we want to show that if we assume $\pi_1(\mathrm{Cact}(n)) = PB_n$ then also $\pi_1(\mathrm{Cact}(n+1)) = PB_{n+1}$. As the tool to do that observe the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(PB_{n+1} \to PB_n) \arrow[r] \arrow[d] & PB_{n+1} \arrow[r] \arrow[d] & PB_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(\mathrm{Chord}(c_n)) \arrow[r] & \pi_1(\mathrm{Cact}(n+1)) \arrow[r] & \pi_1(\mathrm{Cact}(n)) \arrow[r] & 0
\end{tikzcd}$$

The upper row in the diagram comes from the fibration $p\colon \mathrm{Conf}_{n+1} \to \mathrm{Conf}_n$ where $\bigvee_n S^1$ is the fiber. We have used this in the proof of proposition 2.16 previously as it is part of the long exact sequence (2.3). We can also think of left most part of that sequence as the kernel of the projection $PB_{n+1} \to PB_n$ - the free group on n variables denoted F_n. We display it as $\ker(PB_{n+1} \to PB_n)$ to get a better sense of the elements.

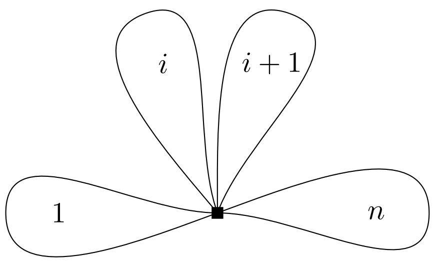
The bottom row comes from the long exact sequence (3.4), where we plug in that both $\pi_2(\mathrm{Cact}(n))$ and $\pi_0(\bigvee_n S^1)$ are trivial. In this case it makes more sense to display the fiber as the chord diagram over the basepoint as it emphasizes what kind of elements we are to talk about in the following. All in all, the diagram is a diagram of short exact sequences.

The groups PB_n and $\pi_1(\mathrm{Cact}(n))$ are assumed to be isomorphic. If we can specify an isomorphism between them, such that the right square of the diagram commutes, and also point out the isomorphism between the groups on the left side of the diagram such that also the left square commutes, then also the middle arrow will be an isomorphism, and that will prove the proposition.

In other words we have some constructing and checking to do.

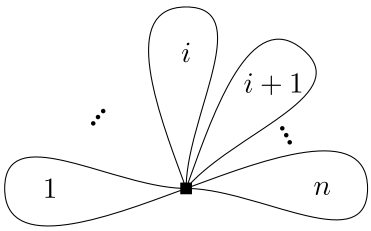
We are to map elements to the fundamental group of $\mathrm{Cact}(n)$, so we are to specify what will happen to homotopy classes of loops in $\mathrm{Cact}(n)$.

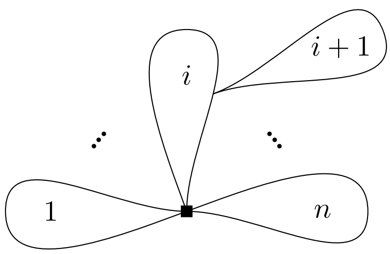
First pick a corolla cacti of n lobes where all the lobes are are of unit length and are labelled $1, 2, \ldots, n$, c_n, that sits in $\mathrm{Cact}(n)$. An illustration is given below. We let that be the basepoint of $\mathrm{Cact}(n)$.

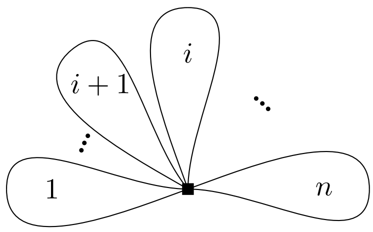


Then by a path in $\mathrm{Cact}(n)$ we will understand series of deformation of a cacti, one for each $t \in I$, such that it complies with the standard notion of a path. In this way we can reach any element in $\mathrm{Cact}(n)$ by a path from the basepoint by deforming the basepoint by moving the lobes around by moving there intersection points.

Now the path below, that is pictured below in a series from illustrations to be read from left to right, illustrates a path from the basepoint to the point that is the basepoint acted on by a transposition from the symmetric group, $\tau_i \in \Sigma_n$. We will call the path described α_i.







Then we notice that if we quotient $\mathrm{Cact}(n)$ by the symmetric group, we have $\mathrm{Cact}(n)/\Sigma_n$. We can think of $\mathrm{Cact}(n)/\Sigma_n$ as the analogous to the unlabelled configuration space. In $\mathrm{Cact}(n)$ the path α_i starts and ends at different points, but in $\mathrm{Cact}(n)/\Sigma_n$ α_i starts and ends at the same point and hence it is a loop in there.

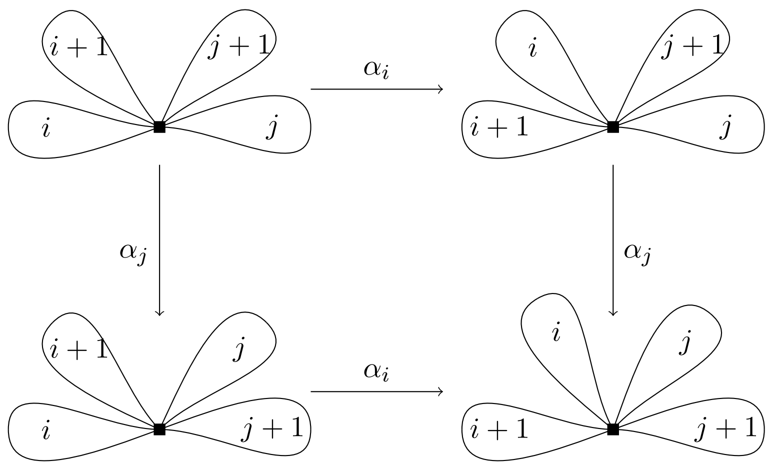
This inspires to claim that the map that sends $\sigma_i \in B_n$ to $[\alpha_i] \in \pi_1(\mathrm{Cact}(n)/\Sigma_n)$ is an isomorphism. It is not the one we sought for, but we will deal with that later. First we need to see that this map is well-defined. We have just seen how α_i can be thought of as a generator of the braid group and thus describes the braid action on $\mathrm{Cact}(n)/\Sigma_n$. Then what we need to do is to check that the relations on the generators of the braid group are respected when mapped into paths in $\mathrm{Cact}(n)/\Sigma_n$. Hence we want to check that following relations on homotopy classes

(i) $[\alpha_i][\alpha_j] = [\alpha_j][\alpha_i]$ for $|i - j| \geq 2$ and

(ii) $[\alpha_i][\alpha_{i+1}][\alpha_i] = [\alpha_{i+1}][\alpha_i][\alpha_{i+1}]$

holds.

For the first requirement to be satisfied observe the following illustrative diagram of loops in $\mathrm{Cact}(n)/\Sigma_n$ where $j = i + 2$:



The relation must hold when the distance between i and j are greater than or equal to 2. So the illustration above deals with the border case. But for greater distance between i and j lobes between the i'th and the j'th will not be touched at all, so as such the illustration scales.

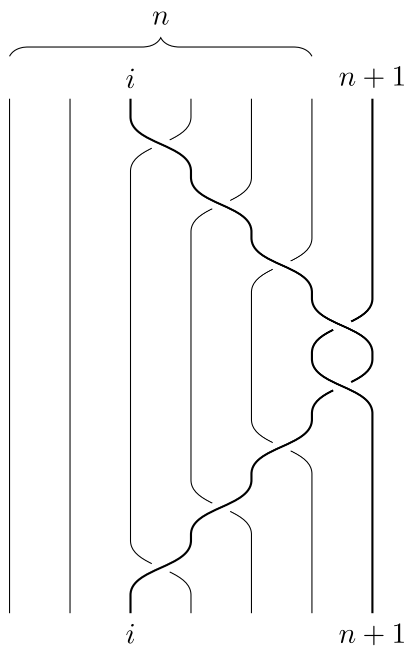
That the second requirement, (ii), is satisfied is verified in the following illustration that shows that composition of paths are path-homotopic as both paths from c_n to $\tau_i\tau_{i+1}\tau_i(c_n)$ and $\tau_i\tau_{i+1}\tau_i(c_n)$ ends in the same point.

$$\begin{tikzcd}
 & \tau_i(c_n) \arrow[r, "\tau_i(\alpha_{i+1})"] & \tau_i\tau_{i+1}(c_n) \arrow[dr, "\tau_i\tau_{i+1}(\alpha_i)"] & \\
c_n \arrow[ur, "\alpha_i"] \arrow[urr] \arrow[dr, "\alpha_{i+1}"'] & & & \tau_i\tau_{i+1}\tau_i(c_n) \\
 & \tau_{i+1}(c_n) \arrow[r, "\tau_{i+1}(\alpha_i)"'] \arrow[urr] & \tau_i\tau_i(c_n) \arrow[ur, "\tau_{i+1}\tau_i(\alpha_i)"'] &
\end{tikzcd}$$

In conclusion it is well-defined. Also we see that it is surjective and the inverse is also surjective. Any cacti in $\mathrm{Cact}(n)/\Sigma_n$ can be hit by a suitable composition of generators of the braid group. And then, as seen, we can hit all generators in the braid group by paths of cacti that satisfy the braid relations.

We can now go on to define the map from PB_n to $\mathrm{Cact}(n)$ from this isomorphism. Since $p\colon B_n \to \Sigma_n$ is surjective, then Σ_n is isomorphic $B_n/\ker p = B_n/PB_n$, thus we can quotient the isomorphism by Σ_n to induce an isomorphism from $PB_n \to \pi_i(\mathrm{Cact}(n))$. This is the same as restricting to PB_n, so we map PB_n generator $\chi_{i,j}$ to $[\alpha_{i,j}] \in \pi_i(\mathrm{Cact}(n))$. See definition 2.14 for the specification of the pure braid group. The middle map is then defined as the map that sends generator $\chi_{i,j+1}$ to $[\alpha_{i,j+1}] \in \pi_i(\mathrm{Cact}(n+1))$, and these will make the right square commute by assumption.

On to the left square. Take an element $\chi_{i,n+1} \in \ker(PB_{n+1} \to P_n)$, as illustrated below.



Then when we map it along the inclusion into PB_{n+1} and then down by the map just described then we get to same point $[\alpha_{i,n+1}]$. When we map down to $\mathrm{Chord}(c_n)$, we need first to give the isomorphism. The map given by $\chi_{i,n+1} \mapsto [\alpha_{i,n+1}]$ is exactly that isomorphism as generators of $pi_1(\mathrm{Chord}(c_n))$ are exactly the paths $\alpha_{i,n+1}$ as we recall that the fiber is the cacti with the $n+1$th lobe attached. And $\alpha_{i,n+1}$ is the path where the $n+1$ lobe loops around the i'th lobe. So when we further include that in $\pi_1(\mathrm{Cact}(n+1))$, we get to the same point as the other way around. Hence it commutes.

□

3.6 Equivalence with Little Disk

Now we have gone through lengths of issues with the Cact operad. In this final part of the thesis I will wrap it up to conclude that the *Disk* operad and the Cact operad are equivalent – as operad.

We have just seen that they are homotopic, since also Cact is $K(PB_n, 1)$, and they both admit CW complex structure. To ever hope for a morphism of operads between them to also be an operad equivalence, then that is a necessary condition. We will, of cause, make use of the recognition principle to establish this statement, and not give a direct morphism. We will use the same procedure as we went through in detail in proposition 2.28 and 2.29, hence therefore also be brief here.

To apply the recognition principle we need to have a suboperad CC of Cact that is E_1. This first fact is show in lemma 3.47 and the later in proposition 3.48.

As we let $\widetilde{\mathrm{Cact}}$ be the universal cover of Cact we can thus use the CC as a basepoint such that we can lift the operadic composition on Cact to a composition on $\widetilde{\mathrm{Cact}}$. As we have seen in the proof of proposition 3.78, that from the Σ action on Cact we can construct paths in Cact that correspond to the generators of the braid group. And hence be the path lifting property we can lift those to a braid action on $\widetilde{\mathrm{Cact}}$. The situation can thus be illustrated in the following diagram, that as such is fundamental for this thesis:

$$\begin{tikzcd} B_n \arrow[r, phantom, "\curvearrowright"] & \widetilde{\mathrm{Cact}}(n) \arrow[d] & \simeq * \\ \Sigma_n \arrow[r, phantom, "\curvearrowright"] & \mathrm{Cact}(n) & = \widetilde{\mathrm{Cact}}(n)/PB_n = K(PB_n, 1) \end{tikzcd}$$

This means that the universal cover of the Cact operad admits a B_∞ operad structure, and that means by theorem 2.22 – or just proposition 2.27 – and 2.29 that:

Theorem 3.79. *The* Cact *and* Disk *operads are equivalent as operads.*

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